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# Ordem Topológica em Sistemas Tridimensionais e Simetria de 2-Gauge

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# Topological Order in Three-dimensional Systems and 2-Gauge Symmetry

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## Resumo

Ordem topológica é um novo paradigma para fases quânticas da matéria desenvolvido para explicar transições de fase que não se encaixam no esquema de classificação de fases da matéria por quebra de simetria. Estas fases são caracterizadas por padrões de emaranhamento que levam a uma degenerescência de estado fundamental topológica e a excitações anyônicas. Uma abordagem comum para o estudo de tais fases em sistemas bidimensionais é através de modelos Hamiltonianos exatamente solúveis de rede como os modelos duplos quânticos e modelos de String-Nets. O primeiro pode ser entendido como a formulação Hamiltoniana de teorias de gauge na rede e, desta maneira, é definido por um group de gauge finito. Entretanto, pouco é conhecido a respeito de fases topológicas em sistemas tridimensionais. Motivado por isso nós desenvolvemos uma nova classe de modelos tridimensionais exatamente solúveis que vai além de modelos duplos quânticos pelo uso de módulos cruzados finitos no lugar de grupos de gauge. Esta abordagem se baseia numa implementação em redes de teoria de 2-gauge para obter modelos com uma estrutura topológica mais rica. Nós construímos o modelo Hamiltoniano explicitamente e fornecemos uma demonstração rigorosa de que a degenerescência de estado fundamental é um invariante topológico e que os estados fundamentais só podem ser caracterizados por parâmetros de ordem não locais.

**Palavras-chave:** Mecânica Quântica, Teoria de Gauge, Topológica Algébrica

## Abstract

Topological order is a new paradigm for quantum phases of matter developed to explain phase transitions which do not fit the symmetry breaking scheme for classifying phases of matter. They are characterized by patterns of entanglement that lead to topologically depended ground state degeneracy and anyonic excitations. One common approach for studying such phases in two-dimensional systems is through exactly solvable lattice Hamiltonian models such as quantum double models and String-Net models. The former can be understood as the Hamiltonian formulation of lattice gauge theories and, as such, it is defined by a finite gauge group. However, not much is known about topological phases in tridimensional systems. Motivated by this we develop a new class of three-dimensional exactly solvable models which go beyond quantum double models by using finite crossed modules instead of gauge groups. This approach relies on a lattice implementation of 2-gauge theory to obtain models with a richer topological structure. We construct the Hamiltonian model explicitly and provide a rigorous proof that the ground state degeneracy is a topological invariant and that the ground states can only be characterized with nonlocal order parameters.

**Keywords:** Quantum Mechanics, Gauge Theory, Algebraic Topology

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# Chapter 1

## Introduction

All the matter in the universe is made up of just a few basic building blocks such as electrons and atoms. This might be considered a trivial statement nowadays given the remarkable discoveries of the twentieth century but we would argue that it is still, at the very least, a bit counterintuitive. After all, it seems unlikely that all the different substances we handle on our daily lives are all made of the same elementary constituents given that they display such incredibly diverse behavior. On the other hand, if one holds a piece of copper it is tempting to disregard it as a trivial object and yet we now that behind its deceitful simplicity lies the rich behavior of a huge number of electrons. How can these two perspectives be reconciled so that we can make sense of the world in light of our knowledge of the fundamental laws of physics? The answer to this question lies on the notion of emergence which has become central to modern physics.

From this point of view, the sheer number of particles that make up any macroscopic system is sufficient to give rise to the enormous complexity we observe and also acts as an barrier for us to make sense of its basic properties. It is ironic that even though we know all the basic building blocks of matter and their fundamental laws we can't seem to wrap our heads around a couple of strongly interacting electrons going about their lives. Hence, it is no surprise that after almost a hundred years of quantum mechanics we are still struggling to predict all the states of matter that can emerge in nature. In particular, the problem of classifying the different phases of matter is still, and most likely will remain for some time, one of the greatest challenges of physics even though we can trace it back to ancient times.

As physicist worked to advance our understanding of this issue it has become clear that, due to the huge number of degrees of freedom involved, direct approaches are bound to fail. Instead it is necessary to find an effective description of the different systems which throws away unnecessary microscopic details and keeps only what is essential to describe the macroscopic properties. Landau's symmetry breaking theory [1, 2, 3] was a major breakthrough in this direction and has led to great advances. It relies on the idea that the phases of a given system can be specified by its symmetries so that there is a correspondence between the different symmetries and the phases. In particular, it allows one to understand these phases and how it is possible to transit between each other through the means of a symmetry breaking mechanism and a corresponding local order parameter. In this context, we say that a system has a symmetry breaking order to refer to the internal microscopic structure that describe how its constituents are organized and the associated properties.

It is possible to describe a large variety of phenomena using Landau's paradigm and over the years

it has proved itself to be an extremely successful theory. In fact, for a long time it was believed that it provided a solution to the problem of classifying all phases of matter as there was no known counter example. However, nature always ends up frustrating our apparently perfect theories and Landau's theory was no different. This realization came in the 1980's with the discovery of the Fractional Quantum Hall effect (FQHE) by Tsui et al [4] as it showed the limitations of the symmetry breaking scheme. This effect is observed on a strongly interacting two-dimensional electron gas subjected to strong magnetic field at low temperatures and it is characterized by plateaus of the Hall conductance with fractional filling factors. This phenomena marks the end of the dominance of symmetry breaking theory as it was soon realized it could not be explained using the underlying symmetries and requires the development of a new theoretical framework capable of describing it. From a historical perspective this phenomena was the first discovery of a topological phase of matter [5] which comes from a different type of order not characterized by the usual local observables.

The word topological refers to the fact that, since the order does not come from symmetry breaking, there is no local order parameter to distinguish between the different phases and one must use global quantities instead. An important example of one such quantity is the ground state degeneracy [6] which is controlled by the underlying topology where the system lives and leads to physical properties that have a global nature and can not be understood from a local perspective. Entanglement [7] also provides a way to characterize systems in a topological phase and one can talk about intrinsic topological order when there is long range entanglement [8]. This means that the ground states cannot not be deformed by local unitary transformations into a non entangled state. It is also possible to have topological order with short range entanglement due to symmetry protection, that is, when it is not possible to deform states into to non entangled states without breaking a given symmetry. Another important feature of these systems is the presence of anyonic excitations [9] and the related phenomena of fractionalization first observed in the FQHE and described theoretically by Laughlin [10]. We refer to [5] and [11] for an more detailed overview of the general properties of topological phases of matter.

Since the first measurements of the FQHE a variety of systems not described by a symmetry breaking mechanisms have been discovered and this gave rise to a exponential growth in interest on the subject of topological phases. Also, the potential applications to quantum computation through topological quantum computation [12] have galvanized this field and brought together ideas from quantum information and condensed matter in innovative manner. However, the rapid growth has some disadvantages and it is important clarify what we mean by a topological phase since the term is often use to describe completely different systems. Of course, this is a consequence of the fact that there is no complete understanding of what is the correct definition and the term is often applied to any phase which is not described by a symmetry breaking mechanism.

For our purposes we are going to say a quantum many body system has intrinsic topological order whenever the ground states subspace is made up of long range entangled states characterized by the topology of the underlying space where the system lives. This leaves out some topological states, such as SPT's, but is necessary to have a rigorous approach. From a practical point of view we focus on the condition that the system must have topological degeneracy, that is, that the ground state degeneracy (gsd) is a topological invariant of the space. This is also closely related to the presence of anyonic excitations similar to the ones observed in the FQHE as these excitations are responsible for creating the topologically non trivial ground states.

One the most famous example of a system with topological order is the toric code [13] first introduced by Alexei Kitaev as a quantum error correction code and a potential platform for implementing a quantum memory. It is a many body quantum system defined by assigning spins to the edges of a

finite square lattice with periodic boundary conditions. The name toric code comes from the fact the space underlying the lattice is a torus and this the source of the non trivial topological properties of the model. In particular, there is a 4-fold degeneracy at the level of the ground states which is related to the presence of non contractible curves that give rise to ground states whose quantum numbers are non local quantities. Moreover, these states are highly entangled due to the presence of a local  $\mathbb{Z}_2$  symmetry enforced dynamically by the Hamiltonian which allows one to interpret the toric code as a lattice  $\mathbb{Z}_2$  gauge theory. This point of view lead to the development of a large class of models known as quantum double models [14, 15, 16] which are Hamiltonian realization of lattice gauge theories and are also topologically ordered. These models are defined by an exactly solvable Hamiltonian of the following form:

$$H = - \sum_v A_v - \sum_p B_p \quad (1.1)$$

where  $A_v$  is a local operator enforcing gauge invariance under the action of a gauge group  $G$  and  $B_p$  is a projector which maps into the subspace of states with trivial holonomy. Just like the toric code the states of the system are obtained by assigning local degrees of freedom to the edges of a finite square lattice with periodic boundary conditions. Now, these degrees of freedom belong to the gauge group and each basis state corresponds to the data specifying a gauge connection on the lattice. It is necessary to assume an orientation for the lattice to define the operators in the Hamiltonian and below we describe their action on the basis states using the standard convention:

$$A_v \left| \begin{array}{c} \uparrow g_2 \\ \leftarrow g_1 \quad \rightarrow g_3 \\ \downarrow g_4 \end{array} \right\rangle = \frac{1}{|G|} \sum_{h \in G} \left| \begin{array}{c} \uparrow hg_2 \\ \leftarrow hg_1 h^{-1} \quad \rightarrow hg_3 \\ \downarrow hg_4 h^{-1} \end{array} \right\rangle \quad (1.2)$$

$$B_v \left| \begin{array}{c} \rightarrow g_2 \\ \uparrow g_1 \quad \downarrow g_3 \\ \rightarrow g_4 \end{array} \right\rangle = \delta(g_1 g_2 g_3^{-1} g_4^{-1}, 1) \left| \begin{array}{c} \rightarrow g_2 \\ \uparrow g_1 \quad \downarrow g_3 \\ \rightarrow g_4 \end{array} \right\rangle \quad (1.3)$$

The ground states of quantum double models are related to non contractible curves on the torus which is why the ground state degeneracy is related to the topology of the surface. Moreover, it is possible to define the model for arbitrary compact surfaces and demonstrate that the ground state degeneracy only depends on the first homotopy group of the surface. The observables which can distinguish the ground states are Wilson loops [17] calculated along non contractible curves and, as such, are necessarily non local. This can be understood through a partition function of a lattice gauge theory [18, 19] which is given by:

$$Z(L) = \frac{1}{V|G|} \sum_{\mathfrak{U}} \omega(\mathfrak{U}) \quad (1.4)$$

where  $L$  is some lattice realizing a closed three-dimensional manifold  $M$ ,  $V$  is the number of vertexes in the lattice and the sum runs over all gauge configurations  $\mathfrak{U}$  living on the lattice. The weights  $\omega(\mathfrak{U})$  give one for configurations which have trivial curvature everywhere and zero otherwise so that the partition function just counts the number of flat connections modulo gauge equivalence. This partition function gives a topological invariant of the manifold [20] and it was shown in [16] that the ground state degeneracy of a quantum double model defined on a compact surface  $\Sigma$  is controlled by this invariant:

$$\text{gsd} = Z(\Sigma \times S^1) \quad (1.5)$$

This is a consequence of the fact that the partition function in equation 1.5 comes from a topological quantum field theory [21, 22] which controls the low energy properties of quantum double models. This is a general feature of topologically ordered system and can be used to define a more general class of models known as String-Net models [23] which are related to the Turaev-Viro invariants [24] and are said to classify all topological phases for two-dimensional systems in the sense that they realize all types of anyonic excitations. In fact, the great advances in understanding the two-dimensional case are a direct consequence of the existence of a powerful theory that describes topological quantum field theories in dimension  $(2 + 1)d$  using tensor categories[25].

While the theory of topological phases in two-dimensional systems is well developed, not much is known for the three-dimensional case and this is in part due to difficulties in studying topological quantum field theory in dimension  $(3 + 1)d$ . Nonetheless, in recent years some advances have been made [26, 27, 28] motivated in part by the search ofr new quantum error correction codes based on higher dimensional systems [29, 30]. Moreover, it has been realized that three-dimensional system with topological order can have extended excitations as their lowest energy excitations and this opens up many interesting possibilities [31, 32, 33] both a physical and mathematical point of view.

The goal of this dissertation is to contribute to this ongoing effort by introducing a generalization of quantum double models for the three-dimensional setting which provides a Hamiltonian realization of Lattice 2-Gauge theories [34]. Our construction is inspired by the existence of a topological field theory due to Yetter [35, 36] that generalizes the partition function 1.4 and provides a more sensitive topological invariant which, from the point of view of physics, leads to a model with richer structure. To give the reader an overall view of the whole dissertation we present a quick outline of the content present in each chapter:

- Chapter 2 is a review of the basic mathematical tools we are going to use. It is organized in two sections with the first one handling the geometrical requirements and the second one the algebraic needs. In particular, it provides a short introduction to the theory of simplicial complexes and triangulations of three-dimensional manifolds since these structures will provide the lattices where our models will be defined. On the algebraic side we develop some basic notions of two-dimensional algebra, crossed modules and 2-groups, which will provide the algebraic machinery required for 2-gauge theory.
- Chapter 3 is the most important as it is where we construct the Hamiltonian model and demonstrate that it is exactly solvable. We define the operators of the model using ideas from higher gauge theory and try to do so in away that emphasize the connection with quantum double models. For the most part we try to keep everything as straightforward as possible by providing explicit formulas for everything so that the reader can easily follow the computations.
- The discussion in Chapter 4 revolves around the properties of the ground states and contains all of our main results. In particular, we show that the model has topological order by demonstrating that the ground state degeneracy is a topological invariant and that there is no local order parameter. Also, we provide formulas for the ground state degeneracy using techniques from homological algebra and homotopy theory.
- Finally we conclude with a summary of the whole dissertation with the main results obtained in Chapter 5 and discuss some potential directions for further research structure.

The work presented here was conducted within the research group of Prof. Paulo Teotônio Sobrinho and, as such, was a collective effort. In particular, the core of the work was developed in close

collaboration with Kazuo Teramoto as he was focused on studying the space-time point of view and the topological quantum field theory whereas my focus was the Hamiltonian model realization of the field theories. Also, at the later stages Javier Lorca Espiro and Pablo Ibieta Jimenez joined the effort to understand the abelian models and this led to the cohomology formula for the ground state degeneracy. Currently we are writing a paper which summarizes some of these results.

We should also mention that the work of Bullivant et al [37] has great overlap with what we present in Chapter 3 but was developed independently from us and we were only made aware of it once it was published. Hence, while we have developed our formalism independently we make no claim of originality with regards to material which intersects with their work. Nonetheless, we emphasize that our approach is more general due to the extended 2-group of a crossed module which we introduce in Chapter 2 and the results presented in Chapter 4 concerning the characterization of the ground states are original.



# Chapter 2

## Mathematical Background

This chapter aims to quickly review the mathematical tools required for Chapters 3 and 4 in an effort to keep this text self contained and introduce some basic notation. Our goal is to showcase the main ideas in a straightforward manner so the reader can get the overall picture without getting bogged down in technical details. Hence, there is no ambition of providing a full account of any of the topics discussed and we refer to the references we provide for a more in depth discussion.

Our models are going to be defined over triangulations of compact 3-manifolds so we need to develop the formalism required for working with them in a general setting. This is the goal of the first section which deals with simplicial complexes and their topological properties. It starts with a geometrical motivation and then moves on to some formal definitions that rely on a combinatorial description of simplexes. We then discuss the connection between the abstract and the geometrical point of view by describing how to construct the geometric realization of a complex. Afterwards we describe orientations and the role played by enumerations of simplicial complexes and use this to introduce the simplicial homology and cohomology groups. Finally we conclude with a discussion of what is most important for our purposes, namely, triangulations of three-manifolds and how a deep theorem due to Pachner [38] can be used to characterize them. The basic reference for the first section is [39] but we also rely on [40, 41] in our discussion of triangulations of manifolds and use [42] for results coming from algebraic topology.

In the second section we shift gears and move to an algebraic discussion. The reason is that the models we are going to construct in the next chapter are a Hamiltonian realization of 2-gauge theory [34] and this requires the use of some exotic algebraic structures. We begin with a brief discussion revolving around the interpretation of groups as a one-dimensional algebraic structure and the limitations that arise from this. Having done so we move on to our main goal which is to introduce crossed modules and show how they provide a convenient framework for working with 2-groups. These objects are two-dimensional analogues of groups defined using the ideas of higher category theory [43] and encode the internal symmetries of a 2-gauge theory. We follow [44] for the most part but we encourage the reader to take a look at [45] and [46] if he wants to take a deep dive into the subject.

## 2.1 Simplicial Complexes and Triangulations

A common way of describing surfaces, and manifolds in general, is through some kind discretization procedure. For instance, in Chapter 1 we've shown an example where the torus is realized by a square lattice with periodic boundary conditions but it is also possible to achieve the same result with different kinds of decompositions. In particular, one can obtain a torus, or any surface for that matter, by gluing triangles as described in Figure 2.1 and this has great technical advantages when compared to the square lattice. In particular, there is a natural generalization of such triangulations to higher dimensions which allows us to describe more general manifolds in a simple manner.

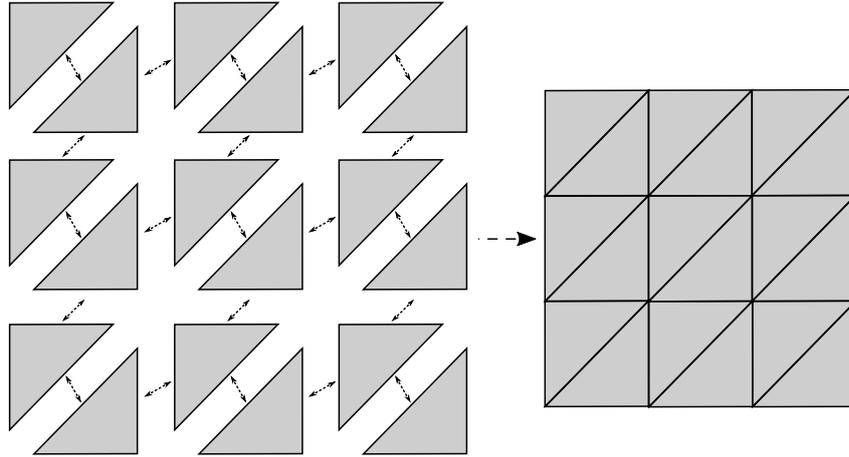


Figure 2.1: Triangulation of a torus obtained by gluing triangles.

In order to work with such triangulations in higher dimensions we must first define the analogues of triangles for arbitrary dimensions. For dimensions 0 and 1 it is clear that we need to use points and line segments, respectively, as there are no other alternatives. Also, notice that points, line segments and triangles can all be recovered by taking the convex hull of the vertexes that define them. Motivated by this we consider the following definition for their higher dimensional counterparts:

**Definition 2.1.** *Given points  $v_0, \dots, v_n \in \mathbb{R}^m$  in general linear position, we define the  $n$ -dimensional geometrical simplex  $(v_0 \dots v_n)$  as the subspace obtained by taking the convex hull of such points. More concretely, it is the set made up of points  $x \in \mathbb{R}^m$  such that:*

$$x = \sum_{i=0}^n x_i v_i$$

where  $x_0, \dots, x_n \in [0, 1]$  are numbers satisfying  $\sum x_i = 1$ . These numbers are called the barycentric coordinates of the point and they completely specify any element of the geometrical simplex.

The condition that the points are in general linear position means that there is no  $(n - 1)$ -dimensional hyperplane that contains all of them simultaneously and it ensures that the geometrical simplex is a  $n$ -dimensional subspace. Formally this is equivalent to saying that the vectors

$$v_1 - v_0, v_2 - v_0, \dots, v_n - v_0$$

are linearly independent and it guarantees that the barycentric coordinates of a point are unique. In particular, the points  $v_i$  always have barycentric coordinates given by  $x_j = \delta_{ij}$  and we refer to them as the vertexes of the simplex.

If we consider the three-dimensional case we see that  $(v_0v_1v_2v_3)$  is the tetrahedron obtained by taking the point  $v_3$  and drawing all the lines connecting it to points in the triangle formed by  $(v_0v_1v_2)$ . Similarly, we obtain a pentachoron  $(v_0v_1v_2v_3v_4)$  if we take a fifth point  $v_4$ , not contained in the three-dimensional hyperplane defined by the tetrahedron, and draw the lines connecting it to the tetrahedron. In Figure 2.2 we depict this situation with some low dimensional examples.

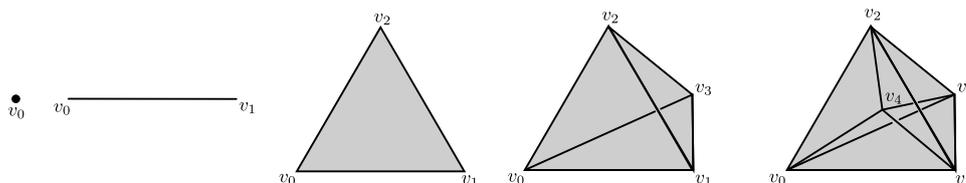


Figure 2.2: Geometrical description of simplexes with dimensions ranging from 0 to 4.

It should be clear from the pictures that the boundary of a geometrical simplex  $(v_0 \dots v_n)$  is made up of all the lower dimensional simplexes defined by its vertexes. That is, by all geometrical simplexes  $(w_0 \dots w_k)$  such that  $\{w_0, \dots, w_k\}$  is a proper subset of  $\{v_0, \dots, v_n\}$ . We call these simplexes the faces of  $(v_0 \dots v_n)$  and we denote this relation by  $(w_0 \dots w_k) \prec (v_0 \dots v_n)$ . A convenient notation for representing the faces of a geometrical simplex is to write

$$(v_0 \dots \hat{v}_i \dots v_n)$$

for the face opposite to  $v_i$ , that is, the face defined by all the vertexes  $v_0, \dots, v_n$  except  $v_i$ . If we iterate this notation by adding more vertexes with "hats" we can specify the faces of a simplex in a concise manner. In Figure 2.3 we show a 2-simplex with all of its faces.

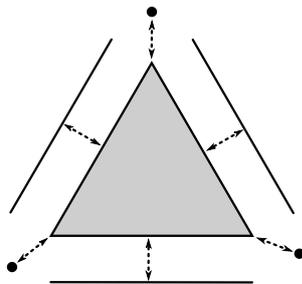


Figure 2.3: The different faces of a 2-dimensional geometrical simplex.

Now that we've defined geometrical simplexes we can introduce the spaces obtained by gluing them together. To do so we refer to Figure 2.1 and observe that we can generalize the procedure described there to construct spaces using geometrical simplexes. Roughly speaking, we can obtain topological spaces by considering a bunch of geometrical simplexes in such a way that their union provide a decomposition of the space. These simplexes must intersect along their boundaries just like the triangles of a triangulation as this ensures that the process is consistent. The spaces obtained this way are called polyhedrons and we provide a formal definition below:

**Definition 2.2.** A polyhedron is a subspace  $K \subset \mathbb{R}^m$  such that there is a collection of geometrical simplexes in  $\mathbb{R}^m$  satisfying the following conditions:

- The polyhedron  $K$  is equal to the union of all the simplexes of the collection
- The faces of a simplex that belongs to the collection also belong to the collection
- Two simplexes of the collection are either disjoint or intersect at a common face

We often abuse the notation by using  $K$  to denote both the actual polyhedron and the corresponding collection. Also, we write  $K_n$  for all the  $n$ -dimensional geometrical simplexes belonging to the collection.

It should be clear from the definition that any geometrical simplex is an example of a polyhedron if we consider the collection formed by the simplex and all of its faces. Explicitly the collection of simplexes corresponding to this polyhedron is:

$$\begin{aligned} K_n &= \{(v_0, \dots, v_n)\} \\ K_{n-1} &= \{(v_0, \dots, \hat{v}_i, \dots, v_n) \mid 0 \leq i \leq n\} \\ K_{n-2} &= \{(v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n) \mid 0 \leq i < j \leq n\} \\ &\dots \\ K_{n-k} &= \{(v_0, \dots, \hat{v}_{i_1}, \dots, \hat{v}_{i_k}, \dots, v_n) \mid 0 \leq i_1 < i_2 < \dots < i_k \leq n\} \\ &\dots \\ K_0 &= \{(v_0), \dots, (v_n)\} \end{aligned}$$

where it is implicit that  $n$  is the top dimension so there are no simplexes with dimension beyond it. It is easy to convince oneself that this polyhedron is homeomorphic to the  $n$ -ball  $B^n$  so we say it is a triangulation of it. Moreover, since the boundary of  $B^n$  is the  $(n-1)$ -sphere  $S^{n-1}$  we can obtain a triangulation of  $S^{n-1}$  by considering the polyhedron associated to the boundary of a geometrical  $n$ -dimensional simplex (see Figure 2.4).

