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Introdução às Anomalias Conformes e os Teoremas C & F

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Introduction to Conformal Anomalies and the C & F Theorems

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To my loving mother, Cristina.

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"[...]
Tyger Tyger, burning bright,
In the forests of the night:
What immortal hand or eye,
Dare frame thy fearful symmetry?"
-William Blake, "The Tyger"

Resumo

As ideias fundamentais sobre entropia de emaranhamento e fluxos de renormalização são expostas, assim como uma introdução a CFTs e sua ligação com a estrutura do espaço de parâmetros. A anomalia de traço é calculada em uma abordagem semi-clássica usando o método de "heat kernel" e regularização por função ζ. É mostrado que os coeficientes de Seeley-DeWitt são responsáveis pela quebra de simetria conforme em um espaço-tempo curvo de dimensão par, com isso alcançamos uma definição geométrica para as cargas centrais. A inexistência de anomalias no caso de dimensões ímpares também é mostrado.

O "C-theorem", que prova a monotonicidade das cargas centrais sob o fluxo de renormalização, é demonstrado como feito por Zamolodchikov [17] por meio de uma abordagem euclideana assumindo unitariedade, positividade por reflexão e condições de renormalizabilidade. A análise feita por Cardy também é demonstrada, nela considera-se os mesmos ingredientes de [17]. Por fim, a prova tecida por Casini & Huerta em [12] é demonstrada com detalhes, essa prova utiliza das propriedades de "SSA" da entropia de emaranhamento, unitariedade e invariância sob o grupo de Poincaré. Com isso, uma conexão com informação quântica é feita naturalmente.

No último capítulo generalizamos o conceito de carga central para dimensões ímpares as definindo como o termo universal na entropia de emarahamento de uma esfera. As considerações geométricas feitas para provar o "C-theorem" são estendidas para um espaço-tempo de Minkowski com três dimensões. Como consequência temos a prova do "F-theorem" que é o análogo em três dimensões do "C-theorem".

Palavras-Chave: "Anomalias de Traço", "C-Theorem", "F-Theorem", "Seeley-DeWitt", "Entropia de Emaranhamento".

Abstract

The fundamental ideas of entanglement entropy and RG flows are laid out, as well as the basics of CFTs and its connection to the framework of RG flows. The trace anomaly is calculated in a semi-classical fashion by using the heat kernel method and ζ-function regularization. It is shown that the Seeley-DeWitt coefficients are responsible for the breaking of conformal symmetry in a curved even-dimensional background, which also achieves a geometrical definition of a central charge. The absence of anomalies in odd space-time dimensions is also contemplated.

The C-theorem, which proves the monotonicity of the two-dimensional central charge under RG flows, is demonstrated as first done by Zamolodchikov [17] in an Euclidean approach assuming unitarity, reflection positivity, and renormalizability conditions. Cardy's analysis is also demonstrated by considering the same conditions as [17]. And at last the proof via entanglement entropy by Casini & Huerta [12] which relies on the SSA property of EE, unitarity and Poincaré invariance is explained in detail, providing a quantum information approach to the problem.

In the last chapter a generalization of central charges to odd-dimensional space-times is given through the universal term of the EE of a sphere. We provide the extension of the geometrical setup considered in the proof of the C-theorem to a three-dimensional Minkowski space-time, which ultimately yields the F-theorem, constituting the three-dimensional analog of the C-theorem.

Key-Words: "Trace Anomaly", "C-Theorem", "F-Theorem", "Seeley-DeWitt", "Entanglement Entropy".

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Chapter 1

Introduction

During the development of Quantum Electrodynamics (QED) in the mid 1930's physicists stumbled upon divergent quantities arising from the loops in Feynman graphs, and the regularization procedure was implemented as a way of bypassing the UV catastrophe. However, until the ideas of Kenneth Wilson, renormalization was not but a mathematical trick which consisted in "dressing" the "bare" parameters with vacuum polarization effects in order to produce finite observables. This was first responsible for the well succeeded predictions of QED.

The idea that the relevant physics of a given system changes with the scale one observes is perhaps intuitive, as probing the smaller distances often reveal undiscovered features to the macroscopic observation. The Nobel prize of 1982 was awarded to Ken Wilson [1] for his contributions to the dynamics of the renormalization group flow, which precisely combines the idea of scale transformations and renormalization of QFTs, thus providing a physical interpretation for the once mysterious renormalization scheme of QED.

The framework of perturbative quantum field theory describes the dynamics of interacting fields by means of an action functional of a range of local operators built from the fundamental fields φ_j of the theory, so in general the action functional can be written as a sum of local operators Φ_i

$$S[\varphi_j] = \int d^D x g_i \Phi_i(x). \tag{1.0.1}$$

The constants g_i are called coupling constants, φ_j is a collective label for a list of independent fields, and Φ_i for the possible local operators that one can build from them. For example, in QED one has the gauge field A_μ interacting with matter fields ψ, so those are the independent fields. To construct an arbitrary local operator from the fields φ_j of QED one could consider derivatives and combinations of A_μ and ψ, for example $\Phi_1 = \partial_\nu A_\mu \partial^\nu A^\mu$.

The symmetries of the theory impose constraints over the operators Φ_i. As one is mainly interested in relativistic field theories, Lorentz invariance must be implemented and in such case the Φ_i operators must be Lorentz scalars.[1]

The list of operators $\{\Phi_i\}$ can be thought of as the building blocks for action functionals with the desired symmetry, as by simply assigning new values of the

[1] Certain symmetries do not require the operators to be invariant by themselves, but the combination of all them must be an invariant in order to the action functional to be invariant too.

coupling constants g_i one generates a different action functional. In order to compute observables one makes use of the generating functional

$$\mathcal{Z} = \int [\mathcal{D}\varphi_j] \exp\left(-iS[\varphi_j]\right). \tag{1.0.2}$$

The N-point functions are obtained by coupling the operators with external sources and taking functional derivatives. From the construction, we see that defining a set of values g_i is equivalent to defining a theory described by an action functional (1.0.1). Defining a different set of couplings g_i' one defines a different theory, or equivalently a theory described by a different action with couplings $g_i \to g_i'$. This means that one can parameterize the space of QFTs by the couplings g_i and the geometric nature of the parameter space of QFTs becomes very clear, as we can regard such space as a manifold and the coordinates in the chart $\{\Phi_i\}$ are given by the parameters g_i themselves.

QFTs must be understood as effective descriptions endowed with a cutoff Λ in momentum space. Suppose the generating functional describing the system with resolution Λ_1

$$\mathcal{Z}[\Lambda_1, g_{i_1}] = \int [\mathcal{D}\varphi_j] \exp\left(-iS[\varphi_j, g_{i_1}]\right). \tag{1.0.3}$$

This notation means one is supposed to integrate the modes of the fields φ_j only up to the value of momentum Λ_1, and the action has couplings $\{g_{i_1}\}$. With the given generating functional one is able to compute N-point functions of insertions of operators with resolution bounded by the cutoff Λ_1, or in position space by distances $\delta \sim \Lambda_1^{-1}$.

The cutoff Λ_1 is an artificial parameter, and must not influence the behaviour of N-point functions that respect the resolution of the system. The coupling constants must be regarded as functions of the cutoff Λ_1. For a generic $\mathcal{Z}[\Lambda, g_i]$ it must hold

$$\Lambda \frac{d}{d\Lambda} \mathcal{Z}[\Lambda, g_i] = \left(\Lambda \frac{\partial}{\partial \Lambda}\Big|_{g_i} + \Lambda \frac{\partial g_i(\Lambda)}{\partial \Lambda} \frac{\partial}{\partial g_i}\Big|_{\Lambda} \right) \mathcal{Z}[\Lambda, g_i] = 0. \tag{1.0.4}$$

This is the Callan-Symanzik equation, relating the change in the scale Λ with the change in the coupling constants g_i of the action. Given a generating functional $\mathcal{Z}[\Lambda_1, g_{i_1}(\Lambda_1)]$, it is possible to define a continuum of generating functionals $\mathcal{Z}[\Lambda, g_i(\Lambda)]$ with resolution $\Lambda \leq \Lambda_1$ by requiring that (1.0.4) holds. The new functionals produce the same observables by consruction, but now with a lower resolution.

Equation (1.0.4) when locally integrated generates the Renormalization Group flows (RG flows) in the coupling constants, which can be seen as generated by a vector field $\beta_i(\Lambda)$ in parameter space. The initial conditions are given by a specific theory described by $\mathcal{Z}[\Lambda_1, g_{i_1}(\Lambda_1)]$, and the flow defines a curve $g_i(\Lambda)$ such that $g_i(\Lambda_1) = g_{i_1}$. Some of these curves might have endpoints, where no matter the scale transformation in momentum the couplings do not change. These fixed points define conformal field theories, since the space-time symmetry is enhanced from Poincaré to the conformal group.

Generally speaking, given two points in parameter space one could try to find an RG flow which connects them. If such a flow exists it means that these two

points are descriptions of the same system, but in different length scales. It is of enormous physical interest to check the low energy behaviour of theories defined in the UV spectrum, and the RG flow presents a great tool for that because it produces effective descriptions by reducing the degrees of freedom of the UV limit.

The present work aims at describing no-go theorems about the relativistic RG flows[2] by constraining flows that interpolate between two fixed points of parameter space. These theorems impose a notion of order in parameter space, pointing to a direction in which the degrees of freedom are decreasing, and forbidding that a more fundamental description in the UV limit be obtained via an RG flow from a less fundamental one in the IR limit.

The fixed points represent systems with conformal invariance, and a direct consequence is the vanishing of the trace of the stress-energy tensor as an operator

$$T^{\mu}_{\mu}(x) = 0, \tag{1.0.5}$$

which can be directly connected by the invariance of the action under Weyl scalings $\delta g^{\mu\nu} \propto g^{\mu\nu}$.

For CFTs there is a parameter called the *central charge* which is a measure of the degrees of freedom in the theory. In the case of a two-dimensional CFT the central charge is also the coefficient of the central extension of the De-Witt algebra, and it is sensible to modifications in the field content of the CFT. The central charge is shown to be the coefficient of the *trace anomaly* that occurs when a given CFT is minimally coupled to gravity in a semi-classical manner. The trace anomaly means a breaking of conformal symmetry by the acquisition of a non-vanishing expectation value of the trace

$$T^{\mu}_{\mu}(x) = C\frac{R(x)}{24\pi}. \tag{1.0.6}$$

The central charge is precisely the coefficient C, and $R(x)$ is the Ricci scalar of the manifold in which one places the field content of the CFT. For four-dimensional systems the trace anomaly is shown to be

$$T^{\mu}_{\mu}(x) = A\frac{1}{16\pi^2}\left(\frac{R^2(x)}{120} + \frac{R^{\mu\nu}R_{\mu\nu}(x)}{60}\right), \tag{1.0.7}$$

where the central charge is the coefficient A. For odd dimensions the trace anomaly is absent, and the natural measure of the degrees of freedom comes from the universal part of the Entanglement Entropy (EE) of a circle.[3] As we shall see chapter 6, for a three-dimensional CFT the EE of a circle D is given by

$$S(D) = \mu_1\frac{r}{\epsilon} - F, \tag{1.0.8}$$

where μ_1 is a multiplicative constant, r is the radius of the circle D, and ϵ is a UV cutoff.[4] The coefficient F is the universal part of the EE, and it is the three-dimensional analog of C and A.

[2]RG flows that preserve the Poincaré space-time symmetries.

[3]In all even dimensions the universal part of the EE coincides with the trace anomalies.

[4]EE will be shown to be divergent in a general QFT.

The theorems we are going to focus on prove the existence of dimensionless non-increasing functions $C(\Lambda)$ under the RG flow, such that at the fixed points, where conformal symmetry is restored, the function $C(\Lambda)$ is stationary and takes the value of the central charges: C in the two-dimensional case, F in three dimensions and A in four dimensions. As the function is non-increasing the central charges of RG connecting CFTs must follow the order relation

$$C_{UV} \geq C_{IR}. \tag{1.0.9}$$

For this reason the central charges are called *ordered parameters* under the RG flow. The central charges are usually easily computable quantities, and the result (1.0.9) means that one can discard a relativistic RG flow between CFTs that do not obey the theorem. Furthermore, more exotic RG flows such as closed loops are also discarded.

The main point is that these order theorems help to classify the structure of the parameter space from a very geometric perspective. The theorems also have a deep connection with the information loss that occurs during an RG flow, also emphasizing the irreversibility of the process in accordance with the physical intuition that a theory built from more fundamental degrees of freedom is capable of giving rise to the lower energy dynamics of the system, but it is impossible to do the contrary.

This thesis is organized as follows: an introduction to the concept of RG flows is given in chapter 2, along with the fundamental tools of entanglement entropy which are needed for the proof of the theorems. In chapter 3 the general theory of conformal symmetry is developed focusing on the results that are needed in order to derive the trace anomaly and the formula for the EE in a two-dimensional CFT.

Chapter 4 is a comprehensive demonstration of the trace anomalies from a semi-classical approach to QFT in curved space-times. Detailed calculations of the heat-kernel method and zeta-function renormalization are provided, thus setting the stage for the proof of the theorems.

Chapter 5 presents the main result of this work: the proof of the C-Theorem. It is there proven by three distinct ways, two of them are done using the Callan-Simanzik equations, Poincaré invariance of two-point functions and unitarity (reflection positivity), as done by A.B.Zamolodhikov in [17] and later by J.Cardy. The third way presents a majestic proof originally proposed by Casini et al. In [12], using the strong subadditivity of EE, Lorentz invariance and unitarity.

Chapter 6 presents the proof of the rather recent F-theorem as proposed by Casini et al. In [32], proving that the same three properties of SSA, Poincaré invariance and unitarity are enough to prove an order relation of the F parameter under RG flows in three dimensions.

Chapter 2

RG flows and Entanglement Entropy

2.1 RG Flows in QFT

The importance of RG flows cannot be overstressed, as it provides a powerful analysis tool for the behaviour of quantum field theories as the energy scale of observed phenomena changes. It is often usual for one to have an action describing the system under study, and one could ask what is the relevant physics described by this action in a certain energy scale. In order to produce effective descriptions from the original action, one has to trace out the irrelevant degrees of freedom from such an action, and this process is contemplated by an RG flow. For example: we know that water is made of atoms, and these atoms have subatomical particles that are described by quantum dynamical equations. But if we wish to describe the classical flow of water in a pipe, we don't need to care about the complicated subatomical structure of the fluid, instead we may use the well known effective theory of Navier-Stokes to describe the system. In this case, the microscopic degrees of freedom give rise to effective dynamics in the low energy scale, and this is all we need to care about in this regime. This is the central idea behind the mechanics of RG flows.

Given a QFT (call it QFT_0) which describes a system up to a momentum cutoff Λ_0, one is supposed to evaluate the contribution of modes of the fields with momentum $|p| \leq \Lambda_0$. It is possible to construct an effective description of the system at a smaller energy scale $\Lambda_1 = a\Lambda_0$, with $0 < a < 1$ (call it QFT_1) by integrating out the modes with momentum in between $\Lambda_1 \leq |p| \leq \Lambda_0$. Pictorially:

$$QFT_0 \quad \underbrace{\Longrightarrow}_{\text{Integrate out high momentum modes}} \quad QFT_1. \tag{2.1.1}$$

During this process, the fast modes (higher momenta modes) may not simply disappear, but they might be translated into new low energy effective couplings, thus changing the functional form of the action that describes the system. The change on the couplings with the energy scale is precisely described by the RG flow.

The RG flow can be formulated in a series of steps:

- Definition of a momentum shell: defining what is the new cutoff for the effective theory.

- The RG step: when the action of integrating out the modes takes place.
- Rescaling: rescale the momentum, so that the quantum fluctuation of fields normalize to the original fields.

A general action for the QFT_0 can be written as

$$S = \sum_{a=1}^{n} g_a \mathcal{O}_a[\phi], \tag{2.1.2}$$

where $\mathcal{O}_a[\phi]$ are local operators. The generating function can be written as

$$\mathcal{Z}_{\Lambda_0} = \int_{\Lambda_0} \mathcal{D}[\Phi] e^{-S[\Phi]}, \tag{2.1.3}$$

the subscript Λ_0 reminds us that we must integrate over configurations of the fields Φ such that

$$\Phi(x) = \int_{|p|\leq\Lambda_0} \frac{d^d p}{(2\pi)^d} e^{-ip\cdot x} \tilde{\Phi}(p). \tag{2.1.4}$$

One could define a momentum shell to be integrated out, splitting the fields into fast modes Φ_f and slow modes Φ_s

$$\underbrace{\int_{|p|\leq\Lambda_1} \frac{d^d p}{(2\pi)^d} e^{-ip\cdot x} \tilde{\Phi}(p)}_{\Phi s} \quad \text{and} \quad \underbrace{\int_{\Lambda_1\leq|p|\leq\Lambda_0} \frac{d^d p}{(2\pi)^d} e^{-ip\cdot x} \tilde{\Phi}(p)}_{\Phi_f}. \tag{2.1.5}$$

The generating function is rewritten as

$$\mathcal{Z}_{\Lambda_0} = \int \mathcal{D}[\Phi_s]\mathcal{D}[\Phi_f] e^{-S[\Phi_s+\Phi_f]}. \tag{2.1.6}$$

Finally the integration over $\mathcal{D}[\Phi_f]$ generates an effective theory only containing slow modes, given by

$$\mathcal{Z}_{\Lambda_1} = \int_{\Lambda_1} \mathcal{D}[\Phi_s] e^{-S'[\Phi_s]}, \tag{2.1.7}$$

with

$$S' = \sum_{a=1}^{n} g'_a \mathcal{O}'_a[\Phi_s]. \tag{2.1.8}$$

Notice the subscript s, meaning that the momentum $|p|$ goes up to Λ_1. We can make use of a *scale transformation* of the fields to recover the original fields $\Phi(x)$ which have modes up to Λ_0

$$x \to \lambda x, \tag{2.1.9}$$
$$p \to \lambda^{-1} p. \tag{2.1.10}$$

we chose λ such that the rescaled fields are integrated up to the previous cutoff value Λ_0 (in this case $\lambda = a$)

$$\Phi_s(x) \to \lambda^{-\Delta_\phi} \Phi(x), \tag{2.1.11}$$

where Δ_ϕ is called the classical *scaling dimension* of the field. Now we use the fact that concerning the path integration, the fields $\Phi(x)$ are integration variables, so we can rescale the fields in order to keep the kinetic term invariant under the process, while all other operators on the action might have a change on the couplings.[1] We are left with

$$S' = \sum_{a=1}^{N} g'_a \mathcal{O}_a[\Phi]. \tag{2.1.12}$$

We can regard this whole process as a map in the couplings $g_a \to g'_a$.

Consider a free massless scalar field as an example. In this case our action is given by

$$S[\phi] = \int \frac{1}{2}(\partial^\mu \phi(x)\partial_\mu \phi(x))d^D x. \tag{2.1.13}$$

Let's analyze the effect of the whole process in this action. First one splits fast and slow modes

$$\begin{aligned} S[\phi_s + \phi_f] = \int \frac{1}{2} \, &(\partial^\mu \phi_s(x)\partial_\mu \phi_s(x) + \partial^\mu \phi_f(x)\partial_\mu \phi_s(x) \\ &+\partial^\mu \phi_s(x)\partial_\mu \phi_f(x) + \partial^\mu \phi_f(x)\partial_\mu \phi_f(x)) \, d^D x. \end{aligned} \tag{2.1.14}$$

If we simply throw out the last three terms[2] and focus on the first kinetic term, meaning the zeroth order expansion to the integration of fast modes, we would be ignoring the quantum corrections to the scaling dimension of the term $\frac{1}{2}(\partial^\mu \phi_s(x)\partial_\mu \phi_s(x))$. So the action would change to

$$S'[\phi_s] = \int \frac{1}{2}(\partial^\mu \phi_s(x)\partial_\mu \phi_s(x))d^D x. \tag{2.1.15}$$

One can rescale the coordinates using (2.1.10) and (2.1.9), so that the slow the slow modes transform back to the original fields ϕ. By doing so, the integration measure transforms as $d^D x \to \lambda^{-D} d^D x$, the derivatives $\partial \to \lambda \partial$. As we have two of them, it gives us a factor of λ^2, and the fields ϕ_s pick factors of scaling dimension, yielding

$$S'[\phi] = \int \frac{1}{2}(\lambda^{-D+2+2\Delta_\phi} \partial^\mu \phi(x)\partial_\mu \phi(x))d^D x. \tag{2.1.16}$$

The generating functional for the theory reads

$$\mathcal{Z} = \int_{\Lambda_0} \mathcal{D}[\phi] e^{-S'[\phi]}. \tag{2.1.17}$$

We can rescale the fields as $\phi \to \lambda^{\frac{D-2-2\Delta_\phi}{2}} \phi$ to absorb the change in the kinetic term. However the action must be a scalar, so the mass dimension of the kinetic term must fix the scaling dimension to be $\Delta_\phi = \frac{D-2}{2}$ rendering the field rescaling

[1] The step where one integrates out (2.1.6) the fast modes has to be done perturbatively in most cases, and this may introduce anomalies in the classical *scaling dimension* of the fields, so that the scaling dimension is: $\Delta = \Delta_\phi + \gamma_\phi$.

[2] In fact the crossed terms $\phi_f \phi_s$ does not contribute, because the modes have support in different momentum regions by definition.

trivial. Setting this particular example a *scale invariant theory*, at least classically. One can show that further integration of fast modes leads to no quantum corrections to the scaling dimension Δ_ϕ, meaning the scale invariance is preserved at a quantum level.

The running of the coupling constants with the scale Λ is the main effect of RG flows, so one defines the β-function to be the logarithimic derivative

$$\beta_i = \Lambda \frac{\partial g_i}{\partial \Lambda}. \tag{2.1.18}$$

In order to make things simpler in the study of the β-functions , we can define dimensionless coupling constants by multiplying the operators $\mathcal{O}_a$ by powers of some energy scale Λ. We rewrite the action as

$$S = \int d^D x \left(\frac{1}{2} \partial^\mu \phi(x) \partial_\mu \phi(x) + \sum_a \Lambda^{D-d_a} g_a \mathcal{O}_a(x) \right), \tag{2.1.19}$$

where again, the kinetic term fixes the classical *scaling dimension* of the fields, and the factors of Λ ensure that all terms in the Lagrangian have the right mass dimension. With this formulation we can easily check the effect of the rescaling step of the RG flow in zeroth order of fast mode integration

$$S[\phi_s(x)] = \int d^D x \left(\frac{1}{2} \partial^\mu \phi_s(x) \partial_\mu \phi_s(x) + \sum_a \Lambda^{D-d_a} g_a \mathcal{O}_a(x) \right). \tag{2.1.20}$$

Consider rescaling momentum and space coordinates as: $x' = ax$ and $p' = a^{-1}p$, and also define the scaling dimension of an operator $\mathcal{O}_a$ to be $\mathcal{O}'_a(x') = a^{-d_a}\mathcal{O}_a(x)$. Using the scale transformation of the fields, we can write this action in terms of the original fields

$$S = \int d^D x' \left(\frac{1}{2} \partial'^\mu \phi'(x') \partial'_\mu \phi'(x') + \sum_a \Lambda^{D-d_a} \underbrace{a^{d_a - D} g_a}_{g'_a} \mathcal{O}'_a(x') \right). \tag{2.1.21}$$

This way the β-function can be written as the sum of two contributions, one coming from dimensional analysis, and a correction coming from the integration of modes ($\beta_a^{quant}(g_j)$). Note that the rescaling means a change in the cutoff $\Lambda_0 \to \lambda\Lambda_0$, allowing us to write

$$\beta_a(g_j(\Lambda)) = (d_a - D) g_a(\Lambda) + \beta_a^{quant}(g_j). \tag{2.1.22}$$

The first term dictates the leading behaviour of an operator under the RG flow, we begin by noticing that if $d_a < D$ the coupling in the infrared (IR) g'_a is larger then the coupling in the ultraviolet (UV) g_a, so that the operator becomes strongly coupled under the RG flow. Such operators are called *Relevant Operators*. If $d_a > D$ the opposite happens, and the coupling assymptotically vanishes in the IR limit. These are called *Irrelevant Operators*. The remaining class of operators, for $d_a = 0$, is called *Marginal Operators*, and in this case the quantum corrections play an important role on defining the relevance of these operators in higher loop contributions. If the operator remains marginal at all loops, we call it *Exactly Marginal*.

In general, the relevance of an operator depends on the dimensions of the space-time, take for example the scalar ϕ^4 theory in D dimensions, whose action functional is

$$S[\phi] = \int d^D x \left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}g_4\phi^4 \right). \tag{2.1.23}$$

The scaling dimension of the kinetic term fixes $\Delta_\phi = \frac{D-2}{2}$, so that the scaling dimension of the ϕ^2 operator is $d_2 = 2\Delta_\phi = D-2 \to d_2 < D$ for any D, so this operator is *always* relevant. The operator ϕ^4 has dimension $d_4 = 2D-4$, so it is *irrelevant* for $D > 4$, marginal for $D = 4$, and *relevant* for $D < 4$.

The β-functions can be understood as the vector field that generates flows in the space of couplings $[g_a]$, given that it accounts for the local coupling variation as the scale Λ changes. The vector field acts on a point of the space of couplings by changing its coordinates to a neighboring point. For a practical example, take the field of velocities in a fluid which acts on a fluid particle and transports it to a neighboring position. Just as in the fluid scenario, the flow β-functions generate flows in parameter space.

A special class of points in this space is the set of points g_a^* such that $\beta(g_a^*) = 0$, called the critical points of the RG flow. These points can be seen as the endpoints of some line of flow, meaning that these theories with couplings g_a^* are invariant under change of scale. Theories with this feature are called Scale Invariant Field Theories (SFTs). The scale invariance of these theories gives them a fractal-like structure, in the sense that "zooming in" or "zooming out" does not reveal any undiscovered feature.

If one considers RG flows that preserves Poincaré space-time symmetries of a given action, it follows that the critical points have enhanced space-time symmetry when compared to other points in coupling space. As we are mainly interested in relativistic QFTs, it is natural to consider the original space-time symmetries to be the Poincaré group, so that the critical points must enjoy the extra symmetry under scale transformations.

In the vast majority of cases, the scale symmetry is enhanced even further to *conformal symmetry*, meaning that SFTs are frequently also symmetric under the *special conformal transformations*, and this set of symmetries (Poincaré, scale and special conformal) forms the well known *conformal group*. If this enhanced symmetry holds true, the theory is said to be a *Conformal Field Theory* or for short, a CFT. In fact, if not for a few pathological cases,[3] one can always enhance a SFT to a CFT, and in this spirit we say that the critical points g_a^* are actually CFTs.

Within this language of RG flows some geometrical objects appear naturally. That is the case of a *critical surface*, which is a manifold locally parameterized by the couplings of irrelevant operators to a given CFT (represented by a point g_a^*). One can consider an arbitrary fixed point g_a^*, and then introduce irrelevant perturbations in the vicinity of this point by adding irrelevant operators to the action. The new theory (perturbed action) is represented by another point g_a', and since the new theory will flow back to the original CFT, it means that the RG flow can be seen geometrically as a curve on parameter space which ends on g_a^*. We could perturbate the CFT as we wish (with different couplings to added operators). As

[3] See [28].

long as the perturbations are irrelevant, The "perturbed theory" will flow back to the unperturbed point g_a^*.

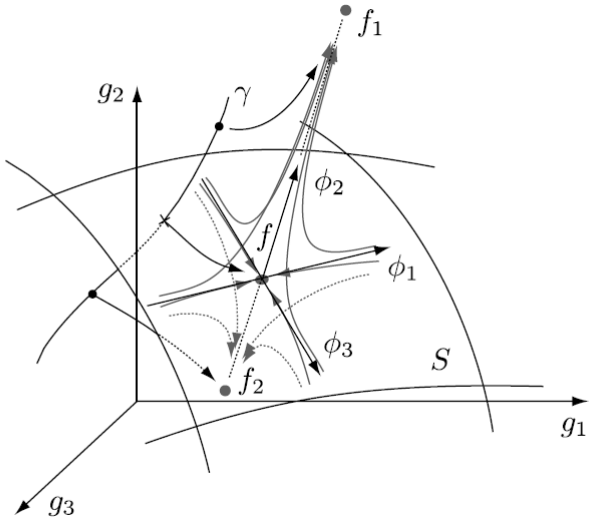

Figure 2.1: Critical Surface for two irrelevant operatores ϕ_1 and ϕ_3, and one relevant operator ϕ_3. It is locally parameterized by the couplings g_1, g_2, and g_3 in the vicinity of the RG fixed point f. f_1 and f_2 are other fixed points outside the critical surface. [35]

As the couplings on the irrelevant perturbations vary, a set of curves in parameter space is defined. The set of all curves define the *Critical Surface*. Given that we have N irrelevant operators $\tilde{\mathcal{O}}_i$, the most generic irrelevant perturbation to a CFT is

$$S_{CFT} \Longrightarrow S_{CFT} + \int d^D x \sum g_i \tilde{\mathcal{O}}_i. \tag{2.1.24}$$

Which means that the critical surface is a manifold of dimension N embedded in parameter space (2.1), and locally parameterized by the coupling constants g_i of the irrelevant operators.

2.2 Holographic RG-flows

This subsection is based on the content of chapter 9 of [4]. The AdS/CFT *correspondence*, sometimes called the *Holographic Principle*, states that the information contained in a theory of gravity in a D-dimensional AdS space-time can be encoded in a CFT which lives in a $(D-1)$-dimensional Minkowski space-time. The observables that can be computed in the theory of gravity in AdS have a correspondence in terms of observables that live in the CFT side.

AdS space-times are asymptotically Minkowski in the radial direction, so that the boundary of an D-dimensional AdS space-time is a $(D-1)$-dimensional Minkowski space-time.[4] For this reason it is often said that the CFT is defined in the boundary

[4]Appart from a global conformal factor.

of the AdS space, in which lives its holographic dual theory of gravity. The AdS space-time itself is sometimes referred to as the *Bulk* of the duality.

In the light of the *AdS/CFT correspondence* [2, 3], one could construct a Bulk configuration which is conjectured to be the holographic dual of an RG-flow. Such configuration is expected to naturally extend emergent objects in the Wilsonian renormalization scheme (field theory side) to the gravity side of the duality.

The *Gauge/Gravity duality* extends the holographic principle even further, stating that some gauge theories in the boundary, which are not necessarily CFTs, have a holographic theory of gravity dual in a bulk which is not pure AdS [4].[5] We are mainly interested in describing the RG-flow picture in the gravity side, and the *Domain Wall Flow* is of particular interested since it provides a candidate for the holographic dual of a flow connecting two CFTs.

The main idea consists of a toy model for *Super Gravity* (SUGRA), namely the truncated action containing only the scalar fields. The SUGRA configuration is conjectured to be the holographic dual of an RG-flow connecting two CFTs in the gauge side of the duality.[6]

The D-dimensional Anti de-Sitter space (AdS_D) with length scale L is defined as the set of $(X^0, ..., X^D) \in \mathcal{M}^{D-1,2}$, such that

$$(X^0)^2 - \sum_{i=1}^{D-1} (X^i)^2 + (X^D)^2 = L^2, \tag{2.2.1}$$

where $\mathcal{M}^{D-1,2}$ is the $(D+1)$-dimensional Minkowski space with the metric

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + \ldots + (dX^{D-1})^2 - (dX^D)^2. \tag{2.2.2}$$

The induced metric in the AdS_D space, emerging from the elimination of the X^D coordinate using the defining condition, reads

$$ds^2 = \left(\eta_{\mu\nu} - \frac{\eta_{\mu\lambda}\eta_{\nu\rho}}{X \cdot X + L^2}\right) dX^\mu dX^\nu, \tag{2.2.3}$$

where $\mu = 0, ..., D-1$. This is the global chart of AdS_D. The AdS_D space is a maximally symmetric space-time, having $\frac{1}{2}D(D+1)$ linearly independent Killin vector fields. The Riemann, Ricci and Einstein tensors are

$$R_{\mu\nu\lambda\sigma} = -\frac{1}{L^2}(g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda}), \tag{2.2.4}$$

$$R_{\mu\nu} = -\frac{(D-1)}{L^2} g_{\mu\nu} \quad \text{, and} \quad G_{\mu\nu} = \frac{(D-1)(D-2)}{2L^2} g_{\mu\nu}. \tag{2.2.5}$$

The Ricci scalar is $R = -\frac{D(D-1)}{L^2}$.

[5] An AdS space-time with deformations in its geometry.

[6] Consider a $\mathrm{CFT}_{\mathrm{UV}}$ perturbed by a relevant operator. Such perturbation may produce a flow which ends in another CFT in the IR limit, the $\mathrm{CFT}_{\mathrm{IR}}$.

A more useful parametrization of AdS_D is given by defining the coordinates $\vec{x} = (x^1, ..., x^{D-1}) \in \mathbb{R}^{D-1}$, $t \in \mathbb{R}$, and $r \in \mathbb{R}_+$ such that

$$X^1 = \frac{L}{2r}\left(1 + \frac{r^2}{L^4}\left(\vec{x}^2 - t^2 + L^2\right)\right), \tag{2.2.6}$$

$$X^2 = \frac{L}{2r}\left(1 - \frac{r^2}{L^4}\left(-\vec{x}^2 + t^2 + L^2\right)\right), \tag{2.2.7}$$

$$X^i = \frac{rx^i}{L} \text{ , for i} = 2, \ldots, D, \tag{2.2.8}$$

$$X^{D+1} = \frac{rt}{L}. \tag{2.2.9}$$

These local coordinates $(r, t, \vec{x})$ are called the Poincaré patch coordinates, and the metric in these coordinates is

$$ds^2 = \frac{L^2}{r^2}dr^2 + \frac{r^2}{L^2}(dx^\mu dx^\nu \eta_{\mu\nu}), \tag{2.2.10}$$

with the ordinary $\eta_{\mu\nu} = diag(-1, +1, ..., +1)$, $x^0 = t$ and $x^i = \vec{x}$. We could also define a transformation for the r coordinate $w = \frac{L^2}{r}$ so that

$$ds^2 = \frac{L^2}{w^2}(dx^\mu dx^\nu \eta_{\mu\nu} + dw^2). \tag{2.2.11}$$

Finally, it will be particularly useful for domain wall flows to use the coordinate transformation $w = Le^{\frac{u}{L}}$, with $u \in \mathbb{R}$. The metric takes the form

$$ds^2 = e^{\frac{-2u}{L}}(dx^\mu dx^\nu \eta_{\mu\nu}) + du^2. \tag{2.2.12}$$

Notice how w induces a natural momentum scale in the AdS space, as w increases we start to probe the UV. The same occurs for $u \to -\infty$ in (2.2.10). This direction may be identified as a natural momentum scale for RG flows in the bulk.

For the gravity side we consider a model that greatly illustrates the desired properties of holographic flows, a SUGRA toy model given by the *Einstein-Hilbert Action* in $D = d + 1$ dimensions and a scalar field $\phi(x)$ with a general potential $V(\phi)$. We further require that $V'(\phi) = 0$ for at least two distinct field configurations, defining a potential with two stationary points, as considered in [5]. This model can also be seen as the scalar bosonic part of a SUGRA [6]. Consider the action for the toy model

$$S = \int dx^d dr \sqrt{-g}\left(\frac{R}{16\pi G} - \frac{1}{2}g^{mn}\partial_m\phi\partial_n\phi - V(\phi)\right). \tag{2.2.13}$$

The constant G is the gravitational constant in $d + 1$ dimensions. The variation of S with respect to ϕ yields

$$S[g, \phi + \delta\phi] = \int d^{d+1}x\sqrt{-g}\left(\frac{R}{16\pi G} - g^{mn}\partial_m\phi\partial_n\phi - g^{mn}\partial_m\delta\phi\partial_n\phi - V(\phi) - V'(\phi)\delta\phi + (\mathcal{O}(\delta\phi^2))\right) \tag{2.2.14}$$

$$= S[g,\phi] + \int d^{d+1}x\sqrt{-g}\left(-g^{mn}\partial_m\delta\phi\partial_n\phi - V'(\phi)\delta\phi + (\mathcal{O}(\delta\phi^2))\right). \tag{2.2.15}$$

Integrating by parts leads to

$$\partial_n(\sqrt{-g}g^{mn}\partial_m(\phi)\delta\phi) = \partial_n(\sqrt{-g}g^{mn}\partial_m(\phi))\delta\phi + \sqrt{-g}g^{mn}\partial_m(\phi)\partial_n(\delta\phi), \tag{2.2.16}$$

the equation of motion for ϕ then reads

$$\frac{1}{\sqrt{-g}}\partial_n(\sqrt{-g}g^{mn}\partial_m(\phi)) - V'(\phi) = 0. \tag{2.2.17}$$

Varying S with respect to the metric gives us

$$\begin{aligned}\frac{\delta S}{\delta g^{ab}} = \int d^D x \Bigg\{ \frac{\delta\sqrt{-g}}{\delta g^{ab}}\left(\frac{R}{16\pi G} - \frac{1}{2}g^{mn}\partial_m\phi\partial_n\phi - V(\phi)\right) \\ +\sqrt{-g}\Big(\frac{1}{16\pi G}\frac{\delta R}{\delta g^{ab}} - \partial_a\phi\partial_b\phi\Big)\Bigg\}.\end{aligned} \tag{2.2.18}$$

Using Jacobi's identity

$$\delta(\det(g_{\alpha\beta})) = \det(g_{\alpha\beta})Tr[g^{\alpha\beta}\delta g_{\beta\nu}], \tag{2.2.19}$$

we have the variation of $\sqrt{-g}$ as

$$\delta\sqrt{-g} = \frac{-1}{\sqrt{-g}}g^{\alpha\beta}\delta g_{\alpha\beta}. \tag{2.2.20}$$

Using

$$g^{\mu\nu}g_{\nu\beta} = \delta^\mu_\beta \quad \rightarrow \quad g^{\alpha\beta}\delta g_{\alpha\beta} = -g_{\alpha\beta}\delta g^{\alpha\beta}, \tag{2.2.21}$$

we have

$$\frac{\delta\sqrt{-g}}{\delta g^{ab}} = \frac{-1}{2}\sqrt{-g}g_{ab}. \tag{2.2.22}$$

For the variation of the Ricci scalar it is necessary to write the variation of the Riemann tensor, which is by definition

$$R^\rho_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}. \tag{2.2.23}$$

The variation $\delta R^\rho_{\sigma\mu\nu}$ is

$$\begin{aligned}\delta R^\rho_{\sigma\mu\nu} = \partial_\mu(\delta\Gamma^\rho_{\nu\sigma}) - \partial_\nu(\delta\Gamma^\rho_{\mu\sigma}) + (\delta\Gamma^\rho_{\mu\lambda})\Gamma^\lambda_{\nu\sigma} + \Gamma^\rho_{\mu\lambda}(\delta\Gamma^\lambda_{\nu\sigma}) \\ -(\delta\Gamma^\rho_{\nu\lambda})\Gamma^\lambda_{\mu\sigma} - \Gamma^\rho_{\nu\lambda}(\delta\Gamma^\lambda_{\mu\sigma}).\end{aligned} \tag{2.2.24}$$

Notice that we can write the variation in terms of covariant derivatives

$$\nabla_\mu(\delta\Gamma^\rho_{\nu\sigma}) = \partial_\mu(\delta\Gamma^\rho_{\nu\sigma}) + (\delta\Gamma^\rho_{\mu\lambda})\Gamma^\lambda_{\nu\sigma} - \Gamma^\lambda_{\mu\nu}(\delta\Gamma^\rho_{\lambda\sigma}) - \Gamma^\lambda_{\mu\sigma}(\delta\Gamma^\rho_{\nu\lambda}). \tag{2.2.25}$$

This way the variation reads

$$\delta R^\rho_{\sigma\mu\nu} = \nabla_\mu(\delta\Gamma^\rho_{\nu\sigma}) - \nabla_\nu(\delta\Gamma^\rho_{\mu\sigma}), \tag{2.2.26}$$

the induced variation on the Ricci tensor is

$$\delta R_{\sigma\nu} = \nabla_\mu(\delta\Gamma^\mu_{\nu\sigma}) - \nabla_\nu(\delta\Gamma^\mu_{\mu\sigma}), \tag{2.2.27}$$

so that the Ricci scalar variation reads

$$\delta R = \nabla_\mu(\delta\Gamma^\mu_{\nu\sigma}) - \nabla_\nu(\delta\Gamma^\mu_{\mu\sigma}) + R_{\sigma\nu}\delta g^{\sigma\nu}. \tag{2.2.28}$$

The metric compatibility condition of the Levi-Civita connection $\nabla_\mu g^{\nu\sigma} = 0$ allows us to write the first term on the last equation as a total covariant derivative

$$\delta R = \nabla_\alpha(g^{\sigma\nu}(\Gamma^\alpha_{\sigma\nu}) - g^{\sigma\alpha}(\Gamma^\mu_{\mu\sigma})) + \delta g^{\sigma\nu}R_{\sigma\nu}. \tag{2.2.29}$$

Define $B^\alpha = g^{\sigma\nu}(\Gamma^\alpha_{\sigma\nu}) - g^{\sigma\alpha}(\Gamma^\mu_{\mu\sigma})$, so the first term on (2.2.29) becomes $\nabla_\alpha B^\alpha$, this term will multiply the $\sqrt{-g}$ factor and produce $\sqrt{-g}\nabla_\alpha B^\alpha$, which can be written as a total derivative term $\partial_\mu(\sqrt{-g}B^\mu)$, so we have that the only contribution to the δR term is given by $R_{\sigma\nu}\delta g^{\sigma\nu}$.

