Universidade de São Paulo
Instituto de Física

Teoria de cordas, invariância conforme e simetria BRST

Héctor Arturo Benítez del Águila
Orientador: Prof. Dr. Victor de Oliveira Rivelles

Dissertação de mestrado apresentada ao Instituto de Física para a obtenção do título de Mestre em Ciências.

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Prof. Dr. Victor de Oliveira Rivelles (IFUSP)
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Resumo

O principal objetivo deste trabalho é estudar a quantização covariante da corda bosônica e da supercorda RNS, explorando as simetrias envolvidas, ou seja, as simetrias BRST e conforme no caso da corda bosônica e as generalizações correspondentes para a corda fermiônica. Em particular, discutimos alguns aspectos perturbativos da teoria bosônica e a construção de operadores de vértice da corda fermiônica.

Abstract

The main goal of this work is to study the covariant quantization of the bosonic and RNS string theories by exploiting the involved symmetries, namely, the BRST and conformal invariance for the bosonic string and the corresponding supersymmetric generalizations for the fermionic case. In particular, we discuss some perturbative aspects of the bosonic theory and the construction of vertex operators for the fermionic string.

Contents

Chapter 1

Introduction

Our understanding of the electromagnetic, weak and strong interactions is well described by local quantum field theories of the particles experimentally observed. Nature presents another force, gravity, which is described at classical level by the theory of general relativity. However, gravity has not been successfully included in this picture as a quantum theory. Since the early ages of string theory it is known that its spectrum contains a massless spin-2 particle coupling as the graviton does in general relativity. In string theory, particles are represented by one dimensional objects with infinitesimal thickness interacting by joining and splitting.

The evolution of a string develops a two dimensional surface known as the world-sheet which is embedded in a Minkowski target space. The incoming and outgoing strings indicate the merging of initial and the emergence of final states, possibly accompanied by creation and annihilation of virtual string pairs. A theory of quantum gravity, such as string theory, must be able to provide a manifestly Lorentz covariant frame for calculating scattering amplitudes in flat space-time. In order to accomplish this purpose we will make use of the formulation of Polyakov. In analogy to the well known Feynman diagrams, the propagator lines are replaced by strings propagating in time and loop diagrams by the number of handles of the two dimensional surfaces. These surfaces are endowed with an intrinsic metric g_{ab}. From now on we will be concerned with closed oriented strings whose evolution sweeps out an oriented compact surface Σ_g of genus g. The string dynamics is codified in the reparametrization invariant action

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{ab} \partial_a x^\mu \partial_b x_\mu \,. \tag{1.1}$$

where σ^i are local coordinates on Σ_g and x^μ maps the two dimensional surface into the target space-time. In this formulation, the quantization is performed by summing in the

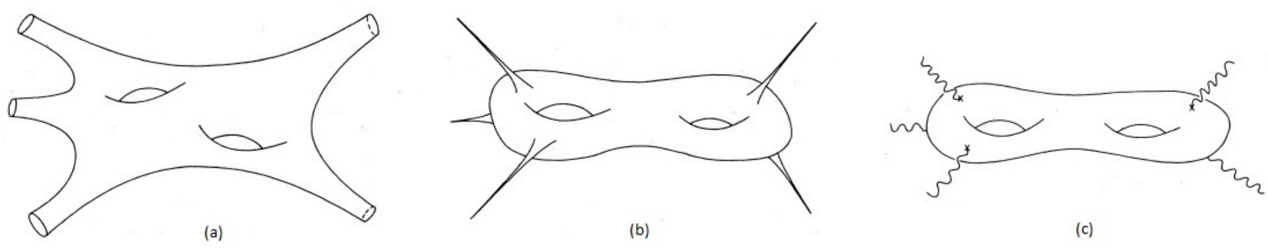

Figure 1.1: the five-point function to two-loop order (g=2)

functional integral over all closed compact surfaces, treating both x^μ and g_{ab} as two dimensional quantum fields.

When calculating S-matrix with n external states, they are represented by the incoming and outgoing external handles of the surface (as it is shown in 1.1). For on-shell external states, the boundary of these handles can be set at large space-time distance (a). As we will see, the relevant structure of the surface for string theory is the conformal structure. This allows us to use a conformal map that locally acts as $z = e^\varsigma$ on the external handles, transforming such world-sheet to a compact surface with n points removed (punctures)(b). Conversely, we can treat the worl-sheet as a compact surface without punctures but with local insertions of vertex operators which introduce the quantum numbers of the external states onto the surface (c). The relation between these two pictures will be revisited in section 2.5.

For a two dimensional manifold of genus g and n punctures $\Sigma_{g,n}$, endowed with a metric g_{ab}, in any local patch we can pick a complex parametrization $z = \sigma_1 + i\sigma_2$, $\bar{z} = \sigma_1 - i\sigma_2$ such that

$$ds^2 = 2g_{z\bar{z}}dzd\bar{z} \tag{1.2}$$

It shows the residual symmetry due to conformal transformations: $z \to f(z)$ for f an analytic function. Therefore, g_{ab} determines a complex structure on $\Sigma_{g,n}$ defining a Riemann surface.

Another metric g'_{ab} which is not related to g_{ab} by a globally defined diffeomorphism and Weyl transformation, defines another complex structure parametrized by the complex coordinate w. In this case, the transformation relating z and w produces a change in the complex structure. This kind of deformations of the complex structure are produced by discontinuous reparametrizations called quasiconformal transformations. We will use this tool widely throughout the text.

After gauge-fixing the local symmetries, the Faddeev-Popov determinant is introduced into the functional integral. This determinant can be computed as an integral over Grassmann ghost fields corresponding to variations of the gauge condition and to infinitesimal reparametrizations of the local coordinates. One may fix a covariant gauge which leaves conformal invariance as a residual symmetry. This remaining symmetry will be useful to study the local aspects of the theory. In this way, the operator product of the fields and the conformal algebra determine all the correlation functions of the theory.

The full action of the system which includes both the matter and the ghost actions, possesses a residual fermionic symmetry, the so called BRST symmetry. The BRST invariance of string theory gives us an alternative approach to quantize the theory. The physical states are BRST invariant states modulo the null states. Moreover, BRST invariance gives us useful information in order to build the covariant vertex operators which produce the physical states of the theory by acting on the ground state. Furthermore, BRST invariance is a powerful tool to study the unitarity of the scattering amplitudes, in particular the decoupling of non-physical states.

In order to successfully treat string theory as a conformal field theory, it must fulfil some important properties. On the one hand, all states in the two dimensional Hilbert space should have non-negative norm. What is more, negative norm states must decouple from scattering amplitudes in order to have a unitary time evolution as required by quantum mechanics.

On the other hand, the operator product of two fields, which produces their corresponding algebra, should be well-defined on the complex plane, which means that it should not change when any field is carried continuously around any point, which means that the correlation function must be single valued.

Another strong constraint to be imposed is modular invariance. In the path integral quantization of the string we will have to integrate over all the possible conformal inequivalent surfaces which are parametrized by complex parameters called *moduli*. However, there are large reparametrizations called modular transformations. They are diffeomorphisms not connected to the identity which change the values of the moduli but not the shape of the surface. Therefore, any correlation function on the surface should be invariant under modular transformations. Moreover, modular invariance in string perturbation theory is responsible for the absence of ultraviolet divergences.

Bosonic string theory can not represent a complete quantum model of nature because of two crucial aspects. Firstly, bosonic strings present a tachyon in their spectrum which is unacceptable for the stability of the theory. Furthermore, the spectrum only describes bosons and we need to come up with a novelty to include fermions if we desire a complete phenomenological description. This is solved by formulating a theory with local supersymmetry on the world-sheet (the Ramond-Neveu-Schwarz string).

In the same way conformal invariance was a remaining symmetry of the reparametrization invariance in conformal gauge, superconformal invariance is the remaining symmetry of local supersymmetry in the superconformal gauge, for the fermionic string. Because of the boundary conditions of the two dimensional fermionic fields, the Hilbert space of a superconformal field theory split into two sectors, the Neveu-Schwarz (NS) and Ramond (R) sectors with periodic and anti periodic boundary conditions respectively.

We focus on the local properties of the superconformal structure of the superstring and on the calculation of tree amplitudes. The main issue is to obtain covariant vertex operators which represent the fermionic and bosonic states of the theory. They are constructed from fields of the NS and R sectors which have the correct superconformal transformations. The BRST symmetry plays a crucial role in order to construct the fermionic vertex operators. The Faddeev-Popov ghosts for local supersymmetry also enter in this construction in a essential way. Although the RNS string does not have manifest space-time supersymmety, we will see that a supersymmetric spectrum is required in order to have a consistent theory, free of anomalies. In particular, we will see that GSO projection, which yields a supersymmetric spectrum, is imposed by demanding modular invariance in loop diagrams.

The present text is organized as follows. Chapter 2 is devoted to the quantization of the bosonic string. Section 2.1 illustrates the basics of conformal field theory. In sections 2.2 and 2.3 we obtain the correct measure for the bosonic string in the critical dimension. The sections 2.4 and 2.5 are concerned with the BRST symmetry of the string in order to construct BRST invariant vertex operators. After an examination of the BRST invariance of the scattering amplitudes we examine the sources of possible anomalies. In section 2.6 we calculate explicitly the Virasoro-Shapiro amplitude and in the section 2.7 the bosonic amplitude is factorized isolating the divergence term. The one-loop partition function and the modular invariance of this amplitude is studied in section 2.8.

Chapter 3 is concerned with the covariant quantization of the RNS superstring. In section 3.1 we formulate the theory as a two dimensional supergravity coupled to a matter system. Superconfomal field theory is presented in section 3.2. Matter and ghost system are revisited in superconformal formalism in sections 3.3 and 3.4. The sections 3.5 and 3.6 are about the super BRST current, first order systems and bosonization, preparing the tools to construct the correct vertex operators for the femionic string in 3.7. The one-loop partition function for the superstring is obtained in Section 3.8. Finally, the conclusions of the present work are included in chapter 4.

Chapter 2

Quantization of the bosonic string

This chapter is devoted to study the quantization of the bosonic string and some of its most important properties as a two dimensional conformal field theory (CFT)[1][2] [3]. Treating string theory as a CFT on a Riemann surface is a natural formulation that will provide us with the necessary tools to implement the so called path integral quantization. For this reason, in the first section of this chapter, we review the basics of CFTs which will be widely used throughout this text.

2.1 Basics of two dimensional conformal field theory

2.1.1 Two dimensional conformal transformation

We consider a two-dimensional space $\mathcal{M}$ endowed with an Euclidean metric $g_{\mu\nu}$. We define a conformal transformation as a change of coordinates $x \to x'$ such that it leaves the metric invariant up to a scaling factor

$$g_{\mu\nu}(x) = g'_{\mu\nu}(x') = \frac{\partial\, x^{\alpha}}{\partial\, x'^{\mu}} \frac{\partial\, x^{\beta}}{\partial\, x'^{\nu}} g_{\alpha\beta}(x) = \Omega(x) g_{\mu\nu}(x)\,. \tag{2.1}$$

For the two dimensional case it is convenient to rewrite this in complex coordinates, $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$. Let us define the complex derivatives as

$$\partial_z = \frac{1}{2}(\partial_x + i\partial_y)\,, \qquad \partial_{\bar{z}} = \frac{1}{2}(\partial_x - i\partial_y)\,, \tag{2.2}$$

in order to have $\partial_z z = \partial_{\bar{z}} \bar{z} = 1$ and $\partial_z \bar{z} = \partial_{\bar{z}} z = 0$. [1]

[1]Let us obtain the metric tensor in complex coordinates. We can see that

$$g_{\bar{z}z} = g_{z\bar{z}} \quad = \quad \partial_z\, x^{\mu} \partial_{\bar{z}} x^{\nu} g_{\mu\nu} = \partial_z\, x^1 \partial_{\bar{z}} x^1 + \partial_z\, x^2 \partial_{\bar{z}} x^2 = \frac{1}{2}\,.$$

The metric takes the form $ds^2 = dzd\bar{z}$. In this coordinate system, the condition (2.1) consists of all transformations of the form [2]

$$z \to f(z), \qquad \bar{z} \to \bar{f}(\bar{z}), \tag{2.3}$$

where f and $\bar{f}$ are arbitrary analytical functions. For these kind of transformations, the rescaling factor is $\Omega = |\partial f/\partial z|^2$. [3] The generators of infinitesimal conformal transformations are [4]

$$l_n = -z^{n+1}\partial_z \qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}\,,$$

produce the infinite dimensional local algebra

$$[l_n, l_m] = (m-n)l_{m+n} \qquad \left[\bar{l}_n, \bar{l}_m\right] = (m-n)\bar{l}_{m+n}\,, \tag{2.5}$$

and $[l_n, \bar{l}_m] = 0$.

There is an important point to remark about this algebra; the generators (2.5) are not well defined globally on the Riemann sphere $S^2 = \mathbb{C} \cup \infty$. [5]

Demanding analyticity at $z = 0$, implies that l_n is globally defined only for $n \geq -1$. In order to describe the behaviour of functions at $z \to \infty$ we make use of the inversion map

$$z = -\frac{1}{u}\,, \tag{2.6}$$

In the same way, we can obtain that $g_{zz} = g_{\bar{z}\bar{z}} = 0$. Therefore $g^{z\bar{z}} = g^{\bar{z}z} = 2$.

[2]Let $z'^{\mu}(z, \bar{z})$ be a reparametrization of the $(z, \bar{z})$ coordinates. Then, in order this change of coordinates to be a conformal transformation the components of the metric tensor must be $g'_{11} = g'_{22} = 0$. We can impose this condition, by using (2.1), getting the constraints

$$\partial_z z'^1 \partial_z z'^2 = 0\,, \qquad \partial_{\bar{z}} z'^1 \partial_{\bar{z}} z'^2 = 0\,.$$

The solution for this system is $z'^1 = z'^1(z)$ and $z'^2 = z'^2(\bar{z})$. The choice of z^1 or z^2 is totally arbitrary. Then, we see that the two-dimensional conformal transformations remains in analytical transformations of the form

$$z \to z'(z)\,, \qquad \bar{z} \to \bar{z}'(\bar{z})\,.$$

[3]This follows from $ds'^2 = |dz'|^2 = |\partial z'/\partial z|^2 |dz|^2$

[4]These generators expand the infinitesimal conformal transformations $z \to z + \epsilon(z)$ and $\bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z})$, in the basis

$$\epsilon(z) = -\epsilon_n z^{n+1}\,, \qquad \bar{\epsilon}(\bar{z}) = -\bar{\epsilon}_n \bar{z}^{n+1}\,, \qquad n \in \mathbb{Z}\,. \tag{2.4}$$

[5]An important subset of two-dimensional conformal transformations consists in the invertible mapping $z \to 1/z$ globally defined on $\mathbb{C} \cup \infty$, that is, the Riemann sphere, the complex plane plus a point at infinity.

and imposing analyticity for $u = 0$,

$$l_n = -z^{n+1}\partial_z = \left(-\frac{1}{u}\right)^{n+1}\left(\frac{1}{u^2}\right)\partial_u = \left(-\frac{1}{u}\right)^{n-1}\partial_u\,, \tag{2.7}$$

it is easy to see that only the generators l_n, for $n \leq 0$ are well defined at $u = 0$ and we note the only conformal transformations defined globally are generated by $\{l_0, l_1, l_{-1}\}$ (global conformal transformations), producing translations $l_{-1}, \bar{l}_{-1}$, dilations $l_0 + \bar{l}_0$, rotations $i(l_0 - \bar{l}_0)$ and special conformal transformations $l_1, \bar{l}_1$. The finite form of this can be obtained from successive infinitesimal transformations and takes the form

$$z \to \frac{az+b}{cz+d}, \qquad ad - bc = 1\,, \qquad a, b, c, d \in \mathbb{C}\,. \tag{2.8}$$

This is the $SL(2,\mathbb{C})$ group of transformations. The transformation above remains unaltered when changing the sign of all the coefficients, a, b, c and d, and we need to incorporate a quotient of $\mathbb{Z}_2$. These group of transformation is known as the group of projective conformal transformations $SL(2,\mathbb{C})/\mathbb{Z}_2$.

Moreover, since the total algebra is the direct sum of the holomorphic and anti-holomorphic algebra (2.5), the variables z and $\bar{z}$ can be treated as independent coordinates. The physical condition $z^* = \bar{z}$ is left to be imposed at some convenient point.