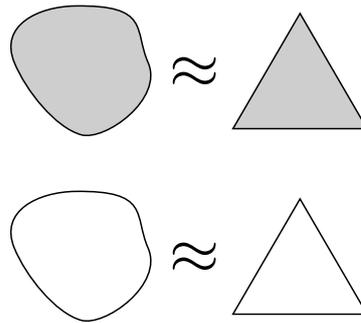


Figure 2.4: Triangulation of  $B^2$  and  $S^1$  coming from a 2-dimensional simplex and its boundary.

In order to keep things rigorous we state a formal definition of triangulations:

**Definition 2.3.** Given a finite dimensional manifold  $M$ , a triangulation of  $M$  is a polyhedron  $K$  together with a homeomorphism  $h : K \rightarrow M$  where we regard the polyhedron as a topological space with the topology induced by the underlying euclidean space.

This definition is fairly straightforward and it is easy to see that the examples we gave, such as the torus in Figure 2.1, are in accordance with it. However, we have no guarantee it is actually going to be useful in general since we don't know when a given manifold  $M$  will admit a triangulation or not. Luckily, this question has been studied extensively and for smooth manifolds there is a theorem due to Whitehead [47] which implies that all manifolds admit a triangulation. Moreover, whenever the manifold is compact its triangulations are always finite in the sense that they are defined by a finite number of geometrical simplexes.

Of course we can easily build more complicated examples by putting simplexes together such as the polyhedron described in Figure 2.5 which is obtained from two tetrahedrons  $(v_0v_1v_2v_3)$  and  $(v_0v_1v_2v_4)$ . Notice that depending on the position of the vertexes it might not be possible to glue the tetrahedrons since they will not intersect along their boundaries. However when the process is successful we end up with a triangulation of  $B^3$  which is different from the one obtained from just one tetrahedron. This is a general phenomena as any triangulable manifold, a manifold that admits a triangulation, has an infinite number of different triangulations.

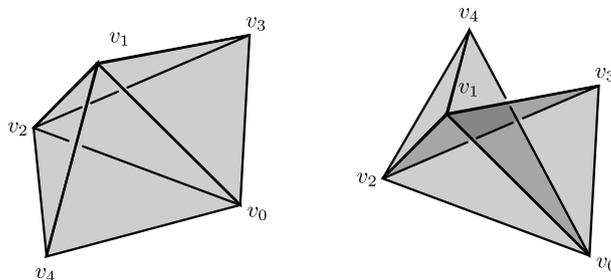


Figure 2.5: Gluing two tetrahedrons along a common triangle in a way that defines a polyhedron(left) and one that doesn't(right).

## Combinatorial perspective

We've defined geometrical simplexes and showed how to use them to decompose manifolds through our definition of polyhedrons and now we are going to describe them from a different point of view. This new approach emphasizes the combinatorial properties of simplexes and, while being more abstract, is far more efficient and practical.

Let us start with the observation that each geometrical simplex is completely determined by its vertexes and that, from the point of view of topology, details regarding how the vertexes are embedded in the euclidean space, such as the distance between them, are irrelevant. This is crucial as it allows us to work with a definition of simplexes which is completely abstract and only uses combinatorial data by ignoring the geometry. Essentially, we can abstract away the underlying euclidean space and work with the following definition of simplexes:

**Definition 2.4.** *A  $n$ -dimensional simplex, a  $n$ -simplex for short, is a set with  $n + 1$  elements which are called the vertexes of the simplex. Any  $n$ -simplex  $\sigma$  is completely defined by its vertexes  $v_0, \dots, v_n$  and we use a special notation  $\sigma = [v_0 \dots v_n]$  to emphasize that we are thinking of it as a simplex and not just a set.*

This definition has a direct correspondence with Definition 2.1 as one can imagine that the abstract vertexes  $v_0, \dots, v_n$  are actually points embedded in some euclidean space. As an example of how the geometrical perspective translates nicely into the combinatorial description let's consider the notion of a face of a simplex. Given a simplex  $\sigma = [v_0 \dots v_n]$ , we say a simplex  $\tau = [w_0, \dots, w_m]$  is a face of  $\sigma$  if  $\{w_0, \dots, w_m\} \subset \{v_0, \dots, v_n\}$  and denote this by  $\tau \prec \sigma$ . Even though this is a purely combinatorial concept, as there is no geometry involved, it still captures the geometrical intuition in a straightforward manner without having to carry unnecessary overhead.

Also, notice that since there is no geometry we don't need to worry about conditions on the vertexes such as them being in general linear position. This highlights why it is more convenient to work with simplexes instead of geometrical simplexes since we don't need to worry about details which are not relevant for the topological properties.

Just as we've abstracted the notion of a geometrical simplex to define a simplex we can abstract the idea of a polyhedron. The resulting spaces are called simplicial complexes:

**Definition 2.5.** *A simplicial complex is a  $\mathbb{N}$ -graded set  $K = \bigcup_n K_n$  with the following properties:*

- *The elements of  $K_n$  are  $n$ -simplexes.*
- *If a simplex belongs to  $K$  then so does any of its faces.*

Each set  $K_n$ , just like its geometrical counterpart in the definition of a polyhedron, specifies the  $n$ -simplexes used to assemble this abstract space. The second condition ensures the complex has all the faces required to glue the simplexes in analogy with the geometrical version. More precisely, any simplexes  $\sigma, \tau \in K$  of the complex are either disjoint, so they are not glued, or intersect at some common face  $\xi = \sigma \cap \tau \in K$ , in which case we say they are glued along this face. This shows why the second condition is important as it guarantees that  $\xi$  will also belong to the complex. An important feature is that we don't need to worry about simplexes having to intersect along their boundaries since this holds vacuously for the abstract simplexes.

Below we provide some basic terminology and notations which will be useful later on:

- A subcomplex  $S \subset K$  is a subset that is also a simplicial complex.
- The boundary of a simplex  $\sigma \in K$  is the subcomplex  $\partial(\sigma) \subset K$  defined by:

$$\partial(\sigma) = \{\tau \in K \mid \tau \prec \sigma\}$$

- The closure of a subset  $S \subset K$ , denoted  $\text{cl}(S)$ , is the smallest subcomplex that contains  $S$ . It is given by the union of the elements of  $S$  and their boundaries:

$$\text{cl}(S) = \bigcup_{\sigma \in S} (\partial(\sigma) \cup \{\sigma\})$$

- The star of a simplex  $\sigma \in K$ , denoted  $\text{star}(\sigma)$ , is the subset formed by all  $\tau \in K$  such that  $\sigma$  is either a face of  $\tau$  or equal to it.
- A simplex  $\sigma \in K$  is called a facet if it is not in the boundary of any simplex of the complex.
- We say  $K$  is pure if all of its facets have the same dimension.

If a simplicial complex is pure then we can define its dimension to be the dimension of its facets and say it is a  $n$ -complex where  $n$  is the dimension of the facets. Whenever that is the case, all the simplexes of the complex  $K$  are either  $n$ -simplexes or faces of them so that we have

$$K = \text{cl}(K_n) \quad (2.1)$$

which will often be a convenient way of describing such complexes.

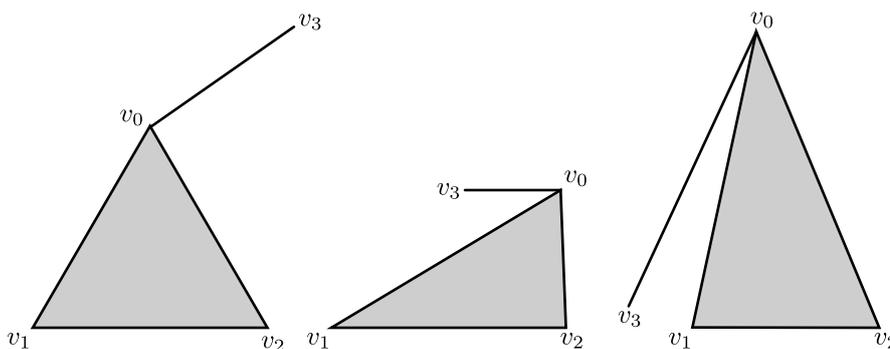


Figure 2.6: Different, but equivalent, polyhedrons that give rise to the same complex.

It should be clear that any polyhedron gives rise to a simplicial complex, called its carrier, if we just associate a simplex  $[v_0 \dots v_n]$  to each geometrical simplex  $(v_0 \dots v_n)$  by forgetting about the geometrical structure. This implies that all the polyhedrons we've defined previously give rise to simplicial complexes. However, it isn't too hard to realize that different polyhedrons can generate the same simplicial complex. For instance, consider the polyhedrons in Figure 2.6 and observe that while they are all different they give rise to the same complex specified below:

$$\begin{aligned} K_2 &= \{[v_0 v_1 v_2]\} \\ K_1 &= \{[v_0 v_1], [v_0 v_2], [v_1 v_2], [v_0 v_3]\} \\ K_0 &= \{[v_0], [v_1], [v_2], [v_3]\} \end{aligned}$$

It seems that even though the polyhedrons in Figure 2.6 are different they are equivalent to each other in the sense that they are all homeomorphic. This happens because the simplicial complex only cares about the combinatorial information whereas each geometrical implementation also keeps track of details which do not affect the topology. In fact, this is the reason simplicial complexes as we've defined are enough to encode all the topological information and provide the most efficient description since they get rid of any unnecessary information. However, to demonstrate all of this we need to give a recipe for building a topological space out of a simplicial complex  $K$  and then show that the space obtained does not depend on the geometrical details of the procedure.

## Geometric realization

The topological space associated to a simplicial complex is called its geometrical realization and we need to describe it so we can take full advantage of the combinatorial definition. As we've seen, it is always possible to associate a simplicial complex to a polyhedron so a possible strategy might be to

define an inverse procedure to obtain a polyhedron from a simplicial complex. However, since there are different polyhedrons corresponding to the same complex we are going to have to make some choices to remove the ambiguity. We should also note that, even though, Definition 2.5 makes no reference to any actual geometry or topology it is heavily inspired by geometrical notions and we can use this to make our job easier.

First of all let's give a formal definition of what we mean by a geometric realization:

**Definition 2.6.** *Consider a simplicial complex  $K$ , we say that a function  $f : K_0 \rightarrow \mathbb{R}^m$  is a geometric realization of  $K$  in  $\mathbb{R}^m$  if it maps simplexes into geometrical simplexes of the same dimension and respects intersections. The first condition means that if  $\sigma = [v_0 \dots v_n]$  is a  $n$ -simplex of  $K$  then the points  $\{f(v_0), \dots, f(v_n)\}$  must be in general linear position so that  $f(\sigma) = (f(v_0) \dots f(v_n))$  is a geometrical simplex of the same dimension. The second condition ensures that given simplexes  $\sigma, \tau \in K$  we always have  $f(\sigma \cap \tau) = f(\sigma) \cap f(\tau)$  so that  $f$  respects the way simplexes are glued in the complex.*

It should be clear from this definition that any geometric realization  $f : K_0 \rightarrow \mathbb{R}^m$  gives rise to a polyhedron

$$|K|_f = \bigcup_{\sigma \in K} f(\sigma) = \bigcup_{\sigma \in K} (f(v_i) \mid v_i \in \sigma)$$

and it follows by construction that the carrier of  $|K|_f$  is  $K$  so each geometric realization associates a topological space to  $K$  in a consistent fashion. However to make use of this we need to show that there is at least on such function for every simplicial complex and that the topological space obtained this ways does not depend on the chosen function.

The lemma below ensures the existence of geometric realizations for finite simplicial complexes:

**Lemma 2.1.** *Any finite simplicial complex  $K$  has at least one geometrical realization in  $\mathbb{R}^{|K_0|-1}$ .*

*Proof.* Consider the points

$$p_0 = (0, 0, \dots, 0) \quad p_1 = (1, 0, \dots, 0) \quad p_2 = (0, 1, \dots, 0) \quad \dots \quad p_{|K_0|-1} = (0, 0, \dots, 1)$$

and fix an enumeration  $v_0 < v_1 < v_2 \dots < v_{|K_0|-1}$  of the vertexes of  $K$ . We can then define a function that maps  $v_i$  into  $p_i$  and note that it is automatically a geometrical realization since the points  $\{p_0, \dots, p_{|K_0|-1}\}$  are in general linear position.  $\square$

There is a similar result for the infinite case but it is slightly more complicated, all though not much, as it might require using euclidean spaces of infinite dimension. However, since we will only be concerned with simplicial complexes which can be embedded into euclidean spaces of finite dimension we will not concern ourselves with such complications. Also, we should mention that this does not provide the most efficient realization of  $K$  in principle as in the examples of Figure 2.6 it would use  $\mathbb{R}^3$  when the pictures clearly show that there is a way to embedded it in  $\mathbb{R}^2$ .

Now that we've established the existence of geometric realizations we must show that they are consistent with each other i.e. that different geometric realizations define the same topological space.

**Lemma 2.2.** *Given two different geometrical realizations  $f : K_0 \rightarrow \mathbb{R}^m$  and  $g : K_0 \rightarrow \mathbb{R}^n$ , the polyhedrons  $|K|_f$  and  $|K|_g$  are homeomorphic.*

*Proof.* Consider a point  $x \in |K|_f$  and notice that by construction there is a simplex  $[v_0 \dots v_n] \in K$  such that  $x$  belongs to the geometrical simplex  $(f(v_0) \dots f(v_n))$ . Since these points are in general linear position we can use the barycentric coordinates  $x_0, \dots, x_n$  to write  $x = \sum x_i f(v_i)$  and define a function

$$x = \sum x_i f(v_i) \in |K|_f \mapsto \sum x_i g(v_i) \in |K|_g$$

This function provides the required homeomorphism so that  $|K|_f \approx |K|_g$  are indeed homeomorphic. Therefore, we are justified in writing  $|K|$  for the topological space obtained by taking any geometrical realization of  $K$  and leave the function used to construct it implicit.  $\square$

This lemma concludes our discussion about the geometric realization of a simplicial complex as we now see that, up to homeomorphism, any complex has a well defined polyhedron associated to it. This implies that, as far as topology goes, the whole combinatorial versus geometric dichotomy is irrelevant as we can safely move from one point of view to the other. Hence, from now on we work with simplicial complexes as in Definition 2.5 but talk about their topologies and draw pictures when necessary. Of course, whenever we do so we are actually talking about the geometric realization but often this will be implicit. As always, the geometrical picture should always be kept in mind as a source of intuition and inspiration.

## Orientations and enumerations

Since each simplex is a triangulation of a ball it is possible to assign an orientation to it. It turns out that there is a simple and straightforward way to specify these orientations directly from the combinatorial description using the definition given below:

**Definition 2.7.** *An orientation of a simplex  $[v_0 \dots v_n]$  is a choice of an enumeration for its vertexes and two such choices are considered equivalent if they are related by an even permutation.*

Each simplex always has two inequivalent orientations and it is easy to see how this definition captures the geometrical intuition by looking at some lower dimensional examples. If we consider a 1-simplex  $[v_0 v_1]$  we see that there are two possibilities  $v_0 < v_1$  or  $v_1 < v_0$  and each case fixes a starting vertex (the smallest one) and a final vertex (the largest one) so that the 1-simplex gets an orientation. The situation is a bit more interesting if we look at a 2-simplex  $[v_0 v_1 v_2]$  and list all the two possible orientations with the corresponding enumerations:

Orientation (1)	Orientation (2)
$v_0 < v_1 < v_2$	$v_0 < v_2 < v_1$
$v_2 < v_0 < v_1$	$v_2 < v_1 < v_0$
$v_1 < v_2 < v_0$	$v_1 < v_0 < v_2$

To see how each orientation appears we need to make the convention that the chosen orientation comes from applying the right hand rule to  $[v_0 v_1 v_2]$  in way that starts at the smallest vertex and then goes through the other vertexes in ascending order (see Figure 2.7).

A convenient way to choose an orientation for all simplexes of a simplicial complex  $K$  is to fix an enumeration for all of its vertexes and whenever that is the case we say that the orientation induced by

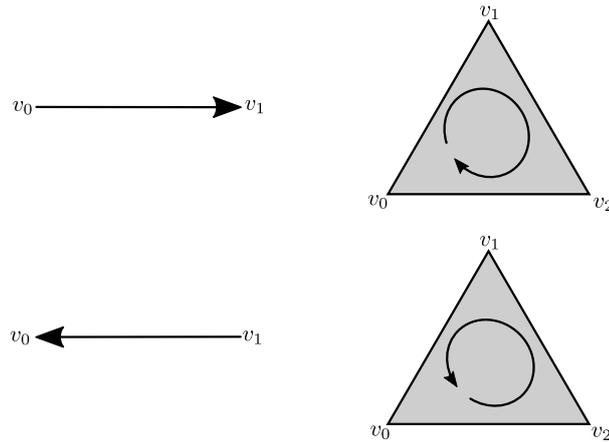


Figure 2.7: The possible orientations of a 1-simplex and a 2-simplex.

the chosen enumeration is the standard orientation, also called the positive orientation with the other being the negative orientation. Not only does an enumeration provides an orientation for the simplexes but it also removes the redundancy of the notation since every simplex  $\sigma \in K$  gets a unique description as  $\sigma = [v_0 \dots v_n]$  where  $v_0 < v_1 < \dots < v_n$  are the vertexes that define it written in a way that respects the enumeration. We should mention that we usually use the Latin alphabet with lexicographic order  $a < b < c \dots < z$  as a kind of generic label for simplicial complexes with an enumeration. Since we will never have to work with more then 26 vertexes at once this will be more then enough so, unless we say otherwise, one should always assume the lexicographic order for vertexes labeled with the Latin alphabet.

Fixing an enumeration also allow us to define some special functions, known as face maps:

**Definition 2.8.** *Given simplicial complex  $K$  with an enumeration, the face maps are the functions defined below:*

$$\begin{aligned} \partial_i : K_n &\rightarrow K_{n-1} \\ \partial_i[v_0 \dots v_n] &= [v_0 \dots \hat{v}_i \dots v_n] \end{aligned}$$

where  $i$  runs from 0 to  $n$  and the  $\hat{v}_i$  notation means we removed the vertex  $v_i$  from the original  $n$ -simplex to obtain a  $(n - 1)$ -simplex.

These functions are called face maps because given a  $n$ -simplex  $\sigma \in K_n$ , all of its  $(n - 1)$ -faces are of the form  $\partial_i(\sigma)$  for some  $i = 0, \dots, n$ . As such, we can use these maps to generate any face of  $\sigma$  by successively applying face maps to  $\sigma$ . A neat feature of the face maps is that they give us information about how the different orientations of simplexes relate to each other. What we mean by this is that the orientation of a  $n$ -simplex  $\sigma$  induces an orientation on all of its  $n - 1$ -faces and whether this induced orientation agrees with standard one or not depends on which face maps must be applied to obtain a given face. More precisely, if  $\tau = \partial_i\sigma$  then the orientation of  $\tau$  induced by  $\sigma$  agrees with the standard one if, and only, if  $i$  is even.

The following relation, called the simplicial identity, holds for all  $0 < i < j < n$ :

$$\partial_i\partial_j = \partial_{j-1}\partial_i \tag{2.2}$$

and implies that there are always exactly two ways of obtaining a  $(n - 2)$ -face from a  $n$ -simplex, each one coming from a different  $(n - 1)$ -simplex. It follows from this identity that the standard orientation of each  $(n - 2)$ -face will agree with the one induced by one 1-face and disagree with other one. To see this concretely consider a 2-simplex  $[abc]$  and notice how each of its 0-faces  $[a], [b], [c]$  appears as the initial vertex of a 1-face and the final vertex of another one:

$$\begin{aligned} \partial_0\partial_0[abc] &= \partial_0[\hat{a}bc] = \partial_0[bc] = [\hat{b}c] = [c] = [\hat{a}c] = \partial_0[ac] = \partial_0[a\hat{b}c] = \partial_0\partial_1[abc] \\ \partial_1\partial_1[abc] &= \partial_1[a\hat{b}c] = \partial_1[ac] = [a\hat{c}] = [a] = [\hat{a}\hat{b}] = \partial_1[ab] = \partial_1[ab\hat{c}] = \partial_1\partial_2[abc] \\ \partial_1\partial_0[abc] &= \partial_1[\hat{a}bc] = \partial_1[bc] = [b\hat{c}] = [b] = [\hat{a}b] = \partial_0[ab] = \partial_0[ab\hat{c}] = \partial_0\partial_2[abc] \end{aligned}$$

These seemly trivial observations about orientations are actually really important as they hold the key for defining an important class of topological invariants of simplicial complexes. These invariants, known as simplicial homology and cohomology groups, will appear in Chapter 4 so we quickly review them. The basic idea is to define an algebraic structure that allows us to talk about the objects obtained by considering many simplexes simultaneously. Formally we do so using the following concept:

**Definition 2.9.** *Given a simplicial complex  $K$ , we define  $C_n(K) = \mathbb{Z}K_n$  as the free abelian group generated by the set of  $n$ -simplexes. Its elements are  $n$ -chains, that is, formal sums of  $n$ -simplexes with coefficients in  $\mathbb{Z}$ . A generic  $n$ -chain can be written as*

$$\sum_{\sigma \in K_n} c(\sigma)\sigma$$

with the group operation being induced by addition of coefficients.

The  $n$ -simplexes provide a basis for  $C_n(K)$  and we always assume there is a fixed enumeration so that we can think of the chain defined by a simplex  $\sigma \in C_n(K)$  as representation of this simplex with the standard orientation. The inverse chain  $-\sigma \in C_n(K)$  represents the same simplex with the opposite orientation. Inspired by this we use the face maps to define the boundary morphism

$$d_n : C_n(K) \rightarrow C_{n-1}(K) \tag{2.3}$$

$$d_n(\sigma) = \sum_{i=0}^n (-1)^i \partial_i(\sigma) \tag{2.4}$$

that map an  $n$ -simplex into the  $(n - 1)$ -chain associated to its boundary. The factors  $(-1)^i$  ensure that the boundary has the orientation induced by  $\sigma$  and thanks to them we can use relation 2.2 to obtain the following lemma:

**Lemma 2.3.** *The composition of  $d_{n-1}$  with  $d_n$  is the trivial morphism.*

*Proof.* All we need to check is that  $d_{n-1}(d_n(\sigma)) = 0$  for all  $n$ -simplexes:

$$\begin{aligned} d_{n-1}(d_n(\sigma)) &= \sum_{ij} (-1)^{i+j} \partial_i \partial_j \sigma \\ &= \sum_{i < j} (-1)^{i+j} \partial_i \partial_j \sigma + \sum_{j \leq i} (-1)^{i+j} \partial_i \partial_j \sigma \\ &= \sum_{i < j} (-1)^{i+j} \partial_{j-1} \partial_i \sigma + \sum_{j \leq i} (-1)^{i+j} \partial_i \partial_j \sigma \\ &= \sum_{j \leq i} (-1)^{i+j+1} \partial_i \partial_j \sigma + \sum_{j \leq i} (-1)^{i+j} \partial_i \partial_j \sigma \\ &= 0 \end{aligned}$$

where we've used  $\partial_i \partial_j = \partial_{j-1} \partial_i$  for  $i < j$  from the second line to third one and then did a change of variables to get to the fourth line.  $\square$

This result is a direct manifestation of the fact that the  $(n-2)$ -faces of a  $n$ -simplex always appears twice but with opposite orientations as Figure 2.8 shows for a 2-simplex. Geometrically, this relation holds because the boundary of a  $n$ -simplex is a triangulation of  $S^{n-1}$  and spheres have empty boundary.

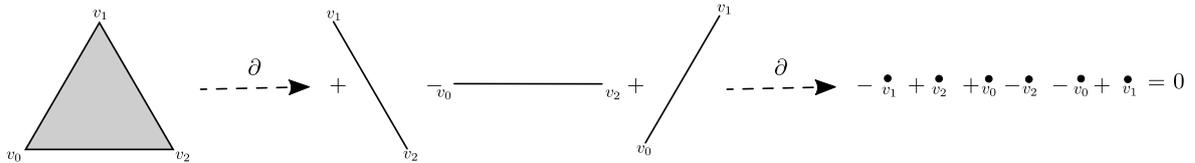


Figure 2.8: The result of applying the boundary morphism twice.

From an algebraic point of view Lemma 2.3 provides solutions to the equation  $d_n(m) = 0$  and one can ask whether there are more solutions which are inequivalent. Formally this information is captured by the simplicial homology group  $H_n(K) = \ker(d_n)/\text{im}(d_{n+1})$  and it turns out that these groups are actually topological invariants of  $K$ . We will not provide a demonstration of this since it is a standard result from algebraic topology but we mention that this invariant detects  $n$ -dimensional holes in the space. What we mean by this is that solutions to  $d_n(m) = 0$  are  $n$ -chains that look like the boundary of a ball so that when we quotient out the actual boundaries of balls, that correspond to  $\text{im}(d_{n+1})$ , we are left with a group describing the holes in the space formed by these hollow spheres.