Plugging all the pieces produces

$$\begin{aligned}\delta S = \int d^{d+1}x \Bigg\{ -\frac{\sqrt{-g}}{2} g_{ab} \left(\frac{R}{16\pi G} - \frac{1}{2}\partial_l\phi\partial^l\phi - V(\phi) \right) \delta g^{ab} \\ + \sqrt{-g} \left(\frac{R_{ab}}{16\pi G} - \frac{1}{2}\partial_a\phi\partial_b\phi \right) \delta g^{ab} \Bigg\},\end{aligned} \tag{2.2.30}$$

so we finally have the equations of motion for the metric

$$R_{ab} - \frac{R}{2}g_{ab} = 8\pi G \left(\partial_a\phi\partial_b\phi - \frac{1}{2}g_{ab}\partial_l\phi\partial^l\phi - g_{ab}V(\phi) \right), \tag{2.2.31}$$

also defining the energy-momentum tensor as $T_{ab} \doteq \partial_a\phi\partial_b\phi - \frac{1}{2}g_{ab}\partial_l\phi\partial^l\phi - g_{ab}V(\phi)$.

An immediate and trivial solution to the equations of motion is achieved by setting $\phi(x) = \phi_i$ stationary point of the potential, equation (2.2.31) yields an AdS space

$$G_{ab} = -8\pi g_{ab}V(\phi_i). \tag{2.2.32}$$

We can compare with the solution to the Einstein field equations of an AdS space

$$G_{ab} + \Lambda g_{ab} = 0 \quad \longrightarrow \quad \Lambda = -\frac{d(d-1)}{2L^2}, \tag{2.2.33}$$

identifying the cosmological constant

$$\Lambda = 8\pi V(\phi_i) = -\frac{d(d-1)}{2L^2}. \tag{2.2.34}$$

A generalization of such solution is provided by the *Domain Wall Ansatz*, which is given in terms of the metric

$$ds^2 = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + dr^2, \tag{2.2.35}$$

with the conditions

$$A(r) = \frac{r}{L} \quad \text{and} \quad \phi(r) = \phi_i, \text{ for} \quad r \to \pm\infty. \tag{2.2.36}$$

These conditions ensure that the Bulk geometry is AdS at the boundary and at the deep interior, which is expected from an interpolating flow since the CFTs have AdS spaces as holographic duals. As discussed previously, we identify the r direction with a natural momentum scale in the bulk. It relates to the momentum scale on the field theory side as

$$\mu = \mu_0 e^{\frac{r}{L}}. \tag{2.2.37}$$

One can probe the UV (IR) limit by taking $r \to \infty$ ($r \to -\infty$). This particular construction is extremely useful in defining *holographic c-theorems* [33].

Solving the Einstein tensor for the ansatz yields

$$G^{\mu}_{\nu} = (d-1)\delta^{\mu}_{\nu}\left(\frac{d^2A(r)}{dr^2} + \frac{d}{2}\left(\frac{dA(r)}{dr}\right)^2\right), \tag{2.2.38}$$

$$G^{r}_{r} = \frac{d(d-1)}{2}\left(\frac{dA(r)}{dr}\right)^2. \tag{2.2.39}$$

The imposition of *Null Energy Condition*[7] (NEC) leads to a constraint in the $A(r)$ function. Choosing a null vector of the form $\xi^n = (e^{-A(r)}, 0, ..., 0, 1)$, and noticing that the G_{mn} tensor is diagonal (from the equation of motion we also see that T^{mn} is diagonal) the constraint becomes

$$T^r_r - T^t_t \geq 0 \quad \longrightarrow \quad G^r_r - G^t_t \geq 0. \tag{2.2.40}$$

The $A'(r)$[8] factors cancel out, resulting in

$$-d(d-1)\frac{d^2A(r)}{dr^2} \geq 0 \quad \longrightarrow \quad \frac{d^2A(r)}{dr^2} \leq 0. \tag{2.2.41}$$

This equation means that the length scale of interpolated AdS spaces must satisfy the order relation

$$L_{UV} \geq L_{IR}. \tag{2.2.42}$$

This construction contains the desired elements of an interpolating RG flow: the domain wall with the given conditions (2.2.36) fixes the geometry of the bulk with the one of an AdS space at the end of the flow, at the same time the RG flow has CFTs as endpoints on the boundary. Furthermore, we can identify the momentum scale of the RG flow with a bulk coordinate by the map (2.2.37). However, the identification is scheme dependent, meaning that different coordinate choices in the bulk will correspond to different renormalization schemes at the boundary. At last one extracts a constraint on length scales of the AdS slices of the bulk, which is fruit of the null energy condition. This constraint is important on the discussion of holographic C-theorems.

[7] $T^{mn}\xi_m\xi_n \geq 0$, for any null vector ξ_n.

[8] Notice that $A'(r)$ is the holographic C-function.

2.3 Entanglement Entropy

Entanglement entropy plays an important role in the modern discussion of monotonically decreasing functions along RG flows. It is the main ingredient in the recent proofs of entanlgement C-theorem and A-theorem by Casini and Huerta in [12, 14]. With that in mind, this section is dedicated to the fundamentals of entanglement entropy (EE), mainly based on [7].

Entanglement entropy provides a natural tool for measuring the entanglement between degrees of freedom in a quantum system. Generally speaking, the ground state of QFTs are highly entangled states. If it were not the case, correlators between space-like separated points would always vanish due to causality considerations. It is usual to ask for space-like separated field operators to commute in order to preserve causality

$$[\phi(x), \phi(y)] = 0 \quad \text{, for x and y space-like separated,} \tag{2.3.1}$$

and even with such imposition, there is non-vanishing correlation between space-like separated degrees of freedom in a general QFT. Take for example a scalar field two-point function

$$\langle 0| \phi(x)\phi(y) |0\rangle \sim \exp\left(-m|x-y|\right) \quad \text{, for large separations.} \tag{2.3.2}$$

The non-vanishing of not causally connected operators is due to entanglement between the degrees of freedom of the field ϕ. Note also that such correlator diverges when the points x and y are brought close together, so that neighboring points are infinitely correlated, and contribute at leading order to the entanglement entropy of the system.

2.3.1 Basic Definitions and Results

The system is said to be represented by an ensemble $\{p_i, |\psi_i\rangle\}$ if it has probability p_i of being in the state $|\psi_i\rangle$. The *density operator* or *density matrix* is defined as

$$\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i| , \tag{2.3.3}$$

with

$$\sum_i p_i = 1. \tag{2.3.4}$$

The density matrix is said to define a *pure state* if $p_i = 1$ for some i, and zero otherwise. If the density matrix is not describing a pure state, we say the state is mixed. There is a simple criterion on defining if a state is pure or mixed:

$$Tr(\rho^2) = 1 \quad \longrightarrow \quad \text{pure state,} \tag{2.3.5}$$

$$Tr(\rho^2) < 1 \quad \longrightarrow \quad \text{mixed state.} \tag{2.3.6}$$

The set of ensembles are not in one-to-one correspondence to density operators, in the sense that a given density operator can be generated by different ensembles. As an example one can consider a two level system described by a density matrix

$$\rho = \frac{3}{4} |0\rangle \langle 0| + \frac{1}{4} |1\rangle \langle 1| . \tag{2.3.7}$$

One can easily check that such density matrix could be generated by the ensemble

$$\left\{p_0 = \frac{3}{4}, p_1 = \frac{1}{4}; |\psi_0\rangle = |0\rangle\,, |\psi_1\rangle = |1\rangle\right\}. \tag{2.3.8}$$

Performing a simple change of basis

$$|a\rangle = \sqrt{\frac{3}{4}}\,|0\rangle + \sqrt{\frac{1}{4}}\,|1\rangle\,, \tag{2.3.9}$$

$$|b\rangle = \sqrt{\frac{3}{4}}\,|0\rangle - \sqrt{\frac{1}{4}}\,|1\rangle\,, \tag{2.3.10}$$

the ensemble is rewritten as $\left\{p_0 = \frac{1}{2}, p_1 = \frac{1}{2}; |\psi_0\rangle = |a\rangle\,, |\psi_1\rangle = |b\rangle\right\}$.

The *Von Neumann entropy* for a density matrix ρ is defined as

$$S(\rho) = -Tr(\rho \log(\rho)). \tag{2.3.11}$$

Once a basis for the Hilbert space is chosen we can rewrite the entropy as the eigenvalues of the density matrix, which is what is usually done in practice

$$S(\rho) = -\sum_i \lambda_i \log(\lambda_i). \tag{2.3.12}$$

A pure state in the Hilbert space will generate a density matrix whose entropy vanishes. In contrast, an ensemble with $p_i = p_j$ for all i, j corresponds to the state of complete ignorance about a system: since all quantum states are equally probable, one is unable to extract information about it, thus yielding maximum entropy. The last ensemble provides an upper bound to the entropy

$$S(\rho) \leq \log(dim(\mathcal{H})), \tag{2.3.13}$$

where $\dim(\mathcal{H})$ is the dimension of the Hilbert space of the theory. Entanglement Entropy follows from an application of Von Neumann entropy to a subspace of a Hilbert space, and the notion of *reduced density matrix*.

Consider a separable Hilbert space

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}. \tag{2.3.14}$$

The *reduced density matrix* ρ_A is an operator acting on the subspace $\mathcal{H}_A$, defined for a pure state $|\psi\rangle$ as

$$\rho_A = Tr_{A^c}(\rho) = Tr_{A^c}(|\psi\rangle\,\langle\psi|). \tag{2.3.15}$$

The density matrix ρ is defined for an ensemble of global states $|\psi_i\rangle \in \mathcal{H}$, but we are generally interested in the case when the global state is the vacuum (generally a *pure* state). The A^c subscript means a partial trace over the degrees of freedom in $\mathcal{H}_{A^c}$.

The *Entanglement Entropy* is defined in terms of the reduced density matrix as

$$S(A) = -Tr(\rho_A \log(\rho_A)). \tag{2.3.16}$$

As an illustrative example, consider a coupled two level quantum system in a pure state given by $|\psi\rangle = \cos(\theta)\,|1\rangle_A\,|0\rangle_{A^c} - \sin(\theta)\,|0\rangle_A\,|1\rangle_{A^c}$. The global density matrix is

$$\begin{aligned}\rho &= |\psi\rangle\langle\psi| \\ &= \cos^2(\theta)\,(|1\rangle\langle 1|)_A\,(|0\rangle\langle 0|)_{A^c} - \sin(\theta)\cos(\theta)\,(|1\rangle\langle 0|)_A\,(|0\rangle\langle 1|)_{A^c} \\ &- \sin(\theta)\cos(\theta)\,(|0\rangle\langle 1|)_A\,(|1\rangle\langle 0|)_{A^c} + \sin^2(\theta)\,(|0\rangle\langle 0|)_A\,(|1\rangle\langle 1|)_{A^c}\,.\end{aligned} \tag{2.3.17}$$

Choosing the canonical product base for $\mathcal{H}$, and taking the partial trace yields

$$Tr_{A^c}(\rho) = \sum_{(1,0)} \langle i|_{A^c}\,\rho\,|i\rangle_{A^c}\,. \tag{2.3.18}$$

The reduced matrix ρ_A reads

$$\rho_A = \begin{bmatrix} cos^2(\theta) & 0 \\ 0 & sin^2(\theta) \end{bmatrix}.$$

Notice that for $\theta = \frac{\pi}{4}$ we have $p_0 = p_1 = \frac{1}{2}$, thus maximum entropy

$$S(A) = \log(2) = \log(dim(\mathcal{H})). \tag{2.3.19}$$

This example illustrates how a pure entangled global state can generate a mixed reduced state when partially traced. By generating a mixed state in the A partition of $\mathcal{H}$, it produces a non-vanishing entanglement entropy $S(A)$. By evaluating $S(A)$, one retrieves information about entanglement of the global state $|\psi\rangle$. Taking $\theta = 0 \pmod{2\pi}$, we can see the disentangling of the state $|\psi\rangle$ and the consequent production of a pure reduced density matrix ρ_A, leading to a vanishing entanglement entropy.

A global pure state $|\psi\rangle \in \mathcal{H}$ can be written as $|\psi\rangle = \sum_{i,j} c_{ij}\,|a_i\rangle\,|b_j\rangle$, where $|a_i\rangle$ is a basis for $\mathcal{H}_A$, and $|b_j\rangle$ for the A^c subspace. Define local unitary transformation $\mathcal{U}$ and $\mathcal{V}$ in $\mathcal{H}_A$ and $\mathcal{H}_{A^c}$ respectively, and consider the change of basis

$$\begin{aligned}|\tilde{a}_i\rangle &= \mathcal{U}\,|a_i\rangle\,, \\ |\tilde{b}_j\rangle &= \mathcal{V}\,|b_j\rangle\,.\end{aligned} \tag{2.3.20}$$

Writing the identity operator on $\mathcal{H}$ as $\mathbb{1} = \mathbb{1}_A \otimes \mathbb{1}_{A^c}$, and using the completion relation

$$\mathbb{1} = \sum_{l,m} (|a_l\rangle\langle a_l|)\,(|b_m\rangle\langle b_m|)\,, \tag{2.3.21}$$

we can rewrite the global state $|\psi\rangle$ in the new basis (2.3.20) and apply the identity operator (2.3.21). Denoting the matrix elements of the unitary operators $\mathcal{U}$ and $\mathcal{V}$ in the basis $|a_i\rangle$ and $|b_j\rangle$ by $\langle a_l|\,\mathcal{U}\,|a_i\rangle = \mathcal{U}_{li}$ and $\langle a_m|\,\mathcal{V}\,|a_j\rangle = \mathcal{V}_{mj}$, one reaches

$$|\psi\rangle = \sum_{l,m,i,j} \mathcal{U}_{li}\tilde{c}_{ij}\mathcal{V}_{jm}\,(|a_l\rangle\,|b_j\rangle)\,, \tag{2.3.22}$$

identifying a matrix multiplication

$$|\psi\rangle = \sum_{l,m} [\mathcal{U}\tilde{c}\mathcal{V}]_{lm}\,(|a_l\rangle\,|b_j\rangle)\,. \tag{2.3.23}$$

It is always possible to find unitary matrices $\mathcal{U}$ and $\mathcal{V}$ such that the complex matrix $[\mathcal{U}\tilde{c}\mathcal{V}]_{lm}$ is diagonal, allowing us to finally write

$$|\psi\rangle = \sum_i \lambda_i \left(|a_i\rangle\,|b_i\rangle\right). \tag{2.3.24}$$

This is known as the *Schmidt decomposition*. We can immediately see that the reduced density matrices ρ_A and ρ_{A^c} have the same eigenvalues, leading to an important property of EE: if the Hilbert space decomposes as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$, then $S(A) = S(A^c)$ for pure states.

When dealing with the continuum limit, it is useful to define the *Renyi entropies*

$$S_A^q = \frac{1}{1-q}\log\left(Tr_A({\rho_A}^q)\right) = \frac{1}{1-q}\log\left(\sum_i \lambda_i^q\right), \tag{2.3.25}$$

where λ_i are the eigenvalues of ρ_A. By taking the analytic continuation $q \in \mathbb{R}$ with $q > 1$, we can identify

$$S(A) = \lim_{q\to 1} S_A^q. \tag{2.3.26}$$

The eigenvalues are such that $\lambda_i \in [0,1]$ and $\sum \lambda_i = 1$, so that the series $\sum \lambda_i^q$ is absolutely convergent for all real values of $q > 1$, therefore this analytic continuation exists and is unique [21].

2.3.2 Continuum Limit

The application of entanglement entropy to a quantum field theory requires the construction of a Hilbert space based on the ideas of Cauchy surfaces, while locality and causality lead to important geometric properties of entanglement entropy. We can use the definitions already given for density matrices and entropies in finite-dimensional spaces, and then take the continuum limit of a lattice QFT to extend those concepts to a continuum version. By doing so, we anticipate that divergences occur and need to be regulated by a cutoff. In order to make precise statements about the causality of a given spacetime $\mathcal{M}$, we make use of the following definitions [15]:

- A causal curve is a *past(future)-inextendable* curve if it has no past(future) endpoint in $\mathcal{M}$
- A *partial Cauchy surface* of $\mathcal{M}$, denoted Σ, is a subset of $\mathcal{M}$ which is intersected by all causal curves no more than once.
- The *future domain of dependence* of a subset $\mathcal{A} \subset \mathcal{M}$, denoted $\mathcal{D}^+[\mathcal{A}]$, is the set of points $p \in \mathcal{M}$ such that every past-inextendable causal curve through p intersects $\mathcal{A}$.
- The *past domain of dependence* of a subset $\mathcal{A} \subset \mathcal{M}$, denoted $\mathcal{D}^-[\mathcal{A}]$, is the set of points $p \in \mathcal{M}$ such that every future-inextendable causal curve through p intersects $\mathcal{A}$.

- A *partial Cauchy surface* Σ is said to be a *Cauchy surface* if

$$\mathcal{D}^+[\Sigma] \cup \mathcal{D}^-[\Sigma] = \mathcal{M}.$$

- If a lorentzian manifold $\mathcal{M}$ has a Cauchy surface, it is said to be *globally hyperbolic*

Consider a quantum field theory defined on a globally hyperbolic d-dimensional Lorentzian manifold $\mathcal{M}$. We choose a Cauchy surface Σ to define a moment of simultaneity for the QFT. Define a Hilbert space $\mathcal{H}_x$ for each point $x \in \Sigma$. The total Hilbert space $\mathcal{H}$ (or global Hilbert space) is constructed as the product space of all point-wise Hilbert spaces: $\mathcal{H} = \otimes_{x\in\Sigma}\mathcal{H}_x$.

Consider splitting Σ into two disjoint sets as $\Sigma = A \cup A^c$, and suppose that the total space is separable as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$,[9] where $\mathcal{H}_A = \otimes_{x\in A}\mathcal{H}_x$, and $\mathcal{H}_{A^c} = \otimes_{x\in A^c}\mathcal{H}_x$.

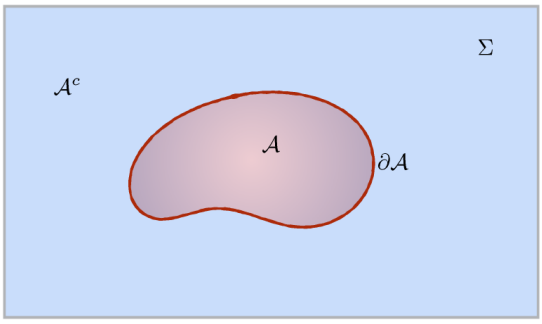

Figure 2.2: A Cauchy surface Σ and the regions A and A^c sharing a boundary ∂A [8]

As in the discrete case, one can now calculate the reduced density matrix $\rho_A = Tr_{A^c}(\rho)$, creating an operator that acts on $\mathcal{H}_A$, and the EE is given by $S_A = -Tr_A(\rho_A \log \rho_A)$. This is technically challenging since ρ_A is a continuum spectrum operator, so generally one uses (2.3.25) and (2.3.26) instead. For a general interacting QFT, it is computationally hard to calculate the eigenvalues λ_i of the reduced density matrix, so that the replica trick is a more suitable approach.

Locality and causality come into play: from the previous construction we have $\mathcal{D}^+[\Sigma] \cup \mathcal{D}^-[\Sigma] = \mathcal{M}$, which means the information contained in Σ is unitarily evolved to all other points of space-time, so that we only need to specify the QFT data in a time-slice defined by the Cauchy surface Σ, and the Hamiltonian operator generates time translations evolving the data unitarily from an initial Cauchy surface to another Cauchy surface that defines a "later time" configuration. This is analogous to providing initial conditions (QFT data in Σ) and evolving some differential equation (action of the Hamiltonian) to obtain the behaviour of the system at a later time.

The information in $\mathcal{D}^+[A]$ is uniquely determined by the information contained in $A \subset \Sigma$, because $\mathcal{D}^+[A]$ is the locus of all points of space-time on the causal

[9]This decomposition is not possible in gauge theories for example, because there is no way of decomposing the Hilbert space in a gauge invariant way.

future of A. The same goes for $\mathcal{D}^-[A]$: it is the locus of all space-time points on the causal past of A, so that all information in A is also unitarily evolved from $\mathcal{D}^-[A]$. The union $\mathcal{D}[A] = \mathcal{D}^+[A] \cup \mathcal{D}^-[A]$ is called the *Causal Diamond*, and it is a region in which all information is self-contained, so that we can unitarily evolve the ρ_A operator by the time evolution operator in A.

This way if we consider another region in another slice $A' \subset \Sigma'$ such that A and A' share their boundaries ($\partial A = \partial A'$), or equivalently $\mathcal{D}[A] = \mathcal{D}[A']$, the operator ρ_A must be related with $\rho_{A'}$ by a unitary transformation U

$$\rho_{A'} = U^\dagger \rho_A U. \tag{2.3.27}$$

This equation says that ρ_A and $\rho_{A'}$ must have the same eigenvalues once a basis is picked, leading to an important property of entanglement entropy: $S_A = S_{A'}$ if $\mathcal{D}[A] = \mathcal{D}[A']$(or $\partial A = \partial A'$ equivalently), proving that the EE is a function of the boundary of the entangling region [13]. This property is to be expected, since EE accounts for entanglement between degrees of freedom inside A and outside A. It is natural to expect that the leading order behaviour should be governed by the boundary since that's where the degrees of freedom from inside and outside are the closest, leading to a highly correlated region.

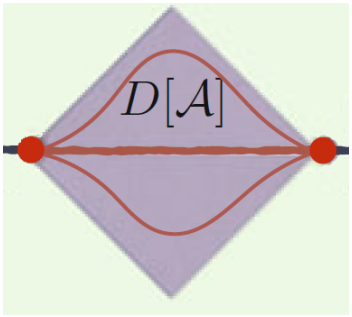

Figure 2.3: A Cauchy surface A as the line segment connecting the red dots, and other partial Cauchy surfaces A' with same boundary. The causal diamond $\mathcal{D}[A]$ is displayed in purple. [8].

Another non-trivial property of EE that will prove useful later on is the strong subadditivity (SSA) [38, 37],

$$S(A \cup B \cup C) + S(B) \leq S(A \cup B) + S(B \cup C). \tag{2.3.28}$$

SSA property and unitarity considerations are the main ingredients for the derivations of C-theorems and A-theorems from an EE point of view, as we will see in the next chapters.

2.4 Holographic Entanglement Entropy

This section is dedicated to the basic definitions of holographic entanglement entropy and important results. The work presented here is mainly contained in [7] and [8].

The search for a holographic dual of entanglement entropy is greatly inspired by the black hole entropy given by the Beckenstein-Hawking formula (2.4.1), which relates the entropy with a geometric quantity in the bulk, namely the area H of the black hole horizon

$$S_{\text{BH}} = \frac{Area(H)}{4G}. \tag{2.4.1}$$

The Ryu-Takayanagi construction for Holographic Entanglement Entropy (HEE) [9, 10] is guided by this parallel, and generalizes the relation between the entropy of a subsystem and a geometric quantity in the AdS side of the correspondence.

Being a non-local quantity, HEE is a nice tool for understanding the AdS/CFT correspondence in a way independent of the specific field content of a given field theory, as opposed to the information contained in correlation functions of local operators or Wilson loops. Instead, it gives a full picture of holography in terms of a universal observable which is always well defined for quantum field theories.

We already know how to compute the entanglement entropy of a given subregion on the QFT side, by means of the Von Neumann entropy. We can regard this quantity as a measure of degrees of freedom in the QFT, in fact, the infinite number of degrees of freedom in a QFT is responsible for UV divergences in the computations. A direct field theory computation for 2-d CFTs reveals that the EE scales with the central charge[10] C (central extends the Virasoro algebra), providing us with a hint that C must somehow be a measure of the degrees of freedom.

The calculation of S_A in the CFT side involves a geometric separation already discussed (2.3.2). The Ryu-Takayanagi proposal for HEE makes possible the calculation of S_A via bulk defined quantities.

In the AdS/CFT picture (we take the AdS to be $(d+2)$-dimensional), the CFT lives on the boundary of the AdS space, and a theory of gravity in the interior of the bulk gets mapped to operators in the CFT. As before, we define a cauchy surface Σ in the boundary, then we pick a region $A \subset \Sigma$ which is a d-dimensional subspace. In principle, one could use the previous definition (2.3.16) to compute S_A non-holographically, but this is in general very hard to do.

The Ryu-Takayanagi (RT) proposal defines a d-dimensional surface γ_A *living inside the AdS space*, such that the boundary of γ_A coincides with the boundary of A ($\partial\gamma_A = \partial A$). There are, of course, infinite surfaces γ_A that meet the criterion, but the proposal states that we should pick the *minimal area surface* as our surface γ_A. The HEE is calculated via

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{d+2}}, \tag{2.4.2}$$

where $4G_N^{d+2}$ is Newton's constant in $d+2$ dimensions.

One can be guided by the intuition given by the $\text{AdS}_3/\text{CFT}_2$ case to build an argument to support that the RT proposal leads to a strong subadditive (SSA) quantity in the bulk. The time-slice in the CFT_2 side defines a line (1-dimensional space), in this line we consider a line segment divided into three non-empty connected regions A, B and C. Following the RT recipe, we have for the regions $A \cup B$ a minimal

[10] See section 4 of chapter 3 for more details about the central charge.

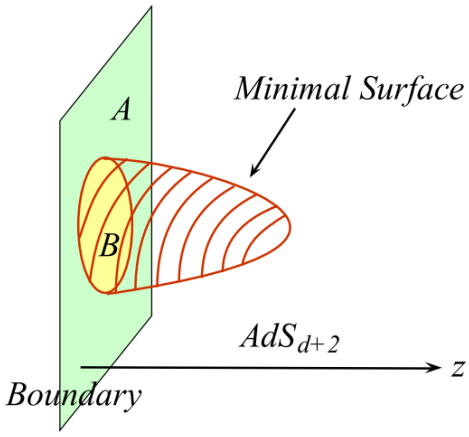

Figure 2.4: A minimal surface γ_A and the region A sharing a boundary ∂A. From [8]

surface $\gamma_{A\cup B}$ in the bulk, and also $B \cup C$ defines $\gamma_{B\cup C}$. With the given geometric construction, we see that the union of the surfaces $\gamma_{A\cup B} \cup \gamma_{B\cup C}$ can be decomposed in a surface γ'_B (the subscript B means $\partial\gamma'_B = \partial B$) and another surface $\gamma'_{A\cup B\cup C}$. Since for a given region X in the line, there is a unique γ_X which is minimal, the surfaces γ'_B and $\gamma'_{A\cup B\cup C}$ are not, in general, minimal.

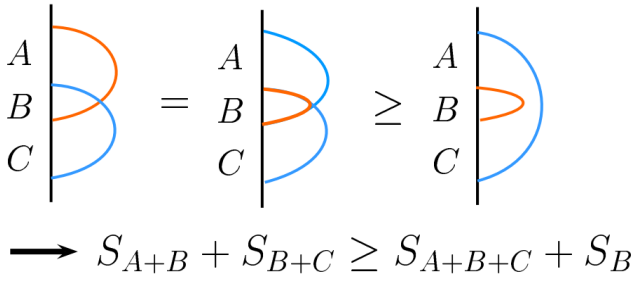

Figure 2.5: The line on the left of each diagram represents the Cauchy surface in the CFT_2 side, and the curves are the minimal surfaces embedded in a time-slice of AdS_3. [8]

$$\mathrm{Area}(\gamma_{A\cup B}) + \mathrm{Area}(\gamma_{B\cup C}) = \mathrm{Area}(\gamma'_{A\cup B\cup C}) + \mathrm{Area}(\gamma'_B). \tag{2.4.3}$$

As the primed surfaces are not minimal, $\mathrm{Area}(\gamma'_{A\cup B\cup C}) \geq \mathrm{Area}(\gamma_{A\cup B\cup C})$ and $\mathrm{Area}(\gamma'_B) \geq \mathrm{Area}(\gamma_B)$, and we have

$$\mathrm{Area}(\gamma_{A\cup B}) + \mathrm{Area}(\gamma_{B\cup C}) \geq \mathrm{Area}(\gamma_{A\cup B\cup C}) + \mathrm{Area}(\gamma_B), \tag{2.4.4}$$

which is precisely the SSA property, formulated in terms of RT formula.

The RT proposal has been applyed to higher-dimensional CFTs considering the entangling region A to have a smooth and compact boundary, and it predicts the following behaviour:

$$S_A = c_1\left(\frac{l}{a}\right)^{d-1} + c_3\left(\frac{l}{a}\right)^{d-3} + \ldots + c_{d-1}\frac{l}{a} + c_d \qquad \text{for d even,} \tag{2.4.5}$$

$$S_A = c_1\left(\frac{l}{a}\right)^{d-1} + c_3\left(\frac{l}{a}\right)^{d-3} + \ldots + c_{d-2}\left(\frac{l}{a}\right)^{2} + \tilde{c}log\left(\frac{l}{a}\right) \qquad \text{for d odd.} \quad (2.4.6)$$

The parameter a is a lattice UV cutoff, and l is a typical length scale of the region A.

The power of the holographic proposal can be contemplated through (2.4.5) and (2.4.6). It predicts the result for S_A in arbitrary dimensions, while the approach of Von Neumann entropy often leads to very cumbersome calculations.[11]

The results for CFT_2 in this configuration of entangling region is known through direct calculation on the CFT side, and matches the RT prescription. The results for CFT_3 are also known to match the (2.4.6) formula as proved Ryu and Takayanagi in [9].

[11] See section 3 of chapter 5.

Chapter 3

CFT Basics

3.1 Introduction

The work that follows is mainly covered in [16]. In this chapter the basics of Conformal Field Theories are developed in a self-contained way, the goal being to present all results necessary to prove the C-theorems and its higher-dimensional versions. It is also of our interest the controlled symmetry breaking through trace anomalies that occurs when a CFT is defined in a curved background, such symmetry breaking provides an elegant and formal definition of the central charges for even-dimensional CFTs.

Upon the analysis of RG trajectories in parameter space, one stumbles upon fixed points which define theories with scale invariance. In most cases the scale invariance is accompanied by a symmetry called *special conformal transformation* [28, 27], so that these extra symmetries along with the Poincaré group form a bigger symmetry group called the *conformal group*. That is the natural connection between CFTs and RG flows. We begin by defining the conformal group, along with its irreducible field representations.

Definition: A conformal transformation is an invertible map: $x \to x'$ such that

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x), \tag{3.1.1}$$

where $g_{\mu\nu}(x)$ is the metric tensor, which under coordinate changes transforms as

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x). \tag{3.1.2}$$

Some authors define the conformal transformations as angle preserving coordinate changes. The group property of conformal transformations can be explicitly seen by making two conformal transformations $x \to x'$ and $x' \to x''$, yielding

$$g''_{\mu\nu}(x'') = \Lambda(x') g_{\mu\nu}(x') = \underbrace{\Lambda(x)\Lambda'(x)}_{\Lambda''(x)} g_{\mu\nu}(x). \tag{3.1.3}$$

Straight from the definition we can already see that the Poincaré group is contained in the conformal group, as the special case of (3.1.1) for $\Lambda(x) = 1$.

Let's derive the constraints in an infinitesimal arbitrary coordinate change imposed by (3.1.1). Consider an infinitesimal transformation $x'^\mu(x) = x^\mu + \epsilon^\mu(x)$ and

its inverse $x^\mu(x) = x'^\mu - \epsilon^\mu(x') + \mathcal{O}(\epsilon^2)$. Under such transformation, the metric tensor transforms as (3.1.2). Evaluating the derivatives, we have

$$\frac{\partial x^\alpha}{\partial x'^\mu} = \delta^\alpha_\mu - \partial'_\mu \epsilon^\mu(x'), \tag{3.1.4}$$

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial}{\partial x^\mu} = (\delta^\alpha_\mu + \mathcal{O}(\epsilon))\partial_\alpha, \tag{3.1.5}$$

and also

$$\epsilon^\alpha(x') = \epsilon(x+\epsilon) = \epsilon^\alpha(x) + \mathcal{O}(\epsilon^2). \tag{3.1.6}$$

Up to $\mathcal{O}(\epsilon^2)$ one evaluates

$$\frac{\partial x^\alpha}{\partial x'^\mu} = \delta^\alpha_\mu - \partial_\mu \epsilon^\mu(x), \tag{3.1.7}$$

which yields

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x) - \partial_\nu \epsilon_\mu(x) - \partial_\mu \epsilon_\nu(x). \tag{3.1.8}$$

The infinitesimal version of the defining condition of the conformal group $\Lambda(x) = 1 + f(x)$, with $f(x)$ small, constraints the $\epsilon(x)$ vector field to obey

$$\partial_\nu \epsilon_\mu(x) + \partial_\mu \epsilon_\nu(x) = f(x) g_{\mu\nu}(x), \tag{3.1.9}$$

taking the trace yields

$$f(x) = \frac{d}{2}\partial_\rho \epsilon^\rho. \tag{3.1.10}$$

Applying ∂_ρ in (3.1.8) and permuting indexes, one obtains

$$\begin{aligned} g_{\mu\nu}\partial_\rho &= \partial_\rho\left(\partial_\nu\epsilon_\mu + \partial_\mu\epsilon_\nu\right) - f\partial_\rho g_{\mu\nu}, \\ g_{\mu\rho}\partial_\nu &= \partial_\nu\left(\partial_\rho\epsilon_\mu + \partial_\mu\epsilon_\rho\right) - f\partial_\nu g_{\mu\rho}, \\ g_{\nu\rho}\partial_\mu &= \partial_\mu\left(\partial_\nu\epsilon_\rho + \partial_\rho\epsilon_\nu\right) - f\partial_\mu g_{\nu\rho}. \end{aligned} \tag{3.1.11}$$

Specializing to the case of $g_{\mu\nu} = \eta_{\mu\nu}$, and adding the three equations of (3.1.11) we reach

$$g_{\mu\nu}\partial_\nu f + g_{\nu\rho}\partial_\mu f - g_{\mu\nu}\partial_\rho f = 2\partial_\mu\partial_\nu\epsilon\rho. \tag{3.1.12}$$

Contracting with $g^{\mu\nu}$ yields

$$\partial^2\epsilon_\rho = \frac{(2-d)}{2}\partial_\rho f. \tag{3.1.13}$$

The first interesting feature of the conformal group is the critical case of two dimensions, which will turn out to be the most interesting case. Consider for now the case $d \neq 2$, applying ∂_μ gives us

$$\partial_\mu(\partial^2\epsilon_\rho) = \frac{(2-d)}{2}\partial_\mu\partial_\rho f. \tag{3.1.14}$$

Applying ∂^2 on (3.1.9) yields

$$\partial^2\left(\partial_\nu\epsilon_\mu(x) + \partial_\mu\epsilon_\nu(x)\right) = \partial^2\left(f(x)g_{\mu\nu}(x)\right). \tag{3.1.15}$$

Plugging (3.1.14) in (3.1.15) , one reaches

$$\partial_\mu \partial_\nu f = \frac{1}{2-d} \partial^2 f(x) g_{\mu\nu}(x). \tag{3.1.16}$$

Taking the trace gives us

$$(d-1)\partial^2 f(x) = 0. \tag{3.1.17}$$

Equations (3.1.16) and (3.1.17) are the most useful to see the constraints of conformal invariance. In $d = 1$ it leads to no restrictions in $f(x)$, as it was expected, since there is no notion of angle to be preserved. The case $d = 2$ will be treated later, and for the case $d \geq 3$, we read off

$$\partial^2 f(x) = 0 \qquad \text{and} \qquad \partial_\mu \partial_\nu f(x) = 0. \tag{3.1.18}$$

The solutions are functions at most linear in x,

$$f(x) = A + B_\mu x^\mu. \tag{3.1.19}$$

$f(x)$ relates to the parameters ϵ through

$$f(x)\eta_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu, \tag{3.1.20}$$

which implies ϵ is at most quadratic in x,

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho, \tag{3.1.21}$$

where a_μ, $b_{\mu\nu}$, and $c_{\mu\nu\rho}$ parameterize the conformal transformations. $c_{\mu\nu\rho}$ is symmetric in the last two indexes. We can directly identify the parameter a_μ as a rigid translation, part of the Poincaré group. Setting $a_\mu = c_{\mu\nu\rho} = 0$, and imposing the constraint (3.1.20), we have

$$b_{\mu\nu} + b_{\nu\mu} = f(x)\eta_{\mu\nu}. \tag{3.1.22}$$

Again taking the trace we reach

$$f(x) = \frac{2}{d} b^\mu_\mu. \tag{3.1.23}$$

This means that the symmetric part of $b_{\mu\nu}$ is proportional to the flat metric $\eta_{\mu\nu}$, so we can decompose the tensor as

$$b_{\mu\nu} = \alpha \eta_{\mu\nu} + m_{\mu\nu}, \tag{3.1.24}$$

where $m_{\mu\nu} = -m_{\nu\mu}$ is the generator of *Lorentz transformations*, and the symmetric part $\alpha\eta_{\mu\nu}$ generates *scale transformations*.

Repeating the process for $a_\mu = b_{\mu\nu} = 0$ defines the *special conformal transformations*

$$x'^\mu = x^\mu + 2(x \cdot b)x^\mu - b^\mu x^2, \tag{3.1.25}$$

where $b_\mu = \frac{1}{d} c^\sigma_{\sigma\mu}$. Direct exponentiation of the infinitesimal form of the transformation yields:

$$\text{(Translations)} \qquad x'^\mu = x^\mu + a^\mu, \tag{3.1.26}$$

$$\text{(Rotations)} \qquad x'^\mu = M^\mu_\nu x^\nu, \tag{3.1.27}$$

$$\text{(Dilatations)} \qquad x'^\mu = \alpha x^\mu, \tag{3.1.28}$$

$$\text{(SCT)} \qquad x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2}. \tag{3.1.29}$$

3.2 Representations of the Conformal Algebra

In order to build representations for the conformal group, we need to derive its Lie algebra. For this purpose we define a field that is a scalar under the conformal transformations. In general, under a transformation on the coordinates $x^\mu \to x'^\mu$, the field transforms as

$$\Phi(x) \qquad \Longrightarrow \qquad \Phi'(x') = \mathcal{F}\{\Phi(x)\}. \tag{3.2.1}$$

A scalar field under the conformal group means a special case where $\mathcal{F}\{\Phi(x)\} = \Phi(x)$. This way we can define the generators of the conformal group by watching the pure orbital behaviour, and later on include the functional changes in the field. Recall that for an infinitesimal transformation, the fields change as

$$x'^\mu = x^\mu + w_a \frac{\delta x^\mu}{\delta w_a}, \tag{3.2.2}$$

$$\Phi'(x') = \Phi(x) + w_a \frac{\delta \mathcal{F}(x)}{\delta w_a}, \tag{3.2.3}$$

where $\{w_a\}$ is a set of infinitesimal parameters for the transformations. We define the variation on a field $\Phi(x)$ through a transformation parameterized by $\{w_a\}$, and the generator G_a, by

$$\delta_w \Phi(x) := \Phi'(x) - \Phi(x) := -i w_a G_a \Phi(x). \tag{3.2.4}$$

Using (3.2.2) and (3.2.3), we can write the generators as

$$iG_a\Phi = \frac{\delta x^\mu}{\delta w_a}\partial_\mu \Phi - \frac{\delta \mathcal{F}}{\delta w_a}. \tag{3.2.5}$$

Applying these definitions for a scalar field under the conformal group, we have for the translations and rotations part, the generators of the Poincaré group

$$\text{(Translations)} \qquad P_\mu = -i\partial_\mu, \tag{3.2.6}$$

$$\text{(Rotations)} \qquad L_{\mu\nu} = -i(x_\nu \partial_\mu - x_\mu \partial_\nu). \tag{3.2.7}$$

A dilatation is parameterized by w_g as $x'^\mu = (1 + w_g)x^\mu$, so we have

$$\begin{aligned} \Phi'(x') = &\quad \Phi(x), \\ \Phi'(x) = &\quad \Phi(x(1 - w_g)), \\ \Phi'(x) = &\quad (1 - w_g x^\mu \partial_\mu)\Phi(x). \end{aligned}$$

With this, we reach the generator in space representation, where we Taylor expanded the field Φ from the second to the third line up to linear order in w_g, yielding

$$D = -ix^\mu \partial_\mu. \tag{3.2.8}$$

Applying the exact same procedure to special conformal transformations, where the parameter is b^μ, one is able to complete the table of generators for the conformal

group, with only the coordinate dependent part. It reads

$$\text{(Translations)} \qquad P_\mu = -i\partial_\mu, \tag{3.2.9}$$

$$\text{(Rotations)} \qquad L_{\mu\nu} = -i(x_\nu\partial_\mu - x_\mu\partial_\nu), \tag{3.2.10}$$

$$\text{(Dilatations)} \qquad D = -ix^\mu\partial_\mu, \tag{3.2.11}$$

$$\text{(SCT)} \qquad K_\mu = -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu). \tag{3.2.12}$$

With the generators in hands, one easily calculates the Lie algebra of the conformal group. Direct calculation yields

$$[P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu), \tag{3.2.13}$$

$$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}), \tag{3.2.14}$$

$$[D, P_\mu] = iP_\mu, \tag{3.2.15}$$

$$[D, K_\mu] = -iK_\mu, \tag{3.2.16}$$

$$[K_\mu, P_\mu] = 2i(\eta_{\mu\nu}D - L_{\mu\nu}), \tag{3.2.17}$$

$$[K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu). \tag{3.2.18}$$

Notice how similar is the role of P_μ and K_μ generators, they have the same behaviour under rotations and act like creation and annihilation operators under the D operator. Furthermore, we see that the first two commutation relations define the Lie algebra of the Poincaré group, and the inclusion of the dilatation operator to this Lie algebra does not spoil closure of the algebra, meaning the generators P_μ, $L_{\mu\nu}$, and D define a subalgebra of the conformal group. But the same is not true for the inclusion of K_μ in the Poincaré group, since a dilatation operator appears in the commutation (3.2.17).