2.1.2 Conformal fields in two dimensions

From now on, we shall generalize the above ideas to the quantum realization of the conformal algebra. In order to define a field theory with conformal invariance we require a set of fields $\{A_j\}$, in general infinite, which includes the derivatives of all the fields A_j. The complete set of states of the theory can be obtained by the action of the conformal fields on the ground state. [6] The vacuum $|0\rangle$, by definition, is invariant under the action of the global conformal group $SL(2,\mathbb{C})$ invariant). There is a subset of fields $\{\phi_j\}$, called *quasi-primaries*, which under *global* conformal transformations $x \to x'$ transform as

$$\phi_i(x) \to \left|\frac{\partial x'}{\partial x}\right|^{d_i/2} \phi_i(x')\,, \tag{2.9}$$

[6] This is an important fact of conformal field theories quantized on a circle, namely, the state-operator correspondence. An asymptotic state propagating from $\tau \to -\infty$ can be conformally mapped to the origin by using a map that locally acts as e^w (where $w = \tau + i\sigma$). This map defines a local operator at the origin of the complex plane, known as the *vertex operator* (subsection 2.3.1). For more details [4].

where d_i is the scaling dimension of ϕ_i. [7]

We can obtain the total set of fields by linear combinations of the elements of $\{\phi_i\}$ and their derivatives.
The correlation functions of quasi-primary fields transform according to

$$\langle \phi_1(x_1) \dots \phi_n(x_n) \rangle = \left| \frac{\partial x'}{\partial x} \right|_{x=x_1}^{d_1/2} \dots \left| \frac{\partial x'}{\partial x} \right|_{x=x_n}^{d_n/2} \langle \phi_1(x'_1) \dots \phi_n(x'_n) \rangle \,. \tag{2.11}$$

Conformal invariance imposes strong constraints on the correlator functions of quasi-primary fields, specially for the two and three point functions. According to (2.11), the 2-point function correlator must transforms as

$$G(x_1, x_2) = \langle \phi_1(x_1) \phi_2(x_2) \rangle = \left| \frac{\partial x'}{\partial x} \right|_{x=x_1}^{d_1/2} \left| \frac{\partial x'}{\partial x} \right|_{x=x_2}^{d_2/2} \langle \phi_1(x'_1) \phi_2(x'_2) \rangle \,. \tag{2.12}$$

Invariance under translations and rotations requires that

$$G(x_1, x_2) \equiv G(|x_1 - x_2|) \,, \tag{2.13}$$

and transformation by dilations $x' = \lambda\, x$ tell us that

$$G(|x_1 - x_2|) = \lambda^{d_1+d_2} G(\lambda |x_1 - x_2|) \,, \tag{2.14}$$

which implies

$$G(|x_1 - x_2|) = \frac{C_{12}}{|x_1 - x_2|^{d_1+d_2}} \,. \tag{2.15}$$

Finally, special conformal transformation which are given by

$$x' = \frac{x + bx^2}{1 + 2b\dot{} x + b^2 x^2} \,, \tag{2.16}$$

forces that $d_1 = d_2$ in order to have a non-zero correlator, thus

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \begin{cases} \frac{C_{12}}{|x_1 - x_2|^{2d}} & \text{for} \quad d_1 = d_2 = d \\ 0 \quad \text{for} \quad d_1 \neq d_2 \end{cases} \,. \tag{2.17}$$

The three-point function can be obtained in a similar way

$$\langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \rangle = \frac{C_{123}}{|x_1 - x_2|^{d_1+d_2-d_3} \, |x_2 - x_3|^{d_2+d_3-d_1} \, |x_1 - x_3|^{d_1+d_3-d_2}} \,, \tag{2.18}$$

[7] The scaling dimension of ϕ defines the transformation of the field under a rescaling of coordinates.

$$x \to \lambda\, x \,, \qquad \phi(x) \to \lambda^{-d} \phi(x) \,. \tag{2.10}$$

On the other hand, there is a subset of quasi-primary fields $\{\Phi_i(z,\bar{z})\}$ which, under the conformal map $z \to w = f(z)$, transforms as

$$\Phi'(z,\bar{z}) = (\frac{\partial w}{\partial z})^h (\frac{\partial \bar{w}}{\partial \bar{z}})^{\bar{h}} \Phi_i(f(z), \bar{f}(\bar{z}))\,, \tag{2.19}$$

these fields are called $(h, \bar{h})$ primary fields, where h and $\bar{h}$ are defined as the holomorphic and anti-holomorphic conformal dimensions of Φ which are related to the scaling dimension d and the planar spin s as [8]

$$h + \bar{h} = d\,, \qquad h - \bar{h} = s\,, \tag{2.20}$$

We remark that transformations of the form $w = f(z)$ are not restricted to global conformal transformations $SL(2,\mathbb{C})$, but to any conformal transformation. Under an infinitesimal coordinate transformation $z \to z + \epsilon(z)$, (2.19) gives

$$\delta\Phi(z,\bar{z}) = (h\partial\epsilon + \epsilon\partial + \bar{h}\,\bar{\partial}\bar{\epsilon} + \bar{\epsilon}\bar{\partial})\,\Phi(z,\bar{z})\,. \tag{2.21}$$

2.1.3 Energy-momentum tensor

The energy-momentum tensor $T_{\alpha\beta}$ is a $(2,2)$ quasi-primary field ($SL(2,\mathbb{Z})$ primary) which plays a central role in conformal field theory. For a two-dimensional field theory defined by an action $S(\phi_i, g_{\alpha,\beta})$ we define the symmetric tensor $T_{\alpha\beta}$ as

$$T_{\alpha\beta} = \frac{-2\pi}{\sqrt{g}} \frac{\delta\, S(\phi_i, g_{\alpha\beta})}{\delta\, g^{\alpha\beta}}\,. \tag{2.22}$$

For an action invariant under Weyl transformations $g_{\alpha\beta} \to g'_{\alpha\beta} = \Omega\, g_{\alpha\beta}$, the energy-momentum tensor is traceless, that is [9]

$$0 = \frac{-2\pi}{\sqrt{g}} \frac{\delta\, S\,(\phi_i, g_{\alpha\beta})}{\delta\Omega} = \frac{-2\pi}{\sqrt{g}} \frac{\delta\, S\left(\phi_i, g'_{\alpha\beta}\right)}{\delta\, g'^{\alpha\beta}} \frac{\delta\, g'^{\alpha\beta}}{\delta\Omega} = T^{\alpha}_{\alpha}\,. \tag{2.23}$$

In order to rewrite this expression in complex coordinates, we use tensor transformations

[8] These values will be defined below as the eigenvalues of the operators of dilation and rotation $L_0 + \bar{L}_0$ and $L_0 - \bar{L}_0$ respectively.

[9] Taking into account that $S\,(\phi_i, g_{\alpha\beta}) = S(\phi_i, g'_{\alpha\beta} = \Omega\, g_{\alpha\beta})$

and calculate[10]

$$
\begin{aligned}
T_{z\bar{z}} &= \frac{\partial\sigma^\alpha}{\partial z}\frac{\partial\sigma^\beta}{\partial\bar{z}}T_{\alpha\beta} \\
&= \frac{\partial\sigma^1}{\partial z}\frac{\partial\sigma^1}{\partial\bar{z}}T_{11} + \frac{\partial\sigma^2}{\partial z}\frac{\partial\sigma^1}{\partial\bar{z}}T_{21} + \frac{\partial\sigma^1}{\partial z}\frac{\partial\sigma^2}{\partial\bar{z}}T_{12} + \frac{\partial\sigma^2}{\partial z}\frac{\partial\sigma^2}{\partial\bar{z}}T_{22} \\
&= \frac{1}{4}(T_{11}+T_{22}) = 0\,. \qquad (2.24)
\end{aligned}
$$

Let us consider that the action $S(\phi_i, g_{\alpha,\beta})$ possess conformal invariant. Let ϵ^α be a vector field generating translations of the form $\sigma^\alpha \to \sigma^\alpha + \epsilon^\alpha$, then

$$
\begin{aligned}
0 &= \delta_\epsilon S(\phi_i, g_{\alpha\beta}) \\
&= \int d^2\sigma \left(\frac{\delta\, S}{\delta\, g^{\alpha\beta}}\delta_\epsilon g^{\alpha\beta} + \frac{\delta\, S}{\delta\phi}\delta_\epsilon\phi\right) \\
&= \frac{1}{2\pi}\int d^2\sigma\sqrt{g}\,T_{\alpha\beta}(\nabla^\alpha\epsilon^\beta + \nabla^\beta\epsilon^\alpha)\,, \qquad (2.25)
\end{aligned}
$$

where in the second line we made use of the equation of motion of ϕ. After integrating by parts we obtain the conservation law of the energy-momentum tensor.

$$\nabla^\alpha T_{\alpha\beta} = 0\,. \qquad (2.26)$$

The last result together with (2.24) can be used to show that

$$\partial_{\bar{z}}T_{zz} = 0, \qquad \partial_z T_{\bar{z}\bar{z}} = 0\,. \qquad (2.27)$$

Then, the two remaining non-zero components of the the energy-momentum tensor are the holomorphic $T_{zz} = T(z)$ and the anti-holomorphic $T_{\bar{z}\bar{z}} = \bar{T}(\bar{z})$ components.

In general, (anti)holomorphic fields $\mathcal{O}(z)$ of weight h ($\bar{h}$) can be expanded in Laurent series as follows

$$\mathcal{O}(z) = \sum_{n=-\infty}^{\infty}\frac{\mathcal{O}_n}{z^{n+h}}, \qquad \bar{\mathcal{O}}(\bar{z}) = \sum_{n=-\infty}^{\infty}\frac{\bar{\mathcal{O}}_n}{\bar{z}^{n+\bar{h}}}\,. \qquad (2.28)$$

The Laurent modes satisfy

$$\mathcal{O}_n = \frac{1}{2i\pi}\oint dz\, z^{n+h-1}\mathcal{O}(z)\,. \qquad (2.29)$$

[10]You should note that we renamed the x coordinates as σ. We leave x to label the matter field system of the bosonic string.

The components of the energy-momentum tensor T_{zz} and $T_{\bar{z}\bar{z}}$ have weights $(2,0)$ and $(0,2)$ respectively. This is obtained by a rescaling $z \to \lambda z$. Then, components of the energy-momentum tensor has the Laurent expansions

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}}, \qquad \bar{T}(\bar{z}) = \sum_{n=-\infty}^{\infty} \frac{\bar{L}_n}{\bar{z}^{n+2}}\,. \tag{2.30}$$

We have obtained the energy-momentum tensor as a conserved charge due to two-dimensional translations, therefore $T_{\alpha\beta}$ is the generator of these variations but not conformal transformations. We need to introduce a conserved current associated with conformal transformations. Given a vector field v^α, let us consider

$$J^\alpha = v_\beta T^{\alpha\beta}\,. \tag{2.31}$$

We note that a vector field v^α produces a change in the metric of the form

$$\begin{aligned} \delta_v g_{\alpha\beta} &= \nabla_\alpha v_\beta + \nabla_\beta v_\alpha = (P_1 v)_{\alpha\beta} + (\nabla . v) g_{\alpha\beta}\,, \\ (P_1 v)_{\alpha\beta} &= \nabla_\alpha v_\beta + \nabla_\beta v_\alpha - (\nabla . v) g_{\alpha\beta} \end{aligned} \tag{2.32}$$

The value of $(P_1 v)_{\alpha\beta}$ is only zero if v^α is a conformal Killing vector(CKV), leaving invariant the metric up to a scaling factor. It follows that

$$\begin{aligned} \nabla_\alpha J^\alpha &= \nabla_\alpha (v_\beta T^{\alpha\beta}) \\ &= (\nabla_\alpha T^{\alpha\beta}) v_\beta + (\nabla_\alpha v_\beta) T^{\alpha\beta} \\ &= \frac{1}{2}(\nabla_\alpha v_\beta + \nabla_\beta v_\alpha) T^{\alpha\beta} \end{aligned} \tag{2.33}$$

where we have used the conservation and symmetry of the energy-momentum tensor. Now, if the reparametrization $\sigma^\alpha \to \sigma^\alpha + v^\alpha$ is a conformal transformation, it follows that

$$\nabla_\alpha J^\alpha = \frac{1}{2}(\nabla_\rho v^\rho) g_{\alpha\beta} T^{\alpha\beta} = 0\,. \tag{2.34}$$

We expect this quantity to be the conserved charge which produces conformal transformations.

2.1.4 Radial quantization

Until now we have been considering complex coordinates of the form $\varsigma\,, \bar{\varsigma} = \sigma^1 \pm i\sigma^2$

for the two dimensional euclidean time σ^1 and space σ^2. This define a cylinder after considering the spatial coordinate periodic $\sigma^2 \equiv \sigma^2 + 2\pi$. The conformal transformation

$$z = e^{\varsigma} \qquad \bar{z} = e^{\bar{\varsigma}} \tag{2.35}$$

maps the cylinder to the complex plane whose origin ($z = 0$) is related to the distant past ($\sigma^1 \to -\infty$). The euclidean time points radially outward from the origin; that is the reason why quantum version of the dilation operator in the complex plane, namely, $L_0 + \bar{L}_0$, is considered as the Hamiltonian. The charge associated with J^α can be written using complex coordinates as follows

$$\begin{aligned} \mathcal{Q} &= \frac{1}{2\pi i} \oint_{\mathcal{C}} dz^\alpha \, T_{\alpha\beta} \epsilon^\beta \\ &= \frac{1}{2\pi i} \oint_{\mathcal{C}} \left[dz \, T(z) \epsilon(z) + d\bar{z} \, \bar{T}(\bar{z}) \bar{\epsilon}(\bar{z}) \right] . \end{aligned} \tag{2.36}$$

where $\mathcal{C}$ is a contour that surrounds the origin. This conserved charge produces conformal transformations on the fields $\phi(w, \bar{w})$ in the following way

$$\delta\phi(w,\bar{w}) = [\mathcal{Q}, \phi(w,\bar{w})] = \frac{1}{2\pi i} \oint_{\mathcal{C}} dz \, \left[T(z)\epsilon(z), \phi(w,\bar{w}) \right] + \frac{1}{2\pi i} \oint_{\mathcal{C}} d\bar{z} \, \left[\bar{T}(\bar{z})\bar{\epsilon}(\bar{z}), \phi(w,\bar{w}) \right] . \tag{2.37}$$

After being compactified σ^2 direction, we choose the σ^1 direction as the quantization direction. In the complex plane this means that the operators must be radially ordered. The prescription for radial ordering is

$$\mathcal{R}\left(\mathcal{A}(z)\mathcal{B}(w)\right) := \begin{cases} \mathcal{A}(z)\mathcal{B}(w) & \text{for} \quad |z| > |w| \\ \mathcal{B}(w)\mathcal{A}(z) & \text{for} \quad |w| > |z| \end{cases} . \tag{2.38}$$

Taking this into account, conformal field transformations (2.37) can be expressed as (holomorphic part)

$$\begin{aligned} & \oint_{|z|>|w|} dz \, T(z)\epsilon(z)\phi(w,\bar{w}) - \oint_{|z|<|w|} dz \phi(w,\bar{w}) T(z)\epsilon(z) \\ = & \oint_{\mathcal{C}(w)} dz \, \mathcal{R}(T(z)\epsilon(z)\phi(w)) . \end{aligned} \tag{2.39}$$

We have to keep in mind that $T(z)$ is analytic everywhere except at the point where ϕ is inserted, this allows us to deform the contour into a small circle surrounding w (figure 2.1).

As we can see from the above result, conformal transformations only take into consideration the short distance behaviour of energy-momentum tensor close to an operator insertion $\phi(w, \bar{w})$. When two local operators approach one another, singularities can be

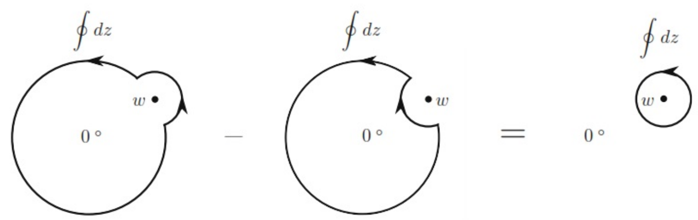

Figure 2.1: Contour of integration

characterized by their Operator Product Expansion (OPE).

$$\mathcal{A}_i(z)\,\mathcal{B}_j(w) \sim \sum_k c_{ij}^k\,(z-w)\,\mathcal{O}_k(w)\,, \tag{2.40}$$

where $\{\mathcal{O}_k\}$ is a complete set of local operators and c_{ij}^k are singular coefficients.

The singular part of the OPE between $T(z)$ and $\phi(w,\bar{w})$ can be easily obtained by comparing equations (2.39) and (2.21), and can be written as

$$T(z)\phi(w,\bar{w}) \sim \frac{h}{(z-w)^2}\,\phi(w,\bar{w}) + \frac{1}{z-w}\,\partial_w\phi(w,\bar{w}) \tag{2.41}$$

$$\bar{T}(\bar{z})\phi(w,\bar{w}) \sim \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\,\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\,\partial_{\bar{w}}\phi(w,\bar{w})\,. \tag{2.42}$$

2.1.5 Central charge and Virasoro algebra

The variation of the energy-momentum tensor can be expressed as a linear function of T and their derivatives. Following [5] wee write the expression for the variation $\delta\,T(z)$ under infinitesimal conformal transformation [11]

$$\delta T(w) = \frac{c}{12}\,\partial^3\epsilon(w) + 2\,\partial_w\epsilon(w)\,T(w) + \epsilon(w)\partial_w T(w)\,, \tag{2.43}$$

[11] The most general expression for this variation should include a second derivative term, however the coefficient of this contribution must be zero as will be explained below.

the factor $\frac{c}{12}$ has been chosen for convenience.[12] This variation, as stated in the discussion above, is produced by action of (2.36) on $T(w)$

$$\delta T(w) = \frac{1}{2\pi i}\oint_{\mathcal{C}_w} dz\, \epsilon(z) T(z) T(w)\,,$$

then we see that (2.43) holds if the OPE $T(z)T(w)$ is given by

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}\,. \tag{2.45}$$

At this point it is instructive to say that the OPE of two arbitrary fields $\mathcal{A}(z_1)\mathcal{B}(z_2)$ must be symmetric in z_1 and z_2 [6]. This explains why terms with second order derivatives in $\epsilon(w)$ does not appear in $\delta\, T(w)$. The same considerations apply for the anti-holomorphic component

$$\bar{T}(\bar{z})\bar{T}(\bar{w}) \sim \frac{\bar{c}/2}{(\bar{z}-\bar{w})^4} + \frac{2\,\bar{T}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\bar{T}(\bar{w})}{(\bar{z}-\bar{w})}\,. \tag{2.46}$$

By treating $T(z)$ as an operator, the Laurent modes are promoted to operators too. Their algebra can be obtained by application of radial ordering and using the OPE (2.45). The modes L_n of energy-momentum tensor are

$$L_n = \oint \frac{dz}{2i\pi} T(z) z^{n+1}\,, \tag{2.47}$$

considering the discussion above we obtain the algebra

$$\begin{aligned}
[L_n, L_m] &= \oint_{\mathcal{C}_0} \frac{dw}{2i\pi} \oint_{\mathcal{C}_w} \frac{dz}{2i\pi} w^{n+1} z^{m+1} \mathcal{R}(T(z)T(w)) \\
&= \oint_{\mathcal{C}_0} \frac{dw}{2i\pi} \oint_{\mathcal{C}_w} \frac{dz}{2i\pi} w^{n+1} z^{m+1} \left(\frac{c/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}\right) \\
&= \oint_{\mathcal{C}_0} \frac{dw}{2i\pi} w^{n+1} \left(\frac{c}{12}(m+1)m(m-1)w^{m-2} + 2(m+1)w^m T(w) + w^{m+1}\partial_w T(w)\right)\,.
\end{aligned} \tag{2.48}$$

[12] As we will see, the energy-momentum tensor is not a primary field beacause it does not transform as (2.19), but a quasi-primary field, namely, a $SL(2,\mathbb{C})$ primary. Equation (2.43) corresponds to the transformation under finite conformal transformation (2.157)

$$T(z) \to T'(f(z)) = \left(\frac{df(z)}{dz}\right)^{-2} \left(T(z) - \frac{c}{12}\{f(z); z\}\right)\,,$$

where the term $\{f(z); z\}$ is the Schwartzian derivative

$$\{f(z); z\} = \frac{d^3 f(z)/dz^3}{df(z)/dz} - \frac{3}{2}\left(\frac{d^2 f(z)/dz^2}{df(z)/dz}\right)^2\,. \tag{2.44}$$

In particular, for the map $z = f(w) = e^w$, $T(z)$ transforms as, $T(z) = T(w) + c/24$.