One can also obtain another topological invariant using a dual construction by fixing an abelian group  $G$  and defining:

**Definition 2.10.** We define  $C^n(K, G)$  as the abelian group formed by morphisms  $C_n(K) \rightarrow G$ . Since  $C_n(K)$  is a free abelian group,  $C^n(K, G)$  is generated by morphisms such as  $g\sigma^*$  where  $g \in G$ ,  $\sigma \in K_n$  and  $(g\sigma^*)(\tau) = g$  if  $\sigma = \tau$  and 0 otherwise for any  $\tau \in K_n$ . Therefore, elements of  $C^n(K, G)$ , called  $n$ -cochains, can be thought of as formal sums of elements of  $K$  with coefficients in  $G$ .

It is straightforward to get a coboundary morphism  $d^n : C^n(K, G) \rightarrow C^{n+1}(K, G)$  defined by pre-composition with  $d_{n+1}$  i.e.  $d^n(f) = f d_{n+1}$  where  $f$  is a morphism from  $C_n(K)$  to  $G$ . It follows from Lemma 2.3 that  $d^{n+1}(d^n(\sigma)) = 0$  and this leads to the simplicial cohomology groups with coefficients in  $G$  defined as  $H^n(K, G) = \ker(d^n)/\text{im}(d^{n-1})$  which are also topological invariants.

## Pachner Moves

To conclude this section we go back to the study of triangulations of three-dimensional manifolds and the problem of classifying them. For clarity we should mention that whenever we say a simplicial complex is a triangulation of some manifold what we mean is that its geometric realization is a triangulation of this manifold.

As we've mentioned earlier any triangulable manifold admits an infinite number of triangulation and for our purposes we need to understand how different triangulations of a given manifold relate to each other. This issue is extremely complicated for general topological manifolds and it is known that there are manifolds with inequivalent triangulations and even manifolds which do not admit a triangulation at all. These complexities lead to the development of the theory of piecewise linear manifolds [41], a refinement to the definition of triangulations, which allows one to get a better handle on this issue. In particular, we are interested in a theorem due to Pachner [38] which allows one to characterize triangulations which are piecewise linear equivalent using a series of mutations known as Pachner moves. Roughly speaking, one can always connect two equivalent triangulations of a given manifold by applying Pachner moves to one of the triangulations to obtain the other. Below we state Pachner's theorem:

**Theorem 2.4.** *Given two triangulations  $K, K'$ , if  $K$  is piecewise linear equivalent to  $K'$  then it is possible to obtain  $K$  from  $K'$  by applying a finite sequence of Pachner moves. Moreover, two triangulations which are connected by a finite sequence of Pachner moves must be piecewise linear equivalent.*

Of course for this theorem to actually mean anything we have to explain the whole piecewise linear business and describe what Pachner moves are. The latter will be the focus of the rest of this section but we won't discuss the technical issues concerning the piecewise linear structure. The reason we won't do this is that these details would require a long digression into the theory of piecewise linear manifolds which, at the end of the day, won't be relevant for our purposes. They won't be relevant for us because our goal is to use triangulations to describe three-dimensional manifolds and these subtleties are not important for dimensions 1, 2 and 3 as they only affect higher dimensional manifolds. In particular, a theorem due to Radó [48] states that any two-dimensional topological manifold is triangulable and their triangulations are all equivalent to each other so that the distinction between piecewise linear manifolds and triangulations is irrelevant in dimension two. An analogous theorem due to Moise [49] ensures that the same holds for the three-dimensional case so we can safely proceed without worries. Nonetheless, for the reader interested in these issues we refer to [40, 50].

From a practical point of view Theorem 2.4 tells us that two triangulations of a three manifold are always connected by a finite sequence of Pachner moves. This will be important for us in Chapter 4 where we will need a tool to decide whether some quantities depend on the chosen triangulation or are intrinsic to the manifold and, thanks to Pachner's theorem, it will be sufficient to check whether they are invariant under Pachner moves.

To describe Pachner moves we start with a smooth version which will provide some intuition. From this point of view the basic strategy comes from the following observation:

$$\partial B^{n+1} = S^n = B_i^n \cup B_f^n \tag{2.5}$$

where  $B_i^n$  and  $B_f^n$  are two copies of the  $n$ -dimensional ball that intersect along their boundary and make up the boundary of the  $(n + 1)$ -dimensional ball. Using this we can define an internal mutation

of a  $n$ -manifold  $M$  by selecting some  $n$ -ball in its interior and identifying it with  $B_i^n$ . If we sweep  $B_i^n$  across the interior of  $B^{n+1}$  we can interpolate between the two parts of the boundary and mutate  $M$  by replacing  $B_i^n$  with  $B_f^n$ . It is also possible to define a border move by gluing one half of the boundary of a  $n$ -ball to  $\partial M$  to grow the boundary or, for the opposite procedure, carve out a  $n$ -ball from the boundary. It is possible to show that these mutations do not change the underlying manifold (under appropriate circumstances) and we provide a pictorial description of them in Figure 2.9.

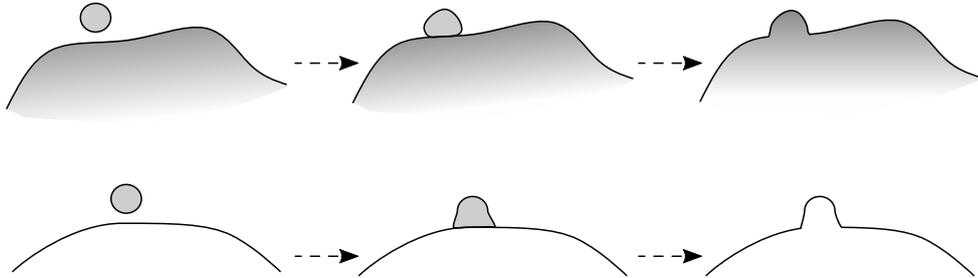


Figure 2.9: Mutation on the boundary a 2-manifold and on the interior of a 1-manifold.

On a discrete scenario there are analogues of equation 2.5 since we can always decompose the boundary a  $(n + 1)$ -simplex, the smallest triangulation of a  $B^{n+1}$ , into two triangulations of  $B^n$ . However, as shown in Figure 2.10 there are many inequivalent ways of doing so and each one of them is going to define a discrete mutation that can change a triangulation without modifying the underlying manifold. These discrete mutations are the Pachner moves and since any triangulation of a  $n$ -manifold is a  $n$ -complex (a simplicial complex whose facets are  $n$ -simplexes) we can describe them in dimension  $n$  by saying which  $n$ -simplexes are added and/or removed. We can get this information from Figure 2.10 by adapting the procedure described in Figure 2.9 to the discrete setting so we now proceed to list all the Pachner move for dimensions two and three.

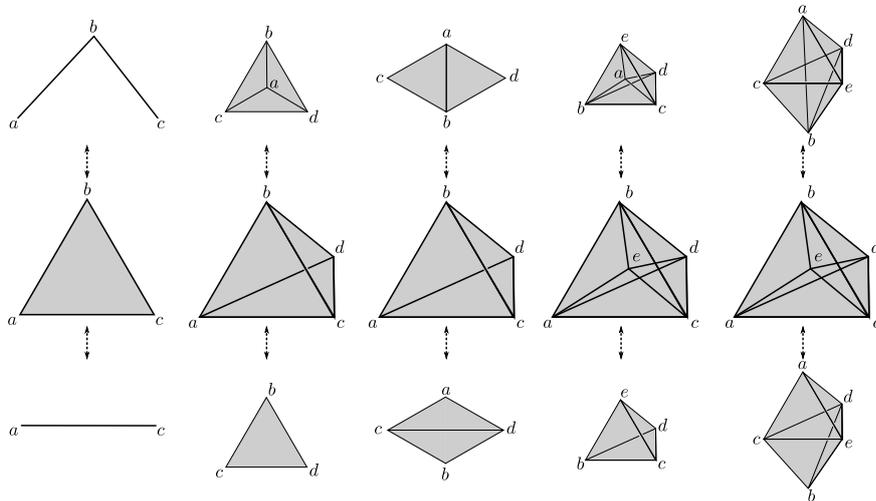
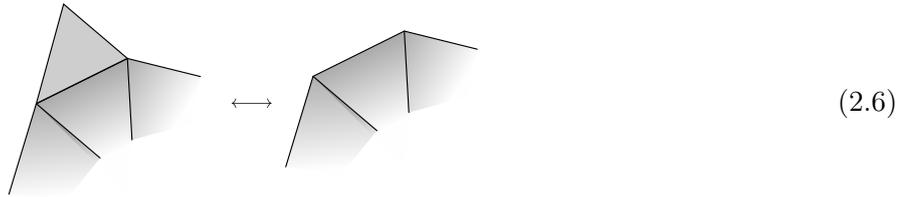


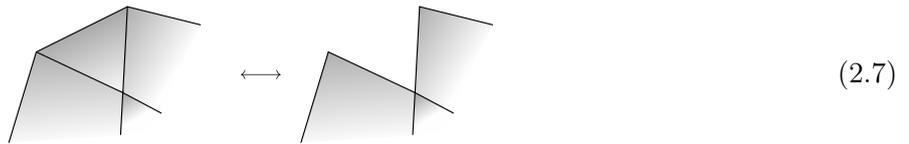
Figure 2.10: Decompositions of the boundary of low dimensional simplexes.

The only way to break the boundary of a 2-simplex  $[abc]$  into two triangulations of the 1-ball is to divide it into  $[ab], [bc] \leftrightarrow [ac]$  or a similar move with a different choice of vertexes. This leads to

the only internal Pachner move in dimension one and to the two types of border Pachner moves in dimension two:



(2.6)



(2.7)

To get the internal moves in dimension two we note that there are four 2-simplexes in the boundary of a 3-simplex  $[abcd]$  so there are two types of decompositions:

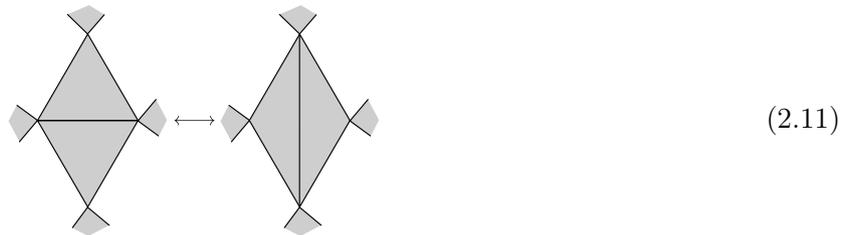
$$[bcd] \leftrightarrow [abc], [abd], [acd] \tag{2.8}$$

$$[acd], [bcd] \leftrightarrow [abc], [abd] \tag{2.9}$$

Therefore we get the following internal moves in dimension two:

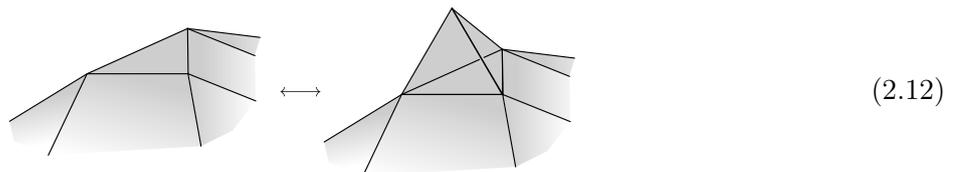


(2.10)

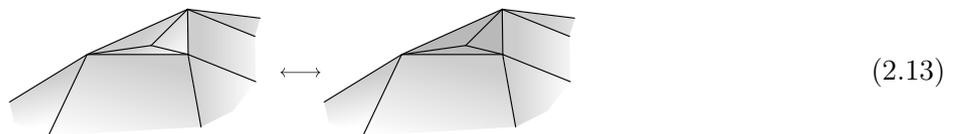


(2.11)

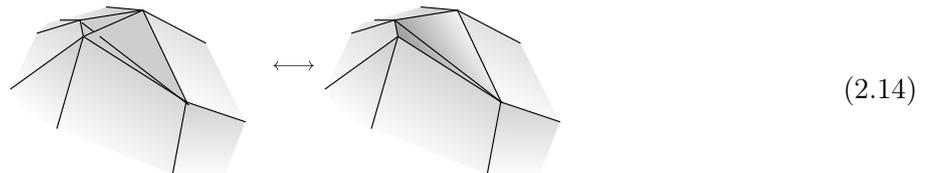
The different decompositions of a 3-simplex also give rise to the border moves in dimension three:



(2.12)



(2.13)



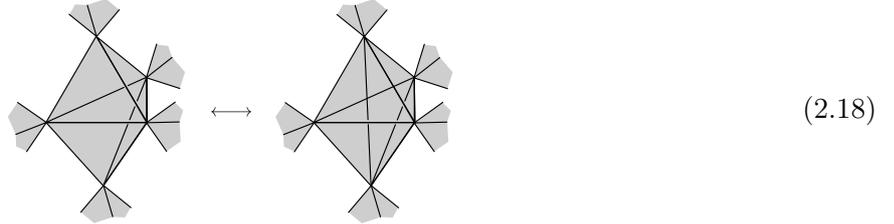
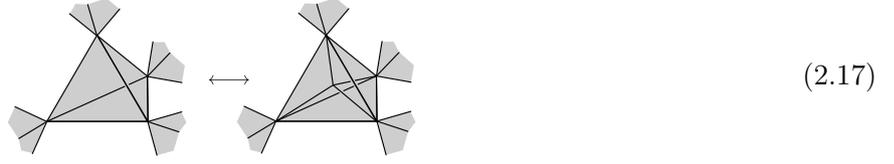
(2.14)

Finally we list the two different ways of decomposing the boundary of a 4-simplex  $[abcde]$

$$[bcde] \leftrightarrow [abce], [abde], [acde], [abcd] \quad (2.15)$$

$$[acde], [bcde] \leftrightarrow [abce], [abde], [abcd] \quad (2.16)$$

to get the internal moves in dimension three:



As our final word on Pachner moves we should say that we refer to the internal moves in dimension  $n$  obtained from breaking the  $(n + 1)$ -simplex into two pieces, one with  $k$  faces and the other with  $n - k$  faces, as the  $(k \leftrightarrow n - k)$  move. Similarly, the border moves in dimension  $n + 1$  coming from the same decomposition are referred to as the  $(k \rightarrow n - k)$  move and the  $(n - k \rightarrow k)$  move. As an example, we can list all the moves in dimension three:

- Equation 2.12 is the  $(1 \rightarrow 3)$  move.
- Equation 2.13 is the  $(3 \rightarrow 1)$  move.
- Equation 2.14 is the  $(2 \rightarrow 2)$  move.
- Equation 2.17 is the  $(1 \leftrightarrow 4)$  move.
- Equation 2.18 is the  $(2 \leftrightarrow 3)$  move.

## 2.2 Groups, 2-Groups and Crossed Modules

Groups are by far the most important algebraic structure in physics as they can be used to describe the different symmetries of a given system. However, they are not without their limitations as they are, in a sense, a one-dimensional algebraic structure. To explain what we mean by this consider a group  $G$  and assign a curve to each of its elements as shown below:

$$g \in G \mapsto \begin{array}{c} \text{---} g \text{---} \\ \curvearrowright \\ \cdot \end{array} \quad (2.19)$$

If we assume that the curves are loops starting and ending at the same point we can describe the group multiplication as the outcome of going through two loops in a sequence and regarding it as a new loop:

$$\begin{array}{c}
 \overset{g}{\curvearrowright} \quad \overset{h}{\curvearrowright} \\
 \cdot \quad \quad \quad \cdot \quad \quad \quad \cdot \\
 \end{array} = \begin{array}{c}
 \overset{gh}{\curvearrowright} \\
 \cdot \\
 \end{array} \tag{2.20}$$

From this point of view the associativity rule is a natural consistency relation saying that we can combine many loops in a well defined manner:

$$\begin{array}{c}
 \overset{g_1}{\curvearrowright} \quad \overset{g_2}{\curvearrowright} \quad \dots \quad \overset{g_{n-1}}{\curvearrowright} \quad \overset{g_n}{\curvearrowright} \\
 \cdot \quad \quad \quad \cdot \quad \quad \quad \cdot \quad \quad \quad \cdot \quad \quad \quad \cdot \\
 \end{array} = \begin{array}{c}
 \overset{g_1 g_2 \dots g_{n-1} g_n}{\curvearrowright} \\
 \cdot \\
 \end{array} \tag{2.21}$$

The unit of the group can be seen as the loop that just sits still at the point so that it does not change any loops when combined to them and the inverse of a group element correspond to the curve going backwards:

$$\begin{array}{c}
 \overset{1}{\curvearrowright} \\
 \cdot \\
 \end{array} = \cdot \tag{2.22}$$

$$\begin{array}{c}
 \overset{g^{-1}}{\curvearrowright} \\
 \cdot \\
 \end{array} = \begin{array}{c}
 \overset{g}{\curvearrowleft} \\
 \cdot \\
 \end{array} \tag{2.23}$$

We conclude that the entire group structure can be captured and described using curves which is why we say it is one-dimensional. Perhaps it might be possible to mimic this procedure in a two dimensional setting by using a set  $G$  with two different group structures so that one can assign surfaces to its elements:

$$g \in G \mapsto \begin{array}{c}
 \curvearrowright \\
 \cdot \\
 \text{---} \\
 \cdot \\
 \curvearrowleft \\
 \end{array} \tag{2.24}$$

These kinds surfaces are called bigons as they are a polygon with just two sides and they can be understood as a "path of paths" connecting the two curves in the boundary. This idea can be made precise in the context of homotopy theory where such a surface is related to a homotopy between two loops but we won't go into too many details regarding this. More important for us is the fact that

these surfaces come equipped with two types of composition, a horizontal and a vertical one:

$$(2.25)$$

$$(2.26)$$

where  $\circ$  and  $\bullet$  are the two different group multiplications of  $G$ . There are natural consistency relations for the two compositions which come from the geometrical description the most important being the middle four exchange rule:

$$(2.27)$$

This condition is the analogue of the associativity in two dimensions and it is the reason groups are not an adequate structure for describing bigons. To see this we consider the lemma below which is a version of the Eckmann-Hilton argument[51]:

**Lemma 2.5.** *Consider a set  $G$  with two group structures, if the two group multiplications obey equation 2.27 then the group structures are the same and they are abelian.*

*Proof.* Given  $g, h, t, w \in G$  we have, from the exchange rule, that  $(g \circ h) \bullet (t \circ w) = (g \bullet t) \circ (h \bullet w)$  so that if we take  $g = w = 1_\circ$  and  $h = t = 1_\bullet$ , where  $1_\circ$  and  $1_\bullet$  are units of the group structures, we obtain:

$$(1_\circ \circ 1_\bullet) \bullet (1_\bullet \circ 1_\circ) = (1_\circ \bullet 1_\bullet) \circ (1_\bullet \bullet 1_\circ) \implies 1_\bullet \bullet 1_\bullet = 1_\circ \circ 1_\circ \implies 1_\bullet = 1_\circ$$

so there is only one unit for both multiplications and we denote this by  $1 = 1_\bullet = 1_\circ$ . If we take  $h = t = 1$  we get:

$$(g \circ 1) \bullet (1 \circ w) = (g \bullet 1) \circ (1 \bullet w) \implies g \bullet w = g \circ w$$

so both multiplications are equal. Finally, if we take  $g = w = 1$  we see that this multiplication is abelian:

$$(1 \circ h) \bullet (t \circ 1) = (1 \bullet t) \circ (h \bullet 1) \implies h \bullet t = t \circ h = t \bullet h$$

□

The fact that both group structures are equal and abelian suggests that we must consider something more general if we want to work with two dimensional objects using algebra. This was first realized in the context of homotopy theory and it led to the development of a rich theory revolving around higher dimensional analogues of groups.

## Crossed modules

We would like to find a better algebraic structure to describe the bigon we've introduced since, as we've seen, groups are not adequate for this task. The correct structure for this purpose is something called a 2-group which, in an informal sense, is the algebraic structure obtained by taking "bigons seriously" by which we mean that the axioms for a 2-group are a direct implementation of the rules for manipulating bigons. However, we won't provide a formal definition to avoid having to develop the theory of 2-categories [43] and because we only use 2-groups indirectly in our work. Instead we are going to use crossed modules and show how they can be used to construct 2-groups while providing a more concrete framework for manipulating them. We should mention that this is a valid strategy because there is a theorem due to Janelidze, Brown and Spencer [52] that says that the category of crossed modules is equivalent to that of (strict) 2-groups. Here, the word strict refers to the fact that we are not considering the most general type of 2-group as for our purposes this is not required so whenever we talk about 2-groups we let the strictness implied from now on.

Crossed modules were first introduced by Whitehead [53] as a tool for calculating homotopy groups and his basic example provides the intuition of why crossed modules are two-dimensional in a sense. Essentially, what he realized is that the first homotopy group  $\pi_1(M, x_0)$ , a group that encodes information about loops on a topological space  $M$ , and the second homotopy group  $\pi_2(M, x_0)$ , the two-dimensional version of  $\pi_1$  which describes surfaces, carry an additional structure which comes from the underlying geometry (see [42] for details on the definition of homotopy groups). This structure can be formalized and leads to the definition of crossed modules:

**Definition 2.11.** *A crossed module is a quadruple  $G = (G_1, G_2, \partial, \triangleright)$  where  $G_1$  and  $G_2$  are groups,  $\partial$  is a group morphism from  $G_2$  to  $G_1$  and  $\triangleright$  is a group action of  $G_1$  on  $G_2$ . They obey the following conditions:*

$$\partial(g \triangleright \alpha) = g \partial(\alpha) g^{-1} \tag{2.28}$$

$$\partial(\beta) \triangleright \alpha = \beta \alpha \beta^{-1} \tag{2.29}$$

The idea is to use the group  $G_1$  to describe the initial and final curves of bigons and  $G_2$  to label their surfaces. The morphism  $\partial$  captures what happens to a curve when it is transported through a bigon and the action  $\triangleright$  provides a twisted version of the product of  $G_2$  which allows one to bypass the Eckmann-Hilton argument. Conditions 2.28 and 2.29 are crucial as they ensure the consistency of the operations we are going to define later on.

It is easy to come with examples of crossed modules and we list some below:

- Given a group  $G$ , there is a crossed module with  $G_1 = G$  and  $G_2, \partial, \triangleright$  trivial. This captures the structure of the group and allows us to think of groups as a particular type of crossed module.

- Using the action by conjugation we can define a crossed module from a group  $G$  and a normal subgroup  $N \subset G$  by taking  $G_1 = G$ ,  $G_2 = N$ ,  $\partial(g) = g$  and  $g \triangleright h = ghg^{-1}$ .
- An alternative way to use the action by conjugation is to think of it as morphism  $G \rightarrow \text{Aut}(G)$  to obtain a crossed module that has  $G_1 = \text{Aut}(G)$ ,  $G_2 = G$ ,  $\partial$  is the morphism coming from the action by conjugation and  $f \triangleright \alpha = f(\alpha)$ .
- If we consider abelian groups  $G_1$  and  $G_2$  then any morphism  $\partial : G_2 \rightarrow G_1$  gives rise to a crossed module with the trivial action.

Crossed module come with a natural notion of morphism which we define below:

**Definition 2.12.** *Given crossed modules  $G = (G_1, G_2, \partial, \triangleright)$  and  $H = (H_1, H_2, \tilde{\partial}, \tilde{\triangleright})$  a morphism  $\phi : G \rightarrow H$  is a pair of morphism  $\phi_1 : G_1 \rightarrow H_1$  and  $\phi_2 : G_2 \rightarrow H_2$  that satisfy the following:*

$$\phi_1(\partial(\alpha)) = \tilde{\partial}(\phi_2(\alpha)) \quad (2.30)$$

$$\phi_1(g)\tilde{\triangleright}\phi_2(\beta) = \phi_2(x \triangleright \beta) \quad (2.31)$$

Also, motivated by Whitehead's example we can define the homotopy groups of a crossed module  $G = (G_1, G_2, \partial, \triangleright)$  as

$$\pi_1(G) = \text{coker } \partial = G_1 / \text{im } \partial \quad (2.32)$$

$$\pi_2(G) = \ker \partial \quad (2.33)$$

$$\pi_n(G) = 1 \quad n \neq 1, 2 \quad (2.34)$$

This might seem a bit weird since crossed modules are not topological spaces but it is actually related to the fact they provide a complete model for homotopy 2-types. In particular, any morphism  $\phi : G \rightarrow H$  induces group morphisms:

$$\pi_1(\phi) : \pi_1(G) \rightarrow \pi_1(H) \quad [g] \in \text{coker } \partial \mapsto [\phi_1(g)] \in \text{coker } \tilde{\partial} \quad (2.35)$$

$$\pi_2(\phi) : \pi_2(G) \rightarrow \pi_2(H) \quad \alpha \in \ker \partial \mapsto \phi_2(\alpha) \in \ker \tilde{\partial} \quad (2.36)$$

and we say it is a weak homotopy equivalence if it induces isomorphisms in the homotopy groups. If that is the case we say that the crossed modules are homotopic.