Defining the particular linear combination of the generators

$$J_{\mu\nu} = L_{\mu\nu}, \tag{3.2.19}$$

$$J_{-1\mu} = \frac{1}{2}(P_\mu - K_\nu), \tag{3.2.20}$$

$$J_{-10} = D, \tag{3.2.21}$$

$$J_{0\mu} = \frac{1}{2}(P_\mu + K_\nu), \tag{3.2.22}$$

we see that the new generators J_{ab} with $a, b \in \{-1, 0, 1, \cdots, d\}$ satisfy the $SO(d+1, d)$ algebra, proving an isomorphism between the conformal group and $SO(d+1, d)$,

$$\text{Conformal Group in d-dimensions} \quad \simeq \quad SO(d+1, 1). \tag{3.2.23}$$

With the algebra in hands, it is possible to build the irreducible representations of the conformal group for *quasi-primary fields*.

Definition: A *quasi-primary field* $\Phi(x)$ is a field that under a conformal transformation $x \to x'$, transforms as

$$\Phi'^A(x') = \left|\frac{\partial x'}{\partial x}\right|^{\frac{-\Delta}{d}} \left[\exp(\frac{-i}{2}w^{\mu\nu}S_{\mu\nu})\right]^A_B \Phi^B(x). \tag{3.2.24}$$

The piece $|\frac{\partial x'}{\partial x}|$ is the Jacobian of the transformation, Δ is called the scaling dimension of the field, and $[S_{\mu\nu}]^A_B$ is a representation of the rotation operators $L_{\mu\nu}$ (a set of matrices satisfying the commutation (3.2.14)) regarding the field transformation under the Lorentz subgroup. For example, if the field is a Lorentz scalar, it has no indexes A, B, and $S_{\mu\nu} = 1$ is the trivial representation. In this case the scalar quasi-primary transforms as

$$\Phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta}{d}} \Phi(x). \tag{3.2.25}$$

For a spinor representation, the matrices are proportional to the anti-commutator of the Dirac matrices, $[S_{\mu\nu}]^A_B \sim [\{\gamma_\mu, \gamma_\nu\}]^A_B$.

The generators P_μ, K_μ, $L_{\mu\nu}$, and D can be expressed in terms of the scaling dimension Δ and the matrices $S_{\mu\nu}$. In order to do this, one has to come back to the definition of a generator, and apply infinitesimal conformal transformations for the definition of a quasi-primary field.

Starting with the Poincaré subgroup, which is the subset of conformal trasnformations that generates no Jacobian, $|\frac{\partial x'}{\partial x}| = 1$, one sees that the generator of translations is the same as before, $P_\mu = -i\partial_\mu$. To realize the representation of the rotation generators, it is useful to define an infinitesimal rotation

$$x'^\mu = x^\mu + w^\mu_\nu x^\nu \Longrightarrow \Phi'^A(x') = \left(1 - \frac{i}{2} w_{\rho\nu} S^{\rho\nu}\right)^A_B \Phi^B(x), \tag{3.2.26}$$

where w^μ_ν is an anti-symmetric infinitesimal parameter. Using the inverse transformation $x' \to x$, we write

$$\Phi'^A(x) = \left(1 - \frac{i}{2} w_{\rho\nu} S^{\rho\nu}\right)^A_B \Phi^B(x^\mu - w^\mu_\nu x^\nu), \tag{3.2.27}$$

Taylor expanding $\Phi^B(x^\mu - w^\mu_\nu x^\nu)$ and collecting the linear terms in w^μ_ν, one reaches

$$\Phi'^A(x) = \Phi^A(x) - \frac{1}{2} w_{\mu\nu}(x^\nu \partial^\mu - x^\mu \partial^\nu)\Phi^A(x) - \frac{i}{2} w_{\rho\nu}[S^{\rho\nu}]^A_B \Phi^B. \tag{3.2.28}$$

Finally, using the definition of a generator, one writes

$$[L^{\mu\nu}]^A_B = -i(x^\nu \partial^\mu - x^\mu \partial^\nu)\delta^A_B + [S^{\mu\nu}]^A_B, \tag{3.2.29}$$

where the diagonal part in the indexes A, B is called *orbital part*, because it comes from the coordinates dependence of the transformations, while the non-diagonal part is called *spin part* of the generator.

The same strategy is applied to a dilatation through

$$x'^\mu = x^\mu + \alpha x^\nu \Longrightarrow \Phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{\frac{-\Delta}{d}} \Phi(x). \tag{3.2.30}$$

No mixing of indexes occur, because the rotation parameter is $w^\mu_\nu = 0$. This is a pure dilatation, parameterized by an infinitesimal α. Keeping the Jacobian linear in α, we have $|\frac{\partial x'}{\partial x}|^{\frac{-\Delta}{d}} = 1 - \alpha\Delta$, and the equation reads

$$\Phi'(x) = (1 - \alpha\Delta)(1 - \alpha x^\mu \partial_\mu)\Phi(x), \tag{3.2.31}$$

which yields the dilatation generator

$$D = -i(\Delta + x^\mu \partial_\mu). \tag{3.2.32}$$

The only missing generator is the one of an SCT, which turns out to be the trickier to get. b^μ is the infinitesimal parameter controlling the SCT. Notice that hidden inside a SCT there actually are a dilatation and a rotation,

$$\frac{\partial x'_\mu}{\partial x^\nu} = \eta_{\mu\nu}(1 + \underbrace{2(x \cdot b)}_{rescaling}) + \underbrace{2(b_\nu x_\mu - b_\mu x_\nu)}_{antisymmetric}, \tag{3.2.33}$$

where one identifies a rotation parameterized by $w_{\mu\nu}$ and dilatation α through

$$w_{\mu\nu} = 2(b_\nu x_\mu - b_\mu x_\nu) \qquad \text{and} \qquad \alpha = 2(x \cdot b). \tag{3.2.34}$$

In the definition of quasi-primary, it yields

$$\Phi'^A(x) = \underbrace{(1 - 2\Delta(b \cdot x))}_{Jacobian} \underbrace{[1 - i(b_\nu x_\mu - b_\mu x_\nu) S^{\mu\nu}]^A_B}_{Spin} \underbrace{(-2(b \cdot x)x^\mu \partial_\mu + b^\mu x^2 \partial_\mu + 1)}_{Orbital} \Phi^B(x). \tag{3.2.35}$$

Collecting linear terms in b^μ defines the generator of SCTs

$$[K^\mu]^A_B = i(x^2 \partial^\mu - 2x^\mu x^\nu \partial_\nu - 2\Delta x^\mu)\delta^A_B - 2x_\nu [S^{\mu\nu}]^A_B. \tag{3.2.36}$$

The expressions (3.2.9), (3.2.29), (3.2.32), (3.2.36) define the generators of the conformal group in a primary field representation.

3.3 N-Point Functions

The set of N-point functions of quasi-primary operators transforming under the conformal group is heavily constrained for the cases $N = 1, 2$ and 3. The information contained in them is completely encoded in the *dilatation* and *spin* quantum numbers (Δ and s), and some numerical coefficients called structure constants. From now on, only spinless fields are considered for the sake of simplicity.

Consider a collective label Φ denoting the set of independent fields of a given theory. We are interested in the correlators of primary fields $\phi_i(x_i)$, which are defined as insertions in the path integral

$$\langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle \equiv \frac{1}{\mathcal{Z}} \int [\mathcal{D}\Phi] \phi_1(x_1) \cdots \phi_n(x_n) e^{-S[\Phi]}. \tag{3.3.1}$$

A CFT may be defined without the reference of an action functional or a path integral. In this case, one denotes as fields the basic local operators that transform among themselves under conformal transformations.[1] Denote by Δ_i the scaling dimension of the field ϕ_i. The 2-point function takes the form

$$\langle \phi_1(x_1)\phi_2(x_2) \rangle = \frac{1}{\mathcal{Z}} \int [\mathcal{D}\Phi] \phi_1(x_1)\phi_2(x_2) e^{-S[\Phi]}. \tag{3.3.2}$$

[1] This is a direct statement from [16].

Under the conformal map $x \to x'$, the action functional is invariant if the theory is a CFT : $S[\Phi] \to S[\Phi]$, and the integration measure is also assumed to be invariant $[\mathcal{D}\Phi] \to [\mathcal{D}\Phi]$, thus making the generating functional $\mathcal{Z}$ an invariant quantity. The quasi-primary fields ϕ_i transform as (3.2.25), so the 2-point function obeys

$$\langle \phi_1'(x_1')\phi_2'(x_2')\rangle = \Big|\frac{\partial x'}{\partial x}\Big|_{x'=x_1}^{-\frac{\Delta_1}{d}} \Big|\frac{\partial x'}{\partial x}\Big|_{x'=x_2}^{-\frac{\Delta_2}{d}} \langle \phi_1(x_1)\phi_2(x_2)\rangle. \tag{3.3.3}$$

Further constraints arise from the Poincaré subgroup, which in this case consists of $SO(d)$ symmetry plus translation invariance (because we are working in Euclidean signature), so we may write

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = f(|x_1 - x_2|). \tag{3.3.4}$$

Defining the dilatation $x' = \lambda x$, and plugging in (3.3.3) and (3.3.4), yields

$$\begin{aligned} & f(\lambda|x_1 - x_2|) = \lambda^{-(\Delta_1+\Delta_2)} f(|x_1 - x_2|) \\ \implies & \frac{f(\lambda|x_1 - x_2|)}{f(|x_1 - x_2|)} = \lambda^{(\Delta_1+\Delta_2)}. \end{aligned} \tag{3.3.5}$$

Therefore the 2-point function must satisfy

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}, \tag{3.3.6}$$

where C_{12} is a structure constant. If a given relativistic theory is scale invariant, then it follows that the 2-point functions must satisfy (3.3.6).

There is still invariance under special conformal transformations to be implemented. Consider the SCT map

$$x'^{\mu} = \frac{x^{\mu} - b^{\mu}x^2}{1 - 2b \cdot x + b^2x^2}. \tag{3.3.7}$$

The modulus of the difference transforms as

$$|x_1' - x_2'| = \frac{|x_1 - x_2|}{(1 - 2b \cdot x_1 + b^2x_1^2)^{\frac{1}{2}}(1 - 2b \cdot x_2 + b^2x_2^2)^{\frac{1}{2}}}. \tag{3.3.8}$$

The Jacobian for a SCT is

$$\Big|\frac{\partial x'}{\partial x}\Big| = \frac{1}{(1 - 2b \cdot x + b^2x^2)} \frac{1}{(1 - 2b \cdot x + b^2x^2)}. \tag{3.3.9}$$

Plugging in (3.3.3) yields

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{1}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}} \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} (\gamma_1\gamma_2)^{\frac{\Delta_1+\Delta_2}{2}}, \tag{3.3.10}$$

where $\gamma_i = (1 - 2b \cdot x_i + b^2x_i^2)$. Using (3.3.6) in LHS, we get

$$\frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} = \frac{1}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}} \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} (\gamma_1\gamma_2,)^{\frac{\Delta_1+\Delta_2}{2}}, \tag{3.3.11}$$

so we conclude that if $\Delta_1 = \Delta_2 \equiv \Delta$,

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{2\Delta}}, \tag{3.3.12}$$

and it vanishes otherwise.

Also important in general, but irrelevant for the present work, is the constraint on 3-point functions. Applying the same transformations, a more clumsy calculation yields

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{13}^{\Delta_1+\Delta_3-\Delta_2} x_{23}^{\Delta_2+\Delta_3-\Delta_1}}, \tag{3.3.13}$$

where $x_{ij} \equiv |x_i - x_j|$ and C_{123} is a structure constant.

3.4 Two-Dimensional CFTs

3.4.1 Basic Definitions

During the derivation of the conformal algebra, one learns that the case of two dimensions requires special attention as it can be seen from (3.1.14). This is because in such critical dimension, the power of conformal symmetry gets enhanced, the algebra has infinite generators, and the conformal transformations correspond to the set of all holomorphic maps in the extended complex plane.[2]

By demanding the map $w^\mu(z)$ to be a conformal transformation, we have

$$\tilde{g}^{\mu\nu} = \underbrace{\left(\frac{\partial w^\mu}{\partial z^\alpha}\right)\left(\frac{\partial w^\nu}{\partial z^\beta}\right)}_{\Omega^2(x)} g^{\alpha\beta}. \tag{3.4.1}$$

The map must satisfy

$$\left(\frac{\partial w^1}{\partial z^0}\right) = \left(\frac{\partial w^0}{\partial z^1}\right) \quad \text{and} \quad \left(\frac{\partial w^0}{\partial z^0}\right) = -\left(\frac{\partial w^1}{\partial z^1}\right), \tag{3.4.2}$$

or,

$$\left(\frac{\partial w^1}{\partial z^0}\right) = -\left(\frac{\partial w^0}{\partial z^1}\right) \quad \text{and} \quad \left(\frac{\partial w^0}{\partial z^0}\right) = \left(\frac{\partial w^1}{\partial z^1}\right). \tag{3.4.3}$$

The transformation (3.4.2) defines the *holomorphic maps*, and the transformation (3.4.3) defines the *anti-holomorphic maps*.[3] Originally, the CFT is defined in the Euclidean space with real coordinates σ^μ, but in order to make use of the power of complex analysis we may promote the real variables to complex values variables. To do so we define the coordinates

$$z = \sigma^0 + i\sigma^1 \quad \text{and} \quad \bar{z} = \sigma^0 - i\sigma^1. \tag{3.4.4}$$

[2]The extended complex plane, denoted $\hat{\mathbb{C}}$ is defined as the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

[3]The maps are defined for complex valued variables w^μ and z^μ. Equations (3.4.2) and (3.4.3) define Cauchy Riemann equations for analytic functions in $\mathbb{C}$.

An important detail is that we treat z and $\bar{z}$ as two *independent* complex variables, even though we have defined one as the complex conjugate of the other. This means in practice that we have doubled the number of original coordinates from two to four. Bear in mind that one needs to restrict $\bar{z} = z^*$ in order to restore physical meaning to the system.

In this new coordinate system, the derivatives are given by

$$\partial \equiv \partial_z = \frac{1}{2}(\partial_0 - i\partial_1) \quad \text{, and} \quad \bar{\partial} \equiv \partial_{\bar{z}} = \frac{1}{2}(\partial_0 + i\partial_1). \tag{3.4.5}$$

The metric reads $g_{zz} = g_{\bar{z}\bar{z}} = 0$ and $g_{\bar{z}z} = g_{z\bar{z}} = \frac{1}{2}$. The defining conditions for a holomorphic map $w(z, \bar{z})$ can be expressed in terms of derivatives as

$$\bar{\partial} w(z, \bar{z}) = 0, \tag{3.4.6}$$

and an anti-holomorphic as

$$\partial \bar{w}(z, \bar{z}) = 0. \tag{3.4.7}$$

These equations tell us that the conformal transformations are the ones in which we do not mix z and $\bar{z}$: $(z, \bar{z}) \to (w(z), \bar{w}(\bar{z}))$. In the original definition of conformal transformations, we have required the maps to be invertible. However, not all holomorphic maps are invertible, so one needs to make a distinction between the maps that are invertible[4] and the ones that are not. The first ones are called the *global conformal transformations*, and can be written as

$$f(z) = \frac{az + b}{cz + d} \quad \text{with} \quad ad - bc = 1. \tag{3.4.8}$$

One can represent the map (3.4.8) with a 2×2 matrix

$$f(z) \to A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{3.4.9}$$

such that $\det(A) = 1$. This subspace of 2×2 matrices form the group of special linear transformations $SL(2, \mathbb{C})$, which is isomorphic to $SO(1, 3)$. The conformal group in two dimensions is the set of invertible maps as defined before, but this time there are extra transformations which are not global but still satisfy the requisite of being conformal. These extra transformations compose the *local conformal transformations*.

The generators may be obtained by considering an arbitrary infinitesimal conformal transformation $(z, \bar{z}) \to (z + \epsilon(z), \bar{z} + \bar{\epsilon}(\bar{z}))$, with

$$\epsilon(z) = \sum_{n \in \mathbb{Z}} c_n z^{n+1} \quad \text{and} \quad \bar{\epsilon}(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{c}_n \bar{z}^{n+1}. \tag{3.4.10}$$

[4] In general, authors work in the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. A holomorphic function in $\hat{\mathbb{C}}$ is not necessarily holomorphic in $\mathbb{C}$.

The variation on a scalar field is

$$\begin{aligned} \phi'(z', \bar{z}') &= \phi(z, z') \qquad (3.4.11) \\ \phi'(z + \epsilon(z), \bar{z} + \bar{\epsilon}(\bar{z})) &= \phi(z, z') \\ \implies \delta\phi &= -\epsilon(z)\partial\phi(z, \bar{z}) - \bar{\epsilon}(\bar{z})\bar{\partial}\phi(z, \bar{z}) \\ &= -\left(\sum_{n\in\mathbb{Z}} c_n z^{n+1}\partial\right)\phi(z, \bar{z}) - \left(\sum_{n\in\mathbb{Z}} \bar{c}_n \bar{z}^{n+1}\bar{\partial}\right)\phi(z, \bar{z}) \\ &\equiv \left(\sum_{n\in\mathbb{Z}} c_n l_n + \bar{c}_n \bar{l}_n\right)\phi(z, \bar{z}), \qquad (3.4.12) \end{aligned}$$

where we have defined the generators

$$l_n \equiv -z^{n+1}\partial \quad \text{and} \quad \bar{l}_n \equiv -\bar{z}^{n+1}\bar{\partial}. \qquad (3.4.13)$$

The algebra reads

$$[l_m, l_n] = (m - n)l_{m+n}, \qquad (3.4.14)$$
$$[\bar{l}_m, \bar{l}_n] = (m - n)\bar{l}_{m+n}, \qquad (3.4.15)$$
$$[l_m, \bar{l}_n] = 0. \qquad (3.4.16)$$

The algebra of generators decouples in two exact copies (3.4.14) and (3.4.15). There is an infinite number of generators ($n \in \mathbb{Z}$). This algebra contains the global conformal transformations as well as the local ones. The $SL(2, \mathbb{C})$ subgroup corresponding to the global part is generated by l_0, l_{-1} and l_1, while all others generate the non-invertible conformal tranformations.

We define the conformal dimensions as

$$h \equiv \frac{1}{2}(\Delta + s) \quad \text{and} \quad \bar{h} \equiv \frac{1}{2}(\Delta - s). \qquad (3.4.17)$$

In two-dimensional CFTs a more convenient definition of quasi-primary field is given by:

Definition: A quasi-primary field $\phi(z, \bar{z})$ is a field which under the coordinate change $(z, \bar{z}) \to (w(z), \bar{w}(\bar{z}))$ transforms as

$$\phi'(w, \bar{w}) = \left(\frac{dw}{dz}\right)^{-h}\left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}}\phi(z, \bar{z}), \qquad (3.4.18)$$

where h and $\bar{h}$ define the conformal dimensions of the field ϕ. Definition (3.4.18) defines a quasi-primary field as an object that behaves as a $(0, h + \bar{h})$-rank tensor with "h" z-indexes and "$\bar{h}$" $\bar{z}$-indexes under conformal transformations. Recall the generic tensor transformation rule

$$A'_{\mu\cdots\nu}(x') = \frac{\partial x^\mu}{\partial x'^\alpha}\cdots\frac{\partial x^\nu}{\partial x'^\beta}A_{\alpha\cdots\beta}(x), \qquad (3.4.19)$$

with $x^\mu = (z, \bar{z})$. As conformal transformations are maps that do not mix z with $\bar{z}$, we take the new coordinates $x'^\nu = (w(z), \bar{w}(\bar{z}))$. This means that the transformation is always diagonal

$$\frac{\partial x^\mu}{\partial x'^\nu} \sim \delta^\mu_\nu. \qquad (3.4.20)$$

For example the (w, w) component of a rank (0,2) transforms as

$$A'_{ww}(w, \bar{w}) = \left(\frac{\partial w}{\partial z}\right)^{-2} A_{zz}(z, \bar{z}), \tag{3.4.21}$$

thus defining a quasi-primary field of weight $h = 2$. It is straightforward to see that a quasi-primary of weight h and $\bar{h}$ transforms as

$$A'_{\underbrace{w \cdots w}_{h}\underbrace{\bar{w} \cdots \bar{w}}_{\bar{h}}}(w, \bar{w}) = \left(\frac{\partial w}{\partial z}\right)^{-h} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{-\bar{h}} A'_{\underbrace{z \cdots z}_{h}\underbrace{\bar{z} \cdots \bar{z}}_{\bar{h}}}(z, \bar{z}). \tag{3.4.22}$$

If (3.4.18) holds true for *all* conformal transformations, the field is called a *primary field*. However if it holds only for global conformal transformations, the field is called a *secondary field*. All primaries are quasi-primaries, but when a quasi-primary is not a primary, it is called a secondary field.

3.4.2 Ward Identities

We have already covered the constraints of conformal symmetry in N-point functions, but there are extra constraints coming from the Ward identities associated with the conformal group. Such identities impose powerful constraints in the correlation functions involving the insertion of components of the energy-momentum tensor. By analyzing the correlators, it is possible to determine the short range interaction of operator insertions by introducing the Operator Product Expansion (OPE).

Consider the N-point function of arbitrary primary fields Φ,

$$\langle \Phi(x_1) \cdots \Phi(x_n) \rangle = \frac{1}{\mathcal{Z}} \int [\mathcal{D}\Phi] \Phi(x_1) \cdots \Phi(x_n) e^{-S[\Phi]}. \tag{3.4.23}$$

An infinitesimal variation $x^\mu \to x^\mu + w_a \frac{\partial x^\mu}{\partial w_a}$ leads to a variation on the fields $\Phi(x) \to \Phi'(x') = \mathcal{F}(\Phi(x))$. The fields Φ in (3.4.23) can be regarded as integration variables in respect to the measure $\mathcal{D}\Phi$. Performing the change of variables $\Phi(x) \to \Phi'(x)$, we can write

$$\langle \Phi(x_1) \cdots \Phi(x_n) \rangle = \frac{1}{\mathcal{Z}} \int [\mathcal{D}\Phi'] \Phi'(x_1) \cdots \Phi'(x_n) e^{-S[\Phi']}. \tag{3.4.24}$$

One may transform the fields in the RHS of (3.4.24) back to Φ. We assume no anomalies are induced by the change of variables, so that $[\mathcal{D}\Phi] = [\mathcal{D}\Phi']$. Defining $S'[\Phi] \equiv S[\Phi']$, we write

$$\begin{aligned} &\langle \Phi(x_1) \cdots \Phi(x_n) \rangle = \\ &\frac{1}{\mathcal{Z}} \int [\mathcal{D}\Phi] \left\{ \Phi(x_1) \cdots \Phi(x_n) - i \left(\sum_{i=1}^{n} w_a(x_i) \Phi(x_1) \cdots G_a^{(i)} \Phi(x_i) \cdots \Phi(x_n) \right) \right\} e^{-S'[\Phi]}, \end{aligned} \tag{3.4.25}$$

where $G_a^{(i)}$ is the generator of the w_a transformation acting on the i^{th} field. As w_a is an infinitesimal parameter, we may keep only the linear order. The transformed action $S'[\Phi]$ will also contain orders of w_a.

Consider that the action is invariant under a set of transformations parameterized by rigid parameters w_a.[5] We now consider a local version of the parameters $w_a \to w_a(x)$, which will no longer correspond to a symmetry of the system. The action reads

$$S'[\Phi] \equiv S[\Phi'] = \int d^D x \mathcal{L}(\Phi'(x), \partial_\mu \Phi'(x)) \tag{3.4.26}$$

$$= \int d^D x' \mathcal{L}(\mathcal{F}[\Phi(x)], \partial'_\mu \mathcal{F}[\Phi(x)]) \tag{3.4.27}$$

$$= \int d^D x \left|\frac{\partial x'^\mu}{\partial x^\nu}\right| \mathcal{L}(\mathcal{F}[\Phi(x)], \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu \mathcal{F}[\Phi(x)]). \tag{3.4.28}$$

From the first to the second line, a relabel of integration variable $x \to x'$ has been made, and from the second to the third, a change of variable $x' \to x$. For an infinitesimal transformation one may approximate the Jacobian of (3.1.4) by

$$\left|\frac{\partial x'^\mu}{\partial x^\nu}\right| = 1 + \partial_\mu \left(w_a \frac{\delta x^\mu}{\delta w_a} \right). \tag{3.4.29}$$

Using (3.2.2), (3.2.3) and (3.1.4), we can rewrite (3.4.28) as

$$\begin{aligned} S'[\Phi] = \int d^D x \left(1 + \partial_\mu \left(w_a \frac{\delta x^\mu}{\delta w_a} \right)\right) \mathcal{L} \Bigg\{ \Phi + w_a \frac{\delta \mathcal{F}}{\delta w_a}, \left[\delta^\nu_\mu - \partial_\mu \left(w_a \frac{\delta x^\nu}{\delta w_a} \right)\right] \\ \times \left(\partial_\nu \Phi + \partial_\nu \left(w_a \frac{\delta \mathcal{F}}{\delta w_a} \right)\right) \Bigg\}. \end{aligned} \tag{3.4.30}$$

Expanding in linear order of w_a we have

$$\begin{aligned} S'[\Phi] = \int d^D x \Bigg\{ \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \Phi} \left(w_a \frac{\delta \mathcal{F}}{\delta w_a} \right) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\mu \left(w_a \frac{\delta \mathcal{F}}{\delta w_a} \right) \\ - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\mu \left(w_a \frac{\delta x^\nu}{\delta w_a} \right) \partial_\nu \Phi + \partial_\mu \left(w_a \frac{\delta x^\mu}{\delta w_a} \right) \mathcal{L} \Bigg\}. \end{aligned} \tag{3.4.31}$$

The arguments of $\mathcal{L}(\Phi, \partial_\mu \Phi)$ are dropped for simplicity. The first term $\mathcal{L}$ gives $S[\Phi]$ again, the third term may be expanded as

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\mu \left(w_a \frac{\delta \mathcal{F}}{\delta w_a} \right) = \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \frac{\delta \mathcal{F}}{\delta w_a} \partial_\mu (w_a)}_{I} + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\mu \left(\frac{\delta \mathcal{F}}{\delta w_a} \right) w_a}_{II}. \tag{3.4.32}$$

Proceeding in the same way for the last two terms, and collecting all terms with $\partial_\mu(w_a)$ dependence, one writes

$$\delta S = - \int d^D x j^\mu_a \partial_\mu (w_a) + \mathcal{S}, \tag{3.4.33}$$

[5] This will become necessary below.

with

$$j_a^\mu = \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\nu \Phi - \delta^\mu_\nu \mathcal{L} \right\} \frac{\delta x^\nu}{\delta w_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \frac{\delta \mathcal{F}}{\delta w_a}, \tag{3.4.34}$$

and

$$\begin{aligned} \mathcal{S} = \int d^D x \Bigg\{ & \underbrace{- \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\nu \partial_\mu \left(\frac{\delta x^\nu}{\delta w_a} \right) w_a}_{I} + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \left(w_a \partial_\mu \frac{\delta \mathcal{F}}{\delta w_a} \right)}_{II} \\ & \underbrace{+ w_a \partial_\mu \left(\frac{\delta x^\mu}{\delta w_a} \right) \mathcal{L}}_{III} + w_a \frac{\partial \mathcal{L}}{\partial \Phi} \frac{\delta \mathcal{F}}{\delta w_a} \Bigg\}. \end{aligned} \tag{3.4.35}$$

Integrating I, II, and III by parts yields

$$\mathcal{S} = - \int d^D x w_a(x) \left\{ \frac{\delta \mathcal{F}}{\delta w_a} \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \mathcal{L})} \right)}_{0 \quad \text{by EOM}} + \frac{\delta x^\nu}{\delta w_a} \partial_\mu \underbrace{\left(\delta^\mu_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right)}_{T^\mu_\nu} \right\} = 0, \tag{3.4.36}$$

where we have used the conservation of the $T_{\mu\nu}$ and the equations of motion, leading to the result

$$S'[\Phi] = S[\Phi] - \int d^D x j_a^\mu \partial_\mu w_a. \tag{3.4.37}$$

Integrating by parts yields

$$S'[\Phi] = S[\Phi] + \int d^D x \partial_\mu j_a^\mu w_a, \tag{3.4.38}$$

back in (3.4.25), it gives us

$$\begin{aligned} & \langle \Phi(x_1) \cdots \Phi(x_n) \rangle = \\ & \frac{1}{\mathcal{Z}} \int [\mathcal{D}\Phi] \left(\Phi(x_1) \cdots \Phi(x_n) - i \left(\sum_{i=1}^n w_a(x_i) \Phi(x_1) \cdots G_a^{(i)} \Phi(x_i) \cdots \Phi(x_n) \right) \right) \\ & \times e^{-S[\Phi] - \int d^D x w_a(x) \partial_\mu j^\mu}. \end{aligned} \tag{3.4.39}$$

Expanding the exponential, and keeping linear order in w_a leads to

$$\begin{aligned} \langle \Phi(x_1) \cdots \Phi(x_n) \rangle = \quad & \langle \Phi(x_1) \cdots \Phi(x_n) \rangle \\ & - i \int d^D x \sum_{i=1}^n w_a(x) \delta(x - x_i) \langle \Phi(x_1) \cdots G_a^{(i)} \Phi(x_i) \cdots \Phi(x_n) \rangle \\ & - \int d^D x w_a(x) \partial_\mu \langle j_a^\mu(x) \Phi(x_1) \cdots \Phi(x_n) \rangle. \end{aligned} \tag{3.4.40}$$

Finally, the general Ward identity reads

$$\partial_\mu \langle j_a^\mu(x)\Phi(x_1)\cdots\Phi(x_n)\rangle = -i\sum_{i=1}^{n}\delta(x-x_i)\langle\Phi(x_1)\cdots G_a^{(i)}\Phi(x_i)\cdots\Phi(x_n)\rangle. \quad (3.4.41)$$

It is easy to check the constraints imposed in $T_{\mu\nu}$, because the conserved current j_a^μ is usually written in terms of a parameter contracted with $T_{\mu\nu}$. Substituting the corresponding generator G_a on RHS of (3.4.41) yields the corresponding identity.

Conformal symmetry is composed of invariance under translation, rotation, scale, special conformal transformations. One expects to collect four independent Ward identities, nonetheless, a special conformal transformation can be seen as a composition of rotations and scale transformations, leaving only three independent identities, given by

$$\partial_\mu \langle T_\nu^\mu(x)\Phi(x_1)\cdots\Phi(x_n)\rangle = -\sum_{i=1}^{n}\delta(x-x_i)\frac{\partial}{\partial x_i^\nu}\langle\Phi(x_1)\cdots\Phi(x_n)\rangle, \quad (3.4.42)$$

$$\epsilon_{\mu\nu}\langle T^{\mu\nu}(x)\Phi(x_1)\cdots\Phi(x_n)\rangle = -i\sum_{i=1}^{n}\delta(x-x_i)s_i\langle\Phi(x_1)\cdots\Phi(x_n)\rangle, \quad (3.4.43)$$

$$\langle T_\mu^\mu(x)\Phi(x_1)\cdots\Phi(x_n)\rangle = -\sum_{i=1}^{n}\delta(x-x_i)\Delta_i\langle\Phi(x_1)\cdots\Phi(x_n)\rangle, \quad (3.4.44)$$

where the $\epsilon_{\mu\nu}$ is the totally anti-symmetric tensor, s_i is the spin of the field Φ_i, and Δ_i its scaling dimension.

In order to have the identities in the complex coordinates $(z,\bar{z})$, one needs to chose a representation of the δ distribution and the anti-symmetric tensor $\epsilon_{\mu\nu}$. The whole point of working in complex coordinates is to transform the integrals over d^2x in contour integrals of analytic functions, and use the residue theorem to easily solve them.

Stoke's theorem in real coordinates (σ^0,σ^1) reads

$$\int_R d^2\sigma\partial_\mu J^\mu = \oint_{\partial R}\left(J_0 d\sigma^1 - J_1 d\sigma^0\right), \quad (3.4.45)$$

where the measure is $d^2\sigma = d\sigma^0 d\sigma^1$, and the integration on RHS is counter-clockwise oriented. Transforming (3.4.45) to complex coordinates yields

$$\int_R d^2\sigma\partial_\mu J^\mu = \frac{i}{2}\oint_{\partial R}\left(J^z d\bar{z} - J^{\bar{z}} dz\right). \quad (3.4.46)$$

The delta distribution can be defined as

$$\delta(x) = \frac{1}{\pi}\partial\left(\frac{1}{\bar{z}}\right) = \frac{1}{\pi}\bar{\partial}\left(\frac{1}{z}\right). \quad (3.4.47)$$

One can chose the convenient representation depending on the calculation.

The translation Ward identity (3.4.42) splits in two, one for each value of "ν". We chose the representation of δ with a ∂ derivative when it multiplies a anti-holomorphic function $f(\bar{z})$, so that we can commute $[\partial, f(\bar{z})] = 0$, and the contrary

for a holomorphic function. (3.4.43) becomes

$$2\partial_{\bar{z}}\langle T_{zz}\Phi(w_1)\cdots\Phi(w_n)\rangle + 2\partial_z\langle T_{\bar{z}z}\Phi(w_1)\cdots\Phi(w_n)\rangle = \\ -\frac{1}{\pi}\sum_i \partial_{\bar{z}}\left(\frac{1}{z-w_i}\right)\partial_{w_i}\langle\Phi(w_1)\cdots\Phi(w_n)\rangle, \tag{3.4.48}$$

and

$$2\partial_{\bar{z}}\langle T_{z\bar{z}}\Phi(x_1)\cdots\Phi(x_n)\rangle + 2\partial_z\langle T_{\bar{z}\bar{z}}\Phi(x_1)\cdots\Phi(x_n)\rangle = \\ -\frac{1}{\pi}\sum_i \partial_{z}\left(\frac{1}{\bar{z}-\bar{w}_i}\right)\partial_{\bar{w}_i}\langle\Phi(w_1)\cdots\Phi(w_n)\rangle. \tag{3.4.49}$$

The rotation Ward identity (3.4.43) becomes [6]

$$-2\langle T_{z\bar{z}}\Phi(w_1)\cdots\Phi(w_n)\rangle + 2\langle T_{\bar{z}z}\Phi(w_1)\cdots\Phi(w_n)\rangle = -\sum_i s_i\delta(x-x_i)\Phi(x_1)\cdots\Phi(x_n). \tag{3.4.50}$$

Analogously, (3.4.44) becomes

$$2\langle T_{z\bar{z}}\Phi(w_1)\cdots\Phi(w_n)\rangle + 2\langle T_{\bar{z}z}\Phi(w_1)\cdots\Phi(w_n)\rangle = -\sum_i \Delta_i\delta(x-x_i)\Phi(x_1)\cdots\Phi(x_n). \tag{3.4.51}$$

By adding (3.4.50) with (3.4.51), we reach

$$2\pi\langle T_{\bar{z}z}\Phi(w_1)\cdots\Phi(w_n)\rangle = -\sum \partial_{\bar{z}}\left(\frac{1}{z-w_i}\right)h_i\langle\Phi(w_1)\cdots\Phi(w_n)\rangle. \tag{3.4.52}$$

Subtracting (3.4.50) from (3.4.51) yields

$$2\pi\langle T_{z\bar{z}}\Phi(w_1)\cdots\Phi(w_n)\rangle = -\sum \partial_{z}\left(\frac{1}{\bar{z}-\bar{w}_i}\right)\bar{h}_i\langle\Phi(w_1)\cdots\Phi(w_n)\rangle. \tag{3.4.53}$$

After substituting (3.4.52) in (3.4.48), and defining $T(z)\equiv -2\pi T_{zz}$, one reaches

$$\partial_{\bar{z}}\left\{\langle T\Phi(w_1)\cdots\Phi(w_n)\rangle - \sum_i\left(\frac{h_i}{(z-w_i)^2}+\frac{\partial_{w_i}}{(z-w_i)}\right)\langle\Phi(w_1)\cdots\Phi(w_n)\rangle\right\} = 0. \tag{3.4.54}$$

This means that the expression in between the brackets is holomorphic, which in turn implies that the most divergent piece of a correlator between $T(z)$ and primary fields is of order z^{-2},

$$\langle T\Phi(w_1)\cdots\Phi(w_n)\rangle = \sum_i\left(\frac{h_i}{(z-w_i)^2}+\frac{\partial_{w_i}}{(z-w_i)}\right)\langle\Phi(w_1)\cdots\Phi(w_n)\rangle + \text{reg in } z. \tag{3.4.55}$$

The regular terms are functions of z which contain no singularities as $z\to w_i$. The same analysis for the anti-holomorphic part yields

$$\langle \bar{T}\Phi(w_1)\cdots\Phi(w_n)\rangle = \sum_i\left(\frac{\bar{h}_i}{(\bar{z}-\bar{w}_i)^2}+\frac{\partial_{\bar{w}_i}}{(\bar{z}-\bar{w}_i)}\right)\langle\Phi(w_1)\cdots\Phi(w_n)\rangle + \text{reg in } \bar{z}. \tag{3.4.56}$$

[6] We interchanged notation $\Phi(x_i)$ and $\Phi(w_i)$, the latter stresses the fact that we are using complex coordinates, and actually means $\Phi(w_i,\bar{w}_i)$.

3.4.3 Operator Product Expansion

Commonly in quantum field theory, a correlator of multiple field insertions will present divergences as the position of the insertions converge to the same coordinate. For example, one can see that (3.4.55) is singular whenever the position of the insertion $T(z)$ coincides with one of the other Φ_i operators. This is nothing more than the UV divergences generated by infinite quantum fluctuations of the fields at the same point.

Consider the expectation value of arbitrary insertions of a complete set of operators,

$$\langle \Phi_1(z_1, \bar{z}_1) \cdots \Phi_n(z_n, \bar{z}_n) \rangle. \tag{3.4.57}$$

It is of most interest, in general, to study the short range interactions between the operators, meaning the behaviour of correlators like (3.4.57) as $(z_i, \bar{z}_i) \to (z_j, \bar{z}_j)$ for some i and some j. The Operator Product Expansion (OPE) is particularly useful in this situation, because it states that we can approximate pairwise interactions of operators, with arbitrary accuracy, by an infinite series of the complete set of operators

$$\Phi_i(x_i)\Phi_j(x_j) = \sum_k C_{ij}^k (x_i - x_j)\Phi_k(x_j), \tag{3.4.58}$$

such that the operators $\Phi_k(x_j)$ are well defined in x_j. The functions $C_{ij}^k(x_i - x_j)$ may be divergent as $x_i \to x_j$. The easiest example comes from the Ward identity (3.4.55) when one considers the correlation function between $T(z)$ and a primary field $\Phi(w_1)$,

$$\langle T(z)\Phi(w_1)\rangle = \left(\frac{h_1}{(z-w_1)^2} + \frac{\partial_{w_1}}{(z-w_1)} \right) \langle \Phi(w_1)\rangle + \text{regular terms.} \tag{3.4.59}$$

In this case, the first C_{ij}^k functions are $C(z-w_1) = \frac{h_1}{(z-w_1)^2}$ and $C'(z-w_1) = \frac{1}{(z-w_1)}$, which are multiplying the operators $\Phi(w_1)$ and $\partial_{w_1}\Phi(w_1)$ respectively. The "regular terms" correspond to an infinite tower of operators multiplied by functions $C(z-w_1)$ that contain no divergences as $z \to w_1$.