After integration on w and considering (2.47) we get

$$[L_n, L_m] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n} \,. \tag{2.49}$$

The $\bar{L}_n$ modes satisfy the same algebra and L_m commutes with $\bar{L}_n$. This is the Virasoro algebra. It is essential to remark that every conformal field theory defines a representation of this algebra, characterized by the central charge c. On the other hand, global conformal transformations $SL(2,\mathbb{C})$ generated by $L_{-1,0,1}$ and $\bar{L}_{-1,0,1}$ form a subalgebra without the anomalous term c (2.5).

The effect of L_n on the vacuum can be determined as follows

$$L_n \left|0\right\rangle \cong \int \frac{dz}{2i\pi} z^{n+1} T(z) \left|0\right\rangle \,. \tag{2.50}$$

By requiring the regularity of $T(z)$, the integral vanishes for $n > -1$. We state this important result as

$$L_n \left|0\right\rangle = 0, \qquad n > -1 \tag{2.51}$$

To define $\langle 0|$ at $z \to \infty$, we use the same reasoning but now for the map $u = -1/z$ and demanding regularity at $u = 0$. Similarly, we obtain for out-states

$$\langle 0| L_{-n}, \qquad n > 1. \tag{2.52}$$

The generators $L_{1,0,-1}$ and $\bar{L}_{1,0,-1}$ annihilates both $|0\rangle$ and $\langle 0|$, which means that the vacuum is $SL(2,\mathbb{C})$ invariant. The same applies for $\bar{L}_n$.

2.1.6 Highest weight states and descendant fields

In a conformal field theory quantized on a circle, there is an important isomorphism between the set of local operators and the space of states of the theory. For an S-matrix calculation, one must specify initial and final states. Initial states ($\sigma^0 \to -\infty$) can be created by a field acting on the vacuum. On the complex plane this means to insert a local operator at the origin $z = 0$

$$\left|\phi\right\rangle \equiv \phi(z=0) \left|0\right\rangle \,. \tag{2.53}$$

Let ϕ be a primary field. By using the OPE with the energy-momentum tensor we can obtain the commutator of the modes L_n with ϕ

$$\begin{aligned} [L_n, \phi(0)] &= \oint \frac{dz}{2i\pi} z^{n+1} T(z)\phi(w) \\ &= \oint \frac{dz}{2i\pi} (h z^{n-1}\phi(0) + z^n \partial\phi(0)), , \end{aligned} \tag{2.54}$$

we easily see that for $n > 0$ the integrand is holomorphic at $z = 0$, as a consequence the integral vanishes. However, for $n = 0$ we have a single pole which yields to a non zero answer. Taking into account that L_n annihilates the vacuum for $n > -1$, we can summarize

$$L_n |\phi\rangle = \begin{cases} 0 & \text{for} \quad n \geq 1, \\ h |\phi\rangle & \text{for} \quad n = 0 \end{cases} . \tag{2.55}$$

States satisfying this condition are known as the *highest weight states*. Furthermore, noting that $[L_0, L_n] = -nL_n$ we see that for $n > 0$, the action of L_n on $\phi |0\rangle$ lowers its weight by n.
States of the form

$$L^{i_n}_{-n} \ldots L^{i_2}_{-2} L^{i_1}_{-1} |\phi\rangle \tag{2.56}$$

are known as *descendant states*. We note that every primary field gives rise to an infinite set of descendant fields. In this sense, Virasoro algebra organizes the fields of a conformal field theory by grouping them in families, each of these, labelled by a primary field. For bosonic strings, there are two important conformal systems to take into consideration; the ghost system bc and the matter system x^μ.

2.1.7 Matter system

Our principal motivation in this chapter is to obtain the quantization of the Polyakov action (1.1). After fixing the conformal gauge, it means, $\hat{g}_{ab} = e^{\phi(\sigma_1, \sigma_2)} \delta_{ab}$, we have to deal with the conformal symmetry which remains unfixed. The gauge fixed Polyakov action can be written as [13]

$$S = \frac{1}{2\pi\alpha'} \int d^2 z\, \partial_z x^\mu \partial_{\bar{z}} x_\mu \,, \tag{2.57}$$

with equations of motion

$$\partial_z(\partial_{\bar{z}} x^\mu) = \partial_{\bar{z}}(\partial_z x^\mu) = 0 \,. \tag{2.58}$$

We can see that ∂x^μ and $\bar{\partial} x^\mu$ are holomorphic and antiholomorphic fields of weight

[13] After fixing the conformal gauge in Polyakov's action, we can rewrite this in complex coordinates by considering the non-zero components of the metric tensor $\hat{g}_{\bar{z}z} = \hat{g}_{z\bar{z}} = \frac{e^{\phi(z,\bar{z})}}{2}$ and $\hat{g}^{\bar{z}z} = \hat{g}^{z\bar{z}} = 2e^{-\phi(z,\bar{z})}$. Now, it is easy too see that

$$\hat{g}^{1/2} d^2\sigma = e^{\phi(z,\bar{z})} \frac{dz d\bar{z}}{2} \,, \qquad \hat{g}^{ab} \partial_a x^\mu \partial_b x^\mu = 4e^{-\phi(z,\bar{z})} \partial_z x^\mu \partial_{\bar{z}} x_\mu \,.$$

$(1,0)$ and $(0,1)$ respectively, and have Laurent expansions of the form

$$\partial_z x^\mu(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \frac{\alpha_n^\mu}{z^{n+1}}, \qquad \partial_{\bar{z}} x^\mu(\bar{z}) = -i\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \frac{\bar{\alpha}_n^\mu}{\bar{z}^{n+1}} \tag{2.59}$$

Thus, the mode expansion for $x^\mu(z,\bar{z})$ split into a sum of holomorphic and antiholomorphic part and is obtained by integrating (2.59)

$$x^\mu(z,\bar{z}) = x^\mu(z) + x^\mu(\bar{z}) \tag{2.60}$$

$$\begin{aligned} x^\mu(z) &= \frac{x_0^\mu}{2} - i\frac{\alpha'}{2} p^\mu \ln z + i\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \frac{\alpha_n^\mu}{n} z^{-n}\,, \\ x^\mu(\bar{z}) &= \frac{x_0^\mu}{2} - i\frac{\alpha'}{2} p^\mu \ln \bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \frac{\bar{\alpha}_n^\mu}{n} \bar{z}^{-n}\,, \end{aligned} \tag{2.61}$$

here, x_0^μ and $p^\mu = \left(\frac{2}{\alpha'}\right)^{1/2} \alpha_0 = \left(\frac{2}{\alpha'}\right)^{1/2} \bar{\alpha}_0$, represent the center of mass-momentum position and momentum of the string. The algebra generated by these modes can be obtained by using the propagator of the matter system. We obtain this by following [4], [14]

$$\begin{aligned} 0 &= \int \mathcal{D}x \, \frac{\delta}{\delta x_\mu(z,\bar{z})} \left[\exp\left(-S\right) x^\nu(w,\bar{w})\right] \\ &= \int \mathcal{D}x \, \exp\left(-S\right) \left[\eta^{\mu\nu} \delta^2(z-w, \bar{z}-\bar{w}) + \frac{1}{\pi\alpha'} \, \partial_z \partial_{\bar{z}} x^\mu(z,\bar{z}) x^\nu(w,\bar{w})\right] \\ &= \eta^{\mu\nu} \left\langle \delta^2(z-w,\bar{z}-\bar{w}) \right\rangle + \frac{1}{\pi\alpha'} \, \partial_z \partial_{\bar{z}} \left\langle x^\mu(z,\bar{z}) x^\nu(w,\bar{w}) \right\rangle \,, \end{aligned} \tag{2.62}$$

which holds as an operator equation. Solving (2.62) we get

$$\langle x^\mu(z,\bar{z}) x^\nu(w,\bar{w}) \rangle = -\eta^{\mu\nu} \frac{\alpha'}{2} \ln |z-w|^2\,. \tag{2.63}$$

where we have made use of the identity

$$\partial_z \partial_{\bar{z}} \ln |z|^2 = \partial_{\bar{z}} \frac{1}{z} = 2\pi \, \delta^2(z,\bar{z})\,. \tag{2.64}$$

This result will be useful as the starting point to define the *conformal normal ordering* of an operator. The conformal normal ordering of a field $\mathcal{O}$ is obtained by subtracting all

[14] Demanding that the path integral of a total derivative vanishes.

self contractions. For the matter field x^μ we have that

$$
\begin{aligned}
:x^\mu(z,\bar z): &= x^\mu(z,\bar z)\,, && (2.65)\\
:x^\mu(z,\bar z)x^\nu(w,\bar w): &= x^\mu(z,\bar z)x^\nu(w,\bar w) - \langle x^\mu(z,\bar z)x^\nu(w,\bar w)\rangle \\
&= x^\mu(z,\bar z)x^\nu(w,\bar w) + \frac{\alpha'}{2}\,\eta^{\mu\nu}\ln|z-w|^2\,. && (2.66)
\end{aligned}
$$

As we will use frequently in this text, we obtain the energy-momentum tensor for the matter system by the Noether method. The energy-momentum tensor is the conserved charge related to symmetry under translations $\delta z = \epsilon\,,\ \delta\bar z = \bar\epsilon$. Under this variations, matter fields transform as $\delta x^\mu(z,\bar z) = \epsilon\,\partial_z x^\mu + \bar\epsilon\,\partial_{\bar z}x^\mu$ and varying the action

$$
\begin{aligned}
\delta S &= \frac{1}{2\pi\alpha'}\int d^2z\,\Big[-:\partial_z x^\mu\partial_z x_\mu:\partial_{\bar z}\epsilon - :\partial_{\bar z}x^\mu\partial_{\bar z}x_\mu:\partial_z\bar\epsilon + \\
&\qquad + \partial_z(\partial_{\bar z}x_\mu\partial_z x^\mu\epsilon) + \partial_{\bar z}(\partial_z x_\mu\partial_{\bar z}x^\mu\bar\epsilon)\Big]\,. \qquad (2.67)
\end{aligned}
$$

The last two terms are total derivatives and can be dropped, and we obtain the energy-momentum tensor which splits in holomorphic and antiholomorphic components, as expected
[15]

$$
T(z) = -\frac{1}{\alpha}\ :\partial_z x^\mu\partial_z x_\mu:\,,\qquad \bar T(\bar z) = -\frac{1}{\alpha}\ :\partial_{\bar z}x^\mu\partial_{\bar z}x_\mu:\ . \qquad (2.72)
$$

[15] In particular, we can obtain the transformation law (2.120) for $T(z) = -\frac{1}{\alpha}\ :\partial_z x^\mu\partial_z x_\mu:$ by rewriting it as

$$
T(z) = \lim_{\epsilon\to 0}\left[-\frac{1}{\alpha}\,\partial_z x^\mu(z+\epsilon)\partial_z x_\mu + \frac{1}{\alpha}\langle\partial_z x^\mu(z+\epsilon)\partial_z x_\mu\rangle\right] \qquad (2.68)
$$

Let us consider two points z_1 and z_2 such that $z_1 - z_2 = \epsilon$. For a conformal transformation $z \to f(z)$ we see that $f(z_1) - f(z_2) = \eta$. We are concerned with the variation of the singular part

$$
\begin{aligned}
\frac{1}{\alpha}\langle\partial_z x^\mu(z+\epsilon)\partial_z x_\mu\rangle &= f'(z+\epsilon)f'(z)\frac{1}{\alpha}\frac{-\alpha\,D/2}{(f(z+\epsilon)-f(z))^2}\\
&= f'(z)\left(f'(z)+\epsilon\,f''(z)+\frac{\epsilon^2}{2}f'''(z)\right)\left(-\frac{D}{2}\frac{1}{(\epsilon\,f'(z)+\frac{\epsilon^2}{2}f''(z)+\frac{\epsilon^3}{6}f'''(z))^2}\right)\\
&= -\frac{D}{2}f'(z)\left(f'(z)+\epsilon\,f''(z)+\frac{\epsilon^2}{2}f'''(z)\right)\frac{\left(1+\frac{\epsilon}{2}\frac{f''(z)}{f'}+\frac{\epsilon^2}{6}\frac{f'''(z)}{f'(z)}\right)^{-2}}{\epsilon^2 f'(z)^2}\,,
\end{aligned}
\qquad (2.69)
$$

expanding the polynomial and taking the limit $\epsilon \to 0$ we get

$$
\frac{1}{\alpha}\lim_{\epsilon\to 0}\langle\partial_z x^\mu(z+\epsilon)\partial_z x_\mu\rangle = -\lim_{\epsilon\to 0}\frac{D/2}{\epsilon^2} = -\lim_{\eta\to 0}(f'(z))^2\left(\frac{D/2}{\eta^2}\right) - \frac{D}{12}\left(\frac{f'''(z)f'(z)+f''(z)^2}{f'(z)^2}\right)\,. \qquad (2.70)
$$

And taking into consideration the variation of the non-singular part of the transformation of (2.68) we

And the commutation relation of the mode expansion (2.61) are given by

$$\begin{aligned} [\alpha^\mu_m, \alpha^\nu_n] &= [\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_n] = m\delta_{m+n}\eta^{\mu\nu}\,, \\ [x^\mu, p^\nu] &= i\eta^{\mu\nu}\,. \end{aligned} \tag{2.73}$$

The representation for this algebra can be constructed by defining a ground state with momentum k^μ, $|0,k\rangle$ that are annihilated by lowering operators, that is, all the modes α^μ_n with $n > 0$. The complete Hilbert space of the theory is obtained by acting with the raising operators, the modes α^μ_n with $n < 0$, on the ground state.

2.1.8 Ghost system

We consider two holomorphic anti-commutating fields b and c with conformal weights $(\lambda, 0)$ and $(1-\lambda, 0)$ with the following action

$$S_g = \frac{1}{2\pi}\int d^2z\, b\,\partial_{\bar{z}}\, c\,. \tag{2.74}$$

We can see that the action is invariant under conformal transformations. The equations of motion for the fields of this system are given by

$$\bar{\partial} b = \bar{\partial} c = 0\,. \tag{2.75}$$

These equations tell us that the fields we are considering are indeed holomorphic, so we can now write down how they transform under infinitesimal conformal transformations.

$$\delta b(z) = \lambda\,\partial_z\epsilon\, b(z) + \epsilon\,\partial_z b(z)\,, \tag{2.76}$$

$$\delta c(z) = (1-\lambda)\,\partial_z\epsilon\, c(z) + \epsilon\,\partial_z c(z)\,. \tag{2.77}$$

---

obtain the variation of $T(z)$

$$T(z) \to f''(z)T(f(z)) + \frac{D}{12}\left[\frac{f'''(z)f'(z) + f''(z)^2}{f'(z)^2}\right] \tag{2.71}$$

We can use these transformations to apply Noether's procedure in order to find the energy-momentum tensor of the ghost system

$$\begin{aligned}
\delta S_g &= \frac{1}{2\pi}\int d^2z\,[\delta b\,\partial_{\bar z}c + b\,\partial_{\bar z}\delta c] \\
&= \frac{1}{2\pi}\int d^2z\;[(\lambda\,\partial_z\epsilon\,b + \epsilon\,\partial_z b)\,\partial_{\bar z}c + b\,\partial_{\bar z}((1-\lambda)\partial_z\epsilon c + \epsilon\,\partial_z c)] \\
&= \frac{1}{2\pi}\int d^2z\;[b(1-\lambda)\,\partial_z\partial_{\bar z}\epsilon\,c + \bar\epsilon\,b\partial_z c] \\
&= \frac{-1}{2\pi}\int d^2z\,[(1-\lambda)\,\partial_z b\,c - \lambda\,b\,\partial_z c]\,\partial_{\bar z}\epsilon\,.
\end{aligned} \tag{2.78}$$

hence,

$$T^g(z) = (1-\lambda) : \partial b\,c : -\lambda : b\,\partial c : \qquad , \tag{2.79}$$

We calculate the propagator for the bc CFT in order to compute relevant OPE's by proceeding as we did for the matter system .

$$\begin{aligned}
0 &= \frac{\delta}{\delta c(z)}\int \mathcal{D}\,b\,\mathcal{D}\,c\,[\exp(-S_g)\,c(w)] \\
&= \int \mathcal{D}\,b\,\mathcal{D}\,c\,\exp(-S_g)\left[-\frac{1}{2\pi}\,\partial_{\bar z}b(z)\,c(w) + \delta^2(z-w,\bar z-\bar w)\right]\,.
\end{aligned} \tag{2.80}$$

Since the equations above hold inside path integrals, we can write them as operator equations,

$$\partial_{\bar z}\langle b(z)\,c(w)\rangle = 2\pi\delta^2(z-w,\bar z-\bar w)\,. \tag{2.81}$$

Solving the equation above, we find the propagator for the bc CFT which is given by

$$\langle b(z)\,c(w)\rangle = \frac{1}{z-w}\,. \tag{2.82}$$

Now, it is straightforward to obtain the OPEs of T^g with the ghost fields

$$T^g(z)c(w) \sim (1-\lambda)\frac{c(w)}{(z-w)^2} + \frac{\partial c(w)}{(z-w)}\,, \tag{2.83}$$

$$T^g(z)b(w) \sim \lambda\frac{b(w)}{(z-w)^2} + \frac{\partial b(w)}{(z-w)}\,, \tag{2.84}$$

and the Laurent expansions of the ghost fields

$$b(z) = \sum_{m=-\infty}^{\infty}\frac{b_m}{z^{m+\lambda}}\,, \qquad c(z) = \sum_{m=-\infty}^{\infty}\frac{c_m}{z^{m+1-\lambda}}\,. \tag{2.85}$$

These modes satisfy

$$b_n = \oint \frac{dz}{2i\pi} z^{n+1} b(z)\,, \qquad c_n = \oint \frac{dz}{2i\pi} z^{n-2} c(z)\,. \tag{2.86}$$

By imposing analyticity on the $SL(2,\mathbb{C})$ invariant ghost vacuum $|0\rangle_{bc}$, the oscillators act as

$$\begin{aligned} b_n|0\rangle_{bc} &= 0 \qquad \text{for} \qquad n \geq -1\,, \\ c_n|0\rangle_{bc} &= 0 \qquad \text{for} \qquad n \geq 2\,. \end{aligned} \tag{2.87}$$