Hence, each homotopy group defines a functor  $\pi_n : XMod \rightarrow Grp$  between the category of crossed modules and the category of groups. The topological interpretation of these functors comes from a connection with the ordinary functors  $\pi_n : Top_* \rightarrow Grp$  mapping pointed topological spaces into their homotopy groups. The connection comes from a functor  $B : XMod \rightarrow Top_*$  that sends each crossed module  $G$  into its classifying space  $BG$  which is the space with the property that  $\pi_n(BG) = \pi_n G$ . There is a sophisticated homotopy theory of crossed modules based on these ideas and we refer the reader to [54] and [55] for details regarding it and the construction of the classifying space of a crossed module.

### 2-Groups

Now we must explain how crossed modules can be used as an algebraic model for bigons and the different ways of manipulating them. The first step is to introduce bigons labeled by elements of the crossed module:

$$(g, \alpha) \in G_1 \times G_2 \mapsto \begin{array}{c} \begin{array}{ccc} & g & \\ & \curvearrowright & \\ & \downarrow \alpha & \\ & \curvearrowleft & \\ & h & \end{array} \\ h = \partial(\alpha)^{-1}g \end{array} \quad (2.37)$$

We say  $s(g, \alpha) = g$  is the source of the bigon and  $t(g, \alpha) = \partial(\alpha)^{-1}g$  the target so that  $(g, \alpha)$  maps the curve corresponding to  $g$  into the one corresponding to  $h = \partial(\alpha)^{-1}g$ . In particular, if  $\alpha = 1$  then the bigon corresponds to the "trivial" surface connecting  $g$  to itself, that is, the surface that just sits still at the curve corresponding to  $g$ . This is similar to the way we interpreted the identity of a group as a point (a "trivial" curve) and we can use it to establish a correspondence  $g = (g, 1)$  which implies the following in terms of diagrams:

$$\begin{array}{c} \curvearrowright \\ g \end{array} = \begin{array}{c} \begin{array}{ccc} & g & \\ & \curvearrowright & \\ & \downarrow 1 & \\ & \curvearrowleft & \\ & g & \end{array} \end{array} \quad (2.38)$$

Of the course the goal is to define horizontal and vertical compositions for bigons and this is where conditions 2.28 and 2.29 shine as they ensure the consistency of these operations. In terms of the diagrams the horizontal composition of two bigons is given by:

$$\begin{array}{c} \begin{array}{ccc} & g & \\ & \curvearrowright & \\ & \downarrow \alpha & \\ & \curvearrowleft & \\ & \partial(\alpha)^{-1}g & \end{array} & \begin{array}{ccc} & h & \\ & \curvearrowright & \\ & \downarrow \beta & \\ & \curvearrowleft & \\ & \partial(\beta)^{-1}h & \end{array} & = & \begin{array}{c} \begin{array}{ccc} & gh & \\ & \curvearrowright & \\ & \downarrow (g \triangleright \beta)\alpha & \\ & \curvearrowleft & \\ & \partial(\alpha)^{-1}g\partial(\beta)^{-1}h \\ \underline{\underline{\partial(g \triangleright \beta)(\alpha)^{-1}gh}} & & \end{array} \end{array} \end{array} \quad (2.39)$$

and we denote it as  $(g, \alpha) \circ (h, \beta) = (gh, (g \triangleright \beta)\alpha)$ . The vertical composition is only defined for two bigons  $(g, \alpha), (h, \beta)$  with matching sources and targets  $s(h, \beta) = t(g, \alpha)$  and it is given by:

$$(2.40)$$

and denoted  $(g, \alpha) \bullet (h, \beta) = (g, \alpha\beta)$ . Whenever we write a vertical composition of two bigons without saying that they have matching sources and targets it should be understood that this is implicit.

For these operations to be well defined they need to respect sources and targets in the natural way suggested by the diagrams, that is:

$$s((g, \alpha) \circ (h, \beta)) = s(g, \alpha)s(h, \beta) \quad (2.41)$$

$$t((g, \alpha) \circ (h, \beta)) = t(g, \alpha)t(h, \beta) \quad (2.42)$$

$$s((g, \alpha) \bullet (h, \beta)) = s(g, \alpha) \quad (2.43)$$

$$t((g, \alpha) \bullet (h, \beta)) = t(h, \beta) \quad (2.44)$$

The equations for the sources follow immediately from the fact that  $G_1, G_2$  are groups and the target condition for the vertical composition is a direct consequence of  $\partial$  being a group morphism. Hence, we just need to check the consistency of the target of the horizontal composition as it is the only non trivial relation:

$$\begin{aligned} t((g, \alpha) \circ (h, \beta)) &= t(gh, (g \triangleright \beta)\alpha) = \partial((g \triangleright \beta)\alpha)^{-1}gh \\ &= \partial(\alpha)^{-1}\partial((g \triangleright \beta^{-1}))gh = \partial(\alpha)^{-1}g\partial(\beta)^{-1}g^{-1}gh = t(g, \alpha)t(h, \beta) \end{aligned}$$

where we had to use equation 2.28 to write  $\partial(x \triangleright \beta^{-1}) = g\partial(\beta^{-1})g^{-1}$ .

Of course, there are many additional consistency relations for the two compositions which follow from the diagrams and we list them below:

- Identities:

$$g \bullet (g, \alpha) = (g, 1) \bullet (g, \alpha) = (g, \alpha) = (g, \alpha) \bullet (\partial(\alpha)^{-1}g, 1) = (g, \alpha) \bullet (\partial(\alpha)^{-1}g) \quad (2.45)$$

$$1 \circ (g, \alpha) = (1, 1) \circ (g, \alpha) = (g, \alpha) = (g, \alpha) \circ (1, 1) = (g, \alpha) \circ 1 \quad (2.46)$$

- Vertical inverse:

$$(g, \alpha)^{-1} = (\partial(\alpha)^{-1}g, \alpha^{-1}) \quad (2.47)$$

$$(g, \alpha) \bullet (g, \alpha)^{-1} = (g, \alpha\alpha^{-1}) = (g, 1) = g \quad (2.48)$$

$$(g, \alpha)^{-1} \bullet (g, \alpha) = (\partial\alpha^{-1}g, \alpha^{-1}\alpha) = (\partial(\alpha)^{-1}g, 1) = \partial(\alpha)^{-1}g \quad (2.49)$$

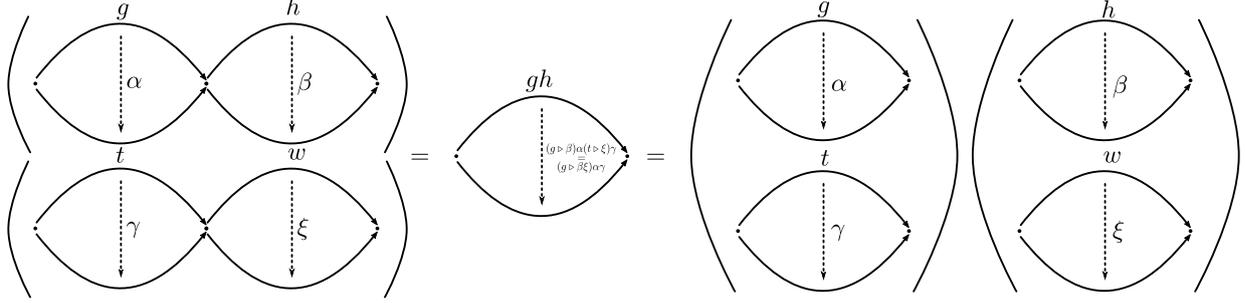
- Associativity:

$$(g, \alpha) \circ ((h, \beta) \circ (t, \gamma)) = ((g, \alpha) \circ (h, \beta)) \circ (t, \gamma) \quad (2.50)$$

$$(g, \alpha) \bullet ((h, \beta) \bullet (t, \gamma)) = ((g, \alpha) \bullet (h, \beta)) \bullet (t, \gamma) \quad (2.51)$$

- Exchange rule:

$$((g, \alpha) \circ (h, \beta)) \bullet ((t, \gamma) \circ (w, \xi)) = ((g, \alpha) \bullet (t, \gamma)) \circ ((h, \beta) \bullet (w, \xi)) \quad (2.52)$$



These are all fairly straightforward with the exception of the exchange rule. Moreover, it is the most restrictive conditions imposed by the bigons and it shows the importance of condition 2.29 so we must verify it. For starters let us expand both sides of the equation:

$$\begin{aligned} ((g, \alpha) \circ (h, \beta)) \bullet ((t, \gamma) \circ (w, \xi)) &= (gh, (g \triangleright \beta)\alpha) \bullet (tw, (t \triangleright \xi)\gamma) \\ &= (gh, (g \triangleright \beta)\alpha(t \triangleright \xi)\gamma) \\ ((g, \alpha) \bullet (t, \gamma)) \circ ((h, \beta) \bullet (w, \xi)) &= (g, \alpha\gamma) \circ (h, \beta\xi) \\ &= (gh, (g \triangleright \beta\xi)\alpha\gamma) \end{aligned}$$

We know that  $g, \alpha$  and  $t, \gamma$  have matching sources and targets since they are being composed vertically and this implies  $t = \partial(\alpha)^{-1}g$  so

$$\alpha(t \triangleright \xi) = \alpha((\partial(\alpha)^{-1}) \triangleright (g \triangleright \xi)) = \alpha\alpha^{-1}(g \triangleright \xi)\alpha = (g \triangleright \xi)\alpha$$

where we've used equation 2.29 to expand the action of  $\partial(\alpha)^{-1}$ . We conclude that

$$(gh, (g \triangleright \beta)\alpha(t \triangleright \xi)\gamma) = (gh, (g \triangleright \beta\xi)\alpha\gamma)$$

from which the exchange rule follows as desired.

All of this structure obtained from a crossed module  $G = (G_1, G_2, \partial, \triangleright)$  defines a 2-group which we call the standard 2-group of  $G$  and denote as  $\overline{G}$ . This is the 2-group that appears in the standard literature and it is the one used in the proof of Janelidze's Brown-Spencer Theorem [52] which implies that any 2-group is isomorphic to  $\overline{G}$  for some crossed module  $G$ . The theorem also specifies an inverse procedure which allows one to obtain a crossed module from a 2-group and this is the main step in proving that the categories of crossed modules and 2-groups are equivalent.

For our purposes we also need to define another 2-group, the extended 2-group  $\overline{\overline{G}}$ , which extends  $\overline{G}$  in the sense that  $\overline{G}$  is a 2-subgroup of  $\overline{\overline{G}}$ . This construction was introduced by us to meet the

requirements of our model and it is the maximal 2-group with the required properties. To do so we first note that an alternative way to label the bigons of  $\overline{G}$  is to use triples  $(g, \alpha, h) \in G_1 \times G_2 \times G_1$  satisfying the following condition:

$$\partial(\alpha)^{-1}gh^{-1} = 1 \quad (2.53)$$

We can make this condition more flexible by using a subgroup  $N \subset G_1$  to define a 2-group  $\overline{G}_N$  whose bigons are labeled by triples  $(g, \alpha, h) \in G_1 \times G_2 \times G_1$  that satisfy:

$$F(g, \alpha, h) = \partial(\alpha)^{-1}gh^{-1} \in N \quad (2.54)$$

All of the operations of  $\overline{G}_N$  can be defined using the same formulas used for  $\overline{G}$  so that we have the following for the horizontal and vertical composition:

$$(g, \alpha, t) \circ (h, \beta, w) = (gh, (g \triangleright \beta)\alpha, tw) \quad (2.55)$$

$$(g, \alpha, t) \bullet (t, \beta, w) = (g, \alpha\beta, w) \quad (2.56)$$

However,  $\overline{G}_N$  is not a 2-group for arbitrary subgroups since it does not obeys all the required consistency relations. For instance, the compositions are only well defined if the bigons obtained from them also belong to  $\overline{G}_N$  and this is not true in general. To see what assumption we need to make on  $N$  in order for this to hold let us apply condition 2.54 to a bigon obtained from vertical composition:

$$F((g, \alpha, t) \bullet (t, \beta, w)) = F(g, \alpha\beta, w) = F(t, \beta, w)wt^{-1}F(g, \alpha, t)tw^{-1} \quad (2.57)$$

Since  $(g, \alpha, t)$  and  $(t, \beta, w)$  belong to  $\overline{G}_N$  we know that  $F(g, \alpha, t), F(t, \beta, w) \in N$  so equation 2.57 implies that  $N$  needs to be a normal subgroup for  $F((g, \alpha, t) \bullet (t, \beta, w)) \in N$  to hold in general. This assumption also ensures that the bigons obtained from horizontal composition belong to  $\overline{G}_N$  as well but it still not enough to guarantee all the relations a 2-group needs to obey. In particular, consider the following expressions:

$$\begin{aligned} ((g, \alpha, t) \circ (h, \beta, w)) \bullet ((t, \gamma, z) \circ (w, \xi, p)) &= (gh, (g \triangleright \beta)\alpha, tw) \bullet (tw, (t \triangleright \xi)\gamma, zp) \\ &= (gh, (g \triangleright \beta)\alpha(t \triangleright \xi)\gamma, zp) \\ ((g, \alpha, t) \bullet (t, \gamma, z)) \circ ((h, \beta, w) \bullet (w, \xi, p)) &= (g, \alpha\gamma, z) \circ (h, \beta\xi, p) \\ &= (gh, (g \triangleright \beta\xi)\alpha\gamma, zp) \end{aligned}$$

It follows that the middle four exchange rules is valid if, and only if,  $(g \triangleright \xi)\alpha = \alpha(t \triangleright \xi)$  and this is equivalent to:

$$(g \triangleright \xi)\alpha = \alpha(t \triangleright \xi) \Leftrightarrow \xi = (g^{-1}\partial(\alpha)t) \triangleright \xi = (tF(g, \alpha, t)t^{-1}) \triangleright \xi$$

so we get the additional constraint that  $tF(g, \alpha, t)t^{-1}$  must act trivially. Since  $N$  is a normal subgroup we see that  $F(g, \alpha, t) \Leftrightarrow tF(g, \alpha, t)t^{-1} \in N$  and it follows that the exchange only holds in general if all the elements of  $N$  act trivially. Moreover, if we think of the action as as a morphism  $\triangleright : G_1 \rightarrow \text{Aut}(G_2)$  then the elements that act trivially are those that belong to  $\ker \triangleright$  and the constraint on  $N$  becomes that it must be a normal subgroup of  $G_1$  contained in  $\ker \triangleright$ . It turns out that this is sufficient to verify the remaining conditions and we don't need to impose any more limitations.

It should be clear that given normal subgroups  $N, N' \subset G_1$  such that  $N \subset N' \subset \ker \triangleright$  it follows that  $\overline{G}_N$  is a 2-subgroup of  $\overline{G}_{N'}$  under the obvious inclusion. This provides a partial order on the set of 2-groups of the form  $\overline{G}_N$  and since  $\ker \triangleright$  itself is normal we have that  $\overline{G}_{\ker \triangleright}$  is a maximal element of this set. Motivated by this we define the extended 2-group  $\overline{\overline{G}}$  of a crossed module to be  $\overline{G}_{\ker \triangleright}$  or, equivalently, as the 2-group whose bigons are triples  $(g, \alpha, h)$  such that

$$F(g, \alpha, h) = \partial(\alpha)^{-1}gh^{-1} \in \ker \triangleright \quad (2.58)$$

In the next chapter we will see that the extended 2-group labels provides the local degrees of freedom of the configurations of our model whereas the standard 2-group labels configurations living on the ground state.



## Chapter 3

# Hamiltonian Models from 2-Gauge Theory

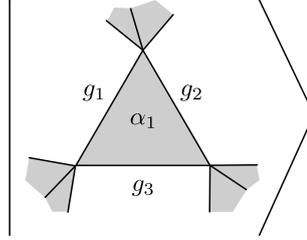
In this chapter we develop the core material of this text, namely, we show how to construct a new class of exactly solvable models that provides a Hamiltonian realization of lattice 2-gauge theories. These models arise as quantum many body systems which generalize the usual quantum double models by replacing groups with 2-groups so we've attempted to present it in a way that emphasizes this. For concreteness, we always work with a fixed enumeration for the vertexes of the lattice as this allows us to give explicit formulas for everything and, hopefully, makes things clearer. However this also means that there will be a lot of repetitive computations going on so we've tried to strike a balance by allowing ourself to omit some calculations when we fill they don't add anything to the reader.

We start by describing the Hilbert space of the model and we try to do so in two ways, as a sort of generalized spin model [56] and also a lattice 2-gauge theory, since both these perspective will be useful. The first section also includes a fairly technical discussion about the role played by the choice of an enumeration and why the model is independent of this. Afterwards we move on to define the first operator of the model, the projector associated to the fake 1-holonomy, and we do so by explaining how it is a generalization of the usual holonomy projector. The next operator is also related to a generalization of holonomy, that is, it is the operator associated to the 2-holonomy, a higher dimensional version of the usual holonomy. We then describe the operators implementing the 2-gauge symmetry, namely, the 2-gauge transformations. Finally, we conclude with the definition of the Hamiltonian operator and a demonstration that it is exactly solvable.

The material presented in this chapter was developed independently but we should reiterate that similar results were obtained and published in [37] by another research group. For details concerning quantum double models we refer to [13, 15, 16] and to [34] for ideas related to 2-gauge theory. Also, we should highlight that [57, 58] provided the basis for the procedure we used to define the operators corresponding to 2-gauge transformations.

### 3.1 Hilbert Space

Let us fix a simplicial complex  $K$  which is a triangulation of some compact 3-dimensional manifold  $M$ . This complex is the lattice of our model so, in analogy with quantum double models, we define a quantum many body system by assigning local degrees of freedom to it. The degrees of freedom come from a finite crossed module  $G = (G_1, G_2, \partial, \triangleright)$  by attaching  $G_1$ -spins to the edges of  $K$  and  $G_2$ -spins to the triangles. Therefore, the quantum states of our system are described by basis states such as:

$$|g, \alpha\rangle = |g_1 \dots g_E, \alpha_1 \dots \alpha_T\rangle =$$


where  $g_i \in G_1$  specifies the value sitting at the edge  $l_i \in K_1$  and  $\alpha_j \in G_2$  the value assigned to the triangle  $p_j \in K_2$ . Also,  $E = |K_1|$  and  $T = |K_2|$  denote the number of edges and triangles, respectively, of the triangulation which we assume to be finite from now on.

This description of the states is fairly straightforward but it requires us to enumerate all edges and triangles in a specific manner so it is not so convenient. It is much better to fix an enumeration for the vertices of  $K$  so that our labels  $(g, \alpha)$  can be described by functions mapping edges and triangles to the associated groups:

$$\begin{aligned} \mathbf{u} : l \in K_1 &\mapsto \mathbf{u}(l) \in G_1 \\ \mathbf{u} : p \in K_2 &\mapsto \mathbf{u}(p) \in G_2 \end{aligned}$$

where we've used the same symbol for both functions in order to simplify the notation. As a consequence we have a basis of states  $|\mathbf{u}\rangle$  labeled by functions  $\mathbf{u}$  and defined by:

$$|\mathbf{u}\rangle = \left( \bigotimes_{[ab] \in K_1} |u[ab]\rangle \right) \otimes \left( \bigotimes_{[abc] \in K_2} |u[abc]\rangle \right)$$

The assumption we made that the complex has an enumeration is a technical requirement since we want to think of the group elements as the values a 2-gauge connection assigns to curves. There are two possible curves corresponding to an edge  $[ab] \in K_1$ , one goes from  $a$  to  $b$  and the other goes in the opposite direction, so we need to decide to which curve the value  $u[ab]$  is related. In principle, given some basis state  $|\mathbf{u}\rangle$ , we would have no way of doing this systematically but, thanks to the enumeration, we can make a convention to get rid of the ambiguity. Hence, from now on  $u[ab]$  will be assigned to the curve which agrees with the standard orientation induced by the enumeration, that is, the curve that goes from the smallest index,  $a$  in our example, to the one with the larger index,  $b$  in our example.

A similar issue arises when we consider the values assigned to triangles. In fact, the ambiguity is even worse since the values  $\mathbf{u}$  assigns to them should correspond to bigons associated to the surface of the triangles. Once again, this is inspired by the fact that we want to consider a 2-gauge configuration associated to  $\mathbf{u}$  and this implies it should assign values of a 2-group to bigons living in the lattice. To remedy this we can list all possibilities, as shown in Figure 3.1, and fix a convention just as we did for

the edges. Our convention is that the value  $u[abc]$  corresponds to the case described in Figure 3.1a i.e. to the bigon whose initial curve goes through the edge  $[ab]$  and then  $[bc]$  and the final curve corresponds to  $[ac]$ .

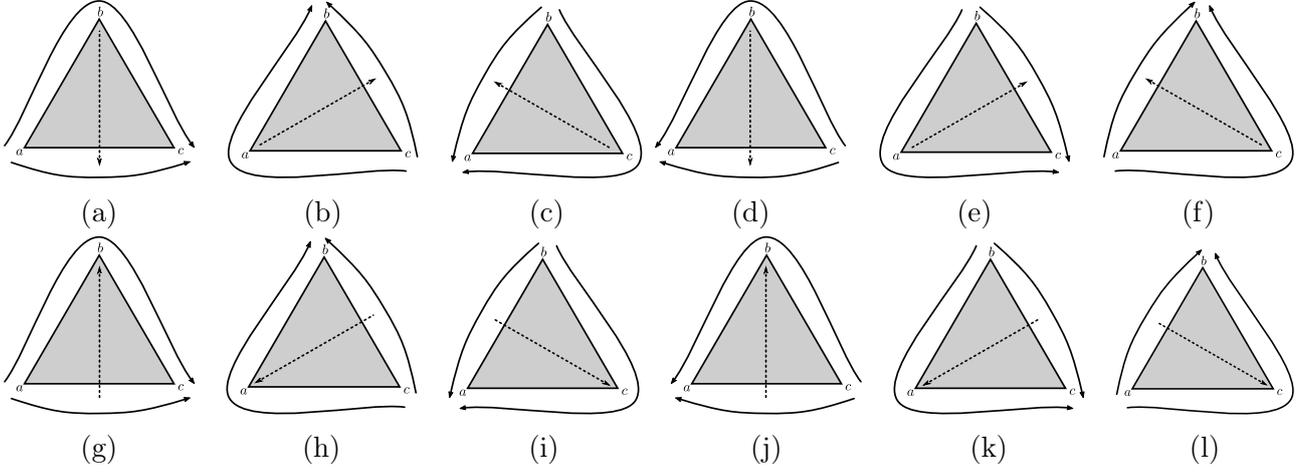


Figure 3.1: Possible bigons associated to a triangle

Now that we've mentioned that  $u$  should be understood as the data specifying a 2-gauge connection we need to say what is the actual 2-group underlying the model. After all, in Section 2.2, we saw that there are more than one 2-group related to a crossed module  $G$  so we must choose which one we are going to use.

If we consider condition 2.53, that defines the bigons of the standard 2-group  $\overline{G}$ , and apply it to the values  $u$  assigns to a triangle  $[abc] \in K_2$  we end up with:

$$\partial(u[abc])^{-1}u[ab]u[bc]u[ac]^{-1} = 1 \tag{3.1}$$

However, if we assume  $G_2$  is the trivial group we see that equation 3.1 implies

$$u[ab]u[bc]u[ac]^{-1} = 1 \tag{3.2}$$

which is a problem because a trivial  $G_2$  means we are just doing ordinary gauge theory so we should reproduce the entire Hilbert space of the quantum double model. Constraint 3.2 restricts us to only using flat gauge configurations so it throws away states which are physically relevant. The best alternative is to use the extended 2-group  $\overline{\overline{G}}$  instead, whose bigons are defined by condition 2.58 which leads to a more flexible constraint:

$$\partial(u[abc])^{-1}u[ab]u[bc]u[ac]^{-1} \in \ker(\triangleright) \tag{3.3}$$

Now if we go back to the case of trivial  $G_2$  we see that  $\triangleright$  is also trivial so  $\ker(\triangleright) = G_1$  and 3.3 becomes void for the case of ordinary gauge theory. Therefore, the extended 2-group  $\overline{\overline{G}}$  allow us to obtain the full Hilbert space of the quantum double and we interpret this as evidence that it is the correct choice.