As the regular terms are often of no interest, one writes the OPE between $T(z)$ and the field $\Phi(w_1, \bar{w}_1)$ as

$$T(z)\Phi(w_1, \bar{w}_1) \sim \left(\frac{h_1}{(z-w_1)^2} + \frac{\partial_{w_1}}{(z-w_1)} \right) \Phi(w_1, \bar{w}_1), \tag{3.4.60}$$

which is to be understood as an operator statement, meaning it is valid inside a correlation function. Also the symbol "$\sim$" means the OPE modulo regular terms. Similarly, one would have for $\bar{T}(\bar{z})$

$$\bar{T}(\bar{z})\Phi(w_1, \bar{w}_1) \sim \left(\frac{\bar{h}_1}{(\bar{z}-\bar{w}_1)^2} + \frac{\partial_{\bar{w}_1}}{(\bar{z}-\bar{w}_1)} \right) \Phi(w_1, \bar{w}_1). \tag{3.4.61}$$

The OPE between $T(z)$ and itself encodes information about the degrees of freedom of the underlying theory, defining the *central charge* of the CFT. The central charge is a real number that varies from CFT to CFT and it is sensitive to the field

content on those. As the field content of a given CFT increases, e.g. more fields are considered in a Lagrangian, the central charge increases too. The simplest example is provided by the CFT of a single free bosonic field. Consider an action for the free boson

$$S_b[\phi] = \frac{g}{2}\int d^2x \partial_\mu\phi(x)\partial^\mu\phi(x). \tag{3.4.62}$$

This action is invariant under the conformal group. The generating functional is

$$\mathcal{Z}_b = \int [\mathcal{D}\phi] e^{-S_b[\phi]}. \tag{3.4.63}$$

As the path integral of a total functional derivative is zero, we write

$$\int [\mathcal{D}\phi] \frac{\delta}{\delta\phi(x)} \left(e^{-S_b[\phi]}\phi(x')\right) = 0 \tag{3.4.64}$$

$$\int [\mathcal{D}\phi] e^{-S_b[\phi]} \left(\delta(z-z',\bar{z}-\bar{z}') + 4g\partial\bar{\partial}(\phi(z,\bar{z})\phi(z',\bar{z}'))\right) = 0, \tag{3.4.65}$$

defining the operators $\partial = \partial_z$ and $\bar{\partial} = \partial_{\bar{z}}$. This leads to the propagator

$$\langle \partial\bar{\partial}\phi(z,\bar{z})\phi(z',\bar{z}')\rangle = -\frac{1}{4g}\delta(z-z',\bar{z}-\bar{z}') \tag{3.4.66}$$

$$\langle \phi(z,\bar{z})\phi(z',\bar{z}')\rangle = -\frac{1}{4\pi g}\ln|z-z'|^2 + \text{Const.} \tag{3.4.67}$$

So the OPE reads

$$\phi(z,\bar{z})\phi(z',\bar{z}') \sim -\frac{1}{4\pi g}\ln|z-z'|^2 = -\frac{1}{4\pi g}\left(\ln(z-z') + \ln(\bar{z}-\bar{z}')\right). \tag{3.4.68}$$

As well known, the energy-momentum tensor of a free scalar field theory is

$$T_{\mu\nu} = g\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial^\rho\phi\partial_\rho\phi\right). \tag{3.4.69}$$

In complex coordinates, the metric is off diagonal $g^{z\bar{z}} = g^{\bar{z}z} = 2$, so we write

$$T_{zz}(z) = g(\partial\phi\partial\phi), \tag{3.4.70}$$

$$T_{\bar{z}\bar{z}}(\bar{z}) = g(\bar{\partial}\phi\bar{\partial}\phi). \tag{3.4.71}$$

Recalling the definition of $T(z)$ and $\bar{T}(\bar{z})$, one writes

$$T(z) = -2\pi g(\partial\phi\partial\phi), \tag{3.4.72}$$

$$\bar{T}(\bar{z}) = -2\pi g(\bar{\partial}\phi\bar{\partial}\phi). \tag{3.4.73}$$

As we are interested in the divergent pieces of correlation functions, we introduce a normal ordering on the definitions of $T(z)$ and $\bar{T}(\bar{z})$ in order to use Wick's theorem to calculate the correlation functions. [7] The OPE between $T(z)$ and itself takes the form

$$T(z)T(z) = 4\pi^2 g^2 : \partial\phi\partial\phi :: \partial\phi\partial\phi : . \tag{3.4.74}$$

[7] For more details see [26].

The contraction between $\partial\phi$'s is easily read off from the propagator (3.4.67). The action of ∂_z and $\partial_{z'}$ yields

$$\begin{aligned}
\partial_z\partial_{z'}\langle\phi(z,\bar{z})\phi(z',\bar{z}')\rangle &= \langle\partial_z\partial_{z'}\phi(z,\bar{z})\phi(z',\bar{z}')\rangle \\
\implies \partial_z\partial_{z'}\langle\phi(z,\bar{z})\phi(z',\bar{z}')\rangle &= -\frac{1}{4\pi g}\partial_z\partial_{z'}\left(\ln|z-z'|^2\right) \\
&= -\frac{1}{4\pi g}\partial_{z'}\left(\frac{1}{z-z'}\right) \\
&= -\frac{1}{4\pi g}\frac{1}{(z-z')^2} + \text{Const},
\end{aligned} \tag{3.4.75}$$

so we may write the OPE as

$$\partial_z\phi(z,\bar{z})\partial_{z'}\phi(z',\bar{z}') \sim -\frac{1}{4\pi g}\frac{1}{(z-z')^2}. \tag{3.4.76}$$

Performing all contractions on (3.4.74) yields

$$T(z)T(z') \sim \frac{1/2}{(z-z')^4} + \frac{2T(z')}{(z-z')^2} + \frac{\partial_{z'}T(z')}{(z-z')}. \tag{3.4.77}$$

One can conclude that the $T(z)$ field is not a primary operator, because its OPE with $T(z')$ gives rise to a $(z-z')^{-4}$ divergence, [8] not satisfying the Ward identities of $T(z')$ with primary operators (3.4.60).

As another example, we can consider the bosonic string [26]. The action for the string in D-dimensions consists of D copies of a bosonic field $X(z,\bar{z})$,

$$S_{\text{b-string}}[X] = \frac{1}{4\pi\alpha'}\int d^2x\partial_\alpha X^\mu\partial^\alpha X_\mu, \tag{3.4.78}$$

where the constant α' is called the Regge slope. The calculation for the OPE of $T(z)$ with itself yields

$$T_{\text{b-string}}(z)T_{\text{b-string}}(z') \sim \frac{D/2}{(z-z')^4} + \frac{2T(z')}{(z-z')^2} + \frac{\partial_{z'}T(z')}{(z-z')}. \tag{3.4.79}$$

Notice that the coefficient in front of the most divergent piece has scaled exactly by the number of copies of the bosonic field in the action, so that each bosonic field contributes with $1/2$ to the coefficient.

During the string quantization, there are gauge symmetries consisting of diffeomorphism invariance of the string action that need to be fixed. In the process of gauge fixing, ghost fields need to be implement in order to properly quantize the system. The ghost fields are added via the action

$$S_g[b,c] = \frac{1}{4\pi\alpha'}\int d^2x b_{\mu\nu}\partial^\mu c^\nu, \tag{3.4.80}$$

[8] Some authors define primary operators as the ones that have maximum divergence of z'^{-2} in the OPE with $T(z')$.

the ghost fields $b_{\mu\nu}$ and c_μ are Grassman variables, and b is symmetric traceless. The OPE of $T_g T_g$ is

$$T_{\mathrm{g}}(z)T_{\mathrm{g}}(z') \sim \frac{-26/2}{(z-z')^4} + \frac{2T(z')}{(z-z')^2} + \frac{\partial_{z'}T(z')}{(z-z')}. \tag{3.4.81}$$

As one can see, the ghost fields contribute negatively to the coefficient, and in fact, it is due to this negative contribution that one can make the trace anomalies vanish in the superstring action.

Repeating the same thought for a fermionic field, which is governed by the action

$$S[\psi] = \frac{g}{2}\int d^2x \psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \psi, \tag{3.4.82}$$

the OPE yields

$$T_{\mathrm{f}}(z)T_{\mathrm{f}}(z') \sim \frac{1/4}{(z-z')^4} + \frac{2T(z')}{(z-z')^2} + \frac{\partial_{z'}T(z')}{(z-z')}. \tag{3.4.83}$$

A fermion contributes half as much as a bosonic degree of freedom. In general, one could construct a generic action containing an arbitrary number of bosons, fermions and ghost fields. The OPE of $T(z)$ with itself would take a generic form

$$T(z)T(z') \sim \frac{C/2}{(z-z')^4} + \frac{2T(z')}{(z-z')^2} + \frac{\partial_{z'}T(z')}{(z-z')}, \tag{3.4.84}$$

where the coefficient C is called the *central charge* of the CFT, and it is a measure of the field content of the theory. The fact that it is a good measure of the degrees of freedom can also be attributed to its monotonic behaviour under the RG equations, as consequence of the C-theorem. Coarse-graining the system and integrating out degrees of freedom will make C decrease.

The central charge C is also connected with the central extension of the DeWitt algebra (3.4.14), (3.4.15). In quantum mechanics there is a natural $U(1)$ "gauge symmetry", because physical configurations are the same for representatives of the equivalence class of states that are the same up to a phase factor $e^{i\phi}$ (therefore the $U(1)$ "gauge symmetry"). In this sense, one is generally interested in representations of a gauge group acting on "rays" of the Hilbert space, which are the equivalence classes themselves.

The DeWitt algebra as it was first presented in (3.4.14) and (3.4.15) is a projective represntation of the conformal algebra, and a central extension of such an algebra provides a *true representation* of the algebra. The central extension is given by

$$[l_m, l_n] = (m-n)l_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n,0}, \tag{3.4.85}$$

$$[\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n} + \frac{\bar{C}}{12}(m^3 - m)\delta_{m+n,0}, \tag{3.4.86}$$

$$[l_m, \bar{l}_n] = 0. \tag{3.4.87}$$

The last important result to be presented is the vanishing of the trace T^{μ}_{μ} as an operator. There are several ways to demonstrate such a result. A simple, yet elegant one is provided by noticing that one can induce a Weyl scaling of the metric using a dilatation.

A change of coordinates $x'^{\mu} = \Omega^{-1}(x)x^{\mu}$ defines a conformal map. Under such mapping, the metric transforms as $g'_{\mu\nu} = \Omega^2(x)g_{\mu\nu}$. Taking an infinitesimal version $\Omega(x) = 1+\delta\Omega(x)$, it produces $g'_{\mu\nu} = (1+2\delta\Omega(x))g_{\mu\nu}$, up to linear order in $\delta\Omega$. Classical conformal invariance implies that the action is invariant under the conformal group, so one writes

$$\delta g^{\mu\nu}\frac{\delta S}{\delta g^{\mu\nu}} = 0, \tag{3.4.88}$$

for $\delta g_{\mu\nu} = 2\delta\Omega(x)g_{\mu\nu}$ it follows

$$\begin{aligned}\delta g^{\mu\nu}\frac{\delta S}{\delta g^{\mu\nu}} &= \int d^D x \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}}\delta g_{\mu\nu} \\ &= \int d^D x T^{\mu\nu}(x) 2\delta\Omega(x) g_{\mu\nu} \\ &= \int d^D x T^{\mu}_{\mu}(x) 2\delta\Omega(x).\end{aligned} \tag{3.4.89}$$

For an non-vanishing arbitrary function $\Omega(x)$, it yields

$$\int d^D x (T^{\mu}_{\mu}(x))\delta\Omega(x) = 0, \tag{3.4.90}$$

so that $T^{\mu}_{\mu} = 0$. If the conformal symmetry is not broken upon quantization, it then follows that it must hold as an operator equation

$$T^{\mu}_{\mu}(x) = 0. \tag{3.4.91}$$

By studying the two-point function

$$S_{\mu\nu\rho\sigma}(x) = \langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle, \tag{3.4.92}$$

one may use conformal and parity invariance, along with the conservation of energy $\partial^{\mu}T_{\mu\nu} = 0$ to constraint $S_{\mu}{}^{\mu}{}_{\rho}{}^{\rho}(x)$ to vanish identically . Which means in particular that

$$\langle T^{\mu}_{\mu}(x)T^{\rho}_{\rho}(0)\rangle = 0. \tag{3.4.93}$$

By taking $x \to 0$ we conclude that

$$\langle (T^{\mu}_{\mu}(x))^2\rangle = 0. \tag{3.4.94}$$

The trace of the energy-momentum tensor is an operator with vanishing expectation value and zero standard deviation.

Chapter 4

Conformal Anomalies

4.1 Introduction

The main point of this work is to study ordered parameters under RG flows. It happens that these so-called ordered parameters are coefficients arising from the renormalization of the stress-energy tensor of a quantum field in a curved space-time geometry. In the two-dimensional case, the parameter is shown to be the central charge C of the CFT. In order to fully understand these coefficients, one must first understand the framework in which it is developed, that is of a *semi-classical description* of gravity.

In such a description, one considers quantum matter fields defined over an arbitrary Lorentzian geometry. The effects of gravity are contemplated via the metric tensor treated as a classical field. No loop expansions are considered for gravitational interactions, as opposed to the quantum fields of matter which will actually be quantized through the path integration procedure. This framework relies on a description of quantized fields interacting with classical fields, and several mathematical complications arise in doing so. No attempt of mathematical rigor is made in the present work.

As already presented, conformal theories in flat space have vanishing trace of $T_{\mu\nu}$ as an operator. However, when such theory is placed in a curved background, the trace acquires a non-vanishing expectation value, therefore breaking conformal symmetry

$$\langle T^{\mu}_{\mu} \rangle = \sum_i \Big(c_i I_i - (-1)^{\frac{d}{2}} a_d E_d \Big) \qquad \text{for even d.} \tag{4.1.1}$$

Although broken, the conformal symmetry is not completely lost, in the sense that we can control the expectation value by means of the Euler density E_d and invariants of the Weyl tensor I_i, which are local functions of the metric tensor $g_{\mu\nu}(x)$.

This chapter is devoted to not only the reproduction of the formula (4.1.1) in the case of two and four space-time dimensions, but also as an introduction to the framework of semi-classical gravity, which is by itself an interesting subject. The main references are [23, 24].

The dynamics of classical fields in a curved geometry are contemplated by Einstein's equations, where the energy-momentum tensor acts as the source of the gravitational field, and vice-versa. Upon the quantization of the matter fields, we

expect the source of gravity to be the expectation value of the $T_{\mu\nu}$,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = 8\pi G\langle T_{\mu\nu}\rangle. \tag{4.1.2}$$

The coupling between the matter fields and the metric gives rise to a source term in the equations of motion of the matter fields, and as the fields are basically harmonic oscillators at each point of the space-time, much insight can be learned from the analysis of the driven harmonic oscillator. In this case, the concept of vacuum and particle creation easily generalizes to the more realistic picture of scalar fields coupled to gravity.

Furthermore, the driven harmonic oscillator allows the introduction of the concept of *effective action* for a semi-classical system. Such an action formalism accounts for the quantization of the matter fields while the classicality of the gravitational field is preserved. Through this formalism, one can readily derive the semi-classical version of Einstein's equation, and with extra machinery compute counter-terms and renormalize correlation functions.

The renormalization of fields in curved space-time is implemented via ζ-function renormalization. The nature of UV divergences lies on the divergent behaviour of the Feynman propagator in curved space. Such propagator appears when one calculates the effective action, and the heat kernel method for calculating this action is explained.

4.2 The Driven Harmonic Oscillator : A Toy Problem

One key aspect of the semi-classical formalism consists in the interaction of a classical field J with a quantized field q. In the actual problem of interest, the source J is to be identified with the metric tensor $g_{\mu\nu}$, and the fields q may be whatever matter content defined in the theory. The simplest toy model of such system consists of a quantized driven harmonic oscillator [23], in which the driving force J acts as a source and is a classical field, while the position q is a quantized field obeying proper commutation relations. This might seem a little too over simplified, but in fact the bosonic or fermionic[1] fields defined over a manifold are simply copies of the harmonic oscillator at each point of space-time, in the sense that our toy model consists of the dynamics of the full system reduced to one lattice point.

Consider a one-dimensional harmonic oscillator subjected to a time-dependent external force $J(t)$. The displacement coordinate being $q(t)$, the Lagrangian for the system reads

$$L(t, \dot{q}, q) = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 - J(t)q. \tag{4.2.1}$$

Through a Legendre transformation, we reach the Hamiltonian

$$H(t, \dot{q}, q) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 - J(t)q, \tag{4.2.2}$$

[1]For fermionic fields the right toy model consists of a fermionic driven harmonic oscillator, which is to be implemented by the right anti-comuutation relations.

where the conjugate momentum is $p = \frac{\partial L}{\partial \dot{q}}$. The quantization process is implemented by imposing standard commutation relation between the coordinate $q(t)$ and its conjugate momentum $p(t)$. We are working in Heisenberg's picture, so the operators are governed by the equations of motion

$$\frac{dq(t)}{dt} = p(t), \tag{4.2.3}$$

$$\frac{dp(t)}{dt} = -\omega^2 q(t) + J(t). \tag{4.2.4}$$

The operators follow the equal time commutation relation[2]

$$[q(t), p(t)] = i. \tag{4.2.5}$$

Introducing the generic annihilation and creation operators as the specific linear combinations

$$a^-(t) = \sqrt{\frac{\omega}{2}}\left(q(t) + \frac{i}{\omega}p(t)\right), \qquad a^+(t) = \sqrt{\frac{\omega}{2}}\left(q(t) - \frac{i}{\omega}p(t)\right), \tag{4.2.6}$$

and explicitly substituting the definitions in the equations of motion, it yields

$$\frac{da^\pm(t)}{dt} = \pm a^\pm(t) \mp \frac{i}{\sqrt{2\omega}}J(t). \tag{4.2.7}$$

To solve these equations, we define the driving force $J(t)$ to act only for $t \in [0, T]$, meaning $J(t) = 0$ for $t \notin [0, T]$. Direct integration gives us the solution for the a operators

$$a^-(t) = \left(a^-_{in} + \frac{i}{\sqrt{2\omega}}\int_0^t e^{i\omega t'}J(t')dt'\right)e^{-i\omega t}, \tag{4.2.8}$$

$$a^+(t) = \left(a^+_{in} - \frac{i}{\sqrt{2\omega}}\int_0^t e^{-i\omega t'}J(t')dt'\right)e^{i\omega t}, \tag{4.2.9}$$

where $a^\pm_{in}$ are the initial conditions for the integrated solutions. By choosing the source to vanish for $t \to \mp\infty$, one mimics the behaviour of a space-time which is asymptotically Minkowski. As will be clear in the next pages, the curvature of space acts like a source which makes it more difficult to define asymptotic vacuum states.

The vacuum is defined to be the unique state annihilated by the $a^-(t)$ operator, so that the notion of a vacuum state is time-dependent when a source is present. With the given conditions for the driving force, we see that the operator $a^-(t)$ evolves as a plane wave for $t \notin [0, T]$

$$a^-(t) = a^-_{in}e^{-i\omega t} \qquad \text{for } t < 0, \tag{4.2.10}$$

$$a^-(t) = a^-_{out}e^{-i\omega t} \qquad \text{for } t > T. \tag{4.2.11}$$

[2]Using natural units $\hbar = 1$.

We can define two asymptotic vacuum states, one for the region $t < 0$, called $|0_{in}\rangle$, and another one called $|0_{out}\rangle$ for the region $t > T$, "after" the driving force acted in the system . By definition we have

$$a_{in}^{-} |0_{in}\rangle = 0, \tag{4.2.12}$$
$$a_{out}^{-} |0_{out}\rangle = 0. \tag{4.2.13}$$

The difference between the *in* and *out* operators is due to the existence of $J(t)$. If it set to zero at all times, the two vacua collapse to the vacuum of the simple harmonic oscillator. The very presence of an interacting classical background field $J(t)$ introduces particle creation. In fact, we can also see that the vacuum in the two regions are distinct by writing the Hamiltonian in terms of the number operator

$$N(t) = a^{+}(t)a^{-}(t). \tag{4.2.14}$$

The Hamiltonian reads

$$H(t) = \left(a_{in}^{+}a_{in}^{-} + \frac{1}{2}\right)\omega \qquad \text{for in region,} \tag{4.2.15}$$
$$H(t) = \left(a_{out}^{+}a_{out}^{-} + \frac{1}{2}\right)\omega \qquad \text{for out region.} \tag{4.2.16}$$

The presence of the source creates particles if the initial state is $|0_{in}\rangle$, thus raising the value of the energy measured by $H(t)$. Defining

$$J_0 = \frac{i}{\sqrt{2\omega}} \int_0^t e^{i\omega t'} J(t')dt', \tag{4.2.17}$$

a quick calculation shows that the number of particles measured in the $|0_{in}\rangle$ state goes from 0 to $|J_0|^2$ for $t > T$. This agrees with the classical intuition about the system: if one begins with a harmonic oscillator in the state of lowest energy, which classically corresponds to a harmonic oscillator in the relaxed position with no momentum, and then shake it with a driving force during a finite period of time, it will produce an excited state with energy greater than the initial one. The quantum mechanical analog is the production of a coherent state by the driving force.

Right from the definitions of the asymptotic vacuums $|0_{in}\rangle$ and $|0_{out}\rangle$, we notice that the operator $a_{out}^{-} = a_{in}^{-} + J_0$, so that the vacuum state $|0_{out}\rangle$ is eigenstate of a_{in}^{-} with eigenvalue $-J_0$, therefore a coherent state.

The strategy is now to express the matrix elements of different operators by the path integral formulation, and to do so one introduces a Wick rotation to Euclidean time in order to make things simpler (boundary conditions for path integrals and Feynman propagators), and then analytically continue the solutions to Lorentzian time to obtain the physical solution to the system.

Defining the Wick rotation $t = -i\tau$, with $\tau \in \mathbb{R}$, we defining the *Euclidean driven harmonic oscillator*

$$-\frac{d^2q(\tau)}{d\tau^2} + \omega^2 q(\tau) = J(\tau). \tag{4.2.18}$$

The solutions behave as $q(\tau) = e^{\pm\tau}$ for large τ. As by construction the action of $J(\tau)$ occurs only for a finite interval, it should not inject an infinite amount of energy in the system. This gives a natural boundary condition that the disturbance in the coordinate $q(\tau)$ should not grow increasingly large as Euclidean time evolves

$$\lim_{\tau\to\pm\infty} q(\tau) = 0. \tag{4.2.19}$$

This boundary condition makes it very easy to find the Euclidean Green's function $G_E(\tau, \tau')$, since we can write the solution $q(\tau)$ as

$$q(\tau) = \int_{-\infty}^{\infty} d\tau' G_E(\tau, \tau') J(\tau'), \tag{4.2.20}$$

where the boundary condition on G_E is $\lim_{\tau\to\pm\infty} G(\tau, \tau') = 0$. The next task is to find the solution for the equation

$$\left(-\frac{\partial^2}{\partial\tau^2} + \omega^2\right) G_E(\tau, \tau') = \delta(\tau - \tau'), \tag{4.2.21}$$

which respects the Euclidean boundary conditions. It yields

$$G_E(\tau, \tau') = \frac{1}{2\omega} e^{-\omega|\tau-\tau'|}. \tag{4.2.22}$$

Rotating back to Lorenztian time, by making $\tau = it$, one obtains the Feynman Green's function

$$G_F(t, t') = iG_E(\tau, \tau')|_{\tau=it,\tau'=it'}. \tag{4.2.23}$$

With these definitions, one can construct an Euclidean path integral,[3] calculate correlators by inserting operators in Euclidean time, and at the end of the day Wick rotate back to Lorentzian time to obtain the system in Lorentzain signature.

As we originally defined the vacuum states to be annihilated by operators in the Heisenberg picture, the *in* vacuum respects

$$a^-(t_i)\,|0_{in}\rangle = 0, \tag{4.2.24}$$

where $t_i < 0$. The *out* annihilation operator is given by a time translation of the $a^-(t_i)$ to a later time $t_f > T$,

$$a^-(t_f) = U^{-1}(t_f, t_i) a^-(t_i) U(t_f, t_i). \tag{4.2.25}$$

Starting with the definition of *out* vacuum, it yields

$$\begin{aligned} a^-(t_f)\,|0_{out}\rangle &= 0 \\ U^{-1}(t_f, t_i) a^-(t_i) U(t_f, t_i)\,|0_{out}\rangle &= 0 \\ a^-(t_i)\underbrace{U(t_f, t_i)\,|0_{out}\rangle}_{|0_{in}\rangle} &= 0, \end{aligned} \tag{4.2.26}$$

[3]The use of Euclidean path integrals is standard in the presented formalism, mainly because they are well defined in most cases as opposed to the Lorentzian path integrals.

so we have

$$|0_{out}\rangle = U^{-1}(t_f, t_i)\,|0_{in}\rangle\,. \tag{4.2.27}$$

Assuming that time evolution is unitary, the transition amplitude from $|0_{in}\rangle$ to $|0_{out}\rangle$ can be rewritten as the action of the evolution operator, therefore a path integral with vacuum boundary conditions,

$$\begin{aligned}\langle 0_{out}|0_{in}\rangle &= \langle 0_{in}|\,U(t_f, t_i)\,|0_{in}\rangle \\ &= \int_{q(t_i)=0}^{q(t_f)=0} \mathcal{D}q e^{iS[q(t),J(t)]}.\end{aligned} \tag{4.2.28}$$

The action is

$$S[q(t), J(t)] = \int dt \frac{1}{2}\dot{q}^2(t) - \frac{1}{2}\omega^2 q^2(t) + J(t)q(t). \tag{4.2.29}$$

A Wick rotation $t = -i\tau$ produces the Euclidean action $S_E[q(\tau), J(\tau)]$, defined as

$$\begin{aligned}S[q(t), J(t)]|_{t=-i\tau} &= -i \int d\tau - \frac{1}{2}\dot{q}^2(\tau) - \frac{1}{2}\omega^2 q^2(\tau) + J(\tau)q(\tau) \\ &= i \int d\tau \frac{1}{2}\dot{q}^2(\tau) + \frac{1}{2}\omega^2 q^2(\tau) - J(\tau)q(\tau) \\ &= iS_E[q(\tau), J(\tau)].\end{aligned} \tag{4.2.30}$$

The Euclidean path integral from vacuum to vacuum reads

$$\int_{q(\tau_i\to\infty)=0}^{q(\tau_f\to-\infty)=0} [\mathcal{D}q] e^{-S_E[q(\tau),J(\tau)]}. \tag{4.2.31}$$

This is a standard description of the system in terms of a path integral containing a quantized field $q(\tau)$ and a source $J(\tau)$. In usual formulations of QFT, one considers sources as a mathematical trick to take functional derivatives in order to generate time ordered correlation functions, later taking the sources to vanish. Beware that this is different than the present case, where the sources are non-vanishing local fields that are not quantized.

In the presented toy model, the source field $J(t)$ is whatever function we choose it to be, like an external parameter that we can tune and compute observables with. We are interested in the picture reported by a distinct set of interacting fields, of which some are quantized,[4] and some are not. This means that in general, the classical source fields are not to be chosen, but instead follow classical equations of motion coming from the principle of least action. The driven harmonic oscillator case does not contemplate the full picture, because there are no equations of motion for the $J(t)$ field, instead it appears as an external parameter. In realistic situations there should be a "piece" of the action containing the kinetic and potential terms of the source field $J(t)$, which would lead to equations of motions to be satisfied by $J(t)$.

[4]In the sense that there is a path integral over these fields.

The standard formalism to treat such "partially quantized" systems is known as the *Effective Action Formalism*[24]. Notice that the path integral (4.2.31) presents the field $q(\tau)$ as an "integration variable", as opposed to the $J(\tau)$ field which is taken to be a "free variable". Once the integration with respect to the measure $\mathcal{D}q$ is performed, one is left with a functional of the field $J(\tau)$. This functional defines the Euclidean effective action $\Gamma_E[J]$ via the expression

$$e^{-\Gamma_E[J]} \equiv \int [\mathcal{D}q] e^{-S_E[q(\tau),J(\tau)]}, \tag{4.2.32}$$

where the integration is taken over all Euclidean times. The real problem is to actually compute the effective action via the integration. One can readily see that UV divergences will appear, since we are integrating over all configurations. But once computed, we can use it to generate all correlation functions just by taking derivatives in respect to the fields. As an example, we calculate the effective action for the toy model, and then generalize for the case of matter fields coupled to gravity.

As the field q is to be integrated over, we must sum over all possible field configurations. The Euclidean action decays very fast for configurations far from the saddle point of the action (classical configuration). One can reach all configurations of the field q by parameterizing an arbitrary configuration as a perturbation around the classical solution

$$q(\tau) = q_{cl}(\tau) + p(\tau). \tag{4.2.33}$$

An arbitrary configuration $q(\tau)$ can be achieved if one considers arbitrary $p(\tau)$. The classical configuration is defined to be the solution to

$$\frac{\delta S_E[q,J]}{\delta q(\tau)} = 0. \tag{4.2.34}$$

As we parameterized the field by $p(\tau)$, the integration measure reduces to $\mathcal{D}p$, so we can write the Euclidean action as

$$e^{-\Gamma_E[J]} = \int [\mathcal{D}p] e^{-S_E[q_{cl}+p,J]}. \tag{4.2.35}$$

In order to maintain the boundary conditions on Euclidean signature (4.2.19), one must only integrate over perturbations $p(\tau)$ that satisfies $p(\tau \to \pm\infty) = 0$, given that the classical solution q_{cl} also satisfies the boundary conditions. Using the equations of motion and boundary conditions, it is possible to reduce the action to a simple form [5],

$$S_E[q_{cl}+p,J] = \frac{1}{2}\int d\tau \left(\dot{p}^2 + \omega^2 p^2\right) - \frac{1}{2}\int d\tau q_{cl} J. \tag{4.2.36}$$

The important point is that the integral decouples the classical field J from the quantum field p

$$e^{-\Gamma_E[J]} = e^{\frac{1}{2}\int d\tau q_{cl} J} \int [\mathcal{D}p] e^{-\left(\frac{1}{2}\int d\tau \left(\dot{p}^2+\omega^2 p^2\right)\right)}. \tag{4.2.37}$$

[5] See [23], page 153.

The integration over p can be seen as the quantum contribution of the fields q, and the argument in the exponential can be rearranged as the action of a differential operator. Dropping boundary terms due to (4.2.19), it reads

$$-\frac{1}{2}\int d\tau\left(\dot{p}^2+\omega^2p^2\right)=-\frac{1}{2}\int d\tau q(\tau)\left(\underbrace{-\frac{\partial^2}{\partial\tau^2}+\omega^2}_{\mathcal{O}_E}\right)q(\tau). \tag{4.2.38}$$

The integration may be carried out[6] to yield

$$\int[\mathcal{D}p]e^{-\left(\frac{1}{2}\int d\tau\left(\dot{p}^2+\omega^2p^2\right)\right)}\propto\sqrt{det({\mathcal{O}_E}^{-1})}. \tag{4.2.39}$$

The inverse operator ${\mathcal{O}_E}^{-1}$ is just the Euclidean Green's function G_E. As usual, (4.2.39) is a divergent quantity that needs to be regularized. At the end of the day we can see that the integration in p in expression (4.2.37) contributes as an infinite numerical constant $\mathcal{N}$, allowing us to write

$$e^{-\Gamma_E[J]}=\mathcal{N}e^{\frac{1}{2}\int d\tau q_{cl}J}. \tag{4.2.40}$$

As $q_{cl}(\tau)$ is a solution to the equations of motion of the Euclidean harmonic oscillator (4.2.18) satisfying Euclidean boundary conditions, it can be written in terms of G_E as

$$q_{cl}(\tau)=\int_{-\infty}^{\infty}J(\tau')G_E(\tau,\tau')d\tau'. \tag{4.2.41}$$

Finally taking the logarithm of expression (4.2.40) gives us the effective action

$$\Gamma_E[J(\tau)]=-\frac{1}{2}\int_{-\infty}^{\infty}J(\tau')J(\tau)G_E(\tau,\tau')d\tau d\tau'-\log(\mathcal{N}). \tag{4.2.42}$$

The Lorenztian effective action $\Gamma_L[J]$ is obtained by rotating the Euclidean effective action back to Lorentzian time

$$\Gamma_L[J(t)]\equiv i\Gamma_E[J(\tau)]|_{\tau=it}, \tag{4.2.43}$$

so we have

$$\begin{aligned}\Gamma_E[J(\tau)]&=-\frac{1}{2}\int_{-\infty}^{\infty}J(\tau')J(\tau)G_E(\tau,\tau')d\tau d\tau'-\log(\mathcal{N})\\&=\frac{1}{2}\int_{-\infty}^{\infty}J(t')J(t)G_E(\tau,\tau')|_{\tau=it,\tau'=it'}dtdt'-\log(\mathcal{N})\\&=-\frac{i}{2}\int_{-\infty}^{\infty}J(t')J(t)G_F(t,t')dtdt'-\log(\mathcal{N})\\\implies\Gamma_L[J(t)]&=\frac{1}{2}\int_{-\infty}^{\infty}J(t')J(t)G_F(t,t')dtdt'-i\log(\mathcal{N}).\end{aligned} \tag{4.2.44}$$

[6]Standard QFT trick, take the continuum limit of a lattice QFT where $\mathcal{O}_E$ has matrix representation.

With the effective action in this form, it is now easy to calculate the N-point correlators for the $q(t)$ fields, one just needs to write the $\Gamma_L[J(t)]$ in terms of the Lorenzian path integral. As $\Gamma_L[J(t)]$ is the analytic continuation of $\Gamma_E[J(\tau)]$, the latter is directly identified through (4.2.32) with the path integral in Lorenztian signature

$$e^{i\Gamma_L[J(t)]} = \int [\mathcal{D}q] e^{iS[q(t),J(t)]}. \tag{4.2.45}$$

It is implicit that the path integral runs from $t \to -\infty$ to $t \to \infty$, which is the path integral representation of the transition amplitude $\langle 0_{out}|0_{in}\rangle$. The time ordered N-point functions are insertion of operators in the path integral. For example, the 1-point function reads

$$\begin{aligned}\frac{\langle 0_{out}|\, q(t_1)\, |0_{in}\rangle}{\langle 0_{out}|0_{in}\rangle} &= \frac{\int [\mathcal{D}q] q(t_1) e^{iS[q(t),J(t)]}}{\int [\mathcal{D}q] e^{iS[q(t),J(t)]}} \\ &= e^{-i\Gamma_L[J]}(-i)\frac{\delta}{\delta J(t_1)} e^{i\Gamma_L[J]} \\ &= \frac{\delta \Gamma_L[J]}{\delta J(t_1)}.\end{aligned} \tag{4.2.46}$$

Analogously, the 2-point function is written as

$$\frac{\langle 0_{out}|\, T[q(t_1)q(t_2)]\, |0_{in}\rangle}{\langle 0_{out}|0_{in}\rangle} = \frac{\delta \Gamma_L[J]}{\delta J(t_1)} \frac{\delta \Gamma_L[J]}{\delta J(t_2)} - i \frac{\delta^2 \Gamma_L[J]}{\delta J(t_2) \delta J(t_1)}, \tag{4.2.47}$$

where T is the standard time ordering operator.

This concludes the analysis of the driven harmonic oscillator. In summary: we have defined the effective action for a system with one quantized field and one classical field as the path integration over the quantum fields q. The integration leads to an action functional of the classical field J, called effective action. The next step consists in accounting for the dynamics of the classical fields J. As the equations of motion of the J field will generally be coupled to the quantized fields, there will be *backreaction*, so that the quantum fields will play a role in determining the solution for the classical fields J and vice-versa.

4.3 Effective Action in Gravitational Background

Having developed the machinery for the effective action formalism, we generalize for the case of a dynamical source J, which in the case of semi-classical gravity is the classical field $g_{\mu\nu}$. The dynamics of the background metric is given by the Einstein Hilbert action

$$S_{\rm EH}[g_{\mu\nu}] = -\frac{1}{16\pi G} \int d^D x \sqrt{-g}\, (R + 2\Lambda)\,. \tag{4.3.1}$$

When no matter fields are present, $S_{\rm EH}$ is the total action of the system and the solutions are given by

$$\frac{\delta S_{\rm EH}[g_{\mu\nu}]}{\delta g_{\mu\nu}} = 0. \tag{4.3.2}$$

Suppose now that matter fields are present and coupled to the gravitational background via the action $S_m[\phi, g_{\mu\nu}]$. Such an action typically contains the kinetic structure for the field ϕ, and a potential term. The coupling to gravity is implemented by making the action covariant.[7] The total action is given by

$$S[\phi, g_{\mu\nu}] = S_{\text{EH}}[g_{\mu\nu}] + S_m[\phi, g_{\mu\nu}]. \tag{4.3.3}$$

The quantization of matter fields is implemented via the path integration over the quantum degrees of freedom ϕ, and the field $g_{\mu\nu}$ remains untouched. The total Lorenztain affective action $S_{\text{eff}}[g_{\mu\nu}]$ is defined as

$$\begin{aligned} e^{iS_{\text{eff}}[g_{\mu\nu}]} &\equiv \int [\mathcal{D}\phi] e^{iS_{\text{EH}}[g_{\mu\nu}] + iS_m[\phi, g_{\mu\nu}]} \\ &= e^{iS_{\text{EH}}[g_{\mu\nu}]} \int [\mathcal{D}\phi] e^{iS_m[\phi, g_{\mu\nu}]} \\ &= e^{iS_{\text{EH}}[g_{\mu\nu}]} e^{i\Gamma_L[g_{\mu\nu}]} \\ \implies S_{\text{eff}}[g_{\mu\nu}] &= S_{\text{EH}}[g_{\mu\nu}] + \Gamma_L[g_{\mu\nu}]. \end{aligned} \tag{4.3.4}$$

As $g_{\mu\nu}$ is a classical field, it should satisfy the equation of motion

$$\begin{aligned} \frac{\delta S_{\text{eff}}[g_{\mu\nu}]}{\delta g^{\alpha\beta}} &= 0 \\ \frac{\delta S_{\text{EH}}[g_{\mu\nu}]}{\delta g^{\alpha\beta}} + \frac{\delta \Gamma_L[g_{\mu\nu}]}{\delta g^{\alpha\beta}} &= 0, \end{aligned} \tag{4.3.5}$$

and

$$\begin{aligned} \frac{\delta \Gamma_L[g_{\mu\nu}]}{\delta g^{\alpha\beta}} &= (-i) \frac{\delta}{\delta g^{\alpha\beta}} \log \left(\int [\mathcal{D}\phi] e^{iS_m[\phi, g_{\mu\nu}]} \right) \\ &= (-i) \frac{\int [\mathcal{D}\phi] (i) \frac{\delta S_m}{\delta g^{\alpha\beta}} e^{iS_m[\phi, g_{\mu\nu}]}}{\int [\mathcal{D}\phi] e^{iS_m[\phi, g_{\mu\nu}]}} \\ &= \frac{\sqrt{-g}}{2} \frac{\int [\mathcal{D}\phi] T_{\alpha\beta} e^{iS_m[\phi, g_{\mu\nu}]}}{\int [\mathcal{D}\phi] e^{iS_m[\phi, g_{\mu\nu}]}} \\ &= \frac{\sqrt{-g}}{2} \langle T_{\alpha\beta} \rangle. \end{aligned} \tag{4.3.6}$$

plugging back back in (4.3.5), we reach the semi-classical version of Einstein's equation.

