University of São Paulo
“Luiz de Queiroz” College of Agriculture

**The impact of choosing different meshes under INLA/SPDE
framework for geostatistical modelling**

Ana Julia Righetto

Thesis presented to obtain the degree of Doctor in Science. Area: Statistics and Agricultural Experimentation

Piracicaba
2017

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Degree in Statistics

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To my parents,
Edison Righetto and **Luzia Francisca Pedrazzi Righetto**, for all support and encouragement.

To my boyfriend,
Thiago Gentil Ramires for always being by my side.

To them,
I dedicate this work.

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RESUMO

O impacto na escolha de diferentes malhas em modelagem geoestatística sob a abordagem INLA/SPDE

Métodos de estatística espacial são amplamente utilizados, uma vez que várias áreas do conhecimento, como ciências ambientais, geologia, agronomia, entre outros, envolvem a compreensão da distribuição espacial de processos a partir de dados referenciados espacialmente. Com o avanço dos Sistemas de Informação Geográfica e dos Sistemas de Posicionamento Global, esse uso foi ampliado. Muitos métodos utilizados na estatística espacial são computacionalmente exigentes e, portanto, o desenvolvimento de métodos mais eficientes recebeu muita atenção nos últimos anos. Um desenvolvimento importante foi a introdução do método de aproximação de Laplace aninhado integrado, capaz de realizar análises Bayesianas de forma mais eficiente. O uso deste método para dados geoestatísticos é comumente feito considerando a abordagem de equações diferenciais parciais estocásticas que requer a criação de uma malha que cobre a área de estudo. Este é o primeiro e um importante passo, pois todos os resultados dependerão da escolha desta malha. Como não existe uma maneira formal e direta de especificar a malha, investigamos possíveis diretrizes sobre como uma malha adequada é escolhida para um problema específico. Através de estudos de simulações, tentamos criar diretrizes para a construção da malha para conjunto de dados aleatórios, regulares e de cluster e aplicamos essas diretrizes em conjunto de dados reais.

Palavras-chave: Geoestatística; INLA; SPDE; Malhas

ABSTRACT

The impact of choosing different meshes under INLA/SPDE framework for geostatistical modelling

Spatial statistics methods are widely used since several areas of knowledge such as environmental sciences, geology, agronomy, among others, involve the understanding of the spatial distribution of processes from spatially referenced data. With the advancement of Geographic Information Systems and the Global Positioning Systems this use has been extended. Many methods used in spatial statistics are computationally demanding, and therefore, the development of more computationally efficient methods has received a lot of attention in recent years. One such important development is the introduction of the integrated nested Laplace approximation method which is able to carry out Bayesian analysis in a more efficient way. The use of this method for geostatistical data is commonly done considering the stochastic partial differential equation approach that requires the creation of a mesh overlying the study area. This is the first and an important step since all results will depend on the choice of this mesh. As there is no formal and close way to specify the mesh, we investigate possible guidelines on how a suitable mesh is chosen for a specific problem. Through simulations studies, we tried to create guidelines for the construction of the mesh for random, regular and cluster data set and we apply this guidelines in real data set.

Keywords: Geostatistics; INLA; SPDE; Meshes

1 INTRODUCTION

The understanding of the spatial distribution of data from phenomena occurring in space is a major challenge to clarify key issues in various areas of knowledge, such as health, environment, geology, agronomy, among many other areas. Such studies are becoming increasingly common, due to advances in Geographic Information Systems (GIS) and Global Positioning Systems (GPS), which are related to spatial information systems, as well as advances in computational procedures that enable and facilitate the analysis or representation of space and phenomena that occur in it. Thus, there is growing interest in analyzing and fit models to georeferenced data.

Spatial statistics methods are intended to measure properties and relationships, taking into account the spatial location of the studied phenomenon. This area of statistics is didactically divided into three sub-areas: geostatistics (point referenced data), area data (area referenced data) and point process. Different statistical methods are used to analyze and describe the data for each of these sub-areas. This work will emphasize the sub-area of geostatistics, which incorporates the interpretation of the statistical distribution and the spatial correlation of the samples. According to Matheron (1963), geostatistics is the application of the formalism of random functions for the recognition and estimation of natural phenomena.

In geostatistical analysis, the data consist of measurements $y_1, \dots, y_n$ obtained in locations $\mathbf{s}_1, \dots, \mathbf{s}_n$ sampled from a spatially continuous region $\mathbf{A} \subset R^d$. These measurements are associated to a spatial phenomenon, treated as the realization of a stochastic process $X(\mathbf{s})$, $\mathbf{s} \in R^d$ which is not directly observed, given by any set of random variables $Y(\mathbf{s})$, $\mathbf{s} \in R^d$, defined in a common probability space.

Considering a latent Gaussian random field $X(\mathbf{s})$ over the study region $\mathbf{A}$, it is necessary to perform a sampling, in n locations, to obtain values of the variable of interest, $\mathbf{Y}(\mathbf{s})$ of dimension n , from the values observed in the sampled locations $\mathbf{s} = (s_1, \dots, s_n)$. The basic model is given by

$$Y(\mathbf{s}) = \mu + X(\mathbf{s}) + \epsilon, \quad (1.1)$$

in which ϵ is the white noise, $\mu = \mathbf{D}\beta$, where $\mathbf{D}$ is a matrix of covariates and β is a q -dimensional vector of parameters and $X(\mathbf{s})$ is a Gaussian random field, with vector of null mean and covariance matrix Σ taking into account some correlation function.

With geostatistical techniques it is possible to describe the spatial dependence structure of the data, establishing a spatial correlation function that depends only on the distance of the position of the samples and represents the behavior of the spatial variability of the data, using a model. In order to analyze the spatial dependence of a random variable, the geostatistical model is based on the covariance or semivariance function and, according to Mello et al. (2005), these functions are important for the definition of the spatial variability structure and the interpolation of values by kriging, obtaining minimal errors in the interpolation and enabling the construction of thematic maps. The covariances are functions of the distance between observations and usual models for the covariance functions are the spherical, exponential, gaussian, Matérn, among others.

The log-likelihood for the model (1) is given by

$$L(\boldsymbol{\theta}) = -\frac{1}{2}(n \log(2\pi) + \log(|\Sigma|) + (y - \mu)' \Sigma^{-1} (y - \mu)).$$

The computations that involves the covariance matrix is costly as it depends on the covariance function, which depends on the distances of the locations. This cost also appears on the predictions, so it is necessary to choose wisely the covariance function.

According to Bolin and Lindgren (2013), traditional methods in spatial statistics were typically developed without any consideration of computational efficiency and, in many applications, the cost of taking measurements limits the sample size in which the required computational cost is not a problem.

However, with the increasing use of GIS and GPS, which produce large data sets, computational efficiency is of the utmost importance.

Sometimes the inference, as is usually done in the classical approach, may present problems especially in the estimation of variance components, since the inference process is based on the asymptotic distribution of the maximum likelihood estimators. As a solution to this, it is common to propose the use of the Bayesian approach, which allows the updating of information. However the Bayesian approach still requires the demanding computations related to the covariance matrix in the likelihood and specification of more realistic priori in order to make the model identifiable. In recent years, a new method has been proposed by Rue, Martino and Chopin (2009), called Integrated Nested Laplace Approximation (INLA), which is a computationally effective alternative to methods based on Markov Chain Monte Carlo (MCMC) algorithms.

According to Lindgren and Rue (2015), INLA is designed for latent Gaussian models, a very broad and flexible class of models, ranging from a mixed (generalized) linear model to spatial and temporal-spatial models. In the spatial setup it can be combined with the approach of Stochastic Partial Differential Equation (SPDE) (Lindgren, Rue and Lindstrom, 2011), one can accommodate georeferenced data from all sub-areas of spatial statistics.

A Gaussian field (GF) is the key component in geostatistical modeling and it is important when specifying hierarchical spatial models. According to Lindgren, Rue and Lindstrom (2011), the covariance function specification of an intuitive interpretation of the field properties, as regards computational efficiency, impairs the GF because the cost of factoring dense matrices is cubic in dimension. When a GF satisfies the Markov property (conditional independence), it will be considered a Gaussian Markov random field (GMRF).

Consider the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, in which $\mathcal{V} = \{1, \dots, n\}$ and $\mathcal{E}$ is defined by $x_i \perp x_j | \mathbf{x}_{-i,j}$, meaning that there is no edge between the knots i and j . Rue and Held (2005) define a GMRF as a random vector $\mathbf{x} = (x_1, \dots, x_n)' \in R^n$ which follows a multivariate normal distribution and satisfies the Markov property. Also, the GMRF, with mean vector $\boldsymbol{\mu}$ and precision matrix $\mathbf{Q} > 0$, in which $\mathbf{Q} = \boldsymbol{\Sigma}^{-1}$, will be with respect to the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with the condition that $Q_{ij} \neq 0 \Leftrightarrow i, j \in \mathcal{E}$ for all $i \neq j$ and that its density will be:

$$\Pi(\mathbf{x}) = (2\pi)^{-\frac{n}{2}} |\mathbf{Q}|^{\frac{1}{2}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})' \mathbf{Q}(\mathbf{x} - \boldsymbol{\mu})\right). \quad (1.2)$$

According to Lindgren, Rue and Lindstrom (2011), GMRF is a discretely indexed Gaussian field in which the complete conditionals of $\Pi(x_i | \mathbf{x}_{-i})$, which $\mathbf{x}_{-i}$ represents all elements of $\mathbf{x}$ except the i th, depend only on a set of neighbors ∂_i for each place i , $i = 1, \dots, n$ and this generates a computational gain due to the zero standard in the precision matrix $\mathbf{Q}$. The GMRF is widely used in hierarchical models because it allows stochastic dependence among a set of unknown parameters. This hierarchy is done in three steps, in which the unknown hyperparameters $\boldsymbol{\theta}$ specify a GMRF. In this way, $\mathbf{x}$ is connected to the data $\mathbf{y}$ and it is usually assumed that $\mathbf{y}$ is conditionally independent of $\mathbf{x}$. Considering a simple case, each y_i depends only on x_i and, in this way, the two vectors have the same dimension. The three steps of the hierarchy are given by:

$$\begin{aligned} \boldsymbol{\theta} &\sim \Pi(\boldsymbol{\theta}) \\ \mathbf{x} &\sim \Pi(\mathbf{x} | \boldsymbol{\theta}) \\ y_i &\sim \Pi(y_i | x_i), \end{aligned} \quad (1.3)$$

thus, a posteriori distribution is given by $\Pi(\mathbf{x}, \boldsymbol{\theta} | \mathbf{y}) \propto \Pi(\boldsymbol{\theta}) \Pi(\mathbf{x} | \boldsymbol{\theta}) \prod_{i=1}^n \Pi(y_i | x_i)$.

Empirically, Rue and Tjelmeland (2002) have demonstrated that it can be found a GMRF which can approximate a GF for most of the common covariance functions used in geostatistical analysis and have proposed using GMRF instead of GF precisely because of computational issues. This proposal has shown problems since the adjustment of GMRF to GF was restricted to the regular grid. In this way, this approach implies some restrictions because, according to Lindgren, Rue and Lindstrom (2011), the modeling must be done with a GF in a set of locations $\{s_i\}$ and one must construct a GF with matrix of covariance Σ of which there is a GMRF with local neighborhood and precision matrix Q , such that Q^{-1} is close to Σ and computations should be performed with the GMRF by numerical methods for sparse matrices.

Following the requirements from the proposal of Rue and Tjelmeland (2002), Lindgren, Rue and Lindstrom (2011) have demonstrated that these it can be satisfied for some GF members when using the Matérn covariance function, where the GMRF representation is explicit. The GMRF construct can be done using a certain SPDE that has GF with Matérn covariance function as the solution when triggered by a Gaussian white noise. The result is given by a basic function representation with parts-based linear functions and Gaussian weights with Markov dependencies determined by general domain triangulation. This triangulation is called mesh.

Currently, the INLA method has received considerable importance due to its precision and computational speed. When used in conjunction with SPDE for geostatistical models, it is necessary to generate a mesh, formed by junctions of triangles, under the study region and, in addition, the precision of the parameter estimates, as well as the prediction of the final model, will depend on this mesh. A set of attributed needs to be carefully chosen on order to create a suitable mesh: set values for the edges of the triangles inside the mesh (which encompasses the study region), for the edges of the outer mesh (outside the study region, to avoid edge effect) and for the smallest permitted distance between the edge junctions (knots). Each change in these values implies more or less knots in the mesh, which implies more or less computational burden.

In this work we named by a the maximal allowed lengths of triangle edges in the inner domain, by b the maximal allowed lengths of triangle edges in the outter domain and by c the shortest distance allowed between points. Different combinations of this three values, generate different meshes, as we can see in Figure 1.1. where meshes were generate with $b = 0.03$ and with $a = (0.02, 0.08, 0.14)$ and $c = (0.002, 0.03)$ As we increase the value of a and c , we get meshes with less triangle and different mesh gives different results in the analysis.

Bolin and Lindström (2014), raise concerns about the importance of using the appropriate mesh and how a simple change in the mesh can affect the result of the analysis. Then, following up on these issues, the main objective here is to try to find a way to create the more suitable mesh for different types of georeferenced data (regular grid, irregular grid or data with clusters), through a simulation study.

In this context, in Chapter 2 we reports results of a simulation study, for random data with different sample sizes, to investigate the most appropriate mesh, in which the meshes were generated taking into account the sample locations and changing some parameters for the creation. In Chapter 3, we fixed the sample size and focus on different types of data, being them regular, irregular and cluster data. For each type of data, were considered different parameter values to generate the meshes as well as different ways to define the knots. Chapter 4 shows an overall conclusion.

References

Bolin, D.; Lindgren, F. A comparison between Markov approximations and other methods for large spatial data sets. **Computational Statistics & Data Analysis**, v. 61, p. 7-21, 2013.

Figure 1.1. Generate meshes with $b = 0.03$ different values of (a, c) given by I) $(0.02, 0.002)$, II) $(0.08, 0.002)$, III) $(0.08, 0.03)$ and IV) $(0.14, 0.03)$.

Bolin, D. Lindstrom, J. Spatial Statistics using R-INLA and Gaussian Markov random fields. **Pan-American Advanced Study Institute**, 15:29:57Z, rev.3691, 2014.

Lindgren, F.; Rue, H. Bayesian Spatial Modelling with R-INLA. **Journal of Statistical Software**, v. 63, 2015.

Lindgren, F.; Rue, H.; Lindstrom, J. An explicit link between Gaussian fields and Gaussian Markov random fields: the stochastic partial differential equation approach. **Journal of the Royal Statistical Society**, v. 73, p. 423-498, 2011.

Matheron, G. Principles of geostatistics. **Economic Geology**, Austin, v. 58, p. 1246-1266, 1963.

Mello, J. M.; Batista, J. L. F.; Ribeiro Jr, P. J.; Oliveira, M. S. Ajuste e seleção de modelos espaciais de semivariograma visando à estimativa volumétrica de *Eucalyptus grandis*. **Scientia Florestalis**, Piracicaba, v.1, n. 69, p. 25-37, 2005.

Rue, H.; Martino, S.; Chopin, N. Approximate Bayesian inference for latent Gaussian models by using integrated nested Laplace approximations (with discussion). **Journal of the Royal Statistical Society - Series B**, v.71, Part 2, p.319-392, 2009.

Rue, H.; Tjelmeland, H. Fitting Gaussian Markov random fields to Gaussian fields. **Scand. J. Statist**, v. 29, p. 31-50, 2002.

2 ON THE CHOICE OF THE MESH FOR THE ANALYSIS OF GEOSTATISTICAL DATA USING R-INLA

Abstract: Spatial statistics are useful to measure properties and relationships of a geographically studied phenomenon. Many methods used in spatial statistics are computationally demanding, and therefore, the development of more computationally efficient methods has received a lot of attention in recent years. One such important development is the introduction of the integrated nested Laplace approximation method which is able to carry out Bayesian analysis in a more efficient way. The use of this method for geostatistical data is commonly done considering the SPDE approach that requires the creation of a mesh overlying the study area. This is an important step in the analysis since the obtained results depend on the choice of this mesh. Here, the impact of the mesh on inference and prediction is investigated through simulation studies. As there is no formal procedure to specify the mesh, we investigate possible guidelines on how an optimal mesh is chosen. Through a simulation study, with random data set, different meshes were created, considering the sample locations as starting point for the knots of the mesh. An application using rainfall data in Paraná state, Brazil, is presented.

Keywords: Geostatistics; Integrated nested Laplace approximation; Mesh; Stochastic partial differential equation.

2.1 Introduction

Models in spatial statistics are used to measure properties and relationships of a geographically measured outcome, explicitly taking into account the spatial locations of the studied phenomenon. Traditional methods in spatial statistics were developed without the consideration of computational efficiency, as in many applications the sample size was limited due to the cost involved in obtaining the measurements (Bolin and Lindgren, 2013). Nowadays, however, with the increasing use of geographic information systems and global positioning systems, large data sets are becoming available, and computational efficiency has become extremely important.

Geostatistical data are widely analyzed using the kriging method, an optimal interpolation method employing the spatial autocorrelation to derive predictions at unmeasured locations (Gelfand et al., 2010). A popular choice to estimate the statistical dependence is the Matérn family of covariance functions, which includes, for example, the exponential and Gaussian covariance functions, as particular cases. For inference, it is common to use the Bayesian framework, which is based on the observed data only and easily allows for sequential updating of information. Sampling methods such as Markov Chain Monte Carlo (MCMC) are well known in Bayesian inference, but demand intense computation time.

As an alternative to the computational intensive MCMC methods, Rue, Martino and Chopin (2009) proposed the use of integrated nested Laplace approximations (INLA). INLA is an approximate method designed for inference of latent Gaussian models, a very wide and flexible class of models ranging from (generalized) linear mixed model to spatial and spatio-temporal models (Lindgren and Rue, 2015). When combined with stochastic partial differential equation (SPDE) (Lindgren, Rue and Lindstrom, 2011), it can accommodate georeferenced data from all sub-areas of spatial statistics (geostatistics, lattice data and point processes). Recently, INLA has received considerable attention due to its accuracy and computational speed (Karagiannis et al., 2013; Lindgren, 2012).

The SPDE approach approximates the Gaussian field using a Gaussian Markov random field. For geostatistical data, the spatial field is described using a weighted sum of piece-wise linear basis function that is usually defined from a triangularisation of the study area, called the mesh. The accuracy of the estimates as well as the predictions will depend on the mesh specification. The idea is that smaller

triangles will increase the accuracy, though when the spatial range is larger, larger triangles can be used. The mesh can be constructed using different principles. A common method, using R-INLA, is to use the locations of the measurements as initial triangulation vertices, and add or remove knots according to the size for the edges of the triangles of the inside mesh (which includes the study region), for the edges of the outer mesh (outside the study area, to avoid border effects) and to the shortest distance allowed between the junctions of the edges (knots). Each combination of these three values implies a certain number of knots on the mesh, and more knots can increase in the computational cost. While it has been indicated that the choice of the mesh must be done carefully (Lindgren and Rue, 2015), no conclusive way to create the best mesh has been described.

As the analysis depends on the chosen mesh, the objective of this work is to extensively study the parameters for the creation of the mesh, to see their impact on the analysis and set guidelines to construct the optimal mesh in two dimensional spatial domains, for irregular geostatistical data, solving some issues raised by Bolin and Lindstrom (2014). In Section 2.2, we give a brief introduction to INLA and the SPDE approach and discuss some criteria to evaluate the mesh. A simulation study is performed in Section 2.3 and an application with real data set is provided in Section 2.4. A discussion is provided in Section 2.5.

2.2 INLA and SPDE

INLA is a deterministic method to perform Bayesian inference of latent Gaussian models, avoiding computational intensive Markov chain Monte Carlo simulations (Rue, Martino and Chopin, 2009). The method is designed for a large class of Gaussian Markov random fields (GMRF), including many of the hierarchical models with latent Gaussian processes. If $\mathbf{y}$ represents the data, $\mathbf{x}$ the GMRF and $\boldsymbol{\theta}$ the vector of hyperparameters, the idea of INLA is to approximate the posterior density $\Pi(\boldsymbol{\theta}|\mathbf{y})$ via Gaussian approximation of the posterior of the latent field $\Pi(\mathbf{x}|\boldsymbol{\theta}, \mathbf{y})$ and a Laplace approximation in the latent field marginal posterior calculations. Details can be found in Rue, Martino and Chopin (2009).

Lindgren, Rue and Lindstrom (2011) showed that the use of INLA is not restricted to discrete spatial domains, given there is a connection between continuous domains and Markov models. Indeed, a large class of random field models can be expressed as a solution of a continuous domains stochastic partial differential equation (SPDE). Whittle (1954) showed that the Matérn covariance function can be seen as a solution of the SPDE. This connection allows one to carry out the calculations for a continuous random field using the (discrete) GMRF representation. This is done by approximating the spatial field as a basis function representation:

$$\mathbf{x}(s) = \sum_{k=1}^K \Psi_k(s) \mathbf{x}_k, \quad (2.1)$$

where $\Psi_k(s)$ are piece-wise linear basis function defined on a triangulation of the study domain and $\{\mathbf{x}_1, \dots, \mathbf{x}_K\}$ are Gaussian weights chosen to approximate the solutions to the SPDE on the domain. The triangulation consists in subdividing the study area into a set of non-intersecting triangles (Cameletti et al., 2012), with linear basis function defined with barycentric interpolation. This SPDE/GMRF analysis is available in the R-package INLA (Rue, Martino and Chopin, 2009).

The first step of the analysis is the construction of a triangulated mesh on which the SPDE/GMRF representation is built (Lindgren, 2012). As an initial step, we can choose to place the triangle vertices at the locations where measurements were taken, and after that, add or remove additional vertices, creating the triangulation for the analysis. As the solution may present a border effect, it is necessary to extend the mesh also outside the study area to avoid it. According to Lindgren (2012), the effect in constructing triangles is that we can obtain small ones (and so obtain high accuracy to represent the field) where the

locations are dense, and bigger ones when the locations are sparse. As the analysis depends on the mesh, it is very important to create a mesh suited for the data at hand.

In the R-INLA implementation, the function `inla.mesh.2d()` is available to create the mesh. Several parameters in this function can be chosen as to construct a mesh for the problem at hand, e.g. the `max.edge` parameters define the maximal allowed lengths of triangle edges in the inner domain and in the outer extension, the `cutoff` parameter corresponds with the shortest distance allowed between points (Krainski and Lindgren, 2014; Krainski and Rue, 2013). A change in one of these values, will create a different mesh, making it more or less dense (with more or less knots).

We will work with the above three arguments for building the convex mesh: the maximal allowed edge length in the inner domain (a) and outer domain (b), and the shortest allowed distance between points (c). We will always consider larger values of the maximal allowed edge length for the outer domain as for the inner domain ($b > a$), as the outer domain is needed only to avoid border effects.

Figure 2.1. Example of meshes with $b = 0.40$ and (a, c) given by (I) $(0.03, 0.001)$, (II) $(0.07, 0.001)$, (III) $(0.11, 0.001)$, (IV) $(0.07, 0.01)$, (V) $(0.07, 0.03)$, (VI) $(0.07, 0.06)$

For exploratory reasons, we illustrated how the choice of these parameters determine the aspect of the mesh in Figure 2.1. We consider 100 randomly generated locations of a spatial process on the unit square. Starting with these locations as initial vertices of the triangulation, we create different meshes, depending on the parameters a , b and c . In the left panel, we fix the maximal allowed edge length ($b = 0.4$) and shortest distance allowed between points ($c = 0.001$), but change the value of the maximal allowed edge length in the inner domain ($a = (0.03, 0.07, 0.11)$). By increasing the maximal allowed edge length, the triangulation becomes less dense, with the number of vertices decreasing from 3701 to 890 and 633, respectively. The right panel in Figure 2.1 shows the impact of increasing the minimal allowed distance between vertices ($c = 0.01, 0.03, 0.06$). The edge length for the inner and outer domain was kept fix ($a = 0.07, b = 0.4$). While changing c from 0.001 (panel II) to 0.01 (panel IV) has only little impact, a further increase of c has a bigger one, changing the number of vertices to 859, 753 and 266, respectively.

By increasing the value of c , less clustered vertices are observed, making the mesh more regular over the whole domain. Even a small change in one parameter to construct the mesh can generate a complete different mesh and the results will be affected. Then, the focus of this survey is to investigate the impact of the mesh in the analysis.

2.2.1 Model fitting and predictive information criteria

To evaluate and compare the performance of different meshes, different predictive information criteria can be used. Three information criteria are considered in this study: the deviance information criterion (DIC), the Watanabe-Akaike information criterion (WAIC) and the conditional predictive ordinate (CPO). In Section 2.3, we investigate the performance of these criteria to perform mesh selection. These criteria are briefly explained below.

The deviance information criterion, described by Spiegelhalter et al. (2000), is defined as

$$DIC = -2 \log p(\mathbf{y}|\hat{\boldsymbol{\theta}}) + 2p_{DIC},$$

where $2 \log p(\mathbf{y}|\hat{\boldsymbol{\theta}})$ is the posterior expected deviance measuring the suitability of the model and p_{DIC} the estimated effective number of parameters of the model, representing the complexity of the model. In this way, DIC takes into consideration both the suitability and complexity of the model.

Proposed by Watanabe (2010), WAIC can be viewed as an improvement on the DIC for Bayesian models (Vehtari and Gelman, 2014) and is given by

$$WAIC = -2 \log \int p(\mathbf{y}|\boldsymbol{\theta}) \Pi(\boldsymbol{\theta}|\mathbf{y}) d\boldsymbol{\theta} + \sum_{i=1}^n Var(\log(p(y_i|\boldsymbol{\theta}))).$$

Similar as with DIC, the first term measures the predictive performance, and the second term penalizes for model complexity. Compared to DIC, WAIC averages $p(\mathbf{y}|\boldsymbol{\theta})$ over the posterior distribution of $\boldsymbol{\theta}$, whereas DIC conditions on the point estimate of $\boldsymbol{\theta}$. The complexity term is a sum of the variance for each data point separately, yielding more stable results as DIC (Gelman, Hwang and Vehtari, 2014).

The conditional predictive ordinate is a cross-validation method, estimating the leave-one-out predictive distribution (Geisser and Eddy, 1979). The posterior predictive distribution for an observation i can be expressed as

$$p(\mathbf{y}_i|\mathbf{y}_{-i}) = \int p(\mathbf{y}_i|\boldsymbol{\theta}) \Pi(\boldsymbol{\theta}|\mathbf{y}_{-i}) d\boldsymbol{\theta},$$

where $\mathbf{y}_i$ is the i th observation of $\mathbf{y}$ and $\mathbf{y}_{-i}$ is $\mathbf{y}$ without observation i . So the CPO is the probability a posteriori to observe y_i when the model is adjusted without y_i . A bigger value of CPO indicates a better adjustment. Instead of having to refit the model each time, an approximation of CPO can be obtained by sampling from the posterior distribution:

$$CPO_i = \frac{1}{T^{-1} \sum_{t=1}^T p(\mathbf{y}_i|\boldsymbol{\theta}^t)^{-1}},$$

where $\boldsymbol{\theta}^t$ is a sample from $\Pi(\boldsymbol{\theta}|\mathbf{y}_i)$ and T indicated the number of posterior samples. This is the inverse of the posterior mean of the inverses of the likelihood. This is an efficient method, but requires care since numerical problems in the calculations can occur (Held, Schrodle and Rue, 2010). However, when using INLA it is possible to verify if numerical problems did occur and recompute the CPO using a less efficient method (re-fitting the model for every observation).

In this work we used a summary by calculating the log-score, given by

$$\log CPO = -\frac{1}{n} \sum_{i=1}^n \log(p(\mathbf{y}_i|\mathbf{y}_{-i})).$$

This log-score is an estimator of the log marginal likelihood function and smaller values indicate better model performance. In contrast to DIC and WAIC, the log CPO score does not adjust for the complexity of the model.

2.3 Simulation Study

In this section, we first describe the simulation design, after which results are discussed. The first goal of the simulation study is to investigate the impact of the choice of the mesh on the inference (parameter estimation and prediction). The second goal is to investigate how we can select an optimal mesh for a particular data problem at hand.

2.3.1 Simulation Design

A simulation study, with 200 simulation runs, was set up to compare different meshes. Different simulation scenarios are considered. Spatial geostatistical processes with sample sizes of $n = 300, 100$ and 50 are generated. Observed locations are randomly generated on the unit square. A response vector $\mathbf{y}$ is generated according to the model $\mathbf{y} = \beta_0 + \beta_1 \mathbf{x} + \mathbf{s} + \mathbf{e}$, where $\mathbf{x}$ is a uniformly distributed covariate, $\mathbf{s}$ is a Matérn spatial random field with correlator function given by $M(\mathbf{h}|k, \phi) = \frac{1}{2^{k-1}\Gamma(k)} (\phi \|\mathbf{h}\|)^k K_k(\phi \|\mathbf{h}\|)$ and $\mathbf{e}$ is white noise. Parameters are fixed to $\beta_0 = 1, \beta_1 = 2$. The spatial variability of the Matérn spatial random field is set to $\sigma_s^2 = 0.7$ and Gaussian white noise variability to $\sigma_e^2 = 0.3$. The nominal range of the spatial random field is specified as $\phi^* = \frac{\sqrt{8(\alpha - \frac{d}{2})}}{k} = 0.20$, by setting the smooth parameter $k = 10$, the fractional operator order $\alpha = 1.5$ and $d = 2$, as the study region $\mathbf{A} \in R^d$. Note that the choice of $\alpha = 1.5$ in the Matérn spatial random field corresponds to an exponential random field (Gelfand et al., 2010). For the prior, we used the log-gamma distribution with parameters $c(1.0, 0.00005)$. To prevent effects caused by a single choice of coordinates, we conduct the simulation in five batches of 40 simulations and within one batch the coordinates are kept fixed.