Having clarified what is the 2-group that defines the configurations of the model, we can finally define a 2-gauge configuration over  $K$  with values in  $G$  to be a pair of functions, which we label with

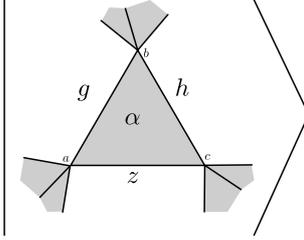
the same symbol, that satisfies the kernel condition in all triangles:

$$\begin{aligned} \mathbf{u} &= (\mathbf{u} : K_1 \rightarrow G_1, \mathbf{u} : K_2 \rightarrow G_2) \\ \forall [abc] \in K_2 &\implies \partial(\mathbf{u}[abc])^{-1} \mathbf{u}[ab] \mathbf{u}[bc] \mathbf{u}[ac]^{-1} \in \ker \triangleright \end{aligned}$$

and we denote the set of all such function by  $\mathfrak{X}(K, G)$ . Formally speaking, the Hilbert space for our model is made of states which are superpositions of 2-gauge configurations so we can write it as:

$$\mathcal{H} = \text{span}(|\mathbf{u}\rangle \mid \mathbf{u} \in \mathfrak{X}(K, G))$$

and any state can be depicted graphically as:

$$|\psi\rangle = \sum_{\mathbf{u} \in \mathfrak{X}(K, G)} \psi(g, h, z, \alpha)$$


where  $g = \mathbf{u}[ab]$ ,  $h = \mathbf{u}[bc]$ ,  $z = [ac]$ ,  $\alpha = \mathbf{u}[abc]$  and we've omitted the degrees of freedom not appearing in the picture. Notice that the group elements are not independent since  $(\partial\alpha)^{-1}ghz^{-1}$  must belong to  $\ker \triangleright$  due to condition 3.3.

The mathematically minded reader might be concerned that what we defined is just a vector space and not a Hilbert space. However, since we've assumed that  $G$  and  $K$  are finite we have:

$$\dim(\mathcal{H}) \leq |G_1|^E |G_2|^T = |G_1|^{|K_1|} |G_2|^{|K_2|} < \infty \quad (3.4)$$

so  $\mathcal{H}$  must be finite dimensional which make it easy to define a Hilbert space structure for it since we just have to specify an inner product. We can do this by imposing that the basis labeled by 2-gauge configurations is orthonormal, meaning  $\langle \mathbf{u}' | \mathbf{u} \rangle = \delta(\mathbf{u}', \mathbf{u})$ , since this completely determines an inner product.

Note that if we consider the case of a trivial group  $G_2$  then the inequality 3.4 becomes an equality and we can write the Hilbert space of the model as:

$$\mathcal{H} = \bigotimes_{[ab] \in K_1} \mathbb{C}G_1$$

which is the Hilbert space of the quantum double model with group  $G_1$  as expected. The Hilbert space  $\mathbb{C}G_1$  is just the Hilbert space associated to the group algebra of  $G_1$ , that is, it is the span over states  $|g\rangle$  where  $g \in G_1$ . The fact that we recover the Hilbert space of the quantum double model for groups is an important feature since we are trying to generalize it and there would be no hope of doing if the kinematics of both model had a mismatch. Of course all of this happens thanks to us using the extend 2-group and its more flexible constraint.

It would be great to write  $\mathcal{H}$  as a tensor product of local Hilbert spaces but this is impossible for arbitrary crossed modules  $G$ . In general, the best we can do is to describe is as subspace of such a product:

$$\mathcal{H} \subset \left( \bigotimes_{[ab] \in K_1} \mathbb{C}G_1 \right) \otimes \left( \bigotimes_{[abc] \in K_2} \mathbb{C}G_2 \right)$$

and they will only be the same when the action of  $G$  is trivial which can only happen if  $G_2$  is abelian. Hence, this inconvenience is a direct consequence of us doing non abelian algebra in a two dimensional setting and these types of difficulties are unavoidable.

Having described the Hamiltonian we can now move on to construct the relevant operator for the dynamics of the mode. However, before doing so we should say a few words to explain why fixing an enumeration for the lattice is not an issue. Since this is a highly technical issue the reader might want to skip directly to the next section or just skim through the calculations. The reason we say this is that, while it is important to clarify this point, this whole business concerning the enumeration looks much more complicated than it actually is and is not particularly interesting.

## Independence on the Enumeration

It would clearly be unphysical for our model to depend on the enumeration we choose for  $K$  so we need to show how to pass from one enumeration to a different one, and later show this does not change the dynamics of the model. One way of doing this is to define a unitary transformation  $U : \mathcal{H} \rightarrow \mathcal{H}$  that corresponds to moving from the description using one enumeration to a different one and, later on, show this commutes with the Hamiltonian. To do so we must describe rules such that, given any basis state  $|\mathbf{u}\rangle$ , we can specify rules to define a new basis state  $|\mathbf{u}'\rangle$  which will be the version of  $|\mathbf{u}\rangle$  in the description using the new enumeration.

Let's consider an edge  $[ab]$  and look at what can happen with it in the new enumeration:

- If  $a < b$  in the new enumeration then  $[ab] \Rightarrow [ab]$  and we define  $\mathbf{u}'[ab] = \mathbf{u}[ab]$
- If  $b < a$  in the new enumeration then  $[ab] \Rightarrow [ba]$  and we define  $\mathbf{u}'[ba] = \mathbf{u}[ab]^{-1}$

Now let's look at what happens with a triangle  $[abc]$ :

- If  $a < b < c$  in the new enumeration then  $[abc] \Rightarrow [abc]$  and we define  $\mathbf{u}'[abc] = \mathbf{u}[abc]$
- If  $a < c < b$  in the new enumeration then  $[abc] \Rightarrow [acb]$  and we define  $\mathbf{u}'[acb] = \mathbf{u}[abc]^{-1}$
- If  $b < a < c$  in the new enumeration then  $[abc] \Rightarrow [bac]$  and we define  $\mathbf{u}'[bac] = \mathbf{u}[ab]^{-1} \triangleright \mathbf{u}[abc]^{-1}$
- If  $b < c < a$  in the new enumeration then  $[abc] \Rightarrow [bca]$  and we define  $\mathbf{u}'[bca] = \mathbf{u}[ab]^{-1} \triangleright \mathbf{u}[abc]$
- If  $c < a < b$  in the new enumeration then  $[abc] \Rightarrow [cab]$  and we define  $\mathbf{u}'[cab] = \mathbf{u}[ac]^{-1} \triangleright \mathbf{u}[abc]$
- If  $c < b < a$  in the new enumeration then  $[abc] \Rightarrow [cba]$  and we define  $\mathbf{u}'[cba] = \mathbf{u}[ac]^{-1} \triangleright \mathbf{u}[abc]^{-1}$

These rules might seem arbitrary but they arise naturally from a more formal description based on the notion of the path 2-groupoid [59] of  $K$ . This formalism allows one to describe 2-gauge connections and it shows that our functions  $\mathbf{u}$  are just a way of specifying such connections. Hence, the choice of an enumeration is really just a tool we employ for labeling the states in a more concrete manner without having to use abstract machinery from higher category theory. Our reason for doing so is that, hopefully, it makes the description of the model more accessible as we don't have to introduce a great deal of additional mathematical concepts.

Nonetheless, it is advisable to verify that the rules we've just prescribed are consistent with each other. For instance, let's say we start with some enumeration, change to a new one, then to an additional one and, finally, come back to the initial choice, do we have any guarantees that we will

go back to the original state? One way of checking this will happen, and to check the consistency in general, is to enforce that the rules for edges define an  $S_2$  representation and those associated to the triangles define a  $S_3$  representation. Here  $S_n$  denotes the group of permutations in  $n$  elements.

Checking the  $S_2$  condition is fairly easy as it has a simple presentation:

$$S_2 = \langle x : x^2 = 1 \rangle$$

where  $x$  corresponds to the only non trivial permutation of two elements i.e.  $a < b \Rightarrow b < a$  so we just need to check that the rule obtained from  $x^2 : a < b \Rightarrow b < a \Rightarrow a < b$  maps any basis state  $|\mathbf{u}\rangle$  into itself:

$$\mathbf{u}[ab] \Rightarrow \mathbf{u}'[ba] = \mathbf{u}[ab]^{-1} \Rightarrow \mathbf{u}''[ab] = (\mathbf{u}'[ab])^{-1} = (\mathbf{u}[ab]^{-1})^{-1} = \mathbf{u}[ab]$$

To check the  $S_3$  conditions we use the following presentation:

$$S_3 = \langle y, z : y^2 = z^2 = (yz)^3 = 1 \rangle$$

where  $y$  corresponds to  $a < b < c \Rightarrow b < a < c$  and  $z$  to  $a < b < c \Rightarrow a < c < b$ . The consistency relations coming from  $y^2 = z^2 = 1$  are pretty straightforward but  $(yz)^3$  is a bit trickier:

$$\mathbf{u}[abc] \Rightarrow \dots \Rightarrow (\mathbf{u}[ab]\mathbf{u}[bc]\mathbf{u}[cb]^{-1}) \triangleright \mathbf{u}[abc]$$

We can use relation 2.29 and the kernel condition 3.3 to simplify this:

$$\begin{aligned} (\mathbf{u}[ab]\mathbf{u}[bc]\mathbf{u}[cb]^{-1}) \triangleright \mathbf{u}[abc] &= (\mathbf{u}[ab]\mathbf{u}[bc]\mathbf{u}[cb]^{-1}) \triangleright \mathbf{u}[abc]^{-1} \mathbf{u}[abc] \mathbf{u}[abc] \\ &= (\partial(\mathbf{u}[abc])^{-1} \mathbf{u}[ab]\mathbf{u}[bc]\mathbf{u}[cb]^{-1}) \triangleright \mathbf{u}[abc] = \mathbf{u}[abc] \end{aligned}$$

The only thing remaining is to check that our rules are actually generated by the rules corresponding to  $y, z$ . For instance,  $a < b < c \Rightarrow b < c < a$  can be written as  $zy$  and we compute the rule associated to this:

$$\begin{aligned} \mathbf{u}[abc] \Rightarrow \mathbf{u}'[bac] &= \mathbf{u}[ab]^{-1} \triangleright \mathbf{u}[abc]^{-1} \Rightarrow \mathbf{u}''[bca] = (\mathbf{u}'[bac])^{-1} \\ &= (\mathbf{u}[ab]^{-1} \triangleright \mathbf{u}[abc]^{-1})^{-1} = \mathbf{u}[ab]^{-1} \triangleright \mathbf{u}[abc] \end{aligned}$$

which is precisely the rule we gave for  $a < b < c \Rightarrow b < c < a$ . We won't write how to get the other rules from the generators as it would get too repetitive but we ensure the reader everything works fine.

The conclusion we arrive from all this discussion is that we can always move from one enumeration to another one in a consistent fashion by applying the rules we've described. These rules allow us to define an unitary operator:

$$\begin{aligned} U : \mathcal{H} &\rightarrow \mathcal{H} \\ U : |\mathbf{u}\rangle &\mapsto |\mathbf{u}'\rangle \end{aligned}$$

which allows us to pass from a description of  $\mathcal{H}$  based on one enumeration to a different enumeration.

## 3.2 Fake 1-Holonomy

Now we are going to define our first type of operator but to do so we need to quickly review the notion of holonomy coming from ordinary lattice gauge theory. We will describe how this leads to some of the operators of the quantum double model and use it as the starting point to define the operators for our model. For reasons that will become clear in the next section we use the term 1-holonomy instead of holonomy.

Let's assume the crossed module  $G$  has trivial  $G_2$  so that it is fully specified by  $G_1$  i.e. it is just a group. In this scenario, as we've already discussed, the 2-gauge configurations are just gauge configurations defined by maps:

$$\mathbf{u} : K_1 \rightarrow G_1$$

so that, given any curve  $\gamma$  in the lattice, we can compute its 1-holonomy

$$\mathfrak{H}^1_\gamma(\mathbf{u}) = \mathbf{u}[ab]^{\pm 1} \mathbf{u}[bc]^{\pm 1} \mathbf{u}[cd]^{\pm 1} \dots \quad (3.5)$$

where  $[ab], [bc], [cd], \dots$  are the edges that define  $\gamma$  and the  $\pm 1$  signs take into account whether  $\gamma$  goes through an edges in its standard orientation or the opposite way. In Figure 3.2 we show a couple of examples and, in particular, show a curve that goes through the boundary of a triangle. These types of curves are particularly important because their holonomies are related to the curvature of the underlying gauge connection [19] so they determine the field strength.

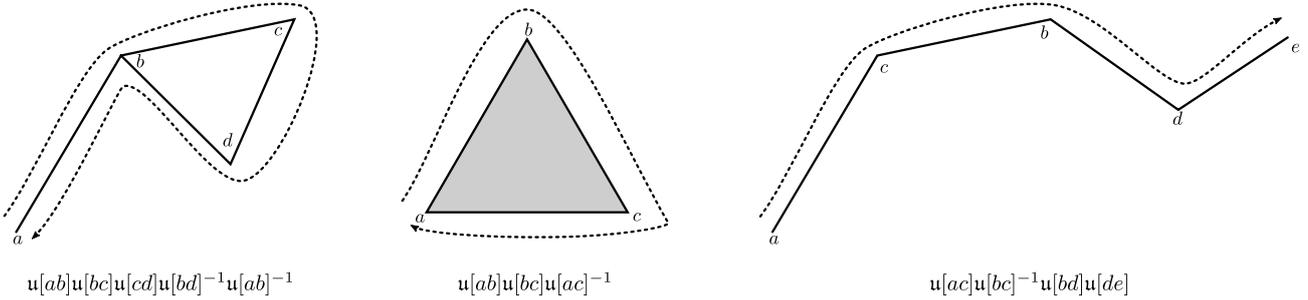


Figure 3.2: 1-Holonomy for different curves on the lattice.

One can define the holonomy associated to a triangle by fixing a curve going along its boundary:

$$\mathfrak{H}^1_{[abc]}(\mathbf{u}) = \mathbf{u}[ab]\mathbf{u}[bc]\mathbf{u}[ac]^{-1} \quad (3.6)$$

where, by convention, we've used the curve that starts at the smallest vertex  $a$  of the triangle  $[abc]$  and goes around it in the standard orientation. Of course we could have chosen a different curve and this would lead to another expression, for instance if the curve starts at  $b$  and also goes in the standard orientation we get:

$$\mathbf{u}[bc]\mathbf{u}[ac]^{-1}\mathbf{u}[ab] = \mathbf{u}[ab]^{-1}\mathfrak{H}^1_{[abc]}(\mathbf{u})\mathbf{u}[ab]$$

This last expression is not an accident since, in general, if we consider some triangle  $p \in K_2$  and a vertex  $v$  in its boundary, the holonomy obtained from a curve  $\gamma$  that starts in  $v$  and goes around

$p$  in some orientation will always be equal to  $\mathfrak{H}^1_p(\mathbf{u})^{\pm 1}$  conjugated by some group element. Here,  $\mathfrak{H}^1_p(\mathbf{u})$  denotes the reference holonomy computed with the curve starting at the smallest vertex and the positive orientation and  $\pm 1$  takes into account whether  $\gamma$  has positive orientation or not. As a consequence, to obtain observables from the 1-holonomies of a gauge configuration without having to worry about specifying the initial point we must use class functions, that is, functions that only depend on the conjugacy classes of the gauge group. This is in accordance with the fact that the partition function of a lattice gauge theory is defined using class functions evaluated at the holonomies.

Therefore, any observable obtained from the values of the 1-holonomies can be decomposed into the following local projectors:

$$B_{[abc]}^{[k]} |\mathbf{u}\rangle = \delta(\mathfrak{H}^1_{[abc]}(\mathbf{u}) \in [k]) |\mathbf{u}\rangle = \begin{cases} |\mathbf{u}\rangle, & \text{if } \mathfrak{H}^1_{[abc]}(\mathbf{u}) \in [k] \\ 0, & \text{otherwise} \end{cases} \quad (3.7)$$

where  $[k] = \{rkr^{-1} : r \in G_1\}$  denotes the conjugacy class of some  $k \in G_1$ . We can use a more explicit expression and depict them graphically as:

$$B_{[abc]}^{[k]} \left| \begin{array}{c} \text{triangle with vertices } a, b, c \text{ and edges } g, h, z \end{array} \right\rangle = \sum_r \delta(\mathfrak{H}^1_{[abc]}(\mathbf{u}), rkr^{-1}) \left| \begin{array}{c} \text{triangle with vertices } a, b, c \text{ and edges } g, h, z \end{array} \right\rangle = \sum_r \delta(ghz^{-1}, rkr^{-1}) \left| \begin{array}{c} \text{triangle with vertices } a, b, c \text{ and edges } g, h, z \end{array} \right\rangle \quad (3.8)$$

In particular, we get the  $B_p$  operator of the quantum double model:

$$B_{[abc]} \left| \begin{array}{c} \text{triangle with vertices } a, b, c \text{ and edges } g, h, z \end{array} \right\rangle = \delta(\mathfrak{H}^1_{[abc]}(\mathbf{u}), 1) \left| \begin{array}{c} \text{triangle with vertices } a, b, c \text{ and edges } g, h, z \end{array} \right\rangle = \delta(ghz^{-1}, 1) \left| \begin{array}{c} \text{triangle with vertices } a, b, c \text{ and edges } g, h, z \end{array} \right\rangle \quad (3.9)$$

which enforces the following condition:

$$\mathfrak{H}^1_{[abc]}(\mathbf{u}) = \mathbf{u}[ab]\mathbf{u}[bc]\mathbf{u}[ac]^{-1} = 1 \quad (3.10)$$

This condition, which we call 1-flatness, is equivalent to saying that the gauge connection has no curvature in the vicinity of the triangle  $[abc]$  so it projects into the vacuum of the theory. As a consequence the projector  $B_p$  appears in the Hamiltonian of the quantum double model to enforce that its ground states doesn't have any flux excitations.

Now that we've reviewed how to obtain the  $B_p$ 's operators of the quantum double model let's go back to general case of an arbitrary  $G$  and define the analogous operators for our model. Since  $G_2$  is no longer trivial there will be a degree of freedom attached to every triangle and we need to take this into account when we compute the holonomy as it contributes to the field strength. The correct way to include this contribution in a way that is invariant under 2-gauge symmetries is the so called fake 1-holonomy [34]:

$$\mathfrak{F}^1_{[abc]}(\mathbf{u}) = \partial(\mathbf{u}[abc])^{-1} \mathbf{u}[ab]\mathbf{u}[bc]\mathbf{u}[ac]^{-1} \quad (3.11)$$

where we are using the same convention we used for  $\mathfrak{H}^1_{[abc]}(\mathbf{u})$  so we compute it with the curve that starts in  $a$ , goes to  $b$ , then  $c$  and goes back to  $a$ .

The fake 1-holonomy behaves just like the 1-holonomy when we change the starting point or the orientation so the corresponding observables are also class functions and we get operators  $B_p^{[k]} : \mathcal{H} \rightarrow \mathcal{H}$  for each conjugacy class

$$B_{[abc]}^{[k]} |u\rangle = \delta(\mathfrak{F}_{[abc]}^1(u) \in [k]) |u\rangle = \begin{cases} |u\rangle, & \text{if } \mathfrak{F}_{[abc]}^1(u) \in [k] \\ 0, & \text{otherwise} \end{cases} \quad (3.12)$$

More explicitly we have

$$B_{[abc]}^{[k]} \left| \begin{array}{c} \text{triangle with } g, h, z, \alpha \end{array} \right\rangle = \sum_r \delta(\mathfrak{F}_{[abc]}^1(u), rkr^{-1}) \left| \begin{array}{c} \text{triangle with } g, h, z, \alpha \end{array} \right\rangle = \sum_r \delta(\partial(\alpha)^{-1}ghz^{-1}, rkr^{-1}) \left| \begin{array}{c} \text{triangle with } g, h, z, \alpha \end{array} \right\rangle \quad (3.13)$$

Notice that condition 3.3 implies  $\mathfrak{F}_p^1(u) \in \ker(\triangleright)$  so we only need to consider conjugacy classes of the subgroup  $\ker \triangleright$  as these provide all the possible values.

Just like in the quantum double model there is a  $B_p$  operator which projects into the trivial sector:

$$B_{[abc]} \left| \begin{array}{c} \text{triangle with } g, h, z, \alpha \end{array} \right\rangle = \delta(\mathfrak{F}_{[abc]}^1(u), 1) \left| \begin{array}{c} \text{triangle with } g, h, z, \alpha \end{array} \right\rangle = \delta(\partial\alpha^{-1}ghz^{-1}, 1) \left| \begin{array}{c} \text{triangle with } g, h, z, \alpha \end{array} \right\rangle \quad (3.14)$$

The condition it enforces, which we call fake 1-flatness, implies a 2-gauge configuration has trivial field strength around  $p$  or, equivalently, that it has trivial fake curvature. From another points of view, if a configuration is fake 1-flat in some triangle then the bigon it assigns to this triangle belongs to the standard 2-group  $\overline{G}$  contained in the extended 2-group  $\overline{\overline{G}}$  so that the  $B_p$ 's project into configurations assigning values in the standard 2-group.

To conclude this section we derive a discrete version of the Bianchi identities [59] for the fake curvatures of a 2-gauge configuration. Hence, let us consider a tetrahedron  $[abc]$  and compute the values of fake 1-holonomy associated to its faces:

$$\begin{aligned} \mathfrak{F}_{[abc]}^1(u) &= \partial(u[abc])^{-1}u[ab]u[bc]u[ac]^{-1} \\ \mathfrak{F}_{[abd]}^1(u) &= \partial(u[abd])^{-1}u[ab]u[bd]u[ad]^{-1} \\ \mathfrak{F}_{[acd]}^1(u) &= \partial(u[acd])^{-1}u[ac]u[cd]u[ad]^{-1} \\ \mathfrak{F}_{[bcd]}^1(u) &= \partial(u[bcd])^{-1}u[bc]u[cd]u[bd]^{-1} \end{aligned}$$

If we manipulate this formulas we obtain the following quantities:

$$\partial((u[ab] \triangleright u[bcd])u[abd]u[acd]^{-1}u[abc]^{-1})u[ab]\mathfrak{F}_{[bcd]}^1(u)u[ab]^{-1}\mathfrak{F}_{[abd]}^1(u)\mathfrak{F}_{[acd]}^1(u)^{-1}\mathfrak{F}_{[abc]}^1(u)^{-1} = 1 \quad (3.15)$$

To make sense of this expression let us go back to the case where  $G_2$  is trivial and see what we get:

$$u[ab]\mathfrak{H}_{[bcd]}^1(u)u[ab]^{-1}\mathfrak{H}_{[abd]}^1(u)\mathfrak{H}_{[acd]}^1(u)^{-1}\mathfrak{H}_{[abc]}^1(u)^{-1} = 1 \quad (3.16)$$

Equation 3.16 can be interpreted as a conservation law for the fluxes associated to the holonomies. More precisely, it implies that if three faces of a tetrahedron have trivial curvature then the fourth one

must be trivial as well which implies there is no source of flux. If we go back to the general case 3.15 we see that the term  $\partial(\dots)^{-1}$  is precisely such a source since it allows one of the faces to have non trivial curvature even when all the others are trivial. Hence, we can think of this contribution as a kind of monopole sitting inside of the tetrahedron. In the next section we will show that this contribution arises naturally and can be interpreted as a different type of curvature of the 2-gauge connection.

### 3.3 2-Holonomy

The next operator we need to define is associated to an observable called the 2-holonomy which is a two dimensional version of the 1-holonomy, hence its name. Intuitively, one can picture it by considering two curves with matching end points, such that there is a surface between them that works as path moving one curve into the other. This is analogous to the way a 1-holonomy calculation relies on the idea of a curve realizing the parallel transport between two points. Therefore, a 2-holonomy is a way of assigning values on the bigons 2-group to surfaces realizing a 2-parallel transport between curves. In Figure 3.3 we describe this idea graphically in a way that highlights how it is similar to the usual 1-holonomy.

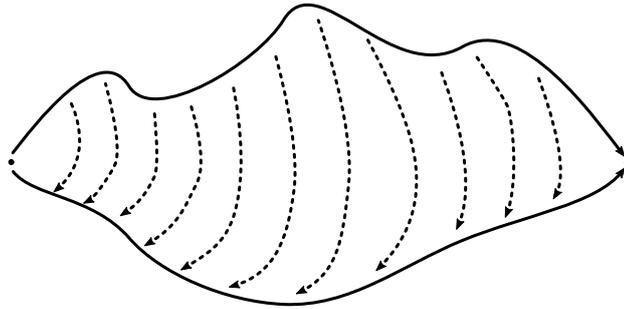


Figure 3.3: A curve undergoing 2-parallel transport through a 2-path.