For the sake of simplicity, consider a scalar field minimally coupled[8] as the matter action S_m. The calculations that follow can be generalized for fields with arbitrary spin, but are greatly simplified in the scalar case. The matter action reads

$$S_m[\phi, g_{\mu\nu}] = \int d^{2\omega} x \sqrt{-g} \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(x) \phi^2 \right), \tag{4.3.7}$$

[7] By making the action covariant, the integration measure $\int d^D x \to \int d^D x \sqrt{-g}$, and $\partial_\mu \to \nabla_\mu$. The introduction of factors of $g_{\mu\nu}$ couples the field ϕ to $g_{\mu\nu}$.

[8] The generic coupled scalar field is given by $S_m = \int d^D x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) - \frac{\eta}{2} R \phi^2 \right)$, for minimal couplings take $\eta = 0$.

where 2ω is the number of space-time dimensions, $V(x)$ is an arbitrary potential. In what follows, it is convenient to define the notation $\partial_\mu \phi \equiv \phi_{,\mu}$.

Recall that in order to find $\Gamma_L[g]$, one must analytically extend $\Gamma_E[g]$, which is obtained through the Euclidean path integral over the fields ϕ. The first step is to use a Wick rotation to Euclidean time: the original signature of the metric is $(+,-,\cdots,-)$. After the rotation it goes to $(-,-,\cdots,-)$, so we define a metric $g_{\mu\nu} \equiv -g_{\mu\nu}|_{t=-i\tau}$. The Euclidean action is

$$S_E[g,\phi] = \frac{1}{2}\int d^{2\omega}x\sqrt{g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(x)\phi^2\right). \tag{4.3.8}$$

Partially integrating and throwing away boundary terms yields

$$S_E[g,\phi] = \frac{1}{2}\int d^{2\omega}x\sqrt{g}\left(\phi(x)\underbrace{\left(-\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu) + V(x)\right)}_{\hat{F}}\phi(x)\right). \tag{4.3.9}$$

It is useful to express the action as the inner product of a field configuration $\phi(x)$ with a field configuration $\phi'(x) = \hat{F}\phi(x)$. Equation (4.3.9) strongly suggests that we take the inner product of the space of configurations to be

$$(\phi(x),\phi'(x)) = \int d^{2\omega}x\sqrt{g}\phi(x)\phi'(x). \tag{4.3.10}$$

This picture is the continuum analog of a finite-dimensional vector space endowed with an inner product and a linear operator $\hat{F}$. The vectors of such space are the field configurations $\phi(x)$, and this space is infinite-dimensional. In the case of a finite-dimensional vector space (imagine putting the QFT on a lattice), the $\hat{F}$ operator has a matrix representation F once a basis is picked. The space of configurations would be represented by finite-dimensional vectors $\vec{x} = (x_1,\cdots,x_n)$. Of course, as F is linear operator in the vector space, its application in an arbitrary vector $\vec{x}$ yields another vector in the space $\vec{x'} = F\vec{x}$. So the action is

$$S_E[\vec{x}] = \frac{1}{2}(\vec{x},F\vec{x}). \tag{4.3.11}$$

The path integral measure represents a summation over all possible states of ϕ. In the toy model it is simply an integration over all entries of the vector $(x_1,\cdots,x_n)$, and the standard result is

$$\int dx_1\cdots dx_n e^{-\frac{1}{2}(\vec{x},F\vec{x})} = \left(\frac{(2\pi)^n}{\det(F)}\right)^{\frac{1}{2}}. \tag{4.3.12}$$

The continuum case is a direct analogy of (4.3.12), where we will need to calculate the determinant of the operator $\hat{F}$, and that's when the heat kernel method becomes useful. One can mimic this behaviour in the case of the space of configurations for the field ϕ, appart for some mathematical difficulties [25].[9]

[9]The $\hat{F}$ operator needs to be self-adjoint, and the domain of $\hat{F}$ and its adjoint must coincide.

Consider the eigenvalue equation

$$\hat{F}\phi_n(x) = \lambda_n \phi_n(x). \tag{4.3.13}$$

Matrices M in linear spaces can be diagonalized as $M = U \cdot D \cdot U^{-1}$, where D is a diagonal matrix containing the eigenvalues λ_n, and U is a unitary matrix. As U is unitary, it can be seen as implementing a change of basis, in the sense that acting with M is the same as changing the basis from an arbitrary basis $g_n(x)$ to the basis of eigenfunctions of M, denoted $f_n(x) = U^{-1}g_n(x)$, then acting with a diagonal operator D, and finally changing back to the original basis through the action of U^{-1}. This means that through the change of basis represented by U, one can make the matrix M to be diagonal. Similarly, there exists a change of basis in the configuration space $\phi(x)$ such that the differential operator $\hat{F}$ is diagonal.

Consider ϕ_n, eigenfunctions of $\hat{F}$, as an orthonormal basis for the space of configurations.[10] We can write any configuration $\phi(x)$ as

$$\phi(x) = \sum_n c_n \phi_n(x). \tag{4.3.14}$$

The Euclidean action can be written as

$$\begin{aligned} S_E[g,\phi] &= \frac{1}{2}\int d^{2\omega}x\sqrt{g}\left(\phi(x)\hat{F}\phi(x)\right) \\ &= \frac{1}{2}\sum_{m,n}\int d^{2\omega}x\sqrt{g}c_m c_n \lambda_n \phi_m(x)\phi_n(x) \\ &= \frac{1}{2}\sum_{m,n} c_m c_n \lambda_n \underbrace{(\phi_m,\phi_n)}_{\delta_{m,n}} \\ &= \frac{1}{2}\sum_n (c_n)^2\lambda_n. \end{aligned} \tag{4.3.15}$$

The information of the action of the operator $\hat{F}$ is entirely contained in λ_n. The path integral can be seen as

$$\int [\mathcal{D}\phi] e^{-\frac{1}{2}(c_n)^2\lambda_n}, \tag{4.3.16}$$

where the measure $[\mathcal{D}\phi]$ instructs us to sum over the space of configurations. Once we picked a basis, in this case the eigenfunctions ϕ_n, we can reach any configuration by summing over the coordinates c_n. The measure is $[\mathcal{D}\phi] = \prod_n \frac{dc_n}{\sqrt{2\pi}}$, where the π factor is just a convenient normalization constant. The integral becomes

$$\begin{aligned} \int [\mathcal{D}\phi] e^{-\frac{1}{2}(c_n)^2\lambda_n} &= \int \prod_n \frac{dc_n}{\sqrt{2\pi}} e^{-\frac{1}{2}(c_n)^2\lambda_n} \\ &= \left(\prod_n \lambda_n\right)^{-\frac{1}{2}}. \end{aligned} \tag{4.3.17}$$

[10] In fact, it is not always that the set of eigenfucntions of a differential operator forms an orthonormal basis of the space, see [25].

Which yields the Euclidean action

$$\Gamma_E[g_{\mu\nu}] = \frac{1}{2}\ln\left(\det(\hat{F})\right). \tag{4.3.18}$$

This is clearly a divergent quantity, the eigenvalues λ_n scale with the momentum squared, and notice that we have a $\Box^g = \nabla_\mu\nabla^\mu$ in the definition of the $\hat{F}$ operator, which gives us a dependence on p^2 when $\hat{F}$ acts on a free wave $e^{p\cdot x}$. Since the effective action consists in summing over all modes, we already expect UV divergences.

4.4 Regularization and Heat Kernel

The strategy for the calculation of (4.3.18) is to impose a UV cutoff and regularize the functional determinant $\hat{F}$ using ζ-function regularization. However, with an arbitrary potential $V(x)$ in $\hat{F}$, it is only possible to approach the problem via a perturbation series, defining the heat kernel method.

The function $\zeta_{\hat{O}}(s)$, for a differential operator $\hat{O}$, is defined as

$$\zeta_{\hat{O}}(s) \equiv Tr(\hat{O}^{-s}). \tag{4.4.1}$$

Being λ_n the eigenvalues of $\hat{O}$, we have

$$\zeta_{\hat{O}}(s) = \sum_n \left(\frac{1}{\lambda_n}\right)^s. \tag{4.4.2}$$

One relates the $\det(\hat{O})$ with $\zeta_{\hat{O}}$ via

$$\begin{aligned}
\frac{d}{ds}\zeta_{\hat{O}}(s) &= \frac{d}{ds}\sum_n \left(\lambda^{-s}\right) \\
&= \frac{d}{ds}\sum_n \left(e^{-s\ln(\lambda_n)}\right) \\
&= \sum_n \left(-\ln(\lambda_n)e^{-s\ln(\lambda_n)}\right) \\
\implies \frac{d}{ds}\zeta_{\hat{O}}(s)\Big|_{s=0} &= \sum_n \left(-\ln(\lambda_n)\right).
\end{aligned} \tag{4.4.3}$$

Also

$$\begin{aligned}
\ln\det\hat{O} &= \ln\left(\prod_n \lambda_n\right) \\
&= \sum_n \left(\ln\lambda_n\right),
\end{aligned} \tag{4.4.4}$$

which yields

$$\ln\det\hat{O} = -\frac{d}{ds}\zeta_{\hat{O}}(s)\Big|_{s=0}. \tag{4.4.5}$$

Suppose the existence of an operator $\hat{O}$, with same eigenvalues of $\hat{F}$, which acts on a Hilbert space $\mathcal{H}$ as

$$\hat{O}\,|\psi_n\rangle = \lambda_n\,|\psi_n\rangle\,. \tag{4.4.6}$$

We define the inner product of $\mathcal{H}$ as [11]

$$\langle\alpha|\beta\rangle = \int d^{2\omega}x\alpha(x)\beta(x). \tag{4.4.7}$$

Being $|\psi_n\rangle$ eigenstates of $\hat{O}$, they form a basis of $\mathcal{H}$. The eigenvalue equation (4.4.6) reduces to (4.3.13), given that

$$\langle x|\psi_n\rangle = \psi_n(x) = g^{\frac{1}{4}}\phi_n(x), \tag{4.4.8}$$

with

$$\langle x'|\,\hat{O}\,|x\rangle = g^{\frac{1}{4}}(x)\left(-\Box_x^g + V(x)\right)\left(g^{-\frac{1}{4}}\delta(x-x')\right). \tag{4.4.9}$$

To see that, insert the identity operator in terms of the basis $|x\rangle$ in (4.4.6):

$$\begin{aligned}
\hat{O}\,|\psi_n\rangle &= \lambda_n\,|\psi_n\rangle \\
\int d^{2\omega}x'\,\langle x|\,\hat{O}\,|x'\rangle\,\langle x'|\psi_n\rangle &= \lambda_n\langle x|\psi_n\rangle \\
\int d^{2\omega}x' g^{\frac{1}{4}}(x)\left(-\Box_x^g + V(x)\right)\left(g^{-\frac{1}{4}}\delta(x-x')\psi_n(x')\right) &= \lambda_n\psi_n(x) \\
g^{\frac{1}{4}}(x)\left(-\Box_x^g + V(x)\right)\left(\phi_n(x)\right) &= \lambda_n g^{\frac{1}{4}}(x)\phi_n(x) \\
\implies \hat{F}\phi_n(x) &= \lambda_n\phi_n(x).
\end{aligned} \tag{4.4.10}$$

The Heat Kernel method is a trick that allows us to obtain the functional determinant $\det(\hat{O})$ by solving differential equations for matrix elements of powers of $\hat{O}$. For a Hermitian operator $\hat{O}$ with positive eigenvalues λ_n, and $|\psi_n\rangle$ a complete basis of the auxiliary Hilbert space $\mathcal{H}$, we define the *heat kernel operator*

$$\hat{K}(\tau) \equiv \exp\left(-\hat{O}\tau\right), \tag{4.4.11}$$

where $\tau \in \mathbb{R}_+$ is called *proper time*. This parameter is a bookeeping device that allows us to collect powers of $\hat{O}$ when a perturbative solution for $\hat{K}$ is needed . The idea consists in constructing a relation between $\zeta_{\hat{O}}(s)$ and $\hat{K}(\tau)$.

Inserting the projector of the basis $|\psi_n\rangle$ on (4.4.11) yields

$$\hat{K}(\tau) = \sum_n e^{-\lambda_n\tau}\,|\psi_n\rangle\,\langle\psi_n|\,. \tag{4.4.12}$$

[11] Notice that there is no complex conjugation of one of the entries of the inner product, because the Euclidean path integration sums over real valued functions $\phi(x)$.

We define the Euler's gamma function $\Gamma(s)$, for $Re(s) > 0$

$$\begin{aligned}
\Gamma(s) &\equiv \int_0^\infty e^{-\tau}\tau^{s-1}d\tau \qquad (4.4.13)\\
&= (\lambda_n)^s \int_0^\infty e^{-\lambda_n\tau}\tau^{s-1}d\tau\\
(\lambda_n)^{-s} &= \frac{1}{\Gamma(s)}\int_0^\infty e^{-\lambda_n\tau}\tau^{s-1}d\tau\\
\underbrace{\sum_n(\lambda_n)^{-s}}_{\zeta_{\hat{O}}(s)} &= \frac{1}{\Gamma(s)}\int_0^\infty \underbrace{\sum_n e^{-\lambda_n\tau}}_{Tr(\hat{K}(\tau))}\tau^{s-1}d\tau\\
\implies \zeta_{\hat{O}}(s) &= \frac{1}{\Gamma(s)}\int_0^\infty Tr(\hat{K}(\tau))\tau^{s-1}d\tau. \qquad (4.4.14)
\end{aligned}$$

From the first to the second line, a change o variables $\tau \to \lambda_n\tau$ was implemented. From the third to the fourth line, we added the expression for all eigenvalues. The $Tr(\hat{K}(\tau))$ is obtained directly from (4.4.12) by tracing over the basis $|\psi_n\rangle$. Finally, we used the definition of $\zeta_{\hat{O}}(s)$ in LHS of the fourth line. Equation (4.4.14) means that *the problem is solved once one knows* $Tr(\hat{K}(\tau))$.

From the definition (4.4.11), it follows

$$\frac{d\hat{K}(\tau)}{d\tau} = -\hat{O}\hat{K}(\tau). \qquad (4.4.15)$$

We define the matrix element $K(x, x', \tau) = \langle x|\, \hat{K}(\tau)\, |x'\rangle$. Rewriting the differential equation and applying the unit operator yields

$$\frac{dK}{d\tau}(x, x', \tau) = -\int d^{2\omega}x''\, \langle x|\, \hat{O}(\tau)\, |x''\rangle\, K(x'', x', \tau). \qquad (4.4.16)$$

Directly from (4.4.15) we have the initial condition $\hat{K}(\tau = 0) = \mathbb{1}$, so that the matrix element $K(x, x', 0) = \delta(x - x')$. In terms of the matrix elements, the trace reads

$$Tr(\hat{K}(\tau)) = \int d^{2\omega}x K(x, x, \tau). \qquad (4.4.17)$$

Solving (4.4.16) gives us $K(x, x', \tau)$, consequently giving us $Tr(\hat{K}(\tau))$, thus solving the problem. Expanding the $\Box^g = g^{\mu\nu}\nabla_\mu\nabla_\nu$, the matrix elements of (4.4.9) read

$$\langle x|\, \hat{O}\, |x'\rangle = -g^{-\frac{1}{4}}\partial_\nu\left(g^{\mu\nu}g^{\frac{1}{2}}\partial_\mu(g^{-\frac{1}{4}}\delta(x - x'))\right) + V(x)\delta(x - x'). \qquad (4.4.18)$$

It is impossible to solve the Heat Kernel for an arbitrary potential $V(x)$ and metric $g_{\mu\nu}$, so we take a perturbative approach by making $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$, with small $h_{\mu\nu}$, and a small potential $V(x)$.[12] We now develop (4.4.18). Noting that

$$\partial_\mu(g^n) \equiv g^n g_{,\mu} = g^n g^{\alpha\beta}g_{\alpha\beta,\mu}, \qquad (4.4.19)$$

[12]Small when compared to the spectrum of the "flat space" Laplacian $\partial^\mu\partial_\mu$.

and omitting the argument of delta functions, one reaches

$$\begin{aligned}\langle x|\,\hat{O}\,|x'\rangle &= -g^{-\frac{1}{4}}\partial_\nu\left(g^{\mu\nu}g^{\frac{1}{2}}\partial_\mu(g^{-\frac{1}{4}}\delta(x-x'))\right)+V(x)\delta(x-x')\\ &= -g^{-\frac{1}{4}}\partial_\nu\left(g^{\mu\nu}g^{\frac{1}{4}}\left[-\frac{1}{4}g_{,\mu}\delta+\delta_{,\mu}\right]\right)+V(x)\delta(x-x')\\ &= h^{\mu\nu}_{,\nu}\left[-\frac{1}{4}g_{,\mu}\delta+\delta_{,\mu}\right]-\frac{1}{16}g^{\mu\nu}g_{,\mu}g_{,\nu}\delta+\frac{1}{4}g^{\mu\nu}g_{,\nu}\delta_{,\mu}\\ &\quad-\frac{1}{4}\partial_\nu(g^{\mu\nu}g_{,\mu})\delta-\frac{1}{4}g^{\mu\nu}g_{,\mu}\delta_{,\nu}+g^{\mu\nu}\delta_{,\mu\nu}+V(x)\delta.\end{aligned}$$

Noticing that $g^{\alpha\beta}_{,\mu}=h^{\alpha\beta}_{,\mu}$, we have

$$\begin{aligned}\langle x|\,\hat{O}\,|x'\rangle &= \frac{1}{4}h^{\mu\nu}_{,\nu}g^{\alpha\beta}h_{\alpha\beta,\mu}\delta-\underbrace{h^{\mu\nu}_{,\nu}\partial_\mu\delta}_{\langle x|\hat{\Gamma}|x'\rangle}+\frac{1}{16}g^{\mu\nu}g^{\alpha\beta}h_{\alpha\beta,\nu}g^{\kappa\lambda}h_{\kappa\lambda,\mu}\delta\\ &\quad+\frac{1}{4}g^{\mu\nu}g^{\alpha\beta}h_{\alpha\beta,\mu\nu}\delta+\frac{1}{4}g^{\mu\nu}g^{\alpha\beta}_{,\nu}h_{\alpha\beta,\mu}\delta-\underbrace{\delta^{\mu\nu}\partial_\mu\partial_\nu\delta}_{\langle x|\Box|x'\rangle}-\underbrace{h^{\mu\nu}\partial_\mu\partial_\nu\delta}_{\langle x|\hat{h}|x'\rangle}+V(x)\delta.\end{aligned}\tag{4.4.20}$$

Defining the matrix elements of the operators $\hat{\Gamma}$ and $\hat{\Box}$ as indicated, and further defining the operator $\hat{P}$ yields

$$\begin{aligned}&\langle x|\,\hat{O}\,|x'\rangle =\\ &\underbrace{\frac{1}{4}h^{\mu\nu}_{,\nu}g^{\alpha\beta}h_{\alpha\beta,\mu}\delta+\frac{1}{16}g^{\mu\nu}g^{\alpha\beta}h_{\alpha\beta,\nu}g^{\kappa\lambda}h_{\kappa\lambda,\mu}\delta+\frac{1}{4}g^{\mu\nu}g^{\alpha\beta}h_{\alpha\beta,\mu\nu}\delta+\frac{1}{4}g^{\mu\nu}g^{\alpha\beta}_{,\nu}h_{\alpha\beta,\mu}\delta}_{\equiv-\langle x|\hat{P}|x'\rangle}\\ &-\langle x|\,\hat{\Gamma}\,|x'\rangle-\langle x|\,\hat{h}\,|x'\rangle-\langle x|\,\hat{\Box}\,|x'\rangle\,,\end{aligned}\tag{4.4.21}$$

so we can write

$$\begin{aligned}\hat{O}&\equiv-\hat{\Box}-\hat{s}[h_{\mu\nu},V]\\ &=-\hat{\Box}-\hat{h}-\hat{\gamma}-\hat{P}.\end{aligned}\tag{4.4.22}$$

The operator $\hat{s}[h_{\mu\nu},V]\equiv\hat{h}+\hat{\gamma}+\hat{P}$ contains the factors of $h_{\mu\nu}$ and $V(x)$, these are the perturbation parameters we need to solve the heat kernel equation. We may think of the operator $\hat{O}$ as the leading operator $\hat{\Box}$ (the Laplacian in a flat space) plus curvature and potential-like corrections coming from the piece $\hat{s}[h_{\mu\nu},V]$. With that in mind, we may propose the expansion of $\hat{K}(\tau)$ in orders of $\hat{s}[h_{\mu\nu},V]$, thereafter abbreviated to $\hat{s}$:

$$\hat{K}(\tau)=\hat{K}_0(\tau)+\hat{K}_1(\tau)+\hat{K}_2(\tau)+\cdots\tag{4.4.23}$$

The heat kernel equation reads

$$\frac{d\hat{K}}{d\tau}(\tau)=\left(\hat{\Box}+\hat{s}\right)\hat{K}(\tau).\tag{4.4.24}$$

Collecting in powers of $\hat{s}$ yields

$$\frac{d\hat{K}_0}{d\tau}(\tau) = \hat{\Box}\hat{K}_0(\tau), \tag{4.4.25}$$

$$\frac{d\hat{K}_1}{d\tau}(\tau) = \hat{\Box}\hat{K}_1(\tau) + \hat{s}\hat{K}_0(\tau), \tag{4.4.26}$$

$$\dots$$

One can perturbatively solve the equations by first solving $\hat{K}_0$ and plugging in the first order equation, then solving for $\hat{K}_1$ and so on. Let's solve the zeroth order equation in $\hat{s}$ (4.4.25) : notice that the initial equations are $\hat{K}_0(0) = \mathbb{1}$ and $\hat{K}_1(0) = 0$. From now on we drop the "hat" notation for simplicity.

We have for order zero

$$K_0(\tau) = e^{\tau\Box}. \tag{4.4.27}$$

In order to find K_1 we take the ansatz

$$K_1(\tau) = K_0(\tau)C(\tau), \tag{4.4.28}$$

where $C(\tau)$ is an operator to be restricted by demanding that $K_1(\tau)$ satisfies (4.4.26). In this case

$$\begin{aligned}
\underbrace{\frac{dK_1}{d\tau}}_{\Box K_1(\tau)+sK_0(\tau)} &= \frac{dK_0}{d\tau}C(\tau) + K_0\frac{dC}{d\tau} \\
&= \Box\underbrace{K_0C(\tau)}_{K_1} + K_0\frac{dC}{d\tau} \\
\implies K_0\frac{dC}{d\tau} &= sK_0 \\
\implies C(\tau) &= \int_0^\tau d\tau' K_0^{-1}(\tau')sK_0(\tau').
\end{aligned} \tag{4.4.29}$$

To write (4.4.29) in a more convenient way, we notice the property

$$K_0(\tau) = e^{\tau\Box} \implies K_0(-\tau) = K_0^{-1}(\tau). \tag{4.4.30}$$

Multiplying (4.4.29) on the left by $K_0(\tau)$ yields

$$K_1(\tau) = \int_0^\tau d\tau' K_0(\tau - \tau')sK_0(\tau'). \tag{4.4.31}$$

As we are ultimately interested in solving $Tr(K(\tau))$, we actually need the matrix elements of K_0 and K_1 with respect to the $|x\rangle$ basis. For the zeroth order, it reads

$$\begin{aligned}
\langle x| K_0(\tau) |y\rangle &= e^{\tau\Box_x}\delta(x-y) \\
&= e^{\tau\Box_x}\int \frac{d^{2\omega}}{(2\pi)^{2\pi}}e^{ik\cdot(x-y)} \\
&= \frac{1}{(4\pi\tau)^\omega}\exp\left(-\frac{(x-y)^2}{4\tau}\right).
\end{aligned}$$

The first order contains a contribution of $\hat{s} = \hat{h} + \hat{\gamma} + \hat{P}$, and we may treat the terms separately as

$$\begin{aligned} K_1(\tau) &= \int_0^\tau d\tau' K_0(\tau - \tau') s K_0(\tau') \\ &= \underbrace{\int_0^\tau d\tau' K_0(\tau - \tau') h K_0(\tau')}_{\equiv K_1^h} + \underbrace{\int_0^\tau d\tau' K_0(\tau - \tau') \Gamma K_0(\tau')}_{\equiv K_1^\Gamma} \\ &\quad + \underbrace{\int_0^\tau d\tau' K_0(\tau - \tau') P K_0(\tau')}_{\equiv K_1^P}. \end{aligned} \tag{4.4.32}$$

Solving the three different contributions relies on the same strategy, which consists in apply projection operators $\mathbb{1} = \int d^{2\omega}x \, |x\rangle \langle x|$ in between the operators to obtain expressions in terms of the already known matrix elements. As an example, we solve K_1^P:

$$\begin{aligned} \langle x| K_1^P |x'\rangle &= \int_0^\infty d\tau' d^{2\omega}y d^{2\omega}z \, \langle x| K_0 |y\rangle \langle y| (\tau - \tau') P |z\rangle \langle z| K_0(\tau') |x'\rangle \\ &= \int_0^\infty d\tau' d^{2\omega}y \frac{\exp\left(-\frac{\tau}{4(\tau-\tau')\tau}(x-y)^2\right)}{(4\pi(\tau-\tau'))^\omega (4\pi\tau')^\omega} P(y), \end{aligned} \tag{4.4.33}$$

where by definition $P(y)$ is such that

$$\langle y| \hat{P} |y'\rangle \equiv P(y)\delta(y - y'). \tag{4.4.34}$$

One can write the exponential in (4.4.33) using the Laplacian operator, by noticing that

$$e^{\tau\Box_x} P(x) = \int d^{2\omega}y \frac{1}{(4\pi\tau)^\omega} \exp\left(-\frac{(x-y)^2}{4\tau}\right) P(y), \tag{4.4.35}$$

so that

$$\langle x| K_1^P |x\rangle = \frac{1}{(4\pi\tau)^\omega} \int_0^\tau d\tau' \exp\left(\frac{\tau'(\tau-\tau')}{\tau}\Box_x\right) P(x). \tag{4.4.36}$$

In a similar fashion one obtains for the remaining two pieces

$$\langle x| K_1^\Gamma |x\rangle = -\frac{1}{(4\pi\tau)^\omega} \int_0^\tau d\tau' \exp\left(\frac{\tau'(\tau-\tau')}{\tau}\Box_x\right) \left(\frac{\tau-\tau'}{\tau} h^{\mu\nu}_{,\mu\nu}(x)\right), \tag{4.4.37}$$

$$\langle x| K_1^h |x\rangle = \frac{1}{(4\pi\tau)^\omega} \int_0^\tau d\tau' \exp\left(\frac{\tau'(\tau-\tau')}{\tau}\Box_x\right) \left(-\frac{\delta_{\mu\nu}h^{\mu\nu}(x)}{2\tau} + \left(\frac{\tau-\tau'}{\tau}\right)^2 h^{\mu\nu}_{,\mu\nu}\right). \tag{4.4.38}$$

Plugging all the pieces together yields the diagonal part of $K_1(\tau)$

$$\begin{aligned} \langle x| K_1(\tau) |x'\rangle = \frac{1}{(4\pi\tau)^\omega} \int_0^\tau d\tau' \exp\left(\frac{\tau'(\tau-\tau')}{\tau}\Box_x\right) \Big(P(x) \\ -\frac{1}{2\tau}\delta^{\mu\nu}h^{\mu\nu}(x) - \frac{\tau-\tau'}{\tau^2} h^{\mu\nu}_{,\mu\nu}(x)\Big). \end{aligned} \tag{4.4.39}$$

Recall that we are expanding the heat kernel in orders of $\hat{s}[V, h_{\mu\nu}]$. The solution up to $\mathcal{O}(s)$ contains the polynomial $P(x)$ which has terms like $h_{\mu\nu}^2$ as well as lower order terms like $h_{\mu\nu}$. Higher order corrections $K_n(\tau)$ with $n \geq 2$ will correspond to order $\hat{s}^2$, so that the lowest order in $h_{\mu\nu}$ is $\mathcal{O}(h_{\mu\nu}^2)$. This allows us to expand $P(x)$ in $\mathcal{O}(h_{\mu\nu})$ to reach an expression up to linear order in $h_{\mu\nu}$ for $Tr\left(K(\tau)\right)$.

The only linear order contribution from $P(x)$ (4.4.21) is given by $\frac{1}{4}g^{\mu\nu}g^{\alpha\beta}h_{\alpha\beta,\mu\nu}$. Expanding $g^{\mu\nu} = \delta^{\mu\nu} + h^{\mu\nu}$ leads to $P(x) = \frac{1}{4}\delta_{\mu\nu}\Box h^{\mu\nu}(x) - V(x) + \mathcal{O}(h^2)$. Expanding $\exp\left(\frac{\tau'(\tau-\tau')}{\tau}\Box_x\right)$ as a series, we get

$$\langle x|\, K_1(\tau)\, |x\rangle = \frac{1}{(4\pi\tau)^\omega}\left[-\frac{1}{2}\delta_{\mu\nu}h^{\mu\nu}(x) - \tau V(x) + \frac{\tau}{6}(\delta_{\mu\nu}\Box h^{\mu\nu}(x) - h^{\mu\nu}_{,\mu\nu}(x)) + \Box(\cdots)\right].$$

The $\Box(\cdots)$ term will give rise to vanishing boundary terms upon integration over $\int d^{2\omega}x$ from the trace, so one can drop its contribution. The trace reads

$$Tr(K(\tau)) = \int d^{2\omega}x \frac{1}{(4\pi\tau)^\omega}\left(1 - -\frac{1}{2}\delta_{\mu\nu}h^{\mu\nu}(x) - \tau V(x) + \frac{\tau}{6}(\delta_{\mu\nu}\Box h^{\mu\nu}(x) - h^{\mu\nu}_{,\mu\nu}(x))\right).$$

Recall the expansions

$$\sqrt{g} = 1 - \frac{1}{2}\delta_{\mu\nu}h^{\mu\nu} + \mathcal{O}(h)^2 \text{ and } R = \delta_{\mu\nu}\Box h^{\mu\nu} - h^{\mu\nu}_{,\mu\nu}. \tag{4.4.40}$$

We assumed that both $h_{\mu\nu}$ and $V(x)$ were small, so the product $hV(x)$ can be ignored at linear order in h, allowing the result to be written as

$$Tr(K(\tau)) = \frac{1}{(4\pi\tau)^\omega}\int d^{2\omega}x\sqrt{g}\left(1 + \tau\left[\frac{R}{6} - V\right] + \mathcal{O}(h^2)\right), \tag{4.4.41}$$

which contains the terms K_0 and K_1. The strategy may be repeated for higher order terms, and by considering up to K_2 one may calculate [29] (for $V = 0$)

$$Tr(K(\tau)) = \int \frac{d^{2\omega}x\sqrt{g}}{(4\pi\tau)^\omega}\left(1 + \frac{\tau}{6}R + \tau^2 R f_3(-\tau\Box^g)R + \tau^2 R_{\mu\nu}f_2(-\tau\Box^g)R^{\mu\nu} + \mathcal{O}(R^3, \cdots)\right), \tag{4.4.42}$$

with the functions

$$f_1(x) \equiv \int_0^1 e^{-xu(1-u)}du, \tag{4.4.43}$$

$$f_2(x) \equiv \frac{f_1(x) - 1 + \frac{1}{6}x}{x^2}, \tag{4.4.44}$$

$$f_3(x) \equiv \frac{f_1(x)}{32} + \frac{f_1(x) - 1}{8x} - \frac{f_4(x)}{8}. \tag{4.4.45}$$

At the end of the day, one is interested in the integral $\int_0^\infty d\tau Tr(K(\tau))$. The τ^ω in the denominator of (4.4.42) introduces divergences from the lower integration limit. In order to regularize the integral, one must expand the integral in powers of τ. These divergences at $\tau \to 0$ are actually the UV divergences of the theory, since $\tau \sim x^2$ where x is a length scale. The $f_i(\tau\Box^g)$ functions contain all orders of τ, by

expanding it to order zero (notice that we already have a τ^2 factor on those terms) it defines the *Seeley-DeWitt expansion*

$$Tr(K(\tau)) = \int \frac{d^{2\omega}x\sqrt{g}}{(4\pi\tau)^{\omega}} \left(1 + a_1(x)\tau + a_2(x)\tau^2 + \mathcal{O}(\tau^3)\right), \tag{4.4.46}$$

where the $a_i(x)$ are the *Seeley-DeWitt coefficients*. They are local functions of the curvature tensors given by

$$a_1(x) = \frac{1}{6}R(x), \tag{4.4.47}$$

$$a_2(x) = \frac{1}{60}\left(\frac{1}{2}R^2(x) + R^{\mu\nu}R_{\mu\nu}(x)\right). \tag{4.4.48}$$

Notice that the Seeley-DeWitt expansion is only well-behaved for small τ, so it will naturally diverge when the integral over τ is computed. The IR divergences coming from the integration as $\tau \to \infty$ are not physical, because they are out of the well-behaved regime of the expansion. As this expansion contemplates the UV divergences of the theory, it is perfect for the renormalization of the effective action. The local functions $a_i(x)$ contain the local effects of vacuum polarization, since they probe very small distances $\tau \to 0$.

4.5 Renormalization and Conformal Anomalies

In the previous subsections we have been analyzing the construction of an effective action for the massless scalar minimally coupled to a gravitational background. We have managed to solve the heat kernel operator for the kinetic term in curved space in a perturbative fashion. It is now a matter of computing the $\zeta(s)$ function through (4.4.14) and analytically extending the result to Lorentzian signature. As one should expect, we obtained a UV divergent result, and (4.4.46) organizes the divergences in polynomial order for the $Tr(K(\tau))$.

The Seeley-DeWitt expansion was derived by expanding the non-local functions $f_i(\tau\Box^g)$ in orders of τ and keeping divergent terms only. This means that such an expansion contemplates the "infinite part" of the action, namely the UV divergent sector that must be renormalized using the coupling constants of the theory. Had we not expanded the f_i functions, it would also lead to the finite pieces of the action, which will shortly be addressed.

The ζ-function for 4 space-time dimensions ($\omega = 2$) reads

$$\begin{aligned}\zeta(s) = \frac{1}{\Gamma(s)}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\left(\int_0^{\tau_1} d\tau\tau^{s-3} + \frac{R}{6}\int_0^{\tau_1} d\tau\tau^{s-2}\right.\\ \left. + \frac{1}{60}\left(\frac{R^2}{2} + R^{\mu\nu}R_{\mu\nu}\right)\int_0^{\tau_1} d\tau\tau^{s-1}\right),\end{aligned} \tag{4.5.1}$$

where the factor factor $\tau^{-\omega}$ on (4.4.42) plays a crucial role of defining the highest order correction necessary to account for all UV divergences, in this case we only

need to include up to the a_2 term, since the limit $s \to 0$ will be taken.[13] The IR cutoff τ_1 is not physical, it is present only to assure that the *Seeley-DeWitt* expansion is well defined.

The effective action reads

$$\Gamma_E[g_{\mu\nu}] = -\frac{1}{2}\frac{d}{ds}\zeta(s)\Big|_{s=0}, \tag{4.5.2}$$

for small s

$$\frac{1}{\Gamma(s)} \sim s + O(s^2). \tag{4.5.3}$$

The derivative in s on (4.5.2) picks up the linear contribution of $\Gamma(s)^{-1}$ since we take the limit $s \to 0$. Introducing a UV cutoff τ_0, and integrating (4.5.1) in τ we have

$$\Gamma_E[g_{\mu\nu}] = -\frac{1}{32\pi^2}\int d^4x\sqrt{g}\left(\frac{1}{2\tau_0^2} + \frac{1}{6\tau_0}R + \frac{1}{60}\left(\frac{R^2}{2} + R^{\mu\nu}R_{\mu\nu}\right)\ln|\tau_0|\right). \tag{4.5.4}$$

Rotating back to Lorenztian time (4.2.43), we have: $d^4x \to id^4x$, $\sqrt{g} \to \sqrt{-g}$ and $R \to -R$, yielding

$$\Gamma_L[g_{\mu\nu}] = \frac{1}{32\pi^2}\int d^4x\sqrt{-g}\left(\frac{1}{2\tau_0^2} - \frac{1}{6\tau_0}R + \frac{1}{60}\left(\frac{R^2}{2} + R^{\mu\nu}R_{\mu\nu}\right)\ln|\tau_0|\right). \tag{4.5.5}$$

The effective action for the total system $S_{\rm tot}[g_{\mu\nu}]$ consists of the integrated modes of the scalar field $\Gamma_L[g_{\mu\nu}]$ and the action for the background metric S_B^g, the subscript "B" standing for "Bare". It is very interesting to notice that Einstein gravity is not enough to renormalize the scalar fields, since we need counterterms for the R^2 part of the divergence. In order to renormalize our theory, we must in principle begin with Einstein gravity plus higher derivatives corrections.

Consider the action S_B^g to be

$$S_B^g[g_{\mu\nu}] = \int d^4x\sqrt{-g}\left(-\frac{R + 2\Lambda_B}{16\pi G_B} + \alpha_B\left[\frac{R^2}{120} + \frac{R^{\mu\nu}R_{\mu\nu}}{60}\right]\right), \tag{4.5.6}$$

then

$$\begin{aligned} S_{\rm tot}[g_{\mu\nu}] &= S_B^g[g_{\mu\nu}] + \Gamma_L[g_{\mu\nu}] \\ &= \int d^4x\sqrt{-g}\left(\left[-\frac{1}{2\tau_0^2} - \frac{\Lambda_B}{8\pi G_B}\right] + R\left[-\frac{1}{192\tau_0} - \frac{1}{16\pi G_B}\right] + \right. \\ &\qquad \left. \left[\alpha_B - \frac{\ln|\tau_0|}{32\pi^2}\right]\left[\frac{R^2}{120} + \frac{R^{\mu\nu}R_{\mu\nu}}{60}\right]\right). \end{aligned} \tag{4.5.7}$$

We define the renormalized coupling constants

$$\frac{\Lambda}{8\pi G} = -\frac{1}{2\tau_0^2} - \frac{\Lambda_B}{8\pi G_B}, \tag{4.5.8}$$

$$\frac{1}{16\pi G} = -\frac{1}{192\tau_0} - \frac{1}{16\pi G_B}, \tag{4.5.9}$$

$$\alpha = \alpha_B - \frac{\ln|\tau_0|}{32\pi^2}. \tag{4.5.10}$$

[13] Had we continued the expansion, we would end up with terms $\sim \tau^{s+n}$ with $n \geq 0$, which contribute finitely.

In order to calculate the finite terms of the effective action, we consider (4.4.14) in two dimensions ($\omega = 1$), and later generalize the result to $\omega = 2$. First notice that in two dimensions we have

$$R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R. \tag{4.5.11}$$

The ζ function reads

$$\zeta(s) = \frac{1}{4\pi\Gamma(s)} \int d^2x\sqrt{g} \left[\int_0^\infty d\tau \left(\underbrace{\frac{1}{\tau^{2-s}} + \frac{R}{6\tau^s}}_{\text{UV divergent}} + \underbrace{\tau^s R f_3(-\tau\Box^g)R + \tau^s R_{\mu\nu} f_4(-\tau\Box^g)R^{\mu\nu}}_{\text{Finite terms}} \right) \right].$$

We have already absorbed the contribution of the UV divergent in the bare couplings, so we don't need to consider them. With (4.5.11), we can write Γ_E as

$$\begin{aligned}
\Gamma_E &= -\frac{1}{2}\frac{d}{ds}\zeta(s)\Big|_{s=0} \\
&= \frac{1}{8\pi} \int d^2x\sqrt{g} \left(\int_0^\infty d\tau R f_3(-\tau\Box^g)R + R_{\mu\nu} f_4(-\tau\Box^g)R^{\mu\nu} \right) \\
&= \frac{1}{8\pi} \int d^2x\sqrt{g} \left(R \underbrace{\left(\int_0^\infty d\tau f_3(-\tau\Box^g) + \frac{1}{2} f_4(-\tau\Box^g) \right)}_{\hat{\mathcal{K}}} R \right).
\end{aligned}$$

The $\hat{\mathcal{K}}$ operator is defined through the Taylor expansion of the functions $f_2(x)$ and $f_3(x)$, with its argument replaced by $-\tau\Box^g$. We can write the expression in terms of the propagator $(\Box^g)^{-1}$ by defining $\xi = -\tau\Box^g$ and performing the change of variables

$$\begin{aligned}
\Gamma_E[g_{\mu\nu}] &= \frac{1}{8\pi} \int d^2x\sqrt{g} \left(R \left(\int_0^\infty d\tau f_3(-\tau\Box^g) + \frac{1}{2} f_4(-\tau\Box^g) \right) R \right) \\
&= \frac{1}{8\pi} \int d^2x\sqrt{g} \left(R(\Box^g)^{-1} \left(\int_0^\infty d\xi f_3(\xi) + \frac{1}{2} f_4(\xi) \right) R \right) \\
&= \frac{1}{96\pi} \int d^2x\sqrt{g} R(\Box^g)^{-1} R,
\end{aligned} \tag{4.5.12}$$

where we have used the result

$$\int_0^\infty d\xi \left(f_3(\xi) + \frac{1}{2} f_4(\xi) \right) = \frac{1}{12}. \tag{4.5.13}$$

In the position representation, the effective action reads

$$\Gamma_E[g_{\mu\nu}] = \frac{1}{96\pi} \int d^2x d^2y \sqrt{g(x)} R(x) G_E(x,y) R(y). \tag{4.5.14}$$

Analytically continuing to Lorentzian time, we finally reach the effective action

$$\Gamma_L[g_{\mu\nu}] = \frac{1}{96\pi}\int d^2x d^2y \sqrt{-g(x)}R(x)G_F(x,y)R(y). \tag{4.5.15}$$

With the effective action in hands, it is a matter of taking functional derivatives to obtain n-point functions of the energy-momentum tensor (4.3.6), which means we are finally in position to evaluate the *trace anomaly* suffered by a scalar field in a gravitational background .