To investigate the impact of meshes, different meshes are constructed for the generated processes. The initial triangulation taken is given by the locations of the generated process. Note that, as we keep the coordinates fixed within batches, also the meshes will remain the same within a batch. Different meshes are considered by allowing the `max.edge=(a,b)` and `cutoff=c` parameters to change. The set of values considered for the maximal allowed edge length in the inner domain is $a = (0.05, 0.09, 0.13, 0.17, 0.19, 0.20)$, for the maximal edge length in the outer domain is $b = (0.4, 0.3, 0.2)$ and for the shortest allowed distance between two points is $c = (0.001, 0.005, 0.01, 0.03, 0.05, 0.07, 0.10, 0.13)$. Combinations of these three values leads to a total of 144 different meshes. These values were chosen such that the value of a is smaller or equal to the range ϕ^* , and the value of b , controlling the outer boundaries to avoid boundary effects, should be larger than a . The results are shown for $b = 0.3$, as we notice that there is almost no impact for the different values of b .

Table 2.1 summarizes, for the different combination of the parameters a and c , the average number of vertices (rounded to integer) in the mesh, for a sample size of $n = 100$ and a maximal edge length in the outer domain of $b = 0.3$. A more dense triangulation is obtained with smaller values of both a and c . When the `cutoff` parameter is large, the number of vertices in the triangulation is lower than the number of observed locations. In addition, we observe that the influence of a decreases as c increases. Results in the next sections focus on the scenario of a sample size of $n = 100$. Results for $n = 50$ and $n = 300$ are given in the Appendix A.

2.3.2 Impact of mesh on inference

To investigate the impact of the mesh on the inference, we calculated, for each mesh, the coverage (based on the 95% Confidence Interval), bias and mean squared error (MSE) for each of the parameters in the model and for the predictions.

Table 2.2 summarizes the simulation coverage for the covariance function parameters ($\sigma_e^2, \sigma_s^2, k$ and ϕ^*) and for β_0 and β_1 , for meshes with varying maximal allowed edge length in the inner domain (a)

Table 2.1. Average number of knots for different combinations of the parameters a and c , with parameter $b = 0.3$ and sample size $n = 100$.

	$c=0.001$	$c=0.005$	$c=0.01$	$c=0.03$	$c=0.05$	$c=0.07$	$c=0.10$	$c=0.13$
$a=0.05$	1398	1394	1390	989	400	208	120	86
$a=0.09$	523	517	507	487	400	206	117	84
$a=0.13$	315	307	293	251	236	206	116	84
$a=0.17$	272	265	250	203	174	159	115	84
$a=0.19$	266	259	243	192	160	145	115	84
$a=0.20$	263	256	241	189	155	140	113	84

and varying shortest allowed distance between points (c), but with fixed maximal length in the external area (namely, $b = 0.3$). Figure A.1 in the Appendix A shows the coverage for all considered meshes.

Table 2.2. Coverage, in %, of the 95% credible intervals of the parameters β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* for different meshes with $b = 0.3$.

Mesh	c	a	Coverage in % for:					
			β_0	β_1	σ_e^2	σ_s^2	k	ϕ^*
1	0.001	0.05	90.50	90.00	13.50	52.50	30.00	30.00
2		0.09	92.50	90.00	8.50	17.50	21.50	21.50
3		0.13	93.00	90.50	6.50	9.00	19.00	19.00
4		0.17	92.50	90.00	6.50	8.00	21.00	21.00
5		0.19	92.50	90.50	8.00	9.00	21.50	21.50
6	0.005	0.20	92.00	90.00	7.50	8.50	21.50	21.00
7		0.05	88.50	85.00	17.00	56.50	34.00	34.00
8		0.09	89.50	84.00	13.00	26.50	26.00	26.00
9		0.13	89.50	84.00	12.00	16.00	25.50	25.50
10		0.17	89.50	84.00	12.00	16.00	27.00	27.00
11	0.01	0.19	89.00	84.50	12.00	15.00	26.50	26.50
12		0.20	88.50	84.50	12.50	15.50	26.50	26.00
13		0.05	92.00	86.00	36.50	70.50	52.00	52.00
14		0.09	94.00	90.00	38.50	50.00	48.50	48.50
15		0.13	92.50	89.50	37.50	39.50	46.50	46.50
16	0.03	0.17	92.50	89.00	39.00	38.00	47.00	47.00
17		0.19	92.50	89.00	37.00	37.00	48.00	48.50
18		0.20	92.00	89.00	37.50	37.50	48.50	48.00
19		0.05	95.50	90.00	79.00	91.00	89.50	88.50
20		0.09	97.00	92.00	83.50	94.00	92.50	92.50
21	0.05	0.13	98.00	92.50	81.50	94.00	91.50	92.00
22		0.17	98.00	93.50	80.50	92.00	89.50	90.00
23		0.19	97.50	94.00	78.00	90.50	90.00	90.50
24		0.20	97.50	94.00	78.50	91.00	90.50	90.50
25		0.05	97.50	91.00	83.50	96.00	91.00	91.00
26	0.07	0.09	98.50	92.00	80.50	96.00	90.00	90.00
27		0.13	99.50	92.50	74.00	97.50	89.50	89.50
28		0.17	99.50	93.00	66.00	97.50	88.50	88.50
29		0.19	99.50	93.00	67.00	96.00	88.00	88.00
30		0.20	99.00	93.50	65.50	96.00	88.50	88.50
31	0.10	0.05	99.00	92.50	60.50	96.00	87.00	86.50
32		0.09	99.50	92.00	59.00	95.50	86.50	86.50
33		0.13	99.50	92.50	59.50	95.50	86.50	86.50
34		0.17	99.00	94.00	44.00	94.50	82.00	82.00
35		0.19	99.00	92.50	39.50	96.00	85.00	85.00
36	0.13	0.20	99.50	92.50	36.50	96.50	84.00	84.00
37		0.05	99.50	93.50	20.50	93.50	83.50	83.50
38		0.09	99.50	92.50	19.50	92.50	82.00	82.00
39		0.13	99.50	93.50	19.00	93.50	82.50	82.50
40		0.17	99.50	93.50	19.50	93.00	84.00	84.00
41	0.20	0.19	99.50	93.50	19.50	92.50	85.00	85.00
42		0.20	99.50	93.00	18.50	93.00	85.50	85.50
43		0.05	98.50	93.50	2.50	94.50	86.00	86.00
44		0.09	98.50	93.00	2.00	94.50	89.00	89.00
45		0.13	98.50	93.50	3.00	94.00	89.50	90.00
46	0.20	0.17	98.50	93.50	3.00	94.00	89.50	90.00
47		0.19	98.50	93.00	3.00	94.00	89.50	90.50
48		0.20	98.50	93.50	3.00	94.50	89.50	90.00

It can be observed that the largest impact on the coverage is observed for the `cutoff` parameter c . The parameters β_0 and β_1 have good coverage specially for meshes with high values of c . Increasing the shortest allowed distance, improves the coverage for the parameters σ_s^2 , k and ϕ^* , no matter much the value of a . However, we also observe an undercoverage for the parameter σ_e^2 , especially if c gets too small or too large. Overall, we can also observe that an increase of the maximal allowed edge length in the inner domain (a) has a negative impact on the coverage of most parameters. These results show that, indeed, the choice of the mesh is very important to obtain good coverage for the covariance parameters. Mesh 20, with $a = 0.09$, $b = 0.3$ and $c = 0.03$, has the best coverage of σ_e^2 , k and ϕ^* , meshes 27 and 28 have the best results for σ_s^2 and meshes with $c = 0.10$ have the best coverage for β_0 and β_1 . So there does

not seem to be one mesh that is outperforming all other meshes for all parameters together. Considering the coverage of all parameters together, mesh 20 has the best results overall.

Results given in the Appendix A with sample size of $n = 50$ or $n = 300$ are very similar. When the value of the `cutoff` parameter c is very small, the coverage for all covariance parameters is too low. Increasing the value of c , decreases the number of knots in the mesh, but leads to better nominal coverages. However, when c is too large, the coverage for σ_e^2 drops again. The impact of changing the maximal allowed length in the inner domain (a) has much less impact on the nominal coverage and changing the value of b , the maximal allowed length in the outer domain, has no impact. The mesh with best coverage depends on the parameter of interest and the sample size: for σ_e^2 , mesh 29 has a slightly better results for coverage when $n = 50$ and for all the others parameters, mesh 45 is outstanding when $n = 50$; when $n = 300$, σ_e^2 has the best coverage for mesh 13, while for the others parameters, meshes with higher values of c have better results.

Table 2.3 presents the absolute bias of the estimated parameters, for all meshes with $b = 0.3$. We observe again that the impact of the `cutoff` parameter is largest, compared to the impact of the `max.edge` parameter. The bias of the estimated parameters is highest when the minimal allowed distance between points is very small. Note that when c is very small, the mesh can have a highly clustered pattern, whereas when c is very large, the mesh has only a very limited number of knots. Decreasing the number of knots by increasing the maximal allowed edge in the inner domain, slightly decreases the bias when c is small. We can observe that for β_0 and β_1 , the values of bias is really small, and quite often zero, for σ_e^2 , mesh 20 has the minimal absolute bias. For the parameter σ_s^2 , mesh 21 performs well, for k meshes with $c = 0.07$ have the small bias and for ϕ^* , the minimum value of bias is obtained by $c = 0.01$. Looking all the parameters together, we can conclude that meshes with $c = 0.03$ would yield optimal results in this simulation setting.

Table 2.3. Absolute bias of the parameters β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* for different meshes with $b = 0.3$.

Mesh	c	a	Bias for:						Mesh	c	a	Bias for:					
			β_0	β_1	σ_e^2	σ_s^2	k	ϕ^*				β_0	β_1	σ_e^2	σ_s^2	k	ϕ^*
1	0.001	0.05	0.00	0.03	0.26	0.27	13.92	0.08	25	0.05	0.05	0.01	0.02	0.06	0.02	0.20	0.10
2		0.09	0.01	0.02	0.28	0.45	13.06	0.09	26		0.09	0.01	0.02	0.07	0.01	0.54	0.09
3		0.13	0.00	0.03	0.28	0.56	11.06	0.08	27		0.13	0.01	0.02	0.10	0.03	0.62	0.11
4		0.17	0.00	0.03	0.28	0.62	9.76	0.08	28		0.17	0.01	0.02	0.12	0.01	0.27	0.11
5	0.005	0.19	0.00	0.03	0.28	0.63	9.22	0.08	29	0.07	0.19	0.01	0.02	0.13	0.02	0.20	0.11
6		0.20	0.00	0.03	0.28	0.63	8.97	0.07	30		0.20	0.01	0.02	0.13	0.01	0.01	0.10
7		0.05	0.00	0.03	0.25	0.26	13.14	0.07	31		0.05	0.01	0.02	0.15	0.01	0.51	0.13
8		0.09	0.00	0.04	0.26	0.42	12.06	0.08	32		0.09	0.01	0.02	0.15	0.01	0.64	0.13
9	0.01	0.13	0.00	0.04	0.26	0.52	10.07	0.07	33	0.10	0.13	0.01	0.02	0.15	0.01	0.73	0.14
10		0.17	0.00	0.04	0.26	0.58	8.80	0.06	34		0.17	0.01	0.02	0.17	0.01	0.50	0.15
11		0.19	0.00	0.04	0.26	0.59	8.49	0.06	35		0.19	0.01	0.02	0.18	0.05	0.03	0.14
12		0.20	0.00	0.04	0.26	0.59	8.27	0.06	36		0.20	0.01	0.02	0.18	0.06	0.24	0.13
13	0.03	0.05	0.01	0.02	0.20	0.19	9.18	0.03	37	0.13	0.05	0.01	0.02	0.23	0.28	1.73	0.13
14		0.09	0.01	0.02	0.18	0.28	7.79	0.02	38		0.09	0.01	0.02	0.23	0.31	1.78	0.13
15		0.13	0.00	0.03	0.19	0.37	6.79	0.02	39		0.13	0.01	0.02	0.23	0.33	1.75	0.13
16		0.17	0.00	0.03	0.18	0.42	6.03	0.02	40		0.17	0.01	0.02	0.23	0.32	1.68	0.13
17	0.03	0.19	0.00	0.03	0.18	0.43	5.74	0.02	41	0.13	0.19	0.01	0.02	0.23	0.32	1.65	0.13
18		0.20	0.00	0.03	0.18	0.43	5.57	0.02	42		0.20	0.00	0.03	0.24	0.29	1.33	0.13
19		0.05	0.01	0.02	0.05	0.02	1.94	0.05	43		0.05	0.00	0.03	0.29	0.50	2.77	0.12
20		0.09	0.01	0.02	0.01	0.01	0.30	0.08	44		0.09	0.00	0.03	0.29	0.53	2.75	0.12
21	0.03	0.13	0.01	0.02	0.03	0.00	0.18	0.09	45	0.13	0.13	0.00	0.03	0.30	0.43	2.16	0.12
22		0.17	0.01	0.02	0.04	0.03	0.06	0.09	46		0.17	0.00	0.03	0.30	0.44	2.20	0.12
23		0.19	0.01	0.02	0.04	0.04	0.10	0.09	47		0.19	0.00	0.03	0.30	0.45	2.22	0.12
24		0.20	0.01	0.02	0.04	0.04	0.10	0.08	48		0.20	0.00	0.03	0.30	0.44	2.24	0.12

The mean square error (MSE) was calculated for each parameter and results are presented in Table A.1 in the Appendix A. As we can notice, the value of MSE is high for k in all meshes, especially for meshes with small values of c and a . For β_0 almost all meshes have the same results and for β_1 meshes with $c = 0.07$ and $c = 0.10$ the results are the same and with the minimum value. For σ_e^2 the best results for MSE occur for $c = 0.03$ and for σ_s^2 and k small values for MSE occur for mesh 21, and for ϕ^* meshes 4 and 5 have the minimum value for MSE. Similar results are observed when $n = 50$ and $n = 300$.

Finally, we also investigated the impact of the mesh on the predictive performance of the model. For each mesh, the coverage and the mean square error (MSE) of the predictions in the sampled locations

Table 2.4. Nominal coverage, in %, and MSE for the variable Y calculated for each mesh with $b = 0.3$, considering 200 simulations.

Mesh	c	a	Coverage(%)	MSE	Mesh	c	a	Coverage(%)	MSE
1	0.001	0.05	18.75	0.28	25	0.05	0.05	90.97	0.21
2		0.09	13.11	0.29	26		0.09	90.79	0.21
3		0.13	11.41	0.29	27		0.13	89.83	0.22
4		0.17	12.24	0.29	28		0.17	89.04	0.22
5		0.19	12.51	0.29	29		0.19	88.89	0.22
6	0.005	0.20	12.82	0.29	30	0.07	0.20	88.72	0.22
7		0.05	26.48	0.28	31		0.05	87.76	0.23
8		0.09	23.54	0.28	32		0.09	87.59	0.23
9		0.13	23.25	0.28	33		0.13	87.61	0.23
10		0.17	23.79	0.28	34		0.17	85.91	0.25
11	0.01	0.19	23.82	0.28	35	0.10	0.19	85.42	0.25
12		0.20	23.98	0.28	36		0.20	84.98	0.25
13		0.05	45.17	0.25	37		0.05	82.06	0.28
14		0.09	48.95	0.25	38		0.09	81.81	0.28
15		0.13	48.67	0.25	39		0.13	81.33	0.28
16	0.03	0.17	49.01	0.25	40	0.13	0.17	81.28	0.28
17		0.19	48.74	0.25	41		0.19	81.33	0.28
18		0.20	48.85	0.25	42		0.20	81.06	0.28
19		0.05	85.56	0.21	43		0.05	75.89	0.32
20		0.09	90.49	0.20	44		0.09	75.53	0.32
21		0.13	90.83	0.20	45		0.13	75.28	0.32
22		0.17	90.84	0.20	46		0.17	75.28	0.32
23		0.19	90.95	0.20	47		0.19	75.33	0.32
24		0.20	90.92	0.21	48		0.20	75.36	0.32

are calculated. Results are presented in Table 2.4 (as well as Table A.5 and Table A.11 in the Appendix A). Within the groups of meshes with the same value of c , changing the value of a has little impact on the MSE and coverage of the predictions. While meshes with the smallest c value perform the worst in term of coverage, meshes with $c = 0.03$ have the best coverage and MSE. The results for $b = 0.4$ and $b = 0.2$ are similar. In Table 2.4 we can notice that mesh 20 is one of the meshes with better results and was the mesh that in Table 2.2 has the best results.

The predictive performance of the model was also investigated for locations not used in the analysis. For $n = 100$ the predictions was made in 25 locations, and for $n = 50$ and $n = 300$ in 13 and 75 locations, respectively. Table 2.5 presents the results and we can notice that as the value of c increases, the coverage is smaller and the MSE has a slight decrease and gets bigger again with $c = 0.10$. Within the groups, the mesh with $a = 0.05$ has the best results. For $n = 50$ and $n = 300$ the same results are presented in Table A.6 and Table A.12 in the Appendix A.

Boxplots of bias for each mesh, considering the 25 extra locations are presented in Figure 2.2 (as well as in Figure A.3 and Figure A.7 in the Appendix A) and we can notice that there is no difference within and between the groups.

2.3.3 Selecting the best mesh

Results in the previous section showed that there is a large impact of the mesh on the inference (parameter estimation and prediction). Also, it was observed that there is not a single mesh which is optimal. Therefore, in this section, we investigate whether we can use the predictive information criteria as described in Section 2.1, to select an optimal mesh.

To perform mesh selection, we calculate DIC, WAIC and logCPO, for each simulation run. To simplify the comparison, we fix $b = 0.3$ and a and compare meshes for different values of c . Figure 2.3 shows the number of simulations in which the mesh with a particular value of c was chosen as the best according to DIC (panel I), WAIC (panel II) or logCPO (panel III). In settings where two meshes perform equally good, the lowest value was considered for the two meshes. Looking at Figure 2.3, we observe an

Table 2.5. Nominal coverage, in %, and MSE for the variable Y calculated for each mesh with $b = 0.3$, considering 200 simulations, for 25 locations not in the analysis.

Mesh	c	a	Coverage(%)	MSE	Mesh	c	a	Coverage(%)	MSE
1	0.001	0.05	90.80	0.45	25	0.05	0.05	82.60	0.44
2		0.09	80.26	0.45	26		0.09	82.54	0.44
3		0.13	71.40	0.46	27		0.13	78.60	0.44
4		0.17	62.86	0.47	28		0.17	74.56	0.44
5		0.19	61.14	0.47	29		0.19	73.02	0.45
6	0.005	0.20	59.64	0.47	30	0.20	0.20	71.58	0.45
7		0.05	90.76	0.45	31		0.07	77.78	0.45
8		0.09	80.62	0.45	32		0.09	77.62	0.45
9		0.13	72.50	0.46	33		0.13	77.62	0.45
10		0.17	64.68	0.47	34		0.17	73.82	0.45
11	0.01	0.19	63.04	0.47	35	0.19	0.19	71.82	0.45
12		0.20	61.50	0.47	36		0.20	70.46	0.46
13		0.05	89.94	0.44	37		0.10	71.32	0.46
14		0.09	81.30	0.45	38		0.09	71.06	0.47
15		0.13	73.94	0.45	39		0.13	70.62	0.47
16	0.03	0.17	67.38	0.46	40	0.17	0.17	70.22	0.47
17		0.19	65.92	0.46	41		0.19	70.12	0.47
18		0.20	65.26	0.46	42		0.20	69.72	0.47
19		0.05	87.98	0.44	43		0.13	65.98	0.49
20		0.09	83.10	0.44	44		0.09	65.98	0.49
21	0.03	0.13	78.66	0.45	45	0.13	0.13	65.94	0.49
22		0.17	74.88	0.45	46		0.17	65.66	0.49
23		0.19	73.60	0.45	47		0.19	65.72	0.49
24		0.20	72.68	0.45	48		0.20	65.96	0.49

Figure 2.2. Bias for Y in 25 extra locations, for all 200 simulations, for $b = 0.3$.

apparent difference between the DIC and WAIC measured on the one hand, and the logCPO on the other hand. Results based on DIC and WAIC are very similar, with smaller values of c to be preferred in most simulation runs (no matter which a is selected). LogCPO, on the other hand, seem to select meshes with a larger value of c , and meshes with $c = 0.03$ (blue color) seems to stand out.

The mean of DIC, WAIC and logCPO are reported in Table 2.6 for $b = 0.3$. We can observe

Figure 2.3. For each value of a , minimum value of: (I) DIC; (II) WAIC and (III) logCPO, for all 200 simulations, for $b = 0.3$.

that meshes with smaller value of c are selected using DIC and WAIC. For the mean of logCPO, there are multiple meshes with good results and a way to choose one can be the minimum number of knots, which occurs with higher values of c and a . If we consider all these three results together, we can choose

mesh 20 as the best one, which is the same conclusion as in section 2.3.2. As we have seen before that the meshes with smaller value of c lead to larger MSE and bias for inference of covariance parameters, it can be argued that those selection methods are not well suited if interest is in the covariance function.

Table 2.6. Mean values of DIC, WAIC and logCPO for meshes with $b = 0.3$.

Mesh	c	a	DIC	WAIC	logCPO	Mesh	c	a	DIC	WAIC	logCPO
1	0.001	0.05	-442.09	-448.64	2.07	25	0.05	0.05	230.73	231.21	1.28
2		0.09	-490.14	-500.73	2.09	26		0.09	231.33	231.77	1.28
3		0.13	-504.62	-515.26	2.13	27		0.13	237.21	237.70	1.28
4		0.17	-495.78	-506.18	2.12	28		0.17	240.72	241.35	1.28
5		0.19	-496.70	-506.81	2.11	29		0.19	241.05	241.69	1.28
6		0.20	-492.78	-503.08	2.11	30		0.20	241.55	242.19	1.28
7	0.005	0.05	-369.05	-370.31	1.98	31	0.07	0.05	242.94	243.85	1.28
8		0.09	-389.08	-395.66	1.96	32		0.09	243.20	244.13	1.28
9		0.13	-387.74	-395.32	1.97	33		0.13	243.30	244.24	1.28
10		0.17	-384.21	-389.65	1.96	34		0.17	246.10	247.05	1.28
11		0.19	-383.24	-389.03	1.95	35		0.19	246.78	247.72	1.29
12		0.20	-381.55	-387.75	1.95	36		0.20	247.02	247.92	1.28
13	0.01	0.05	-211.80	-205.13	1.78	37	0.10	0.05	251.05	252.20	1.29
14		0.09	-162.39	-163.09	1.69	38		0.09	250.94	252.10	1.29
15		0.13	-161.75	-166.19	1.70	39		0.13	251.48	252.61	1.29
16		0.17	-156.79	-160.54	1.68	40		0.17	251.50	252.63	1.29
17		0.19	-162.57	-163.38	1.68	41		0.19	251.39	252.54	1.29
18		0.20	-162.23	-162.97	1.68	42		0.20	251.62	252.79	1.29
19	0.03	0.05	155.35	161.35	1.34	43	0.13	0.05	255.85	257.13	1.30
20		0.09	213.02	213.13	1.28	44		0.09	256.22	257.47	1.30
21		0.13	221.37	221.18	1.28	45		0.13	256.55	257.78	1.30
22		0.17	223.23	223.12	1.28	46		0.17	256.55	257.78	1.30
23		0.19	223.54	223.35	1.28	47		0.19	256.49	257.73	1.30
24		0.20	223.71	223.56	1.28	48		0.20	256.57	257.80	1.30

We now evaluate the logCPO, DIC and WAIC criteria for mesh selection over all values of a , b and c . Figure 2.4 shows boxplots of parameter estimates based on the mesh with lowest DIC, WAIC or logCPO, obtained in each simulation. For comparison, we also present in this figure the boxplots of the estimates obtained for each parameter with mesh 20, the mesh chosen as the one with best overall performance for the chosen simulation settings (based on bias, MSE, coverage of parameter estimates and predictions). We can observe that the boxplots obtained with mesh 20 are similar to the boxplots obtained for the results of logCPO. Selection of the mesh using DIC and WAIC has bad performance, seen in the large bias of all covariance parameters. Indeed, DIC and WAIC tends to select the mesh with smaller `cutoff` values, and thus the more dense meshes, though it was observed in the previous section that a mesh with smallest c value leads to larger bias and worse coverage for estimation of the covariance parameters. The logCPO tends to select the mesh with larger `cutoff` values, and thus the more sparse/regular meshes. While the difference between the DIC, WAIC and logCPO would be expected to behave similarly (as they are asymptotically the same), it is important to realize that all methods are only approximations of the predictive accuracy. While DIC and WAIC give reasonable answers in many cases, they only attempt to capture the predictive behavior within the sample, whereas the logCPO captures also out-of-sample prediction error.

Similar results are observed for other sample sizes. For $n = 50$ and $n = 300$, the best values of DIC and WAIC also occurs for smaller values of c , as we can see in Table A.7 and Table A.13 of the Appendix A, respectively. For logCPO with $n = 50$, the best results occur for larger value of c , especially for $c = 0.10$. In the case of $n = 300$, meshes with smaller values of c , mainly $c = 0.01$, have the best values of logCPO. Considering the analysis, we can assume that mesh 27 is a good one for $n = 50$ and mesh 13 for $n = 300$. These meshes have plausible and satisfactory results. Figure A.5 and Figure A.9 of the Appendix A are figures similar to Figure 2.4 above and in all this three figures, the boxplot of the chosen mesh is more alike with the boxplot of logCPO, so as told before, logCPO is the best criteria for mesh selection.

Figure 2.4. Estimated β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* , considering the meshes with minimal DIC, WAIC and logCPO, and based on mesh 20.

Based on these simulation results, it is confirmed that the logCPO criterion is better suited for mesh selection.

2.4 Data Analysis

The data analysed was the so called Paraná dataset, which is freely available in the geoR package (Ribeiro and Diggle, 2001) in R (R Core Team, 2015). The data considers average precipitation levels throughout Paraná state, Brazil. Data is available on 143 locations and the response represents average winter (dry season) rainfall.

To perform the analysis we take out the linear trend from the variable, because we noticed that this trend lead to misleading results. To model these data, we assume the model $y = \beta_0 + \mathbf{s} + \epsilon$, where $\mathbf{s}$ is a Matérn spatial random field with $\alpha = 1.5$. Considering the range as 380, assumed by a semivariogram analysis, we create eighteen meshes to analyse this data set, namely $a = c(20, 35, 55, 100, 240, 380)$, $b = 570$ and $c = c(13, 35, 51)$. The value of a was chosen taking into account the value of range, allowing for small values until we reach the expected value of the range. The value of b is chosen larger than a and for c values it was considered random values in the case that all the coordinates are the inner mesh. In Table 2.7 we present the number of knots for each of these meshes. We can observe that the number of knots almost does not change for values of a larger than 100.

Table 2.7. Number of knots and values for DIC, WAIC and logCPO for each mesh for Paraná data.

Mesh	c	a	Number knots	DIC	WAIC	logCPO
1	13	20	1235	1305.94	1306.35	4.58
2		35	678	1307.34	1307.46	4.58
3		55	362	1306.53	1306.47	4.58
4		100	234	1307.85	1307.78	4.58
5		240	230	1307.97	1307.91	4.58
6		380	230	1307.97	1307.91	4.58
7	35	20	197	1307.69	1306.64	4.58
8		35	195	1308.14	1307.04	4.58
9		55	194	1307.90	1306.81	4.58
10		100	134	1309.51	1308.59	4.58
11		240	122	1309.71	1308.65	4.58
12		380	122	1309.71	1308.65	4.58
13	51	20	100	1308.18	1307.65	4.58
14		35	100	1308.44	1307.93	4.58
15		55	97	1307.65	1307.09	4.58
16		100	94	1307.60	1307.08	4.58
17		240	77	1307.83	1307.05	4.57
18		380	77	1307.83	1307.05	4.57

The posterior distributions for β_0 , σ_s^2 , k and ϕ^* are not very different across the considered meshes, as we can observe in Figures 13-16 of the Supplementary Material. Table 2.8 shows the posterior mean estimates for the parameters and we can observe that the choice of the mesh does not have a large impact on the estimates, though mesh 16 was the one that estimates the value for ϕ^* closer to the one that we obtained from the exploratory analysis and also has good results for σ_s^2 . The values of DIC, WAIC and logCPO was computed and added into Table 2.7. As for the posterior estimates, these three criteria also only show minor difference between the meshes. The smallest results for these three criteria occurs for meshes 1, 17 and 18, in which meshes 17 and 18 are exactly the same. Comparing the results of this meshes with mesh 16 we can say that mesh 16 is better although it has more knots than meshes 17 and 18. So we selected mesh 16 and computed the prediction mean and prediction standard deviation over the study area. This is given in Figure 2.5.

Table 2.8. Estimates of the parameters for each mesh.

Mesh	c	a	β_0	σ_s^2	k	ϕ^*
1	13	20	-3.04	1004.25	4×10^{-3}	530.36
2		35	-2.90	1019.22	3×10^{-3}	568.74
3		55	-2.88	1026.20	3×10^{-3}	573.44
4		100	-2.87	1020.58	3×10^{-3}	588.12
5		240	-2.85	1007.17	3×10^{-3}	582.86
6		380	-2.85	1007.17	3×10^{-3}	582.86
7	35	20	-2.80	1002.03	3×10^{-3}	555.72
8		35	-2.75	1003.72	3×10^{-3}	570.49
9		55	-2.77	987.82	3×10^{-3}	556.80
10		100	-2.60	1018.22	3×10^{-3}	588.27
11		240	-2.37	1063.74	3×10^{-3}	618.14
12		380	-2.37	1063.74	3×10^{-3}	618.14
13	51	20	-3.25	994.48	4×10^{-3}	520.95
14		35	-3.14	979.35	4×10^{-3}	512.67
15		55	-3.24	979.76	4×10^{-3}	504.03
16		100	-3.14	979.48	4×10^{-3}	494.87
17		240	-2.11	1263.75	3×10^{-3}	666.64
18		380	-2.11	1263.75	3×10^{-3}	666.64

Figure 2.5. (I) Histogram, (II) plot of sample and corresponding values, (III) predicted surface and (IV) prediction standard deviation considering mesh 16.