To formalize this idea we define a 2-path as a subcomplex of the lattice which is the triangulation of a simply connected surface (a surface without holes) that connects two open curves, one being the initial curve and the other the final curve. We denote a 2-path as  $(\gamma_i, \sigma, \gamma_f)$  where  $\gamma_i(\gamma_f)$  denotes the initial(final) curve and  $\sigma$  is the surface connecting the curves. Given a 2-gauge configuration  $\mathbf{u}$  our goal is to define a map assigning bigons in the 2-group to 2-paths:

$$\mathfrak{H}^2_{(\gamma_i, \sigma, \gamma_f)}(\mathbf{u}) = (\mathfrak{H}^2_{\gamma_i}(\mathbf{u}), \mathfrak{H}^2_{\sigma}(\mathbf{u}), \mathfrak{H}^2_{\gamma_f}(\mathbf{u})) \quad (3.17)$$

where  $\mathfrak{H}^2_{\gamma_i}(\mathbf{u}), \mathfrak{H}^2_{\gamma_f}(\mathbf{u}) \in G_1$  and  $\mathfrak{H}^2_{\sigma}(\mathbf{u}) \in G_2$  are the values labeling the bigon. It is easy to compute  $\mathfrak{H}^2_{\gamma_i}(\mathbf{u}), \mathfrak{H}^2_{\gamma_f}(\mathbf{u})$  since they are just the usual values of 1-holonomy given by equation 3.5. However, it can get a bit tricky to actually compute  $\mathfrak{H}^2_{\sigma}(\mathbf{u})$  in general so we are going to describe some simple examples first and gradually build up until we can obtain the quantities of interest.

For starters, notice that each possibilities described in figure 3.1 is a potential 2-path corresponding to the surface of a triangle and these are the elementary 2-paths we are going to use to compute the more elaborate 2-paths. Our chosen convention for the triangles is essentially a choice of a standard



a tetrahedron  $[abcd] \in K_3$  and an open curve in its boundary in such a way that we can use the surface of the boundary  $\partial([abcd])$  as a 2-path going from this curve to itself. One can visualize this process by picturing the boundary of the tetrahedron as a sphere so that when we cut along the curve what remains is a disc mapping the curve into itself. If we use the curve that goes through all the vertexes in ascending order we get:

$$(3.24)$$

The term  $(g \triangleright \alpha)\beta\gamma^{-1}\sigma^{-1} = (\mathbf{u}[ab] \triangleright \mathbf{u}[bcd])\mathbf{u}[acd]\mathbf{u}[acd]^{-1}\mathbf{u}[abc]^{-1}$  is precisely the monopole-like contribution that appeared in the Bianchi identity 3.15 of the fake 1-holonomy and it is what determines the 2-curvature of the 2-gauge connection corresponding to  $\mathbf{u}$ . As such it is related to the field strength which is why we are going to use it to obtain an operator just like we did for the fake 1-holonomy. Therefore, we define:

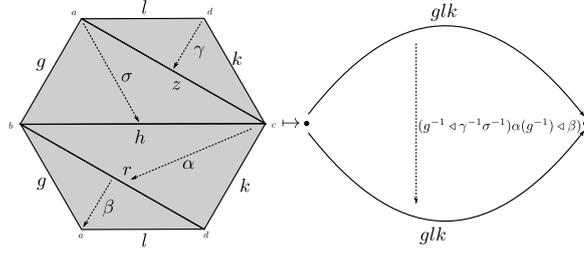
$$\mathfrak{H}^2_{[abcd]}(\mathbf{u}) = (\mathbf{u}[ab] \triangleright \mathbf{u}[bcd])\mathbf{u}[acd]\mathbf{u}[acd]^{-1}\mathbf{u}[abc]^{-1} \quad (3.25)$$

Of course this uses a series of conventions just like 3.11. For instance, we had to fix a curve connecting  $a$  to  $d$  but there are additional choices such as:

$$(3.26)$$

It should be clear that this has the same behavior we saw when we changed the initial vertex in the 1-holonomy calculation, namely, it is just a conjugation of  $\mathfrak{H}^2_{[abcd]}(\mathbf{u})$  by an element of  $G_2$ . This is true in general, as long as the end points remain the same, so we can always use a fixed curve if we only consider values up to conjugation. However, we should also be able to change the end points as there

is no preferred choice so let's consider the 2-path below that starts at  $b$  and ends at  $c$ :


(3.27)

We see that the new value can be obtained from  $\mathfrak{H}^2_{[abcd]}(\mathbf{u})$  by acting with an element of  $G_1$ :

$$(g^{-1}\gamma^{-1}\sigma^{-1})\alpha(g^{-1}\triangleright\beta) = (\partial(\sigma\gamma)g)^{-1}\triangleright((g\triangleright\alpha)\beta\gamma^{-1}\sigma^{-1}) = (\partial(\sigma\gamma)g)^{-1}\triangleright\mathfrak{H}^2_{[abcd]}(\mathbf{u})$$

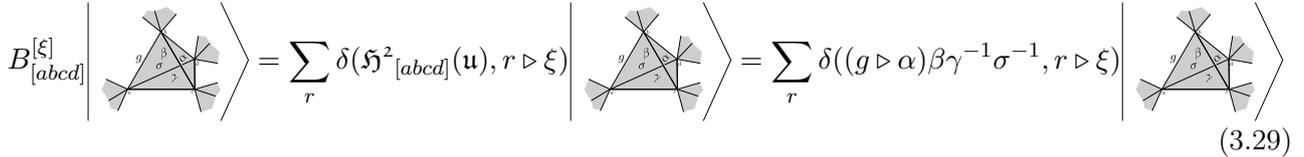
so that if we consider values up to the action of elements of  $G_1$  we also don't need to worry about the choice of initial and final vertex. Conveniently enough, this takes care of the issue of conjugation for free since:

$$\alpha\beta\alpha^{-1} = \partial(\alpha)\triangleright\beta$$

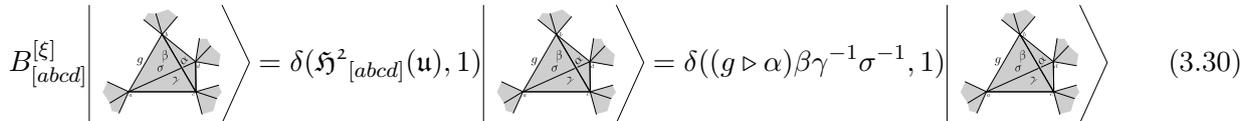
Therefore the observables related to 2-holonomies need to be functions which are invariant by the action  $\triangleright$  and we can define the following operators indexed by orbits of the action:

$$B_{[abcd]}^{[\xi]}|\mathbf{u}\rangle = \delta(\mathfrak{H}^2_{[abcd]}(\mathbf{u}) \in [\xi])|\mathbf{u}\rangle = \begin{cases} |\mathbf{u}\rangle, & \text{if } \mathfrak{H}^2_{[abcd]}(\mathbf{u}) \in [\xi] \\ 0, & \text{otherwise} \end{cases} \quad (3.28)$$

where  $[\xi] = \{r\triangleright\xi : r \in G_1\}$  denotes the orbit of some  $\xi \in G_2$ . We can express them graphically as:


(3.29)

In particular, we get the  $B_t$  operator that projects into the subspace of 2-flat configurations:


(3.30)

by enforcing the following condition:

$$\mathfrak{H}^2_{[abcd]}(\mathbf{u}) = (\mathbf{u}[ab]\triangleright\mathbf{u}[bcd])\mathbf{u}[acd]\mathbf{u}[acd]^{-1}\mathbf{u}[abc]^{-1} = 1 \quad (3.31)$$

As we've seen, the 2-holonomy appears naturally in 3.15 as the contribution the 2-curvature provides for the curvatures (the 1-curvatures) in the boundary of a tetrahedron. In particular, we can write:

$$\mathfrak{F}^1_{[bcd]}(\mathbf{u}) = \mathbf{u}[ab]^{-1}\partial(\mathfrak{H}^2_{[abcd]}(\mathbf{u}))^{-1}\mathfrak{F}^1_{[abc]}(\mathbf{u})\mathfrak{F}^1_{[acd]}(\mathbf{u})\mathfrak{F}^1_{[abd]}(\mathbf{u})^{-1}\mathbf{u}[ab] \quad (3.32)$$

which implies the following for the operators:

$$B_{[bcd]} = B_{[abcd]}B_{[abc]}B_{[acd]}B_{[abd]} \quad (3.33)$$

and similar results if we change which operator appears in the left hand side or consider the operators for arbitrary conjugacy classes.

There is also a Bianchi equation for the 2-holonomy [59] and we conclude this section with a derivation of it. In order to do that let's fix a pentachoron  $[abcde] \in K_4$  and consider the 2-holonomies associated to the tetrahedrons in its boundary:

$$\begin{aligned} \mathfrak{H}^2_{[abcd]}(\mathbf{u}) &= (\mathbf{u}[ab] \triangleright \mathbf{u}[bcd])\mathbf{u}[abd]\mathbf{u}[acd]^{-1}\mathbf{u}[abc]^{-1} \\ \mathfrak{H}^2_{[abce]}(\mathbf{u}) &= (\mathbf{u}[ab] \triangleright \mathbf{u}[bce])\mathbf{u}[abe]\mathbf{u}[ace]^{-1}\mathbf{u}[abc]^{-1} \\ \mathfrak{H}^2_{[abde]}(\mathbf{u}) &= (\mathbf{u}[ab] \triangleright \mathbf{u}[bde])\mathbf{u}[abd]\mathbf{u}[ade]^{-1}\mathbf{u}[abd]^{-1} \\ \mathfrak{H}^2_{[acde]}(\mathbf{u}) &= (\mathbf{u}[ac] \triangleright \mathbf{u}[cde])\mathbf{u}[abd]\mathbf{u}[ade]^{-1}\mathbf{u}[acd]^{-1} \\ \mathfrak{H}^2_{[bcde]}(\mathbf{u}) &= (\mathbf{u}[bc] \triangleright \mathbf{u}[cde])\mathbf{u}[bce]\mathbf{u}[bde]^{-1}\mathbf{u}[bcd]^{-1} \end{aligned}$$

which we can manipulate to obtain the following:

$$(x \triangleright \mathfrak{H}^2_{[bcde]}(\mathbf{u}))(\mathfrak{H}^2_{[abcd]}(\mathbf{u}))(y \triangleright \mathfrak{H}^2_{[abde]}(\mathbf{u})) = (z \triangleright \mathfrak{H}^2_{[acde]}(\mathbf{u}))(t \triangleright \mathfrak{H}^2_{[abce]}(\mathbf{u})) \quad (3.34)$$

where we've defined

$$\begin{aligned} x &= \mathbf{u}[ab] \\ z &= \partial(\mathbf{u}[abc]) \\ y &= \partial(\mathbf{u}[abc][acd]\mathbf{u}[abd]^{-1}) \\ t &= \partial(\mathbf{u}[abc]\mathbf{u}[acd]\mathbf{u}[ade]\mathbf{u}[ace]^{-1}\mathbf{u}[abc]^{-1}) \end{aligned}$$

Notice, in particular, that equation 3.34 is the kind of conservation constraint we've seen earlier for the 1-holonomy with 3.16 as it implies it is enough to know the 2-curvature associated to four tetrahedrons to infer the value for the fifth one. This leads to the following equation for the operators:

$$B_{[abcd]} = B_{[abce]}B_{[abde]}B_{[acde]}B_{[bcde]} \quad (3.35)$$

and similar results if we change which operator appears in the left hand side or consider operators for arbitrary orbits.

## 3.4 2-Gauge Transformations

In this section we are going to define the operators that implement the 2-gauge symmetry of our model and generalize the operators corresponding to gauge transformations used in quantum double models. To define them we are going to use a procedure inspired by a technique for obtaining the operators for the quantum double models which relies on the existence of a topological quantum field theory underlying the model(see [16] for instance). However, our focus is the Hamiltonian picture so we do

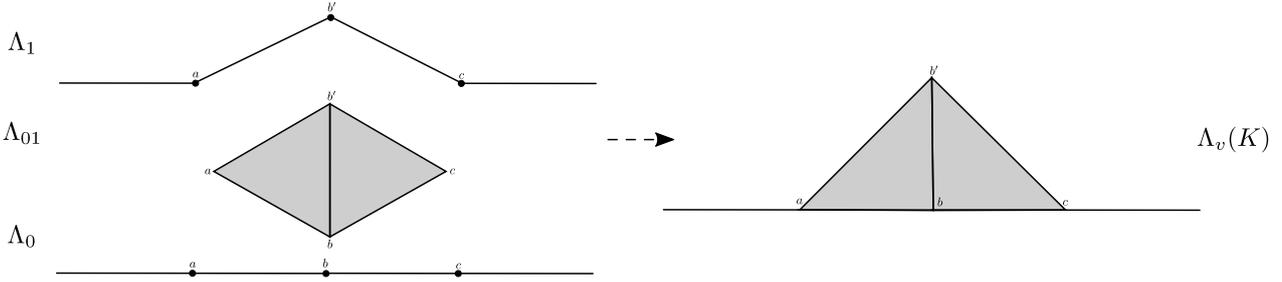


Figure 3.4: Lift procedure in lower dimensions.

not want to go into details regarding the partition function since it would divert us from our goals. Luckily, these details are not necessary since to define our operators we don't need information about the full topological quantum field theory but just the operators related to the cylinder and these can be constructed by hand. This approach was used in [58] to obtain the same kind of operator for the twisted quantum double models and we use it as inspiration as our similar conventions highlight.

Let us fix a vertex  $v$  in the triangulation which we will refer to as the base point. We can build a new complex  $\Lambda_v(K)$ , called the lift over  $v$ , by adding an additional copy of the base point  $v'$  called the lifted point as follows:

$$\Lambda_v(K) = \Lambda_0 \cup \Lambda_{01} \cup \Lambda_1 \quad (3.36)$$

where  $\Lambda_0 = K$  is the initial triangulation and  $\Lambda_1 = K'$  is the final triangulation obtained by replacing  $v$  with  $v'$  in the simplexes of  $K$ . The middle triangulation  $\Lambda_{01}$  is the closure of the set of all vertical simplexes connecting the base point to the lifted point. That is, it is generated by simplexes obtained by adding the lifted point  $v'$  to each simplex  $\sigma \in \text{star}(v)$  in the star of  $v$  to obtain a higher dimensional simplexes  $\Lambda_v(\sigma)$  which has the vertexes of  $\sigma$  and the lifted point as its vertexes. We can describe this explicitly if we choose an enumeration for  $\Lambda_v(K)$  by fixing the convention that the lifted point  $v'$  will be immediately behind  $v$  in the enumeration so that  $v' < v$  and  $w < v \implies w < v'$  and we can write:

$$\Lambda_v([v_0 \dots v_n]) = \Lambda_v[v_0 \dots v_n] = [v_0 \dots v_{i-1} v' v_i \dots v_n] \quad (3.37)$$

where  $[v_0 \dots v_n] \in \text{star } v_n$  is a  $n$ -simplex in the star of the base point  $v$  with  $v_i = v$  and  $\Lambda([v_0 \dots v_n], v)$  is a vertical  $(n+1)$ -simplex.

We can think of the lift  $\Lambda_v(K)$  as a transformation mapping the initial triangulation  $\Lambda_0$  into the final triangulation  $\Lambda_1$  which is equal to the initial one (minus the different label for the vertexes) by sweeping the middle triangulation. This is similar to the way Pachner moves can be understood as transformations mapping one triangulation into another by interpolating them through the interior of a ball. The difference is that for the Pachner moves we used a triangulation of the ball coming from a simplex since we wanted the initial and final triangulations to be different but equivalent and related by a particular decomposition of the boundary of the simplex. Here, we want the initial and final triangulations to be the same so we use a different triangulation of the ball, namely the prism  $\Lambda_{01}$ , whose boundary can be decomposed into equal pieces. In Figure 3.4 we depict lower dimensional analogues since we can't actually draw an example of  $\Lambda_v(K)$  because  $\Lambda_{01}$  is a four-dimensional complex.

Now we employ a little trick to extend any 2-gauge configuration  $\mathbf{u} \in \mathfrak{X}(K, \mathbf{G})$  into a 2-gauge configuration living on the lift of the complex. To do so we fix a parametrization  $\mathbf{p} = (\mathbf{p}_0, \mathbf{p}_1) \in \mathfrak{P}_v$

where

$$\mathfrak{p}_0 \in G_1 \quad (3.38)$$

$$\mathfrak{p}_1 : \text{star}(v)_1 \rightarrow G_2 \quad (3.39)$$

are the parameters used to specify the value at the only vertical 1-simplex  $\Lambda_v(v) = [v'v]$  and the vertical 2-simplexes sitting above the edges. Using this we can define a 2-gauge configuration  $\mathbf{u}_v^{\mathfrak{p}}$  that belongs to  $\mathfrak{X}(\Lambda_v(K), \mathbb{G})$  using the following rules:

- Its restriction to the initial complex  $K$  agrees with  $\mathbf{u}$ :

$$\forall [ab] \in K_1 \quad \mathbf{u}_v^{\mathfrak{p}}[ab] = \mathbf{u}[ab] \quad (3.40)$$

$$\forall [abc] \in K_2 \quad \mathbf{u}_v^{\mathfrak{p}}[abc] = \mathbf{u}[abc] \quad (3.41)$$

- Its values on the vertical simplexes are given by the parameters:

$$\mathbf{u}_v^{\mathfrak{p}}\Lambda_v[v] = \mathbf{u}_v^{\mathfrak{p}}[v'v] = \mathfrak{p}_0 \quad (3.42)$$

$$\forall [ab] \in \text{star}(v)_1 \quad \mathbf{u}_v^{\mathfrak{p}}\Lambda_v[ab] = \begin{cases} \mathbf{u}_v^{\mathfrak{p}}[a'ab] = \mathfrak{p}_1[ab]^{-1}, & \text{if } v = a \\ \mathbf{u}_v^{\mathfrak{p}}[ab'b] = \mathfrak{p}_1[ab], & \text{if } v = b \end{cases} \quad (3.43)$$

- All vertical 2-simplexes are fake 1-flat and all vertical 3-simplexes are 2-flat:

$$\forall [ab] \in \text{star}(v)_1 \quad \mathfrak{F}^1_{\Lambda_v[ab]}(\mathbf{u}_v^{\mathfrak{p}}) = 1 \quad (3.44)$$

$$\forall [abc] \in \text{star}(v)_2 \quad \mathfrak{H}^2_{\Lambda_v[abc]}(\mathbf{u}) = 1 \quad (3.45)$$

The reason these rules completely determine  $\mathbf{u}_v^{\mathfrak{p}}$  is that we can use the flatness conditions on the vertical simplexes to obtain its values on all simplexes not specified by the first two rules. The missing simplexes are those that contain the lifted point but do not contain the base point i.e. the copies of the simplexes of  $\text{star}(v)$  obtained by replacing the base point with the lifted point. Below we list all the possibilities with the explicit formulas for the values of  $\mathbf{u}_v^{\mathfrak{p}}$  at these simplexes.

- Given a 1-simplex  $[ab] \in \text{star } v$  we either have  $v = a$  or  $v = b$ . In the first case we can apply the fake 1-flatness condition to the vertical 2-simplex  $[a'ab]$  to obtain:

$$\mathfrak{F}^1_{a'ab}(\mathbf{u}_v^{\mathfrak{p}}) = \partial(\mathbf{u}_v^{\mathfrak{p}}[a'ab])^{-1} \mathbf{u}_v^{\mathfrak{p}}[a'a] \mathbf{u}_v^{\mathfrak{p}}[ab] \mathbf{u}_v^{\mathfrak{p}}[a'b]^{-1} = 1 \quad (3.46)$$

which implies:

$$\mathbf{u}_v^{\mathfrak{p}}[a'b] = \partial(\mathbf{u}_v^{\mathfrak{p}}[a'ab])^{-1} \mathbf{u}_v^{\mathfrak{p}}[a'a] \mathbf{u}_v^{\mathfrak{p}}[ab] = \partial(\mathfrak{p}_1[ab]) \mathfrak{p}_0 \mathbf{u}[ab] \quad (3.47)$$

Similarly, for the second case we get:

$$\mathbf{u}_v^{\mathfrak{p}}[ab'] = \partial(\mathbf{u}_v^{\mathfrak{p}}[ab'b])^{-1} \mathbf{u}_v^{\mathfrak{p}}[ab] \mathbf{u}_v^{\mathfrak{p}}[b'b] = \partial(\mathfrak{p}_1[ab]) \mathbf{u}[ab] \mathfrak{p}_0^{-1} \quad (3.48)$$

- Given a 2-simplex  $[abc] \in \text{star } v$  we can use the 2-flatness condition to get the value on the lifted simplex. If  $v = a$  we apply it to the vertical 3-simplex  $[a'acb]$  to obtain:

$$\mathfrak{H}^2_{a'abc}(\mathbf{u}_v^{\mathfrak{p}}) = (\mathbf{u}_v^{\mathfrak{p}}[a'a] \triangleright \mathbf{u}_v^{\mathfrak{p}}[abc])\mathbf{u}_v^{\mathfrak{p}}[a'ac]\mathbf{u}_v^{\mathfrak{p}}[a'bc]^{-1}\mathbf{u}_v^{\mathfrak{p}}[a'ab]^{-1} = 1 \quad (3.49)$$

and it follows:

$$\mathbf{u}_v^{\mathfrak{p}}[a'bc] = \mathbf{u}_v^{\mathfrak{p}}[a'ab]^{-1}(\mathbf{u}_v^{\mathfrak{p}}[a'a] \triangleright \mathbf{u}_v^{\mathfrak{p}}[abc])\mathbf{u}_v^{\mathfrak{p}}[a'ac] = \mathfrak{p}_1[ab](\mathfrak{p}_0 \triangleright \mathbf{u}[abc])\mathfrak{p}_1[ac]^{-1} \quad (3.50)$$

Using the same procedure we get the formulas for  $v = b$  and  $v = c$ :

$$\mathbf{u}_v^{\mathfrak{p}}[ab'c] = \mathbf{u}_v^{\mathfrak{p}}[ab'b](\mathbf{u}_v^{\mathfrak{p}}[ab]\mathbf{u}_v^{\mathfrak{p}}[b'b]^{-1} \triangleright \mathbf{u}_v^{\mathfrak{p}}[b'bc]^{-1})\mathbf{u}_v^{\mathfrak{p}}[abc] = \mathfrak{p}_1[ab](\mathbf{u}[ab]\mathfrak{p}_0^{-1} \triangleright \mathfrak{p}_1[bc])\mathbf{u}[abc] \quad (3.51)$$

$$\mathbf{u}_v^{\mathfrak{p}}[abc'] = (\mathbf{u}_v^{\mathfrak{p}}[ab] \triangleright \mathbf{u}_v^{\mathfrak{p}}[bc'c])\mathbf{u}_v^{\mathfrak{p}}[abc]\mathbf{u}_v^{\mathfrak{p}}[ac'c]^{-1} = (\mathbf{u}[ab] \triangleright \mathfrak{p}_1[bc])\mathbf{u}[abc]\mathfrak{p}_1[ac]^{-1} \quad (3.52)$$

Using  $\mathbf{u}_v^{\mathfrak{p}}$  we can define a 2-gauge configuration  $\mathfrak{a}_v^{\mathfrak{p}}\mathbf{u} \in \mathfrak{X}(K, \mathfrak{G})$  as the restriction of  $\mathbf{u}_v^{\mathfrak{p}}$  to the final triangulation  $K'$  since it is equal to the initial triangulation up to a different labeling of the vertexes. Formally we can do this if we denote the simplex obtained from  $\sigma \in K$  by replacing  $v$  with  $v'$  as  $\sigma'$  and define  $\mathfrak{a}_v^{\mathfrak{p}}\mathbf{u}(\sigma)$  to be equal to  $\mathbf{u}_v^{\mathfrak{p}}(\sigma')$  for all  $\sigma \in K_1$  and  $\sigma \in K_2$ . Of course this only affects edges and triangles contained in the star of  $v$  and the equations obtained from the flatness conditions lead to:

$$\mathfrak{a}_v^{\mathfrak{p}}\mathbf{u}[ab] = \begin{cases} \partial(\mathfrak{p}_1[ab])\mathfrak{p}_0\mathbf{u}[ab], & \text{if } v = a \\ \partial(\mathfrak{p}_1[ab])\mathbf{u}[ab]\mathfrak{p}_0^{-1}, & \text{if } v = b \\ \mathbf{u}[ab], & \text{otherwise} \end{cases} \quad (3.53)$$

$$\mathfrak{a}_v^{\mathfrak{p}}\mathbf{u}[abc] = \begin{cases} \mathfrak{p}_1[ab](\mathfrak{p}_0 \triangleright \mathbf{u}[abc])\mathfrak{p}_1[ac]^{-1}, & \text{if } v = a \\ \mathfrak{p}_1[ab](\mathbf{u}[ab]\mathfrak{p}_0^{-1} \triangleright \mathfrak{p}_1[bc])\mathbf{u}[abc], & \text{if } v = b \\ (\mathbf{u}[ab] \triangleright \mathfrak{p}_1[bc])\mathbf{u}[abc]\mathfrak{p}_1[ac]^{-1}, & \text{if } v = c \\ \mathbf{u}[abc], & \text{otherwise} \end{cases} \quad (3.54)$$

We can use the functions  $\mathfrak{a}_v^{\mathfrak{p}} : \mathfrak{X}(K, \mathfrak{G}) \rightarrow \mathfrak{X}(K, \mathfrak{G})$  to define operators  $A_v^{\mathfrak{p}} : \mathcal{H} \rightarrow \mathcal{H}$  given by:

$$A_v^{\mathfrak{p}}|\mathbf{u}\rangle = |\mathfrak{a}_v^{\mathfrak{p}}\mathbf{u}\rangle \quad (3.55)$$

Notice that the set of parameters  $\mathfrak{P}_v$  has a group structure given by the product  $\mathfrak{t} = \mathfrak{p}\mathfrak{h}$  where  $\mathfrak{t}_0 = \mathfrak{p}_0\mathfrak{h}_0$  and  $\mathfrak{t}_1[ab] = \mathfrak{p}_1[ab](\mathfrak{p}_0 \triangleright \mathfrak{h}_1[ab])$  and it follows from equations 3.53 and 3.54 by direct computation that the operators  $A_v^{\mathfrak{p}}$  are a representation of the group  $\mathfrak{P}_v$ :

$$A_v^{\mathfrak{p}}A_v^{\mathfrak{h}} = A_v^{\mathfrak{p}\mathfrak{h}} \quad (3.56)$$

If we compare the expression for the product in  $\mathfrak{P}_v$  with the definition of the horizontal composition for the bigons of  $\overline{\mathfrak{G}}$  we see that they are related and the operators  $A_v^{\mathfrak{p}}$  are related to a representation of this 2-group implementing the corresponding symmetry. This is a consequence of the fact that these are the 2-gauge transformations operators responsible for realizing the 2-gauge symmetry of the model.