In order to explicitly obtain the trace of the energy-momentum tensor, we may proceed with the standard trick of inducing a Weyl transformation in the metric to produce a variation on the action proportional to T^μ_μ. As we are dealing with the effective action, it will naturally give rise to the expectation value of the trace.

From (4.3.6), we have

$$\begin{aligned}\frac{\delta\Gamma_L[g_{\mu\nu}]}{\delta g^{\mu\nu}} &= \frac{\sqrt{-g}}{2}\langle T_{\mu\nu}\rangle \\ \Longrightarrow \Gamma_L[g_{\mu\nu}+\delta g_{\mu\nu}] - \Gamma_L[g_{\mu\nu}] &= \frac{1}{2}\int d^2x\sqrt{-g}\langle T_{\mu\nu}\rangle \delta g^{\mu\nu}.\end{aligned} \tag{4.5.16}$$

Inducing a metric variation through a Weyl transformation $g'_{\mu\nu} = \Omega^2 g_{\mu\nu}$, which takes the infinitesimal form $g'_{\mu\nu} = (1+2\delta\Omega)g_{\mu\nu} + O(\delta\Omega^2)$, we may write $\delta g_{\mu\nu} = 2\delta\Omega g_{\mu\nu}$, and (4.5.16) yields

$$\Gamma_L[g_{\mu\nu}+\delta g_{\mu\nu}]\Big|_{\delta g_{\mu\nu}=2\delta\Omega g_{\mu\nu}} - \Gamma_L[g_{\mu\nu}] = -\int d^2x\sqrt{-g}\langle T^\mu_\mu\rangle\delta\Omega(x). \tag{4.5.17}$$

The problem reduces to the calculation of the effective action $\Gamma_L[g_{\mu\nu}+\delta g_{\mu\nu}]$. Recall that the core object defining the effective action is $\hat{O}$, defined in (4.4.18), specialized to the case of a massless minimally coupled field ($V(x)=0$). This operator transforms under the Weyl scaling as

$$\begin{aligned}\langle x|\,\hat{O}_{g'}\,|x'\rangle &= -g'^{-\frac{1}{4}}\partial_\nu\left(g'^{\mu\nu}g'^{\frac{1}{2}}\partial_\mu(g'^{-\frac{1}{4}}\delta(x-x'))\right) \\ &= -\Omega^{-1}g^{-\frac{1}{4}}\partial_\nu\left(\Omega^{-2}g^{\mu\nu}\Omega^2 g^{\frac{1}{2}}\partial_\mu(\Omega^{-1}g^{-\frac{1}{4}}\delta(x-x'))\right) \\ &= \Omega^{-1}\hat{O}_g\Omega^{-1} \\ &= \hat{O}_g - \delta\Omega\hat{O}_g - \hat{O}_g\delta\Omega + O(\delta\Omega^2),\end{aligned} \tag{4.5.18}$$

where we have dropped the matrix element notation because all operators are diagonal. $\zeta_{\hat{O}}(s)$ transforms, as consequence of (4.5.18), as

$$\zeta_{\hat{O}_{g'}}(s) = Tr\left[\left(\hat{O}_g - \delta\Omega\hat{O}_g - \hat{O}_g\delta\Omega\right)^{-s}\right] \tag{4.5.19}$$

$$= \zeta_{\hat{O}_g}(s) + \underbrace{2sTr\left(\delta\Omega\hat{O}_g^{-s}\right)}_{\delta\zeta}. \tag{4.5.20}$$

Relating $\delta\zeta$ with the matrix elements, one evaluates the trace as

$$\delta\zeta = 2s\int d^2x\delta\Omega(x)\,\langle x|\,\hat{O}^{-s}\,|x\rangle\,. \tag{4.5.21}$$

In addition, we have shown that

$$\hat{O}^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \hat{K}(\tau) \tau^{s-1} \tag{4.5.22}$$

$$\implies \langle x| \hat{O}^{-s} |x\rangle = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \, \langle x| \hat{K}(\tau) |x\rangle \, \tau^{s-1}. \tag{4.5.23}$$

With the Seeley-DeWitt expansion (4.4.46) and assuming $\omega = 1$, the variation $\delta\zeta$ reads

$$\delta\zeta = 2s \frac{1}{4\pi\Gamma(s)} \int d^2x \int d\tau \sqrt{-g} \left(1 + a_1(x)\tau + a_2(x)\tau^2 + O(\tau^3)\right) \delta\Omega(x) \tau^{s-2}. \tag{4.5.24}$$

We know that the expansion diverges in the upper limit of integration in τ because it is only well defined at small values of τ, so we foresee the need to introduce an IR regulator further on. The extra piece $\delta\zeta$ contributes to the effective action by adding a piece $\delta\Gamma_L$

$$\begin{aligned} \Gamma_L[g_{\mu\nu} + \delta g_{\mu\nu}] &= -\frac{1}{2} \frac{d\zeta_{g+\delta g}(s)}{ds}\Big|_{s=0} \\ &= -\frac{1}{2} \frac{d}{ds} \left(\zeta_g(s) + \delta\zeta(s)\right) \Big|_{s=0} \\ &= \Gamma_L[g_{\mu\nu}] \underbrace{-\frac{1}{2} \frac{d}{ds} \delta\zeta(s)\Big|_{s=0}}_{\delta\Gamma_L}. \end{aligned} \tag{4.5.25}$$

Let's develop (4.5.24) by performing the $d\tau$ integration and collecting orders of τ. One may write the pieces as

$$\frac{2s}{4\pi} \int d^2x \sqrt{-g} \delta\Omega(x) \left\{ \int_0^\infty d\tau \frac{1}{\Gamma(s)} \left(1 + a_1(x)\tau + a_2(x)\tau^2\right) \tau^{s-2} \exp(-\alpha\tau) \right\}, \tag{4.5.26}$$

where the cutoff $\exp(-\alpha\tau)$ was introduced. Expression (4.5.26) is to be computed before setting $\alpha = 0$. Recall the definition (4.4.13) to calculate the contributions of the coefficients a_i separately. The $a_0 = 1$ piece is

$$\begin{aligned} \frac{1}{\Gamma(s)} \int_0^\infty d\tau \left(\tau^{s-2} \exp(-\alpha\tau)\right) &= \frac{1}{\alpha^{s-1}} \frac{1}{\Gamma(s)} \int_0^\infty d\tau' \tau'^{s-2} \exp(-\tau') \\ &= \frac{1}{\alpha^{s-1}} \frac{1}{\Gamma(s)} \Gamma(s-1) \qquad (4.5.27) \\ &= \frac{\alpha^{1-s}}{s-1}. \qquad (4.5.28) \end{aligned}$$

Repeating the calculation for $a_1(x)$ and $a_2(x)$ means substituting $s \to s+1$ and $s \to s+2$ in (4.5.27) respectively. So we evaluate the contributions as

$$\frac{1}{\Gamma(s)} \int_0^\infty d\tau a_1(x) \left(\tau^{s-1} \exp(-\alpha\tau)\right) = \alpha^{-s} a_1(x), \tag{4.5.29}$$

and

$$\frac{1}{\Gamma(s)}\int_0^\infty d\tau a_2(x)\left(\tau^s\exp(-\alpha\tau)\right)=\frac{s}{\alpha^{s+1}}a_2(x). \tag{4.5.30}$$

Plugging in (4.5.26), it yields

$$\delta\zeta=\frac{1}{2\pi}\int d^2x\sqrt{-g}\delta\Omega(x)\left\{\underbrace{\frac{s\alpha^{1-s}}{s-1}}_{I}+\underbrace{s\alpha^{-s}a_1(x)}_{II}+\underbrace{\frac{s^2}{\alpha^{s+1}}a_2(x)}_{III}\right\}. \tag{4.5.31}$$

We just need to evaluate (4.5.25): the piece I does not contribute because there is an α dependence after we take the limit $\frac{d}{ds}|_{s=0}$. The piece III also does not survive due to the quadratic contribution in s. The only surviving piece is II, yielding

$$\delta\Gamma_L=-\frac{1}{4\pi}\int d^2x\sqrt{-g}\delta\Omega(x)a_1(x), \tag{4.5.32}$$

comparing with (4.5.17) immediately gives us

$$\langle T^\mu_\mu\rangle=\frac{1}{4\pi}a_1(x). \tag{4.5.33}$$

Expanding the coefficient we reach the trace anomaly in two dimensions

$$\langle T^\mu_\mu\rangle=\frac{1}{24\pi}R(x). \tag{4.5.34}$$

The generalization for $\omega\neq 1$ is almost trivial. For $\omega=2$ we begin by noticing that the expression (4.5.24) picks up a global factor of τ^{-2} coming from the heat kernel expansion factor $\frac{1}{(4\pi\tau)^\omega}$. It reads

$$\delta\zeta=2s\frac{1}{16\pi^2\Gamma(s)}\int d^4x\int d\tau\sqrt{-g}\left(1+a_1(x)\tau+a_2(x)\tau^2+O(\tau^3)\right)\delta\Omega(x)\tau^{s-3}. \tag{4.5.35}$$

The regulated expression is

$$\delta\zeta=2s\frac{1}{16\pi^2\Gamma(s)}\int d^4x\sqrt{-g}\int d\tau\left(1+a_1(x)\tau+a_2(x)\tau^2+O(\tau^3)\right)\times\delta\Omega(x)\tau^{s-3}\exp(-\alpha\tau). \tag{4.5.36}$$

The $a_0=1$ contribution remains zero: substituting $s\to s-1$ in (4.5.27) yields

$$\delta\zeta=2s\frac{1}{16\pi^2\Gamma(s)}\int d^4x\sqrt{-g}\int d\tau\delta\Omega(x)\tau^{s-3}\exp(-\alpha\tau) \tag{4.5.37}$$

$$\sim\alpha^{2-s}\frac{s}{(s-1)(s-2)}. \tag{4.5.38}$$

It is a regular polynomial in $s\to 0$ multiplied by an α dependence, so it does not contribute. The case of a_1 in $\omega=2$ is the same of a_0 in $\omega=1$, for the τ dependence

is τ^{s-2}, so it also vanishes. The only non-vanishing contribution comes from $a_2(x)$, so the *trace anomaly in four dimensions* is given by

$$\langle T^{\mu}_{\mu} \rangle = \frac{1}{16\pi^2} a_2(x). \tag{4.5.39}$$

Expanding the $a_2(x)$ coefficient yields

$$\langle T^{\mu}_{\mu} \rangle = \frac{1}{60} \frac{1}{16\pi^2} \left(\frac{1}{2} R^2(x) + R^{\mu\nu} R_{\mu\nu}(x) \right). \tag{4.5.40}$$

The generic case of arbitrary ω is given by the Seeley-DeWitt coefficient a_ω. For $\omega \geq 2$ one needs to solve the heat kernel equations up to order $\tau \geq 2$. No trace anomalies are present in odd space-time dimensions, as it can be seen from (4.5.26), since the regularization factor $\exp(-\alpha\tau)$ will always introduce a dependence in α which makes the trace vanish.[14]

The anomaly can be understood as a symmetry breaking by the quantization of the theory, meaning that a conformal theory at the classical level is led to a *non-scale invariant* quantum theory. As T^{μ}_{μ} is the generator of scale transformations, the non-vanishing of the trace leads to a non-conformal quantum theory.

More formally, during the quantization procedure we chose to integrate over the eigenvalues of the kinetic operator $\hat{F}$, substituting the measure for $\mathcal{D}[c_n]$, which is a generally covariant measure, since it is unaffected by coordinate transformations. However, $\hat{F}$ is not invariant under Weyl scalings, so its eigenvalues change and as consequence the measure $\mathcal{D}[c_n]$ is not invariant.[15] We conclude that it is a case of anomaly coming from the impossibility of defining a generally covariant and scale invariant measure for the path integration.

The anomaly can also be understood from a more physical point of view: we have first defined a field theory which is conformal at the classical level. Supposing that the conformal symmetries remain untouched during the quantization process in flat space, we would end up with a conformal quantum field theory, meaning $\langle T^{\mu}_{\mu} \rangle = 0$. The main feature of scale invariant theories is the absence of natural length scales in the system, which is perfectly compatible with flat space quantization since no natural scales exist in Minkowski space. Nonetheless, upon the substitution of the flat background metric for a generic curved manifold metric $\eta_{\mu\nu} \to g_{\mu\nu}$, a natural length scale given by the curvature scalar $R(x)$ is introduced in the system. As one may expect, the introduction of such scale smoothly breaks the conformal symmetry.

Back when the action functionals were defined, there could be in principle an overall multiplication constant A, since the equations of motion would still be the same. The quantization process is also invariant under $S_{tot} \to AS_{tot}$, and this overall constant is inherited by the effective action $\Gamma_L[g_{\mu\nu}] \to A\Gamma_L[g_{\mu\nu}]$. Equation (4.3.6) gives us the energy-momentum tensor as functional derivatives of $\Gamma_L[g_{\mu\nu}]$, so we can recover the two-point function

$$A \frac{\delta \Gamma_L[g_{\mu\nu}]}{\delta g^{\alpha\beta} \delta g^{\lambda\sigma}} \sim \langle T_{\alpha\beta} T_{\lambda\sigma} \rangle. \tag{4.5.41}$$

[14] See chapter 6 for the complete discussion.

[15] Since it accounts for an integration of functions spanned by the eigenfunction space of $\hat{F}$.

By restricting the metric to be flat $g_{\mu\nu} \to \eta_{\mu\nu}$ one expects to recover a CFT because the Ricci scalar vanishes, thus making the trace anomaly vanish. For CFTs, the two-point function of the energy-momentum tensor is proportional to the central charge in leading order (3.4.74)

$$\langle T_{\mu\nu} T_{\lambda\sigma} \rangle \sim C, \tag{4.5.42}$$

which means that by taking the flat space limit, we should have

$$A \frac{\delta \Gamma_L[g_{\mu\nu}]}{\delta g^{\alpha\beta} \delta g^{\lambda\sigma}} \Big|_{g_{\mu\nu}=\eta_{\mu\nu}} \sim \langle T_{\alpha\beta} T_{\lambda\sigma} \rangle \sim C \tag{4.5.43}$$

The proportionality constant A is then related to the central charge C (in a 2-d CFT). A more lengthy calculation could show that in fact they should be the same in order for the flat limit to be compatible with a CFT. This sets the trace anomalies to be *proportional* to the coefficients $a_1(x)$ and $a_2(x)$, and the constants of proportionality are defined to be the central charge of the CFT limit.[16]These constants are the candidates for ordered parameters under RG-flows. To summarize, one may rewrite the trace anomalies as

$$\langle T^{\mu}_{\mu} \rangle = \frac{C}{24\pi} R(x), \tag{4.5.44}$$

and the four-dimensional case, with central charge A

$$\langle T^{\mu}_{\mu} \rangle = A \frac{1}{16\pi^2} \frac{1}{60} \left(\frac{1}{2} R^2(x) + R^{\mu\nu} R_{\mu\nu}(x) \right). \tag{4.5.45}$$

[16]In 2-D the central charge coincides with the one of the extension of the DeWitt algebra.

Chapter 5

The C-Theorem

5.1 Introduction

This chapter is devoted to the C-theorem, mainly presented in [17, 12]. Conformal invariance at a quantum level implies that the trace of the energy-momentum tensor vanishes as an operator

$$\langle T^{\mu}_{\mu} \rangle = 0. \tag{5.1.1}$$

However, when a given CFT is placed in a curved background, a natural length scale is imposed by the curvature of the background, and conformal symmetry is broken in a controlled fashion. This results in a violation of equation (5.1.1), the trace of $T_{\mu\nu}$ acquires a non-vanishing vacuum expectation value, defining the *trace anomaly*

$$\langle T^{\mu}_{\mu} \rangle = \sum_i \Big(c_i I_i - (-1)^{\frac{d}{2}} a_d E_d \Big) \qquad \text{for even d,} \tag{5.1.2}$$

where I_i are invariants built from the Weyl tensor of the considered manifold in which one places the CFT, and E_d is its Euler density, a_d and c_i are real constants. In two space-time dimensions ($d = 2$) all Weyl invariants vanish, and the Euler density is proportional to the Ricci scalar

$$\langle T^{\mu}_{\mu}(x) \rangle = \frac{C}{24\pi} R(x), \tag{5.1.3}$$

the constant $c = a_2$ is the *central charge* of the CFT.

Consider connecting two CFTs by an RG-flow in the space of couplings g_i, meaning that there exists a curve $g_i(\mu)$ parameterized by an energy scale μ which has as endpoints a CFT in the ultraviolet limit, call it CFT_{UV}, and another CFT in a lower energy scale, call it CFT_{IR}. The C-theorem states that there exists a function defined in the curve $C\{g_i\}$, called *C-function*, that is non-increasing under the RG trajectories, and it is proportional to the central charge C at the fixed points of the flow. The existence of such function constraints the central charge of CFT_{UV} to be greater or equal to the central charge of CFT_{IR},

$$C_{UV} \geq C_{IR}. \tag{5.1.4}$$

It is impossible to reach a CFT with greater central charge than the one we started with via an RG flow, introducing the concept of irreversibility of RG flows. For unitary CFTs, it also introduces the idea that the central charges are precise measures of the degrees of freedom of the theory.

Parameters that satisfy an order relation (5.1.4) under RG trajectories are called ordered parameters. The C-theorem proves the two-dimensional case of Cardy's conjecture, which states that for d-dimensions, the coefficient a_d of the trace anomaly is an ordered parameter. The first proof of the C-theorem was discovered by Zamolodchikov [17] in 1986, using pure operatorial strategy. Later in 2006, Casini and Huerta [12] proved the C-theorem using entanglement entropy as a fundamental quantity for building the C-function, providing a completely different view on the matter of ordered parameters.

5.2 The C-theorem

The proof of the C-theorem in [17] relies on renormalizability conditions, reflection positivity, and preservation of the Poincaré space-time symmetries under an RG flow. We will first prove the theorem by a set of differential equations generated by the dilatation operator, which is John Cardy's strategy. After that we present the proof as Zamolodchikov originally proposed, by using the Callan-Symanzik equation for the two-point functions of two-dimensional quantum fields.

5.2.1 RG setup of the C-theorem

Consider the space of QFTs as parameterized by the coupling constants $\{g_i\}$ of the operator expansion of the action functional $S[\{g_i\}, \Lambda]$,[1] where Λ is a UV cutoff. The renormalizability conditions assure that it is possible to achieve effective actions $S_{\text{eff}}[\{g_i'\}, \Lambda']$ by an RG flow, which has the effect of mapping $\{g_i\} \to \{g_i'\}$ and lowering the cutoff to $\Lambda' \leq \Lambda$. The observables described by S_{eff} and S should agree whenever the cutoff is respected, and the two-point functions should be well-behaved and well-defined as we change momentum scales, or equivalently length scales.

To set up the stage, consider starting with a fixed point $\{g^*_{UV}\}$ in parameter space, which is described by a CFT_{UV}. Consider a perturbation by a relevant operator to the action that describes CFT_{UV},

$$S = S_{CFT_{UV}} + \int d^2x \mathcal{O}(x)_{\text{Relevant}}. \tag{5.2.1}$$

The relevant perturbations give a non-zero value to the β functions, which starts an RG trajectory in parameter space, defining a curve $g_i(\mu)$, where μ is a momentum scale parameterizing the flow.

[1] See (5.2.43).

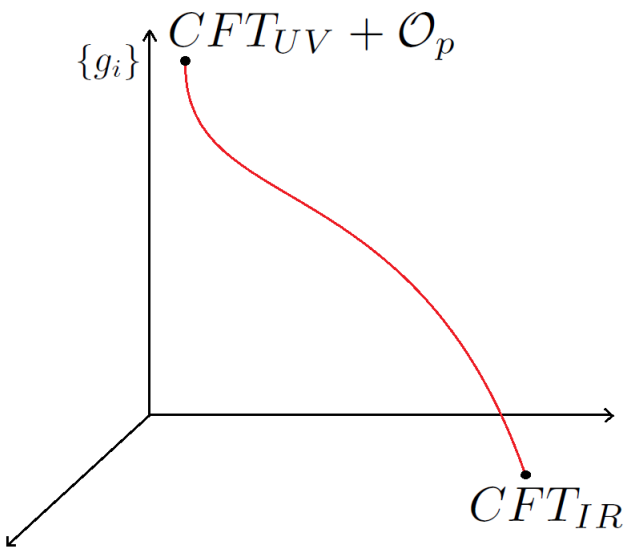

Figure 5.1: An RG flow $\{g_i(\mu)\}$, represented by a red curve, connecting a CFT_{UV} fixed point, perturbed by a relevant operator $\mathcal{O}_p$, to a lower energy scale CFT_{IR}. The axes are the couplings $\{g_i\}$.

Suppose that the couplings of the perturbed action could be fine tuned in a way that the curve $g_i(\mu)$ has another fixed point $\{g^*_{\text{IR}}\}$, defining a lower energy CFT_{IR}. This supposition is a question of imposing the right initial conditions on a set of differential equations generated by the β functions. The evolution of this system is nothing more than the local integration of the β functions generating the flow. Granted the existence of such initial conditions, the flow takes you from a CFT_{UV} to a CFT_{IR}.

The theorem states that there exists a function of the couplings $C(\{g_i\}) \geq 0$, such that:

- It is non-increasing under the RG flow,

$$\beta^i \frac{\partial}{\partial g^i} C(\{g_i(\mu)\}) \leq 0. \tag{5.2.2}$$

  The β functions are the generators of the vector field of the flows in parameter space. This equation is the RG flow acting on the C-function.

- At the RG fixed points, conformal invariance is restored, so the β functions vanish and $C\{g^*_i\}$ is stationary:

$$\frac{\partial C(g^*_i)}{\partial g^i} = 0. \tag{5.2.3}$$

- The C-function takes the value of the *central charge* of the $\{g^*_i\}$ CFTs at the fixed points, where it is stationary:

$$C(\{g^*_i\}) = \tilde{C}\{g^*_i\}. \tag{5.2.4}$$

Recall the central charge for a generic two-dimensional CFT ($\{g_i^*\}$) extends the algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{\tilde{C}\{g_i^*\}}{12}(n^3 - n)\delta_{n+m,0}. \tag{5.2.5}$$

The notation $\tilde{C}\{g_i^*\}$ means that central charges are different for different CFTs, that's why it has a dependence on the coordinates of the fixed points.

5.2.2 Cardy's Proof

The proof of the C-theorem starts with defining complex coordinates $(z, \bar{z}) = (x^1 + ix^2, x^1 - ix^2)$. The Euclidean two-dimensional QFT symmetries consist of translations and rotations. In the two-dimensional case, the rotation group is implemented by $SO(2) \simeq U(1)$. For such symmetries the energy-momentum tensor is symmetric and conserved

$$\partial^\mu T_{\mu\nu} = 0 \qquad \text{and} \qquad T_{\mu\nu} = T_{\nu\mu}. \tag{5.2.6}$$

We define the following notation:[2]

$$T = T_{zz} \qquad \Theta = T_{\bar{z}\bar{z}}. \tag{5.2.7}$$

The conservation equation in complex coordinates reads

$$\bar{\partial}T + \frac{1}{4}\partial\Theta = 0. \tag{5.2.8}$$

Define the following two-point functions

$$\langle T(z,\bar{z})T(0)\rangle,\ \langle T(z,\bar{z})\Theta(0)\rangle,\ \langle \Theta(z,\bar{z})\Theta(0)\rangle. \tag{5.2.9}$$

Imposing translation invariance, along with the behaviour of $T_{\mu\nu}$ under translations, constraints the functional dependence of each correlator. First, we notice that the action of the rotations $U(1)$ is

$$z' = e^{i\phi}z \qquad \bar{z}' = e^{-i\phi}\bar{z}. \tag{5.2.10}$$

Translation invariance for a generic two-point function imposes

$$\langle \Phi(z,\bar{z})T(0)\rangle = \frac{F(z\bar{z})}{z^a \bar{z}^b}, \tag{5.2.11}$$

where a, b must be fixed by the rotation behaviour of the LHS.

A rank 2 tensor transforms as

$$T_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}T_{\alpha\beta}(x). \tag{5.2.12}$$

This translates in complex coordinates to

$$T_{z'z'} = e^{-2i\phi}T_{zz} \text{ and } T_{z'\bar{z}'} = e^{-i\phi}e^{i\phi}T_{z\bar{z}}, \tag{5.2.13}$$

[2]This notation is different from the one in chapter 3.

leading to

$$\langle T_{z'z'}(z',\bar{z}')T(0)\rangle = e^{-4i\phi}\langle T_{zz}(z,\bar{z})T(0)\rangle. \tag{5.2.14}$$

For each z index down, the functional behaviour picks up a z^{-1} dependence, and for each $\bar{z}$ it picks up a $\bar{z}^{-1}$. This behaviour is expected, since $T_{\mu\nu}$ is a *spin 2* object, and its trace is a *spin 0* object. So we write

$$\langle T_{zz}(z,\bar{z})T(0)\rangle = \frac{F(z\bar{z})}{z^4}. \tag{5.2.15}$$

Noticing that the trace part $T^{\mu}_{\mu} = \Theta$ can be written in complex coordinates as

$$T^{\mu}_{\mu} = T^{z}_{z} + T^{\bar{z}}_{\bar{z}} = 2(T_{z\bar{z}} + T_{\bar{z}z}) = 4T_{z\bar{z}}. \tag{5.2.16}$$

In the last step the symmetric property is used. Imposing the same constraints to the other two correlators yields

$$\langle \Theta(z,\bar{z})T_{zz}(0)\rangle = \langle T_{zz}(z,\bar{z})\Theta(0)\rangle = \frac{G(z\bar{z})}{z^3\bar{z}}, \tag{5.2.17}$$

$$\langle \Theta(z,\bar{z})\Theta(0)\rangle = \frac{H(z\bar{z})}{z^2\bar{z}^2}. \tag{5.2.18}$$

The equation of conservation for $T_{\mu\nu}$ generates a set of two differential equations for the functions $F(z\bar{z})$, $G(z\bar{z})$, and $H(z\bar{z})$ as follows: applying $\partial_{\bar{z}}$ in (5.2.15) yields

$$\langle \partial_{\bar{z}} T_{zz}(z,\bar{z})T(0)\rangle = \frac{F'(z\bar{z})}{z^3}, \tag{5.2.19}$$

where $F'(z\bar{z}) = \frac{dF(z\bar{z})}{d(z\bar{z})}$. Applying ∂_z to (5.2.17), one reaches

$$\langle \partial_z \Theta(z,\bar{z})T_{zz}(0)\rangle = \frac{-3G(z\bar{z})}{z^4\bar{z}} + \frac{G'(z\bar{z})}{z^3}. \tag{5.2.20}$$

Using the conservation equation in complex coordinates, we may write

$$\frac{F'(z\bar{z})}{z^3} + \frac{1}{4}\left(\frac{-3G(z\bar{z})}{z^4\bar{z}} + \frac{G'(z\bar{z})}{z^3}\right) = 0. \tag{5.2.21}$$

Defining $\dot{F} = F'(z\bar{z})z\bar{z}$, we finally reach

$$\dot{F} + \frac{1}{4}\Big(\dot{G} - 3G\Big) = 0. \tag{5.2.22}$$

The second differential equation is achieved by applying $\partial_{\bar{z}}$ to (5.2.17)

$$\langle \partial_{\bar{z}} T_{zz}(z,\bar{z})\Theta(0)\rangle = \frac{G'(z\bar{z})}{z^2\bar{z}} - \frac{G(z\bar{z})}{z^3\bar{z}^2}. \tag{5.2.23}$$

Applying ∂_z to (5.2.18) yields

$$\langle \partial_z \Theta(z,\bar{z})\Theta(0)\rangle = \frac{H'(z\bar{z})}{z^2\bar{z}} - \frac{2H(z\bar{z})}{z^3\bar{z}^2}. \tag{5.2.24}$$

Plugging the last two equations in the conservation equation generates

$$\frac{G'(z\bar{z})}{z^2\bar{z}} - \frac{G(z\bar{z})}{z^3\bar{z}^2} + \frac{1}{4}\left(\frac{H'(z\bar{z})}{z^2\bar{z}} - \frac{2H(z\bar{z})}{z^3\bar{z}^2}\right) = 0. \tag{5.2.25}$$

Again defining $\dot{G} = G'(z\bar{z})z\bar{z}$ and $\dot{H} = H'(z\bar{z})z\bar{z}$, we obtain

$$\dot{G} - G + \frac{1}{4}\Big(\dot{H} - 2H\Big) = 0. \tag{5.2.26}$$

To summarize: the rotations and translation properties of the two-point functions, along with conservation of a symmetric $T_{\mu\nu}$ make it possible to write a system of differential equations, given by

$$\dot{F} + \frac{1}{4}\Big(\dot{G} - 3G\Big) = 0, \tag{5.2.27}$$

$$\dot{G} - G + \frac{1}{4}\Big(\dot{H} - 2H\Big) = 0. \tag{5.2.28}$$

Defining the function

$$C(z\bar{z}) = 2F(z\bar{z}) - G(z\bar{z}) - \frac{3}{8}H(z\bar{z}), \tag{5.2.29}$$

and applying the operator $z\bar{z}\frac{d}{d(z\bar{z})}$, one reaches

$$\dot{C}(z\bar{z}) = 2\dot{F}(z\bar{z}) - \dot{G}(z\bar{z}) - \frac{3}{8}\dot{H}(z\bar{z}). \tag{5.2.30}$$

Isolating $\dot{H}$ in (5.2.28) and $\dot{F}$ in (5.2.17) gives us

$$\dot{C} = -\frac{3}{4}H(z\bar{z}). \tag{5.2.31}$$

As we are working in an Euclidean approach, the QFT can be seen as a QFT in $M^{1,1}$ after a Wick rotation. If one starts with a unitary QFT in $M^{1,1}$, the unitarity translates into a property of the Euclidean QFT called *reflection positivity*. Such property guarantees that the the two-point function $H \geq 0$. With such conditions, we have

$$\dot{C} \leq 0. \tag{5.2.32}$$

Renaming the variable $r = z\bar{z}$ makes explicit the fact that the operator $r\frac{d}{dr}$ is the position representation of the *dilatation operator*, so that the function $C(r)$ is non-increasing under dilatation.

A C-function is a function of the couplings $g_i(\mu)$ under the RG flow, but as the flow is parameterized by a momentum scale, the function is actually a $C(\mu)$. Such function must be non-increasing as it flows to the IR limit. The flow is the process of reducing the momentum scale μ, and the momentum scale μ is related to a length scale $r \sim \mu^{-1}$. As μ decreases under the RG flow, the length scale r increases. This way the C-function can be restated as a function which is non-increasing under the dilatation operator.

There are two other criteria to be met by our defined $C(r)$ so that one can claim it is a C-function. Namely, it has to be stationary at the fixed points, and at these points it must take the value of the central charge.

Once at the fixed points, scale invariance is restored. In two-dimensional space-times, it is known that scale invariance implies conformal invariance [28], and for conformal field theories the trace of the energy-momentum tensor vanishes

$$\Theta(z,\bar{z}) = 0. \tag{5.2.33}$$

As a direct consequence, the two-point functions containing the trace operator vanish identically

$$H(z\bar{z}) = G(z\bar{z}) = 0 \qquad \text{at the fixed points.} \tag{5.2.34}$$

The function $C(z\bar{z})$ gets reduced to

$$C(z\bar{z}) = 2F(z\bar{z}). \tag{5.2.35}$$

Conformal invariance proves to be powerful again, since we can use the OPE of the energy-momentum tensor with itself to compute

$$T(z,\bar{z})T(0) \sim \frac{\tilde{C}/2}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z}. \tag{5.2.36}$$

taking the expectation value for the vacuum yields

$$\langle T(z,\bar{z})T(0\rangle) \sim \langle\frac{\tilde{C}/2}{z^4}\rangle + \langle\frac{2T(0)}{z^2}\rangle + \langle\frac{\partial T(0)}{z}\rangle. \tag{5.2.37}$$

The expectation value of one point functions vanish for CFTs (except for the identity operator), and the only contributing term is

$$\langle T(z,\bar{z})T(0\rangle) = \frac{\tilde{C}/2}{z^4}. \tag{5.2.38}$$

From the definition of the $F(z\bar{z})$ function we have

$$F(z\bar{z}) = \tilde{C}, \text{ at the fixed points.} \tag{5.2.39}$$

In fact, the designed function $C(z\bar{z})$ satisfies

$$\dot{C} = 0 \text{ and } C(z\bar{z}) = \tilde{C}, \text{ at the fixed points.} \tag{5.2.40}$$

This proves that it is always possible to define a C-function for two-dimensional QFTs with rotation and translation invariance under the RG flow. Although very elegant, this proof misses an important object that appeared in the original proof: the *Zamolodchikov metric*.

5.2.3 Zamolodchikov's Proof

We begin by defining an action parameterized by the dimensionless couplings g_i, and equipped with a UV cutoff a in position space,

$$S[g_i, a] = \int d^2x \sigma(x, g_i, a). \tag{5.2.41}$$

Denoting the space of couplings by Q, an RG flow is a map $R_t : Q \to Q$ parameterized by a dimensionless quantity $t > 0$. We assume, as before, that the flow is induced by the rescaling of coordinates $x^\mu(t) = e^{\frac{t}{2}} x^\mu$ (recall that scale transformations are the generators of flows). This defines an effective action, with a new cutoff

$$S_{\text{eff}}[R_t g, e^{\frac{t}{2}} a] = \int d^2x \sigma(x, R_t g_i, e^{\frac{t}{2}} a). \tag{5.2.42}$$

The renormalizability hypothesis assures that correlation functions built from either $S_{\text{eff}}[R_t g, e^{\frac{t}{2}} a]$ or $S[g_i, a]$ have to agree as long as the cutoff is respected, in position space: $x \gg a$. One can obtain the operators of the theory by differentiation with respect to the couplings

$$\Phi_i(x) = \frac{\partial}{\partial g^i} \sigma(x, g_i, a). \tag{5.2.43}$$

The β-functions are defined as the linear response of the couplings to a change in scale

$$\beta^i = \frac{dg^i}{dt}. \tag{5.2.44}$$

Using the previous analysis of rotations and translations of two-point functions, and introducing a cutoff dependence to define the *dimensionless* two-point functions, we have

$$C\left(\frac{z\bar{z}}{a^2}\right) = 2z^4 \langle T(z, \bar{z}) T(0) \rangle, \tag{5.2.45}$$

$$H_i\left(\frac{z\bar{z}}{a^2}\right) = z^3 \bar{z} \langle T(z, \bar{z}) \Phi_i(0) \rangle, \tag{5.2.46}$$

$$G_{ij}\left(\frac{z\bar{z}}{a^2}\right) = z^2 \bar{z}^2 \langle \Phi_i(z, \bar{z}) \Phi_j(0) \rangle. \tag{5.2.47}$$

The conservation of the stress-tensor yields

$$\bar{\partial} T + \partial \Gamma = 0, \tag{5.2.48}$$

with the definitions $\Gamma = T_{\bar{z}z}$, $\bar{\partial} = \partial_{\bar{z}}$, and $\partial = \partial_z$.

Renormalization is generated by scale transformations, which can be seen as Weyl rescalings of the metric. For a linearized version of the RG flow rescale: $x'^\mu = (1 - dt/2) x^\mu$, one can check that the metric rescales correspondingly by a Weyl transformation

$$\delta g_{\mu\nu} = dt g_{\mu\nu}. \tag{5.2.49}$$

The change in the action is evaluated via

$$\delta S = \frac{\delta S}{\delta g_{\mu\nu}} \delta g_{\mu\nu} = -\frac{1}{4} \int d^2x dt T^\mu_\mu(x), \tag{5.2.50}$$

where the energy-momentum tensor was normalized as $-4\frac{\delta\sigma}{\delta g^{\mu\nu}} = T_{\mu\nu}$. On the other hand, one can think of the action as a series of operators

$$S = \int d^2x g_i \Phi^i(x). \tag{5.2.51}$$

The infinitesimal rescaling generates an infinitesimal RG flow, which maps the coupling constants as

$$g_i \to g_i' = g_i + \beta_i dt. \tag{5.2.52}$$

After the RG step, the action is rewritten in terms of the *same set of operators*, but with different coupling constants, so that the RG flow does not produce any other operator as modes are integrated out. This is another way to state the renormalizability condition. The action satisfies

$$S(g_i') = \int d^2x g_i' \Phi^i(x) = S(g_i) + \int d^2x \beta_i \Phi^i(x) dt, \tag{5.2.53}$$

which means the Γ operator satisfies

$$\Gamma = -\beta^i \Phi_i. \tag{5.2.54}$$

Repeated indexes imply summation.

The next step consists in requiring the correlation function $C\left(\frac{z\bar{z}}{a^2}\right)$ to be cutoff independent, satisfying the Callan-Symanzik equation

$$\frac{d}{dt} C\left(\frac{z\bar{z}}{a^2}\right) = 0. \tag{5.2.55}$$

The operator $\frac{d}{dt}$ can be written in terms of the variations of the couplings $g_i(t)$ and the explicit cutoff dependence

$$\begin{aligned} \frac{d}{dt}\left(C\left(\frac{z\bar{z}}{a^2}\right)\right) &= \left(\frac{\partial}{\partial t} + \frac{\partial g^i}{\partial t}\frac{\partial}{\partial g^i}\right) C\left(\frac{z\bar{z}}{a^2}\right) \\ &= \left(\frac{a}{2}\frac{\partial}{\partial a} + \beta^i \frac{\partial}{\partial g^i}\right) C\left(\frac{z\bar{z}}{a^2}\right). \end{aligned} \tag{5.2.56}$$

Notice that

$$\frac{a}{2}\frac{\partial}{\partial a}\left(C\left(\frac{z\bar{z}}{a^2}\right)\right) = \quad -z\partial\left(C\left(\frac{z\bar{z}}{a^2}\right)\right) = -\bar{z}\bar{\partial}\left(C\left(\frac{z\bar{z}}{a^2}\right)\right). \tag{5.2.57}$$

So we have the identity

$$\bar{z}\bar{\partial}\left(C\left(\frac{z\bar{z}}{a^2}\right)\right) = \beta^i \partial_i \left(C\left(\frac{z\bar{z}}{a^2}\right)\right), \tag{5.2.58}$$

where $\partial_i = \frac{\partial}{\partial g^i}$.