On the other hand, the algebra of the bc system can be obtained from the propagator (2.81)

$$\{b_m, c_n\} = \delta_{m,-n}\,. \tag{2.88}$$

We note that the zero modes b_0, c_0 satisfy the anti-commutation relations

$$\{b_0, c_0\} = 1\,, \qquad b_0^2 = c_0^2 = 0\,, \tag{2.89}$$

and they commute with the Hamiltonian L_0^g[16], therefore we have a degenerate ground state $|\downarrow\rangle$ and $|\uparrow\rangle$ satisfying

$$b_0\,|\downarrow\rangle = 0\,, \qquad b_0\,|\uparrow\rangle = |\downarrow\rangle\,, \tag{2.91}$$

$$c_0\,|\downarrow\rangle = |\uparrow\rangle\,, \qquad c_0\,|\uparrow\rangle = 0\,, \tag{2.92}$$

$$b_n\,|\downarrow\rangle = b_n\,|\uparrow\rangle = c_n\,|\downarrow\rangle = c_n\,|\uparrow\rangle = 0\,, \qquad n > 0\,. \tag{2.93}$$

The total spectrum is obtained by acting on theses states with the negative modes of the ghost fields. It is worthwhile noting that instead $|0\rangle_{bc}$ is the ground state of the Virasoro

[16] It clearly follows from the definition of the Virasoro modes (2.47). Explicitly, the commutation with the c_0 mode

$$\begin{aligned} [L_0^g, c_0] &= \oint_{\mathcal{C}_w} \frac{dz}{2i\pi} z \oint_{\mathcal{C}_0} \frac{dw}{2i\pi} w^{-\lambda} \left((1-\lambda)\frac{c(w)}{(z-w)^2} + \frac{\partial c(w)}{(z-w)} \right) \\ &= \oint_{\mathcal{C}_0} \frac{dw}{2i\pi} \left((1-\lambda)c(w)w^{-\lambda} + \partial c(w) w^{1-\lambda} \right) \\ &= \oint_{\mathcal{C}_0} \frac{dw}{2i\pi} \left((1-\lambda)c(w)w^{-\lambda} - (1-\lambda)c(w)w^{-\lambda} \right) = 0 \end{aligned} \tag{2.90}$$

The commutation relation with b_0 follows in the same way.

algebra, it is not the highest weight state of the bc algebra since it is not annihilated by all the the positive modes. It can be cured by noting that $b_{-1}\left|\downarrow\right\rangle$ satisfies (2.87), thus

$$|0\rangle_{bc} = b_{-1}\left|\downarrow\right\rangle \tag{2.94}$$

It follows that $c_1|0\rangle_{bc} = c_1 b_{-1}\left|\downarrow\right\rangle = \left|\downarrow\right\rangle + b_{-1}c_1\left|\downarrow\right\rangle$, and $\left|\downarrow\right\rangle$ can be represented by

$$\left|\downarrow\right\rangle = c_1|0\rangle_{bc} \tag{2.95}$$

and

$$\left|\uparrow\right\rangle = c_0 c_1|0\rangle_{bc} \tag{2.96}$$

Further, we can note that $\langle\downarrow|\downarrow\rangle = 0$ by inserting the commutator (2.89) equal to 1

$$\langle\downarrow|\downarrow\rangle = \langle 0 \mid c_{-1}(b_0 c_0 + b_0 c_0)c_1|0\rangle_{bc} = 0 \tag{2.97}$$

because of b_0 annihilates both $SL(2,\mathbb{C})$ vacua. We can obtain the same result for $\langle\uparrow|\uparrow\rangle = 0$. However, the value of $\langle\downarrow|\uparrow\rangle$ is different to zero and can be normalized such that

$$\langle 0 \mid c_{-1}c_0 c_1|0\rangle_{bc} = 1 \tag{2.98}$$

In this sense, $c_{-1}c_0$ is the dual of c_1.

2.2 Path integral quantization

The evolution of a string generates a two dimensional surface, which is known as the world-sheet, embedded in a target space. The incoming and outgoing strings indicate the merging of initial and the emergence of final states, possibly accompanied by creation and annihilation of virtual string pairs. By using conformal transformations, such a world-sheet can be mapped to a compact surface with a number n of points removed which correspond to external string states (punctures). The path integral quantization is performed by summing in the functional integral over all compact surfaces (Polyakov path integral)[7].

As it is well known, this integration contains an overcounting due to summation over configurations that are related by local symmetries of the action. We will solve this difficulty by calculating the correct integration measure on a particular slice, it means, by using the well known Faddeev-Popov method[8].

On the other hand, after gauge-fixing, the total action, which incorporates Faddeev-Popov ghosts, exhibits a new symmetry known as BRST, which imposes strong constraints over physical states and the amplitudes of the theory. For standard references see [4] [9]. The main objective of this section will be to quantize the Polyakov's action by the path integral method.

$$S = \frac{1}{4\pi\alpha'} \int_{\Sigma_g} d^2\sigma \sqrt{g} g^{ab} \partial_a x^\mu \partial_b x_\mu \tag{2.99}$$

where Σ_g is a Riemann surface with genus g. In this approach we consider as the two-dimensional quantum fields the embedding $x^\mu(\sigma_1, \sigma_2)$ in the Minkowski spacetime and the euclidean metric of the world-sheet $g_{ab}(\sigma_1, \sigma_2)$. The functional integral is computed by integrating over them. As we said, this summation is overvalueted because of local world-sheet symmetries of the action:

2.2.1 Classical local symmetries of the Polyakov's action

The Polyakov's action has the following world-sheet symmetries:
1)Diffeomorphism invariance

$$\sigma^a \to \sigma'^a(\sigma^a)\,, \qquad g_{ab}(\sigma) \to g'_{ab}(\sigma') = \frac{\partial\, \sigma^c}{\partial\, \sigma'^a} \frac{\partial \sigma^d}{\partial \sigma'^b} g_{cd}(\sigma)\,. \tag{2.100}$$

2)Weyl invariance

$$\sigma^a \to \sigma^a\,, \qquad g_{ab}(\sigma) \to \Omega(\sigma) g_{ab}(\sigma)\,, \tag{2.101}$$

Also, the Polyakov's action is invariant under D-dimensional Poincaré transformation

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu\,, \tag{2.102}$$

whit Λ^μ_νa Lorentz transformation and a^μ a translation. We have to note that diffeomorphism generated by continue vector fields: $\sigma'^a = \sigma^a + v^a$ form a subgroup of transformations, Diff_0, which produces the infinitesimal variation:

$$\delta\, x^\mu = v^a \partial_a x^\mu\,, \qquad \delta\, g_{ab} = \nabla_a v_b + \nabla_b v_a\,. \tag{2.103}$$

It is important to remark that a subgroup of diffeomorphisms does leave invariant the metric tensor up to a rescaling factor, producing a conformal transformation. This extra symmetry is still preserved after fixing the gauge (Conformal Symmetry).

2.2.2 The S-matrix

The on-shell scattering amplitude is given by

$$< V_{k_1}, ..., V_{k_n} >= \sum_{h=0}^{\infty} \int \frac{D g_{ab} D x^\mu}{N} e^{-S} \prod_{i=1}^{n} \mathcal{V}_i(k_i)\,. \tag{2.104}$$

In order to obtain the correct measure, we have divided the functional integral by the volume of the local symmetry group, $N = Vol(\mathit{Diff} \otimes \mathit{Weyl})$. The product of n Vertex Operators $\mathcal{V}_i$, represent incoming or outgoing strings joining on the world-sheet. Vertex operators for on-shell physical states must accomplish the symmetries of the full theory. In particular, Poincaré invariance give us crucial information in their construction. Space-time translation invariance requires that the x^μ dependence appears as an exponential factor $e^{ik_\mu x^\mu}$ and its derivatives. On the other hand, Lorentz invariance requires that all the space-time indices must be contracted with a polarization tensor. Then, vertex operators are of the form

$$\mathcal{V}_i(k_i, x^\mu) = P(\xi\,, Dx^\mu) e^{ik.x} \tag{2.105}$$

where ξ is the polarization vector and $P(\xi\,, Dx^\mu)$ is a polynomial expression in the derivatives of x^μ. We will say more about vertex operators in section 2.5. The total action S is defined as

$$S = S_P + \lambda\chi\,, \quad \chi = \frac{1}{4\pi}\int_{\Sigma_g} d^2\sigma\sqrt{g}R\,, \tag{2.106}$$

where λ is a coupling constant and χ is a diffeomorphism and Weyl invariant term which in two dimensions corresponds to the Euler number, $\chi = 2 - 2g - b$ for compact surfaces with genus g and b boundaries. Adding external string states corresponds to reduce the Euler number by 1, so the closed string coupling constant g_c should be proportional to e^λ. The functional measure Dx^μ and Dg_{ab} are defined by the inner product

$$\begin{aligned}\langle\delta\, x, \delta\, x\rangle &= \int d^2\sigma\sqrt{g}\delta\, x^\mu \delta\, x_\mu \\ \langle\delta\, g, \delta\, g\rangle &= \int d^2\sigma\sqrt{g} g^{ac} g^{bd} \delta\, g_{ab} \delta\, g_{cd}\,. \end{aligned} \tag{2.107}$$

The factor $N = Vol(\mathit{Diff} \otimes \mathit{Weyl})$ in the functional integral can be removed by restricting the integral to a gauge slice. The metric g_{ab}, as a symmetric tensor, possesses three degrees of freedom. On the other hand, there are three gauge functions, two reparametrizations and the local scaling of the metric, which are enough gauge freedom to fix the metric to a c-number and forget the integration over the metric. However, the Weyl scaling is a genuine symmetry of the classical action, we will see that quantum mechanically it is lost unless we fix the space-time dimension to 26. Nevertheless we can bring the metric locally to a flat metric in 26 dimensions, it is not correct globally if we are interested to describe surfaces with Euler characteristic different to zero. A convenient choice is to consider the effect of the diffeomorphism group alone and fix the metric to the unit form up to a Weyl rescaling. A good slice is the conformal class $[\hat{g}]$ of metrics conformal to some metric $\hat{g}_{ab}$

$$[g] = \{g_{ab}(\sigma) = e^{\phi(\sigma)}\hat{g}_{ab}\} \tag{2.108}$$

As we will see below, for surfaces with genus $g > 0$ one conformal class is not enough to obtain a correct global gauge slice. These inequivalent conformal classes $[\hat{g}_{ab}(m_1...m_k)]$ are parametrized by a finite number of variables $\{m_i\}$ called the *moduli*. This implies that any metric can be reached by the action of the local group of symmetries over the elements of the equivalence class [10]

$$g'_{ab}(\sigma', m_1, ..., m_k) = \exp(\phi(\sigma)) \frac{\partial \sigma^c}{\partial \sigma'^a} \frac{\partial \sigma^d}{\partial \sigma'^b} \hat{g}_{cd}(\sigma, m_1, ..., m_k) \tag{2.109}$$

Infinitesimal variations of the metric by Weyl and diffeomorphism transformations are given by

$$\delta g_{ab} = \delta\phi(\sigma)\hat{g}_{ab} + \nabla_a v_b + \nabla_b v_a \,.$$

These two variations are not orthogonal between them. As we said, they are related by the subgroup of conformal transformation. In order to factorize this, we split it in two orthogonal components, trace and traceless variations:

$$\begin{array}{l} \delta g_{ab} = \tilde{\phi}(\sigma)\hat{g}_{ab} + P_1(v)_{ab} \\ P_1(v)_{ab} = \nabla_a v_b + \nabla_b v_a - (\nabla.v)\hat{g}_{ab} \qquad \tilde{\phi}(\sigma) = \delta\phi(\sigma) + (\nabla.v))\hat{g}_{ab} \end{array}$$

These variations are associated with the symmetries of the action, this means, they connect configurations representing the same physical system. Furthermore, we can see there exist a variation δg^* on the metric space which is orthogonal to variations by Weyl and diffeomorphism

$$\begin{aligned} \langle \delta g^*, \delta g \rangle &= \langle \delta g^*, P_1(v)_{ab} \rangle + \langle \delta g^*, \tilde{\phi}(\sigma) \rangle & (2.110) \\ &= \langle P_1^\dagger \delta g^*, v \rangle = 0 & (2.111) \end{aligned}$$

then we can see by using (2.107) that $(P_1^\dagger \delta g^*)_a = -2\nabla^b \delta g^*_{ab}$. In order to have an orthogonal transformation between Image of P_1 and δg^*_{ab}, it must belong to the $Ker P_1^\dagger$:

$$\nabla^b \delta g^*_{ab} = 0 \tag{2.112}$$

These elements are called quadratic differentials. For the case of complex coordinates, quadratic differentials can be represented by $(2, 0)$ and $(0, 2)$ forms deforming the metric as

$$\phi_{zz} dz dz + \bar{\phi}_{\bar{z}\bar{z}} d\bar{z} d\bar{z} \tag{2.113}$$

such that $\partial_{\bar{z}}\phi_{zz} = \partial_z \bar{\phi}_{\bar{z}\bar{z}} = 0$, the so-called (anti)holomorphic quadratic differential which transform as $\partial_z \otimes \partial_z$ and $\partial_{\bar{z}} \otimes \partial_{\bar{z}}$ respectively under holomorphic change of variables. On

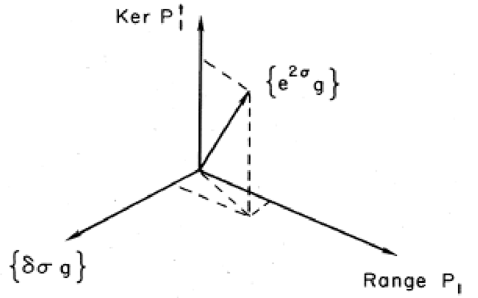

Figure 2.2: Orthogonal descomposition of $e^{2\sigma}\hat{g}$

the other hand, diffeomorphisms of vector fields that fall in the $KerP_1$ are responsible for the extra symmetry, which still remains after fixing the gauge

$$\nabla_a v_b + \nabla_b v_a = (\nabla . v)\hat{g}_{ab} \tag{2.114}$$

This means that any deformation of the metric can be factorized as

$$\{\delta\, g_{ab}\} = \{\delta\, \phi\, g_{ab}\} + \{RangeP_1\} + \{KerP_1^\dagger\} \tag{2.115}$$

The first two terms on the right hand side consist of modes that can be gauged away. Let us explain this better by defining M_g, the space of all the metrics defined on Σ_g, in which we are integrating. However, as we have seen, the integral should be factorized by the volume of the gauge group $N = Vol(Diff_0(\Sigma_g) \otimes Weyl(\Sigma_g))$, restricting the integral over deformations produced by the quadratic differentials. As we shall see in the next section, the Riemann-Roch theorem restricts the dimension of $KerP_1^\dagger$ by

$$dimKerP_1^\dagger = \begin{cases} 0 & h = 0 \\ 2 & h = 1 \\ 6h - 6 & h \geq 2 \end{cases} \tag{2.116}$$

So, we expect to reduce the functional integral to finite-dimensional integrals in every loop expansion. In order to get more insight into the nature of the metric space M_h, in which we are performing the functional integral, we define the Teichmuller Space as

$$T_h = \frac{M_h}{Diff_0(\Sigma_g) \otimes Weyl(\Sigma_g)} \tag{2.117}$$

as we can see from the above discussion, this space has finite dimension (it can be seen from the Riemann-Roch theorem), parametrized by 0, 1 and $3(h-1)$ complex coordinates

m^i for h equal to 0, 1 and $\geq$ 2 respectively. Then, every variation in the Metric Space are of the form $\delta\, m^i \partial_i g_{ab}$ (i running on the moduli coordinates). However, we do not want to gauge away only $Diff_0(\Sigma_g)$ but $Diff$. We define the Mapping Class Group,

$$MCG = \frac{Diff(\Sigma_g)}{Diff_0(\Sigma_g)}\,. \tag{2.118}$$

as the subgroup of diffeomorphisms not connected to the Identity. The space in which we are interested is Moduli Space $\mathcal{M}_h$ and defined as

$$\mathcal{M}_h = \frac{T_h}{MCG} \tag{2.119}$$

It is worthwhile to introduce the dual space to quadratic differentials. An arbitrary metric of a Riemann surface can be represented on a coordinate neighbourhood $U(x, y)$ by

$$ds^2 = A(x, y)dx^2 + B(x, y)dxdy + C(x, y)dy^2\,,$$

here, A, B and C are functions of x and y. Using complex coordinates, it takes the following form

$$ds^2 = \lambda(z, \bar{z})\,|dz + \mu\, d\bar{z}|^2 \tag{2.120}$$

where λ is a positive smooth function on U. On the other hand, it is known from Riemann theorem that, at least locally, we can obtain coordinate system $(w, \bar{w})$ in which the metric ds^2 can be written in isothermal form, $\tilde{\lambda} dw d\bar{w}$. Rewriting this in z language we have that

$$\begin{aligned} ds^2 = \tilde{\lambda}\,|dw|^2 &= |\partial_z w\, dz + \partial_{\bar{z}} w\, d\bar{z}|^2 & (2.121)\\ &= \tilde{\lambda}\,|\partial_z w|^2 \left|dz + \frac{\partial_{\bar{z}} w}{\partial_z w} d\bar{z}\right|^2 & (2.122)\\ &= \tilde{\lambda}\,|\partial_z w|^2\,|dz + \mu_{\bar{z}}^{z} d\bar{z}|^2 & (2.123)\end{aligned}$$

We conclude that if the coordinate system $(w, \bar{w})$ exists on U, it has to be solution of the Beltrami equation:[17]

$$\partial_{\bar{z}} w = \mu_{\bar{z}}^{z} \partial_z w \tag{2.124}$$

In this way, Beltrami differentials $\mu_{\bar{z}}^{z}$ parametrize variations of the metric due to conformal structure and diffeomorphism. Let $z \to w = z + v^z(z, \bar{z})$ be a reparametrization, then $dw = dz + \partial_z v^z\, dz + \partial_{\bar{z}} v^z\, d\bar{z}$, and it is easy to see that changes in the metric due to

[17] Where we have rewritten the μ of (2.120) in components to make clear that it transforms as $dz \otimes \partial_{\bar{z}}$ under holomorphic change of variables.

diffeomorphism come from $\mu_{\bar{z}}^{z} = \partial_{\bar{z}} v^{z}$.[18] Let us define the space of quadratic differentials as $Q(\Sigma_g)$ and the space of Beltrami differentials as $B(\Sigma_g)$. We have seen that deformations of the metric by diffeomorphism are orthogonal to variations respect to quadratic differentials, thus $Q(\Sigma_g)$ is orthogonal to $Diff_0(\Sigma_g)$. Thus, the dual space to $Q(\Sigma_g)$ is the space $B(\Sigma_g)/Diff_0(\Sigma_g)$ and can be taken as the tangent space of M_g, the set of harmonic Beltrami differentials.