2.5 Conclusion

The first step to use the SPDE/GMRF approach in INLA with geostatistical data is to create a triangulated mesh over the study area. Using a simulation study, we have shown that inferential results depend on the considered mesh. The simulation results indicated that the value of b , the maximal edge

length in the outer domain, has almost no impact on the results. On the other hand, the choice of the value of c , the shortest allowed distance between points, has the largest impact. Thirdly, conditional on a value of c , the value of a , the maximal edge length in the inner domain, also leads to some differences.

Pre-specification of the triangulated mesh is not straightforward, since it is possible to create a mesh with very good predictive behavior, but that leads to poor covariance parameter estimates. We have used three predictive information criteria, namely DIC, WAIC and CPO, to investigate whether these criteria can be used to select an optimal mesh for a specific data problem. We noticed that the predictive information criteria DIC and WAIC lead to choosing dense meshes. These meshes indeed perform well in terms of coverage and mean squared error for predictions, but do not perform well in the other parts of the analysis, namely the estimation of the covariance parameters. Thus, DIC and WAIC showed to prefer meshes with more knots. Further, we have noticed that mesh selection using the logCPO leads to better performing meshes overall.

Working with INLA is fast, but selection of the mesh is hindered by the computationally intensive logCPO measurement. While a fast approximating method for CPO calculating is available in INLA, it is very important to ensure that no numerical problems occur. We have observed that in many cases, especially with dense meshes, this approximating method does not work and, therefore, CPO needed to be recalculated by refitting the model, leaving out every observation one at a time.

In this paper, we focused on a simple simulation scenario to investigate the impact of different meshes on prediction and estimation of covariance parameters. Further investigation is still needed towards the mesh triangulation using the SPDE/GMRF approach in INLA. For example, in our simulation study the meshes were created considering the coordinates. However, it is not required that the observed locations are included as nodes in the mesh. Thus, it is also possible to create meshes without the observed coordinates or by using a specific subset of the observed locations. Additionally, situation in which the observed data locations are more clustered or more regular should be investigated in more details.

In the analyzed scenarios, we choose the mesh that stands out from the others, but there is not only one suitable mesh. For $n = 50$, meshes constructed with $c = 0.05$ have good results and comparing with range, it is a ratio of 0.25 ;for $n = 100$, meshes with $c = 0.03$ have good results, generating a proportion of 0.15 with range and for sample size $n = 300$, the proportion is 0.05, with $c = 0.01$.

References

- BOLIN, D.; LINDGREN, F. A comparison between Markov approximations and other methods for large spatial data sets. *Computational Statistics & Data Analysis*, v. 61, p. 7-21, 2013.
- BOLIN, D.; LINDSTRÖM, J. Spatial Statistics using R-INLA and Gaussian Markov random fields. *Pan-American Advanced Study Institute*, 15:29:57Z, rev.3691, 2014.
- CAMELETTI, M.; LINDGREN, F.; SIMPSON, D., RUE, H. Spatio-temporal modeling of particulate matter concentration through the SPDE approach. *Advances in Statistical Analysis*, v. 97, p. 109-131, 2012.
- GEISSER, S.; EDDY, W. A predictive approach to model selection. *Journal of the American Statistical Association*, Boston, v.74, n.365, p.153-160, 1979.
- GELFAND, A E. Model determination using sampling-based methods. In: GILKS, W.; S. RI-CHARDSON, S.; SPIEGELHALTER, D. (Eds.), *Markov chain Monte Carlo in practice*. Florida: Chapman & Hall/CRC, 1996. chap. 9, p. 145-161.

- GELFAND, A. E.; DIGGLE, P. J., FUENTES, M.; GUTTORP, P. (Eds) *Handbook of spatial statistics*. CRC press, 2010.
- GELMAN, A.; HWANG, J.; VEHTARI, A. Understanding predictive information criteria for Bayesian models. *Statistics and Computing*, v. 24, n. 6, p. 997-1016, 2014.
- HELD, L., SCHRODLE, B., RUE, H. Posterior and Cross-validatory Predictive Checks: A comparison of MCMC and INLA. In T. Kneib and G. Tutz, editors, *Statistical Modelling and Regression Structures*, p. 91, 2010.
- KARAGIANNIS-VOULES, D-A, SCHOLTE, R. G. C., GUIMARÉES, L. H., UTZINGER, J., VOUNATSOU, P. Bayesian Geostatistical Modeling of Leishmaniasis Incidence in Brazil. *PLoS Neglected Tropical Diseases*, v. 7, n. 5, e2213. doi:10.1371/journal.pntd.0002213. 2013.
- KRAINSKI, E. T.; RUE, H. A short introduction on how to fit a SPDE model with INLA. 2013.
- KRAINSKI, E. T.; LINDGREN, F. The R-INLA tutorial: SPDE models. 2014.
- LINDGREN, F. Continuous domain spatial models in R-INLA. *The ISBA Bulletin*, v. 19, n. 4, 2012.
- LINDGREN, F.; RUE, H. Bayesian spatial modelling with R-INLA. *Journal of Statistical Software*, v. 63, 2015.
- LINDGREN, F.; RUE, H.; LINDSTRÖM, J. An explicit link between Gaussian fields and Gaussian Markov random fields: The stochastic partial differential equation approach. *Journal of the Royal Statistical Society*, v. 73, p. 423-498, 2011.
- R Core Team (2015). R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. URL <https://www.R-project.org/>.
- RIBEIRO JR., P.J. and DIGGLE, P.J. (2001) geoR: A package for geostatistical analysis. R-NEWS Vol 1, No 2. ISSN 1609-3631. Available for download at: <http://cran.r-project.org/doc/Rnews>.
- RUE, H.; MARTINO, S.; CHOPIN, N. Approximate Bayesian inference for latent Gaussian models by using integrated nested Laplace approximations (with discussion). *Journal of the Royal Statistical Society - Series B*, v.71, Part 2, p.319-392, 2009.
- SPIEGELHALTER, D.J.; BEST, N.G.; CARLIN, B.P.; LINDE, A.V.D. Bayesian measures of model complexity and fit. *Journal of the Royal Statistical Society Series B*, London, v.64, n.4, p.583-639, 2000.
- VEHTARI, A.; GELMAN, A. WAIC and cross-validation in Stan. Submitted. http://www.stat.columbia.edu/gelman/research/unpublished/waic_stan.pdf Accessed, v. 27, n. 2015, p. 5, 2014.
- WATANABE, S. Asymptotic equivalence of Bayes cross validation and widely applicable information criterion in singular learning theory. *Journal of Machine Learning Research*, v. 11, p. 3571-3594, 2010.
- WHITTLE, P. On stationary processes in the plane. *Biometrika*, v. 41, p. 434-449, 1954.

3 A SIMULATION STUDY WITH DIFFERENT MESHES FOR GEOSTATISTICAL DATA WITH RANDOM, CLUSTER AND REGULAR GRIDS.

Abstract: Advanced in Geographic Information Systems and Global Positioning Systems placed increasing demands and challenge for spatial statistics analysis. Most of the methods used in this analysis are computationally demanding. One approach, which is considered a remarkable development, is the introduction of the integrated nested Laplace approximation method which is able to carry out Bayesian analysis in a more efficient way. The use of this method for geostatistical data is commonly done considering the SPDE approach that requires the creation of a mesh over the study area. This is the first and an important step since all results will depend on the choice of this mesh. As there is no formal and close way to specify the mesh, we investigate possible guidelines on how an optimal mesh is chosen for a specific problem. We study the impact of the mesh for random, regular and cluster data set, through a simulation study in which we considered different ways to define the knots. Two applications with real data set, rainfall data in Paraná state, Brazil, and soil data set, is presented.

Keywords: Geostatistics; Clustered data; Random data; Integrated nested Laplace approximation; Mesh

3.1 Introduction

The methods of spatial statistics have the purpose of measuring properties and relationships, taking into account the spatial location of the phenomenon studied in an explicit way. Geostatistics is a sub-area of spatial statistics and according to Matheron (1963) it is the application of the formalism of random functions for the recognition and estimation of natural phenomena.

For geostatistical modeling with Bayesian analysis a method of approach has been increasingly used, which is the integrated nested Laplace approximations (INLA), proposed by Rue, Martino and Chopin (2009). This method is an alternative for Markov Chain Monte Carlo (MCMC) and it is faster (Karagiannis et al., 2013; Lindgren, 2012). INLA is an approximate method designed for inference of latent Gaussian models, a very wide and flexible class of models ranging from (generalized) linear mixed model to spatial and spatio-temporal models (Lindgren and Rue, 2015). When combined with stochastic partial differential equation (SPDE) (Lindgren, Rue and Lindstrom, 2011), it can accommodate georeferenced data from all sub-areas of spatial statistics.

When combining INLA with SPDE, for geostatistical data, the spatial field is best described using a weighted sum of piece-wise linear basis function that is usually defined from a triangularisation of the study area, called the mesh. This triangularisation can be done with small or big triangles or a combination of both. The idea is that smaller triangles will increase the accuracy, though when the spatial range is larger, larger triangles can be used. The mesh can be constructed using different principles. we can create the mesh using all the coordinate or just using a sample of it or we can also use only the domain of the area.

Although it is known that the choice of the mesh must be done carefully (Lindgren and Rue, 2015), since accuracy of the estimates as well as the predictions will depend on the choice of the mesh, there is no conclusive way to create the best one. In this way, the objective of this work is to study the parameters for the creation of the mesh with different type of meshes, to see their impact on the analysis and set guidelines to construct the optimal mesh in two dimensional spatial domains. The study is made with creation of meshes using all coordinate, using a sample of the coordinate and only considering the domain of the study area. With this three types of meshes, we made the study for irregular, regular and cluster geostatistical data. The objective is to try to solve issue raised by Bolin and Lindstrom

(2014), which was how different meshes impact in analysis. In Section 3.2, we give a brief introduction to INLA and the SPDE approach and discuss some criteria to evaluate the mesh, in Section 3.3 we show a simulation study, applications with real data set are provided in Section 3.4 and a discussion is provided in Section 3.5.

3.2 Methodology

In a study that requires the use of the position of the sample elements, traditional statistical techniques are no longer sufficient for analysis. For such studies, one needs the approaches used in spatial statistics. Geostatistics is a sub-area of spatial statistics and, according to Diggle, Ribeiro Jr and Christensen (2003), the data in this sub-area consist of measurements $y_1, \dots, y_n$ obtained in locations $\mathbf{s}_1, \dots, \mathbf{s}_n$, sampled from a region A spatially continuous, in which $A \subset R^d$. The grid of this samples can be in different way, as it can be in a regular grid, meaning that all points of sample are equal spaced, it can be in a random grid or a cluster one.

Diggle, Tawn and Moyeed (1998) introduced the term "model-based geostatistics", in which modeling is performed by specifying a parametric model for the variable(s) under study and using statistical inference methods based on the likelihood function applied to geostatistical studies. With geostatistical techniques it is possible to describe the spatial dependence structure of the data, establishing a spatial correlation function that depends only on the distance of the position of the samples and represents the behavior of the spatial variability of the data, by means of a model. Also, the parameters of the models are estimated and used in the spatial predictions of the variables under study and then, it can be said that there is uncertainty about these parameters and as a consequence, there is uncertainty in the predictions made. When using the classical geostatistical methodology in the estimation of parameters through the maximum likelihood method, we use only the information from the data, the adopted model and the asymptotic theories, but when using the Bayesian methods, besides all this knowledge we also use a priori information about the parameters, that is, there may be an initial knowledge that is expressed through a prior distributions. Under this approach, there is no difference between parameters and observations, both of which are treated as random quantities.

After defining the priori distribution of the parameters it is possible to summarize inferential techniques, usually using probabilities and expectation. In this step, it is often necessary to use intensive computational methods such as Monte Carlo via Markov Chains (MCMC). In geostatistics, the Gibbs sampler and the Metropolis-Hastings algorithm were the most used methods. An alternative approach is the INLA method, which is a deterministic method to perform Bayesian inference in a large class of Gaussian Markov random fields (GMRF), including many of the hierarchical models with latent Gaussian processes. It has been gaining a lot of space on the practice because it is more computationally efficient (Rue, Martino and Chopin, 2009).

If $\mathbf{y}$ represents the data, $\mathbf{x}$ the GMRF and $\boldsymbol{\theta}$ the vector of hyperparameters, the idea of INLA is to approximate the posterior density $\Pi(\boldsymbol{\theta}|\mathbf{y})$ via Gaussian approximation of the posterior of the latent field $\Pi(\mathbf{x}|\boldsymbol{\theta}, \mathbf{y})$ and a Laplace approximation in the latent field marginal posterior calculations. For more details see Rue, Martino and Chopin (2009). Lindgren, Rue and Lindstrom (2011) showed that INLA is not restricted to discrete spatial domains, given there is a connection between continuous domains and Markov models. Also, a large class of random field models can be expressed as a solution of a continuous domains stochastic partial differential equation (SPDE) and it is possible to obtain connections between the parameters of the SPDE and the elements of the precision matrix for weights in representation, using as basis the covariance function Matérn. The Matérn function is given by $M(\mathbf{h}|k, \phi) = \frac{2^{1-k}}{\Gamma(k)}(\phi\|\mathbf{h}\|)^k K_k(\phi\|\mathbf{h}\|)$, in which $\Gamma(\cdot)$ is the Gamma function, K_k is Bessel function with order k and $\|\mathbf{h}\|$ is the distance between two locations. With this connection it is possible to carry out

the calculations for a continuous random field using the (discrete) GMRF representation:

$$\mathbf{x}(s) = \sum_{k=1}^K \Psi_k(s) \mathbf{x}_k, \quad (3.1)$$

where $\Psi_k(s)$ are piece-wise linear basis function defined on a triangulation of the study domain and $\{\mathbf{x}_1, \dots, \mathbf{x}_K\}$ are Gaussian weights chosen to approximate the solutions to the SPDE on the domain. This triangulation consists of partitioning the study area into a set of non-intersecting triangles (Cameletti et al., 2012), with linear basis function defined with barycentric interpolation. This approach is available in the R-package INLA (Rue, Martino and Chopin, 2009).

This partitioning can be done with the construction of a triangulated mesh on which the SPDE/GMRF representation is built (Lindgren, 2012). As an initial step, we can choose to place the triangle vertices at the locations where measurements were taken, and after that, add or remove additional vertices, creating the triangulation for the analysis. As the solution may present a border effect, it is necessary to extend the mesh also outside the study area to avoid it. According to Lindgren (2012), the global effect in constructing triangles is that we can obtain small ones (and so obtain high accuracy to represent the field) where the locations are dense, and bigger ones when the locations are sparse. As the analysis depends on the mesh, this first step is very important, so we can create a mesh suited for the data at hand.

The function `inla.mesh.2d()`, in the R-INLA implementation, is available to create the mesh. Several parameters in this function can be chosen as to construct a mesh for the problem at hand but in this work we are only using three: the `max.edge` parameters, which define the maximal allowed lengths of triangle edges in the inner domain and in the outer extension, the `cutoff` parameter, which corresponds with the shortest distance allowed between points (Krainski and Lindgren, 2014; Krainski and Rue, 2013). A change in one of these values, will create a different mesh, making it with more or less knots.

In this work we named the parameters to construct the mesh as: the maximal allowed edge length in the inner domain is a , the outer domain is represented by b and the shortest allowed distance between points by c . We will always consider larger values of the maximal allowed edge length for the outer domain as for the inner domain ($b > a$), as the outer domain is needed only to avoid border effects.

3.2.1 Goodness of fit

To compare the results obtained from the different meshes, we used some criteria to try to obtain the optimal mesh. The criteria used were: the deviance information criterion (DIC), the Watanabe-Akaike information criterion (WAIC) and the conditional predictive ordinate (CPO).

Spiegelhalter et al. (2000) described DIC as

$$DIC = -2 \log p(\mathbf{y}|\hat{\boldsymbol{\theta}}) + 2p_{DIC},$$

where $2 \log p(\mathbf{y}|\hat{\boldsymbol{\theta}})$ is the posterior expected deviance measuring the suitability of the model and p_{DIC} is the estimated effective number of parameters of the model, representing the complexity of the model. Thus, we can say that DIC takes into consideration both the suitability and complexity of the model.

WAIC was proposed by Watanabe (2010) and can be viewed as an improvement on the DIC for Bayesian models (Vehtari and Gelman, 2014). The formula for WAIC is given by

$$WAIC = -2 \log \int p(\mathbf{y}|\boldsymbol{\theta}) \Pi(\boldsymbol{\theta}|\mathbf{y}) d\boldsymbol{\theta} + \sum_{i=1}^n Var(\log(p(y_i|\boldsymbol{\theta}))).$$

In the same way for DIC, in WAIC the first term measures the predictive performance, and the second term penalizes for model complexity. According to Gelman, Hwang and Vehtari (2014), the complexity term is a sum of the variance for each data point separately, yielding more stable results as DIC.

CPO statistics is a model assessment tool which has been widely used in the statistical literature under various contexts (Ibrahim, Chen and Sinha, 2013) and it is a cross-validation method, estimating the leave-one-out predictive distribution (Geisser and Eddy, 1979). The posterior predictive distribution for an observation i can be expressed as

$$p(\mathbf{y}_i|\mathbf{y}_{-i}) = \int p(\mathbf{y}_i|\boldsymbol{\theta})\Pi(\boldsymbol{\theta}|\mathbf{y}_{-i})d\boldsymbol{\theta},$$

where $\mathbf{y}_i$ is the i th observation of $\mathbf{y}$ and $\mathbf{y}_{-i}$ is $\mathbf{y}$ without observation i . So the CPO is the probability a posteriori to observe y_i when the model is adjusted without y_i . A bigger value of CPO indicates a better adjustment. Instead of having to refit the model each time, an approximation of CPO can be obtained by sampling from the posterior distribution:

$$CPO_i = \frac{1}{T^{-1} \sum_{t=1}^T p(\mathbf{y}_i|\boldsymbol{\theta}^t)^{-1}},$$

where $\boldsymbol{\theta}^t$ is a sample from $\Pi(\boldsymbol{\theta}|\mathbf{y}_i)$ and T indicated the number of posterior samples. This is the inverse of the posterior mean of the inverses of the likelihood. This is an efficient method, but requires care since numerical problems in the calculations can occur (Held, Schrodle and Rue, 2010). However, when using INLA it is possible to verify if numerical problems did occur and recompute the CPO using a less efficient method (re-fitting the model for every observation).

Normally, CPO is used plotting a graphic of CPO's values against the observation index. An alternative to this plots, is to use the average of the summary statistics named as the logarithm of the Pseudomarginal Likelihood (LPML) (Ibrahim, Chen and Sinha, 2013). In this paper we named as logCPO and the formula is given by $\log CPO = -\frac{1}{n} \sum_{i=1}^n \log(p(\mathbf{y}_i|\mathbf{y}_{-i}))$. This log-score is an estimator of the log marginal likelihood function and smaller values indicate better model performance. In contrast to DIC and WAIC, the log CPO score does not adjust for the complexity of the model.

In addition to look only the estimates of the parameters, Zhang and Zimmerman (2007) said that the ratio $\frac{\sigma_s^2}{\phi}$ affects interpolation more than each this parameter separately. So, we will also consider the estimate of this ratio, in which $\phi = \frac{1}{\phi_*}$, in our notation.

3.3 Simulation Study

This section is to describe the simulation design and its results in which we investigate the impact of the choice of the mesh in the parameter estimation and prediction and we also investigate how to select an optimal mesh for a particular data problem considering the design of the sample.

All the analysis was made using R software (R Core Team, 2015) and INLA package (Rue, Martino and Chopin, 2009).

3.3.1 Simulation Design

In this simulation study, we runned 200 simulations and investigate the optimal mesh for three kind of sample design: random, regular and cluster data set, always with sample size $n = 100$, on the unit square. We conduct the simulation in five batches of 40 simulations and within one batch the coordinates are kept fixed, except on regular sample, to avoid possible confounding of the generated coordinates on the simulation results,

A response vector $\mathbf{y}$ is generated according to the model $\mathbf{y} = \beta_0 + \beta_1\mathbf{x} + \mathbf{s} + \mathbf{e}$, where $\mathbf{x}$ is a uniformly distributed covariate, $\mathbf{s}$ is a Matérn spatial random field and $\mathbf{e}$ is white noise. Parameters are fixed to $\beta_0 = 1$, $\beta_1 = 2$. The spatial variability of the Matérn spatial random field is set to $\sigma_s^2 = 0.7$ and Gaussian white noise variability to $\sigma_e^2 = 0.3$. The scale parameter (k) has value 10 and the nominal

range, specified as $\phi^* = \frac{\sqrt{8(\alpha-\frac{d}{2})}}{k}$, has value 0.2. We also fixed the fractional operator order $\alpha = 1.5$ and $d = 2$, as the study region $\mathbf{A} \in R^d$. For the prior, we used the log-gamma distribution with parameters $c(1.0, 0.00005)$.

We worked with 48 meshes for each type of data and they are divided in three groups, of 16 meshes each: the first one (GM1) is the group in which the initial triangulation taken is given by the locations of the generated process; the second group (GM2) is the one that we did not considered the coordinate to create the mesh, we only take into account a specific domain and in the third group (GM3) the meshes was created by half of the coordinate and for the selection of the coordinate we use the space filling algorithm (Johnson, 1990)

The spacial domain used was $[-0.1, 1.1] \times [-0.1, 1.1]$. To create different meshes, we considered the **max.edge** and **cutoff** parameters to change. The values considered for the maximal allowed edge length in the inner domain is $a = (0.05, 0.09, 0.17, 0.20)$, for the maximal edge length in the outer domain is $b = 0.3$ and for the shortest allowed distance between two points is $c = (0.005, 0.03, 0.07, 0.10)$. Combinations of these three values leads to a total of 16 different meshes. These values were chosen such that the value of a is smaller or equal to the range ϕ^* , and the value of b , controlling the outer boundaries to avoid boundary effects, should be larger than a .

The analysis will be presented separately according to the data design.

• Random data set

Table 3.1 summarizes, for the three groups and for the different combination of the parameters a and c , the average number of knots in the mesh. A more dense triangulation is obtained with smaller values of both a and c . We can observe that meshes created with domain and not taking any coordinate, always gets larger numbers of knots.

Table 3.1. Average number of knots for different combinations of the parameters a and c , with parameter $b = 0.3$ for random data.

		c=0.005	c=0.03	c=0.07	c=0.10
GM1	a=0.05	1593	1191	269	148
	a=0.09	598	567	270	146
	a=0.17	347	286	204	147
	a=0.20	333	269	170	143
GM2	a=0.05	2109	1586	370	208
	a=0.09	727	727	369	208
	a=0.17	252	252	252	209
	a=0.20	212	212	212	206
GM3	a=0.05	1599	1198	267	145
	a=0.09	567	565	267	145
	a=0.17	251	248	201	144
	a=0.20	221	218	163	141

We calculated the coverage (based on the 95% Confidence Interval), bias and mean square error (MSE) for each of the parameters in the model and for the predictions and also bias and MSE for the ratio $\frac{\sigma_e^2}{\phi}$, to investigate the impact of the mesh on the inference,

The values of the coverage (in %) for the covariance function parameters (σ_e^2 , σ_s^2 , k and ϕ^*) and for β_0 and β_1 are summarized on Table 3.2. In the Appendix B, Figure B.1 shows the coverage for all considered meshes. As we can observe in this table, all meshes have good results for coverage of β_0 and β_1 . For σ_e^2 , meshes from GM1 have better results, specially the ones with $c = 0.03$ and as the value of a gets higher, the coverage gets small. For the parameters σ_s^2 , k and ϕ^* , all the meshes, except the meshes from GM1 with $c = 0.005$, have similar and good results and this is very clear when looking at the Figure B.1 of the Appendix B. So, considering the coverage of all parameters together, mesh 6 was the one with the best results overall.

Table 3.2. Coverage, in %, of the 95% credible intervals of the parameters β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* for different meshes with $b = 0.3$, for random data set.

Group	Mesh	c	a	Coverage in % for:					
				β_0	β_1	σ_e^2	σ_s^2	k	ϕ^*
GM1	1	0.005	0.050	88.000	86.000	19.000	61.000	35.500	35.500
	2	0.005	0.090	89.000	83.000	11.000	22.000	23.500	23.500
	3	0.005	0.170	89.000	84.000	12.500	16.000	25.000	25.000
	4	0.005	0.200	89.500	84.000	13.000	15.000	27.500	28.000
	5	0.030	0.050	96.000	92.000	81.500	92.500	90.500	90.500
	6	0.030	0.090	97.500	92.000	82.500	94.500	91.000	90.500
	7	0.030	0.170	97.500	93.500	80.000	93.000	90.000	90.000
	8	0.030	0.200	97.500	93.500	80.000	90.500	90.500	90.500
	9	0.070	0.050	99.500	93.000	63.500	96.500	84.500	84.500
	10	0.070	0.090	99.500	93.500	62.500	95.000	84.500	84.500
	11	0.070	0.170	99.000	93.500	47.500	96.000	82.500	82.500
	12	0.070	0.200	99.500	92.000	37.000	96.500	81.500	81.500
	13	0.010	0.050	99.000	93.500	25.500	96.000	84.000	84.000
	14	0.010	0.090	99.000	94.000	26.500	96.000	83.000	83.000
	15	0.010	0.170	99.000	93.500	25.500	97.000	83.500	83.500
	16	0.010	0.200	98.500	93.500	23.500	96.000	85.000	85.000
GM2	17	0.005	0.050	94.000	91.000	63.000	68.500	65.500	65.500
	18	0.005	0.090	96.000	92.500	76.000	90.000	87.500	87.500
	19	0.005	0.170	98.000	92.000	31.500	86.500	84.000	84.000
	20	0.005	0.200	97.000	93.500	16.500	89.000	84.500	84.500
	21	0.030	0.050	96.000	90.000	68.000	75.000	74.500	74.500
	22	0.030	0.090	96.000	92.500	76.000	90.000	87.500	87.500
	23	0.030	0.170	98.000	92.000	31.500	86.500	84.000	84.000
	24	0.030	0.200	97.000	93.500	16.500	89.000	84.500	84.500
	25	0.070	0.050	99.500	92.500	50.500	88.000	86.000	86.000
	26	0.070	0.090	98.500	93.500	50.000	92.000	85.000	85.000
	27	0.070	0.170	98.000	92.000	31.500	86.500	84.000	84.000
	28	0.070	0.200	97.000	93.500	16.500	89.000	84.500	84.500
	29	0.010	0.050	98.500	92.500	23.500	89.000	85.500	85.500
	30	0.010	0.090	99.000	93.500	20.000	88.500	82.000	82.000
	31	0.010	0.170	99.000	94.500	18.000	89.000	85.500	86.000
	32	0.010	0.200	98.500	93.500	18.000	91.000	87.000	87.000
GM3	33	0.005	0.050	96.000	90.500	69.000	78.500	76.500	77.000
	34	0.005	0.090	98.000	91.000	77.500	92.500	89.000	89.000
	35	0.005	0.170	98.500	93.500	39.500	95.000	82.000	82.000
	36	0.005	0.200	98.500	94.000	29.000	94.500	81.500	81.500
	37	0.030	0.050	97.000	91.000	77.000	83.500	83.500	83.500
	38	0.030	0.090	98.000	91.500	77.500	92.500	89.500	89.500
	39	0.030	0.170	98.500	93.500	40.000	95.000	83.000	83.000
	40	0.030	0.200	98.000	94.000	29.000	94.500	82.500	82.500
	41	0.070	0.050	99.000	93.500	56.500	92.500	83.500	83.500
	42	0.070	0.090	99.000	93.500	55.500	93.500	84.000	84.000
	43	0.070	0.170	98.500	93.500	38.000	96.500	82.000	82.500
	44	0.070	0.200	98.000	94.000	25.500	94.000	83.000	83.000
	45	0.010	0.050	99.000	94.000	24.500	92.500	82.500	82.500
	46	0.010	0.090	99.000	94.000	25.000	93.000	83.500	83.500
	47	0.010	0.170	99.000	94.000	23.500	92.500	83.500	83.500
	48	0.010	0.200	99.000	94.000	20.000	92.500	84.000	84.000

Table 3.3 presents the absolute bias of the estimated parameters and for the ratio $\frac{\sigma_s^2}{\phi}$. We can notice that the bias for β_0 and β_1 have small values and similar results for the meshes. For σ_e^2 , the best value for the bias is obtained with mesh 6 and the highest value with mesh 2, but it is not so different from the values from other meshes. The small value of bias for σ_s^2 is with mesh 9 and for these parameter the values are higher than for σ_e^2 , but almost all meshes have the same behavior. The bias for parameter k some meshes obtained high and discrepant values from the others, as for example, mesh 1 has bias with value 11.943 and mesh 10 obtained value 0.051, and we can notice that meshes from GM1 have smaller results. For ϕ^* the values obtained for bias vary from 0.024 to 0.144, so there is no huge difference from each mesh. The results of bias for the ratio $\frac{\sigma_s^2}{\phi}$ also did not have big change between meshes and the lowest values were obtained with $c = 0.005$ from GM1.

Table B.1 from the Appendix B shows the results of MSE for the parameters and the ratio. Similar as the values of bias, we can notice high values of MSE for k and meshes from GM1 with $c = 0.03$ are the ones with small values and these values are really small when comparing with the other meshes. For the others parameters and for the ratio, there are no big difference of the values from all meshes.

Considering the variable, we calculated the coverage and MSE of prediction in the sample locations to investigate the impact of the mesh on the predictive performance of the model. These results are in Table 3.4 and we can notice that the highest values for coverage and the lowest values for MSE are from meshes with $c = 0.03$ from GM1; all the others meshes have similar results for MSE and only meshes with the lowest value of c from GM1 have very small values for coverage.

The predictive performance of the model was also investigated for locations not used in the

Table 3.3. Absolute bias of the parameters $\beta_0, \beta_1, \sigma_e^2, \sigma_s^2, k$ and ϕ^* and for $\frac{\sigma_s^2}{\phi}$, considering all meshes with random data set.