Repeating the same steps for the $H_i\left(\frac{z\bar{z}}{a^2}\right)$ two-point functions, yields the same relation

$$\bar{z}\bar{\partial}\left(H_i\left(\frac{z\bar{z}}{a^2}\right)\right) = \beta^j \partial_j \left(H_i\left(\frac{z\bar{z}}{a^2}\right)\right). \tag{5.2.59}$$

It is important to notice that the two-point functions are also a functions of the couplings, although not explicit in the equations. The Callan-Symanzik equation contains derivatives ∂ or $\bar{\partial}$, when acting on the operator $T(z)$, it is possible to use the conservation equation just as in Cardy's analysis. The CS equation for the $C\left(\frac{z\bar{z}}{a^2}\right)$ reads

$$\begin{aligned}\beta^i\partial_i\left(C\left(\frac{z\bar{z}}{a^2}\right)\right) &= \bar{z}\bar{\partial}\left(C\left(\frac{z\bar{z}}{a^2}\right)\right)\\ &= \bar{z}\bar{\partial}\left(2z^4\langle T(z,\bar{z})T(0)\rangle\right)\\ &= 2z^4\bar{z}\langle\bar{\partial}T(z,\bar{z})T(0)\rangle\\ &= -2z^4\bar{z}\langle\partial\Gamma(z,\bar{z})T(0)\rangle\\ &= -2z^4\bar{z}\partial\langle\Gamma(z,\bar{z})T(0)\rangle. \end{aligned} \tag{5.2.60}$$

It is useful to notice that

$$2z^4\bar{z}\partial\langle\Gamma(z,\bar{z})T(0)\rangle = 2\left(z\partial-3\right)\langle z^3\bar{z}T(z,\bar{z})\Gamma(0)\rangle. \tag{5.2.61}$$

The renormalizability condition, imposed as a constraint in the trace operator Γ, allows us to write

$$\begin{aligned}\beta^i\partial_i\left(C\left(\frac{z\bar{z}}{a^2}\right)\right) &= -2\left(z\partial-3\right)\left(-\beta^i\underbrace{z^3\bar{z}\langle T(z,\bar{z})\Phi_i(0)\rangle}_{H_i}\right)\\ &= 2\left(z\partial-3\right)\left(\beta^iH_i\right)\\ &= 2\underbrace{z\partial\left(\beta^iH_i\right)}_{I}-6\left(\beta^iH_i\right). \end{aligned} \tag{5.2.62}$$

Using the CS equation for I, we can write the action of the $z\partial$ operator in H_i as the aplication of the β functions. Namely

$$\beta^i\partial_i\left(C\left(\frac{z\bar{z}}{a^2}\right)\right) = 2\underbrace{\beta^j\partial_j\left(\beta^iH_i\left(\frac{z\bar{z}}{a^2}\right)\right)}_{I}-6\beta^iH_i\left(\frac{z\bar{z}}{a^2}\right). \tag{5.2.63}$$

This equation tells us that there is a coupling between the evolution of the C and H_i two-point functions under the RG flow. To complete the proof, it is necessary an extra equation coupling the H_i and G_{ij} two-point functions, what will generate a system of differential equations in the same way as in Cardy's proof. Proceeding in the same way by considering the CS equation for β^iH_i, we have

$$\begin{aligned}\beta^i\partial_i\left(\beta^jH_j\left(\frac{z\bar{z}}{a^2}\right)\right) &= \bar{z}\bar{\partial}\left(\beta^jz^3\bar{z}\langle T(z,\bar{z})\Phi_j(0)\rangle\right)\\ &= \beta^j\left(z^3\bar{z}\langle T(z,\bar{z})\Phi_j(0)\rangle+z^3\bar{z}^2\langle\bar{\partial}T(z,\bar{z})\Phi_j(0)\rangle\right)\\ &= \beta^j\left(z^3\bar{z}\langle T(z,\bar{z})\Phi_j(0)\rangle-z^3\bar{z}^2\langle\partial\Gamma(z,\bar{z})\Phi_j(0)\rangle\right)\\ &= \beta^j\underbrace{z^3\bar{z}\langle T(z,\bar{z})\Phi_j(0)\rangle}_{H_j}+\beta^j\beta^i\underbrace{z^3\bar{z}^2\partial\left(\langle\Phi_i(z,\bar{z})\Phi_j(0)\rangle\right)}_{II}. \end{aligned}$$

Again, we notice that II can be written as

$$z^3\bar{z}^2\partial\left(\langle\Phi_i(z,\bar{z})\Phi_j(0)\rangle\right) = (z\partial - 2)\left(z^2\bar{z}^2\langle\Phi_i(z,\bar{z})\Phi_j(0)\rangle\right), \tag{5.2.64}$$

and using the CS equation yields

$$\beta^k\partial_k\left(\beta^j H_j\right) = \beta^i H_i + \left(\beta^k\partial_k - 2\right)\left(\beta^i\beta^j G_{ij}\right). \tag{5.2.65}$$

Relabeling the dummy summation indexes, and expanding the expression one obtains

$$\begin{aligned}&\beta^i(\partial_i\beta^k)H_k + \beta^k\beta^i(\partial_k H_i) - \beta^i H_i = \\ &\beta^j\beta^i(\partial_i\beta^k)G_{kj} + \beta^j\beta^i(\partial_j\beta^k)G_{ik} - 2\beta^i\beta^k G_{ik} + \beta^i\beta^j\beta^k\partial_k G_{ij}.\end{aligned} \tag{5.2.66}$$

Getting rid of the extra β^i contracted with both sides, we reach

$$\begin{aligned}&(\partial_i\beta^k)H_k + \beta^k(\partial_k H_i) - H_i = \\ &\beta^j(\partial_i\beta^k)G_{kj} + \beta^j(\partial_j\beta^k)G_{ik} - 2\beta^k G_{ik} + \beta^j\beta^k\partial_k G_{ij}.\end{aligned} \tag{5.2.67}$$

To summarize, the CS equations for C and H_i yield

$$\beta^i\partial_i C = \beta^k\left(-6H_k + 2\partial_k(\beta^i H_i)\right), \tag{5.2.68}$$

$$\beta^k\left(\partial_k(\beta^i H_i) - H_k\right) = \beta^k\left(\partial_k(\beta^i\beta^j G_{ij}) - 2\beta^i G_{ik}\right). \tag{5.2.69}$$

For the last step, we define the function $c(g_i)$ such that

$$c = C + 4\beta^i H_i - 6\beta^i\beta^j G_{ij}, \tag{5.2.70}$$

where the dependence in the couplings g_i is omitted. To check the behaviour of $c(g)$ under the RG flow, we act with the $\beta^i\partial_i$ operator,

$$\begin{aligned}\beta^i\partial_i c &= \beta^i\partial_i C + 4\beta^k\partial_k(\beta^i H_i) - 6\beta^k\partial_k(\beta^i\beta^j G_{ij}) \\ &= \beta^k\left(-6H_k + 2\partial_k(\beta^i H_i)\right) + 4\beta^k\partial_k(\beta^i H_i) - 6\beta^k\partial_k(\beta^i\beta^j G_{ij}) \\ &= \beta^k\left(-6H_k + 2\partial_k(\beta^i H_i)\right) + 4\beta^k\partial_k(\beta^i H_i) - 6\beta^k(\partial_k(\beta^i H_i) - H_k + 2\beta^i G_{ik}).\end{aligned}$$

Which finally reads

$$\beta^i\partial_i c = -12\beta^i\beta^j G_{ij}. \tag{5.2.71}$$

We are dealing with an Euclidean QFT in two dimensions with reflection positivity condition, which is the Euclidean analog of unitarity for Minkowski space QFTs. Such condition implies that the two-point functions G_{ij} define a *positive definite matrix* in the $\{i,j\}$ indexes. This object is called the *Zamolodchikov metric*, and can be seen as a metric in the space of couplings.

The emergence of this metric reveals the remarkable geometric nature of the moduli space of QFTs, where the $\beta^i(g)$ are components of a vector field defined in the manifold parameterized by the coupling constants, where G_{ij} introduces the notion of a distance. The last equation can be written as a matrix equation

$$\beta^i\partial_i c = -12[\beta][G][\beta^t], \tag{5.2.72}$$

where the matrix $[G]$ is positive definite. In turn, it means that the function $c(g)$ is non-increasing as the theory flows to the infrared limit

$$\beta^i \partial_i c(g_j) \leq 0. \tag{5.2.73}$$

Conformal symmetry is restored at the fixed points, and as before, the trace vanishes as an operator, making the two-point functions H_i and G_{ij} vanish identically. The analysis discussed in the last section is still perfectly good for Zamolodchikov's proof. The defined function $c(g)$ satisfies the properties of being stationary at the fixed points, and it takes the value of the central charge of the CFTs, thus completing the proof that $c(g)$ is a C-function.

5.3 Entanglement Entropy C-theorem

In the last section the C-theorem was introduced as originally proved by Zamolodchikov, supported by a construction of Euclidean $(1+1)$-dimensional QFTs with reflection positivity conditions. In 2006, Casini et al. [12] proved the C-theorem based on a construction of Lorentzian, unitary, $(1+1)$-dimensional QFTs.

This proof consists in defining a C-function in terms of the entanglement entropy of a space-like interval as the theory flows in the renormalization group. Such a clever setup is only possible because the entropy of an interval r of a CFT_2 is known from [18] to be

$$S(r) = \frac{C_V}{3} \log\left(\frac{r}{\epsilon}\right). \tag{5.3.1}$$

The relation between the central charge and the degrees of freedom is given in a very natural way when a C-function is built from the entropy of a system. Since the EE accounts for entangling pairs, the variation of entropy along the RG flow measures the loss of information due to the coarse-graining and integration of modes involved in the process, which will ultimately be perceived in the fixed points by a decrease in the value of the central charge (5.1.4).

The RG setup is exactly the same as in the Zamolodchikov's proof (5.2.1). We first provide a proof of (5.3.1) in order to have a self-contained result, and later state and prove the *entropic C-function* as done by Casini et.al.

5.3.1 Entanglement Entropy in CFT_2

The proof of formula (5.3.1) is obtained by evaluating (2.3.16). The first difficulty comes early if one notice the presence of the "log" of the reduced density matrix. It was not a problem in the discrete calculations of the example (2.3.19), but becomes a major issue when we take the continuum limit. To bypass this problem the *replica trick* was introduced [18].

A reduced density matrix ρ_A can be diagonalized as $\rho_A = UDU^{-1}$, where the matrix D is diagonal with the eigenvalues of ρ_A: $D_{ij} = \delta_{ij}\lambda_i$, for λ_i eigenvalues of ρ_A. We consider that ρ_A is also a normalized operator

$$Tr(\rho_A) = 1 \Longrightarrow \sum_i \lambda_i = 1. \tag{5.3.2}$$

We want to get rid of the "log" inside the trace on the definition (2.3.16), so we look for an alternative expression for the $S(A)$. One can rewrite $S(A)$ as

$$S(A) = -\lim_{n\to 1} \frac{\partial}{\partial n} \left(Tr({\rho_A}^n)\right). \tag{5.3.3}$$

Proof:

$$\left(Tr({\rho_A}^n)\right) = \left(\sum_i {\lambda_i}^n\right).$$

Taking the derivative in respect to n, and taking the limit $n \to 1$ yields

$$\begin{aligned} -\frac{\partial}{\partial n} \left(Tr({\rho_A}^n)\right)\Big|_{n\to 1} &= -\sum_i log(\lambda_i)\lambda_i^n\Big|_{n\to 1} \\ &= -\sum_i log(\lambda_i)\lambda_i \\ &= S(A). \end{aligned}$$

This is known in the literature as the *replica trick*, because it creates n replicas of the reduced density matrix in order to avoid the logarithm. We proceed by defining the reduced density matrices in path integral language.

For an Euclidean QFT, the transition amplitude between two states $|\phi_i\rangle$ and $|\phi_f\rangle$, in a system whose dynamics is governed by a Hamiltonian operator, may be written as a path integral

$$\langle\phi_f| e^{\beta H} |\phi_i\rangle = \int_{\tau=0,\phi=\phi_i}^{\tau=\beta,\phi=\phi_f} \mathcal{D}[\phi(t,x)] e^{-S[\phi]}, \tag{5.3.4}$$

where we integrate over the field configurations. $S[\phi]$ is the euclidean action for the field ϕ, with boundary conditions such that the field at $\tau = 0$ must be in the ϕ_i configuration, and at $\tau = \beta$ it must be in the ϕ_f configuration.

In this language it is very easy to recover the partition function, defined by the *vacuum-to-vacuum amplitude*. It is done by taking amplitudes from $\tau \to -\infty$ to $\tau \to \infty$, because the field configuration at those times must collapse to the vacuum.[3]

The vacuum-to-vacuum amplitude is

$$\langle\psi|\psi\rangle = \int_{\tau\to-\infty}^{\tau\to\infty} \mathcal{D}[\phi(\tau,x)] e^{-S[\phi]}. \tag{5.3.5}$$

We can write it in a more convenient way by splitting the evolution into $(-\infty, 0)$ and $(0, \infty)$,

$$\begin{aligned} &\int_{\tau\to-\infty}^{\tau\to\infty} \mathcal{D}[\phi(\tau,x)] e^{-S[\phi]} = \\ &\int_{\tau\to-\infty}^{\tau=0,\phi=\phi_i} \int_{\tau=0,\phi=\phi_i}^{\tau\to\infty} \mathcal{D}[\phi(\tau\neq 0,x)]\mathcal{D}[\phi_i(\tau=0,x)] e^{-S[\phi]}. \end{aligned} \tag{5.3.6}$$

[3]Diagonalizing in respect to the Hamiltonian, the ground state is dominant when the limit $\tau \to \infty$ is taken.

It is the path integral analog of inserting the operator $\mathbb{1} = \int \mathcal{D}[\phi_i] \, |\phi_i\rangle \, \langle\phi_i|$. The integration over all configurations splits into a sum over configurations for all times different than the time slice $\tau = 0$, and an integration $\mathcal{D}[\phi_i]$ over the configurations at the time slice $\tau = 0$.

The first integration sign is identified with the boundary conditions as a transition amplitude between the vacuum at $\tau \to -\infty$ and an arbitrary state ϕ_i at $\tau = 0$. The second integration sign is identified as the transition amplitude between a state ϕ_i at $\tau = 0$ and the vacuum at $\tau \to \infty$

$$\langle\psi|\psi\rangle = \int_{\tau=0} \mathcal{D}[\phi_i] \langle\psi|\phi_i\rangle\langle\phi_i|\psi\rangle e^{-S[\phi]}. \tag{5.3.7}$$

All these gymnastics make clear that we can see the path integral as an operator, and the imposition of boundary conditions is the analog of choosing a basis to compute matrix elements of such operator. For example, the following transition amplitude defines a complex number

$$\langle\phi_i|\psi\rangle = \int_{\tau\to-\infty}^{\tau=0,\phi=\phi_i} \mathcal{D}[\phi(\tau, x)] \in \mathbb{C}. \tag{5.3.8}$$

This holds for arbitrary boundary condition ϕ_i, if one chooses a different state to compute the transition amplitude to, one generates another $\mathbb{C}$-number. So we can identify a *state* $|\psi\rangle$ at $\tau = 0$ with the path integral with no boundary conditions at $\tau = 0$,

$$|\psi\rangle \, |_{\tau=0} = \int_{\tau\to-\infty}^{\tau=0} \mathcal{D}[\phi(\tau, x)]. \tag{5.3.9}$$

Imposing boundary conditions $\phi = \phi_i$ is the same as acting with a $\langle\phi_i|$ in the LHS of (5.3.9). Analogously, we can identify the second half of the vacuum transition amplitude as the $\langle\psi|$ state at $\tau = 0$

$$\langle\psi| \, |_{\tau=0} = \int_{\tau=0}^{\tau\to\infty} \mathcal{D}[\phi(\tau, x)]. \tag{5.3.10}$$

In the literature it is common to draw schematic diagrams for the path integrals, where the topology of the space-time is stressed, see (5.3.1) for example.

From this representation, we can see that a path integral with no boundary conditions define the *density matrix* ρ. But there is a caveat: the path integral is defined at all times, and we want to generate a density operator in a fixed time slice. In order to get the matrix elements for a density operator in $\tau = \tau'$, one has to specify boundary conditions for the fields at $\tau = \tau'^{+}$ and $\tau = \tau'^{-}$. In order to specify such conditions, we can fix the field configuration via

$$\rho_{ij} = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\phi] e^{-S[\phi]} \delta(\phi(x, \tau^+) - \phi_j) \delta(\phi(x, \tau^-) - \phi_i). \tag{5.3.11}$$

The space-time dependence of fields and boundary times of integration were dropped for simplicity, the integral is defined for $\tau \in (-\infty, \infty)$. The "$\delta$"s present in (5.3.11) are localization operators with respect to the measure $\mathcal{D}[\phi]$.

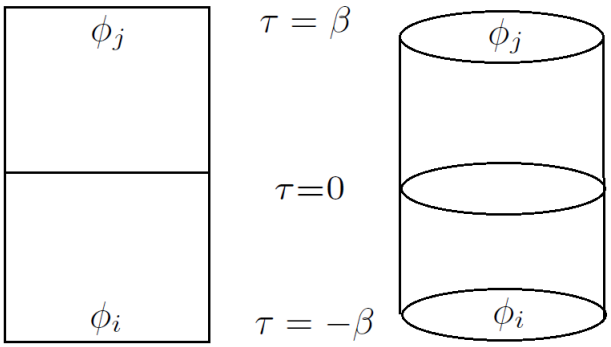

Figure 5.2: Path integral representation of $\langle\phi_j|\, e^{-2\beta H}\, |\phi_i\rangle$. The figure to the left is an $\mathbb{R}^2$ spacetime, and the right figure is representing a $S^1 \times \mathbb{R}$ spacetime. The ϕ_i (ϕ_j) indicate that the field has configuration $\phi = \phi_i$ at $\tau = -\beta$ ($\phi = \phi_j$ at $\tau = \beta$).[36]

In order to use the replica trick, it is necessary to calculate powers of density matrix ρ_{ij}. For instance one writes

$$\rho^2_{ij} = \sum_k \rho_{ik}\rho_{kj}, \tag{5.3.12}$$

and it naturally translates to

$$\rho^2_{ij} = \frac{1}{\mathcal{Z}^2}\int \mathcal{D}[\phi_1]\mathcal{D}[\phi_2]\mathcal{D}[\phi_k] e^{-S[\phi_1]} e^{-S[\phi_2]} \delta(\phi_1(x,\tau^+) - \phi_k)\delta(\phi_1(x,\tau^-) - \phi_i) \\ \times \delta(\phi_2(x,\tau^+) - \phi_j)\delta(\phi_2(x,\tau^-) - \phi_k). \tag{5.3.13}$$

The integration $\int \mathcal{D}[\phi_k]$ is the path integral analog of the summation $\sum_k$ in (5.3.12). Performing the integration yields $\phi_k = \phi_1(x,\tau^+) = \phi_2(x,\tau^-)$,

$$\rho^2_{ij} = \frac{1}{\mathcal{Z}^2}\int \mathcal{D}[\phi_1]\mathcal{D}[\phi_2] e^{-S[\phi_1]} e^{-S[\phi_2]} \delta(\phi_1(x,\tau^+) - \phi_2(x,\tau^-)) \\ \times \delta(\phi_1(x,\tau^-) - \phi_i)\delta(\phi_2(x,\tau^+) - \phi_j). \tag{5.3.14}$$

It is easy to generalize to ρ^n_{ij}: each density matrix will be written in terms of a copy of the field ϕ_i, then all we have to do is perform the integration pertinent to the repeated indexes in the matrix multiplication. At the end, we will have the two delta functions corresponding to the external fields ϕ_i and ϕ_j, and a collection of $(n-1)$ delta functions setting the field configuration $\phi_i(x,\tau^+) = \phi_{i+1}(x,\tau^-)$. As we have n copies of the same field, their actions will add as $\sum_i S[\phi_i] = S[\sum_i \phi_i]$, yielding

$$\rho^n_{ij} = \frac{1}{\mathcal{Z}^n}\int \mathcal{D}[\Phi] e^{-\sum_i S[\phi_i]} \left(\prod_{i=1}^{n-1} \delta(\phi_i(x,\tau^+) - \phi_{i+1}(x,\tau^-))\right) \\ \times \delta(\phi_1(x,\tau^-) - \phi_i)\delta(\phi_2(x,\tau^+) - \phi_j), \tag{5.3.15}$$

where Φ denotes the collection of fields $\{\phi_i\}$, such that $\mathcal{D}[\Phi] = \prod_i \mathcal{D}[\phi_i]$. Finally, taking the trace of ρ^n inserts another delta function, setting $\phi_n(x,\tau^+) = \phi_1(x,\tau^-)$:

$$\begin{aligned} &Tr(\rho^n) = \\ &\frac{1}{\mathcal{Z}^n}\int \mathcal{D}[\Phi] e^{-\sum_i S[\phi_i]} \left(\prod_{i=1}^{n} \delta(\phi_i(x,\tau^+) - \phi_{i+1}(x,\tau^-))\right) \quad , \quad i \equiv i+n. \end{aligned} \tag{5.3.16}$$

This formula still needs a minor upgrade. The Von Neumann entropy is calculated in terms of the reduced density matrix to a subregion A of a Cauchy slice Σ. The boundary conditions imposed by the δ's must be restricted to the desired subregion, while the configurations outside A are free to vary as they please. That can be easily implemented by restricting the delta functions for the configurations of the degrees of freedom *inside* A,

$$Tr({\rho_A}^n) = \frac{1}{\mathcal{Z}^n}\int \mathcal{D}[\Phi] e^{-\sum_i S[\phi_i]} \left(\prod_{i=1, x\in A}^{n} \delta(\phi_i(x,\tau^+) - \phi_{i+1}(x,\tau^-))\right) \quad , \quad i \equiv i+n. \tag{5.3.17}$$

This integration defines an n-sheeted geometry, made of n copies of the original geometry in which $\mathcal{Z}$ is defined. There is an equivalence between summing over configurations of n copies of ϕ_i subjected to twisted boundary conditions over the geometry $\mathcal{M}$, and summing over configurations of one single copy of ϕ_i with no boundary conditions over the geometry $\mathcal{M}_{n,A}$. In this sense of equivalence, we can define the partition function in $\mathcal{M}_{n,A}$ by

$$\mathcal{Z}[\mathcal{L}, \mathcal{M}_{n,A}] = \int \mathcal{D}[\Phi] e^{-\sum_i S[\phi_i]} \left(\prod_{i=1, x\in A}^{n} \delta(\phi_i(x,\tau^+) - \phi_{i+1}(x,\tau^-))\right) \quad , \quad i \equiv i+n. \tag{5.3.18}$$

$\mathcal{L}$ denotes the Lagrangian of a single field which lives in the complicated geometry $\mathcal{M}_{n,A}$. The action is

$$\sum_i S[\phi_i] = \sum_i \int_{\mathcal{M}} \mathcal{L}[\phi_i] \equiv \int_{\mathcal{M}} \mathcal{L}^n[\Phi]. \tag{5.3.19}$$

Here $\mathcal{M}$ denotes the space-time topology over which we first defined our theory. For example: in our case of interest, the CFT_2 lives in $\mathcal{M} = \mathbb{R}^2$. $\mathcal{M}_{n,A}$ denotes the topology of the space obtained by "gluing" n copies of $\mathcal{M}$ through the interval A (5.3.1).

The treatment of fields with twisted boundary conditions is given by [19], and we follow the same strategy. The twisting of the fields can be seen as a symmetry of the lagrangean $\mathcal{L}^n$. Since the trace is invariant under *cyclic permutations*, we naturally have an operator σ that implements the cyclic permutations in the fields ϕ_i via

$$\Phi' = \sigma\Phi, \tag{5.3.20}$$

where Φ can be thought as a column vector of fields ϕ_i, and the operator σ as a $n \times n$ matrix that implements cyclic permutations in the fields ϕ_i. Such operator

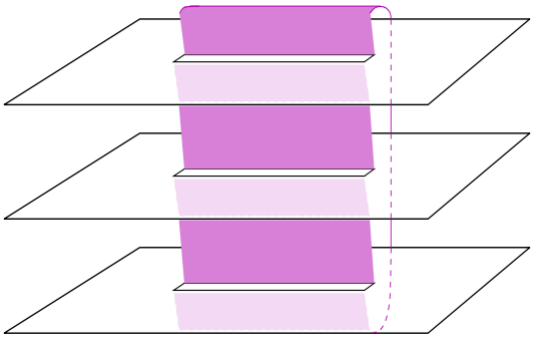

Figure 5.3: The representation of $\mathcal{M}_{n,A}$ in pink, connecting the n copies of $\mathcal{M} = \mathbb{R}^2$ space-times corresponding to the twisted boundary conditions $\phi_i(x, \tau^+) = \phi_{i+1}(x, \tau^-)$. [19].

defines a *global internal symmetry*, since the Lagrangean is invariant,

$$\int_{\mathbb{R}^2} \mathcal{L}^n[\sigma\Phi] = \int_{\mathbb{R}^2} \mathcal{L}^n[\Phi]. \tag{5.3.21}$$

The action of such operator will cyclic permute the boundary conditions, but again the path integral will be invariant because it is the representation of a trace, which is invariant under cyclic permutations assuring no anomalies are present.

Let's define an operator $\mathcal{T}_\sigma$ by its insertion on the path integral

$$\langle \mathcal{T}_\sigma(a,b) \ldots \rangle \propto \int_{\mathcal{B}_\sigma(a,b)} \mathcal{D}[\Phi] \exp\left(- \int_{\mathbb{R}^2} \mathcal{L}^n[\Phi] \right) \ldots \tag{5.3.22}$$

The symbol "..." means other possible insertions in the path integral, and the subscript $\mathcal{B}_\sigma(a,b)$ is a notation for the boundary conditions defined by

$$\mathcal{B}_\sigma(a,b) \Longrightarrow \quad \text{Identify} \quad \Phi(x, b^+) \equiv \sigma\Phi(x, b^-) \quad \text{For} \quad x \in [a, \infty]. \tag{5.3.23}$$

We define the operators that make the cyclic permutations as: $\mathcal{T}$ such that implements $i \to i+1$, and the inverse operator $\bar{\mathcal{T}}$ that implements $i \to i-1$ (recall $i \equiv i+n$) via

$$\mathcal{T} = \mathcal{T}_\sigma \quad , \quad \sigma : i \to i+1 \mod n, \tag{5.3.24}$$

$$\bar{\mathcal{T}} = \mathcal{T}_{-\sigma} \quad , \quad -\sigma : i \to i-1 \mod n. \tag{5.3.25}$$

It is useful to think of $\mathcal{T}(a,b)$ as an operator that "twists" the fields along $x \in [a, \infty)$, and $\bar{\mathcal{T}}(a,b)$ as the operator that "untwists" the fields in the same interval. Choose a region A to lie completely in a constant time slice, defined by $A = \{x \in \mathbb{R}^2 | x = (\tau, r), \tau = 0, \text{ and } r \in [a_1, a_2]\}$. The trace of ${\rho_A}^n$ is given by

$$\begin{aligned} Tr({\rho_A}^n) &\propto \mathcal{Z}[\mathcal{L}, \mathcal{M}_{n,A}] \\ &\propto \int \mathcal{D}[\Phi] e^{-\sum_i S[\phi_i]} \left(\prod_{i=1, x\in A}^{n} \delta(\phi_i(x, \tau^+) - \phi_{i+1}(x, \tau^-)) \right). \end{aligned}$$

This is exactly the boundary conditions implemented by the $\mathcal{T}$ operator restricted to the region A. One can express the action of $\mathcal{T}$ in a region of constant $\tau = b$, and space coordinate $x \in [a_1, a_2]$, by the insertion of $\mathcal{T}(a_1, b)$ followed by an insertion of $\bar{\mathcal{T}}(a_2, b)$. It is easy to see that $\mathcal{T}(a_1, b)$ will implement the twisted boundary conditions to the interval $x \in [a_1, \infty)$, but $\bar{\mathcal{T}}(a_2, b)$ acts untwisting the fields in the region $x \in [a_2, \infty)$, so the result is the implementation of twisted boundary conditions only in the region $x \in [a_1, a_2]$. So we have

$$\int \mathcal{D}[\Phi] e^{-\sum_i S[\phi_i]} \left(\prod_{i=1, x\in A}^{n} \delta(\phi_i(x, \tau^+) - \phi_{i+1}(x, \tau^-)) \right) \propto \langle \mathcal{T}(a_1, b) \bar{\mathcal{T}}(a_2, b) \rangle$$
$$\implies Tr({\rho_A}^n) \propto \langle \mathcal{T}(a_1, b) \bar{\mathcal{T}}(a_2, b) \rangle.$$

The correlation function of operators in the twisted geometry $\mathcal{Z}[\mathcal{L}^n, \mathcal{M}_{n,A}]$ is normalized canonically as

$$\langle \mathcal{O}_i(x, \tau) \rangle_{\mathcal{L}, \mathcal{M}_{n,A}} = \frac{\langle \mathcal{T}(a_1, b) \bar{\mathcal{T}}(a_2, b) \mathcal{O}_i(x, \tau) \rangle_{\mathcal{L}^n, \mathbb{R}^2}}{\langle \mathcal{T}(a_1, b) \bar{\mathcal{T}}(a_2, b) \rangle_{\mathcal{L}^n, \mathbb{R}^2}}. \tag{5.3.26}$$

One can think of this correlator as a single field in a complicated geometry (LHS) being described as n copies of such field, living in a simple geometry, but subjected to the twisted boundary conditions implemented by the $\mathcal{T}$ and $\bar{\mathcal{T}}$ operators. The insertion $\mathcal{O}_i$ means an operator inserted in the i^{th} sheet of the geometry $\mathcal{M}_{n,A}$.

The operators $\mathcal{T}$ and $\bar{\mathcal{T}}$ must be invariant under all symmetries of the system that commute with σ, which means that for a CFT they are *primary operators* with *spin 0* [19]. The next step is to discover the scaling dimension of these operators, and to do so we begin with a CFT_2 in $\mathbb{R}^2$, which has the holomorphic part of its energy-momentum tensor denoted by $T(z)$. The Lagrangian $\mathcal{L}^n$ will also be one of a CFT, and the energy-momentum tensor is just $T^n(z) \equiv \sum_i T_i(z)$, corresponding to n copies of the $T_i(z)$ from original theory.

Let's evaluate the one-point function of $T_i(z)$ in the "glued geometry" $\mathcal{M}_{n,A}$. We can conformally map $\mathcal{M}_{n,A}$ to $\mathbb{R}^2$ via

$$z = \left(\frac{w - a_1}{w - a_1} \right)^{\frac{1}{n}}, \tag{5.3.27}$$

where z are coordinates[4] in $\mathbb{R}^2$, and w coordinates of $\mathcal{M}_{n,A}$. This mapping identifies each copy of $\mathbb{R}^2$ with an angular slice of opening $2\pi/n$ in the complex plane, mapping a_1 to the origin, and a_2 to ∞. Through such change of coordinates, the one-point function of $T_i(z)$ changes as

$$\langle T_i(w) \rangle_{\mathcal{L}, \mathcal{M}_{n,A}} = \underbrace{\langle T_i(z) \rangle_{\mathcal{L}, \mathbb{R}^2}}_{=0} + \frac{c}{12} \frac{z''' z' - (3/2)(z'')^2}{(z')^2}. \tag{5.3.28}$$

The first contribution vanishes because the theory is exactly conformal in $\mathbb{R}^2$, the second term is called the *Schwarzian derivative*, and c is the central charge. Taking

[4] We have complexified the coordinates in $\mathbb{R}^2$ by $z = x + iy$, so we are actually in the extended complex $\bar{\mathbb{C}}$.

the derivatives we have

$$\langle T_i(w)\rangle_{\mathcal{L},\mathcal{M}_{n,A}} = \frac{c(n^2-1)}{24n^2}\frac{(a_1-a_2)^2}{(w-a_1)^2(w-a_2)^2}, \tag{5.3.29}$$

which means the n copies system is n times the RHS of (5.3.29),

$$\langle T^n(w)\rangle_{\mathcal{L},\mathcal{M}_{n,A}} = \frac{c(n^2-1)}{24n}\frac{(a_1-a_2)^2}{(w-a_1)^2(w-a_2)^2}. \tag{5.3.30}$$

We can also calculate the one point function using (5.3.26),

$$\langle T^n(w)(x,\tau)\rangle_{\mathcal{L},\mathcal{M}_{n,A}} = \frac{\langle \mathcal{T}(a_1,b)\bar{\mathcal{T}}(a_2,b)T_i(x,\tau)\rangle_{\mathcal{L}^n,\mathbb{R}^2}}{\langle \mathcal{T}(a_1,b)\bar{\mathcal{T}}(a_2,b)\rangle_{\mathcal{L}^n,\mathbb{R}^2}}. \tag{5.3.31}$$

The twist operators are primaries, so we can use the OPE between $T^n(w)$ and an arbitrary primary $\mathcal{O}(z)$ to write

$$\begin{aligned}&\langle T^n(w)(x,\tau)\rangle_{\mathcal{L},\mathcal{M}_{n,A}} = \\ &\frac{\left(\frac{\partial_{a_1}}{w-a_1}+\frac{h_1}{(w-a_1)^2}+\frac{\partial_{a_2}}{w-a_2}+\frac{h_2}{(w-a_2)^2}\right)\langle \mathcal{T}(a_1,b)\bar{\mathcal{T}}(a_2,b)\rangle_{\mathcal{L}^n,\mathbb{R}^2}}{\langle \mathcal{T}(a_1,b)\bar{\mathcal{T}}(a_2,b)\rangle_{\mathcal{L}^n,\mathbb{R}^2}},\end{aligned} \tag{5.3.32}$$

where h_1 (h_2) is the conformal weight of $\mathcal{T}$ ($\bar{\mathcal{T}}$). As these operators are invariant under rotations, we set $s_i = 0$, and the conformal weight reduces to $h_i = d_i/2$. The two-point function of these operators in a CFT take the form

$$\langle \mathcal{T}(a_1,b)\bar{\mathcal{T}}(a_2,b)\rangle_{\mathcal{L}^n,\mathbb{R}^2} = |a_1-a_2|^{-2d_n}. \tag{5.3.33}$$

To evaluate the numerator in RHS of (5.3.32), we first write

$$|a_1-a_2|^{-2d_n} = ((a_1-a_2)(\bar{a_1}-\bar{a_2}))^{-d_n}, \tag{5.3.34}$$

so we have

$$\left(\frac{\partial_{a_1}}{w-a_1}+\frac{h_1}{(w-a_1)^2}+\frac{\partial_{a_2}}{w-a_2}+\frac{h_2}{(w-a_2)^2}\right)|a_1-a_2|^{-2d_n}$$

$$= |a_1-a_2|^{-2d_n}\left(d_n\frac{1}{w-a_2}\frac{1}{a_1-a_2}-d_n\frac{1}{w-a_1}\frac{1}{a_1-a_2}+\frac{h_1}{(w-a_1)^2}+\frac{h_2}{(w-a_2)^2}\right),$$

plugging in (5.3.32), we finally collect

$$\langle T^n(w)(x,\tau)\rangle_{\mathcal{L}^n,\mathcal{M}_{n,A}} = \frac{d_n}{2}\left(\frac{(a_1-a_2)^2}{(w-a_1)^2(w-a_2)^2}\right). \tag{5.3.35}$$

Equating (5.3.35) and (5.3.30) immediately yields

$$d_n = \frac{c}{12}\left(\frac{1}{n}-n\right). \tag{5.3.36}$$

The EE using the replica trick, in the case $\mathcal{M} = \mathbb{R}^2$, reads

$$\begin{aligned} Tr({\rho_A}^n) &= \frac{\mathcal{Z}[\mathcal{L}, \mathcal{M}_{n,A}]}{\mathcal{Z}[\mathcal{L}, \mathcal{M}]^n} \\ &\propto \frac{\langle \mathcal{T}(a_1, 0)\bar{\mathcal{T}}(a_2, 0)\rangle_{\mathcal{L},\mathbb{R}^2}}{\mathcal{Z}[\mathcal{L}, \mathcal{M}]^n} \\ &\propto \frac{|a_1 - a_2|^{-2d_n}}{\mathcal{Z}[\mathcal{L}, \mathcal{M}]^n} \\ &= c_n \left| \frac{\epsilon}{z - w} \right|^{\frac{c}{6}\left(n - \frac{1}{n}\right)} . \end{aligned} \tag{5.3.37}$$

In the last step a proportionality constant c_n was introduced, and the normalization factor was used to express the trace as a dimensionless quantity by making use of a cutoff ϵ. In fact this method does not provide any insights over the normalization constant c_n, but it is known that $c_1 = 1$ [20]. Applying the replica trick yields

$$\begin{aligned} S(A) &= -\lim_{n\to 1} \partial_n Tr({\rho_A}^n) \\ &= \frac{c}{3} \log\left(\frac{z - w}{\epsilon}\right) \\ &= \frac{C_V}{3} \log\left(\frac{r}{\epsilon}\right), \end{aligned} \tag{5.3.38}$$

where in the last line we just renamed the central charge $c = C_V$, and defined the length of the A interval as $r = z - w$.

5.3.2 A C-function for Entanglement Entropy

With the result (5.3.1), it is possible to build a C-function for a Lorentzian unitary CFT_2 by studying the entanglement entropy of a finite space-like interval of length l [12]. By constructing the right geometrical setup, unitarity and strong subadditivity of EE become strong enough to yield (5.1.4).

Consider the CFT_2 in a vacuum state $|0\rangle$. Assuming Lorentz invariance of the vacuum state, we recover the scalar nature of EE. For the vacuum, the Lorentz transformation is implemented by the unit operator

$$|0'\rangle = U(\Lambda)\,|0\rangle \quad \underbrace{\Longrightarrow}_{U(\Lambda)=\mathbb{1}} \quad |0'\rangle = |0\rangle\,. \tag{5.3.39}$$

$|0'\rangle$ is a notation for the vacuum state observed by a frame boosted by a Lorentz transformation in relation to the original frame where $|0\rangle$ is defined. The direct consequence is the Lorentz invariance of EE, which can be restated as the fact that $S(A)$ is a function of the proper length of the interval A

$$S(A) = S(\sqrt{ds_A^2}). \tag{5.3.40}$$

The vacuum case is even stronger than what one needs to reach (5.3.40), one only needs the Lorentz transformations to be implemented by unitary operators $U(\Lambda)$.

This way, (5.3.40) holds true for all states created by the action of creation and annihilation operators on $|0\rangle$ [22].

Consider two intervals $\mathcal{A}_{\nwarrow}$ and $\mathcal{A}_{\nearrow}$ relatively boosted to each other, lying inside the causal diamond of the interval $\mathcal{A}$, as shown in the picture 5.3.2

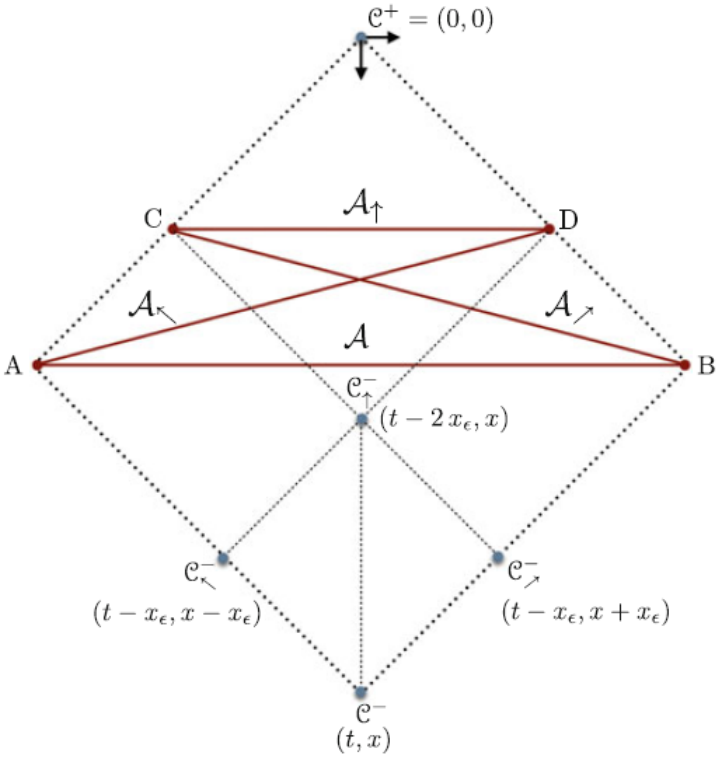

Figure 5.4: The causal diamond of $\mathcal{A}$ of endpoints $\mathcal{C}^+$ and $\mathcal{C}^-$ encloses the relevant intervals $\mathcal{A}_{\nwarrow}$, $\mathcal{A}_{\nearrow}$, $\mathcal{A}_\uparrow$, $\mathcal{A}$ displayed in solid red. Notice that time flows downwards in this diagram. All diagonal dashed lines represent light rays in $M^{1,1}$. [8]

The endpoints of $\mathcal{A}$ are $A = (\frac{t}{2}, x - \frac{t}{2})$, and $B = (\frac{t}{2}, x + \frac{t}{2})$. The endpoints of $\mathcal{A}_\uparrow$ are $D = (\frac{t}{2} - \epsilon, x + \frac{t}{2} - \epsilon)$ and $C = (\frac{t}{2} - \epsilon, x - \frac{t}{2} + \epsilon)$, where ϵ is an infinitesimal parameter.[5] The main idea is to take advantage of the unitarity assumption, which in this case assures that the information contained in $\mathcal{A}$ is self-contained in the causal diamond. With that assumption we can unitarily map an algebra of operators in the $\mathcal{A}$ interval to any Cauchy surface Σ' inside the diamond, as long as $\partial\Sigma' = \partial A$.[6] In this picture, the operators we want to map are the reduced density matrices for the Lorentz invariant vacuum state $|0\rangle$.