For a genus g Riemann surface the complex dimension of those spaces are $3g-3$ and the harmonic Beltrami differentials are parametrized by $\delta\tau^k$ parameters ($k = 1, ..., 3g-3$) yields conformal deformations by acting in the j-th patch of Σ_g as

$$\delta\tau^k(\mu_{k\bar{z}_j}^{z_j} dz_j dz_j + \mu_{k z_j}^{\bar{z}_j} d\bar{z}_j d\bar{z}_j) \tag{2.126}$$

that can be related to local transformations in the j-th patch

$$z_j \to z_j + \delta\tau^k v_k^{z_j}(z_j, \bar{z}_j)\,, \tag{2.127}$$

where $v_k^{z_j}$ is defined only locally in the j-th patch. In this case, Beltrami coefficient takes the form of $\mu_{\bar{z}_j}^{z_j} = \partial_{\bar{z}_j} v_k^{z_j}$.

2.2.3 The measure for moduli

Going back to our initial problem, we will obtain the correct measure for the path integral by fixing the conformal gauge (2.108). and exposing the volume of the local group of symmetries. This is obtained by using the Faddeev-Popov prescription,

$$1 = \int D g_{ab} \delta(g_{ab} - \hat{g}_{ab}) \tag{2.128}$$

By gauge fixing, the functional integral over metrics and positions is converted to an integral over the gauge group and the moduli. For variations near to the identity,

$$1 = \triangle_{FP} \int d\, m_i D\delta\, v\, D\delta\phi \delta(\tilde{\phi}(\sigma)\hat{g}_{ab} + P_1(v)_{ab} + \delta\, m^i \partial_i g_{ab})\,. \tag{2.129}$$

[18]The explicit calculation gives us ,

$$\begin{aligned} ds^2 &= \tilde{\lambda}\,|dw|^2 = ds^2 = \tilde{\lambda}(dz + \partial_z v^z\, dz + \partial_{\bar{z}} v^z\, d\bar{z})(d\bar{z} + \partial_{\bar{z}} \bar{v}^{\bar{z}}\, d\bar{z} + \partial_z \bar{v}^{\bar{z}} dz) \\ &= \tilde{\lambda}\left\{dz d\bar{z}(1 + \partial_z v^z + \partial_{\bar{z}} v^z) + \partial_z v^{\bar{z}} dz dz + \partial_{\bar{z}} v^z d\bar{z} d\bar{z}\right\}\,. \end{aligned} \tag{2.125}$$

At this point we can rewrite the delta function by its integral representation introducing the symmetric tensor field β^{ab}

$$\begin{aligned}&\triangle_{FP}\int d\,m_i D\delta\,v\,D\delta\phi\,D\beta\exp(\langle\beta^{ab},\tilde{\phi}(\sigma)\hat{g}_{ab}+P_1(v)_{ab}+\delta\,m^i\partial_i g_{ab}\rangle)\\ =\;&\triangle_{FP}\int d\,m_i D\delta\,v\,D\beta\,\exp\left(\beta^{ab},(P_1 v)_{ab}+\delta\,m^i\partial_i g_{ab}\right)\delta(\langle\beta^{ab},\hat{g}_{ab}\rangle)\,,\end{aligned}$$

in the last step, we note that β^{ab} has to be a traceless field in order to get a non-zero integral. We have Faddev-Popov determinant

$$\triangle_{FP}^{-1}=\int dm_i D\delta\,v\,D\beta\exp(\langle\beta^{ab},(P_1 v)_{ab}+\delta\,m^i\partial_i g_{ab}\rangle)\,. \tag{2.130}$$

In order to obtain the value of $\triangle_{FP}$ it is possible to invert the path integral by promoting all bosonic variables to Grassmann variables as

$$\beta_{ab}\to\,b_{ab}\,,\qquad\delta\,v_a\to\,c_a\,,\qquad m_i\to\,\tau_i\,, \tag{2.131}$$

$$\begin{aligned}\triangle_{FP}&=\int d\tau_k D\,c\,D\,b\exp(\langle b^{ab},(P_1 c)_{ab}\rangle+\langle b^{ab},\tau^k\partial_k g_{ab}\rangle)\\ &=\int D\,c\,D\,b\exp(\langle b^{ab},(P_1 c)_{ab}\rangle)\prod_{k=1}^{\mu}\langle b^{ab},\partial_k g_{ab}\rangle\,,\end{aligned} \tag{2.132}$$

This is the appropriate measure for integration on the moduli space. The S-matrix takes the form

$$<V_{k_1},...,V_{k_n}>\;=\;\sum_{h=0}^{\infty}\int\frac{D\delta\,v\,D\delta\phi}{N}dm_k Dx^\mu Dc\,Db\exp(\langle b^{ab},(P_1 c)\rangle_{ab})\prod_{k=1}^{\mu}\langle b^{ab},\partial_k g_{ab}\rangle e^{-S}\prod_{i=1}^{n}\mathcal{V}_i(k_i)$$

As we can see, the integration over $\delta\,v$ and $\delta\phi$ produces the volume of local group of symmetries, the differomorphisms and Weyl transformations, and it is dropped out with the normalizing factor N. However we still have to deal with the unfixed residual symmetry due to CKV. We will fix completely these extra degrees of freedom by a gauge choice that fix the correct number of vertex operator coordinates. Thus,

$$<V_{k_1},...,V_{k_n}>=\sum_{h=0}^{\infty}\int dm_k Dx^\mu Dc\,Db\prod_{k=1}^{\mu}\langle b^{ab},\partial_k g_{ab}\rangle e^{-S-S_{bc}}\prod_{i=1}^{n}\mathcal{V}_i(k_i) \tag{2.133}$$

where,

$$S_{bc}=\frac{1}{2\pi}\int d^2\sigma\sqrt{g}\,b^{ab}(P_1 c)_{ab} \tag{2.134}$$

is the ghost system action.

2.2.4 U(1) current

By looking at the action of the bc system we see that it possesses the so called *ghost number* symmetry, that is, it is invariant under the following global transformations. [19]

$$b_{zz} \to e^{i\theta_z} b_{zz}\,, \qquad c^z \to e^{-i\theta_z} c^z. \tag{2.135}$$

We can use Noether's procedure to find the current associated with infinetismal $U(1)$ transformations

$$\begin{aligned}\delta S_g &= \frac{1}{2\pi}\int d^2z\,\left(\delta b\,\bar\partial c + b\,\bar\partial\delta c\right)\\ &= \frac{1}{2\pi}\int d^2z\,\left(i\theta\, b\,\bar\partial c - ib\,\theta\,\bar\partial c - ib\,\bar\partial\theta\, c\right)\\ &= \frac{i}{2\pi}\int d^2z\,(-:bc:)\,\bar\partial\theta\,. \end{aligned}\tag{2.136}$$

We define the *ghost* $U(1)$ *current* as follows

$$j = -:bc:\;. \tag{2.137}$$

Finally, let us compute some OPEs involving the ghost current defined above

$$j(z)b(w) = -:b(z)c(z):b(w) = -\frac{b(w)}{z-w}\,, \tag{2.138}$$

$$j(z)c(w) = -:b(z)c(z):c(w) = -\frac{c(w)}{z-w}\,. \tag{2.139}$$

There are two things to note about the quantum behaviour of the symmetries of the bc system. To visualize this, we remark that the measure is defined by the metric on the ghost field space

$$\|\delta\, c\|^2 = \int d^2z\,\sqrt{g}g_{z\bar z}c^z c^z\,; \qquad \|\delta\, b\|^2 = \int d^2z\sqrt{g}(g^{z\bar z})^2 b_{zz}b_{\bar z\bar z} \tag{2.140}$$

which are diffeomorphism invariant but not Weyl invariant, so we expect the Weyl symmetry to be anomalous. On the other hand, the metric (2.140) is only invariant by $U(1)$ if $\theta_z = \theta_{\bar z}$, therefore, this symmetry is expected to be anomalous quantum mechanically too.

[19] We know that the anti-ghost b_{ab} is g^{ab}-traceless. Thus, in complex coordiantes, their non-zero components are b_{zz} and $b_{\bar z\bar z}$. To simplify the notation we will use $b = b_{zz}$ and $\bar b = b_{\bar z\bar z}$ and the same for the components of the ghost field c^a, $c = c^z$ and $\bar c = c^{\bar z}$.

2.2.5 Weyl anomaly

Although, classically we expect the conformal current to be conserved, quantum mechanically $\nabla^a j_a$ could be different from zero. In general for a current charge $j(z)$ of weight $(1,1)$ we would like obtain the variation of

$$\nabla^a j_a = \nabla^z j_z + \nabla^{\bar{z}} j_{\bar{z}}, \tag{2.141}$$

under Weyl transformations produced by quantum effects. This variation must be proportional to a diffeomorphism invariant term because we expect that this symmetry is still preserved quantum mechanically. On the other hand, this term must be a scalar and vanishes in the flat case. It is clear that the candidate is the dimensionless scalar curvature R, thus

$$\nabla^a j_a \sim R \tag{2.142}$$

In order to compute exactly this variation we need to show some important results from the tensor calculus in complex cordinates. Firstly we take into account the fact that the only non-zero Christoffel symbols in complex coordinates are [11].

$$\Gamma^z_{zz} = g^{z\bar{z}} \partial_z g_{z\bar{z}}\,, \qquad \Gamma^{\bar{z}}_{\bar{z}\bar{z}} = g^{z\bar{z}} \partial_{\bar{z}} g_{z\bar{z}} \tag{2.143}$$

It follows directly that

$$\nabla^z j_z = g^{z\bar{z}} \nabla_{\bar{z}} j_z = g^{z\bar{z}} \partial_{\bar{z}} j_z \tag{2.144}$$

Analogous result we can obtain for $\nabla^{\bar{z}} j_{\bar{z}}$. Another important piece to calculate is the Ricci scalar. We start by noting that the Ricci tensor is given by

$$R_{\lambda\nu} = R^\mu_{\lambda\mu\nu} = \partial_\mu \Gamma^\mu_{\nu\lambda} - \partial_\nu \Gamma^\mu_{\mu\lambda} \tag{2.145}$$

In particular for the metric $g_{z\bar{z}} = g_{\bar{z}z} = e^\phi/2$ we have that

$$\Gamma^{\bar{z}}_{\bar{z}\bar{z}} = \partial_{\bar{z}} \phi\,, \qquad \Gamma^z_{zz} = \partial_z \phi\,, \tag{2.146}$$

and taking into account that the only non-zero components of the metric tensor are $g^{z\bar{z}}$ and $g^{\bar{z}z}$, we just need the value of the components $R_{z\bar{z}}$ and $R_{\bar{z}z}$ to calculate the Ricci scalar

$$R_{z\bar{z}} = R_{\bar{z}z} = -\partial_{\bar{z}} \Gamma^z_{zz} = -\partial_z \Gamma^{\bar{z}}_{\bar{z}\bar{z}} = -\partial_z \partial_{\bar{z}} \phi \tag{2.147}$$

Then

$$R = g^{\mu\nu} R_{\mu\nu} = -2 g^{z\bar{z}} \partial_z \partial_{\bar{z}} \phi = -4 e^{-\phi} \partial_z \partial_{\bar{z}} \phi \tag{2.148}$$

Under the Weyl scaling $g_{ab} = e^{\omega}\hat{g}_{ab}$, the covariant derivative ∇^z, relates to the original as

$$\nabla^z = g^{z\bar{z}}\nabla_{\bar{z}} = e^{-\omega}\hat{g}^{z\bar{z}}\partial_{\bar{z}} = e^{-\omega}\hat{\nabla}^z \,. \tag{2.149}$$

In the same way for the scalar curvature

$$\begin{aligned} R &= -g^{ab}\partial_b\Gamma^k_{ka} \\ &= -g^{ab}\partial_b(g^{\bar{\nu}k}\partial_k g_{a\bar{\nu}}) \\ &= -e^{-\omega}\hat{g}^{ab}\partial_b(e^{-\omega}\hat{g}^{\bar{\nu}k}\partial_k(e^{\omega}\hat{g}_{a\bar{\nu}})) \\ &= -e^{-\omega}\hat{g}^{ab}\partial_b(\hat{\Gamma}^{\bar{\nu}}_{a\bar{\nu}} + \partial_a\omega) \\ &= e^{-\omega}\hat{R} - 2e^{-\omega}\hat{g}^{z\bar{z}}\partial_{\bar{z}}\partial_z\omega \\ &= e^{-\omega}(\hat{R} - 2\hat{\nabla}\partial_z\omega) \end{aligned} \tag{2.150}$$

Now, we compute the variation of $\nabla^a j_a$ by Weyl rescaling

$$\begin{aligned} \nabla^a j_a &= e^{-\omega}\hat{\nabla}^a(\hat{j}_a + \delta_W j_a) \\ &= e^{-\omega}\hat{\nabla}^a\hat{j}_a + e^{-\omega}\hat{g}^{ab}\partial_b\delta_W j_a \end{aligned} \tag{2.151}$$

where $\hat{j}_a$ is the conformal current respect the metric $\hat{g}_{ab}$ and $\delta_W j_a$ is the variation of the current related to Weyl transformation. In order to obtain it, we consider the conformal transformation of a current charge of weight $(1,1)$

$$\begin{aligned} &= \frac{1}{2\pi\, i}\int dw\epsilon(w)T(w)j_z + \frac{1}{2\pi\, i}\int d\bar{w}\bar{\epsilon}(w)T(\bar{w})j_{\bar{z}} \\ &= \frac{1}{2}(Q\,\partial_z^2\epsilon(z) + \bar{Q}\,\partial_{\bar{z}}^2\bar{\epsilon}(\bar{z})) + \text{less singular terms} \end{aligned} \tag{2.152}$$

where Q is the coefficient of z^{-3} in the expansion Tj. In particular for the conformal current $J(z) = v(z)T(z)$, $Q = c/2$ (here c is the central charge).
The leading term in the above variation comes from Weyl transformations generated by the rescaling $e^{\omega(z,\bar{z})}$ and $\omega(z,\bar{z}) = \partial_z\epsilon + \partial_{\bar{z}}\bar{\epsilon}$. If $z \to z' = z + \epsilon(z)$ is a holomorphic transformation (analogous for $\bar{z}'$), then by analyticity $\partial_z\epsilon = \partial_{\bar{z}}\bar{\epsilon}$. The Weyl variation of $\nabla\,.j$ is

$$(\nabla\,.\delta_W j) = g^{z\bar{z}}\frac{1}{2}(c+\bar{c})\partial_{\bar{z}}\partial_z\omega(z,\bar{z}) \tag{2.153}$$

So we have that

$$\nabla^a j_a = e^{-\omega}\left(\hat{\nabla}^a\hat{j}_a + \frac{c+\bar{c}}{2}\hat{g}^{z\bar{z}}\partial_z\partial_{\bar{z}}\omega(z,\bar{z})\right) \tag{2.154}$$

Finally comparing term by term (2.154) with (2.150) we conclude that

$$\nabla^a j_a = -\frac{Q+\bar{Q}}{4} R\,. \tag{2.155}$$

As we can see, the conformal current is Weyl-anomaly free if we set the central charge to zero; it is achieved by demanding that the central charge of the total system $(S_x + S_{bc})$ vanishes. Therefore, we calculate de central charge of every system separately. For the matter system

$$\begin{aligned}
T(z)T(w) &= \frac{1}{\alpha'^2} :\partial_z x^\mu(z,\bar{z})\partial_z x_\mu(z,\bar{z}) :: \partial_w x^\mu(w,\bar{w})\partial_w x_\mu(w,\bar{w}) : \\
&= \frac{4}{\alpha'}\,\partial_w x^\mu(w,\bar{w})\partial_z x_\mu(z,\bar{z})\partial_z\partial_w \ln|z-w|^2 + \frac{1}{2}\frac{1}{(z-w)^4}\eta^\mu{}_\mu + \ldots \\
&= \frac{D/2}{(z-w)^4} - \frac{2}{\alpha'}\frac{1}{(z-w)^2} :\partial_w x^\mu(w,\bar{w})\partial_z x_\mu(z,\bar{z}) : + \ldots \\
&\sim \frac{D/2}{(z-w)^4} - \frac{2}{\alpha'}\frac{1}{(z-w)^2} :\partial_w x^\mu(w,\bar{w})\partial_w x_\mu(w,\bar{w}) : + \\
&\quad - \frac{2}{\alpha'}\frac{1}{(z-w)} :\partial_w x^\mu(w,\bar{w})\partial_w^2 x_\mu(w,\bar{w}) : \\
&\sim \frac{D/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}\,.
\end{aligned} \tag{2.156}$$

And for the bc system

$$\begin{aligned}
T^g(z)T^g(w) &= [(1-\lambda) :\partial b(z)c(z) : -\lambda : b(z)\partial c(z) :]\,[(1-\lambda) :\partial b(w)c(w) : -\lambda : b(w)\partial c(w) :] \\
&\sim (1-\lambda)^2\left[\frac{1}{(z-w)^4} - \frac{:\partial b(z)c(w):}{(z-w)^2} - \frac{:c(z)\partial b(w):}{(z-w)^2}\right] + \\
&\quad -\lambda(1-\lambda)\left[\frac{2}{(z-w)^3}\frac{1}{(z-w)} + \frac{:\partial b(z)\partial c(w):}{(z-w)} + 2\,\frac{:c(z)b(w):}{(z-w)^3}\right] + \\
&\quad -\lambda(1-\lambda)\left[\frac{1}{(z-w)}\frac{2}{(z-w)^3} + 2\,\frac{:b(z)c(w):}{(z-w)^3} + \frac{:\partial c(z)\partial b(w):}{(z-w)}\right] + \\
&\quad +\lambda^2\left[\frac{-1}{(z-w)^2}\frac{-1}{(z-w)^2} - \frac{:b(z)\partial c(w):}{(z-w)^2} - \frac{:\partial c(z)b(w):}{(z-w)^2}\right] \\
&\sim \frac{-(2\lambda-1)^2+1/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}\,.
\end{aligned} \tag{2.157}$$

Our case of interest is $\lambda = 2$ and we remove the anomaly term by fixing the space-time dimension to

$$D = 26 \tag{2.158}$$

that is the critical dimension for the bosonic string for which we can assume the Weyl symmetry holds as a quantum symmetry.