Group	Mesh	c	a	Bias for:						
				β_0	β_1	σ_e^2	σ_s^2	k	ϕ^*	$\frac{\sigma_s^2}{\phi}$
GM1	1	0.005	0.050	0.004	0.030	0.246	0.232	11.943	0.055	0.010
	2	0.005	0.090	0.001	0.035	0.261	0.445	12.328	0.075	0.015
	3	0.005	0.170	0.001	0.040	0.257	0.574	8.716	0.063	0.016
	4	0.005	0.200	0.002	0.039	0.256	0.601	8.127	0.060	0.024
	5	0.030	0.050	0.005	0.025	0.045	0.051	2.395	0.057	0.037
	6	0.030	0.090	0.008	0.018	0.014	0.022	0.139	0.089	0.050
	7	0.030	0.170	0.008	0.019	0.035	0.032	0.263	0.093	0.060
	8	0.030	0.200	0.008	0.020	0.040	0.043	0.384	0.091	0.064
	9	0.070	0.050	0.008	0.017	0.137	0.001	0.072	0.140	0.083
	10	0.070	0.090	0.009	0.017	0.140	0.006	0.051	0.137	0.082
	11	0.070	0.170	0.009	0.018	0.167	0.024	0.054	0.145	0.084
	12	0.070	0.200	0.009	0.017	0.183	0.065	0.308	0.138	0.080
	13	0.010	0.050	0.010	0.020	0.216	0.164	1.185	0.138	0.086
	14	0.010	0.090	0.009	0.021	0.216	0.165	1.306	0.138	0.086
	15	0.010	0.170	0.009	0.021	0.215	0.168	1.197	0.138	0.086
	16	0.010	0.200	0.010	0.019	0.220	0.162	1.289	0.132	0.085
GM2	17	0.005	0.050	0.013	0.012	0.068	0.322	7.567	0.024	0.040
	18	0.005	0.090	0.008	0.021	0.066	0.261	2.707	0.079	0.074
	19	0.005	0.170	0.007	0.024	0.198	0.662	3.907	0.088	0.097
	20	0.005	0.200	0.007	0.025	0.237	0.631	3.753	0.101	0.121
	21	0.030	0.050	0.011	0.010	0.033	0.290	5.703	0.038	0.048
	22	0.030	0.090	0.008	0.021	0.066	0.261	2.707	0.079	0.074
	23	0.030	0.170	0.007	0.024	0.198	0.662	3.907	0.088	0.097
	24	0.030	0.200	0.007	0.025	0.237	0.631	3.753	0.101	0.121
	25	0.070	0.050	0.007	0.022	0.154	0.353	2.582	0.085	0.081
	26	0.070	0.090	0.006	0.025	0.155	0.313	2.456	0.090	0.084
	27	0.070	0.170	0.007	0.024	0.198	0.662	3.907	0.088	0.097
	28	0.070	0.200	0.007	0.025	0.237	0.631	3.753	0.101	0.121
	29	0.010	0.050	0.006	0.024	0.222	0.874	4.695	0.086	0.115
	30	0.010	0.090	0.005	0.023	0.224	0.723	4.108	0.105	0.121
	31	0.010	0.170	0.003	0.028	0.227	0.565	3.250	0.095	0.101
	32	0.010	0.200	0.005	0.026	0.226	0.840	4.039	0.098	0.116
GM3	33	0.005	0.050	0.010	0.013	0.061	0.125	3.618	0.043	0.036
	34	0.005	0.090	0.012	0.013	0.064	0.036	0.293	0.106	0.066
	35	0.005	0.170	0.011	0.016	0.183	0.096	0.674	0.143	0.082
	36	0.005	0.200	0.009	0.018	0.206	0.138	1.154	0.138	0.078
	37	0.030	0.050	0.011	0.012	0.023	0.127	3.561	0.050	0.040
	38	0.030	0.090	0.012	0.012	0.064	0.037	0.312	0.106	0.067
	39	0.030	0.170	0.011	0.016	0.183	0.096	0.679	0.144	0.082
	40	0.030	0.200	0.009	0.018	0.206	0.136	1.145	0.137	0.077
	41	0.070	0.050	0.007	0.018	0.147	0.049	0.034	0.133	0.077
	42	0.070	0.090	0.006	0.018	0.147	0.031	0.298	0.136	0.076
	43	0.070	0.170	0.009	0.019	0.186	0.088	0.572	0.142	0.084
	44	0.070	0.200	0.010	0.018	0.212	0.136	0.701	0.140	0.079
	45	0.010	0.050	0.010	0.016	0.216	0.293	2.343	0.126	0.083
	46	0.010	0.090	0.010	0.017	0.218	0.348	2.457	0.126	0.084
	47	0.010	0.170	0.011	0.016	0.219	0.291	2.197	0.128	0.084
	48	0.010	0.200	0.011	0.015	0.222	0.352	2.367	0.131	0.085

Table 3.4. Nominal coverage, in %, and MSE for the variable Y calculated for each mesh, considering 200 simulations, with random data set.

Group	Mesh	c	a	Coverage(%)	MSE	Group	Mesh	c	a	Coverage(%)	MSE
GM1	1	0.005	0.050	28.955	0.273	GM2	25	0.070	0.050	88.115	0.238
	2	0.005	0.090	23.740	0.281		26	0.070	0.090	87.925	0.238
	3	0.005	0.170	23.385	0.281		27	0.070	0.170	85.045	0.259
	4	0.005	0.200	23.650	0.280		28	0.070	0.200	81.745	0.282
	5	0.030	0.050	87.405	0.205		29	0.010	0.050	82.775	0.273
	6	0.030	0.090	90.485	0.203		30	0.010	0.090	83.005	0.272
	7	0.030	0.170	90.795	0.205		31	0.010	0.170	83.105	0.273
	8	0.030	0.200	90.870	0.205		32	0.010	0.200	82.585	0.275
	9	0.070	0.050	88.270	0.229	GM3	33	0.005	0.050	78.620	0.218
	10	0.070	0.090	88.100	0.230		34	0.005	0.090	89.875	0.211
	11	0.070	0.170	86.455	0.242		35	0.005	0.170	85.515	0.249
	12	0.070	0.200	85.485	0.249		36	0.005	0.200	83.610	0.261
	13	0.010	0.050	82.745	0.268		37	0.030	0.050	86.180	0.209
	14	0.010	0.090	82.710	0.268		38	0.030	0.090	89.875	0.211
	15	0.010	0.170	82.815	0.268		39	0.030	0.170	85.510	0.249
	16	0.010	0.200	82.540	0.270		40	0.030	0.200	83.605	0.261
GM2	17	0.005	0.050	71.860	0.223		41	0.070	0.050	87.790	0.234
	18	0.005	0.090	89.760	0.212		42	0.070	0.090	87.795	0.234
	19	0.005	0.170	85.045	0.259		43	0.070	0.170	85.310	0.251
	20	0.005	0.200	81.745	0.282		44	0.070	0.200	83.090	0.265
	21	0.030	0.050	79.475	0.215		45	0.010	0.050	82.805	0.268
	22	0.030	0.090	89.760	0.212		46	0.010	0.090	82.690	0.269
	23	0.030	0.170	85.045	0.259		47	0.010	0.170	82.740	0.269
	24	0.030	0.200	81.745	0.282		48	0.010	0.200	82.150	0.272

analysis and it was made in 25 locations. Results for coverage and MSE for these 25 locations are presented in Table 3.5. We can notice that the values for MSE are very similar for all the meshes, the results are around 0.4. The values for coverage for the variable in 25 extra locations are not all so high as for 100 locations in the analysis, most meshes have coverage less than 80% and the mesh with highest

coverage is mesh 17 ($a = 0.05$ and $c = 0.005$), from GM2. Mesh 6 obtained the lowest MSE (0.443) and coverage of 83.22, being that the best value was 92.08%.

Table 3.5. Nominal coverage, in %, and MSE for the variable Y_{na} calculated for each mesh in 25 locations, considering 200 simulations, with random data set.

Group	Mesh	c	a	Coverage(%)	MSE	Group	Mesh	c	a	Coverage(%)	MSE
GM1	1	0.005	0.050	90.580	0.451	GM2	25	0.070	0.050	82.680	0.452
	2	0.005	0.090	80.620	0.452		26	0.070	0.090	83.240	0.451
	3	0.005	0.170	65.840	0.467		27	0.070	0.170	80.320	0.459
	4	0.005	0.200	61.900	0.473		28	0.070	0.200	76.000	0.466
	5	0.030	0.050	87.820	0.444		29	0.010	0.050	77.780	0.461
	6	0.030	0.090	83.220	0.443		30	0.010	0.090	77.020	0.462
	7	0.030	0.170	75.700	0.444		31	0.010	0.170	77.260	0.462
	8	0.030	0.200	72.920	0.448		32	0.010	0.200	77.380	0.461
	9	0.070	0.050	77.480	0.447	GM3	33	0.005	0.050	90.020	0.453
	10	0.070	0.090	77.660	0.446		34	0.005	0.090	84.400	0.447
	11	0.070	0.170	74.620	0.452		35	0.005	0.170	74.420	0.454
	12	0.070	0.200	71.060	0.454		36	0.005	0.200	70.680	0.461
	13	0.010	0.050	70.620	0.462		37	0.030	0.050	88.620	0.452
	14	0.010	0.090	70.060	0.463		38	0.030	0.090	84.380	0.447
	15	0.010	0.170	70.620	0.462		39	0.030	0.170	74.460	0.454
	16	0.010	0.200	70.080	0.464		40	0.030	0.200	70.640	0.461
GM2	17	0.005	0.050	92.080	0.463		41	0.070	0.050	78.020	0.447
	18	0.005	0.090	87.720	0.451		42	0.070	0.090	78.240	0.446
	19	0.005	0.170	80.320	0.459		43	0.070	0.170	74.160	0.455
	20	0.005	0.200	76.000	0.466		44	0.070	0.200	70.080	0.461
	21	0.030	0.050	91.700	0.462		45	0.010	0.050	70.900	0.465
	22	0.030	0.090	87.720	0.451		46	0.010	0.090	70.800	0.464
	23	0.030	0.170	80.320	0.459		47	0.010	0.170	70.620	0.465
	24	0.030	0.200	76.000	0.466		48	0.010	0.200	70.680	0.466

The bias for the prediction in these 25 extra locations was calculated and plotted with boxplot, as we can see in Figure 3.1. The 48 boxplot for the values of bias are quite the same and we can notice that there is no difference within and between the groups.

Figure 3.1. Bias for Y in 25 extra locations, with random data set.

These results showed that there is a large impact of the mesh on the inference (parameter estimation and prediction) and it was observed that there is not a single mesh which is optimal in everything. Therefore, we investigate whether we can use the predictive information criteria as described in Section 2.1, to select an optimal mesh.

We calculate the mean of DIC, WAIC and logCPO (Table 3.6) to perform mesh selection. Minimum values of DIC and WAIC occur with meshes from GM1 with $c = 0.005$ and occur negative

values for DIC and WAIC, as we can notice in Table 3.6. Disregarding negative values, the minimum values for DIC and WAIC occur with mesh 17 (35.877 and 43.257) and it is the mesh with more knots (2109 knots, Table 3.1). This does not occur for logCPO. Meshes 7 and 8 have minimum value for logCPO (1.279), followed by mesh 6 and other meshes, with value of 1.281. Mesh 6 has value 213.639 for DIC and 213.871 for WAIC.

Table 3.6. Mean values of DIC, WAIC and logCPO for meshes with random data set .

Group	Mesh	c	a	DIC	WAIC	logCPO	Group	Mesh	c	a	DIC	WAIC	logCPO
GM1	1	0.005	0.050	-348.646	-347.352	1.935	GM2	25	0.070	0.050	242.810	244.572	1.283
	2	0.005	0.090	-387.573	-392.315	1.981		26	0.070	0.090	243.059	244.826	1.283
	3	0.005	0.170	-388.284	-394.420	1.965		27	0.070	0.170	247.626	249.531	1.287
	4	0.005	0.200	-383.974	-390.865	1.949		28	0.070	0.200	251.459	253.263	1.292
	5	0.030	0.050	170.092	176.546	1.316		29	0.010	0.050	249.753	251.548	1.288
	6	0.030	0.090	213.639	213.871	1.281		30	0.010	0.090	250.074	251.837	1.289
	7	0.030	0.170	222.671	222.477	1.279		31	0.010	0.170	250.767	252.612	1.292
	8	0.030	0.200	223.690	223.589	1.279		32	0.010	0.200	250.367	252.118	1.290
	9	0.070	0.050	241.802	242.580	1.281	GM3	33	0.005	0.050	90.609	103.326	1.411
	10	0.070	0.090	242.071	242.810	1.282		34	0.005	0.090	223.217	224.530	1.287
	11	0.070	0.170	245.125	245.932	1.284		35	0.005	0.170	246.560	247.588	1.284
	12	0.070	0.200	246.970	247.811	1.285		36	0.005	0.200	248.739	249.705	1.286
	13	0.010	0.050	249.704	250.704	1.288		37	0.030	0.050	163.988	169.977	1.331
	14	0.010	0.090	249.621	250.623	1.288		38	0.030	0.090	223.214	224.509	1.287
	15	0.010	0.170	249.675	250.683	1.288		39	0.030	0.170	246.600	247.616	1.284
	16	0.010	0.200	250.128	251.117	1.289		40	0.030	0.200	248.768	249.734	1.286
GM2	17	0.005	0.050	35.877	43.257	1.487		41	0.070	0.050	242.290	243.264	1.281
	18	0.005	0.090	219.211	221.541	1.288		42	0.070	0.090	242.284	243.290	1.281
	19	0.005	0.170	247.626	249.531	1.287		43	0.070	0.170	247.018	248.037	1.285
	20	0.005	0.200	251.459	253.263	1.292		44	0.070	0.200	249.300	250.256	1.287
	21	0.030	0.050	110.325	115.101	1.400		45	0.010	0.050	249.623	250.653	1.287
	22	0.030	0.090	219.211	221.541	1.288		46	0.010	0.090	249.764	250.797	1.288
	23	0.030	0.170	247.626	249.531	1.287		47	0.010	0.170	249.805	250.854	1.288
	24	0.030	0.200	251.459	253.263	1.292		48	0.010	0.200	250.144	251.178	1.288

Taking all the results into account (bias, MSE, coverage of parameter estimates and predictions), we can consider that mesh 6 is the optimal one from random data set. In Figure 3.2 we can observe boxplots of parameter estimates based on the mesh with lowest DIC, WAIC or logCPO, obtained in each simulation and for comparison, we also present in this figure the boxplots of the estimates obtained for each parameter with mesh 6.

We can observe that the boxplots obtained with mesh 6 are more similar to the boxplots obtained for the results of logCPO. Also, selection of the mesh using DIC and WAIC has bad performance, seen in the large bias of all covariance parameters. While the difference between the DIC, WAIC and logCPO would be expected to behave similarly, since they are asymptotically the same, it is important to realize that all methods are only approximations of the predictive accuracy. While DIC and WAIC give reasonable answers in many cases, they only attempt to capture the predictive behavior within the sample, whereas the logCPO captures also out-of-sample prediction error. So, it is confirmed that the logCPO criterion is better suited for mesh selection.

Figure 3.2. Estimated β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* , considering the meshes with minimal DIC, WAIC and logCPO, and based on mesh 6 for random data set.

• Regular data set

As previously stated, for regular data set we do not have the batches of 40 simulations, we only have one simulated coordinate. So, Table 3.7 shows the number of knots in the mesh, for the three groups. For regular data, meshes created with part of the coordinate (GM3), obtained smaller numbers of knots, when comparing with GM1 and GM2.

Table 3.7. Number of knots for different combinations of the parameters a and c , with parameter $b = 0.3$ for regular data.

		c=0.005	c=0.03	c=0.07	c=0.10
GM1	a=0.05	2211	1365	309	179
	a=0.09	648	648	310	179
	a=0.17	228	228	201	179
	a=0.20	212	212	182	179
GM2	a=0.05	2109	1586	370	208
	a=0.09	727	727	369	208
	a=0.17	252	252	252	209
	a=0.20	212	212	212	206
GM3	a=0.05	1925	1467	279	167
	a=0.09	614	614	281	167
	a=0.17	227	227	187	167
	a=0.20	198	198	169	167

The values of the coverage (in %) for all the parameters (β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^*) are summarized on Table 3.8. In the Appendix B, Figure B.2 shows the coverage for all considered meshes. As we can

observe in these table and figure, for β_0 and β_1 , all meshes have good coverage. Observing the results for the others parameters, we can notice that meshes from GM1 have bad results. In fact, for σ_e^2 , all the meshes from GM1 have 0% of coverage. Some meshes from GM2 also have 0% coverage for σ_e^2 but meshes 19, 23 and 27 obtained 35.5%. Meshes 35 and 39 (both from GM3) were the ones that obtained the best result for this parameter: 41% for coverage. For σ_s^2 , some meshes from GM1 obtained 0% of coverage, the majority of the meshes of GM2 have good results and the best value for coverage is from mesh 46 (GM3) and mesh 36 (GM2) with 96%. Meshes 19, 23, 27, 35 and 39 all obtained 95% of coverage for σ_s^2 . The results for parameter k are more homogeneous between the groups and mesh 31 was the one with better results, 90% of coverage. In the last column of the table are the results for ϕ^* and we can observe that the results are quite similar to the ones for k . The best value is for mesh 31 (90% of coverage). Meshes 19, 23 and 27 obtained 87% and meshes 35 and 39 obtained 84% both for k and ϕ^* .

Table 3.8. Coverage, in %, of the 95% credible intervals of the parameters β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* for different meshes with $b = 0.3$, for regular data set.

Group	Mesh	c	a	Coverage in % for:					
				β_0	β_1	σ_e^2	σ_s^2	k	ϕ^*
GM1	1	0.005	0.050	95.000	92.000	0.000	65.500	73.500	73.500
	2	0.005	0.090	95.500	91.500	0.000	31.500	63.000	63.500
	3	0.005	0.170	92.500	91.000	0.000	0.000	31.500	31.500
	4	0.005	0.200	93.000	92.500	0.000	0.000	33.500	34.000
	5	0.030	0.050	95.500	91.500	0.000	57.000	73.000	73.000
	6	0.030	0.090	95.500	91.500	0.000	31.500	63.000	63.500
	7	0.030	0.170	92.500	91.000	0.000	0.000	31.500	31.500
	8	0.030	0.200	93.000	92.500	0.000	0.000	33.500	34.000
	9	0.070	0.050	95.500	91.000	0.000	10.000	60.500	63.000
	10	0.070	0.090	95.500	91.500	0.000	9.000	59.500	60.500
	11	0.070	0.170	92.500	91.500	0.000	0.000	31.500	33.000
	12	0.070	0.200	92.500	91.500	0.000	0.500	34.000	36.000
	13	0.010	0.050	93.000	91.000	0.000	0.000	33.000	34.500
	14	0.010	0.090	93.000	91.000	0.000	0.000	33.000	34.500
	15	0.010	0.170	93.000	91.000	0.000	0.000	33.000	34.500
	16	0.010	0.200	93.000	91.000	0.000	0.000	33.000	34.500
GM2	17	0.005	0.050	93.500	92.000	0.000	4.000	55.000	55.500
	18	0.005	0.090	93.500	92.500	0.000	0.000	37.500	37.500
	19	0.005	0.170	97.000	90.500	35.500	95.000	87.000	87.000
	20	0.005	0.200	97.000	91.500	15.500	95.500	87.500	87.000
	21	0.030	0.050	96.000	91.000	0.000	1.500	53.500	53.500
	22	0.030	0.090	93.500	92.500	0.000	0.000	37.500	37.500
	23	0.030	0.170	97.000	90.500	35.500	95.000	87.000	87.000
	24	0.030	0.200	97.000	91.500	15.500	95.500	87.500	87.000
	25	0.070	0.050	92.000	90.500	15.000	27.000	27.500	27.500
	26	0.070	0.090	93.000	90.500	17.500	33.000	33.500	34.000
	27	0.070	0.170	97.000	90.500	35.500	95.000	87.000	87.000
	28	0.070	0.200	97.000	91.500	15.500	95.500	87.500	87.000
	29	0.010	0.050	97.500	92.500	18.500	95.000	86.000	85.500
	30	0.010	0.090	96.000	92.500	24.500	94.000	86.000	86.000
	31	0.010	0.170	96.500	90.500	20.000	95.000	90.000	90.000
	32	0.010	0.200	96.500	91.000	20.500	96.000	88.000	88.000
GM3	33	0.005	0.050	96.000	92.500	0.000	30.500	76.500	76.500
	34	0.005	0.090	96.500	90.000	0.000	1.500	65.500	65.500
	35	0.005	0.170	97.500	92.000	41.000	95.000	84.000	84.000
	36	0.005	0.200	97.500	92.500	35.000	95.000	84.000	84.000
	37	0.030	0.050	96.000	91.500	0.000	20.000	72.500	72.500
	38	0.030	0.090	96.500	90.000	0.000	1.500	65.500	65.500
	39	0.030	0.170	97.500	92.000	41.000	95.000	84.000	84.000
	40	0.030	0.200	97.500	92.500	35.000	95.000	84.000	84.000
	41	0.070	0.050	93.000	87.000	13.000	23.000	34.500	35.500
	42	0.070	0.090	92.500	88.000	8.000	15.500	32.500	34.000
	43	0.070	0.170	97.500	92.000	38.000	95.000	82.500	82.500
	44	0.070	0.200	96.500	92.500	36.000	95.500	83.500	83.500
	45	0.010	0.050	96.500	92.000	37.000	95.000	83.000	83.000
	46	0.010	0.090	97.000	91.500	34.500	96.000	83.500	83.500
	47	0.010	0.170	96.500	92.000	37.000	95.000	83.000	83.000
	48	0.010	0.200	96.500	92.000	37.000	95.000	83.000	83.000

We also calculated the bias for all the parameters and for the ratio $\frac{\sigma_s^2}{\phi}$. These results are presented in Table 3.9 and as we can notice, for β_0 the values are very similar and for the parameter β_1 , meshes from GM2 have slightly better results. For σ_e^2 , all meshes from GM1 have the same results, meshes from GM2 have similar results and meshes 25 and 26 stand out with the lowest values of this group (0.155 and 0.126, respectively) and from GM3, the best one is mesh 41 with value of 0.167 for coverage. The bias for σ_s^2 have some high values and the lowest one is 0.103 from mesh 29. For the parameter k , big values for bias appear for meshes from GM1 and the lowest value is 0.028 of mesh 31. The values for the bias are more homogeneous for parameter ϕ^* and mesh 26 has the lowest value (0.013).

Results for the ratio $\frac{\sigma_s^2}{\phi}$ do not differ much between the meshes. The lowest value of bias (0.008) for the ratio was obtained with mesh 1. Although meshes 19, 23, 27, 35 and 39 did not obtained the best results of bias for either any parameter or the ratio, the results obtained was not so bad, compared with the best ones.

Table 3.9. Absolute bias of the parameters $\beta_0, \beta_1, \sigma_e^2, \sigma_s^2, k$ and ϕ^* and for $\frac{\sigma_s^2}{\phi}$, considering all meshes with regular data set.

Group	Mesh	c	a	Bias for:						
				β_0	β_1	σ_e^2	σ_s^2	k	ϕ^*	$\frac{\sigma_s^2}{\phi}$
GM1	1	0.005	0.050	0.015	0.009	0.300	0.380	8.246	0.044	0.008
	2	0.005	0.090	0.015	0.009	0.300	1.266	10.831	0.049	0.018
	3	0.005	0.170	0.015	0.005	0.300	1.710	11.672	0.085	0.052
	4	0.005	0.200	0.015	0.007	0.300	2.363	13.660	0.086	0.063
	5	0.030	0.050	0.014	0.010	0.300	0.418	7.170	0.038	0.015
	6	0.030	0.090	0.015	0.009	0.300	1.266	10.831	0.049	0.018
	7	0.030	0.170	0.015	0.005	0.300	1.710	11.672	0.085	0.052
	8	0.030	0.200	0.015	0.007	0.300	2.363	13.660	0.086	0.063
	9	0.070	0.050	0.014	0.011	0.300	2.187	10.991	0.042	0.053
	10	0.070	0.090	0.014	0.012	0.300	1.981	9.727	0.041	0.051
	11	0.070	0.170	0.015	0.006	0.300	2.786	13.740	0.085	0.058
	12	0.070	0.200	0.017	0.005	0.300	5.736	15.966	0.084	0.073
	13	0.010	0.050	0.017	0.005	0.300	5.357	17.596	0.086	0.078
	14	0.010	0.090	0.017	0.005	0.300	5.357	17.596	0.086	0.078
	15	0.010	0.170	0.017	0.005	0.300	5.357	17.596	0.086	0.078
	16	0.010	0.200	0.017	0.005	0.300	5.357	17.596	0.086	0.078
GM2	17	0.005	0.050	0.015	0.007	0.300	0.791	9.115	0.067	0.025
	18	0.005	0.090	0.015	0.005	0.300	2.015	13.050	0.083	0.054
	19	0.005	0.170	0.018	0.009	0.248	0.257	0.119	0.164	0.081
	20	0.005	0.200	0.022	0.002	0.319	0.177	0.110	0.182	0.076
	21	0.030	0.050	0.013	0.011	0.300	0.691	8.119	0.070	0.027
	22	0.030	0.090	0.015	0.005	0.300	2.015	13.050	0.083	0.054
	23	0.030	0.170	0.018	0.009	0.248	0.257	0.119	0.164	0.081
	24	0.030	0.200	0.022	0.002	0.319	0.177	0.110	0.182	0.076
	25	0.070	0.050	0.014	0.008	0.155	4.720	19.477	0.036	0.123
	26	0.070	0.090	0.015	0.009	0.126	9.635	21.615	0.013	0.141
	27	0.070	0.170	0.018	0.009	0.248	0.257	0.119	0.164	0.081
	28	0.070	0.200	0.022	0.002	0.319	0.177	0.110	0.182	0.076
	29	0.010	0.050	0.019	0.008	0.316	0.103	0.388	0.208	0.073
	30	0.010	0.090	0.018	0.008	0.301	0.242	0.409	0.185	0.077
	31	0.010	0.170	0.018	0.009	0.306	0.176	0.028	0.168	0.067
	32	0.010	0.200	0.019	0.010	0.305	0.169	0.030	0.175	0.070
GM3	33	0.005	0.050	0.013	0.013	0.300	0.326	4.011	0.044	0.022
	34	0.005	0.090	0.013	0.011	0.300	1.107	6.607	0.052	0.056
	35	0.005	0.170	0.014	0.014	0.238	0.139	2.763	0.241	0.076
	36	0.005	0.200	0.014	0.012	0.258	0.153	2.846	0.246	0.076
	37	0.030	0.050	0.013	0.012	0.300	0.480	6.321	0.046	0.026
	38	0.030	0.090	0.013	0.011	0.300	1.107	6.607	0.052	0.056
	39	0.030	0.170	0.014	0.014	0.238	0.139	2.763	0.241	0.076
	40	0.030	0.200	0.014	0.012	0.258	0.153	2.846	0.246	0.076
	41	0.070	0.050	0.009	0.020	0.167	2.987	12.030	0.018	0.090
	42	0.070	0.090	0.012	0.015	0.210	2.559	11.241	0.026	0.085
	43	0.070	0.170	0.013	0.015	0.249	0.153	2.811	0.245	0.077
	44	0.070	0.200	0.014	0.013	0.256	0.151	2.725	0.244	0.075
	45	0.010	0.050	0.013	0.013	0.254	0.148	2.752	0.243	0.075
	46	0.010	0.090	0.013	0.013	0.263	0.142	2.784	0.262	0.083
	47	0.010	0.170	0.013	0.013	0.254	0.148	2.768	0.243	0.075
	48	0.010	0.200	0.013	0.013	0.254	0.148	2.768	0.243	0.075

Table B.2 from the Appendix B shows the results of MSE for the parameters and the ratio, with regular data set. The results are quite similar to the ones obtained for the bias. There is no mesh that stands out for the values of MSE for β_0 and β_1 . For σ_e^2 , meshes 25 and 26 were the best ones, with MSE of 0.085, followed by meshes 35 and 39, with value of 0.086. The values of MSE for the parameter σ_s^2 are very high for some meshes and the lowest value is 0.120 obtained with meshes 35 and 39. The best value of MSE for k was 34.614 with mesh 33, followed by 37.899 with meshes 35 and 39. Most of all the other meshes have very high values. For the nominal range, the lowest results was 0.004 with mesh 33, followed by 0.005 with several meshes, including meshes 35 and 39. The value of MSE for the ratio was similar between the meshes, the best results (0.004) with meshes 17 and 21. For values of MSE, meshes 35 and 39 have better results than meshes 19, 23 and 27.

We also calculated the coverage (in %) and MSE of prediction in the 100 coordinate of sample locations to investigate the impact of the mesh on the predictive performance of the model. Table 3.10 shows these results and we can notice that the values of coverage for meshes from GM1 are very low, meshes from GM2 with small values of a (0.05 and 0.09) with small values of c (0.005 and 0.03) also have low value for coverage and the same occur for these configuration of meshes from GM3. The best result

for coverage is 86.765% with meshes 35 and 39, meshes 19,23 and 27 obtained 85.720%. For MSE, all meshes have similar results.

Table 3.10. Nominal coverage, in %, and MSE for the variable Y calculated for each mesh, considering 200 simulations, with regular data set.