We define the light-like intervals $\bar{BD} = \gamma_R$ and $\bar{AC} = \gamma_L$. Consider the unions $\mathcal{A}_\uparrow \cup \gamma_R$ and $\mathcal{A}_\uparrow \cup \gamma_L$. Using the strong subbaditivity of EE (2.3.28), it yields

$$S(\mathcal{A}_\uparrow \cup \gamma_R) + S(\mathcal{A}_\uparrow \cup \gamma_R) \geq S(\mathcal{A}_\uparrow) + S(\mathcal{A}_\uparrow \cup \gamma_R \cup \gamma_L). \tag{5.3.41}$$

Notice that $\partial(\mathcal{A}_\uparrow \cup \gamma_R) = \partial(\mathcal{A}_{\nearrow})$, $\partial(\mathcal{A}_\uparrow \cup \gamma_L) = \partial(\mathcal{A}_{\nwarrow})$ and $\partial(\mathcal{A}_\uparrow \cup \gamma_R \cup \gamma_L) = \partial(\mathcal{A})$. As all these sets are contained in the causal domain of $\mathcal{A}$ (see (2.3)), their

[5] The figure shows the situation for an exaggerated ϵ, but in the end we take $\epsilon \to 0^+$.

[6] $\mathcal{A}$ can be seen a special kind of Cauchy surface entirely lying on a constant time.

entanglement entropy must be the same

$$\underbrace{S(\mathcal{A}_\uparrow \cup \gamma_R)}_{S(\mathcal{A}_\nearrow)} + \underbrace{S(\mathcal{A}_\uparrow \cup \gamma_L)}_{S(\mathcal{A}_\nwarrow)} \geq \underbrace{S(\mathcal{A}_\uparrow \cup \gamma_R \cup \gamma_L)}_{S(\mathcal{A})} + S(\mathcal{A}_\uparrow)$$
$$S(\mathcal{A}_\nearrow) + S(\mathcal{A}_\nwarrow) \geq S(\mathcal{A}) + S(\mathcal{A}_\uparrow). \tag{5.3.42}$$

As we are calculating the EE for a Lorentz invariant vacuum, (5.3.40) holds true. We need to express (5.3.42) in terms of the proper length of the intervals. As we have their endpoints by construction, we can trivially see that

$$\begin{aligned} \Delta S_{\mathcal{A}} = l \quad &\Longrightarrow \quad \text{defining} \quad l \equiv t, \\ \Delta S_{\mathcal{A}_\uparrow} &= (l - 2\epsilon)^2, \\ \Delta S_{\mathcal{A}_\nearrow} &= \Delta S_{\mathcal{A}_\nwarrow} = \sqrt{l^2 - 2l\epsilon}. \end{aligned} \tag{5.3.43}$$

In equation (5.3.42), it yields

$$S\left(l - 2\epsilon\right) + S(l) \leq 2S\left(\sqrt{l^2 - 2l\epsilon}\right). \tag{5.3.44}$$

Expanding in $\mathcal{O}(\epsilon^2)$, we have

$$\begin{aligned} S\left(l - 2\epsilon\right) + S(l) &\leq 2S\left(\sqrt{l^2 - 2l\epsilon}\right) \\ &\leq 2S\left(l - \epsilon - \frac{\epsilon^2}{2l} + \mathcal{O}(\epsilon^3)\right), \end{aligned} \tag{5.3.45}$$

redefining $\epsilon \to \frac{\epsilon}{2}$, we write

$$S\left(l - \epsilon\right) + S(l) \leq 2S\left(l - \frac{\epsilon}{2} - \frac{\epsilon^2}{8l}\right). \tag{5.3.46}$$

Up to $\mathcal{O}(\epsilon^2)$, we evaluate the RHS to be

$$2S\left(l - \frac{\epsilon}{2} - \frac{\epsilon^2}{8l}\right) = 2\left(S(l) + S'(l)\left(-\frac{\epsilon}{2} - \frac{\epsilon^2}{8l}\right) + \frac{S''(l)}{2}\frac{\epsilon^2}{4}\right). \tag{5.3.47}$$

The LHS is

$$S(l - \epsilon) + S(l) = S(l) - S'(l)\epsilon + \frac{S''(l)}{2}\epsilon^2 + S(l), \tag{5.3.48}$$

plugging RHS and LHS in (5.3.46) yields

$$lS''(l) + S'(l) \leq 0. \tag{5.3.49}$$

This result is obtained by evoking unitarity of the CFT_2 and the SSA property of EE. A monotonic quantity under dilatations such as (5.3.49) is in general the main ingredient to construct a C-function.

As already discussed in Cardy's proof of the C-theorem (subsection (5.2.2)), we can rephrase the definition of a C-function in terms of a dimensionless function which is non-increasing under dilatation.[7] Define

$$C(l) = 3lS'(l). \tag{5.3.50}$$

[7]The flow is now parameterized by a length scale l which can be identified as the length of the interval A.

By (5.3.49), we have

$$C'(l) \leq 0. \tag{5.3.51}$$

(5.3.50) defines a dimensionless quantity, since the entropy $S(l)$ is dimensionless and the derivative of the length scale l scales as the inverse of l. Outside the fixed points of the flow (CFT_{UV} and CFT_{IR}), one cannot explicitly calculate the value of the C-function, because one does not have a general formula for $S(l)$ of a general QFT_2. However, it is not a problem, since the C-function is monotonic (5.3.51) and can be evaluated at the fixed points.

At the fixed points, conformal symmetry is restored, and the system is again described by a CFT, which means we can explicitly evaluate the C-function via (5.3.1),

$$\begin{aligned} C(l) &= 3l\frac{d}{dl}\left(\frac{C_V}{3}\log\left(\frac{l}{\epsilon}\right)\right) \\ &= C_V. \end{aligned} \tag{5.3.52}$$

The defined *C-function* takes the value of the central charge C_V at the fixed points. Furthermore, it must be stationary at the fixed points, and one can check that indeed it satisfies

$$C'(l) = 0. \tag{5.3.53}$$

Finally we notice that at the UV fixed point, the CFT_{UV} has central charge C_{UV}, and at the end of the flow, the IR fixed point has central charge C_{IR}, which yields the decreasing of the central charge of two CFTs connected by an RG flow,

$$C_{UV} \geq C_{IR}. \tag{5.3.54}$$

This proof of the C-theorem makes no RG calculations, no use of Callan-Symanzik equations or even arguments based on perturbative renormalization. It reveals a more natural and intuitive approach to the irreversibility problem of RG flows, since it connects the notion of entropy to account for the loss information as we change scales.

The *C-function* is nothing more than the dilatation operator acting on the entropy $S(l)$, and it decreases as we go into the IR region. It is expected since the RG step consists of integrating out a portion of modes of the theory, and by integrating those modes out the number of entangling pairs of the system must decrease. $S(A)$ is precisely the quantitative measure of the number of entangling pairs between the A region and the A^c region, so it is sensible to this loss of information occurring as the renormalization group acts in coupling space, getting rid of the most energetic modes once entangled.

Chapter 6

The F-Theorem

6.1 Introduction

The C-Theorem imposes constraints on the possible RG trajectories of two-dimensional QFTs by proving the existence of a C-function monotonic under the RG flow. Similarly, the F-Theorem proves the existence of a monotonic F-function, which is the three-dimensional analog of the C-function.

Although very similar, the fundamental parameters which are ordered under RG flows have distinct properties. In the case of two dimensions, the parameter is identified with the central charge of 2D CFTs, and it can be obtained via two-point functions of local operators. Also, we have shown that it is the constant appearing in the central extension of the conformal algebra.

The three-dimensional case does not share the same features. Recall that the central charge is the coefficient which multiplies the conformal anomaly, which is a function of local curvature tensors. This locality of the trace anomaly can be traced back to the UV behaviour of the QFT, as we can see from the Seeley-DeWitt expansion. The parameter F which is monotonic is not a local quantity as its two-dimensional partner. In fact, we've shown that no trace anomalies exist in odd space-time dimensions, therefore no analog parameter multiplying local curvature tensors exist.

The first issue is to look for a substitute for ordered parameters in odd dimensions. There are two equivalent definitions of the parameter F. The first one is a definition in terms of the free energy of a system placed on a S^3.[1] Just as in the semi-classical analysis provided in chapter 4, consider coupling a field theory with a classical background metric $g_{\mu\nu}$, and take the metric to be the one of a S^3 manifold. The partition function reads

$$\mathcal{Z}_{S^3} = \int [\mathcal{D}\Phi] e^{-S[\Phi, g_{\mu\nu}]}. \tag{6.1.1}$$

The free energy F is defined as

$$F \equiv -\log \left| \mathcal{Z}_{S^3} \right|. \tag{6.1.2}$$

[1] Or S^D for a generalization of the parameter F.

The F-theorem states the existence of a monotonic function, non-increasing under the RG trajectories, which takes the value of the free energy F at the fixed points. Defining the F-coefficient to be the free energy is a smart move computationally. To calculate F, it is a matter of calculating the determinant of the kinetic operator in the action, which might be done in the heat kernel approach, although several complications might occur [30] . We rewrite the action as the inner product between a field configuration and an operator $\hat{O}$ acting on it

$$\begin{aligned}\mathcal{Z}_{S^3} &= \int [\mathcal{D}\Phi] e^{-S[\Phi, g_{\mu\nu}]} \\ &= \int [\mathcal{D}\Phi] e^{-\langle \Phi, \hat{O}\Phi \rangle}.\end{aligned} \tag{6.1.3}$$

Schematically, the free energy is written as

$$\begin{aligned}F &= -\log \left| \mathcal{Z}_{S^3} \right| \\ &= -\log \int [\mathcal{D}\Phi] e^{-\langle \Phi, \hat{O}\Phi \rangle} \\ &= -\log \left(\left(\det \hat{O} \right)^{-\frac{1}{2}} \right) \\ &= \frac{1}{2} \frac{d}{ds} \zeta_{\hat{O}}(s) \Big|_{s=0},\end{aligned} \tag{6.1.4}$$

using the definition (4.4.1). Nonetheless, this is not the only way to define the parameter F.

The task of calculating the EE of the vacuum of an arbitrary QFT in D dimensions is a complicated one. In order to compute the EE, one has to define a state for which the EE is to be calculated, and an entangling region on a Cauchy slice of the space-time in which the QFT is defined. The Cauchy slice of a QFT in D space-time dimensions is a submanifold of codimension 1. Typically, one takes the space-time to be $\mathcal{M}^{1,D-1}$, and a natural Cauchy surface is given by a subspace of constant time, so an Euclidean space $\mathbb{R}^{D-1}$.

We choose the QFT to be in the vacuum state, and for an entangling surface a $D-1$ sphere S^{D-1}. With this setup, the EE will be a complicated function of the radius of the S^{D-1} and a UV cutoff ϵ. But if the QFT is a CFT, one gets the generic result

$$S(r) = \mu_{D-2} r^{D-2} + \mu_{D-4} r^{D-4} + \cdots + f_D(r, \epsilon), \tag{6.1.5}$$

where

$$f_D(r, \epsilon) = (-1)^{\frac{D}{2}-1} A \log\left(\frac{r}{\epsilon}\right) \quad \text{for } D \text{ even}, \tag{6.1.6}$$

$$= (-1)^{\frac{D-1}{2}} F \qquad \text{for } D \text{ odd}. \tag{6.1.7}$$

The constant "A" appearing in (6.1.6) is the central charge (4.5.44) for $D = 2$ and (4.5.45) for $D = 4$. This last piece is also the universal term in the EE of the circles S^D, meaning it is cutoff independent.[2]

[2]Rescaling the cutoff $\epsilon \to \epsilon'$ leaves those terms invariant.

The constant "F" in the odd case seems to naturally extend the even case of conformal anomalies. As we will see further on, this constant is an ordered parameter under the RG trajectories, and it coincides with the free energy given by (6.1.2), as proved by Casini, Myers et al. In [31].

The second definition of F is a statement about EE of three-dimensional QFTs. A natural question is if one can use techniques similar to the ones used to proved the C-theorem, to prove its three-dimensional analog, the F-Theorem. Put it in other words: is unitarity, SSA, and vacuum Lorentz invariance constraining enough to lead to an irreversibility of interpolating RG flows in three dimensions? The answer is yes, but there are several complications that need to be dealt with before one can claim such an answer.

The proof that follows is based on the work of Casini et.al [32], where the authors most beautifully construct a natural extension of the C-Theorem using the same three ingredients of SSA of EE, Lorentz invariance of the vacuum, and unitarity of QFTs in three dimensions.

6.2 Entanglement Entropy F-Theorem

The general strategy for proving the F-Theorem consists in constructing a three-dimensional analog of the geometrical setup of the C-Theorem. In two dimensions, we have defined entangling surfaces inside a light-cone in $\mathcal{M}^{1,1}$, such that the boundaries of those surfaces were completely sitting in the surface of the light-cone. At the time we have named the surfaces $\mathcal{A}_{\nwarrow}$, $\mathcal{A}_{\nearrow}$, and $\mathcal{A}_{\uparrow}$. This setup is in order to guarantee that the information contained in an entangling surface could be unitarily evolved to another one, which in turn implies that the EE of both surfaces are the same. Then one used SSA property to cleverly construct unions and intersections of the entangling surfaces to generate an inequality that would ultimately yield a monotonic C-function. Lorentz invariance of the vacuum made possible to write the C-function in terms of the proper length of the entangling surfaces, which has as immediate result the proof of the C-Theorem.

That setup was immensely simplified by the fact that the entangling surfaces were one-dimensional. The union and intersection of those lines were again lines, which made easy the use of SSA once the EE of lines are a function of its proper length. In three dimensions, we consider circles in $\mathcal{M}^{1,2}$ as the entangling surfaces, and union and intersections of circles are not, in general, circles again. It is even worse, because in the present case the unions and intersections may contain cusps which introduce further divergences on the EE, endangering the inequalities imposed by SSA.

In order to make the F parameter appear in the construction, one has to consider entangling surfaces in a way that, at the end of the process of applying SSA, one ends up with circles in $\mathcal{M}^{1,2}$. By defining the right setup it is possible to work around these problems, and get rid of the extra divergences introduced by the cusps emerging from the combination of entangling surfaces.

Consider a light-cone $\mathcal{L}$, such that its tip defines the origin of our coordinate

system. Define a time-like plane P_t.[3] The intersection of P_t and $\mathcal{L}$ defines an ellipse $X^\mu(\tau)$, for some parameter τ. This curve is by definition in the light-cone as well as in the plane P_t. The curve $X^\mu(\tau)$ divides the surface of the light-cone into two disjoint parts: $\mathcal{L}^+$ which is the set of points in the future of the ellipse, and $\mathcal{L}^-$ the set of points in the past of the ellipse.

The ellipse $X^\mu(\tau)$ also divides the plane into two disjoint sets: $P_t^{(in)}$ which is the set of points of the plane P_t bounded by the curve $X^\mu(\tau)$,[4] and $P_t^{(out)}$ which is the complement $P_t - P_t^{(in)}$. As the P_t plane is a time-like one, we can take $P_t^{(in)}$ as an entangling surface B. Notice that as $\mathcal{L}^-$ and B have the same boundary, defined by $X^\mu(\tau)$, and lie inside the same light-cone $\mathcal{L}$, they define surfaces with the same EE $S(\mathcal{L}^-) = S(B)$. One can see this by noticing that we can deform continuously the region B until it reaches $\mathcal{L}^-$, with no point leaving the inside of the light-cone, so that the region $\mathcal{L}^-$ defines an entangling surface A.

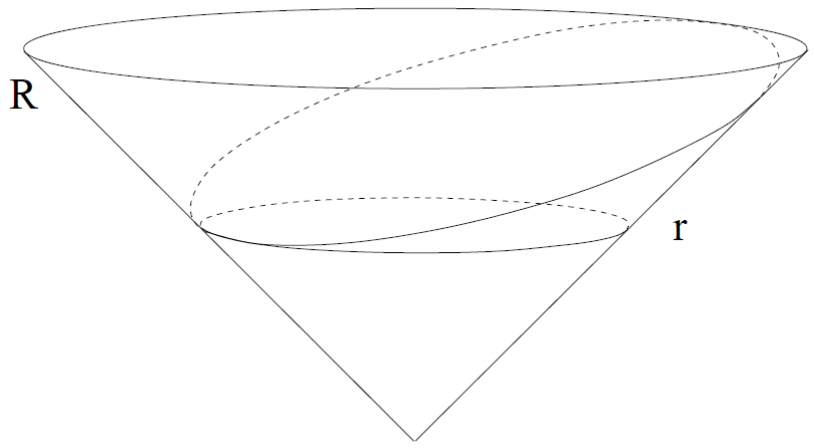

Figure 6.1: The Light-cone $\mathcal{L}$ displayed with its tip pointing "down". The ellipse $X^\mu(\tau)$ is represented by the curve on the surface of $\mathcal{L}$. The plane P_t is tangent to the circles of radii r and R. The bottom part of the cone (from the ellipse to the tip) defines $\mathcal{L}^-$.[31]

The time axis would correspond to a line oriented upwards passing through the tip of $\mathcal{L}$ and the center of the circles of radii r and R, defining the center of the light-cone. Consider rotating the plane P_t around the time axis by an angle $\theta_k \equiv (2\pi)/k$, defining a new plane P_t'. The intersection of P_t' and $\mathcal{L}$ will define a new ellipse $X'^\mu(\tau)$ lying in both the surface of $\mathcal{L}$ and in the plane P_t'. One can see that this new rotated setup defines a rotated copy of the entangling surface B, which we call B' (defined by the set of points in P_t' bounded by $X'^\mu(\tau)$). B' can be continuously deformed to a surface $\mathcal{L}'^-$ completely lying in the bottom part (in relation to the new ellipse $X'^\mu(\tau)$) of the light-cone $\mathcal{L}$, such that $\partial B' = \partial \mathcal{L}'^-$. The surface $\mathcal{L}'^-$ is a rotated version of $\mathcal{L}^-$, and we take it to be a new entangling surface A'. We say that A' is the rotated copy of A by an angle of θ_k.

[3] A time-like plane is a plane whose normal vector is time-like.

[4] We include the points of the ellipse (the points in the surface of $\mathcal{L}$) in the definition of $P_t^{(in)}$.

Consider a set of k entangling surfaces indexed by an integer n as $A_n^{(k)}$, defined by the following procedure. Define as $A_0^{(k)}$ the entangling surface A from the previous construction. Define the surface $A_n^{(k)}$ to be the rotated copy of A by an angle $\theta_n^{(k)} \equiv (2\pi n)/k$, where $n = 0, \cdots, k-1$. The set of all surfaces for a given k is denoted $\{A_n^{(k)}\}$. These surfaces are the building blocks of the proof, and we are ultimately interested in the continuum case when $k \to \infty$.

For each surface $A_n^{(k)}$ there is a correspondent $B_n^{(k)}$ which is lying in a time-like plane, completely inside the light-cone, and has the property $\partial B_n^{(k)} = A_n^{(k)}$, which implies $S(B_n^{(k)}) = S(A_n^{(k)})$.

The SSA property of EE for two arbitrary entangling surfaces A and B is

$$S(A) + S(B) \geq S(A \cap B) + S(A \cup B). \tag{6.2.1}$$

Adding the contribution of a third region C, it yields

$$\begin{aligned} S(A) + S(B) + S(C) &\geq S(A \cap B) + \underbrace{S(A \cup B) + S(C)}_{\text{SSA}} \\ &\geq \underbrace{S(A \cap B) + S((A \cup B) \cap C)}_{\text{SSA}} + S(A \cup B \cup C) \\ &\geq S(A \cup B \cup C) + S(A \cup B \cup C) \\ &\quad + S((A \cap C) \cup (A \cap B) \cup (C \cap B)) + S(A \cap B \cap C). \end{aligned} \tag{6.2.2}$$

We deduce that for an arbitrary number of sets A_i, the sum of entropies satisfies

$$\sum_{i=0}^{N-1} S(A_i) \geq S(\cup_i A_i) + S(\cup_{\{i,j\}}(A_i \cap A_j)) + S(\cup_{\{i,j,k\}}(A_i \cap A_j \cap A_k)) \cdots + S(\cap_i A_i). \tag{6.2.3}$$

The notation $\cup_{\{i,j\}}(A_i \cap A_j)$ means the union of all of pairwise intersections, with n indexes it means the union of all n by n intersections. It is easy to see that there are N terms in each side of (6.2.3). This inequality is the three-dimensional substitute for (5.3.42) of the C-theorem. The treatment given so far defines arbitrary cuts in the light-cone $\mathcal{L}$. In order to proceed, one needs to define the space-like plane P_t, thus defining the first entangling surface A, and the prescription for creating the rotated copies follows immediately.

The light-cone[5] is defined as

$$\mathcal{L} \equiv \{(x, y, t) \quad \text{such that} \quad x^2 + y^2 = t^2\}, \tag{6.2.4}$$

while the plane P_t is defined as

$$P_t \equiv \left\{(x, y, t) \quad \text{such that} \quad t = \frac{x(R-r)}{(R+r)} + \frac{2rR}{(R+r)}\right\}. \tag{6.2.5}$$

For $r, R > 0$ and $R > r$. The intersection of $\mathcal{L}$ and P_t defines the first ellipse $X^\mu(\tau)$, in a non-parametric description we have

$$Rr = \left(x - \frac{R-r}{2}\right)^2 + y^2 - \left(t - \frac{R-r}{2}\right)^2. \tag{6.2.6}$$

[5]Sometimes we use the term inside the light-cone, which means $x^2 + y^2 \leq t^2$.

An ellipse such as (6.2.6) is the Lorentzian analog of an Euclidean sphere S^2, as the latter can be described in an embedding space $\mathbb{R}^3$ by

$$x^2 + y^2 + t^2 = R^2, \tag{6.2.7}$$

where R is the radius of the S^2. In $\mathcal{M}^{1,2}$, the metric picks up a minus sign on the time component, so the natural analog is

$$x^2 + y^2 - t^2 = R^2. \tag{6.2.8}$$

This defines a so-called "boosted sphere", which can also be seen as $\Delta s^2 = R^2$, where Δs is the invariant line element. As it can be defined through the line element Δs, it is obvious that all physical observers describe the same radius R. The radius R is called the proper radius of the boosted sphere. Equation (6.2.6) defines a boosted sphere dislocated from the origin, with proper radius $\sqrt{Rr}$. We can also see from the definition, that rotations around the time axis leave the sphere invariant, for they are implemented by the action of $SO(2)$ in the (x, y) coordinates, and the combination $x^2 + y^2$ is invariant.

The constants r and R play an important role, they will define the radius of the entangling surface defined by the intersection of all $\{A_n^{(k)}\}$ and the union of them all respectively, in the limit $k \to \infty$. To better understand this claim, consider the case $k = 4$, so we have $\{A_0, A_1, A_2, A_3\}$ as the entangling surfaces. Projecting the boosted spheres in the (x, y) plane (6.2), one sees that the intersection of them all approximates a circle with radius ~ 2 in the figure.[6] The union of pairwise entangling surfaces generate the surface which presents the outer set of cusps, such cusps lie on a circle of radius ~ 3.5 in the figure (6.2). The union of three by three intersection give rise to the entangling surface with cusps lying on a circle of radius ~ 2.8.

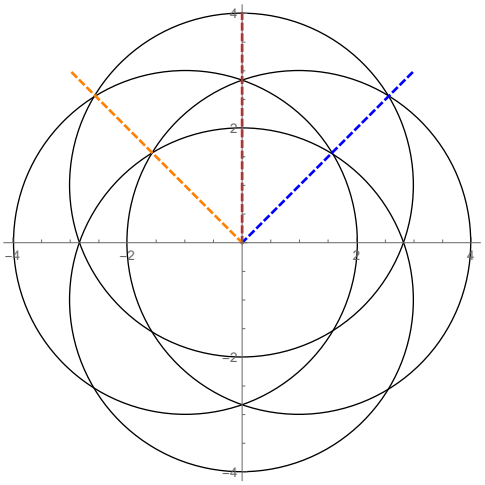

Figure 6.2: The projection of the four rotated boosted spheres in the (x,y) plane are shown in black. The dashed lines are: $y = x \tan(\theta_n^{(k)}/2)$.

The more boosted spheres are considered, the closer to circles the sets on RHS of (6.2.3) become. Consider the case $k = 8$, one can see the sets converging in image (6.2).

[6] We have used $r = 2$ and $R = 4$ to generate the image.

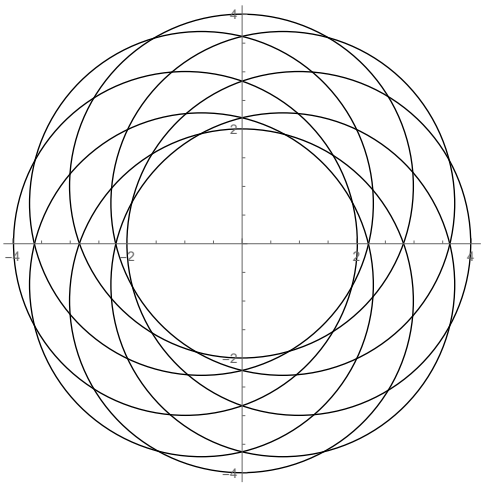

Figure 6.3: The projection of the eight rotated boosted spheres in the (x,y) plane are shown in black. The sets begin to look like wiggly circles.

For arbitrary k, one has k entangling surfaces approaching circles with radius between r and R. The area enclosed by the boundary of the unions of intersections of entangling surfaces do get arbitrarily close to the one of a circle lying in a plane of constant time, but we can see that the perimeter contains an arbitrary number of cusps, and do not approximate the one of a circle in the large k limit. The cusps arising from the intersections present a real problem, for they introduce extra divergences to the entanglement entropy of the approximated circle.

As the resulting entangling surfaces converge to a circle with "wiggly" boundary, one can express its entanglement entropy in terms of the EE of a perfect circle plus an extra term accounting for the defects in the border.

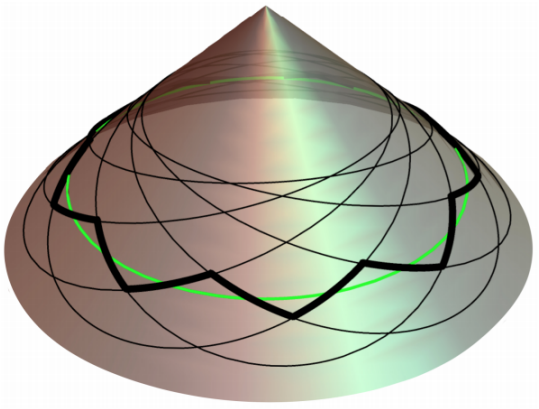

Figure 6.4: The shape in solid black shows the boundary of one of the resulting entangling surfaces. The circle in green approaches the $k \to \infty$ limit of the solid black set. [14]

In a three-dimensional QFT in the vacuum state, the EE of a region A with

smooth boundary follows the behaviour [11]

$$S(A) = \gamma \cdot \frac{|\partial A|}{\epsilon} - F. \tag{6.2.9}$$

The parameter γ is a numerical coefficient, and $|\partial A|$ is the length of the boundary of the entangling region. However, if the boundary of A presents a cusp, which is defined by its angle Ω,[7] there is an extra term in the EE proportional to the log of the cutoff ϵ

$$S(A) = \gamma \cdot \frac{|\partial A|}{\epsilon} + f(\Omega)\log(\epsilon) - F. \tag{6.2.10}$$

Taking the entangling region A to be a perfect circle, the entanglement entropy can be seen from (6.1.7) to be

$$S(A) = \mu_1 \frac{r}{\epsilon} - F. \tag{6.2.11}$$

The "wiggly" circles have its entropy given by

$$S(A) = \mu_1 \frac{r}{\epsilon} + f(\Omega)\log(\epsilon) - F. \tag{6.2.12}$$

This imposes a serious problem to extract useful information from (6.2.3), because the two sides of the inequality have different functional structure of divergences. To see this, consider the case of $k \to \infty$, the LHS of (6.2.3) consists of $k \to \infty$ copies of (6.2.11), and the RHS consists of a sum of entropy of "wiggly" circles with radius between r and R, which schematically gives us

$$k(\mu_1 \frac{r}{\epsilon} - F) \geq \tilde{\mathcal{N}}(\frac{g(r,R)}{\epsilon} + h(r,R)\log(\epsilon) - F), \tag{6.2.13}$$

where $\tilde{\mathcal{N}}$ is a possibly divergent overall constant, $g(r,R)$ and $h(r,R)$ contain the information from the geometrical parameters r and R. The point is that it is very hard to extract any information about this inequality, because there are different orders of divergence on each side.

The cusps that arise from the constructions are only apparent, and do not impose the extra logarithm divergence as in the general case. This is due to the fact that all boundaries of the entangling surfaces are placed in the surface of the light-cone by definition. Which means their intersections, which yield the cusps, are also placed in the surface of the cone. There is no notion of angle in a light-cone, because the tangent vectors of a given point of the light-cone always span a null plane. Therefore we can conclude that the angles Ω that control the cusps are in fact $\Omega = \pi$.

With the vanishing of the cusp contributions, it is now possible to evaluate (6.2.3) for $N = k$ copies of the boosted spheres $\{A_n^{(k)}\}$. From (6.2) we can see that the set $\cap_n A_n^{(k)}$ converges to the inner wiggly circle of radius r, by construction. The next entangling surface from inside out is the set given by $\cup_{\{i_1,\cdots,i_3\}} \left(A_{i_1}^{(k)} \cap \cdots \cap A_{i_3}^{(k)}\right)$. Its cusps lie on a circle, and one of them is also intersected by the line $y = x\tan(\pi/4)$, being $\pi/2$ the rotation angle of the $N = 4$ construction. We can further notice that the cusps of the n^{th} inner entangling surface lie on the line $y = x\tan(n\pi/4)$.

[7] A smooth line corresponds to $\Omega = \pi$, so that $f(\Omega) = 0$.

The generalization for an arbitrary number N of boosted spheres is close to trivial. The rotation angle becomes $\theta = 2\pi/N$ and the cusps of the n^{th} outer entangling surface lie on circles. Also there is at least one cusp of the n^{th} outer entangling surface that lies on the line $y = x\tan(n\pi/N)$. This gives us a system of three equations to find the radius l_n of the n^{th} wiggly circle,

$$t = x\left(\frac{R-r}{R+r}\right) + \frac{2rR}{R+r} \qquad \text{plane,} \tag{6.2.14}$$

$$Rr = \left(x - \frac{R-r}{2}\right)^2 + y^2 - \left(t - \frac{R-r}{2}\right)^2 \qquad \text{boosted-sphere,} \tag{6.2.15}$$

$$y_n = x\tan\left(\frac{n\theta}{2}\right) \qquad \text{line.} \tag{6.2.16}$$

In polar coordinates on (x, y) plane, we have $l_n^2 = x^2 + y_n^2$. Solving the system yields

$$l_n = \frac{2rR}{r + R - (R-r)\cos\left(\frac{\theta n}{N}\right)}. \tag{6.2.17}$$

We have chosen the QFT to be in the vacuum state in order to compute the EE. As the vacuum state is Lorentz invariant, the EE must be a function of the proper radius of the boosted spheres, which is also an invariant quantity under Lorentz transformations. This is the three-dimensional analog of the dependence of the EE on the invariant interval on two-dimensional systems.

Strong subadditivity gives us N copies of a boosted sphere with proper radius $\sqrt{rR}$ in the LHS. And the RHS is a sum of wiggly circles with radii l_n,

$$NS(\sqrt{rR}) \geq \sum_{n=1}^{N} \tilde{S}\left(\frac{2rR}{R + r - (R-r)\cos\left(\frac{\pi n}{N}\right)}\right), \tag{6.2.18}$$

the $\tilde{S}$ is to remind us that the entropy converges to the one of a circle only in the limit $N \to \infty$. Let $z = n\pi/N$, one may take the limit for $N \to \infty$. Taking the continuum version of (6.2.18) yields $\tilde{S} \to S$, and the expression becomes

$$S(\sqrt{rR}) \geq \frac{1}{\pi}\int_0^{\pi} dz S\left(\frac{2rR}{R + r - (R-r)\cos(z)}\right). \tag{6.2.19}$$

Taking the limit where $R = r + \epsilon$ and expanding in order of ϵ^2, the LHS of (6.2.19) reads

$$S(\sqrt{rR}) = S(r) + S'(r)\frac{\epsilon}{2} + \left(-\frac{S'(r)}{8r} + \frac{S''(r)}{8}\right), \tag{6.2.20}$$

where $S'(r)$ denotes $\frac{dS(r)}{dr}$. Expanding the RHS of (6.2.19), we get

$$\begin{aligned}
&\frac{1}{\pi}\int_0^{\pi} dz S\left(\frac{2rR}{R + r - (R-r)\cos(z)}\right) \\
&= \frac{1}{\pi}\int_0^{\pi} dz S\left(r + \frac{1}{2}(1+\cos(z))\epsilon + (\cos^2(z) - 1)\frac{\epsilon^2}{4r}\right) \\
&= \frac{1}{\pi}\int_0^{\pi} dz \left(S(r) + \frac{\epsilon}{2}(1+\cos(\theta))S'(r) + \left(\frac{(\cos^2(z)-1)}{4r}S'(r) + \frac{(1+\cos(z))^2}{8}S''(r)\right)\epsilon^2\right) \\
&= S(r) + \frac{\epsilon}{16r}\left((8r - 2\epsilon)S'(r) + 3r\epsilon S''(r)\right).
\end{aligned} \tag{6.2.21}$$

Plugging (6.2.20) in LHS of (6.2.19), and (6.2.21) in its RHS, we collect

$$S''(r) \leq 0. \tag{6.2.22}$$

Consider the C-function

$$C(r) \equiv rS'(r) - S(r). \tag{6.2.23}$$

It follows from (6.2.22) that

$$\begin{aligned} C'(r) &= rS''(r) + S'(r) - S'(r) \\ &= rS''(r) \leq 0. \end{aligned} \tag{6.2.24}$$

$C(r)$ defines a dimensionless function non-increasing under dilatations. Furthermore, at the stationary points of an RG flow conformal symmetry is restored, and we may use (6.1.7) to write

$$C'(r) = 0. \tag{6.2.25}$$

It is a monotonic function stationary at the fixed points of the RG flow. Given that the CFT in the IR limit is located at r_{IR}, and the UV CFT at r_{UV}. It follows from (6.2.24) that

$$C(r_{\text{UV}}) \geq C(r_{\text{IR}}). \tag{6.2.26}$$

Again, at the fixed points (6.1.7) is valid, and we may write

$$C(r_{\text{UV}}) = F_{UV} \qquad \text{and} \qquad C(r_{\text{IR}}) = F_{IR}. \tag{6.2.27}$$

The immediate consequence is the ordering of the F parameter,

$$F_{UV} \geq F_{IR}. \tag{6.2.28}$$

We are led to the conclusion that, as in the two-dimensional case, strong subadditivity and Lorentz invariance are strong enough to yield the ordering of the universal term of the entanglement entropy of a circle. This proof may be viewed as a generalization of the one presented by Casini in [12], since the central charge in that case is the universal term of EE of a single interval in a two-dimensional field theory.

Chapter 7

Conclusions

Upon investigating the behaviour of the trace anomaly coefficients under the renormalization group equations, we can say that the Poincaré symmetry along with the SSA property of EE imply the existence of a monotonic C-function, which renders the relativistic flow between two fixed points irreversible. This irreversibility is due to a natural preferred direction of the flow, from higher central charges to lower central charges CFTs in both two- and three-dimensional space-times.

The use of EE in the proof of the theorems seems to be the right approach since it naturally entails the ideas of interacting degrees of freedom, spontaneously connecting to the idea of measuring the lost information through the coarse-graining and mode integration steps of the RG flow. Furthermore, in this quantum information language, a natural definition of the ordered parameters arise on equal footing for even and odd space-time dimensional QFTs via the universal term of the EE of an S^D in $\mathcal{M}^{1,D-1}$.

The fact that the C-functions take the value of the central charges at the fixed points makes it safe to say that the charges are a faithful measure of the degrees of freedom of the CTFs. As the RG draws a curve in parameter space, such a function ensures that if the RG reaches an endpoint, it will correspond to a less fundamental description of the system than the QFT at the UV limit, in accordance with intuition.

Although much more elegant, the proofs relying on EE do not make explicit the geometrical character of the parameter space, as does the approach taken by Zamolodchikov by actually applying the Callan-Symanzik equations. This geometrical approach to the space of couplings is of great use when one tries to classify the behaviour of perturbative quantum field theories in a given dimension.

By understanding the parameter space as a manifold, one assigns the status of a "theory" to every point of the space. The β-functions generate a vector field over the parameter space, which are responsible to the existence of RG flows. A flow line correspond to a class of QFTs which have the same correlation functions but different cutoffs, or equivalently, a class of generating functions that describe the same physical system in different length scales.

Given that a $\mathrm{CFT}_{\mathrm{IR}}$ is the endpoint of an RG flow started by a relevant perturbation of a $\mathrm{CFT}_{\mathrm{UV}}$, the order theorems says that $C_{UV} \geq C_{IR}$, meaning that the field content or overall degrees of freedom must not increase as we flow to the IR

fixed point. In the case that $C_{UV} > C_{IR}$, the theorems have as direct consequence the inexistence of Poincaré preserving RG flows that would take a perturbation[1] of the CFT_{IR} back to the CFT_{UV} fixed point. In turn we are left only with flows that take us to even lower central charge CFTs, thus a notion of a "cascade" in parameter space is introduced, where at each node there sits a CFT. The existence of cycles between two fixed points is also ruled out by the theorems, as one is unable to reverse flows.

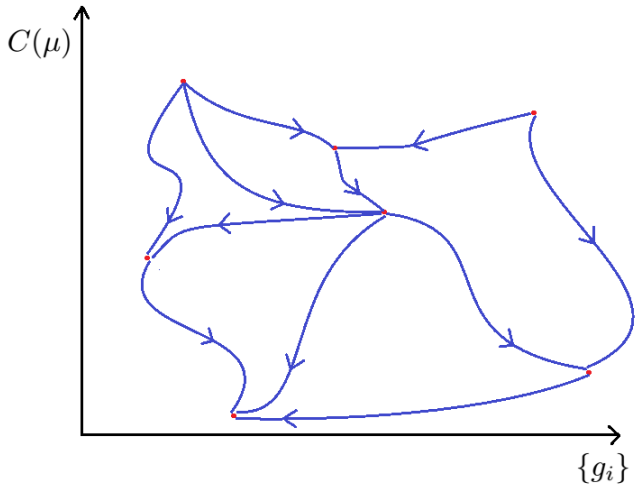

Figure 7.1: The fixed points of the RG flows are displayed in red. The blue lines represent Poincaré preserving RG flows with the direction indicated by the arrows. The whole parameter space is pictorially represented by a single axis.

Figure (7) draws a scenario on the coupling space in which the CFTs with high central charge occupy the higher ground. The flows are started by considering a small relevant perturbation of the CFTs, and a notion of order is given because of the natural direction in which the flows occur as the momentum scale decreases.

This picture might be helpful for clarifying the fixed points of the space of couplings. Furthermore, it may also provide insights over a generic perturbative QFT that could be described by an action functional which is a perturbation of a Gaussian fixed point[2],

$$S[g_i] = S_{\text{gaussian}} + \underbrace{\int d^D x g_i \Phi_i(x)}_{\text{perturbation}} . \tag{7.0.1}$$

This would correspond to theories in the margins of the fixed points. The constraints imposed in the Poincaré preserving RG trajectories by the C and F theorems could help to predict some IR behaviour of such theories, as for example the existence of confinement or IR triviality of the S-matrix.

[1] One generally considers perturbations around the CFTs because the CFT itself is a fixed point where the β-functions vanish due to scale symmetry.

[2] A Gaussian fixed point corresponds to a theory that can be exactly solvable by computing the functional integral. Not all CFTs are Gaussian fixed points.

Finally, it is natural to ask if there exists the analogous of the C-Theorem for arbitrary dimensions. At the present moment, it is still an open problem and it seems that holographic approaches are the best way to go [33]. In four dimensions, the analog is called A-Theorem and has been proved in two different ways so far: by properties of the dilaton two-point functions by Komargodski and Schwimmer [34], and by extending the entanglement entropy proof of the F-Theorem as done by Casini et al. In [14].

It happens that the extension of the geometrical setup of the F-theorem for four space-time dimensions is not trivial, and there are more divergence problems coming from cusps of codimensions 2 and 3. Also it turns out that strong subadditivity of EE, Poincré invariance and unitarity are no longer enough to imply the existence of an A-Theorem. One must make use of the so-called "Markov property" of the CFT vacuum to reach the desired A-Theorem.

A generalization of the setup for the arbitrary-dimensional space-time is still unknown. Nonetheless, the existence of the theorems in any dimensions is to be expected since the ingredients used to build these theorems are not special features of any particular space-time dimension, even less are the mathematical apparatus and physical considerations that were made to prove them.

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