2.2.6 Riemann-Roch theorem

Now, we can use the last analysis for an anomalous current in order to obtain the path integral derivation of The Riemann-Roch theorem. From the equation of motion of the bc system, zero modes of c and b are related to CKV and quadratic differentials respectively and they should be related to a topological invariant value. We will visualize this by calculating the anomaly current $U(1)$. First of all, let us obtain the OPE,

$$\begin{aligned}
T_{bc}(w)j(z) &= \{(1-\lambda)\partial\, b\, c - \lambda\, b\partial\, c)(w)(-bc)(z)\} \\
&= \{\frac{1-\lambda}{(w-z)^3} + \frac{-\lambda}{(w-z)^3} + \frac{(1-\lambda)c(w)b(z)}{(w-z)^2} \\
&+ \frac{(-\lambda)b(w)c(z)}{(w-z)^2} + \frac{(\lambda)\partial\, c(w)b(z)}{w-z} - \frac{(1-\lambda)\partial\,\partial\, b(w)c(z)}{w-z}\} \\
&\sim \frac{1-2\lambda}{(w-z)^3} + \frac{1}{(w-z)^2}\left[(\lambda-1)b(z)c(z) - \lambda\, b(z)c(z)\right] \\
&+ \frac{1}{w-z}\left[(\lambda-1)b(z)\partial_z c(z) - \lambda\, b(z)\partial_z c(z)\right] \\
&+ \frac{1}{w-z}\left[\lambda\partial_z c(z)b(z) - (1-\lambda)\partial_z b(z)c(z)\right] + ... \\
&\sim \frac{1-2\lambda}{(w-z)^3} + \frac{-b(z)c(z)}{(w-z)^2} - \frac{\partial_z b(z)c(z) + b(z)\partial_z c(z)}{w-z} \\
&\sim \frac{1-2\lambda}{(w-z)^3} + \frac{j(z)}{(w-z)^2} + \frac{\partial_z j(z)}{w-z}\,. \qquad (2.159)
\end{aligned}$$

Then, using (2.155) we get

$$\nabla_a j^a = -\frac{Q}{2}R\,. \qquad (2.160)$$

The anomalous charge $Q = 1-2\lambda$, is $Q = -3$ for $\lambda = 2$.
On the other hand, by looking at the path integral of the ghost system with an arbitrary number of insertions μ and κ

$$< ... >_{bc} = \int Db\, Dc\, e^{-S_{bc}} \prod_{i=1}^{\mu} b_i \prod_{j=1}^{\kappa} c_j\,, \qquad (2.161)$$

we obtain the variation of the above expectation value respect $U(1)$ transformations,

$$\begin{aligned}
\delta_{U(1)} < ... >_{bc} &= \int Db\, Dc\, \delta_{U(1)}(e^{-S_{bc}} \prod_{i=1}^{\mu} b_i \prod_{j=1}^{\kappa} c_j) \\
&= \int Db\, Dc \left[(-\frac{i}{2\pi}\epsilon \int d^2z\bar{\partial} j(z)) e^{-S_{bc}} \prod_{i=1}^{\mu} b_i \prod_{j=1}^{\kappa} c_j + e^{-S_{bc}} \delta_{U(1)}(\prod_{i=1}^{\mu} b_i \prod_{j=1}^{\kappa} c_j) \right].
\end{aligned}$$

The variation of the last term on the right hand side can be computed explicitly

$$\begin{aligned}\delta_{U(1)}(\prod_{i=1}^{\mu} b_i \prod_{j=1}^{\kappa} c_j) &= \prod_{i=1}^{\mu} b_i e^{in\epsilon} e^{-im\epsilon} \prod_{j=1}^{\kappa} c_j - (\prod_{i=1}^{\mu} b_i \prod_{j=1}^{\kappa} c_j) \\ &= i\epsilon(\mu-\kappa)(\prod_{i=1}^{\mu} b_i \prod_{j=1}^{\kappa} c_j)\,, \end{aligned} \tag{2.162}$$

and we conclude that

$$\delta_{U(1)} < ... >_{bc} = \int Db\, Dc\, e^{-S_{bc}} \prod_{i=1}^{\mu} b_i \prod_{j=1}^{\kappa} c_j \left((-\frac{i}{2\pi}\epsilon \int d^2z \bar{\partial} j(z)) + i\epsilon(\mu-\kappa) \right) . \tag{2.163}$$

Taking into account the anomaly (2.155) and the Gauss-Bonnet theorem,we obtain the condition

$$\mu - \kappa = \frac{3}{2}\chi\,, \tag{2.164}$$

which must hold for $< ... >_{bc}$ be different from zero. It is easy too see where this constraint comes from; the ghost action does not include the zero modes of c and b,

$$\langle\, b(z), P_1 c_{oj} \rangle = \langle P_1^\dagger b_{oi}, c(z) \rangle = 0\,. \tag{2.165}$$

Fermionic integration of the μ and κ ghost zero-modes vanishes unless we insert exactly μ and κ ghost fields b_i and c_j respectively. Because of this, we can write our version of the Riemann-Roch theorem as

$$\dim KerP_1 - \dim KerP_1^\dagger = 3(1-g) \tag{2.166}$$

2.2.7 Measure of integration as a determinant

After being defined the property that relates the zero modes of the ghost fields in order to get a non-zero amplitude, we can explicitly calculate the measure. We express the ghost fields as expanded by a complete (bosonic) set of basis as

$$c^z = \sum_I c_I C_I^z, \qquad b_{zz} = \sum_J b_J B_{Jzz}\,. \tag{2.167}$$

Rewriting the ghost action in these terms

$$\begin{aligned} &\frac{1}{2\pi}\int d^2z\, \sqrt{g} g^{z\bar{z}} b_{zz} P_1 c^z \\ =\; &\frac{1}{2\pi}\sum_{I,J} c_I b_J \int d^2z\, \sqrt{g} g^{z\bar{z}} B_{Jzz} P_1 C_I^z\,, \end{aligned} \tag{2.168}$$

it is obvious that we can not diagonalize $P_1 : S^n \to S^{n+1}$ but $P_1 P^\dagger$. Then, we can expand the ghost field b as

$$\sum_{J\neq 0} b_J P_1 C_{z,I} + \sum_{K=1}^{\mu} b_{0K} B_{K0zz}\,, \tag{2.169}$$

where B_{K0} describes the μ zero modes of $P_1^\dagger$. The Faddeev-Popov measure is written as

$$\int \prod_{K=0}^{\mu} b_{0K} \prod_{j=1}^{\kappa} c_{0j} \prod_{J\neq 0} b_J \prod_{I\neq 0} c_I \prod_{K=1}^{\mu} \langle b_K, \mu\rangle \prod_{j=1}^{\kappa} c_i \exp(-\frac{1}{2\pi}\langle b, P_1 c\rangle) \tag{2.170}$$

The integration over Grassmann modes yields a nonzero result only if the integrand saturates the number of Grassmann modes. The action does not include zero modes and they must come from enough number of ghost insertions to give a non-zero result. Considering (2.169) the exponential part of the measure can be written as

$$\begin{aligned} & \exp\left(-\frac{1}{2\pi}(\sum_{I,J\neq 0} c_I b_J \langle P_1 C_{z\ J}, P_1 C_I^z\rangle + \sum_{I,K} c_I b_{0K} \langle B_{K0zz}, P_1 C_I^z\rangle)\right) \\ = \;& \exp\left(-\frac{1}{2\pi}(\sum_{I,J\neq 0} c_I b_J \langle C_{z\ J}, P_1^\dagger P_1 C_I^z\rangle + \sum_{I,K} c_I b_{0K} \langle P_1^\dagger B_{K0zz}, C^{z,I}\rangle)\right) \\ = \;& \exp\left(-\frac{1}{2\pi}(\sum_{I,J\neq 0} c_I b_J \langle C_{z\ J}, P_1^\dagger P_1 C_I^z\rangle)\right) \end{aligned} \tag{2.171}$$

It is clear that only the zero modes part of the ghost insertions take part in the final answer. The contribution from the non-zero modes is a gaussian integral and can be computed at this stage,

$$det' \frac{(P_1^\dagger P_1)^{\frac{1}{2}}}{2\pi} \int \prod_{k=0}^{\mu} db_{0k} \prod_{j=1}^{\kappa} dc_{0j} \prod_{k=1}^{\mu} \sum_{k=1}^{\mu} b_{0k} \langle B_{0k}, \mu_k\rangle \prod_{i=1}^{\kappa} \sum_{i=1}^{\mu} c_{0i} C_{0i} \tag{2.172}$$

We calculate explicitly the integration over b_0 modes and the c_0 contribution can be obtained in a similar way

$$\begin{aligned} & \sum_{k''=1}^{\mu} \int \prod_{k=0}^{\mu} db_{0k} \prod_{k'=1}^{\mu} b_{0k'} \langle B_{0k''}, \mu_{k'}\rangle \\ = \;& \sum_{k''=1}^{\mu} \varepsilon_{k'_{01}\ldots k'_{0\mu}} \langle B_{0k''}, \mu_{k'}\rangle \\ = \;& Det\langle B_{0k''}, \mu_{k'}\rangle \end{aligned} \tag{2.173}$$

Putting all together, the correct measure for the bosonic string is [20]

$$\triangle_{FP} = Det' \frac{(P_1^\dagger P_1)^{\frac{1}{2}}}{2\pi}\, Det\langle B_{0k}, \mu_{k'}\rangle\, Det C_{0i} \tag{2.174}$$

2.3 BRST invariance

In the section 2.2 we reduced the functional integral of the Polyakov action to a finite dimensional integral over the moduli space of an combined action $S = S_P + S_{gh}$, in the conformal gauge,

$$S = \frac{1}{2\pi\alpha'} \int d^2z\, \partial X^\mu \bar{\partial} X_\mu + \frac{1}{2\pi} \int d^2z \left(b\, \bar{\partial} c + \bar{b}\, \partial \bar{c} \right) . \tag{2.175}$$

For this action the conformal anomaly for the total stress tensor vanishes in the critical dimension, as we demonstrated for

$$T(z) = T^x(z) + T^g(z)\,, \tag{2.176}$$

As it is usual for theories with local gauge symmetries, the total action, $S = S_P + S_{gh}$, exhibits at classical level, a femionic symmetry, known as BRST symmetry. Classicaly, the BRST charge Q_{BRST}, which produces the desired transformations, is a fermionic nilpotent operator, $Q^2 = 0$. The general form of the BRST operator is:

$$C_I(G_I^x + \frac{1}{2} G_I^{gh}) \tag{2.177}$$

where, G_I^x and G_I^g are the constraints associated with the gauge symmetries of the matter and ghost system respectively. Following this line of reasoning, we define:

$$Q_{BRST} = \frac{1}{2\pi\, i} \oint (dz\, j_{BRST} - d\bar{z} \bar{j}_{BRST}) \tag{2.178}$$

where,

$$\begin{aligned} j_{BRST} &= c(T^x(z)) + \frac{1}{2} T^{gh}(z)) + \frac{3}{2} \partial_z^2 c \\ &= -\frac{1}{2} c \partial_z x^\mu \partial_z x_\mu + bc\partial_z c + \frac{3}{2} \partial_z^2 c\,. \end{aligned} \tag{2.179}$$

The last term is a total derivate and does not contribute to the BRST charge; it has been added for convenience, so that j_{BRST} transforms as a primary field of weight $(1, 0)$.

[20]The prime in the means the omission of zero modes

If j_{BRST} is a holomorphic current that transform covariantely, it follows straightforward that Q_{BRST} is a conserved charge. BRST charge acts on the fields as

$$[Q_{BRST}, O(z)]_{\pm} = \oint \frac{dz}{2\pi\, i} j_{BRST}(w) O(z) \tag{2.180}$$

where the commutattion or anticommutation relation depends of the femionic or bosonic nature of $O(z)$. We compute the BRST transformation of the fields by using their operator product expansions with (2.179).

$$\begin{aligned} [Q_{BRST}, x^{\mu}(z)] &= \oint \frac{dw}{2\pi\, i} c^w \frac{\partial_w x^{\mu}(w)}{w-z} && (2.181)\\ &= c^z \partial_z x^{\mu}(z) && (2.182)\end{aligned}$$

$$\begin{aligned} [Q_{BRST}, c^z]_+ &= \oint \frac{dw}{2\pi\, i} b(w)c(w)\partial\, c(w) + \frac{3}{2}\partial^2 c(w))c(z) && (2.183)\\ &= \oint \frac{dw}{2\pi\, i} \frac{c(w)\partial\, c(w)}{w-z} && (2.184)\\ &= c^z \partial_z\, c^z && (2.185)\end{aligned}$$

$$\begin{aligned} [Q_{BRST}, b_{zz}]_+ &= \oint \frac{dw}{2\pi i} j_w(w) b(z) && (2.186)\\ &= \oint \frac{dw}{2\pi i}[\frac{T^{x+g}}{w-z} + \frac{j^g}{(w-z)^2} + \frac{3}{(w-z)^3} +]\\ &= T^{x+g}(z) \\ & && (2.187)\end{aligned}$$

In the same way we obtained the transformations of the antiholomorphic fields.
We compute the following anti-commutator which will be useful for our computations,

$$\begin{aligned} \{Q_{BRST}, b_m\} &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \{j_B(z), b(w)\}\, w^{m+1}\\ &= \oint \frac{dw}{2\pi i}\, w^{m+1} \oint \frac{dz}{2\pi i} \left[\frac{T^{x+g}}{z-w} + \frac{3}{(z-w)^3} + \frac{j^g}{(z-w)^2}\right]\\ &= \oint \frac{dw}{2\pi i}\, w^{m+1} T^{x+g}(w)\\ &= L^x_m + L^g_m\,. && (2.188)\end{aligned}$$

As we saw earlier, there is an anomaly in the gauge symmetry when $c^x \neq 26$, so we should expect a breakdown in the BRST symmetry. It turns out that the BRST charge Q_{BRST} is not nilpotent unless $c^X = 26$

$$\{Q_{BRST}, Q_{BRST}\} = 0 \quad \text{only if} \quad c^X = 26\,. \tag{2.189}$$

The prove of the BRST nilpotence will be performed in two steps, by makings use of the Jacobi Identity. First at all, we will show that Q_{BRST} is conformal invariant only for the critical dimension.

$$\begin{gathered}\{[Q_{BRST}, L_m], b_n\} - \{[L_m, b_n], Q_{BRST}\} - [\{bn, Q_{BRST}\}, L_m] = 0 \\ \{[Q_{BRST}, L_m], b_n\} - \{(m-n)b_{m+n}, Q_{BRST}\} - [L_n, L_m] \\ \{[Q_{BRST}, L_m], b_n\} - (m-n)L_{m+n} - (m-n)L_{m+n} - \frac{D/2-13}{12}m(m^2-1)\delta_{m+n,0} \\ \{[Q_{BRST}, L_m], b_n\} - \frac{\frac{D}{2}-13}{12}m(m^2-1)\delta_{m+n,0} = 0 \end{gathered} \tag{2.190}$$

We can see the bracket is equal to zero when the total central charge vanishes (critical dimension). The anticommutation of Q_{BRST} with b_n implies that the left hand side of the bracket does not have c_m modes, however $[Q_{BRST}, L_m]$ possess ghost number 1 !, then, we can conclude that Q_{BRST} is conformal invariant only for $D = 26$. Now we see the implication of this outcome in the nilpotence property of Q_{BRST}.

$$\begin{gathered}[\{Q_{BRST}, b_n\}, Q_{BRST}] + [\{b_n, Q_{BRST}\}, Q_{BRST}] + [\{Q_{BRST}, Q_{BRST}\}, b_n] = 0 \\ [\{Q_{BRST}, Q_{BRST}\}, b_n] = -2\,[L_n, Q_{BRST}] \end{gathered} \tag{2.191}$$

By using the same argument, we demonstrate that Q_{BRST} is nilpotent if the right hand of the equality is zero. This implies the central charge of the full theory to be zero. It means that quantum mechanically $Q^2_{BRST} = 0$ holds only for $D = 26$.

- Physical States

  The BRST formalism give us useful information about the unitarity of scattering amplitudes. Let us take a look at the Hilbert space of the complete theory

$$\mathcal{H}_{string} = \mathcal{H}_x \otimes \mathcal{H}_{b,c} \otimes \mathcal{H}_{\bar{b},\bar{c}}\,. \tag{2.192}$$

  We expect that physical states of the theory must be BRST invariant,

$$Q_{BRST}|phys\rangle = 0\,, \tag{2.193}$$

  however, we have some ambiguity due to the nilpotence of Q_{BRST}. States of this form are spurious

$$\langle\psi|(Q_{BRST}|\chi\rangle) = (\langle\psi|Q_B^\dagger)|\chi\rangle = 0\,. \tag{2.194}$$

  These kind of states are called BRST exact and they are expected to decouple from physical processes. Therefore, physical states are represented as elements of the coset

$$\frac{Ker\,Q_{BRST}}{Image\,Q_{BRST}}\,. \tag{2.195}$$

Moreover the BRST invariance condition (2.193), physical states must satisfy the additional condition

$$b_0|phys\rangle = \bar{b}_0|phys\rangle = 0 \tag{2.196}$$

which implies, by using (2.188), the mass-shell condition

$$L_0|phys\rangle = \bar{L}_0|phys\rangle = 0\,, \tag{2.197}$$

In the Polyakov path integral formulation, exact states are represented by vertex operators of the form $[Q_{BRST}, O(z)]$ and unitarity is only guaranteed if these states decouple. We return at this point in the next section after putting the moduli space of the metric and the position of the vertex operator insertions on a more equal footing.

2.3.1 Vertex operators

Until now we have reduced the g-level scattering amplitude to an integral over the moduli space. What we want to do in this section is to look at the g-level scattering amplitude with n vertex operator insertions. The moduli space that parametrizes a Riemann surface of genus g with n punctures, $M_{g,n}$ has dimension $6g-6+2n$, the degrees of freedom over which we have to integrate.