Group	Mesh	c	a	Coverage(%)	MSE	Group	Mesh	c	a	Coverage(%)	MSE
GM1	1	0.005	0.050	3.100	0.301	GM2	25	0.070	0.050	28.510	0.295
	2	0.005	0.090	3.040	0.301		26	0.070	0.090	33.290	0.294
	3	0.005	0.170	2.980	0.301		27	0.070	0.170	85.720	0.307
	4	0.005	0.200	2.980	0.301		28	0.070	0.200	80.830	0.348
	5	0.030	0.050	3.065	0.301		29	0.010	0.050	80.760	0.347
	6	0.030	0.090	3.040	0.301		30	0.010	0.090	81.920	0.339
	7	0.030	0.170	2.980	0.301		31	0.010	0.170	82.300	0.340
	8	0.030	0.200	2.980	0.301		32	0.010	0.200	81.430	0.342
	9	0.070	0.050	2.990	0.301	GM3	33	0.005	0.050	3.125	0.301
	10	0.070	0.090	3.000	0.301		34	0.005	0.090	3.075	0.301
	11	0.070	0.170	2.965	0.301		35	0.005	0.170	86.765	0.300
	12	0.070	0.200	2.980	0.301		36	0.005	0.200	85.730	0.310
	13	0.010	0.050	2.975	0.301		37	0.030	0.050	3.105	0.301
	14	0.010	0.090	2.975	0.301		38	0.030	0.090	3.075	0.301
	15	0.010	0.170	2.975	0.301		39	0.030	0.170	86.765	0.300
	16	0.010	0.200	2.975	0.301		40	0.030	0.200	85.730	0.310
GM2	17	0.005	0.050	3.105	0.301		41	0.070	0.050	25.660	0.301
	18	0.005	0.090	3.035	0.301		42	0.070	0.090	17.400	0.302
	19	0.005	0.170	85.720	0.307		43	0.070	0.170	86.345	0.305
	20	0.005	0.200	80.830	0.348		44	0.070	0.200	85.880	0.309
	21	0.030	0.050	3.085	0.301		45	0.010	0.050	85.870	0.307
	22	0.030	0.090	3.035	0.301		46	0.010	0.090	85.670	0.312
	23	0.030	0.170	85.720	0.307		47	0.010	0.170	85.875	0.307
	24	0.030	0.200	80.830	0.348		48	0.010	0.200	85.875	0.307

The predictive performance of the model was also investigated for locations not used in the analysis and it was made in 25 locations, randomly selected. These results for coverage and MSE are presented in Table 3.11. We can observe that the values for coverage for these 25 extra locations are more homogeneous than for the coverage for the sample set. The best value (90.5%) was obtained with mesh 17. Meshes 19, 23 and 27 obtained 75.580% while meshes 35 and 39 obtained 70.720%. For MSE values, there is no mesh that stands out from the rest.

Table 3.11. Nominal coverage, in %, and MSE for the variable Y_{na} calculated for each mesh in 25 locations, considering 200 simulations, with regular data set.

Group	Mesh	c	a	Coverage(%)	MSE	Group	Mesh	c	a	Coverage(%)	MSE
GM1	1	0.005	0.050	88.480	0.435	GM2	25	0.070	0.050	72.420	0.599
	2	0.005	0.090	78.840	0.447		26	0.070	0.090	71.960	0.605
	3	0.005	0.170	19.720	0.504		27	0.070	0.170	75.580	0.463
	4	0.005	0.200	19.840	0.504		28	0.070	0.200	72.880	0.460
	5	0.030	0.050	84.720	0.439		29	0.010	0.050	74.500	0.463
	6	0.030	0.090	78.840	0.447		30	0.010	0.090	75.880	0.461
	7	0.030	0.170	19.720	0.504		31	0.010	0.170	73.980	0.460
	8	0.030	0.200	19.840	0.504		32	0.010	0.200	74.840	0.465
	9	0.070	0.050	60.300	0.467	GM3	33	0.005	0.050	89.680	0.440
	10	0.070	0.090	60.220	0.467		34	0.005	0.090	78.180	0.499
	11	0.070	0.170	19.860	0.504		35	0.005	0.170	70.720	0.445
	12	0.070	0.200	19.840	0.504		36	0.005	0.200	70.140	0.447
	13	0.010	0.050	19.820	0.504		37	0.030	0.050	87.860	0.447
	14	0.010	0.090	19.820	0.504		38	0.030	0.090	78.180	0.499
	15	0.010	0.170	19.820	0.504		39	0.030	0.170	70.720	0.445
	16	0.010	0.200	19.820	0.504		40	0.030	0.200	70.140	0.447
GM2	17	0.005	0.050	90.500	0.466		41	0.070	0.050	60.040	0.560
	18	0.005	0.090	85.280	0.493		42	0.070	0.090	59.320	0.571
	19	0.005	0.170	75.580	0.463		43	0.070	0.170	70.800	0.444
	20	0.005	0.200	72.880	0.460		44	0.070	0.200	69.840	0.446
	21	0.030	0.050	89.320	0.468		45	0.010	0.050	69.960	0.446
	22	0.030	0.090	85.280	0.493		46	0.010	0.090	70.140	0.448
	23	0.030	0.170	75.580	0.463		47	0.010	0.170	69.960	0.446
	24	0.030	0.200	72.880	0.460		48	0.010	0.200	69.960	0.446

The bias for the prediction in these 25 extra locations was calculated and plotted with individual boxplot for each mesh, as we can see in Figure 3.3. All the boxplots for the values of bias are quite the same and we can notice that there is no difference between groups and nor among groups. Boxplot of meshes with $c = 0.10$ tends to be more homogeneous.

Figure 3.3. Bias for Y in 25 extra locations, with regular data set.

As for the random data set, we can observe that these results showed that there is a large impact of the mesh on parameter estimation and prediction and there is not a single mesh which is optimal in everything. So, to select an optimal mesh, we calculate the mean of DIC, WAIC and logCPO and these results are in Table 3.12. We can notice that, despite only meshes 1 and 5 have more than one thousand knots, all meshes from GM1 have similar results for DIC and WAIC and are among the lowest values. But, meshes from this group did not have any good results. Considering logCPO, the best value was 1.361 from meshes 30 and 32. Meshes 19, 23 and 27 obtained 254.947(DIC), 258.455 (WAIC) and 1.364 (logCPO). Meshes 35 and 39, obtained 1.367 for logCPO and for DIC and WAIC, this two meshes obtained 256.932 and 258.662, respectively. The three meshes from GM2 have better results.

Meshes 19, 23 and 27 have all the same results and in fact, have the same number of knots and the same occur with the meshes 35 and 39 between them. If we consider all the results of bias, MSE, coverage of parameter estimates, predictions and also DIC, WAIC and logCPO we can assume that the three meshes from GM2 are the optimal ones.

Table 3.12. Mean values of DIC, WAIC and logCPO for meshes with regular data set .

Group	Mesh	c	a	DIC	WAIC	logCPO	Group	Mesh	c	a	DIC	WAIC	logCPO
GM1	1	0.005	0.050	-579.196	-592.839	2.312	GM2	25	0.070	0.050	-335.895	-345.227	2.370
	2	0.005	0.090	-580.335	-594.935	2.391		26	0.070	0.090	-286.396	-294.157	2.274
	3	0.005	0.170	-582.154	-598.045	2.474		27	0.070	0.170	254.947	258.455	1.364
	4	0.005	0.200	-582.022	-597.948	2.475		28	0.070	0.200	264.722	266.418	1.363
	5	0.030	0.050	-579.478	-593.444	2.336		29	0.010	0.050	264.149	265.766	1.362
	6	0.030	0.090	-580.335	-594.935	2.391		30	0.010	0.090	262.865	264.600	1.361
	7	0.030	0.170	-582.154	-598.045	2.474		31	0.010	0.170	263.674	265.591	1.363
	8	0.030	0.200	-582.022	-597.948	2.475		32	0.010	0.200	263.287	264.924	1.361
	9	0.070	0.050	-581.576	-597.302	2.502	GM3	33	0.005	0.050	-579.016	-592.257	2.340
	10	0.070	0.090	-581.712	-597.473	2.477		34	0.005	0.090	-579.656	-593.857	2.471
	11	0.070	0.170	-582.055	-598.090	2.467		35	0.005	0.170	256.932	258.662	1.367
	12	0.070	0.200	-582.052	-597.709	2.471		36	0.005	0.200	259.909	261.429	1.367
	13	0.010	0.050	-581.955	-598.152	2.481		37	0.030	0.050	-579.043	-592.596	2.363
	14	0.010	0.090	-581.955	-598.152	2.481		38	0.030	0.090	-579.656	-593.857	2.471
	15	0.010	0.170	-581.955	-598.152	2.481		39	0.030	0.170	256.932	258.662	1.367
	16	0.010	0.200	-581.955	-598.152	2.481		40	0.030	0.200	259.909	261.429	1.367
	17	0.005	0.050	-579.227	-592.826	2.339		41	0.070	0.050	-355.104	-365.382	2.382
	18	0.005	0.090	-580.254	-594.793	2.410		42	0.070	0.090	-436.668	-448.883	2.511
	19	0.005	0.170	254.947	258.455	1.364		43	0.070	0.170	258.444	260.108	1.367
	20	0.005	0.200	264.722	266.418	1.363		44	0.070	0.200	259.794	261.287	1.368
	21	0.030	0.050	-579.273	-593.235	2.336		45	0.010	0.050	259.219	260.790	1.367
	22	0.030	0.090	-580.254	-594.793	2.410		46	0.010	0.090	259.749	261.360	1.368
GM2	23	0.030	0.170	254.947	258.455	1.364		47	0.010	0.170	259.219	260.790	1.367
	24	0.030	0.200	264.722	266.418	1.363		48	0.010	0.200	259.219	260.790	1.367

In Figure 3.4 we can observe boxplots of parameter estimates based on the mesh with lowest DIC, WAIC or logCPO, obtained in each simulation and we also present in this figure the boxplots of

the estimates obtained for each parameter with mesh 19.

We can observe that the boxplots obtained with mesh 19 are more similar to the boxplots obtained for the results of logCPO. Also, selection of the mesh using DIC and WAIC has bad performance, seen in the large bias of all covariance parameters. In this case, DIC and WAIC tends to select not only more denses meshes but also the ones with really bad results.

Figure 3.4. Estimated β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* , considering the meshes with minimal DIC, WAIC and logCPO, and based on mesh 19 for regular data set.

• Cluster data set

As same as for random data set, for cluster data we conduct the simulation in five batches of 40 simulations and within one batch the coordinates are kept fixed, resulting in 200 simulation. Table 3.13 shows the average number of knots for each mesh. .

Table 3.13. Average number of knots for different combinations of the parameters a and c , with parameter $b = 0.3$ for cluster data.

		c=0.005	c=0.03	c=0.07	c=0.10
GM1	a=0.05	1539	1143	265	145
	a=0.09	587	550	265	143
	a=0.17	346	281	199	143
	a=0.20	333	265	166	140
GM2	a=0.05	2109	1586	370	208
	a=0.09	727	727	369	208
	a=0.17	252	252	252	209
	a=0.20	212	212	212	206
GM3	a=0.05	1524	1147	262	143
	a=0.09	550	548	263	142
	a=0.17	250	247	195	142
	a=0.20	227	223	162	138

For β_0 , $\beta_1\sigma_e^2$, σ_s^2 , k and ϕ^* , we calculated values for coverage and summarized the results in Table 3.14 and in the Appendix B, Figure B.3 shows the values of coverage.. The results for cluster data set are very similar to the results for random data. Values of coverage for β_0 and β_1 are very similar amongst the meshes. For σ_e^2 , the parameter more critical for coverage, mesh 5 has the best results of 86.500%, For σ_s^2 , k and ϕ^* , the results are quite homogeneous between meshes, although the meshes with $c = 0.005$ from GM1 stand out with very low results. The best results for σ_s^2 is 98.5% with mesh 5; mesh 31 got the best value of 98.5% for the parameter k and the best results of coverage for ϕ^* was 89.5% with mesh 37.

Excluding β_0 and β_1 , we have four parameters and three meshes with good results: mesh 5, 31 and 37. Looking for the results of these three meshes, mesh 31 is not efficient in covering σ_e^2 . Meshes 5 and 37 have very similar results. It is worth mentioning that both have the same value for $a = 0.05$ and $c = 0.03$.

Table 3.14. Coverage, in %, of the 95% credible intervals of the parameters β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* for different meshes with $b = 0.3$, for cluster data set.

Group	Mesh	c	a	Coverage in % for:					
				β_0	β_1	σ_e^2	σ_s^2	k	ϕ^*
GM1	1	0.005	0.050	89.500	87.000	28.500	59.000	31.000	31.000
	2	0.005	0.090	89.000	84.000	22.500	33.500	25.500	25.500
	3	0.005	0.170	91.500	86.500	23.000	27.500	28.500	28.500
	4	0.005	0.200	93.000	87.000	22.500	27.500	32.000	32.000
	5	0.030	0.050	97.000	93.000	86.500	98.500	86.500	86.000
	6	0.030	0.090	97.500	93.500	84.500	98.000	87.500	87.500
	7	0.030	0.170	96.500	94.000	80.500	96.000	86.000	86.000
	8	0.030	0.200	97.000	93.500	80.000	97.000	88.000	88.000
	9	0.070	0.050	96.000	93.500	56.000	97.000	83.500	83.500
	10	0.070	0.090	95.500	93.500	58.500	97.000	83.000	83.000
	11	0.070	0.170	97.500	92.500	35.500	97.000	80.000	80.000
	12	0.070	0.200	97.000	93.500	33.500	96.000	82.000	82.000
	13	0.010	0.050	98.000	93.500	24.000	95.500	79.000	78.500
	14	0.010	0.090	98.000	93.500	21.000	95.000	79.000	79.000
	15	0.010	0.170	97.500	94.000	23.000	95.500	80.500	80.000
	16	0.010	0.200	98.000	93.500	18.000	93.000	82.500	82.500
GM2	17	0.005	0.050	96.500	92.000	81.000	88.000	82.000	82.000
	18	0.005	0.090	96.500	92.500	73.000	92.000	85.000	85.000
	19	0.005	0.170	98.000	94.500	27.000	86.000	83.000	83.000
	20	0.005	0.200	99.000	94.000	16.500	88.000	85.500	86.000
	21	0.030	0.050	96.500	93.500	81.000	91.500	87.500	87.500
	22	0.030	0.090	96.500	92.500	73.000	92.000	85.000	85.000
	23	0.030	0.170	98.000	94.500	27.000	86.000	83.000	83.000
	24	0.030	0.200	99.000	94.000	16.500	88.000	85.500	86.000
	25	0.070	0.050	97.000	92.000	45.000	88.500	85.500	85.500
	26	0.070	0.090	96.500	92.000	47.500	89.000	84.500	84.500
	27	0.070	0.170	98.000	94.500	27.000	86.000	83.000	83.000
	28	0.070	0.200	99.000	94.000	16.500	88.000	85.500	86.000
	29	0.010	0.050	97.000	93.000	17.000	86.000	82.500	81.500
	30	0.010	0.090	96.500	92.500	21.000	83.500	85.000	85.000
	31	0.010	0.170	98.000	93.000	22.500	87.000	88.500	88.500
	32	0.010	0.200	98.000	92.500	21.500	86.500	83.500	83.500
GM3	33	0.005	0.050	96.500	93.500	83.000	93.000	86.500	86.000
	34	0.005	0.090	98.000	93.000	78.500	97.500	86.500	86.500
	35	0.005	0.170	97.000	93.500	39.500	96.500	84.500	84.000
	36	0.005	0.200	96.000	93.000	35.000	95.500	79.500	79.500
	37	0.030	0.050	97.000	93.500	83.000	96.000	89.500	89.500
	38	0.030	0.090	98.000	93.000	78.500	97.500	86.500	86.500
	39	0.030	0.170	97.000	93.500	39.500	96.500	84.500	84.500
	40	0.030	0.200	96.000	93.000	35.000	96.000	79.000	79.500
	41	0.070	0.050	98.000	92.500	47.000	96.500	81.000	81.000
	42	0.070	0.090	98.000	93.000	48.500	97.000	82.000	81.500
	43	0.070	0.170	97.000	92.500	35.500	95.000	80.500	80.500
	44	0.070	0.200	96.500	93.000	29.500	94.500	81.000	81.000
	45	0.010	0.050	98.500	91.500	18.500	96.000	85.500	85.500
	46	0.010	0.090	98.500	92.000	19.500	95.500	84.500	84.000
	47	0.010	0.170	98.500	91.500	17.500	96.000	85.500	85.500
	48	0.010	0.200	98.500	91.500	17.500	95.000	85.000	85.000

Besides the coverage, we also calculated the values for bias for the six parameters and for the ratio $\frac{\sigma_s^2}{\phi}$. Table 3.15 shows this results and we can observe that no huge difference appears for β_0 and β_1 . For σ_e^2 , the best result occur with mesh 33 with value of 0.001. For σ_e^2 , mesh 5 obtain 0.006 of bias and mesh 37 got 0.018. For the parameter σ_s^2 the best result (0.002) was with mesh 5; mesh 37 obtain 0.039. For parameter k , the values of bias are more different between the meshes. The best result was 0.066 with mesh 34; mesh 5 got 1.034 and for mesh 37 it was 1.274. The value of bias for ϕ^* was best with meshes 3 and 4 with value of 0.027. For this parameter, bias with mesh 5 was 0.071 and with mesh 37 was 0.065. The results for the ratio $\frac{\sigma_s^2}{\phi}$ are not very different between the meshes; meshes from GM1 with $c = 0.005$ obtain better results and the best one was 0.003 with mesh 2. For the ratio, mesh 5 got 0.051 and mesh 37 got 0.050. In general, considering values of bias, mesh 5 was better than mesh 37.

Table B.3 from the Appendix B shows the results of MSE for the parameters and the ratio. This value for β_0 and β_1 are very similar for all meshes, also for σ_e^2 . The best result (0.015) of MSE for σ_e^2 was with mesh 6. Mesh 5 got 0.016 and mesh 37 got 0.017. For the parameter σ_s^2 , the lowest value was 0.052 with mesh 5 and mesh 37 got 0.093. Values of MSE for k are very high and the best one was 60.188 with mesh 7. Meshes 5 and 37 obtained 117.775 and 93.509, respectively. For ϕ^* , the best value was 0.028 with mesh 2. Value for mesh 5 and mesh 37 was 0.037 and 0.042, in this order. MSE for the ratio has little change between the meshes and the lowest value was 0.010 with mesh 4 and for meshes 5 and 37 the values was 0.023 and 0.027, respectively. Again meshe 5 has slightly better results than mesh 37.

Values for coverage and MSE was calculated to evaluate prediction in the 100 coordinate of

Table 3.15. Absolute bias of the parameters β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* and for $\frac{\sigma_s^2}{\phi}$, considering all meshes with cluster data set.

Group	Mesh	c	a	Bias for:						$\frac{\sigma_s^2}{\phi}$
				β_0	β_1	σ_e^2	σ_s^2	k	ϕ^*	
GM1	1	0.005	0.050	0.009	0.027	0.219	0.216	12.153	0.047	0.012
	2	0.005	0.090	0.005	0.029	0.228	0.361	10.745	0.051	0.003
	3	0.005	0.170	0.007	0.029	0.216	0.492	7.078	0.027	0.033
	4	0.005	0.200	0.010	0.025	0.214	0.510	6.286	0.027	0.034
	5	0.030	0.050	0.013	0.016	0.006	0.002	1.034	0.071	0.051
	6	0.030	0.090	0.014	0.015	0.046	0.016	0.313	0.096	0.064
	7	0.030	0.170	0.014	0.016	0.064	0.029	0.350	0.106	0.076
	8	0.030	0.200	0.015	0.017	0.065	0.036	0.508	0.099	0.073
	9	0.070	0.050	0.011	0.024	0.145	0.046	0.505	0.125	0.081
	10	0.070	0.090	0.011	0.025	0.145	0.048	0.603	0.126	0.080
	11	0.070	0.170	0.013	0.022	0.173	0.053	0.522	0.129	0.085
	12	0.070	0.200	0.013	0.023	0.182	0.079	0.512	0.124	0.087
	13	0.010	0.050	0.015	0.024	0.210	0.258	2.124	0.138	0.102
	14	0.010	0.090	0.014	0.024	0.213	0.257	2.032	0.137	0.101
	15	0.010	0.170	0.015	0.023	0.210	0.247	1.986	0.140	0.105
	16	0.010	0.200	0.014	0.023	0.219	0.265	2.094	0.137	0.104
GM2	17	0.005	0.050	0.013	0.011	0.010	0.132	2.746	0.058	0.059
	18	0.005	0.090	0.011	0.019	0.088	0.172	1.828	0.089	0.078
	19	0.005	0.170	0.010	0.019	0.200	0.551	3.424	0.111	0.127
	20	0.005	0.200	0.013	0.016	0.236	0.527	2.780	0.118	0.125
	21	0.030	0.050	0.014	0.011	0.035	0.138	2.383	0.067	0.065
	22	0.030	0.090	0.011	0.019	0.088	0.172	1.828	0.089	0.078
	23	0.030	0.170	0.010	0.019	0.200	0.551	3.424	0.111	0.127
	24	0.030	0.200	0.013	0.016	0.236	0.527	2.780	0.118	0.125
	25	0.070	0.050	0.014	0.017	0.161	0.339	2.267	0.102	0.097
	26	0.070	0.090	0.010	0.022	0.157	0.345	3.052	0.089	0.090
	27	0.070	0.170	0.010	0.019	0.200	0.551	3.424	0.111	0.127
	28	0.070	0.200	0.013	0.016	0.236	0.527	2.780	0.118	0.125
	29	0.010	0.050	0.010	0.021	0.232	0.699	4.141	0.110	0.141
	30	0.010	0.090	0.013	0.015	0.224	0.646	3.723	0.108	0.120
	31	0.010	0.170	0.014	0.016	0.224	0.563	3.705	0.097	0.116
	32	0.010	0.200	0.012	0.019	0.226	0.688	4.129	0.110	0.128
GM3	33	0.005	0.050	0.012	0.019	0.001	0.053	2.364	0.056	0.046
	34	0.005	0.090	0.015	0.017	0.083	0.007	0.066	0.100	0.066
	35	0.005	0.170	0.015	0.018	0.172	0.043	0.542	0.127	0.086
	36	0.005	0.200	0.014	0.019	0.183	0.143	1.348	0.125	0.093
	37	0.030	0.050	0.013	0.019	0.018	0.039	1.274	0.065	0.050
	38	0.030	0.090	0.015	0.017	0.083	0.007	0.091	0.101	0.067
	39	0.030	0.170	0.015	0.018	0.172	0.045	0.574	0.127	0.086
	40	0.030	0.200	0.014	0.019	0.182	0.138	1.285	0.125	0.093
	41	0.070	0.050	0.016	0.019	0.152	0.041	0.402	0.127	0.083
	42	0.070	0.090	0.016	0.018	0.150	0.039	0.412	0.126	0.082
	43	0.070	0.170	0.016	0.019	0.176	0.122	1.548	0.122	0.094
	44	0.070	0.200	0.015	0.019	0.192	0.220	1.886	0.125	0.098
	45	0.010	0.050	0.014	0.023	0.216	0.171	1.278	0.134	0.107
	46	0.010	0.090	0.015	0.022	0.215	0.173	1.272	0.135	0.108
	47	0.010	0.170	0.014	0.024	0.215	0.177	1.330	0.134	0.107
	48	0.010	0.200	0.015	0.023	0.218	0.226	1.829	0.128	0.106

sample locations. Meshes with $c = 0.005$ from GM1 obtained the worst result for coverage. The best value was 91.550 % with mesh 5, mesh 37 obtained 90.600%. For MSE, values were very similar and the best one was 0.189 with mesh 5.

Table 3.16. Nominal coverage, in %, and MSE for the variable Y calculated for each mesh, considering 200 simulations, with cluster data set.

Group	Mesh	c	a	Coverage(%)	MSE	Group	Mesh	c	a	Coverage(%)	MSE
GM1	1	0.005	0.050	36.975	0.263	GM2	25	0.070	0.050	85.855	0.232
	2	0.005	0.090	34.060	0.267		26	0.070	0.090	85.990	0.230
	3	0.005	0.170	34.820	0.265		27	0.070	0.170	82.375	0.255
	4	0.005	0.200	36.060	0.264		28	0.070	0.200	78.720	0.278
	5	0.030	0.050	91.550	0.189		29	0.010	0.050	78.825	0.273
	6	0.030	0.090	91.290	0.191		30	0.010	0.090	79.805	0.269
	7	0.030	0.170	90.525	0.196		31	0.010	0.170	79.650	0.269
	8	0.030	0.200	90.710	0.196		32	0.010	0.200	79.745	0.269
	9	0.070	0.050	86.380	0.223	GM3	33	0.005	0.050	87.985	0.194
	10	0.070	0.090	86.325	0.223		34	0.005	0.090	90.135	0.201
	11	0.070	0.170	84.120	0.237		35	0.005	0.170	83.660	0.238
	12	0.070	0.200	82.995	0.242		36	0.005	0.200	82.725	0.244
	13	0.010	0.050	80.220	0.260		37	0.030	0.050	90.600	0.192
	14	0.010	0.090	79.890	0.262		38	0.030	0.090	90.120	0.201
	15	0.010	0.170	80.180	0.260		39	0.030	0.170	83.650	0.238
	16	0.010	0.200	79.355	0.265		40	0.030	0.200	82.750	0.244
GM2	17	0.005	0.050	89.090	0.195		41	0.070	0.050	85.570	0.227
	18	0.005	0.090	90.030	0.204		42	0.070	0.090	85.815	0.226
	19	0.005	0.170	82.375	0.255		43	0.070	0.170	82.635	0.241
	20	0.005	0.200	78.720	0.278		44	0.070	0.200	81.355	0.249
	21	0.030	0.050	91.115	0.194		45	0.010	0.050	79.820	0.263
	22	0.030	0.090	90.030	0.204		46	0.010	0.090	79.715	0.262
	23	0.030	0.170	82.375	0.255		47	0.010	0.170	79.795	0.263
	24	0.030	0.200	78.720	0.278		48	0.010	0.200	79.510	0.264

For 25 locations not used in the analysis and randomly chosen, we also calculated the predictive performance of the model. Table 3.17 presents the results for coverage and MSE for this 25 extra locations. The best value for coverage was 91.320% with mesh 17 from GM2; meshes 5 and 37 obtained 88.260% and 88.800%, respectively. Values for MSE did not differ greatly between groups nor among groups. The lowest value was 0.458 with meshes 6, 34, 37 and 38; mesh 5 got 0.459.

Table 3.17. Nominal coverage, in %, and MSE for the variable Y_{na} calculated for each mesh in 25 locations, considering 200 simulations, with cluster data set.

Group	Mesh	c	a	Coverage(%)	MSE	Group	Mesh	c	a	Coverage(%)	MSE
GM1	1	0.005	0.050	90.920	0.471	GM2	25	0.070	0.050	82.200	0.471
	2	0.005	0.090	82.560	0.473		26	0.070	0.090	82.400	0.470
	3	0.005	0.170	67.600	0.486		27	0.070	0.170	80.760	0.473
	4	0.005	0.200	63.680	0.493		28	0.070	0.200	78.600	0.480
	5	0.030	0.050	88.260	0.459		29	0.010	0.050	78.180	0.481
	6	0.030	0.090	84.520	0.458		30	0.010	0.090	76.660	0.486
	7	0.030	0.170	77.160	0.461		31	0.010	0.170	77.360	0.488
	8	0.030	0.200	75.420	0.461		32	0.010	0.200	76.960	0.485
	9	0.070	0.050	79.680	0.466	GM3	33	0.005	0.050	89.460	0.460
	10	0.070	0.090	79.820	0.465		34	0.005	0.090	85.220	0.458
	11	0.070	0.170	75.500	0.470		35	0.005	0.170	76.000	0.467
	12	0.070	0.200	73.360	0.469		36	0.005	0.200	73.160	0.467
	13	0.010	0.050	73.620	0.475		37	0.030	0.050	88.800	0.458
	14	0.010	0.090	73.460	0.475		38	0.030	0.090	85.220	0.458
	15	0.010	0.170	73.600	0.475		39	0.030	0.170	75.960	0.467
	16	0.010	0.200	72.960	0.476		40	0.030	0.200	73.140	0.467
GM2	17	0.005	0.050	91.320	0.462		41	0.070	0.050	79.360	0.465
	18	0.005	0.090	87.000	0.460		42	0.070	0.090	79.200	0.464
	19	0.005	0.170	80.760	0.473		43	0.070	0.170	75.980	0.467
	20	0.005	0.200	78.600	0.480		44	0.070	0.200	73.080	0.469
	21	0.030	0.050	90.140	0.467		45	0.010	0.050	73.980	0.475
	22	0.030	0.090	87.000	0.460		46	0.010	0.090	74.200	0.475
	23	0.030	0.170	80.760	0.473		47	0.010	0.170	73.940	0.475
	24	0.030	0.200	78.600	0.480		48	0.010	0.200	73.560	0.474

The bias for the prediction in these 25 extra locations was calculated and plotted with individual boxplot for each mesh (Figure 3.5). As we can notice, all the boxplots for the values of bias are quite the same.

Figure 3.5. Bias for Y in 25 extra locations, with regular data set.

As for the others data set, we can observe that there is a large impact of the mesh on parameter estimation and prediction and there is not a single mesh which is optimal in all results. To try to select an optimal mesh, we calculate the mean of DIC, WAIC and logCPO (Table 3.18). Meshes from GM1 with $c = 0.005$, got negative values for DIC and WAIC and were the meshes with higher values of logCPO. Disregarding this meshes, the lowest value for DIC and WAIC were both with mesh 33 (180.702 and 182.276, respectively). The lowest value for logCPO was 1.253 with meshes 6,7 and 18. Values for

mesh 5 were DIC=215.528, WAIC=216.189 and logCPO= 1.255 and for mesh 37 were DIC=208.811, WAIC=210.748 and logCPO=1.262.

Table 3.18. Mean values of DIC, WAIC and logCPO for meshes with cluster data set .