It is obvious from the definition that  $A_v^{\mathfrak{p}}A_w^{\mathfrak{h}} = A_w^{\mathfrak{h}}A_v^{\mathfrak{p}}$  whenever the stars of the vertexes  $v$  and  $w$  do not intersect. If the stars do intersect then  $v = w$  or they are the end points of some edge and since equation 3.56 takes care of the case  $v = w$  we just need to determine what happens in the other case to

fully specify the algebra of the 2-gauge transformation operators. Hence, assume there is a 1-simplex  $[ab]$  such that  $v = a$  and  $w = b$  so that we can use equations 3.53 and 3.54 to get:

$$A_v^{\mathfrak{p}} A_w^{\mathfrak{h}} = A_w^{\tilde{\mathfrak{h}}} A_v^{\mathfrak{p}} \quad (3.57)$$

where  $\tilde{\mathfrak{h}}$  is given by  $\tilde{\mathfrak{h}}_1[ab] = \partial(\mathfrak{p}_1[ab])(\mathfrak{p}_0 \triangleright \mathfrak{h}_1[ab])\partial(\mathfrak{p}_1[ab])^{-1}$  at  $[ab]$  and it equals all the other values of  $\mathfrak{h}$ .

Equations 3.56 and 3.57 are a bit inconvenient we are going to decompose the 2-gauge transformations operators into simpler operators which implement specific transformations. Not only are these operators convenient from a computational point of view but they are also going to be the ones used in the Hamiltonian because they are the most local 2-gauge transformation. The first type of operator, which we call a pure 1-gauge transformation, is defined by a group element  $g \in G_1$  and a vertex  $v \in K_0$  as  $A_v^g = A_v^{\mathfrak{p}}$  where the parameter  $\mathfrak{p}$  has  $\mathfrak{p}_0 = g$  and  $\mathfrak{p}_1$  trivial. The second type, called a pure 2-gauge transformation, uses a parameter  $\xi \in G_2$  and is localized at an edge  $l \in K_1$  and it is defined as  $A_l^\xi = A_l^{\mathfrak{p}}$  where the parameter  $\mathfrak{p}$  has  $\mathfrak{p}_0 = 1$  and  $\mathfrak{p}_1$  trivial except at  $l$  where it is equal to  $\xi$ . The algebra of these operators follows from the algebra of arbitrary 2-gauge operators and it is:

- Given different vertexes  $v, w \in K_0$  and  $g, h \in G_1$  we have:

$$A_v^g A_w^h = A_w^h A_v^g \quad (3.58)$$

$$A_v^g A_v^h = A_v^{gh} \quad (3.59)$$

- Given different edges  $l, r \in K_1$  and  $\xi, \gamma \in G_2$  we have:

$$A_l^\xi A_r^\gamma = A_r^\gamma A_l^\xi \quad (3.60)$$

$$A_l^\xi A_l^\gamma = A_l^{\xi\gamma} \quad (3.61)$$

- Given an edge  $l \in K_1$ , a vertex  $v \in K_0$  and parameters  $g \in G_1$  and  $\xi \in G_2$  there are two possibilities:

$$A_v^g A_l^\xi = \begin{cases} A_l^{g \triangleright \xi} A_v^g, & \text{if } v \text{ is the initial vertex of } l \\ A_l^\xi A_v^g, & \text{otherwise} \end{cases} \quad (3.62)$$

In particular, we have a representation of  $G_1$  corresponding to the assignment  $g \in G_1 \mapsto A_v^g$  for some vertex  $v$  and in an analogous way a  $G_2$  representation associated to the edges. As a consequence we can define the projectors that enforce invariance under local pure 1-gauge and pure 2-gauge transformations as:

$$A_v = \frac{1}{|G_1|} \sum_{g \in G_1} A_v^g \quad (3.63)$$

$$A_l = \frac{1}{|G_2|} \sum_{\xi \in G_2} A_l^\xi \quad (3.64)$$

and the algebra of pure 1-gauge and 2-gauge operators implies that they commute pairwise:

$$[A_v, A_w] = [A_l, A_r] = [A_v, A_l] = 0 \quad (3.65)$$

It follows from equation 3.56 that pure transformations generate all 2-gauge transformations and, as such, generate the corresponding representation of the 2-group. In particular we have the following formula:

$$A_v^p = A_v^{p_0} \left( \prod_{l \in \text{star}(v)_1} A_l^{p_0^{-1} \triangleright p_1(l)} \right) = \left( \prod_{l \in \text{star}(v)_1} A_l^{p_1(l)} \right) A_v^{p_0} \quad (3.66)$$

so that if we consider a general (non local) 2-gauge transformation obtained as the product of local transformations it is always possible to decompose it as product of pure transformations. By a general 2-gauge transformation we mean an operator  $A^\eta$  defined as:

$$A^\eta = \prod_v^< A_v^{\eta(v)} = A_a^{\eta(a)} A_b^{\eta(b)} A_c^{\eta(c)} \dots \quad (3.67)$$

where  $\eta$  is a function that assigns a parameter  $\eta(v) \in \mathfrak{P}_v$  to each vertex and  $\prod_v^<$  denotes the ordered product defined by the enumeration of the vertexes. If we let  $\mathfrak{P}_K$  be the set of all such functions then we can write the projector that enforces invariance under the 2-gauge symmetry as:

$$\Pi_A = \frac{1}{|\mathfrak{P}_K|} \sum_{\eta} A^\eta = \prod_v \frac{1}{|\mathfrak{P}_v|} \sum_{p \in \mathfrak{P}_v} A_v^p \quad (3.68)$$

and it follows from equation 3.66 that:

$$\Pi_A = \prod_v \sum_{p \in \mathfrak{P}_v} \left( \prod_{l \in \text{star}(v)_1} \frac{1}{|G_2|} A_l^{p_1(l)} \right) \frac{1}{|G_1|} A_v^{p_0} \quad (3.69)$$

$$= \prod_v \left( \prod_{l \in \text{star}(v)_1} \frac{1}{|G_2|} \sum_{\xi \in G_2} A_l^\xi \right) \frac{1}{|G_1|} \sum_{g \in G_1} A_v^g \quad (3.70)$$

$$= \prod_v \left( \prod_{l \in \text{star}(v)_1} A_l \right) A_v = \prod_l A_l \prod_v A_v \quad (3.71)$$

so a state is 2-gauge invariant if, and only if, it is invariant under local pure transformations.

## 3.5 Hamiltonian of the Model

Now that we have defined all the required operators we can finally write the Hamiltonian for our model and show that it is exactly solvable. Our definition is inspired by the Hamiltonian of quantum double model so it is given by a sum of local projectors that commute pairwise. Of course, the observables for our model are the fake 1-holonomies and the 2-holonomies so we need to use their projector in the Hamiltonian. Moreover, since these observables determine the field strength of a 2-gauge configuration we should use an operator such as:

$$\sum_{p \in K_2} \sum_{[h]} E_p(h) B_p^{[h]} + \sum_{t \in K_3} \sum_{[h]} E_t(\xi) B_t^{[\xi]} \quad (3.72)$$

where  $E_p(h)$  is a real-valued function assigning the local energy cost corresponding to the fake 1-holonomy and  $E_t(\xi)$  is the analogous function for the 2-holonomy. These functions need to be invariant by conjugation and  $\triangleright$  since only the conjugacy classes of the fake 1-holonomy and orbits of the 2-holonomy have physical meaning. The simplest choice is to consider:

$$E_p(h) = \begin{cases} -1, & \text{if } h = 1 \\ 0, & \text{otherwise} \end{cases} \quad E_t(\xi) = \begin{cases} -1, & \text{if } \xi = 1 \\ 0, & \text{otherwise} \end{cases} \quad (3.73)$$

so that the ground states correspond to flat 2-gauge configurations, that is, they belong to the following subspace:

$$\mathcal{H}_B = \text{span}(|\mathbf{u}\rangle \mid \forall p \in K_2 \mathfrak{F}^1_p(\mathbf{u}) = 1 \text{ and } \forall t \in K_3 \mathfrak{H}^2_t(\mathbf{u}) = 1) \quad (3.74)$$

We also want to enforce that the ground states are 2-gauge invariant and we do so dynamically, that is, we add the local projectors that ensure 2-gauge invariance directly to the Hamiltonian so that we obtain:

$$H = - \sum_{v \in K_0} A_v - \sum_{l \in K_1} A_l - \sum_{p \in K_2} B_p - \sum_{t \in K_3} B_t \quad (3.75)$$

It is clear that this Hamiltonian is a generalization of the quantum double model since if we take the crossed module with trivial  $G_2$  we see that  $A_v$  is just the gauge invariance projector and  $B_p$  is the projector into states with trivial holonomy. Moreover, we can check that it is also exactly solvable if we prove that all the projectors commute:

$$v, w \in K_0 \quad l, r \in K_1 \quad p, q \in K_2 \quad t, s \in K_3 \quad (3.76)$$

$$[A_v, A_w] = 0 \quad [A_v, A_l] = 0 \quad [A_l, A_r] = 0 \quad (3.77)$$

$$[B_p, B_q] = 0 \quad [B_p, B_t] = 0 \quad [B_t, B_s] = 0 \quad (3.78)$$

$$[A_v, B_p] = 0 \quad [A_v, B_t] = 0 \quad [A_l, B_p] = 0 \quad [A_l, B_t] = 0 \quad (3.79)$$

The commutation relations 3.78 for the "B's" are trivial since they are all diagonal in the same basis and we have shown 3.77 in previous section so we just need to take care of 3.79. To do so it is sufficient to show that  $B_p$  and  $B_t$  are invariant under any 2-gauge transformation:

$$[A_v^{\mathfrak{p}}, B_p] = 0 \quad (3.80)$$

$$[A_v^{\mathfrak{p}}, B_t] = 0 \quad (3.81)$$

where  $\mathfrak{p} \in \mathfrak{P}_v$  is an arbitrary parameter for a 2-gauge transformation localized in a vertex  $v$ . This is trivial when  $p$  and  $t$  do not belong to the star of the base point so we only need to show that the commutation holds when they belong to it. However due to the way we defined our operators it is not too hard to verify it in this case as well. In particular, if we fix a configuration  $\mathbf{u}$  and the lift procedure used to define  $\mathbf{u}_v^{\mathfrak{p}}$  and  $\mathbf{a}_v^{\mathfrak{p}}\mathbf{u}$  requires  $\mathbf{u}_v^{\mathfrak{p}}$  to be flat at all the vertical simplexes so we can apply the Bianchi identities 3.32 and 3.34 to  $\Lambda_v(p)$  and  $\Lambda_v(t)$  to obtain:

$$\mathfrak{F}^1_{p'}(\mathbf{u}_v^{\mathfrak{p}}) \sim \mathfrak{F}^1_p(\mathbf{u}) \implies \mathfrak{F}^1_p(\mathbf{a}_v^{\mathfrak{p}}\mathbf{u}) \sim \mathfrak{F}^1_p(\mathbf{u}) \quad (3.82)$$

$$\mathfrak{H}^2_{t'}(\mathbf{u}_v^{\mathfrak{p}}) \sim \mathfrak{H}^2_t(\mathbf{u}) \implies \mathfrak{H}^2_t(\mathbf{a}_v^{\mathfrak{p}}\mathbf{u}) \sim \mathfrak{H}^2_t(\mathbf{u}) \quad (3.83)$$

$$(3.84)$$

where  $\sim$  denotes equivalence by conjugation for values in  $G_1$  and by the action of  $\triangleright$  for values in  $G_2$ . Hence, we obtain more general commutation relation from which 3.79 follows directly:

$$[A_v^p, B_p^{[h]}] = 0 \implies [A_v^g, B_p^{[h]}] = 0 \quad [A_l^\gamma, B_p^{[h]}] = 0 \quad (3.85)$$

$$[A_v^p, B_t^{[\xi]}] = 0 \implies [A_v^g, B_t^{[\xi]}] = 0 \quad [A_l^\gamma, B_t^{[\xi]}] = 0 \quad (3.86)$$

Hence, we achieved our goal of constructing an exactly solvable model which provides a Hamiltonian realization of lattice 2-gauge theory and generalizes quantum double models based on finite groups. However, before moving on to next chapter where we tackle the issue of understanding the ground states we must explain why this model does not depend on the chosen enumeration. Formally, this amounts to showing that the unitary operators  $U : \mathcal{H} \rightarrow \mathcal{H}$  that implement a change of enumeration commute with the Hamiltonian. This holds because  $U$  commutes with all the local projectors in the Hamiltonian:

$$[A_v, U] = 0 \quad [A_l, U] = 0 \quad [B_p, U] = 0 \quad [B_t, U] = 0 \quad (3.87)$$

because we defined these operators using calculations of holonomies and 2-holonomies which are independent of the enumeration by construction since we've used the rules for changing the enumeration from Section 3.1 to compute them. The enumeration only appears in the explicit formulas through the conventions we made regarding initial and final vertexes/curves, orientations and for the rules to specify the configurations at vertical simplexes using the parametrization. These conventions are washed away when we consider the projector in the Hamiltonian so the model is enumeration independent.



# Chapter 4

## Topological Properties of the Ground States

In this chapter we are going to describe the ground states of the model constructed in the previous section. In particular, we will provide an original demonstration that the dimension of the ground state subspace is a topological invariant and that there is no local order parameter capable of distinguishing the states. Moreover, we also provide a more detailed description of a particular class of models using homological techniques and mention in broad terms how this can be extended to the general case using homotopical techniques.

### 4.1 Ground State Subspace

The Hamiltonian defined in the previous section has a suggestive form since it is a sum of commuting projectors enforcing local constraints and it seems that its ground states are states  $|\psi\rangle \in \mathcal{H}$  such that:

$$\forall v \in K_0 \quad A_v |\psi\rangle = |\psi\rangle \quad (4.1)$$

$$\forall l \in K_1 \quad A_l |\psi\rangle = |\psi\rangle \quad (4.2)$$

$$\forall p \in K_2 \quad B_p |\psi\rangle = |\psi\rangle \quad (4.3)$$

$$\forall t \in K_3 \quad B_t |\psi\rangle = |\psi\rangle \quad (4.4)$$

However, it might happen that the model is frustrated and there is no state which satisfies all the local constraints simultaneously. Notice that this can happen even though the projectors commute because the constraints are not independent (the anti ferromagnetic Ising model on a triangular lattice is an example of this) so we need to show that the model is frustration free. It is enough to construct one state which fulfills all the required properties and to do so we start with the trivial state  $|\mathbf{1}\rangle$  defined by the trivial 2-gauge configuration:

$$\mathbf{1} \in \mathfrak{X}(K, G) \quad (4.5)$$

$$l \in K_1 \mapsto \mathbf{1}(l) = 1 \in G_1 \quad (4.6)$$

$$p \in K_2 \mapsto \mathbf{1}(p) = 1 \in G_2 \quad (4.7)$$

Clearly this state is invariant under the action of  $B_p$  and  $B_t$  but it is not 2-gauge invariant. We can fix this by introducing the gauged state  $|\mathbf{1}_g\rangle$  defined by:

$$|\mathbf{1}_g\rangle = \Pi_A |\mathbf{1}\rangle = \prod_l A_l \prod_v A_v |\mathbf{1}\rangle \quad (4.8)$$

and it is easy to check that it fulfills all the constraints:

$$A_v |\mathbf{1}_g\rangle = A_v \Pi_A |\mathbf{1}\rangle = \Pi_A |\mathbf{1}\rangle = |\mathbf{1}_g\rangle \quad (4.9)$$

$$A_l |\mathbf{1}_g\rangle = A_l \Pi_A |\mathbf{1}\rangle = \Pi_A |\mathbf{1}\rangle = |\mathbf{1}_g\rangle \quad (4.10)$$

$$B_p |\mathbf{1}_g\rangle = B_p \Pi_A |\mathbf{1}\rangle = \Pi_A B_p |\mathbf{1}\rangle = \Pi_A |\mathbf{1}\rangle = |\mathbf{1}_g\rangle \quad (4.11)$$

$$B_t |\mathbf{1}_g\rangle = B_t \Pi_A |\mathbf{1}\rangle = \Pi_A B_t |\mathbf{1}\rangle = \Pi_A |\mathbf{1}\rangle = |\mathbf{1}_g\rangle \quad (4.12)$$

where we've used that the projectors commute to write

$$A_v \Pi_A = \prod_l A_l \prod_{w \neq v} A_w A_v A_v^- \prod_l A_l \prod_{w \neq v} A_w A_v = \Pi_A \quad (4.13)$$

$$A_l \Pi_A = \prod_{s \neq l} A_s A_l A_l \prod_v A_v = \prod_{s \neq l} A_s A_l \prod_v A_v = \Pi_A \quad (4.14)$$

$$[B_p, A_v] = [B_p, A_l] = [B_t, A_v] = [B_t, A_l] = 0 \implies B_p \Pi_A = \Pi_A B_p \quad B_t \Pi_A = \Pi_A B_t \quad (4.15)$$

Of course it might still happen that  $|\mathbf{1}_g\rangle = 0$  which would frustrate our efforts but we can guarantee that this is not the case because

$$|\mathbf{1}_g\rangle = \Pi_A |\mathbf{1}\rangle = \frac{1}{\mathfrak{P}_K} \sum_{\eta} A^{\eta} |\mathbf{1}\rangle \neq 0 \quad (4.16)$$

implies it is nonzero as each state  $A^{\eta} |\mathbf{1}\rangle$  corresponds to some 2-gauge configuration so  $|\mathbf{1}_g\rangle$  is an average over all configurations which are equivalent to the trivial configuration. By equivalent we mean that there is some  $\eta$  which maps one configuration into the other:

$$\forall \mathbf{u}, \mathbf{u}' \in \mathfrak{X}(K, G) \quad \mathbf{u} \sim \mathbf{u}' \Leftrightarrow \exists \eta \in \mathfrak{P}_K \mid |\mathbf{u}'\rangle = A^{\eta} |\mathbf{u}\rangle \Leftrightarrow |\mathbf{u}_g\rangle = \Pi_A |\mathbf{u}\rangle = \Pi_A |\mathbf{u}'\rangle = |\mathbf{u}'_g\rangle \quad (4.17)$$

We can use the same procedure to construct a ground state with any configuration  $\mathbf{u} \in \mathfrak{X}(K, G)$  which is invariant under the action of the  $B$ 's i.e. any flat 2-gauge configuration. We denote the set of these configurations as

$$\mathfrak{X}(K, G)_0 = \{\mathbf{u} \mid \forall p \in K_2 \mathfrak{F}_p^1(\mathbf{u}) = 1 \text{ and } \forall t \in K_3 \mathfrak{H}_t^2(\mathbf{u}) = 1\} \quad (4.18)$$

and observe that they span the subspace  $\mathcal{H}_B$  of flat states:

$$\mathcal{H}_B = \{|\psi\rangle \in \mathcal{H} \mid \forall p \in K_2 B_p |\psi\rangle = |\psi\rangle \text{ and } \forall t \in K_3 B_t |\psi\rangle = |\psi\rangle\} = \text{span}(|\mathbf{u}\rangle)_{\mathbf{u} \in \mathfrak{X}(K, G)_0} \quad (4.19)$$

We can obtain this subspace from the entire Hilbert space using the analogue of  $\Pi_A$  for the  $B$ 's:

$$\Pi_B \mathcal{H} = \prod_p B_p \prod_t B_t \text{span}(|\mathbf{u}\rangle)_{\mathbf{u} \in \mathfrak{X}(K, G)} = \text{span}(|\mathbf{u}\rangle)_{\mathbf{u} \in \mathfrak{X}(K, G)_0} = \mathcal{H}_B \quad (4.20)$$

The ground state subspace  $\mathcal{H}_0$  is formed by states invariant under the action of all the local projectors in the Hamiltonian so it is the result of applying  $\Pi_A$  to  $\mathcal{H}_B$ :

$$\mathcal{H}_0 = \Pi_0 \mathcal{H} = \Pi_A \Pi_B \mathcal{H} = \Pi_A \mathcal{H}_B = \text{span}(|[\mathbf{u}]\rangle)_{\mathbf{u} \in \mathfrak{X}(K, G)_0} \quad (4.21)$$

where  $\Pi_0 = \Pi_A \Pi_B$  is the ground state projector and  $|\mathbf{u}\rangle = |\mathbf{u}_g\rangle = \Pi_A |\mathbf{u}\rangle$  is the state obtained by averaging over equivalent 2-gauge configurations. The labels  $[\mathbf{u}]$  are the corresponding equivalence classes and they provide a basis for the ground state so that any ground state has a unique expression:

$$|G\rangle = \sum_{[\mathbf{u}] \in [\mathfrak{X}(K, G)_0]} G([\mathbf{u}]) |\mathbf{u}\rangle \quad (4.22)$$

where  $[\mathfrak{X}(K, G)_0] = \mathfrak{X}(K, G)_0 / \sim = \{[\mathbf{u}] \mid \mathbf{u} \in \mathfrak{X}(K, G)_0\}$ .

We can now state our main goal for this chapter which is to demonstrate that the ground state degeneracy of the model defined over  $K$  is a topological invariant of the underlying three-dimensional manifold  $M$ . Hence, we must show that the following quantity is a topological invariant:

$$\text{gsd} = \dim(\mathcal{H}_0) = |[\mathfrak{X}(K, G)_0]| \quad (4.23)$$

Before developing our demonstration let us explain the connection of our model with Yetter's topological quantum field theory and why one should expect that the ground state degeneracy is a topological invariant. To do so recall our procedure for defining local 2-gauge transformations using the lift and notice that if we apply it to all the vertexes of triangulation in sequence we get a triangulation of  $M \times [0, 1]$ . Moreover, the operators obtained would be of the form  $A^\eta$  for some  $\eta \in \mathfrak{P}_K$  so that could have defined the projector  $\Pi_A$  as:

$$\Pi_A |\mathbf{u}\rangle = \frac{1}{N} \sum_{\mathfrak{U}} |\mathfrak{U}_f\rangle \quad (4.24)$$

where the sum runs over all 2-gauge configurations on the cylinder which restrict to  $\mathbf{u}$  in the initial copy of  $K$  and  $\mathfrak{U}_f$  denotes the restriction to the final copy. Also, we only consider configurations  $\mathfrak{U}$  which are flat in the interior of the cylinder so that  $\Pi_A$  does not change the values of fake 1-holonomy and 2-holonomy. The number  $N$  is a normalization factor which depends on the triangulation of the cylinder and we won't go into the details concerning it. This procedure can be extended to any triangulated cobordism  $T$  going from a triangulation  $K$  to another triangulation  $K'$  to define an operator  $Z(T) : Z(K) \rightarrow Z(K')$  where  $Z(K)$  and  $Z(K')$  are the ground states subspaces obtained from the Hamiltonian model defined over  $K$  and  $K'$  so that when  $K = K'$  and  $T$  is a cylinder we get  $Z(T) = \Pi_A|_{\mathcal{H}_0} = \mathbb{1}$  as required for  $Z$  to define a (3+1) topological quantum field theory. This idea was made rigorous by Teramoto [50] where it was shown that this procedure leads to Yetter's topological quantum field theory and the following result:

$$\text{tr} \Pi_0 = Z(M \times S^1) \quad (4.25)$$

which, due to  $\text{gsd} = \text{tr} \Pi_0$  implies that the trace of the ground state degeneracy is a topological invariant given by Yetter's partition function.