The main result obtained in this section holds for $g \geq 2$ and some few words about the cases $g = 0, 1$ will be said at the end of the exposition. In our discussion we will make use of the simpler type of Unintegrated Vertex Operators, the BRST invariant vertex of the form $\mathcal{V} = c\bar{c}V$ where V is a $(1,1)$ primary field built only with matter fields. Indeed $\mathcal{V}$ is a prymary field of weight $(0,0)$, for the total Virasoro algebra, so, $[L_n, \mathcal{V}] = 0$ for $n \geq 0$ (Conformal Vertex Operator). As one may have realized, equation (2.166) tell us that only amplitudes satisfying our formulation of the Riemann-Roch theorem survive, therefore, every insertion of ghost fields c,coming from the n unintegrated Vertex operators, has to be counterbalanced with n extra insertions of the b anti-ghost. For the g level scattering amplitude, we write

$$\begin{aligned} < V_{k_1}, ..., V_{k_n} >_g = \prod_{k=1}^{3g-3} \int d^2 m_k \prod_{i=1}^{n} \int d^2 w_i \sqrt{g} \int Dx^\mu Dc\, Db D\bar{c}\, D\bar{b} \prod_{k=1}^{3g-3} \langle\, b, \mu_k \rangle^2 \\ \prod_{k=i}^{n} \langle\, b, \mu_i \rangle^2 \times e^{-\mathcal{S}} \prod_{i=1}^{n} c_i(w_i)\bar{c}_i(\bar{w}_i) V_i(w_i, \bar{w}_i)\,. \end{aligned} \tag{2.198}$$

Just to simplify the calculus, let us focus on a single vertex operator insertion. The final result can be generalized for the n point function. A single insertion introduce the factors

$$\begin{aligned} & \int d^2w_i\sqrt{g}\left(\int d^2z\,\sqrt{g}g^{z\bar{z}}\mu^z_{i\bar{z}}b_{zz}\right)c(w_i)V_i(w_i) \\ = & \sum_{j=1}\int d^2w_i\sqrt{g}\left(\oint_{C_j} d^2z_j\,\sqrt{g}\partial_{\bar{z}_j j}v^{z_j}_k b_{z_j z_j}\right)c(w_i)V_i(w_i) \\ = & \int d^2w_i\sqrt{g}\sum_{j=1}\left(\oint_{C_j} dz\,b_{zz}v^{z_j}_k\right)c(w_i)V_i(w_i)\,. \end{aligned} \tag{2.199}$$

In the first line, we made use of the fact that conformal deformations can be achieved by the action of discontinous vector fields as stated in (2.127). So, we decompose the integral over Riemann surface as summations of the integrations over patchs in which v_j is defined. In the second line we integrated and used the equation of motion for b. Now we have to define what v_j is for this case. To accomplish this, we define in every j-patch the local coordinates z'_j such that, the vertex insertions $V(z = w_i, \bar{z} = \bar{w}_i)$ be inserted in the origin of z'_j, that is, $V(z'_j = 0, \bar{z}'_j = 0)$. The local j-coordinate system can be defined as

$$z'_j = z_j - w_j \tag{2.200}$$

Now we can compare this new coordinates with (2.127); w_i parametrize modular deformations and $v^{z_j} = 1$. Hence, equation (2.199) simplifies to

$$\begin{aligned} & \int d^2w_i\sqrt{g}\sum_{j=1}\left(\oint_{C_j} dz\,b_{zz}c^{w_i}V_i(w_i,\bar{w}_i)\right) \\ = & \int d^2w_i\sqrt{g}V_i(w_i,\bar{w}_i)\,. \end{aligned} \tag{2.201}$$

This result shows that the S-matrix with unintegrad vertex insertions can be can be calculated, as well, with Integrated Vertex Operators (2.201), which are very useful in scattering amplitude computations. This result is in general true. In particular, the 3 point function for $g = 0$ and one point function for $g = 1$ can not be obtained by using this relation because the Riemann-Roch theorem does not allow us to have b insertions to effectuate this manipulation.

2.3.2 Decoupling spurious states

Having obtained enough information about the vertex operators, we go back to our previous scenario, the gauge invariance of the S-matrix, it means, the decoupling of exact

BRST states from the amplitudes, which are created by Vertex operators of the form $\mathcal{W}_1(z) = [Q_{BRST}, \mathcal{V}_1(z)]$. We consider the g-order scattering amplitude for one exact BRST vertex operator and $n-1$ physical fields,

$$
\begin{aligned}
& \prod_{i=1}^{n} \int \sqrt{g} d^2 w_i \int [\ldots] \prod_{i=i}^{3g-3+n} \langle b, \mu_j \rangle e^{-S} \mathcal{W}_1(z) \prod_{i=2}^{n} \mathcal{V}_i(w_i) \\
= & \prod_{i=1}^{n} \int d^2 w_i \sqrt{g} \int [\ldots] e^{-S} \oint_{C_{z_1 \ldots z_n}} j(z) \prod_{i=i}^{3g-3} \langle\, b, \mu_j \rangle \prod_{i=1}^{n} V_i(w_i, \bar{w}_i) \\
= & \prod_{i=1}^{n} \int d^2 w_i \sqrt{g} \int [\ldots] e^{-S} \sum_{j=1}^{n} \oint_{C_{z_1 \ldots z_n}} \prod_{k \neq j}^{3g-3} \langle\, b, \mu_k \rangle \, \langle T^{x+g}, \mu_j \rangle \prod_{i=1}^{n} V_i(w_i, \bar{w}_i) \\
= & \prod_{i=1}^{n} \int d^2 w_i \sum_{j=1}^{n} \frac{\partial}{\partial\, m_j} \left(\int [\ldots] \prod_{k \neq j}^{3g-3} \langle\, b, \mu_k \rangle e^{-S} \prod_{i=1}^{n} V_i(w_i, \bar{w}_i) \right) ,
\end{aligned}
$$

where [...] represents the measure

$$
[...] = \int \prod_{j=1}^{3g-3} d^2 m_j Db\, Dc\, Dx^{\mu} \,. \tag{2.202}
$$

In the first line we wrote the BRST-exact operator as a contour integral and in second line we deformed this contour away from the insertions and pull it off the world-sheet. The $T(z)$ insertion produces a total derivative respect the moduli m_j. Now we might be tempted to apply Stoke's theorem to get a vanishing result for the integral. However, we have to go carefully due to possible singularities which could arise from the boundary of the moduli space. The origin of possible anomalies is related with poles that appear when an intermediate on-shell state propagates for a long time (infrared divergences). We return to this subject in section 2.5 after revisiting the tree level four-point function.

2.4 Tree level amplitudes

In this section we use the techniques we developed earlier to calculate tree-level amplitudes. In particular, the scattering amplitude for four tachyons of the closed string (Virasoro- Shapiro amplitude) As we saw, in the path integral formulation, states are represented by conformal vertex operators, which are built for ghost and matter fields ($c\bar{c}V(z,\bar{z})$). The matter part is a $(1,1)$ conformal field. Vertex operators produces, by acting on the ground state of the string, excited states with momentum k^{μ}. This ground state $|\,k,0,0\rangle \cong e^{ik^\mu x_\mu}\,|0,n=(0,0)\rangle$ corresponds to a tachyonic state. The complete set of states are generated by the action of the raising operators on the vacuum. In particular for the first excited state of the closed string $\alpha_{-1}\bar{\alpha}_{-1}\,|\,k,0,0\rangle$; being α_n^μ the coefficients of the expansion of the $(1,0)$ conformal field $\partial_z x^\mu$

$$\partial_z x^\mu = \sum_{n=1} \frac{\alpha_n^\mu}{z^{n+1}}\,, \qquad \alpha_n^\mu = \int dz\, z^n \partial_z x^\mu\,, \tag{2.203}$$

the procedure is analogous for the antichiral part. So we expect the first excited state be represented by

$$|\,k,1,1\rangle \cong \xi_{\mu\nu}\partial_z x^\mu \partial_{\bar{z}} x^\nu e^{ik^\mu x_\mu}\,|0,n=(0,0)\rangle\;, \tag{2.204}$$

where $\xi_{\mu\nu}$ is a polarization vector. Thus, we can assume that the matter part of vertex operators $V(k^\mu, x^\mu)$ is a polynomial expression in derivatives of x^μ and their polarizations.

Apart from matter fields insertions we will have to deal with ghost insertions, so we will proceed to calculate each contribution separately.

For our propose, we will calculate the scalar expectation value for tachyon-vertex operator insertions.

$$\begin{aligned}
&\langle \prod_{i=1}^n : exp\,(ik_i^\mu x_\mu(\sigma_i,\bar{\sigma}_i)) :\rangle \\
=& \int Dx^\mu : exp\left(i\sum_{i=1}^n k_i^\mu x_\mu(\sigma_i,\bar{\sigma}_i)\right) : exp\left(-\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{ab}\delta_a x^\mu \delta_{\bar{b}} x_\mu\right) \\
=& \int Dx^\mu exp\left(\frac{1}{4\pi\alpha'}\int d^2\sigma\, x^\mu \nabla^2 x_\mu + i\sum_{i=i}^p \int d^2\sigma k^\mu x_\mu(\sigma,\bar{\sigma})\delta^2(\sigma_i,\bar{\sigma}_i)\right).
\end{aligned} \tag{2.205}$$

Expanding x^μ in a complete set of orthonormal basis X_I such that

$$\begin{aligned}
x^\mu &= \sum_{I=1} x_I^\mu X_I \\
\nabla^2 X_I &= -\omega_I^2 x_I \\
\int d^2\sigma\sqrt{g}X_I X_J &= \delta_{IJ}\,,
\end{aligned} \tag{2.206}$$

it is not difficult to obtain the generator function

$$Z[J] = i(2\pi)^{26}\delta^{26}(\sum_i k_i)\prod_{I\neq 0}(\frac{4\pi^2\alpha}{\omega_I^2})^{13} exp\left(-\frac{1}{2}\sum_{i,j} k_i.k_j G'(\sigma_i,\sigma_j)\right)\,, \tag{2.207}$$

where

$$G'(\sigma_i,\sigma_j)=\sum_{I\neq 0}\frac{2\pi\alpha}{\omega_I^2}X_I(\sigma_i)X_I(\sigma_j)\,. \tag{2.208}$$

Using (2.206) we have that

$$-\frac{1}{2\pi\alpha}\nabla\, G'(\sigma_i,\sigma_j)=\sum_{I\neq 0}X_I(\sigma_i)X_I(\sigma_j)\,, \tag{2.209}$$

and the completeness of X_I tell us that

$$\begin{aligned}\delta_{KJ} &= \sum_I \delta_{KI}\delta_{IJ}\\ &= \sum_I \int d^2\sigma_1 g^{\frac{1}{2}}X_K(\sigma_1)X_I(\sigma_1)\int d^2\sigma_2 g^{\frac{1}{2}}X_I(\sigma_2)X_J(\sigma_2)\\ &= \int d^2\sigma_1 g^{\frac{1}{2}}X_K(\sigma_1)\int d^2\sigma_2 g^{\frac{1}{2}}X_J(\sigma_2)(\sum_I X_I(\sigma_1)X_I(\sigma_2))\,, \end{aligned}\tag{2.210}$$

from the equation above, we note that

$$\sum_I X_I(\sigma_1)X_I(\sigma_2)=\delta\left(\sigma_1-\sigma_2\right), \tag{2.211}$$

must hold, then

$$-\frac{1}{2\pi\alpha}\nabla\, G'(\sigma_1,\sigma_2)=g^{-\frac{1}{2}}\delta\left(\sigma_1-\sigma_2\right)-X_0^2\,. \tag{2.212}$$

For complex coordinates $g_{z\bar{z}}=e^{2\omega(z,\bar{z})}$, hence, $\nabla^2 G'(z,\bar{z})=e^{-2\omega(z,\bar{z})}G'(z,\bar{z})$ and (2.209) stays as

$$\partial_z\partial_{\bar{z}}\, G'(z_1,z_2)=-\pi\alpha\left(\delta\left(\sigma_1-\sigma_2\right)-e^{2\omega(z,\bar{z})}X_0^2\right)$$

We can provide a solution for this differential equation by solving term by term. First of all we expect $G'(z_1,z_2)$ to be symmetric in world-sheet coordinates. The delta function produces, as it is expected, a logarithmic solution $(-\frac{\alpha}{2}\ln|z'-z_i|^2)$. The remaining term $f(z_i,z_j)$ has to be symmetric in z_i and z_j and obtained such that $\partial_{z_i}\partial_{\bar{z}_i}g(z_i,z_j)\sim e^{2\omega(z_i,\bar{z}_j)}X_0^2$. Now we can easily see that $g(z_i,z_j)$ splits in two functions $f(z_i)$ and $f(z_j)$ such that

$$\partial_{z_i}\partial_{\bar{z}_i}f(z_i,\bar{z}_i)=\pi\alpha\, X_0^2\int d^2z' e^{2\omega(z',\bar{z}')}\delta^2(z'-z_i)$$

the solution for this equation gives

$$f(z_i,\bar{z}_i)=\frac{X_0^2\alpha}{4}\int d^2z' e^{2\omega(z',\bar{z}')}\ln|z'-z_i|^2\,. \tag{2.213}$$

However, this propagator must be regularized due to possible infinities which indeed appear when two vertex operators meet at the same point ($i = j$). With the correct Green's function we can return to (2.207) and obtain the expectation value.

$$Z[J] = i(2\pi)^{26}\delta^{26}(\sum_i k_i)\left(Det'\frac{-4\pi^2\alpha}{\nabla^2}\right)^{13} exp\left(\frac{1}{2}\sum_{i\neq j} k_i.k_j(\frac{\alpha}{2}ln|z'-z_i|^2 - f(z_i,\bar{z}_i) - f(z_j,\bar{z}_j))\right)$$

As can be seen, the only term which does not vanish at the exponential is the logarithm which relates the i-th and j-th points, while f_i is independently of j variables and the summation of momentums $\sum_j k_j$ vanishes due to the delta function in (2.207). The final result for the tree amplitude of n tachyons as

$$Z[J] = i(2\pi)^{26}\delta^{26}(\sum_i k_i)\left(Det'\frac{-4\pi^2\alpha}{\nabla^2}\right)^{13}\prod_{i>j}^{n}|z_i - z_j|^{\alpha k_i.k_j} \tag{2.214}$$

2.4.1 Four point tachyon amplitude

By making use of the above result, we can compute the amplitude for four tachyon vertex operator insertions. As we saw, the sphere without insertions has three CKV and no moduli parameters,then it follows that for three vertex operator amplitude, one calculates the amplitude with six c insertions ($\langle c(z_1)\bar{c}(\bar{z}_1)c(z_2)\bar{c}(\bar{z}_2)c(z_3)\bar{c}(\bar{z}_3)\rangle$) and no moduli at all. For four-point amplitude, we would insert a fourth vertex insertion $c(z_4)\bar{c}(\bar{z}_4)V(z_4,\bar{z}_4)$ and beacause of this puncture, the b and $\bar{b}$ insertions. Nevertheless we may loose some important local information about unitarity and gauge invariance, it is convenient to treat this insertion as an integrated vertex operator. We proceed to put together the ghost and matter part (2.174) (2.214), and fixing conveniently $z_1 = 0$, $z_2 = 1$, $z_1 = \infty$. The scattering amplitude takes the form

$$\begin{aligned} S(k_1,k_2,k_3,k_4) &= ig_0^4 e^{2\lambda}\left(det'\frac{-4\pi^2\alpha}{\nabla^2}\right)^{13}\left(det'\frac{(P_1^\dagger P_1^{\frac{1}{2}}}{2\pi}\right)(2\pi)^{26}\delta^{26}(\sum_i k_i) \\ &\int d^2z_4\,|z_4|^{\alpha k_1.k_4}\,|1-z_4|^{\alpha k_2.k_4}\,, \end{aligned} \tag{2.215}$$

and can be totally solved by calculating the above integral. It is convenient to work with the Mandelstam's variables:(noting that for the close string tachyon $k^2 = \frac{4}{\alpha}$)

$$\begin{aligned} -s = (k_1+k_2)^2, \qquad -t &= (k_1+k_3)^2, \qquad -u = (k_1+k_4)^2, \\ s+t+u &= -\frac{16}{\alpha}, \end{aligned}$$

The integrand remains as $|z_4|^{-\frac{\alpha}{2}u-4}|1-z_4|^{-\frac{\alpha}{2}t-4}$. It is easier to work with parameters

$$\begin{aligned} 2a = -\frac{\alpha}{2}u - 2, \qquad 2b &= -\frac{\alpha}{2}t - 2, \qquad 2c = -\frac{\alpha}{2}s - 2, \\ a + b + c &= 1\,, \end{aligned} \tag{2.216}$$

Considering the Γ function

$$\Gamma(a) = \int_0^\infty dt e^{-t} t^{a-1}\,, \tag{2.217}$$

independent of the parameter t. Making the change $t = zt'$

$$a^{-u} = \frac{\displaystyle\int_0^\infty dt' e^{-zt'} t'^{(a-1)}}{\Gamma(a)}\,, \tag{2.218}$$

using this property we can rewrite (2.215) as

$$\frac{1}{\Gamma(1-a)\Gamma(1-b)} \int d^2z \int_0^\infty dt e^{-t|z|^2} t^{-a} \int_0^\infty dw e^{-w|1-z|^2} w^{-b}\,,$$

now, we go back to real coordinates for which $|1-z|^2 = |1-x, y|$ and $d^2z = 2dxdy$

$$\frac{2}{\Gamma(1-a)\Gamma(1-b)} \int_0^\infty dt dw \int_{-\infty}^\infty t^{-a} w^{-b} e^{-t(x^2+y^2)} e^{-w((1-x)^2+y^2)}\,,$$

after a little algebra we obtain

$$\begin{aligned} &\frac{2}{\Gamma(1-a)\Gamma(1-b)} \int_0^\infty dt dw t^{-a} w^{-b} \int_0^\infty dx e^{-(t+w)(x-\frac{w}{t+w})^2} \int_0^\infty dy e^{-(t+w)y^2} e^{-\frac{tw}{t+w}} \\ &\frac{1}{\Gamma(1-a)\Gamma(1-b)} \int_0^\infty 2\pi\, dt dw \frac{t^{-a}w^{-b}}{t+w} e^{-\frac{tw}{t+w}}\,. \end{aligned}$$