Group	Mesh	c	a	DIC	WAIC	logCPO	Group	Mesh	c	a	DIC	WAIC	logCPO
GM1	1	0.005	0.050	-277.241	-277.101	1.810	GM2	25	0.070	0.050	240.772	242.300	1.260
	2	0.005	0.090	-294.297	-299.530	1.809		26	0.070	0.090	240.600	242.299	1.260
	3	0.005	0.170	-285.426	-292.638	1.808		27	0.070	0.170	244.745	246.397	1.265
	4	0.005	0.200	-275.800	-279.642	1.795		28	0.070	0.200	248.235	249.763	1.273
	5	0.030	0.050	215.528	216.189	1.255		29	0.010	0.050	247.828	249.316	1.271
	6	0.030	0.090	225.410	225.558	1.253		30	0.010	0.090	246.918	248.395	1.268
	7	0.030	0.170	228.594	228.673	1.253		31	0.010	0.170	247.144	248.646	1.270
	8	0.030	0.200	229.018	229.180	1.254		32	0.010	0.200	247.121	248.697	1.269
	9	0.070	0.050	240.008	240.634	1.259	GM3	33	0.005	0.050	180.702	182.276	1.298
	10	0.070	0.090	240.042	240.651	1.259		34	0.005	0.090	231.515	232.218	1.255
	11	0.070	0.170	243.106	243.729	1.263		35	0.005	0.170	242.657	243.356	1.261
	12	0.070	0.200	243.876	244.484	1.264		36	0.005	0.200	243.752	244.468	1.262
	13	0.010	0.050	246.180	246.919	1.267		37	0.030	0.050	208.811	210.748	1.262
	14	0.010	0.090	246.592	247.284	1.268		38	0.030	0.090	231.541	232.245	1.255
	15	0.010	0.170	246.203	246.946	1.267		39	0.030	0.170	242.633	243.333	1.261
	16	0.010	0.200	247.122	247.812	1.269		40	0.030	0.200	243.701	244.419	1.262
GM2	17	0.005	0.050	195.986	198.941	1.271		41	0.070	0.050	240.637	241.284	1.259
	18	0.005	0.090	230.923	232.265	1.253		42	0.070	0.090	240.352	240.997	1.258
	19	0.005	0.170	244.745	246.397	1.265		43	0.070	0.170	242.941	243.538	1.261
	20	0.005	0.200	248.235	249.763	1.273		44	0.070	0.200	244.638	245.247	1.263
	21	0.030	0.050	217.715	220.822	1.254		45	0.010	0.050	247.072	247.860	1.270
	22	0.030	0.090	230.923	232.265	1.253		46	0.010	0.090	246.952	247.744	1.269
	23	0.030	0.170	244.745	246.397	1.265		47	0.010	0.170	247.051	247.840	1.270
	24	0.030	0.200	248.235	249.763	1.273		48	0.010	0.200	247.217	247.991	1.269

Considering all results, meshes 5 and 37 stand out from the other meshes and mesh 5 have slightly better results than mesh 37.

In Figure 3.6 we can observe boxplots of parameter estimates based on the mesh with lowest DIC, WAIC or logCPO, obtained in each simulation and we also present in this figure the boxplots of the estimates obtained for each parameter with mesh 5. Boxplots obtained with mesh 5 are more similar to the boxplots obtained for the results of logCPO. Also, selection of mesh using DIC and WAIC has bad performance, seen in the large bias of all covariance parameters.

Figure 3.6. Estimated β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* , considering the meshes with minimal DIC, WAIC and logCPO, and based on mesh 5 for cluster data set.

3.4 Data Analysis

- Paraná dataset

One of the analyzed datasets was the so called Paraná dataset, available in the geoR package (Ribeiro and Diggle, 2001) in R. The data were collected in 143 random locations and consider the mean levels of precipitation in the entire state of Paran -Brazil, and the response variable represents the average winter precipitation (dry season).

To perform the analysis we take out the linear trend from the variable, because we noticed that this trend lead to misleading results. To model these data, we assume the model $y = \beta_0 + \mathbf{s} + \epsilon$, where $\mathbf{s}$ is a Matérn spatial random field with $\alpha = 1.5$

Performing a semivariogram analysis, we assumed the range as $380(\phi^*)$ and based on this value, we created 54 meshes being 18 created with all locations (GM1), 18 created with domain (GM2) and 18 created with part of the locations using the space filling algorithm (GM3). To create each of this 18 meshes, for the maximal allowed lengths of triangle edges in the inner domain (a) we used the values $a = c(20, 35, 55, 100, 240, 380)$ taking into account the value of range and allowing for small values until we reach the expected value of the range. For the maximal allowed lengths of triangle edges in the outer domain (b) the value was $b = 570$ and for the shortest distance allowed between points (c) the values were $c = c(13, 35, 51)$. For the domain, we used the bordes available in the data set plus 2.5. Table 3.19 show the number of knots for each mesh in each group and Figures B.4-B.12 of Appendix B show the image of the meshes.

Table 3.19. Number of knots for each mesh for Paran data.

Group	Mesh	c	a	Number knots	Group	Mesh	c	a	Number knots	Group	Mesh	c	a	Number knots
GM1	1	13	20	1235	GM2	19	13	20	1410	GM3	37	13	20	1218
	2	13	35	678		20	13	35	777		38	13	35	679
	3	13	55	362		21	13	55	353		39	13	55	320
	4	13	100	234		22	13	100	125		40	13	100	152
	5	13	240	230		23	13	240	45		41	13	240	137
	6	13	380	230		24	13	380	32		42	13	380	137
	7	35	20	197		25	35	20	209		43	35	20	193
	8	35	35	195		26	35	35	212		44	35	35	194
	9	35	55	194		27	35	55	208		45	35	55	192
	10	35	100	134		28	35	100	126		46	35	100	126
	11	35	240	122		29	35	240	42		47	35	240	104
	12	35	380	122		30	35	380	32		48	35	380	104
	13	51	20	100		31	51	20	119		49	51	20	106
	14	51	35	100		32	51	35	118		50	51	35	105
	15	51	55	97		33	51	55	117		51	51	55	104
	16	51	100	94		34	51	100	114		52	51	100	105
	17	51	240	77		35	51	240	41		53	51	240	80
	18	51	380	77		36	51	380	32		54	51	380	80

Figure B.13 in the Appendix B, shows the posterior distributions for β_0 , σ_s^2 , k and ϕ^* and we can notice that for β_0 , the posterior for mesh 37 behaves differently from other meshes with less variability, for the parameter σ_s^2 the posterior for mesh 37 has a high variability and posterior for meshes 23, 24, 29, 30, 35 and 36 stand out from the others with different means and variability. We can notice that the same occur with the posterior of this six meshes for k and ϕ^* . The parameter estimates are in Table 3.20 and we can observe that the choice of the mesh does not have a large impact on the estimates. Meshes 9, 13 until 16 and 26 have better results for σ_s^2 and for parameter ϕ^* , mesh 23 has the best value, although mesh 16 also gets good results.

Table 3.20. Estimates of the parameters for each meshfor Paran data.

Group	Mesh	c	a	β_0	σ_s^2	k	ϕ^*	Group	Mesh	c	a	β_0	σ_s^2	k	ϕ^*
GM1	1	13	20	-3.041	1004.249	0.004	530.363	GM2	28	35	100	-2.809	1071.426	0.004	564.845
	2		35	-2.899	1019.222	0.004	568.737		29		240	-4.124	2111.668	0.009	226.850
	3		55	-2.883	1026.202	0.003	573.441		30		380	-7.804	3161.480	0.014	144.841
	4		100	-2.867	1020.585	0.003	588.119		31	51	20	-2.695	1085.372	0.004	558.542
	5		240	-2.852	1007.166	0.003	582.856		32		35	-2.726	1092.988	0.004	568.645
	6		380	-2.852	1007.166	0.003	582.856		33		55	-2.961	1070.469	0.004	532.726
	7	35	20	-2.797	1002.032	0.004	555.719		34		100	-3.057	1145.570	0.003	584.626
	8		35	-2.747	1003.724	0.004	570.486		35		240	-4.022	1720.023	0.008	243.698
	9		55	-2.766	987.824	0.004	556.803		36		380	-7.547	3285.426	0.013	152.666
	10		100	-2.605	1018.224	0.003	588.273		37	13	20	-3.907	19759.314	0.318	6.293
	11		240	-2.370	1063.741	0.003	618.143		38		35	-2.781	1016.201	0.004	565.346
	12		380	-2.370	1063.741	0.003	618.143		39		55	-2.865	1009.105	0.004	557.738
	13	51	20	-3.249	994.481	0.004	520.953		40		100	-2.738	1041.134	0.003	638.464
	14		35	-3.145	979.349	0.004	512.673		41		240	-2.608	1033.338	0.003	638.393
	15		55	-3.244	979.765	0.004	504.027		42		380	-2.608	1033.338	0.003	638.393
	16		100	-3.140	979.482	0.004	494.867		43	35	20	-3.068	1025.605	0.003	572.346
	17		240	-2.110	1263.745	0.003	666.639		44		35	-3.048	1039.799	0.003	573.872
	18		380	-2.110	1263.745	0.003	666.639		45		55	-3.062	1045.103	0.003	578.584
GM2	19	13	20	-3.101	1027.246	0.004	516.609	GM3	46		100	-2.892	1080.247	0.003	662.980
	20		35	-3.227	1021.859	0.004	498.400		47		240	-2.763	1103.220	0.003	670.484
	21		55	-3.023	1032.815	0.004	511.843		48		380	-2.763	1103.220	0.003	670.484
	22		100	-2.943	1046.193	0.004	505.889		49	51	20	-2.787	1043.170	0.003	583.009
	23		240	-3.550	1627.018	0.006	318.852		50		35	-2.741	1041.540	0.003	582.401
	24		380	-7.615	3362.882	0.013	155.867		51		55	-2.642	1039.266	0.003	581.089
	25	35	20	-2.986	1010.473	0.004	543.476		52		100	-2.479	1075.265	0.003	614.429
	26		35	-3.058	989.798	0.004	529.683		53		240	-3.091	1032.363	0.004	540.050
	27		55	-3.017	1017.109	0.004	538.467		54		380	-3.091	1032.363	0.004	540.050

The values of DIC, WAIC and logCPO was computed and added into Table 3.21 and we can notice better results for mesh 37 but this mesh has really bad results for estimates. Values for DIC, WAIC and logCPO show minor difference between the meshes. So, we can assume mesh 16 is the optimal one.

Table 3.21. Values for DIC, WAIC and logCPO for each mesh for Paran data.

Group	Mesh	c	a	DIC	WAIC	logCPO	Group	Mesh	c	a	DIC	WAIC	logCPO
GM1	1	13	20	1305.94	1306.35	4.58	GM2	28	35	100	1311.83	1312.10	4.59
	2	35	35	1307.34	1307.46	4.58		29		240	1300.32	1299.51	4.55
	3	55	55	1306.53	1306.47	4.58		30		380	1311.99	1311.71	4.59
	4	100	100	1307.85	1307.78	4.58		31	51	20	1309.11	1308.93	4.58
	5	240	240	1307.97	1307.91	4.58		32		35	1309.31	1309.08	4.58
	6	380	380	1307.97	1307.91	4.58		33		55	1307.69	1307.24	4.58
	7	35	20	1307.69	1306.64	4.58		34		100	1308.75	1308.86	4.58
	8		35	1308.14	1307.04	4.58		35		240	1306.39	1305.56	4.57
	9		55	1307.90	1306.81	4.58		36		380	1316.75	1316.43	4.60
	10		100	1309.51	1308.59	4.58	GM3	37	13	20	-838.05	-862.64	-2.12
	11		240	1309.71	1308.65	4.58		38		35	1307.64	1307.85	4.58
	12		380	1309.71	1308.65	4.58		39		55	1306.69	1306.88	4.58
	13	51	20	1308.18	1307.65	4.58		40		100	1310.78	1310.67	4.59
	14		35	1308.44	1307.93	4.58		41		240	1310.56	1310.49	4.59
	15		55	1307.65	1307.09	4.58		42		380	1310.56	1310.49	4.59
	16		100	1307.60	1307.08	4.58		43	35	20	1308.01	1308.17	4.58
	17		240	1307.83	1307.05	4.57		44		35	1307.69	1307.77	4.58
	18		380	1307.83	1307.05	4.57		45		55	1308.05	1308.17	4.58
GM2	19	13	20	1306.33	1306.97	4.58		46		100	1313.99	1314.40	4.60
	20		35	1305.62	1306.48	4.58		47		240	1313.50	1313.78	4.60
	21		55	1306.63	1306.99	4.58		48		380	1313.50	1313.78	4.60
	22		100	1307.74	1307.51	4.58		49	51	20	1312.86	1313.90	4.60
	23		240	1307.18	1306.44	4.57		50		35	1312.66	1313.39	4.60
	24		380	1316.20	1315.69	4.60		51		55	1313.08	1314.15	4.60
	25	35	20	1309.85	1310.09	4.59		52		100	1313.13	1314.24	4.60
	26		35	1310.06	1310.20	4.59		53		240	1310.72	1311.30	4.59
	27		55	1308.95	1308.97	4.58		54		380	1310.72	1311.30	4.59

Using mesh 16, we computed the prediction mean and prediction standard deviation over the study area (the two figures from the bottom in Figure 3.7). Comparing Figure 3.7 II and Figure 3.7 III we can assume that mesh 16 gives good prediction.

Figure 3.7. (I) Histogram, (II) plot of sample and corresponding values, (III) predicted surface and (IV) prediction standard deviation considering mesh 16.

- Soil dataset

The data set here analyzed is the camg data set, in a regular grid, also available in the geoR package (Ribeiro and Diggle, 2001) in R. We are analyzing the soil samples of magnesium in depth 0-20, named mg020, collected in 178 locations.

To model these data, we assume the model $y = \beta_0 + \mathbf{s} + \epsilon$, where $\mathbf{s}$ is a Matérn spatial random field with $\alpha = 1.5$. The variance of the variable is 39.39 and with the aid of a semivariogram analysis, we assumed the range as 400 (ϕ^*) and so, $k = 0.005$. Based on this value we created 54 meshes. The same as before, we separated the meshes into three groups: 18 meshes created with all locations (GM1), 18 created with domain (GM2) and 18 created with part of the locations using the space filling algorithm (GM3).

To create the meshes, we considered for the maximal allowed lengths of triangle edges in the inner domain (a) the values (20,40,60,120,260,400); for the maximal allowed lengths of triangle edges in the outer domain (b) the value was $b = 600$ and for the shortest distance allowed between points (c) the values were (14,37,54). To create the domain, we use one function of the package R-INLA called `inla.nonconvex.hull` and the domain is given by `inla.nonconvex.hull(coordinate,60,50,resolution=c(90,90))$loc`. Figures B.14-B.22 from Appendix B show the images of created meshes and Table 3.22 below shows the number of knots for each mesh.

Table 3.22. Number of knots for each mesh for soil data.

Group	Mesh	c	a	Number knots	Group	Mesh	c	a	Number knots	Group	Mesh	c	a	Number knots
GM1	1	14	20	3362	GM2	19	14	20	4313	GM3	37	14	20	3215
	2	14	40	1690		20	14	40	2079		38	14	40	1602
	3	14	60	795		21	14	60	986		39	14	60	736
	4	14	120	295		22	14	120	282		40	14	120	208
	5	14	260	246		23	14	260	77		41	14	260	146
	6	14	400	245		24	14	400	51		42	14	400	145
	7	37	20	557		25	37	20	691		43	37	20	548
	8	37	40	553		26	37	40	673		44	37	40	543
	9	37	60	554		27	37	60	686		45	37	60	544
	10	37	120	295		28	37	120	276		46	37	120	208
	11	37	260	246		29	37	260	77		47	37	260	146
	12	37	400	245		30	37	400	51		48	37	400	145
	13	54	20	270		31	54	20	349		49	54	20	267
	14	54	40	270		32	54	40	352		50	54	40	264
	15	54	60	273		33	54	60	351		51	54	60	265
	16	54	120	208		34	54	120	286		52	54	120	196
	17	54	260	147		35	54	260	78		53	54	260	128
	18	54	400	144		36	54	400	49		54	54	400	127

The parameter estimates are in Table 3.23 and we can notice that for β_0 , the estimated values are almost the same. For the parameter σ_s^2 , half of meshes from GM1 overestimated this parameter, the same occur with meshes 24, 25, 30 and 36 from GM2 and meshes 37, 38 and 44 from GM3. The best value for σ_s^2 occur with mesh 27. For k , some meshes, including mesh 27, obtained the exact value. For ϕ^* , the estimate most similar to the stipulated value was with mesh 53 the value of 396.201 and mesh 27 obtained 417.423.

The values of DIC, WAIC and logCPO was computed and added into Table 3.24. We can notice that meshes from GM1 have negative results for this three criteria and those meshes have bad results in all analysis. The small (positive) results for DIC was 1037.02 with mesh 26, for WAIC was 1040.80 also with mesh 26 and for logCPO the best value was 2.97 with meshes 34, 40 and 46. This results with mesh 27 were 1046.39 (DIC), 1048.73 (WAIC) and 2.98 (logCPO), not far from the minors ones.

Table 3.23. Estimates of the parameters for each mesh for Soil data.

Group	Mesh	c	a	β_0	σ_s^2	k	ϕ^*	Group	Mesh	c	a	β_0	σ_s^2	k	ϕ^*
GM1	1	14	20	27.349	225.546	0.311	6.436	GM2	28	37	120	24.790	39.813	0.007	274.361
	2		40	27.249	212.464	0.201	9.943		29		260	24.161	47.518	0.006	356.202
	3		60	25.502	38.138	0.012	168.098		30		400	23.532	112.748	0.010	193.083
	4		120	25.544	41.596	0.015	135.957		31	54	20	24.372	39.492	0.005	375.086
	5		260	26.184	536.623	0.146	13.677		32		40	24.511	37.671	0.006	356.415
	6		400	26.186	533.051	0.147	13.635		33		60	24.431	39.159	0.005	369.717
	7	37	20	27.007	392.239	0.156	12.825		34		120	24.455	40.683	0.005	373.239
	8		40	25.659	39.245	0.014	143.499		35		260	24.056	51.629	0.007	276.123
	9		60	27.006	389.648	0.156	12.857		36		400	22.970	136.836	0.011	187.693
	10		120	25.544	41.596	0.015	135.957	GM3	37	14	20	27.497	205.812	0.229	8.726
	11		260	26.184	536.623	0.146	13.677		38		40	27.204	257.290	0.183	10.957
	12		400	26.186	533.051	0.147	13.635		39		60	25.591	44.040	0.013	150.161
	13	54	20	24.039	38.553	0.003	614.885		40		120	24.483	37.161	0.005	437.492
	14		40	24.033	38.645	0.003	613.556		41		260	24.453	36.714	0.004	484.431
	15		60	23.884	39.873	0.003	610.772		42		400	24.452	36.749	0.004	484.269
	16		120	24.308	36.371	0.004	514.214		43	37	20	25.675	46.431	0.017	119.675
	17		260	24.172	36.932	0.004	567.643		44		40	26.929	496.750	0.161	12.388
	18		400	24.283	35.917	0.004	541.241		45		60	25.765	46.740	0.017	115.848
GM2	19	14	20	25.695	43.909	0.016	126.075		46		120	24.483	37.161	0.005	437.492
	20		40	25.828	48.753	0.017	117.083		47		260	24.453	36.714	0.004	484.431
	21		60	26.146	64.464	0.026	76.259		48		400	24.452	36.749	0.004	484.269
	22		120	24.297	37.278	0.005	423.440		49	54	20	24.370	36.666	0.004	507.665
	23		260	24.501	47.362	0.006	318.026		50		40	24.511	36.060	0.004	445.414
	24		400	23.935	102.094	0.010	204.161		51		60	24.519	35.560	0.005	436.221
	25	37	20	26.357	104.627	0.033	60.693		52		120	24.629	36.447	0.005	386.195
	26		40	24.789	37.422	0.006	327.538		53		260	24.724	34.445	0.005	396.201
	27		60	24.473	39.346	0.005	417.423		54		400	24.743	34.243	0.005	390.551

Table 3.24. Values for DIC, WAIC and logCPO for each mesh for Soil data.

Group	Mesh	c	a	DIC	WAIC	logCPO	Group	Mesh	c	a	DIC	WAIC	logCPO
GM1	1	14	20	-1045.52	-1064.51	-2.09	GM2	28	37	120	1043.60	1048.56	2.98
	2		40	-1051.57	-1090.31	-2.16		29		260	1072.81	1072.84	3.02
	3		60	-1037.86	-1053.63	-2.06		30		400	1067.59	1067.17	3.00
	4		120	-1017.69	-1052.80	-2.06		31	54	20	1049.29	1051.18	2.98
	5		260	-1111.69	-1166.31	-2.38		32		40	1053.02	1055.10	2.99
	6		400	-1112.32	-1166.93	-2.38		33		60	1049.06	1051.17	2.98
	7	37	20	-1046.77	-1101.36	-2.19		34		120	1048.73	1050.50	2.97
	8		40	-1031.61	-1051.38	-2.06		35		260	1066.28	1065.48	3.00
	9		60	-1047.36	-1101.95	-2.20		36		400	1068.88	1068.67	3.00
	10		120	-1017.69	-1052.80	-2.06	GM3	37	14	20	-1066.48	-1119.93	-2.25
	11		260	-1111.69	-1166.31	-2.38		38		40	-1042.52	-1095.65	-2.18
	12		400	-1112.32	-1166.93	-2.38		39		60	-1032.84	-1053.26	-2.06
	13	54	20	1066.26	1066.83	3.02		40		120	1047.06	1047.13	2.97
	14		40	1066.86	1067.50	3.02		41		260	1054.90	1054.56	2.98
	15		60	1064.53	1065.47	3.01		42		400	1054.87	1054.56	2.98
	16		120	1060.81	1060.14	3.00		43	37	20	-1020.86	-1059.65	-2.08
	17		260	1064.89	1063.83	3.00		44		40	-1085.68	-1140.24	-2.30
	18		400	1064.75	1063.63	3.00		45		60	-1031.49	-1052.65	-2.06
GM2	19	14.00	20	-1019.59	-1056.11	-2.07		46		120	1047.06	1047.13	2.97
	20		40	-1030.01	-1049.40	-2.05		47		260	1054.90	1054.56	2.98
	21		60	-1033.28	-1056.65	-2.07		48		400	1054.87	1054.56	2.98
	22		120	1062.98	1066.29	3.01		49	54	20	1056.61	1057.91	3.00
	23		260	1073.61	1073.16	3.02		50		40	1049.74	1050.44	2.98
	24		400	1073.70	1073.41	3.02		51		60	1048.95	1049.86	2.98
	25	37	20	-1044.52	-1062.07	-2.08		52		120	1044.12	1044.07	2.96
	26		40	1037.02	1040.80	2.98		53		260	1054.13	1054.46	2.98
	27		60	1046.39	1048.73	2.98		54		400	1054.03	1054.36	2.98

Using mesh 27, we computed the prediction mean and prediction standard deviation over the study area (the two figures from the bottom in Figure 3.8). Comparing Figure 3.8 I and Figure 3.8 III we can assume that mesh 27 obtained good prediction.

Figure 3.8. (I) Plot of sample and corresponding values, (II) histogram , (III) predicted surface and (IV) prediction standard deviation considering mesh 27.

3.5 Conclusion

When using the SPDE/GMRF approach in INLA in geostatistical analysis, the first step is to create a mesh over and plus the study area. With a simulation study we showed that inferential results depend on this created mesh. In a previous study, we notice that the value of b (the maximal edge length in the outer domain) has almost no impact on the results, so in this study we fixed it. The choice of the value of c , the shortest allowed distance between points, has the largest impact and, conditional on a value of c , the value of a (the maximal edge length in the inner domain) also leads to some differences.

We worked with three predictive information criteria, namely DIC, WAIC and CPO, to investigate whether these criteria can be used to select an optimal mesh for a specific data problem and we notice that DIC and WAIC lead to choosing meshes with poor results in the analysis while CPO has a better sense. The only problem with CPO is that we need to make sure that no numerical problems occur because if it happens, CPO needed to be recalculated by refitting the model, leaving out every observation one at a time and this could take a long time, specially in simulation study.

In this paper, our focus was to investigate different types to create a convex mesh for different geostatistical data set. We worked with three ways to create the mesh and it is possible to notice that working with random and cluster data, best results are obtained with meshes generated with all coordinates, with a relation with ratio of 0.15 between $c = 0.03$ and range. For regular data, results were not good when meshes were created considering the coordinates, better results were obtained with mesh creation by domain. Further investigation with convex mesh is still needed and also to investigate how non-convex meshes worked for each kind of this data set.

References

- BOLIN, D.; LINDSTRÖM, J. Spatial Statistics using R-INLA and Gaussian Markov random fields. *Pan-American Advanced Study Institute*, 15:29:57Z, rev.3691, 2014.
- CAMELETTI, M.; LINDGREN, F.; SIMPSON, D., RUE, H. Spatio-temporal modeling of particulate matter concentration through the SPDE approach. *Advances in Statistical Analysis*, v. 97, p. 109-131, 2012.

- DIGGLE, P.J.; RIBEIRO JR., P.J.; CHRISTENSEN, O.F. An introduction to model-based geostatistics. In: MILLER, J. (Ed.). **Spatial Statistics and Computational Methods**. New York: Springer, 2003. p. 43-86.
- DIGGLE, P. J.; TAWN, J. A.; MOYEED, R. A. Model-based geostatistics. **Journal of the Royal Statistical Society: Series C**, v. 47, p. 299-350, 1998.
- GEISSER, S.; EDDY, W. A predictive approach to model selection. *Journal of the American Statistical Association*, Boston, v.74, n.365, p.153-160, 1979.
- GELMAN, A.; HWANG, J.; VEHTARI, A. Understanding predictive information criteria for Bayesian models. *Statistics and Computing*, v. 24, n. 6, p. 997-1016, 2014.
- HELD, L., SCHRODLE, B., RUE, H. Posterior and Cross-validatory Predictive Checks: A comparison of MCMC and INLA. In T. Kneib and G. Tutz, editors, *Statistical Modelling and Regression Structures*, p. 91, 2010.
- IBRAHIM, J. G.; CHEN, M.; SINHA, D. *Bayesian Survival Analysis*. Ed: Springer Science & Business Media, 2013. 480 p.
- JOHNSON, M.E.; MOORE, L.M.; YLVIKAKER, D. Minimax and maximin distance designs. *Journal of Statistical Planning and Inference*, n. 26, p. 131-148, 1990.
- KARAGIANNIS-VOULES, D.A., SCHOLTE, R. G. C., GUIMARÉES, L. H., UTZINGER, J., VOUNATSOU, P. Bayesian Geostatistical Modeling of Leishmaniasis Incidence in Brazil. *PLoS Neglected Tropical Diseases*, v. 7, n. 5, e2213. doi:10.1371/journal.pntd.0002213. 2013.
- KRAINSKI, E. T.; RUE, H. A short introduction on how to fit a SPDE model with INLA. 2013.
- KRAINSKI, E. T.; LINDGREN, F. The R-INLA tutorial: SPDE models. 2014.
- LINDGREN, F. Continuous domain spatial models in R-INLA. *The ISBA Bulletin*, v. 19, n. 4, 2012.
- LINDGREN, F.; RUE, H. Bayesian spatial modelling with R-INLA. *Journal of Statistical Software*, v. 63, 2015.
- LINDGREN, F.; RUE, H.; LINDSTRÖM, J. An explicit link between Gaussian fields and Gaussian Markov random fields: The stochastic partial differential equation approach. *Journal of the Royal Statistical Society*, v. 73, p. 423-498, 2011.
- MATHERON, G. Principles of geostatistics. *Economic Geology*, Austin, v. 58, p.1246-1266, 1963.
- R Core Team (2015). R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. URL <https://www.R-project.org/>.
- RIBEIRO JR., P.J. and DIGGLE, P.J. (2001) geoR: A package for geostatistical analysis. R-NEWS Vol 1, No 2. ISSN 1609-3631. Available for download at: <http://cran.r-project.org/doc/Rnews>.
- RUE, H.; MARTINO, S.; CHOPIN, N. Approximate Bayesian inference for latent Gaussian models by using integrated nested Laplace approximations (with discussion). *Journal of the Royal Statistical Society - Series B*, v.71, Part 2, p.319-392, 2009.
- SPIEGELHALTER, D.J.; BEST, N.G.; CARLIN, B.P.; LINDE, A.V.D. Bayesian measures of model complexity and fit. *Journal of the Royal Statistical Society Series B*, London, v.64, n.4, p.583-639, 2000.
- VEHTARI, A.; GELMAN, A. WAIC and cross-validation in Stan. Submitted. http://www.stat.columbia.edu/gelman/research/unpublished/waic_stan.pdf Accessed, v. 27, n. 2015, p. 5, 2014.
- WATANABE, S. Asymptotic equivalence of Bayes cross validation and widely applicable information criterion in singular learning theory. *Journal of Machine Learning Research*, v. 11, p. 3571-3594, 2010.

ZHANG, H.; ZIMMERMAN, D. L. Hybrid Estimation of Semivariogram Parameters. **Mathematical Geology**, v. 39, n. 2, February 2007. DOI: 10.1007/s11004-006-9070-8.

4 CONCLUSION

When using the INLA method for geostatistical data, the first step is to create a mesh under the study region and all the analysis will depend on this mesh. This mesh is composed of several triangles in which the junction of the edges (knots) may or may not coincide with the locations. There are several ways to create the mesh and in this work we approach the creation using the maximal allowed edge length in the inner (a) and outer (b) domain and the shortest allowed distance between two points (c). We could notice that the value of b has almost no impact on the results and the choice of the value of c has the largest impact. Also, conditional on a value of c , the value of a also leads to some differences.

Pre-specification of the triangulated mesh is not easy, since it is possible to create a mesh with very good predictive behavior, but that leads to poor covariance parameter estimates. We have used DIC, WAIC and CPO, to investigate whether these criterias can be used to select an optimal mesh for a specific data problem. We noticed that the predictive information criterias DIC and WAIC almost always lead to choosing dense meshes. These meshes indeed perform well in terms of coverage and mean squared error for predictions, but do not perform well in the other parts of the analysis, namely the estimation of the covariance parameters. Thus, DIC and WAIC showed to prefer meshes with more knots. Further, we have noticed that mesh selection using the logCPO leads to better performing meshes overall.

The logCPO criteria needs careful since it is very important to ensure that no numerical problems occur. We have observed that in many cases, especially with dense meshes, this approximating method does not work and, therefore, CPO needed to be recalculated by refitting the model, leaving out every observation one at a time.

First, we study for random data the behavior of the maximal allowed edge length in the inner (a) and outer (b) domain and the shortest allowed distance between two points (c) for different sample values to verify the impact of the mesh with different sample values then we extended the study to different ways to create the mesh with the use of different values of a , b and c . There is still much to study for a better guide on how to create the mesh but with this beginning of study we can notice similarities in the real data applications with the simulation study carried out. It is possible to notice that analysis with regular data set works better with the meshes created by domain, not considering the coordinates. For random and cluster data set, meshes created considering all the locations, generate better results and proportion given by $\frac{c}{\phi_*}$ is equal to 0.15, a proportion to give a special attention when creating the mesh. Also, we notice that the value of b to create the mesh has no impact in the analysis, the attention should be in a and c .