At this point the reader might be wondering why bother with a new demonstration if it was already established that the model has topological degeneracy at the level of the ground state. However, it

is our opinion that this new proof has many advantages when compared to the one which uses the topological quantum field underlying the model which is why we believe it has value. First of all, it is much simpler both from a technical and conceptual point of view. In particular, it provides some intuition on what is going physically speaking and highlights the role played by the 2-gauge symmetry as a source of topological invariance. Moreover, it relies on a new technique which can be used in different settings and provide additional information. As an example of its power we will proof a theorem which generalizes Elitzur's theorem [61] and implies that there is no local order parameter for our model.

## 4.2 Topological Degeneracy

All the results we are going to demonstrate in this section rely on a simple idea, namely, that it is always possible to "gauge away" local degrees of freedom. This means that any observables which is invariant under the 2-gauge symmetry cannot depend on information contained in a local region and, as a consequence, the quantum numbers for the ground state must be non local. To put this idea into a formal footing we first consider the following definition:

**Definition 4.1.** *Given a subcomplex  $S \subset K$ , we say it is 2-gauge trivial if all flat 2-gauge configurations  $\mathbf{u} \in \mathfrak{X}(K, \mathbf{G})_0$  are equivalent to a configuration  $\mathbf{u}'$  such that the restriction of  $\mathbf{u}'$  to  $S$  is trivial.*

This definition is telling us that if a subcomplex is 2-gauge trivial then it cannot contain any non trivial information about a flat configuration since that would break the 2-gauge symmetry. However we need to find out which subcomplexes are 2-gauge trivial to be able to use this concept. As the first step towards this let us show that any tetrahedron is 2-gauge trivial:

**Lemma 4.1.** *The closure  $S = \text{cl}([abcd])$  of a tetrahedron  $[abcd] \in K$  is 2-gauge trivial.*

*Proof.* Consider an arbitrary flat configuration  $\mathbf{u}$  and let us write the flatness constraints coming from the tetrahedron:

$$\begin{aligned} \partial(\mathbf{u}[abc])^{-1} \mathbf{u}[ab] \mathbf{u}[bc] \mathbf{u}[ac]^{-1} &= 1 \\ \partial(\mathbf{u}[abd])^{-1} \mathbf{u}[ab] \mathbf{u}[bd] \mathbf{u}[ad]^{-1} &= 1 \\ \partial(\mathbf{u}[acd])^{-1} \mathbf{u}[ac] \mathbf{u}[cd] \mathbf{u}[ad]^{-1} &= 1 \\ \partial(\mathbf{u}[bcd])^{-1} \mathbf{u}[bc] \mathbf{u}[cd] \mathbf{u}[bd]^{-1} &= 1 \\ (\mathbf{u}[ab] \triangleright \mathbf{u}[bcd]) \mathbf{u}[abd] \mathbf{u}[acd]^{-1} \mathbf{u}[abc]^{-1} &= 1 \end{aligned}$$

Notice that if we fix  $g = \mathbf{u}[ab]$ ,  $h = \mathbf{u}[ac]$ ,  $t = \mathbf{u}[ad]$ ,  $\alpha = \mathbf{u}[abc]$ ,  $\beta = \mathbf{u}[abd]$ ,  $\gamma = \mathbf{u}[acd]$  we can obtain all the other values in  $S$  from the constraints:

$$\begin{aligned} \mathbf{u}[bc] &= g^{-1} \partial(\alpha) h \\ \mathbf{u}[bd] &= g^{-1} \partial(\beta) t \\ \mathbf{u}[cd] &= h^{-1} \partial(\gamma) t \\ \mathbf{u}[bcd] &= g^{-1} \triangleright (\alpha \gamma \beta^{-1}) \end{aligned}$$

and since there are no more constraints the parameters  $g, h, t, \alpha, \beta, \gamma$  are free (ignoring the constraints outside  $S$  of course) so we can write  $|\mathbf{u}\rangle = |g, h, t, \alpha, \beta, \gamma \dots\rangle$  by omitting the degrees of freedom living outside the tetrahedron. Using this labeling it is easy to write down explicit formulas for the pure transformations and below we present some of them:

$$\begin{aligned} A_{[b]}^z |g, h, t, \alpha, \beta, \gamma \dots\rangle &= |gz^{-1}, h, t, \alpha, \beta, \gamma \dots\rangle \\ A_{[c]}^z |g, h, t, \alpha, \beta, \gamma \dots\rangle &= |g, hz^{-1}, t, \alpha, \beta, \gamma \dots\rangle \\ A_{[d]}^z |g, h, t, \alpha, \beta, \gamma \dots\rangle &= |g, h, tz^{-1}, \alpha, \beta, \gamma \dots\rangle \\ A_{[ab]}^\xi |g, h, t, \alpha, \beta, \gamma \dots\rangle &= |\partial(\xi)g, h, t, \xi\alpha, \xi\beta, \gamma \dots\rangle \\ A_{[ad]}^\xi |g, h, t, \alpha, \beta, \gamma \dots\rangle &= |g, h, \partial(\xi)t, \alpha, \beta\xi^{-1}, \gamma\xi^{-1} \dots\rangle \\ A_{[bc]}^\xi |g, h, t, \alpha, \beta, \gamma \dots\rangle &= |g, h, t, (g \triangleright \xi)\alpha, \beta, \gamma \dots\rangle \end{aligned}$$

It is easy to combine these expressions to find a 2-gauge transformation which maps  $|\mathbf{u}\rangle$  into a configuration which is trivial in the tetrahedron. Therefore, all flat configurations are equivalent to a configuration which is trivial at the closure of the tetrahedron so  $S$  is 2-gauge trivial.  $\square$

It is possible to use the same idea to proof that larger subcomplexes are 2-gauge trivial but it quickly becomes cumbersome due to the amount of variables involved. The lemma below solves this problem as it gives us a way to enlarge subcomplexes which are 2-gauge trivial:

**Lemma 4.2.** *If  $S \subset K$  is a 2-gauge trivial subcomplex and  $[abcd] \in K$  is a tetrahedron such that the intersection  $\text{cl}[abcd] \cap S$  is a triangulation of a 2-ball we have that  $S \cup \text{cl}[abcd]$  is also 2-gauge trivial.*

*Proof.* Given any flat configuration we can always pass to a equivalent one which is trivial at  $S$  so it is enough to demonstrate the condition for these states. Moreover, by applying the flatness conditions coming from  $[abcd]$  we can label a flat configuration  $|\mathbf{u}\rangle$  with six degrees of freedom  $g, h, t, \alpha, \beta, \gamma$  just as we did in the proof of Lemma 4.1 and we can use the triviality condition at the intersection with  $S$  to remove more degrees of freedom. Since  $\text{cl}[abcd] \cap S$  is a triangulation of a 2-ball there are only three possibilities:

- The intersection contains one triangle of  $\partial([abcd])$  in which case we only need one element of  $G_1$  and two of  $G_2$  to specify the configuration. Using the only vertex of the tetrahedron which is not contained in  $S$  and two edges which are also not in  $S$  we can gauge away these remaining degrees of freedom without affecting  $S$  so that we obtain an equivalent configuration which is trivial in  $S \cup \text{cl}[abcd]$ .
- The intersection contains two triangles of  $\partial([abcd])$  in which case we only need one element of  $G_2$  to specify the configuration. Using the only edge of the tetrahedron which is not contained in  $S$  we can gauge away the remaining degree of freedom without affecting  $S$  so that we get an equivalent configuration which is trivial in  $S \cup \text{cl}[abcd]$ .
- Finally, if the intersection contains threes triangles of  $\partial([abcd])$  all degrees of freedom became trivial so the configuration already is trivial in  $S \cup \text{cl}[abcd]$  and there is nothing left to do.

As a consequence we conclude that  $\text{cl}[abcd] \cap S$  is 2-gauge trivial.  $\square$

In particular, it follows from Pachner's theorem that any triangulation of the 3-ball can be obtained from a tetrahedron by gluing tetrahedrons to its boundary in the way specified in Lemma 4.2. Hence, we can combine Lemma 4.1 and 4.2 to obtain our main technical tool:

**Theorem 4.3.** *Any subcomplex which is a triangulation of a 3-ball contained in  $M$  is 2-gauge trivial.*

Even though this theorem is a direct consequence of tool easy lemmas it is extremely powerful and, from a physical point of view, it means we can always fix a gauge where everything is trivial inside a ball. In particular it makes the idea that no 2-gauge invariant observable can be local rigorous if we interpret locality as anything contained in the interior of a ball. For instance, if we assume that  $K$  itself is a triangulation of the 3-ball then all of its flat configurations are equivalent to each other and the ground state is non degenerate. This is expected since a ball is trivial from the point of view of topology so there should be no degeneracy in a system with topological order. More importantly, we can now prove our main theorem:

**Theorem 4.4.** *The ground state degeneracy is invariant under Pachner moves.*

*Proof.* Let  $K'$  be a triangulation obtained from  $K$  by applying a internal Pachner move. In this case  $K$  is equal to  $K'$  everywhere except for a region  $S$  which is a triangulation of the 3-ball and gets replaced by a different triangulation  $S'$  with a matching boundary. We can use this to define an operator  $U : \mathcal{H}_0 \rightarrow \mathcal{H}'_0$  which maps ground states of one triangulation into ground state of the other triangulation as follows:

$$U |[u]\rangle = U \Pi_A |\tilde{u}\rangle = \Pi'_A |u'\rangle = |[u']\rangle$$

where  $\tilde{u}$  is an element of  $[u]$  which is trivial at  $S$  and  $u'$  is a flat configuration if  $K'$  which is equal to  $\tilde{u}$  outside  $S'$  and trivial inside it. Clearly this operator is an isomorphism since we can define another operator  $U' : \mathcal{H}'_0 \rightarrow \mathcal{H}_0$  using the same procedure and it is going to be the inverse of  $U$ . A similar procedure also applies for the border moves so that the ground state subspace of models defined on triangulations that differ by Pachner moves are isomorphic and the ground state degeneracy is indeed invariant.  $\square$

As a consequence of Pachner's theorem we see that the ground state degeneracy is a topological invariant and we can use the same technique to obtain a result which gives some intuition into why that is the case:

**Theorem 4.5.** *Consider a local operator  $O : \mathcal{H} \rightarrow \mathcal{H}$ , that is, an operator which only acts non trivially at a subcomplex  $S \subset K$  which is a triangulation of the ball. If this operator is 2-gauge invariant  $[\Pi_A, O] = 0$  and commutes with all  $B_p$  and  $B_t$  then its restriction to the ground state subspace is proportional to the identity.*

*Proof.* Consider a state  $|[u]\rangle \in \mathcal{H}_0$  and a representative of its equivalence class  $u \in [u]$  which is trivial in  $S$ . Since  $O$  is 2-gauge invariant we have  $O |[u]\rangle = \Pi_A O |u\rangle$  so that we just need to show that  $O |u\rangle$  equals  $\lambda |u\rangle$  up to 2-gauge transformations where  $\lambda$  is a fixed complex number. Moreover,  $O$  only affects what is in the interior of  $S$  so it is completely determined by an operator  $O_S$  acting on the model defined on the subcomplex  $S$  which must send ground states into ground states due to the properties of  $O$ . If we denote by  $\Pi_A^S$  the 2-gauge invariance projector for the model defined in  $S$  then we know, since there

is no ground state degeneracy, that  $O_S \Pi_A^S |\mathbf{1}\rangle_S = \lambda \Pi_A^S |\mathbf{1}\rangle_S$  where  $\mathbf{1}_S$  is the trivial configuration at  $S$ . It then follows from the fact that the restriction of  $\mathbf{u}$  to  $S$  is  $\mathbf{1}_S$  that  $O|\mathbf{u}\rangle = \lambda \Pi_A^S |\mathbf{u}\rangle$  where now  $\Pi_A^S$  denotes an operator acting on the full space by applies 2-gauge transformations localized in  $S$ . Using this formula we get  $O|[\mathbf{u}]\rangle = \Pi_A O|\mathbf{u}\rangle = \Pi_A \lambda \Pi_A^S |\mathbf{u}\rangle = \lambda \Pi_A |\mathbf{u}\rangle = \lambda|[\mathbf{u}]\rangle$  so that the restriction of  $O$  to  $\mathcal{H}_0$  is  $\lambda \mathbf{1}$ .  $\square$

This theorem is a generalization of Elitzur's theorem, which applies to lattice gauge theories, to the Hamiltonian realization of lattice 2-gauge theories provided by our model. It is a strong statement as it implies that it is impossible to distinguish ground states using local information and our understanding is that this is the fundamental reason why the model has topological degeneracy. Hopefully it is clear how this theorem implements the idea that is possible to "gauge away" the local properties of a ground states using 2-gauge transformations.

### 4.3 Homological Description

In the previous section we've shown that there is topological degeneracy and that the quantum numbers used to distinguish ground states cannot be local quantities but we haven't found what the degeneracy is or which observables can be used as an order parameter. We hope to remedy this in this section by looking at a specific class of examples which are particularly easy to analyse.

Let us assume that the groups of crossed module  $G$  are abelian and that the action is trivial. Under this assumptions most complications of the general model disappear and it is possible to describe everything using a simple formalism based on homological algebra. For instance, now that the action is trivial the Hilbert space is just a tensor product of local spaces:

$$\mathcal{H} = \left( \bigotimes_{[ab] \in K_1} \mathbb{C}G_1 \right) \otimes \left( \bigotimes_{[abc] \in K_2} \mathbb{C}G_2 \right) \quad (4.26)$$

In this setting a 2-gauge configuration is just a pairs of functions  $K_1 \rightarrow G_1$  and  $K_2 \rightarrow G_2$  and since the groups are abelian this amounts to a 1-cochain with coefficients in  $G_1$  and a 2-cochain with coefficients in  $G_2$ . Hence, the set of 2-gauge configurations becomes the abelian group  $\mathfrak{X}(K, G) = C^1(K, G_1) \times C^2(K, G_2)$  and the Hilbert space is just the group algebra of this group:

$$\mathcal{H} = \mathbb{C}\mathfrak{X}(K, G) = \mathbb{C}C^1(K, G_1) \otimes \mathbb{C}C^2(K, G_2) \quad (4.27)$$

Given a basis state  $|\mathbf{u}\rangle = |u_1, u_2\rangle$  where  $u_1 \in C^1(K, G_1)$  and  $u_2 \in C^2(K, G_2)$  we can write the fake 1-holonomy and 2-holonomy additively as:

$$\begin{aligned} \mathfrak{F}^1_{[abc]}(\mathbf{u}) &= -\partial(u_2[abc]) + u_1[ab] + u_1[bc] - u_1[ac] = -\partial(u_2[abc]) + u_1([bc] - [ac] + [ab]) \\ &= -\partial(u_2[abc]) + u_1 d_2[abc] = (-\partial u_2 + d^1 u_1)([abc]) \end{aligned} \quad (4.28)$$

$$\begin{aligned} \mathfrak{H}^2_{[abcd]}(\mathbf{u}) &= u_2[bcd] + u_2[abd] - u_2[acd] - u_2[abc] = u_2([bcd] - [acd] + [abd] - [abc]) \\ &= u_2 d_3[abcd] = d^2 u_2([abcd]) \end{aligned} \quad (4.29)$$

It is also possible to simplify the expression of the 2-gauge transformations operators. To see this let us consider a parameter  $\eta = (\eta_1, \eta_2)$  where  $\eta_1 \in C^0(K, G_1)$  and  $\eta_2 \in C^1(K, G_2)$  such that there is

a 2-gauge transformation corresponding to it:

$$A^\eta = \prod_v A_v^{\eta_1(v)} \prod_l A_l^{\eta_2(l)} \quad (4.30)$$

and notice we can write the action of this operator in a basis state as:

$$A^\eta |u_1, u_2\rangle = |u_1 + d^0 \eta_1 + \partial \eta_2, u_2 + d^1 \eta_2\rangle \quad (4.31)$$

Motivated by equation 4.28, 4.29 and 4.31 we define two morphisms that encode the algebraic structure of the operators:

$$D^j : C^{1+j}(K, G_1) \times C^{2+j}(K, G_2) \rightarrow C^{2+j}(K, G_1) \times C^{2+j}(K, G_2) \quad (4.32)$$

$$D^0(u_1, u_2) = (-\partial u_2 + d^1 u_1, d^2 u_2) \quad (4.32)$$

$$D^{-1}(\eta_1, \eta_2) = (d^0 \eta_1 + \partial \eta_2, d^1 \eta_2) \quad (4.33)$$

It follows from their definition that we can write the projectors  $\Pi_A$  and  $\Pi_B$  as:

$$\Pi_A |u_1, u_2\rangle = \frac{1}{N} \sum_\eta |(u_1, u_2) + D^{-1}(\eta_1, \eta_2)\rangle \quad (4.34)$$

$$\Pi_B |u_1, u_2\rangle = \begin{cases} |u_1, u_2\rangle, & \text{if } (u_1, u_2) \in \ker D^0 \\ 0, & \text{otherwise} \end{cases} \quad (4.35)$$

where  $N = |G_1|^{|K_0|} |G_2|^{|K_1|} = |C^0(K, G_1) \times C^1(K, G_2)|$  is the number of different 2-gauge transformations. The commutation of the  $A$ 's with  $B$ 's manifests itself algebraically as:

$$D^0 D^{-1}(\eta_1, \eta_2) = D^0(d^0 \eta_1 + \partial \eta_2, d^1 \eta_2) = (-\partial(d^1 \eta_2) + d^1(d^0 \eta_1 + \partial \eta_2), d^2 d^1 \eta_2) \quad (4.36)$$

$$= (d^1 d^0 \eta_1 - \partial d^1 \eta_2 + d^1 \partial \eta_2, d^2 d^1 \eta_1) = (0, 0) = 0 \quad (4.37)$$

so that  $\text{im } D^{-1} \subset \ker D^0$  and we can define a cohomology group  $H^0(K, G) = \ker D^0 / \text{im } D^{-1}$  which describes the ground state subspace. The reason why this group controls the ground state subspace is that the basis states  $[[u]]$  are labeled by equivalence classes of flat configurations which in this situation are just equivalence classes  $[(u_1, u_2)] \in H^0(K, G)$  of  $(u_1, u_2) \in \ker D^0$ . In fact, this implies the ground state subspace is the group algebra of the cohomology group:

$$\mathcal{H}_0 = \mathbb{C}H^0(K, G) \quad (4.38)$$

Cohomology groups such as  $H^0(K, G)$  have been studied and it follows from a theorem due to Brown [62] that it is a topological invariant which can be written using the standard cohomology groups of a simplicial simplex:

$$H^0(K, G) = H^1(K, \pi_1 G) \times H^2(K, \pi_2 G) \quad (4.39)$$

where  $\pi_1 G = G_1 / \text{im } \partial$  and  $\pi_2 G = \ker \partial$  are the homotopy groups of a crossed module previously defined. In particular we get a closed formula for the ground state degeneracy:

$$\text{gsd} = |H^0(K, G)| = |H^1(K, \pi_1 G)| |H^2(K, \pi_2 G)| \quad (4.40)$$

Equation 4.40 is important for two reasons. First of all it tells us that we can use cohomology classes as a source of quantum numbers for our model which opens up a lot of possibilities since there is a well developed theory for computing cohomology groups. More important though is that it also provides a hint to how we can generalize this description using homological techniques to models with a general crossed module. More precisely, the appearance of the homotopy groups of the crossed modules means that the cohomology only depends on the homotopy class of the crossed module, or equivalently, of its classifying space. Hence, we might be tempted to conjecture that there is a generalization of formula 4.40 for the general case which depends on the classifying space of the crossed module. We've actually known what is the correct generalization but to demonstrate it would require us to introduce some additional machinery from the homotopy theory of simplicial sets which is why we've chosen to omit the details regarding it and just describe the abelian case. Nonetheless, we are going to state the general result and make some comments to try to explain why it is true:

**Theorem 4.6.** *Given a crossed module  $G$  and a finite simplicial complex  $K$  the set  $[\mathfrak{X}(K, G)]$  of equivalence classes of flat 2-gauge configurations is equal to the set  $[K, BG]$  formed by homotopy classes of simplicial maps from  $K$  to  $BG$ . As a consequence the ground state degeneracy of corresponding model is given by:*

$$\text{gsd} = |[K, BG]| \quad (4.41)$$

Surprisingly this theorem is much easier to demonstrate than any of the other results we've obtained in this chapter even though it is much stronger. For instance, it implies that the ground state degeneracy is a topological invariant for any finite simplicial complex whereas 4.4 just applies to three-dimensional manifolds (although it can be extended to other dimensions). The reason it is so simple to demonstrate is that, once one has the right definitions in hand, it basically follows by construction. Essentially, it is a direct consequence of the definition that a flat 2-gauge configuration is a map from  $K$  into  $BG$ . Also, if we recall that two maps  $f, g : K \rightarrow BG$  are homotopic if, and only if, there is a map  $F : K \times [0, 1] \rightarrow BG$  such that  $F(\sigma, 0) = f(\sigma)$  and  $F(\sigma, 1) = g(\sigma)$  we see that our definition of 2-gauge transformations using the lift is just a way of parameterizing homotopies. The reason this theorem is so easy to prove is that it takes advantage of the fact that the topological order of the model comes from a homotopy invariant in contrast with techniques that rely on Pachner moves which are unnecessarily sensitive to the topology.



# Chapter 5

## Final Remarks

In this final chapter we present an overview of our main results and mention some potential directions for future research that go beyond what was developed in this work.

The core of our work was presented in chapters 3 and 4 which represent our contributions to the ongoing effort to understand and, eventually, classify topologically ordered for three-dimensional systems. In particular we've constructed an exactly solvable Hamiltonian model for arbitrary triangulations of three-dimensional manifolds using crossed modules as the algebraic input. The models obtained from this construction are Hamiltonian realizations of lattice 2-gauge theories which implement the local 2-gauge symmetry related to a gauge 2-group defined using a crossed module. Moreover, these models are generalizations of quantum double models based on finite groups for dimension three. As we've mentioned before this construction is related to a similar a model developed independently in [37] but our approach is more general due to introduction of the extended 2-group of a crossed module. The ground states of the model were fully described in chapter 4 using our technique that relies on subcomplexes which are 2-gauge trivial and allowed us to provide a much simpler demonstration of the topological invariance of the ground state degeneracy. Our demonstration highlights the role played by the 2-gauge symmetry and we've used this new point of view to extend a version of Elitzur's theorem from lattice gauge theories to lattice 2-gauge theories. We consider this to be a particularly interesting achievement since it puts the often repeated (but never demonstrated) statement that topological degeneracy is equivalent to non local order parameters into a rigorous footing. For abelian crossed models we've shown how to describe the ground states using homological algebra and, in doing so, found a closed formula for the ground state degeneracy based on cohomology groups. An extension to the general non-abelian setting using homotopical algebra was stated although we did not present a detailed demonstration as it would require mathematical background which was not introduced in chapter 2. Hence, we can list the main points of our work as:

- The definition of an exactly solvable Hamiltonian model that generalizes quantum double models:

$$H = - \sum_v A_v - \sum_l A_l - \sum_p B_p - \sum_t B_t$$

- A original demonstration that the ground state degeneracy is a topological invariant and that there is no local order parameter.

- A closed formula for the ground state degeneracy based on homotopy theory:

$$\text{gsd} = |[K, BG]|$$

which reduces to a formula based on cohomology groups for the abelian case:

$$\text{gsd} = |H^1(K, \pi_1 G)| |H^2(K, \pi_2 G)|$$

As for potential lines of research for the future we can list three main themes:

- Explore the full spectrum of the model and characterize the elementary excitations. This can be done in a varying of different ways but the approach we would like to explore is to study the representations of the motion group which control the statistics of the extended excitations of the model. Our hope is that our model has enough structure so that the representations will be non trivial while being concrete enough that we can do explicit computations. A particular instance of this is to study the action of the mapping class on ground states of the models defined for the 3-torus as it is conjectured that this contains information about the statistics of excitations in the form of topological spins. The reality is that there is so little understanding of this issue in the literature that finding any example of a non trivial representation of a motion group would already be an achievement.
- There is also the possibility of looking to specific instances of the models with an eye towards applications in quantum error correction. The reason is that it is easy to obtain quantum codes from our Hamiltonian with the local projectors playing the role of the stabilizers for the code and since the ground states are topologically ordered the corresponding logical qubits will have a built in robustness against local perturbations. Moreover, the combination of a three-dimensional system with a dependence on the second homotopy group might allow us to reverse engineer codes which circumvent some of the issues concerning thermal stability which plague surface codes based on two dimensional systems topologically ordered systems.
- Finally there are many interesting directions from a mathematical point of view such as extending the model for all the different structures that appear in the context of higher gauge theory. We've done this to some extent in a recent work which looks at abelian higher gauge theories based on chain complexes and we've found a generalization of the formula for the ground state degeneracy based on cohomology groups. More interesting though would be to push the relation between homotopy theory and gauge theory to its logical limit and construct models whose ground states are described by the homotopy categories of model categories.

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