APPENDICES

Appendix A: Supplementary Material for Chapter 2

Additional simulations results

In Section 2.3 of Chapter 2, a simulation study is presented with $n = 100$. In this section of Appendix A we provide the results of the simulation study considering $n = 50$ and $n = 300$.

The colours of the graphics in the following sessions are: red for $c=0.001$, orange for $c=0.005$, green for $c=0.01$, blue for $c=0.03$, light blue for $c=0.05$, pink for $c=0.07$, yellow for $c=0.10$ and grey for $c=0.13$. And in each group of these colours, the value of a increases from left to right.

Simulation study with $n=100$

Impact of mesh on inference

Figure A.1. Coverage of the meshes for: β_{00} , β_{01} , σ_e^2 , σ_s^2 , k and ϕ^* with $n=100$.

Table A.1. MSE of the parameters β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* for different meshes with $b = 0.3$.

Mesh	c	a	MSE for:					
			β_0	β_1	σ_e^2	σ_s^2	k	ϕ^*
1	0.001	0.05	0.07	0.12	0.08	0.29	543.04	0.02
2		0.09	0.07	0.12	0.08	0.34	299.48	0.02
3		0.13	0.06	0.12	0.08	0.40	182.82	0.02
4		0.17	0.06	0.12	0.08	0.48	144.96	0.01
5		0.19	0.06	0.12	0.08	0.47	127.04	0.01
6		0.20	0.06	0.12	0.08	0.48	122.49	0.02
7	0.005	0.05	0.07	0.14	0.08	0.30	544.10	0.02
8		0.09	0.07	0.14	0.08	0.33	285.95	0.02
9		0.13	0.07	0.14	0.08	0.38	171.38	0.02
10		0.17	0.07	0.14	0.08	0.44	131.29	0.02
11		0.19	0.07	0.14	0.08	0.46	121.33	0.02
12		0.20	0.07	0.14	0.08	0.45	115.09	0.02
13	0.01	0.05	0.07	0.14	0.06	0.28	424.82	0.04
14		0.09	0.07	0.14	0.06	0.24	202.96	0.04
15		0.13	0.07	0.13	0.06	0.26	120.75	0.04
16		0.17	0.07	0.13	0.06	0.32	100.72	0.04
17		0.19	0.07	0.13	0.06	0.33	91.92	0.04
18		0.20	0.07	0.13	0.06	0.33	86.29	0.04
19	0.03	0.05	0.06	0.12	0.03	0.07	87.72	0.06
20		0.09	0.06	0.11	0.02	0.14	88.41	0.06
21		0.13	0.06	0.11	0.02	0.06	31.88	0.07
22		0.17	0.06	0.11	0.02	0.07	34.07	0.07
23		0.19	0.06	0.11	0.02	0.08	32.10	0.07
24		0.20	0.06	0.11	0.02	0.08	32.60	0.06
25	0.05	0.05	0.06	0.11	0.02	0.08	88.57	0.07
26		0.09	0.06	0.11	0.02	0.12	131.08	0.06
27		0.13	0.06	0.11	0.02	0.07	36.54	0.07
28		0.17	0.06	0.11	0.03	0.08	51.22	0.08
29		0.19	0.06	0.11	0.03	0.10	59.79	0.07
30		0.20	0.06	0.10	0.03	0.09	46.20	0.06
31	0.07	0.05	0.06	0.10	0.03	0.09	45.91	0.10
32		0.09	0.06	0.10	0.03	0.10	46.08	0.10
33		0.13	0.06	0.10	0.03	0.09	41.33	0.11
34		0.17	0.06	0.10	0.04	0.14	49.59	0.12
35		0.19	0.06	0.10	0.05	0.22	66.93	0.11
36		0.20	0.06	0.10	0.05	0.23	69.40	0.11
37	0.10	0.05	0.06	0.10	0.06	1.16	123.40	0.11
38		0.09	0.06	0.10	0.06	1.96	125.43	0.12
39		0.13	0.06	0.10	0.07	2.32	138.98	0.11
40		0.17	0.06	0.10	0.07	1.91	130.67	0.12
41		0.19	0.06	0.10	0.07	1.93	127.96	0.11
42		0.20	0.06	0.11	0.07	2.44	111.41	0.11
43	0.13	0.05	0.06	0.11	0.09	1.87	130.20	0.12
44		0.09	0.06	0.11	0.10	2.20	133.10	0.12
45		0.13	0.06	0.11	0.10	1.33	107.18	0.11
46		0.17	0.06	0.11	0.10	1.38	108.93	0.11
47		0.19	0.06	0.11	0.10	1.42	112.95	0.11
48		0.20	0.06	0.11	0.10	1.37	109.74	0.12

Simulation study with $n=50$

Impact of mesh on inference

Figure A.2. Coverage of the meshes for: β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* with $n=50$.

Table A.2. Coverage, in %, of the 95% credible intervals of the parameters β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* for different meshes with $b = 0.3$, for $n = 50$.

Mesh	c	a	Coverage in % for:					
			β_0	β_1	σ_e^2	σ_s^2	k	ϕ^*
1	0.001	0.05	93.00	93.50	0.50	59.50	51.50	55.00
2		0.09	92.00	93.50	1.00	33.50	43.00	43.00
3		0.13	92.00	94.50	1.00	18.00	41.50	42.00
4		0.17	93.00	94.50	0.50	11.50	44.50	45.50
5		0.19	92.50	95.00	1.00	11.50	43.50	43.50
6	0.005	0.20	92.50	95.50	0.50	11.00	45.00	45.00
7		0.05	93.00	93.50	0.50	59.50	51.50	55.00
8		0.09	92.00	93.50	1.00	33.50	43.00	43.00
9		0.13	92.00	94.50	1.00	18.00	41.50	42.00
10		0.17	93.00	94.50	0.50	11.50	44.50	45.50
11	0.01	0.19	92.50	95.00	1.00	11.50	43.50	43.50
12		0.20	92.50	95.50	0.50	11.00	45.00	45.00
13		0.05	93.50	90.00	5.50	62.50	57.00	60.00
14		0.09	92.50	89.50	7.00	38.00	49.00	49.00
15		0.13	93.00	91.00	7.50	24.00	47.50	48.00
16	0.03	0.17	93.50	91.50	7.00	17.50	50.00	51.00
17		0.19	92.50	92.00	8.00	16.50	50.00	50.00
18		0.20	93.00	92.50	7.50	16.50	51.50	51.50
19		0.05	91.00	87.00	25.50	61.50	62.50	63.00
20		0.09	92.00	85.00	42.00	55.50	65.00	65.00
21	0.05	0.13	89.50	84.50	59.50	67.00	72.00	73.00
22		0.17	90.00	86.00	61.00	66.00	72.00	72.00
23		0.19	90.00	86.50	62.50	65.50	73.50	74.00
24		0.20	92.00	86.50	63.50	64.50	73.00	73.00
25		0.05	92.50	88.50	57.50	66.00	71.00	70.00
26	0.07	0.09	91.50	88.00	52.00	61.50	66.00	66.50
27		0.13	94.50	88.50	72.00	82.50	78.50	78.50
28		0.17	94.00	89.50	74.00	86.50	82.00	82.00
29		0.19	94.50	91.00	74.50	86.50	81.50	81.50
30		0.20	95.50	91.50	74.00	88.00	81.00	81.00
31	0.10	0.05	94.00	91.50	68.00	88.50	81.50	81.50
32		0.09	96.00	91.00	67.50	88.00	82.50	82.50
33		0.13	96.50	92.00	69.00	86.50	83.50	84.00
34		0.17	97.50	92.50	64.50	91.50	82.00	82.00
35		0.19	96.50	93.50	61.00	91.50	81.00	81.00
36	0.13	0.20	97.50	93.50	60.00	90.00	82.00	82.00
37		0.05	96.50	92.50	54.50	89.50	83.00	83.00
38		0.09	97.00	93.00	56.50	90.50	83.00	83.00
39		0.13	96.50	92.50	54.50	92.50	86.00	86.00
40		0.17	96.50	92.50	55.00	92.00	86.00	86.00
41	0.13	0.19	96.50	92.50	53.50	91.00	86.00	86.00
42		0.20	96.00	93.00	54.00	92.00	85.50	85.50
43		0.05	97.50	93.00	35.00	91.00	88.00	88.00
44		0.09	97.00	94.00	29.00	91.00	88.50	88.50
45		0.13	97.50	94.00	29.00	92.50	89.50	89.50
46	0.13	0.17	97.00	93.50	28.50	92.50	89.00	89.00
47		0.19	96.50	94.00	29.00	92.50	89.00	89.00
48		0.20	97.00	93.50	28.50	92.00	89.00	89.00

Table A.3. Absolute bias of the parameters β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* for different meshes with $b = 0.3$, for $n = 50$.

Mesh	c	a	Bias for:					
			β_0	β_1	σ_e^2	σ_s^2	k	ϕ^*
1	0.001	0.05	0.03	0.00	0.30	0.67	18.57	0.08
2		0.09	0.02	0.00	0.30	0.62	14.04	0.08
3		0.13	0.03	0.00	0.30	0.73	11.33	0.08
4		0.17	0.02	0.00	0.30	0.83	9.35	0.07
5		0.19	0.02	0.00	0.30	0.85	8.53	0.07
6		0.20	0.02	0.00	0.30	0.87	8.35	0.07
7	0.005	0.05	0.03	0.00	0.30	0.67	18.57	0.08
8		0.09	0.02	0.00	0.30	0.62	14.04	0.08
9		0.13	0.03	0.00	0.30	0.73	11.33	0.08
10		0.17	0.02	0.00	0.30	0.83	9.35	0.07
11		0.19	0.02	0.00	0.30	0.85	8.53	0.07
12		0.20	0.02	0.00	0.30	0.87	8.35	0.07
13	0.01	0.05	0.02	0.00	0.28	0.62	15.71	0.06
14		0.09	0.02	0.01	0.28	0.56	11.97	0.06
15		0.13	0.02	0.01	0.28	0.66	9.71	0.06
16		0.17	0.02	0.01	0.28	0.75	7.94	0.05
17		0.19	0.02	0.01	0.28	0.77	7.18	0.05
18		0.20	0.02	0.01	0.28	0.79	7.15	0.05
19	0.03	0.05	0.02	0.02	0.21	0.32	9.56	0.00
20		0.09	0.02	0.01	0.16	0.53	9.04	0.05
21		0.13	0.01	0.03	0.10	0.42	4.57	0.10
22		0.17	0.02	0.03	0.09	0.65	5.25	0.09
23		0.19	0.02	0.02	0.09	0.57	4.54	0.09
24		0.20	0.02	0.02	0.08	0.60	4.39	0.11
25	0.05	0.05	0.04	0.02	0.08	0.49	6.61	0.10
26		0.09	0.04	0.02	0.09	0.63	7.91	0.12
27		0.13	0.03	0.01	0.01	0.25	2.76	0.16
28		0.17	0.04	0.02	0.04	0.31	2.16	0.19
29		0.19	0.04	0.03	0.05	0.32	2.00	0.19
30		0.20	0.04	0.03	0.06	0.30	1.72	0.19
31	0.07	0.05	0.03	0.01	0.08	0.29	1.22	0.22
32		0.09	0.03	0.00	0.09	0.46	1.92	0.22
33		0.13	0.03	0.01	0.09	0.21	0.68	0.23
34		0.17	0.03	0.01	0.13	0.21	0.59	0.24
35		0.19	0.03	0.01	0.15	0.33	1.11	0.25
36		0.20	0.03	0.01	0.16	0.38	1.33	0.25
37	0.10	0.05	0.03	0.01	0.19	0.36	1.35	0.24
38		0.09	0.03	0.01	0.20	0.48	1.48	0.25
39		0.13	0.03	0.01	0.19	0.25	0.95	0.25
40		0.17	0.03	0.01	0.19	0.22	0.46	0.25
41		0.19	0.03	0.01	0.20	0.22	0.41	0.26
42		0.20	0.03	0.01	0.20	0.29	1.10	0.25
43	0.13	0.05	0.03	0.01	0.27	0.64	2.64	0.21
44		0.09	0.03	0.01	0.28	0.52	2.37	0.21
45		0.13	0.03	0.01	0.28	0.50	2.08	0.22
46		0.17	0.03	0.01	0.28	0.50	2.07	0.21
47		0.19	0.03	0.01	0.28	0.49	2.21	0.21
48		0.20	0.03	0.01	0.28	0.50	2.06	0.22

Table A.4. MSE of the parameters $\beta_0, \beta_1, \sigma_e^2, \sigma_s^2, k$ and ϕ^* for different meshes with $b = 0.3$, for $n = 50$.

Mesh	c	a	MSE for:					
			β_0	β_1	σ_e^2	σ_s^2	k	ϕ^*
1	0.001	0.05	0.10	0.20	0.09	13.77	2187.42	0.01
2		0.09	0.10	0.19	0.09	1.18	541.03	0.01
3		0.13	0.10	0.19	0.09	0.86	266.87	0.01
4		0.17	0.11	0.19	0.09	1.00	179.91	0.01
5		0.19	0.11	0.19	0.09	0.95	142.91	0.01
6		0.20	0.11	0.19	0.09	1.04	140.73	0.01
7	0.005	0.05	0.10	0.20	0.09	13.77	2187.42	0.01
8		0.09	0.10	0.19	0.09	1.18	541.03	0.01
9		0.13	0.10	0.19	0.09	0.86	266.87	0.01
10		0.17	0.11	0.19	0.09	1.00	179.91	0.01
11		0.19	0.11	0.19	0.09	0.95	142.91	0.01
12		0.20	0.11	0.19	0.09	1.04	140.73	0.01
13	0.01	0.05	0.11	0.23	0.09	13.72	1969.76	0.02
14		0.09	0.11	0.23	0.09	1.10	452.62	0.02
15		0.13	0.11	0.24	0.09	0.75	214.01	0.03
16		0.17	0.11	0.23	0.09	0.89	140.82	0.03
17		0.19	0.12	0.24	0.08	0.85	111.39	0.03
18		0.20	0.12	0.23	0.08	0.95	117.64	0.03
19	0.03	0.05	0.12	0.24	0.07	0.44	406.22	0.09
20		0.09	0.12	0.24	0.06	2.11	591.50	0.12
21		0.13	0.12	0.25	0.05	1.14	230.60	0.16
22		0.17	0.12	0.25	0.05	3.87	283.34	0.16
23		0.19	0.11	0.24	0.05	1.95	214.86	0.16
24		0.20	0.11	0.24	0.05	2.08	215.06	0.24
25	0.05	0.05	0.11	0.24	0.06	1.69	360.94	0.17
26		0.09	0.11	0.25	0.06	3.23	491.41	0.24
27		0.13	0.10	0.22	0.04	0.78	204.33	0.21
28		0.17	0.11	0.22	0.04	1.24	174.91	0.28
29		0.19	0.11	0.22	0.04	1.38	167.62	0.29
30		0.20	0.11	0.22	0.04	1.13	156.68	0.30
31	0.07	0.05	0.10	0.21	0.05	2.45	175.99	0.31
32		0.09	0.10	0.21	0.05	6.31	222.77	0.32
33		0.13	0.10	0.21	0.05	0.92	132.95	0.31
34		0.17	0.10	0.21	0.05	0.70	119.57	0.32
35		0.19	0.10	0.20	0.06	2.35	176.24	0.34
36		0.20	0.10	0.20	0.06	4.00	201.84	0.34
37	0.10	0.05	0.10	0.20	0.07	2.05	140.64	0.33
38		0.09	0.10	0.20	0.07	5.62	145.15	0.36
39		0.13	0.10	0.20	0.07	0.92	100.31	0.36
40		0.17	0.10	0.20	0.07	0.81	85.61	0.37
41		0.19	0.10	0.20	0.07	0.79	86.43	0.38
42		0.20	0.10	0.20	0.07	1.38	107.11	0.36
43	0.13	0.05	0.10	0.20	0.11	5.73	145.46	0.34
44		0.09	0.10	0.20	0.11	3.76	137.79	0.33
45		0.13	0.10	0.20	0.11	3.83	132.40	0.34
46		0.17	0.10	0.20	0.11	3.84	130.75	0.34
47		0.19	0.10	0.20	0.11	2.93	129.65	0.33
48		0.20	0.10	0.20	0.11	3.83	131.23	0.34

Table A.5. Nominal coverage, in %, and MSE for the variable Y calculated for each mesh with $b=0.3$, considering 200 simulations, for $n=50$.

Mesh	c	a	Coverage(%)	MSE	Mesh	c	a	Coverage(%)	MSE
1	0.001	0.05	3.51	0.29	25	0.05	0.05	65.38	0.24
2		0.09	3.94	0.29	26		0.09	60.46	0.25
3		0.13	3.97	0.29	27		0.13	82.89	0.24
4		0.17	3.87	0.29	28		0.17	85.77	0.24
5		0.19	4.01	0.29	29		0.19	87.82	0.24
6		0.20	3.89	0.29	30		0.20	87.87	0.24
7	0.005	0.05	3.51	0.29	31	0.07	0.05	85.83	0.25
8		0.09	3.94	0.29	32		0.09	84.81	0.25
9		0.13	3.97	0.29	33		0.13	85.66	0.25
10		0.17	3.87	0.29	34		0.17	87.21	0.26
11		0.19	4.01	0.29	35		0.19	87.00	0.27
12		0.20	3.89	0.29	36		0.20	86.58	0.27
13	0.01	0.05	10.24	0.29	37	0.10	0.05	84.71	0.29
14		0.09	11.66	0.29	38		0.09	84.74	0.29
15		0.13	13.62	0.29	39		0.13	84.91	0.29
16		0.17	13.73	0.29	40		0.17	84.86	0.29
17		0.19	14.01	0.29	41		0.19	84.76	0.29
18		0.20	13.75	0.29	42		0.20	84.72	0.29
19	0.03	0.05	30.92	0.27	43	0.13	0.05	79.78	0.33
20		0.09	46.52	0.26	44		0.09	79.43	0.34
21		0.13	70.26	0.24	45		0.13	78.93	0.34
22		0.17	71.48	0.24	46		0.17	78.92	0.34
23		0.19	74.28	0.24	47		0.19	79.26	0.34
24		0.20	74.16	0.24	48		0.20	79.01	0.34

Table A.6. Nominal coverage, in %, and MSE for the variable Y calculated for each mesh with $b = 0.3$, considering 200 simulations, for 13 locations not in the analysis.

Mesh	c	a	Coverage(%)	MSE	Mesh	c	a	Coverage(%)	MSE
1	0.001	0.05	93.27	0.54	25	0.05	0.05	84.08	0.57
2		0.09	88.04	0.54	26		0.09	84.42	0.57
3		0.13	79.15	0.54	27		0.13	81.00	0.54
4		0.17	72.12	0.55	28		0.17	75.54	0.54
5		0.19	69.31	0.56	29		0.19	75.00	0.53
6		0.20	65.96	0.56	30		0.20	73.50	0.54
7	0.005	0.05	93.27	0.54	31	0.07	0.05	79.77	0.54
8		0.09	88.04	0.54	32		0.09	79.35	0.54
9		0.13	79.15	0.54	33		0.13	79.15	0.54
10		0.17	72.12	0.55	34		0.17	76.38	0.54
11		0.19	69.31	0.56	35		0.19	75.00	0.54
12		0.20	65.96	0.56	36		0.20	73.73	0.54
13	0.01	0.05	92.62	0.54	37	0.10	0.05	74.38	0.54
14		0.09	87.46	0.54	38		0.09	74.58	0.55
15		0.13	78.62	0.55	39		0.13	74.62	0.54
16		0.17	72.15	0.56	40		0.17	74.54	0.54
17		0.19	69.04	0.56	41		0.19	74.65	0.54
18		0.20	66.19	0.56	42		0.20	74.38	0.54
19	0.03	0.05	90.54	0.55	43	0.13	0.05	71.31	0.54
20		0.09	84.65	0.57	44		0.09	70.92	0.55
21		0.13	80.85	0.54	45		0.13	70.62	0.55
22		0.17	75.42	0.54	46		0.17	70.65	0.55
23		0.19	73.46	0.54	47		0.19	70.81	0.55
24		0.20	71.88	0.54	48		0.20	70.77	0.55

Figure A.3. Bias for Y in 13 extra locations, for all 200 simulations, for $b=0.3$.

Selecting the best mesh

Figure A.4. For each value of a , minimum value of: (I) DIC; (II) WAIC and (III) logCPO, for all 200 simulations, for $b=0.3$ with $n=50$.

Table A.7. Mean values of DIC, WAIC and logCPO for meshes with $b=0.3$ for $n=50$.

Mesh	c	a	DIC	WAIC	logCPO	Mesh	c	a	DIC	WAIC	logCPO
1	0.001	0.05	-288.32	-295.62	2.31	25	0.05	0.05	-9.21	-5.38	1.69
2		0.09	-286.83	-294.16	2.30	26		0.09	-26.82	-25.34	1.74
3		0.13	-286.75	-293.89	2.26	27		0.13	82.06	84.10	1.43
4		0.17	-286.63	-293.64	2.24	28		0.17	97.90	98.59	1.41
5		0.19	-286.25	-293.34	2.26	29		0.19	108.68	109.72	1.37
6		0.20	-286.39	-293.46	2.25	30		0.20	108.39	109.89	1.37
7	0.005	0.05	-288.32	-295.62	2.31	31	0.07	0.05	103.82	105.70	1.39
8		0.09	-286.83	-294.16	2.29	32		0.09	103.59	104.93	1.39
9		0.13	-286.75	-293.89	2.26	33		0.13	104.92	106.40	1.38
10		0.17	-286.63	-293.64	2.24	34		0.17	119.19	120.30	1.35
11		0.19	-286.25	-293.34	2.26	35		0.19	123.92	124.77	1.35
12		0.20	-286.39	-293.46	2.25	36		0.20	124.63	125.44	1.35
13	0.01	0.05	-259.84	-264.27	2.25	37	0.10	0.05	125.45	126.37	1.35
14		0.09	-253.27	-258.52	2.23	38		0.09	126.95	128.03	1.35
15		0.13	-244.01	-246.87	2.16	39		0.13	127.08	128.03	1.35
16		0.17	-244.53	-245.40	2.12	40		0.17	127.17	128.13	1.35
17		0.19	-242.69	-244.77	2.15	41		0.19	127.35	128.31	1.35
18		0.20	-243.27	-244.81	2.14	42		0.20	127.50	128.45	1.35
19	0.03	0.05	-167.43	-166.10	2.02	43	0.13	0.05	130.57	131.41	1.36
20		0.09	-94.18	-92.02	1.86	44		0.09	131.11	131.95	1.36
21		0.13	14.21	21.02	1.55	45		0.13	131.26	132.08	1.36
22		0.17	24.51	26.20	1.55	46		0.17	131.31	132.14	1.36
23		0.19	36.91	40.09	1.51	47		0.19	131.12	131.98	1.36
24		0.20	36.69	40.18	1.50	48		0.20	131.28	132.11	1.36

Figure A.5. Estimates for β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* , considering the mesh with minimal DIC, WAIC and logCPO, and estimates considering mesh 27.

Simulation study with n=300

Impact of mesh on inference

Figure A.6. Coverage of the meshes for: β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* with n=300.

Table A.8. Coverage, in %, of the 95% credible intervals of the parameters β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* for different meshes with $b = 0.3$, for $n = 300$.

Mesh	c	a	Coverage in % for:					
			β_0	β_1	σ_e^2	σ_s^2	k	ϕ^*
1	0.001	0.05	97.00	92.00	80.00	91.00	80.00	80.00
2		0.09	95.50	91.00	69.50	77.00	69.00	69.00
3		0.13	95.00	92.00	68.50	77.00	68.50	69.00
4		0.17	95.00	92.00	72.50	78.50	69.50	70.00
5		0.19	95.00	92.00	72.50	78.50	69.50	70.00
6		0.20	95.00	92.00	72.50	78.50	69.50	70.00
7	0.005	0.05	97.00	91.50	81.50	91.50	80.00	80.00
8		0.09	95.00	92.00	77.00	81.50	75.00	75.00
9		0.13	94.50	91.50	74.50	82.00	72.00	72.00
10		0.17	94.50	91.00	76.50	82.50	74.50	75.00
11		0.19	94.50	91.00	77.00	83.00	75.00	75.00
12		0.20	94.50	91.00	77.00	83.00	75.00	75.00
13	0.01	0.05	98.00	92.00	90.00	95.00	88.50	88.50
14		0.09	97.50	92.50	86.00	94.00	88.50	88.00
15		0.13	97.00	92.50	86.50	93.50	86.00	86.00
16		0.17	97.50	92.50	86.00	93.00	86.50	86.50
17		0.19	97.50	92.50	86.00	93.00	86.50	86.50
18		0.20	97.50	92.50	86.00	93.00	86.50	86.50
19	0.03	0.05	97.50	92.50	75.50	94.50	88.00	88.50
20		0.09	97.50	92.00	55.00	92.00	87.00	87.00
21		0.13	98.00	92.50	45.50	94.00	87.50	87.50
22		0.17	97.50	92.50	45.00	94.50	87.00	87.00
23		0.19	97.50	92.50	45.00	94.50	87.50	87.50
24		0.20	97.50	92.50	45.00	94.50	87.50	87.50
25	0.05	0.05	97.50	93.00	20.00	95.00	86.00	86.00
26		0.09	98.00	93.50	20.00	95.00	86.50	86.50
27		0.13	97.00	94.00	6.50	96.50	87.50	87.50
28		0.17	97.00	93.50	4.00	96.50	88.00	88.00
29		0.19	97.00	93.50	4.00	97.00	86.50	86.50
30		0.20	97.00	93.50	4.00	97.00	86.00	86.00
31	0.07	0.05	97.00	93.50	1.00	96.00	88.50	88.50
32		0.09	97.00	93.50	1.00	97.00	88.00	88.00
33		0.13	97.50	93.50	1.00	97.00	88.50	88.50
34		0.17	97.00	94.00	0.00	96.00	87.50	87.50
35		0.19	96.50	93.50	0.00	96.50	89.00	89.00
36		0.20	96.50	93.50	0.00	97.00	88.50	88.50
37	0.10	0.05	97.00	93.50	0.00	91.00	86.00	85.50
38		0.09	97.00	94.50	0.00	89.50	83.50	83.50
39		0.13	96.50	93.50	0.00	90.00	86.50	86.50
40		0.17	96.50	93.50	0.00	90.00	86.50	86.50
41		0.19	96.50	93.50	0.00	89.50	86.50	86.50
42		0.20	96.00	94.00	0.00	89.00	85.50	85.50
43	0.13	0.05	96.50	93.50	0.00	78.50	82.50	82.50
44		0.09	97.00	92.50	0.00	80.50	82.00	82.00
45		0.13	96.50	92.00	0.00	78.00	81.00	81.00
46		0.17	97.50	93.00	0.00	78.50	81.50	81.50
47		0.19	97.00	93.00	0.00	79.50	80.50	80.50
48		0.20	97.50	93.00	0.00	78.50	81.50	81.50

Table A.9. Absolute bias of the parameters β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* for different meshes with $b = 0.3$, for $n = 300$.

Mesh	c	a	Bias for:					
			β_0	β_1	σ_e^2	σ_s^2	k	ϕ^*
1	0.001	0.05	0.02	0.01	0.03	0.01	1.64	0.01
2		0.09	0.02	0.01	0.05	0.05	3.24	0.00
3		0.13	0.02	0.01	0.05	0.07	3.12	0.00
4		0.17	0.02	0.01	0.05	0.07	2.91	0.00
5		0.19	0.02	0.01	0.05	0.07	2.89	0.00
6		0.20	0.02	0.01	0.05	0.07	2.89	0.00
7	0.005	0.05	0.02	0.01	0.03	0.01	1.34	0.01
8		0.09	0.02	0.01	0.03	0.02	2.29	0.01
9		0.13	0.02	0.01	0.04	0.05	2.35	0.00
10		0.17	0.02	0.01	0.03	0.05	2.14	0.00
11		0.19	0.02	0.01	0.03	0.04	2.05	0.00
12		0.20	0.02	0.01	0.03	0.04	2.05	0.00
13	0.01	0.05	0.02	0.01	0.01	0.04	0.36	0.03
14		0.09	0.02	0.01	0.01	0.03	0.08	0.03
15		0.13	0.02	0.01	0.01	0.02	0.14	0.02
16		0.17	0.02	0.01	0.01	0.01	0.10	0.02
17		0.19	0.02	0.01	0.01	0.01	0.10	0.02
18		0.20	0.02	0.01	0.01	0.01	0.10	0.02
19	0.03	0.05	0.02	0.01	0.06	0.04	1.56	0.06
20		0.09	0.02	0.01	0.08	0.06	1.81	0.07
21		0.13	0.02	0.01	0.09	0.05	1.62	0.06
22		0.17	0.02	0.01	0.09	0.04	1.61	0.06
23		0.19	0.02	0.01	0.09	0.04	1.60	0.06
24		0.20	0.02	0.01	0.09	0.04	1.60	0.06
25	0.05	0.05	0.02	0.01	0.12	0.05	1.71	0.07
26		0.09	0.02	0.01	0.12	0.05	1.69	0.07
27		0.13	0.02	0.01	0.14	0.05	1.46	0.07
28		0.17	0.02	0.01	0.15	0.04	1.38	0.07
29		0.19	0.02	0.01	0.15	0.03	1.35	0.07
30		0.20	0.02	0.01	0.15	0.03	1.36	0.07
31	0.07	0.05	0.02	0.01	0.17	0.00	0.75	0.06
32		0.09	0.02	0.01	0.17	0.00	0.78	0.06
33		0.13	0.02	0.01	0.17	0.00	0.77	0.06
34		0.17	0.02	0.01	0.20	0.02	0.29	0.05
35		0.19	0.02	0.01	0.20	0.04	0.06	0.05
36		0.20	0.02	0.01	0.20	0.04	0.01	0.05
37	0.10	0.05	0.02	0.01	0.24	0.25	1.23	0.04
38		0.09	0.02	0.01	0.25	0.24	1.06	0.04
39		0.13	0.02	0.01	0.24	0.24	1.00	0.04
40		0.17	0.02	0.01	0.24	0.24	1.00	0.04
41		0.19	0.02	0.01	0.25	0.25	1.10	0.04
42		0.20	0.02	0.01	0.25	0.26	1.21	0.04
43	0.13	0.05	0.02	0.01	0.29	0.64	2.85	0.04
44		0.09	0.02	0.01	0.30	0.70	3.14	0.04
45		0.13	0.01	0.01	0.30	$35x10^{100}$	3.00	25.75
46		0.17	0.02	0.01	0.30	$48x10^{100}$	3.11	27107.18
47		0.19	0.02	0.01	0.30	$48x10^{100}$	3.06	27107.18
48		0.20	0.02	0.01	0.30	$48x10^{100}$	3.11	27107.18

Table A.10. MSE of the parameters β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* for different meshes with $b = 0.3$, for $n = 300$.

Mesh	c	a	MSE for:					
			β_0	β_1	σ_e^2	σ_s^2	k	ϕ^*
1	0.001	0.05	0.03	0.03	0.01	0.02	50.01	0.01
2		0.09	0.03	0.03	0.02	0.04	70.98	0.01
3		0.13	0.03	0.03	0.02	0.05	62.53	0.01
4		0.17	0.03	0.03	0.02	0.05	57.79	0.01
5		0.19	0.03	0.03	0.02	0.05	57.34	0.01
6		0.20	0.03	0.03	0.02	0.05	57.34	0.01
7	0.005	0.05	0.03	0.03	0.01	0.03	41.74	0.01
8		0.09	0.03	0.03	0.01	0.04	52.63	0.01
9		0.13	0.03	0.03	0.01	0.04	49.64	0.01
10		0.17	0.03	0.03	0.01	0.04	44.97	0.01
11		0.19	0.03	0.03	0.01	0.04	43.63	0.01
12		0.20	0.03	0.03	0.01	0.04	43.63	0.01
13	0.01	0.05	0.03	0.03	0.01	0.02	11.61	0.01
14		0.09	0.03	0.03	0.01	0.02	13.99	0.01
15		0.13	0.03	0.03	0.01	0.02	14.80	0.01
16		0.17	0.03	0.03	0.01	0.02	13.86	0.01
17		0.19	0.03	0.03	0.01	0.02	13.86	0.01
18		0.20	0.03	0.03	0.01	0.02	13.86	0.01
19	0.03	0.05	0.03	0.03	0.01	0.03	9.27	0.01
20		0.09	0.03	0.03	0.01	0.03	10.96	0.01
21		0.13	0.03	0.03	0.01	0.03	10.20	0.01
22		0.17	0.03	0.03	0.01	0.03	10.37	0.01
23		0.19	0.03	0.03	0.01	0.03	10.30	0.01
24		0.20	0.03	0.03	0.01	0.03	10.30	0.01
25	0.05	0.05	0.03	0.03	0.02	0.03	12.56	0.01
26		0.09	0.03	0.03	0.02	0.03	13.09	0.01
27		0.13	0.04	0.03	0.02	0.03	18.25	0.01
28		0.17	0.04	0.03	0.03	0.03	16.68	0.01
29		0.19	0.04	0.03	0.03	0.03	17.07	0.01
30		0.20	0.04	0.03	0.03	0.03	17.04	0.01
31	0.07	0.05	0.04	0.03	0.03	0.03	23.65	0.01
32		0.09	0.04	0.03	0.03	0.03	24.57	0.01
33		0.13	0.04	0.03	0.03	0.03	23.70	0.01
34		0.17	0.04	0.03	0.04	0.04	41.76	0.01
35		0.19	0.04	0.03	0.04	0.07	55.41	0.01
36		0.20	0.04	0.03	0.04	0.07	50.53	0.01
37	0.10	0.05	0.04	0.03	0.06	0.35	64.42	0.02
38		0.09	0.04	0.03	0.06	0.25	57.28	0.02
39		0.13	0.04	0.03	0.06	0.24	55.05	0.02
40		0.17	0.04	0.03	0.06	0.24	55.05	0.02
41		0.19	0.04	0.03	0.06	0.31	60.95	0.02
42		0.20	0.04	0.03	0.07	0.29	61.35	0.02
43	0.13	0.05	0.04	0.03	0.09	1.42	104.70	0.02
44		0.09	0.04	0.03	0.09	1.73	116.24	0.03
45		0.13	0.04	0.03	0.09	$14x10^{100}$	114.39	$13x10^5$
46		0.17	0.04	0.03	0.09	$14x10^{100}$	127.19	$14x10^{10}$
47		0.19	0.04	0.03	0.09	$14x10^{100}$	125.52	$14x10^{10}$
48		0.20	0.04	0.03	0.09	$14x10^{100}$	127.19	$14x10^{10}$

Table A.11. Nominal coverage, in %, and MSE for the variable Y calculated for each mesh with $b=0.3$, considering 200 simulations, for $n=300$.

Mesh	c	a	Coverage(%)	MSE	Mesh	c	a	Coverage(%)	MSE
1	0.001	0.05	90.08	0.16	25	0.05	0.05	87.74	0.18
2		0.09	86.75	0.17	26		0.09	87.73	0.18
3		0.13	86.99	0.17	27		0.13	84.80	0.19
4		0.17	87.46	0.17	28		0.17	83.81	0.20
5		0.19	87.53	0.17	29		0.19	83.75	0.20
6		0.20	87.53	0.17	30		0.20	83.73	0.20
7	0.005	0.05	90.83	0.16	31	0.07	0.05	81.81	0.21
8		0.09	89.17	0.17	32		0.09	81.69	0.21
9		0.13	88.77	0.17	33		0.13	81.75	0.21
10		0.17	89.53	0.17	34		0.17	78.25	0.22
11		0.19	89.70	0.17	35		0.19	77.31	0.23
12		0.20	89.70	0.17	36		0.20	77.15	0.23
13	0.01	0.05	93.70	0.15	37	0.10	0.05	71.87	0.25
14		0.09	93.42	0.15	38		0.09	71.23	0.26
15		0.13	93.47	0.15	39		0.13	71.49	0.26
16		0.17	93.50	0.15	40		0.17	71.49	0.26
17		0.19	93.49	0.15	41		0.19	71.33	0.26
18		0.20	93.49	0.15	42		0.20	70.73	0.26
19	0.03	0.05	92.30	0.16	43	0.13	0.05	63.44	0.30
20		0.09	90.70	0.16	44		0.09	62.91	0.30
21		0.13	90.06	0.17	45		0.13	62.76	0.30
22		0.17	89.88	0.17	46		0.17	62.47	0.30
23		0.19	89.87	0.17	47		0.19	62.59	0.30
24		0.20	89.87	0.17	48		0.20	62.47	0.30

Table A.12. Nominal coverage, in %, and MSE for the variable Y calculated for each mesh with $b = 0.3$, considering 200 simulations, for 75 locations not in the analysis.

Mesh	c	a	Coverage(%)	MSE	Mesh	c	a	Coverage(%)	MSE
1	0.001	0.05	87.41	0.31	25	0.05	0.05	79.54	0.30
2		0.09	76.05	0.32	26		0.09	79.50	0.30
3		0.13	73.03	0.32	27		0.13	74.19	0.31
4		0.17	73.11	0.32	28		0.17	70.81	0.31
5		0.19	73.03	0.32	29		0.19	70.17	0.31
6		0.20	73.03	0.32	30		0.20	70.12	0.31
7	0.005	0.05	87.45	0.30	31	0.07	0.05	74.32	0.32
8		0.09	80.17	0.31	32		0.09	74.18	0.32
9		0.13	77.83	0.31	33		0.13	74.26	0.32
10		0.17	77.15	0.31	34		0.17	69.95	0.32
11		0.19	77.11	0.31	35		0.19	67.43	0.33
12		0.20	77.11	0.31	36		0.20	66.99	0.33
13	0.01	0.05	87.60	0.30	37	0.10	0.05	65.80	0.34
14		0.09	81.55	0.30	38		0.09	65.20	0.35
15		0.13	79.32	0.30	39		0.13	65.53	0.34
16		0.17	78.88	0.30	40		0.17	65.53	0.34
17		0.19	78.85	0.30	41		0.19	65.38	0.35
18		0.20	78.85	0.30	42		0.20	64.92	0.35
19	0.03	0.05	86.01	0.30	43	0.13	0.05	58.07	0.37
20		0.09	80.59	0.30	44		0.09	57.44	0.38
21		0.13	75.27	0.30	45		0.13	57.19	0.38
22		0.17	73.94	0.30	46		0.17	56.89	0.38
23		0.19	73.82	0.30	47		0.19	56.73	0.38
24		0.20	73.78	0.30	48		0.20	56.89	0.38

Figure A.7. Bias for Y in 75 extra locations, for all 200 simulations, for $b=0.3$.

Selecting the best mesh

Figure A.8. For each value of a , minimum value of: (I) DIC; (II) WAIC and (III) logCPO, for all 200 simulations, for $b=0.3$ with $n=300$.

Table A.13. Mean values of DIC, WAIC and logCPO for meshes with $b=0.3$ for $n=300$.

Mesh	c	a	DIC	WAIC	logCPO	Mesh	c	a	DIC	WAIC	logCPO
1	0.001	0.05	559.42	558.85	1.18	25	0.05	0.05	683.73	684.27	1.17
2		0.09	499.23	494.29	1.20	26		0.09	683.82	684.36	1.17
3		0.13	502.19	495.79	1.20	27		0.13	691.72	692.03	1.18
4		0.17	506.62	502.02	1.20	28		0.17	694.74	695.01	1.18
5		0.19	507.47	502.82	1.20	29		0.19	694.85	695.13	1.18
6		0.20	507.47	502.82	1.20	30		0.20	694.91	695.19	1.18
7	0.005	0.05	581.58	580.48	1.17	31	0.07	0.05	698.96	699.83	1.18
8		0.09	557.66	552.72	1.17	32		0.09	699.29	700.14	1.18
9		0.13	547.11	541.86	1.18	33		0.13	699.05	699.91	1.18
10		0.17	563.49	557.37	1.17	34		0.17	706.11	706.65	1.19
11		0.19	565.44	559.80	1.17	35		0.19	707.55	707.97	1.19
12		0.20	565.44	559.80	1.17	36		0.20	707.72	708.11	1.19
13	0.01	0.05	633.13	634.15	1.16	37	0.10	0.05	718.08	719.39	1.20
14		0.09	634.34	635.34	1.16	38		0.09	719.85	721.11	1.21
15		0.13	633.61	634.30	1.16	39		0.13	719.30	720.53	1.21
16		0.17	634.02	634.72	1.16	40		0.17	719.30	720.53	1.21
17		0.19	634.02	634.70	1.16	41		0.19	719.64	720.86	1.21
18		0.20	634.02	634.70	1.16	42		0.20	720.88	722.01	1.21
19	0.03	0.05	662.41	662.78	1.17	43	0.13	0.05	732.98	733.78	1.23
20		0.09	672.36	672.62	1.17	44		0.09	733.87	734.63	1.23
21		0.13	675.26	675.52	1.17	45		0.13	734.37	735.14	1.23
22		0.17	675.98	676.28	1.17	46		0.17	735.14	735.81	1.23
23		0.19	675.99	676.29	1.17	47		0.19	735.00	735.69	1.23
24		0.20	676.01	676.31	1.17	48		0.20	735.14	735.81	1.23

Figure A.9. Boxplots for estimated β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* , considering the mesh with minimal DIC, WAIC and logCPO, and boxplots of estimated considering mesh 13.

Additional data analysis results

Figure A.10. Meshes with $c = 13$, $b = 570$ and $a = (20, 35, 55, 100, 240, 380)$.

Figure A.11. Meshes with $c = 35$, $b = 570$ and $a = (20, 35, 55, 100, 240, 380)$.

Figure A.12. Meshes with $c = 51$, $b = 570$ and $a = (20, 35, 55, 100, 240, 380)$.

Figure A.13. Posterior density for meshes with $c = 13$.

Figure A.14. Posterior density for meshes with $c = 35$.

Figure A.15. Posterior density for meshes with $c = 51$.

Appendix B: Supplementary Material for Chapter 3

Additional simulations results

In Section 3 of the paper a simulation study is presented. In this section of the supplementary materials we provide some extra figures and tables

The colours of the graphics in the following sessions are: green for $c=0.005$, blue for $c=0.03$, light blue for $c=0.07$ and pink for $c=0.10$ and in each group of these colours, the value of a increases from left to right. First are the meshes from GM1, in the middle the meshes of GM2 and then the meshes from GM3.

Simulation study with random data set

Figure B.1. Coverage of the meshes for: β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* , with random data set.

Table B.1. MSE of the parameters β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* and for $\frac{\sigma_s^2}{\phi}$, considering all meshes with random data set.

Group	Mesh	c	a	MSE for:						
				β_0	β_1	σ_e^2	σ_s^2	k	ϕ^*	$\frac{\sigma_s^2}{\phi}$
GM1	1	0.005	0.050	0.076	0.142	0.074	0.107	342.507	0.042	0.037
	2	0.005	0.090	0.073	0.143	0.078	0.518	334.709	0.020	0.008
	3	0.005	0.170	0.071	0.143	0.078	0.424	124.373	0.022	0.011
	4	0.005	0.200	0.071	0.141	0.078	0.459	109.720	0.023	0.014
	5	0.030	0.050	0.064	0.115	0.027	0.214	234.545	0.068	0.043
	6	0.030	0.090	0.062	0.111	0.020	0.054	36.953	0.074	0.046
	7	0.030	0.170	0.062	0.109	0.019	0.071	25.885	0.085	0.055
	8	0.030	0.200	0.062	0.108	0.020	0.076	24.274	0.075	0.059
	9	0.070	0.050	0.061	0.100	0.031	0.099	77.397	0.118	0.097
	10	0.070	0.090	0.061	0.101	0.032	0.133	73.769	0.112	0.101
	11	0.070	0.170	0.061	0.100	0.041	0.137	64.926	0.121	0.086
	12	0.070	0.200	0.061	0.101	0.045	0.227	79.323	0.113	0.079
	13	0.010	0.050	0.061	0.102	0.059	0.661	91.492	0.134	0.091
	14	0.010	0.090	0.061	0.102	0.059	0.475	92.817	0.135	0.090
	15	0.010	0.170	0.061	0.102	0.059	0.690	92.569	0.136	0.090
GM2	16	0.010	0.200	0.062	0.102	0.060	0.539	88.267	0.134	0.102
	17	0.005	0.050	0.066	0.123	0.039	0.431	303.055	0.082	0.060
	18	0.005	0.090	0.063	0.104	0.024	0.819	135.185	0.101	0.078
	19	0.005	0.170	0.060	0.101	0.052	5.094	209.869	0.103	0.080
	20	0.005	0.200	0.061	0.103	0.067	2.482	150.564	0.131	0.141
	21	0.030	0.050	0.065	0.120	0.033	0.341	178.225	0.088	0.064
	22	0.030	0.090	0.063	0.104	0.024	0.819	135.185	0.101	0.078
	23	0.030	0.170	0.060	0.101	0.052	5.094	209.869	0.103	0.080
	24	0.030	0.200	0.061	0.103	0.067	2.482	150.564	0.131	0.141
	25	0.070	0.050	0.060	0.100	0.038	1.083	122.125	0.086	0.063
	26	0.070	0.090	0.060	0.099	0.038	0.751	115.402	0.095	0.063
	27	0.070	0.170	0.060	0.101	0.052	5.094	209.869	0.103	0.080
	28	0.070	0.200	0.061	0.103	0.067	2.482	150.564	0.131	0.141
	29	0.010	0.050	0.062	0.103	0.063	11.461	233.469	0.112	0.139
	30	0.010	0.090	0.059	0.099	0.062	4.373	194.848	0.144	0.174
GM3	31	0.010	0.170	0.059	0.099	0.063	2.485	136.485	0.097	0.060
	32	0.010	0.200	0.058	0.099	0.063	10.929	212.877	0.131	0.142
	33	0.005	0.050	0.066	0.125	0.036	0.137	114.452	0.074	0.046
	34	0.005	0.090	0.064	0.111	0.023	0.137	68.511	0.086	0.056
	35	0.005	0.170	0.062	0.101	0.045	0.580	112.107	0.115	0.071
	36	0.005	0.200	0.063	0.103	0.054	0.511	111.039	0.115	0.072
	37	0.030	0.050	0.066	0.120	0.027	0.587	416.993	0.057	0.045
	38	0.030	0.090	0.064	0.111	0.023	0.138	68.930	0.087	0.057
	39	0.030	0.170	0.062	0.101	0.046	0.574	112.003	0.116	0.071
	40	0.030	0.200	0.063	0.103	0.054	0.514	109.933	0.114	0.071
	41	0.070	0.050	0.062	0.104	0.035	0.325	82.094	0.104	0.067
	42	0.070	0.090	0.062	0.103	0.035	0.318	68.569	0.109	0.063
	43	0.070	0.170	0.062	0.103	0.047	0.443	98.785	0.123	0.093
	44	0.070	0.200	0.062	0.104	0.057	0.705	101.313	0.117	0.071
	45	0.010	0.050	0.061	0.102	0.059	1.131	142.069	0.127	0.079
	46	0.010	0.090	0.061	0.101	0.060	2.132	164.644	0.122	0.082
	47	0.010	0.170	0.061	0.102	0.060	1.258	139.509	0.129	0.081
	48	0.010	0.200	0.061	0.102	0.061	2.103	161.523	0.133	0.079

Simulation study with regular data set

Figure B.2. Coverage of the meshes for: β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* , with regular data set.

Table B.2. MSE of the parameters β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* and for $\frac{\sigma_e^2}{\phi}$, considering all meshes with regular data set.

Group	Mesh	MSE for:								
		c	a	β_0	β_1	σ_e^2	σ_s^2	k	ϕ^*	$\frac{\sigma_e^2}{\phi}$
GM1	1	0.005	0.050	0.059	0.122	0.090	3.031	1513.227	0.005	0.005
	2	0.005	0.090	0.059	0.123	0.090	55.848	1607.877	0.006	0.005
	3	0.005	0.170	0.061	0.125	0.090	57.665	470.474	0.009	0.007
	4	0.005	0.200	0.061	0.126	0.090	69.132	797.943	0.010	0.010
	5	0.030	0.050	0.059	0.123	0.090	2.272	757.171	0.005	0.005
	6	0.030	0.090	0.059	0.123	0.090	55.848	1607.877	0.006	0.005
	7	0.030	0.170	0.061	0.125	0.090	57.665	470.474	0.009	0.007
	8	0.030	0.200	0.061	0.126	0.090	69.132	797.943	0.010	0.010
	9	0.070	0.050	0.059	0.121	0.090	101.645	1420.148	0.006	0.011
	10	0.070	0.090	0.059	0.121	0.090	111.090	1247.377	0.006	0.010
	11	0.070	0.170	0.060	0.125	0.090	294.339	1147.598	0.010	0.012
	12	0.070	0.200	0.060	0.125	0.090	2146.851	2683.051	0.009	0.032
	13	0.010	0.050	0.060	0.125	0.090	805.068	2435.142	0.010	0.027
	14	0.010	0.090	0.060	0.125	0.090	805.068	2435.142	0.010	0.027
	15	0.010	0.170	0.060	0.125	0.090	805.068	2435.142	0.010	0.027
	16	0.010	0.200	0.060	0.125	0.090	805.068	2435.142	0.010	0.027
GM2	17	0.005	0.050	0.061	0.124	0.090	8.358	825.628	0.006	0.004
	18	0.005	0.090	0.061	0.132	0.090	72.039	1158.467	0.009	0.007
	19	0.005	0.170	0.061	0.128	0.095	1.984	83.765	0.171	0.042
	20	0.005	0.200	0.059	0.125	0.124	1.148	71.887	0.185	0.039
	21	0.030	0.050	0.060	0.125	0.090	1.519	269.073	0.007	0.004
	22	0.030	0.090	0.061	0.132	0.090	72.039	1158.467	0.009	0.007
	23	0.030	0.170	0.061	0.128	0.095	1.984	83.765	0.171	0.042
	24	0.030	0.200	0.059	0.125	0.124	1.148	71.887	0.185	0.039
	25	0.070	0.050	0.067	0.140	0.085	380.776	1497.534	0.072	0.036
	26	0.070	0.090	0.067	0.143	0.085	4458.014	3401.350	0.085	0.086
	27	0.070	0.170	0.061	0.128	0.095	1.984	83.765	0.171	0.042
	28	0.070	0.200	0.059	0.125	0.124	1.148	71.887	0.185	0.039
	29	0.010	0.050	0.059	0.127	0.125	0.830	75.770	0.266	0.040
	30	0.010	0.090	0.059	0.127	0.116	1.661	97.474	0.222	0.040
	31	0.010	0.170	0.059	0.126	0.117	2.239	91.836	0.170	0.032
	32	0.010	0.200	0.059	0.125	0.119	0.950	78.014	0.179	0.031
GM3	33	0.005	0.050	0.060	0.124	0.090	0.131	34.614	0.004	0.005
	34	0.005	0.090	0.064	0.136	0.090	47.797	725.652	0.005	0.008
	35	0.005	0.170	0.060	0.128	0.086	0.120	37.899	0.205	0.044
	36	0.005	0.200	0.059	0.126	0.093	0.126	40.235	0.213	0.047
	37	0.030	0.050	0.061	0.128	0.090	1.674	562.439	0.005	0.005
	38	0.030	0.090	0.064	0.136	0.090	47.797	725.652	0.005	0.008
	39	0.030	0.170	0.060	0.128	0.086	0.120	37.899	0.205	0.044
	40	0.030	0.200	0.059	0.126	0.093	0.126	40.235	0.213	0.047
	41	0.070	0.050	0.063	0.155	0.083	498.872	1652.502	0.107	0.035
	42	0.070	0.090	0.066	0.151	0.087	310.197	932.174	0.041	0.024
	43	0.070	0.170	0.059	0.128	0.090	0.121	41.305	0.212	0.047
	44	0.070	0.200	0.059	0.127	0.093	0.126	42.211	0.211	0.047
	45	0.010	0.050	0.059	0.126	0.092	0.138	44.709	0.208	0.046
	46	0.010	0.090	0.059	0.125	0.098	0.154	46.126	0.256	0.053
	47	0.010	0.170	0.059	0.126	0.092	0.138	43.749	0.208	0.046
	48	0.010	0.200	0.059	0.126	0.092	0.138	43.749	0.208	0.046

Simulation study with cluster data set

Figure B.3. Coverage of the meshes for: β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* , with cluster data set.

Table B.3. MSE of the parameters β_0 , β_1 , σ_e^2 , σ_s^2 , k and ϕ^* and for $\frac{\sigma_s^2}{\phi}$, considering all meshes with cluster data set.

Group	Mesh	c	a	MSE for:						
				β_0	β_1	σ_e^2	σ_s^2	k	ϕ^*	$\frac{\sigma_s^2}{\phi}$
GM1	1	0.005	0.050	0.077	0.122	0.067	0.098	358.855	0.029	0.015
	2	0.005	0.090	0.079	0.137	0.069	0.212	224.366	0.028	0.013
	3	0.005	0.170	0.080	0.140	0.068	0.371	105.996	0.047	0.023
	4	0.005	0.200	0.075	0.123	0.068	0.397	86.291	0.030	0.010
	5	0.030	0.050	0.070	0.096	0.016	0.052	117.775	0.037	0.023
	6	0.030	0.090	0.068	0.095	0.015	0.068	92.372	0.040	0.031
	7	0.030	0.170	0.068	0.094	0.018	0.065	60.188	0.060	0.035
	8	0.030	0.200	0.067	0.093	0.017	0.062	48.392	0.046	0.027
	9	0.070	0.050	0.070	0.097	0.033	0.233	104.191	0.076	0.042
	10	0.070	0.090	0.070	0.098	0.033	0.212	111.425	0.075	0.039
	11	0.070	0.170	0.070	0.096	0.041	0.159	87.497	0.083	0.050
	12	0.070	0.200	0.070	0.098	0.044	0.176	73.412	0.088	0.057
	13	0.010	0.050	0.070	0.097	0.056	0.861	129.622	0.131	0.079
	14	0.010	0.090	0.070	0.098	0.057	0.875	123.825	0.131	0.082
	15	0.010	0.170	0.070	0.097	0.056	0.770	120.767	0.140	0.092
	16	0.010	0.200	0.069	0.097	0.059	0.900	131.053	0.132	0.086
GM2	17	0.005	0.050	0.071	0.096	0.024	0.111	89.881	0.064	0.043
	18	0.005	0.090	0.070	0.092	0.023	0.167	95.257	0.077	0.039
	19	0.005	0.170	0.070	0.096	0.053	2.064	176.069	0.123	0.100
	20	0.005	0.200	0.071	0.097	0.069	1.559	125.360	0.148	0.108
	21	0.030	0.050	0.070	0.094	0.021	0.199	147.394	0.062	0.042
	22	0.030	0.090	0.070	0.092	0.023	0.167	95.257	0.077	0.039
	23	0.030	0.170	0.070	0.096	0.053	2.064	176.069	0.123	0.100
	24	0.030	0.200	0.071	0.097	0.069	1.559	125.360	0.148	0.108
	25	0.070	0.050	0.071	0.098	0.040	1.201	147.496	0.090	0.055
	26	0.070	0.090	0.070	0.098	0.037	0.677	138.932	0.090	0.049
	27	0.070	0.170	0.070	0.096	0.053	2.064	176.069	0.123	0.100
	28	0.070	0.200	0.071	0.097	0.069	1.559	125.360	0.148	0.108
	29	0.010	0.050	0.071	0.096	0.067	2.519	173.810	0.166	0.190
	30	0.010	0.090	0.071	0.096	0.064	2.352	163.539	0.123	0.066
	31	0.010	0.170	0.072	0.102	0.064	1.518	146.115	0.137	0.122
	32	0.010	0.200	0.070	0.097	0.065	2.870	182.690	0.155	0.118
GM3	33	0.005	0.050	0.072	0.103	0.019	0.087	172.062	0.043	0.027
	34	0.005	0.090	0.070	0.095	0.019	0.091	83.856	0.051	0.032
	35	0.005	0.170	0.072	0.097	0.040	0.120	75.647	0.107	0.067
	36	0.005	0.200	0.072	0.099	0.044	0.376	126.372	0.105	0.074
	37	0.030	0.050	0.071	0.096	0.017	0.093	93.509	0.042	0.027
	38	0.030	0.090	0.070	0.095	0.019	0.092	85.093	0.051	0.033
	39	0.030	0.170	0.072	0.097	0.040	0.121	74.818	0.107	0.067
	40	0.030	0.200	0.072	0.099	0.044	0.351	115.666	0.105	0.072
	41	0.070	0.050	0.072	0.097	0.034	0.231	101.557	0.087	0.053
	42	0.070	0.090	0.072	0.098	0.034	0.209	94.262	0.088	0.055
	43	0.070	0.170	0.072	0.098	0.042	0.284	107.271	0.128	0.122
	44	0.070	0.200	0.072	0.100	0.047	0.718	142.403	0.126	0.110
	45	0.010	0.050	0.073	0.101	0.057	0.350	103.119	0.144	0.152
	46	0.010	0.090	0.074	0.101	0.057	0.344	96.259	0.148	0.158
	47	0.010	0.170	0.073	0.101	0.057	0.360	104.559	0.142	0.150
	48	0.010	0.200	0.073	0.100	0.058	0.625	131.008	0.139	0.142

Additional data analysis results

- Paraná dataset

Figure B.4. Meshes with $c = 13$, $b = 570$ and $a = (20, 35, 55, 100, 240, 380)$ of GM1.

Figure B.5. Meshes with $c = 35$, $b = 570$ and $a = (20, 35, 55, 100, 240, 380)$ of GM1.

Figure B.6. Meshes with $c = 51$, $b = 570$ and $a = (20, 35, 55, 100, 240, 380)$ of GM1.

Figure B.7. Meshes with $c = 13$, $b = 570$ and $a = (20, 35, 55, 100, 240, 380)$ of GM2.

Figure B.8. Meshes with $c = 35$, $b = 570$ and $a = (20, 35, 55, 100, 240, 380)$ of GM2.

Figure B.9. Meshes with $c = 51$, $b = 570$ and $a = (20, 35, 55, 100, 240, 380)$ of GM2.

Figure B.10. Meshes with $c = 13$, $b = 570$ and $a = (20, 35, 55, 100, 240, 380)$ of GM3.

Figure B.11. Meshes with $c = 35$, $b = 570$ and $a = (20, 35, 55, 100, 240, 380)$ of GM3.

Figure B.12. Meshes with $c = 51$, $b = 570$ and $a = (20, 35, 55, 100, 240, 380)$ of GM3.

Figure B.13. Posterior density for meshes with Paraná data set.

- Soil data set

Figure B.14. Meshes with $c = 14$, $b = 600$ and $a = (20, 40, 60, 120, 260, 400)$ of GM1.

Figure B.15. Meshes with $c = 37$, $b = 600$ and $a = (20, 40, 60, 120, 260, 400)$ of GM1.

Figure B.16. Meshes with $c = 54$, $b = 600$ and $a = (20, 40, 60, 120, 260, 400)$ of GM1.

Figure B.17. Meshes with $c = 14$, $b = 600$ and $a = (20, 40, 60, 120, 260, 400)$ of GM2.

Figure B.18. Meshes with $c = 37$, $b = 600$ and $a = (20, 40, 60, 120, 260, 400)$ of GM2.

Figure B.19. Meshes with $c = 54$, $b = 600$ and $a = (20, 40, 60, 120, 260, 400)$ of GM2.

Figure B.20. Meshes with $c = 14$, $b = 600$ and $a = (20, 40, 60, 120, 260, 400)$ of GM3.

Figure B.21. Meshes with $c = 37$, $b = 600$ and $a = (20, 40, 60, 120, 260, 400)$ of GM3.

Figure B.22. Meshes with $c = 54$, $b = 600$ and $a = (20, 40, 60, 120, 260, 400)$ of GM3.