The change of coordinates $(t, w) \to (n, m)$

$$\begin{aligned} t = nm, \qquad w &= n(1-m)\,, \\ \triangle_{nm} &= n\,, \end{aligned}$$

yields

$$\begin{aligned} &\frac{1}{\Gamma(1-a)\Gamma(1-b)} \int_0^\infty 2\pi\, dn dm m^{-a}(1-m)^{-b} n^{-(a+b)} e^{-nm(1-m)} \\ = \;& \frac{1}{\Gamma(1-a)\Gamma(1-b)} \int_0^\infty 2\pi\, dm \frac{m^{-a}(1-m)^{-b}}{(m(1-m))^{-(a+b)}} \int_0^\infty (m(1-m))^{-(a+b)} dn n^{-(a+b)} e^{-nm(1-m)} \\ = \;& \frac{2\pi}{\Gamma(1-a)\Gamma(1-b)} \int_0^\infty dm m^{b-1}(1-m)^{a-1}\Gamma(1-a-b) \\ = \;& 2\pi \frac{\Gamma(1-a-b)}{\Gamma(1-a-b)\Gamma(1-a)\Gamma(1-b)} \frac{\Gamma(b)\Gamma(a)}{\Gamma(a+b)}\,. \end{aligned} \tag{2.219}$$

The last step can be demonstrated starting from a product of gamma functions $\Gamma(a)$ and $\Gamma(b)$ and producing the change of coordinates $t = rm$, $s = r(1-m)$,

$$\Gamma(a)\Gamma(b) = \int_0^\infty dr \int_0^1 dm r e^{-r}(rm)^{a-1}(r-rm)^{b-1} \tag{2.220}$$

$$= \int_0^\infty dr e^{-r} r^{a+b-1} \int_0^1 dm m^{b-1}(1-m)^{a-1}, \tag{2.221}$$

therefore,

$$B(b,a) = \int_0^\infty dm m^{b-1}(1-m)^{a-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{2.222}$$

Our final result (2.219) can be rewritten in a more symmetrical way by using (2.216), and recovering the Mandelstam variables the four point function amplitude for the tachyon (Virasoro-Shapiro amplitude) is

$$S(k_1,k_2,k_3,k_4) = ig_0^4 C_{S_2}(2\pi)^{26}\delta^{26}(\sum_i k_i) 2\pi \frac{\Gamma(-1-\alpha\frac{s}{4})\Gamma(-1-\alpha\frac{u}{4})\Gamma(-1-\alpha\frac{t}{4})}{\Gamma(2+\alpha\frac{s}{4})\Gamma(2+\alpha\frac{u}{4})\Gamma(2+\alpha\frac{t}{4})}, \tag{2.223}$$

where

$$C_{S_2} = e^{2\lambda}(Det'\frac{-4\pi^2\alpha}{\nabla^2})^{13}(Det'\frac{(P_1^\dagger P_1)^{\frac{1}{2}}}{2\pi}).$$

We have to say some things about this solution. The integral (2.215) is convergent only if $s,t,u < -\frac{4}{\alpha}$ with poles at $\alpha\, s, \alpha\, t, \alpha\, u = -4, 0, 4, 8...$ (2.223). The amplitude is defined elsewhere by analytic continuation. Poles appears in a channel when the center of mass energy of the process $1+2 \to\ 3+4$ becomes the mass squared of a closed string state. For example, we obtain explicitly the pole at $\alpha\, s = -4$

$$S(\alpha\, s \to\ -4) = ig_0^4 C_{S_2}(2\pi)^{26}\delta^{26}(\sum_i k_i) 2\pi \frac{\Gamma(-1-\alpha\frac{s}{4})}{\Gamma(1-(1+\alpha\frac{s}{4}))} \tag{2.224}$$

$$= ig_0^4 C_{S_2}(2\pi)^{26}\delta^{26}(\sum_i k_i) \frac{2\pi}{-\frac{\alpha\, s}{4}-1}, \tag{2.225}$$

where we used the approximation $\Gamma(a) \underset{a\to 0}{\longrightarrow} \frac{1}{a}$. Thus, we may interpret this pole as a resonance due to an intermediate state propagating, on-shell, for a long distance. If this interpretation is valid, we may factorize a general amplitude and isolate the source of divergences.

2.5 Factorization of amplitudes

From our discussion in the previous section we have inferred that a general amplitude on a surface of genus g can be factored due to the emergence of intermediate states propagating freely through the world sheet. The generalization of this reasoning might seem intuitive. By including a complete set of intermediate states, the amplitude $\Gamma(k_1, ...k_n)$ would be factored in amplitudes $\Gamma(k_1, ...k_{\phi_i})$ and $\Gamma(-k_{\phi_j}, ...k_n)$ connected through the propagator between states ϕ_i and ϕ_j. The path integral of the whole process is given by the product of the functional integrals for each component with defined boundary conditions. The accuracy of this statement rests on two facts, the existence of a full correspondence between states and operators in conformal field theories and the process of cutting and sewing Riemann surfaces where conformal field theory is naturally defined [12][13].

2.5.1 Sewing and cutting Riemann surfaces

Let z be a local coordinate defined around a puncture N on a Riemann surface $\Sigma_1(m_1)$, and w be a local coordinate around a puncture M on $\Sigma_2(m_2)$ (z and w vanishes at N and M respectively). The sewing process of surfaces Σ_1 and Σ_1 is as follows. We extract a region of Σ_1 around the puncture N. The same is done for Σ_2 around M. Now, we proceed to connect Σ_1 and Σ_2 through a tube of length l. In the gluing process, not only the length of the tube is a free parameter, the relative rotation between the two sides of the tube, are parametrized by an angle ϕ. The complex moduli parameter can be written as $\tau = \phi + il$. Let $q = e^{i\tau}$. After removing the disk $|z| < (1-\epsilon)\,|A|^{\frac{1}{2}}$ from Σ_1 and the disk $|w| < (1-\epsilon)\,|A|^{\frac{1}{2}}$ from Σ_2, surfaces remains connected making use of the identification

$$zw = q \tag{2.226}$$

The process of sewing the two surfaces with modulus m_1 and m_2 introduce three more complex moduli in the resulting surface $\Sigma(q)$; local coordinates z and w and the complex moduli q for the handle.

2.5.2 Sewing conformal field theories

We consider an n point function $\Gamma(k_1, ..., k_n)_g$ on a Riemann surface with genus g, After inserting complete sets of fields at the punctures N and M on $\Sigma_{g,n}$, which define coordinates $z = 0$ and $w = 0$ around them respectively, the path integral factorize in three sectors, the functional integrals over surfaces Σ_1 and Σ_2, it means, the amplitudes

$\Gamma(k_1, ..., k)_{g-h}$ and $\Gamma(q, ..., k_n)_g$, and the functional integral over the cylinder connecting the intermediate states

$$\Gamma(k_1, ..., k_n)_g = \sum_a \int \frac{d^{26}q}{(2\pi)^{26}} \int \frac{d^{26}k}{(2\pi)^{26}} \langle \varphi^a(k,z) \mid \varphi_a(q,w)\rangle_{Cyl}$$
$$\langle V(k_1, z_1)...\varphi_a(k,z)\rangle_{g-h} \langle \varphi^a(q,w)...V(k_n,z_n)\rangle_h \tag{2.227}$$

where $\langle \varphi^a(k,z) \mid \varphi_a(q,w)\rangle_{Cyl}$ describes the free-propagator of the state $\varphi^a(k,z)$ from $z = 0$ to the point $w = 0$. Here, φ^a denotes the dual to φ_a defined by

$$\langle \varphi_a(k_1)\varphi^b(k_2)\rangle = \delta_a^b (2\pi)^{26}\delta^{26}(k+q) \tag{2.228}$$

The propagator can be obtained by that calculating the matrix element $e^{-lH+i\phi P}$ between the initial y final stats. Since $L_0 + \bar{L}_0$ and $L_0 - \bar{L}_0$ are the generators of dilation and rotations, the matrix element takes the form of $e^{-l(L_0+\bar{L}_0)+i\phi(L_0-\bar{L}_0)} = q^{L_0}\bar{q}^{\bar{L}_0}$, then

$$\langle \varphi^a(k,z) \mid \varphi_a(q,w)\rangle_{Cyl} = \int d^2q \langle \varphi^a(k,z) \mid \langle b, \mu_q\rangle\langle b, \mu_{\bar{q}}\rangle q^{L_0}\bar{q}^{\bar{L}_0} \mid \varphi_a(q,w)\rangle \tag{2.229}$$

As we have been proceeding trough the text, it is convenient to express the change of the moduli as a change in the coordinate region with fixed metric. For a local change $z \to z' = z + \delta q v^z$, the identification $zw = q$ might change as $z'w = q + \delta q$. Then, the vector field generating the transformation of the complex structure is $v^z = z/q$,

$$\int d^2q \langle \varphi^a(k,z) \mid \int \frac{d^2z}{2i\pi} b_{zz}\frac{z}{q} \int \frac{d^2z}{2i\pi} \bar{b}_{\bar{z}\bar{z}}\frac{\bar{z}}{q} \mid \varphi_a(q,w)\rangle q^{L_0}\bar{q}^{\bar{L}_0}$$
$$= \int d^2q \langle \varphi^a(k,z) \mid \frac{b_0\bar{b}_0}{q\bar{q}} q^{L_0}\bar{q}^{\bar{L}_0} \mid \varphi_a(q,w)\rangle \tag{2.230}$$

Complex moduli $q = e^{-l+i\phi}$, as we said, parametrize length and relative twist between the two sides of the cylinder. In order to visualize where the poles come from, we rewrite the integral as

$$\int d^2q (q\bar{q}) q^{L_0}\bar{q}^{\bar{L}_0} = 2\int_0^\infty ds\, e^{-s(L_0+\bar{L}_0)} \int_0^{2\pi} d\phi\, e^{i\phi(L_0-\bar{L}_0)}$$
$$= (4\pi)\delta_{L_0-\bar{L}_0} \int_0^\infty ds\, e^{-s(L_0+\bar{L}_0)} \tag{2.231}$$

The integral over ϕ give us the constraint that only states with conformal weight $(h, \bar{h})$, such that $h - \bar{h} = 0$, propagate throug the tube. Inserting this result in (2.230) we have that

$$\langle \varphi^a(k,z) \mid \varphi_a(q,w)\rangle_{Cyl} = (4\pi) \int_0^\infty ds \langle \varphi^a(k,z) \mid (2\pi)^{26}\delta^{26}(k+q) b_0\bar{b}_0 \delta_{L_0-\bar{L}_0} e^{-s(L_0+\bar{L}_0)} \mid \varphi_a(q,w)\rangle \tag{2.232}$$

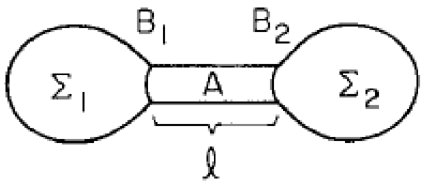

Figure 2.3: Sewing two conformal field theories

Putting the last result in (2.227) and integrating over the momentum q we drop out the delta function. Finally we consider the action of this matrix element over the states φ_a such that $L_0\varphi_a = \alpha\frac{k^2+m_\varphi^2}{4}\varphi_a$. Thus, the closed bosonic propagator is given by

$$\frac{8\pi}{\alpha}\frac{b_0\bar{b}_0\delta_{L_0-\bar{L}_0}}{k^2+m_\varphi^2-i\epsilon} \tag{2.233}$$

and the scattering amplitude remains factorized as

$$\Gamma(k_1,...,k_n)_g = \sum_a\int\frac{d^{26}k}{(2\pi)^{26}}\Gamma(k_1,...,k)_{g-h}\frac{8\pi\delta_{L_0-\bar{L}_0}}{\alpha(k_a^2+m_a^2-i\epsilon)}\Gamma(-k,...,k_n)_h \tag{2.234}$$

We have isolated the potential divergences that uniquely come from intermediate states propagating on-shell, as it was expected. The total calculation of the amplitude (2.227) requires, of course, calculation of path integrals on Σ_h and Σ_{g-h}. Poles appear after integrating over l for the region $l\to\infty$, and can be interpreted as an infrared divergence due to an on-shell particle propagating for a long proper time. It is worthwhile to note that since $b_0^2=\bar{b}_0^2=0$, the operators b_0 and $\bar{b}_0$ projects the string propagator onto states annihilated by them.

2.6 One loop partition function

2.6.1 Moduli space of torus

Torus can represented by a lattice defined by basis w_1 and w_2 and the identification

$$\Omega(w_1,w_2)=\{mw_1+nw_2|m,n\epsilon\mathbb{Z}\} \qquad (\frac{w_1}{w_2}>0) \tag{2.235}$$

we see that it is not actually represented uniquely by this basis. For example the basis $\Omega(w_1+w_2,w_2)$ generates the same lattice

$$\begin{aligned}\Omega(w_1+w_2,w_2) &= \{m(w_1+w_2)+nw_2|m,n\epsilon\mathbb{Z}\}\\ &= \{mw_1+pw_2|m,p\epsilon\mathbb{Z}\}\end{aligned} \tag{2.236}$$

for $p = m + n$. In general, lattices $\Omega(w_1, w_2)$ and $\Omega(u_1, u_2)$ define the same lattice when their basis are connected by an automorphism such that

$$\begin{aligned} u_1 &= aw_1 + bw_2 \\ u_2 &= cw_1 + dw_2 \qquad a, b, c, d\epsilon\mathbb{Z} \end{aligned} \tag{2.237}$$

as can be easily checked. The same can be written as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \qquad a, b, c, d\epsilon\mathbb{Z} \tag{2.238}$$

The automorphism (2.237) has to be invertible, so elements of the inverse matrix

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{2.239}$$

must be integers, therefore, it follows that the condition $ad - bc = \pm 1$ mus holds. We conclude that two lattices are equivalent if they are related by transformations (2.237) such that $detM = \pm 1$. Representing the basis as $w_i = |w_i| e^{i\theta_i}$ we can rewrite the identification of the lattice (2.235) as $mw_1 + nw_2 = \frac{e^{i\theta_1}}{|w_1|}(m + n\tau)$ where $\tau = \frac{w_2}{w_1}$. Then (2.235) is conformal equivalent (equivalent up a rotation and dilation) to $\Omega(1, \tau)$, $Im(\tau) > 0$. Also, we can see that τ is invariant by conformal transformations. τ is the modulus of the basis, and determine a lattice in the upper half-plane. Lattices with the same modulus represent the same configuration. However, not all different modulus represent distinct geometries. Nevertheless, lattices related by transformations (2.239) represent the same torus, the modulus does not remain invariant. Let τ' be the modulus related to the basis (u_1, u_2)

$$\tau' = \frac{u_2}{u_1} = \frac{aw_1 + bw_2}{cw_1 + dw_2} \tag{2.240}$$

$$= \frac{a + b(w_2/w_1)}{c + d(w_2/w_1)} = \frac{a + b\tau}{c + d\tau} \tag{2.241}$$

In principle $ad - bc = \pm 1$ but the restriction $Im(\tau) > 0$ is only fulfilled if $ad - bc = 1$. Additionally τ is unchanged by reversing the signs of a, b, c, d. We can conclude that torus conformal equivalents are related by transformation $PSL(2, \mathbb{Z}) = \frac{SL(2,\mathbb{Z})}{\mathbb{Z}_2} =$, this is the Modular Group for the $g = 1$. We can put everything clearer, in the language we developed. Teichmuller space is generated by modulus τ which are points in the upper-half plane. Moduli space is obtained by the action of $PSL(2, \mathbb{Z})$, the MCG, on the upper-half

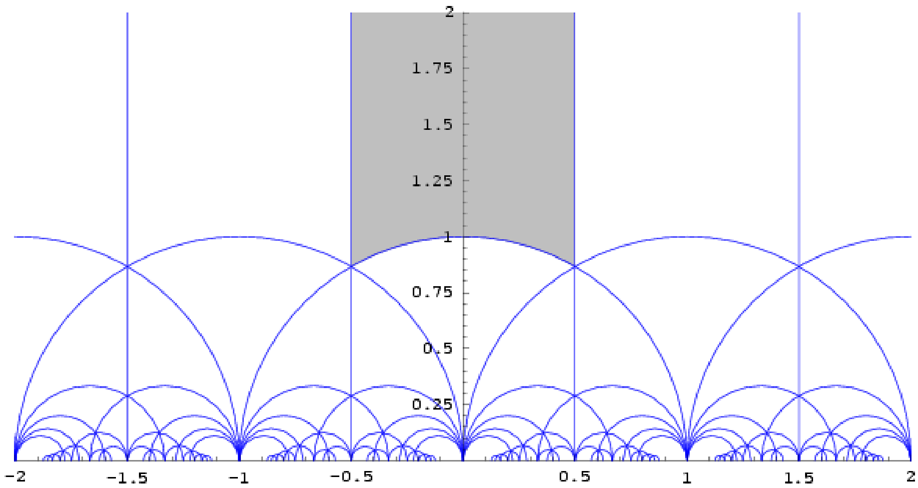

Figure 2.4: Fundamental domain

plane. It is very appropriate to remark that $PSL(2,\mathbb{Z})$ transformations can be generated by repeated application of transformations

$$T : \tau' = \tau + 1, \qquad S : \tau' = -\frac{1}{\tau} \tag{2.242}$$

here, T corresponds to cutting the torus along a circle, twisting one end 2π and gluing them together again. S produce the interchange of the non-contractible circles of the torus (2.235). Points in the upper-half plane related by (2.242) are called to be in the same orbit. We can build a fundamental region F_0, that is, a representation of the Moduli space, as a subset of $\mathbb{C}$ that does not contain two points on the same orbit except possibly on its boundary. F_0 must contain at least one point of each orbit. In the construction of F_0 we note that:

- Wide of F_0 has to be 1 due to T transformation
- $S(\tau)$ take points from $|\tau| < 0$ to the region $|\tau| > 0$.
- Points on the boundary of are identified by $PSL(2,\mathbb{Z})$, and one can think of F_0 as being rolled up and open at $Im(\tau) \to \infty$

$$F_0 = \left\{ \tau \epsilon \mathbb{C} |\, |Re\, \tau| \leq \frac{1}{2}, |\tau| \geq 1 \right\} \tag{2.243}$$

2.6.2 Partition function

Until now we have seen that surfaces with $h \geq 1$ are parametrized by modulus τ. Modular transformation, reparametrizations not connected with the identity, change the value of the moduli but not the shape of the surface. Thus, we expect that physical measurable objects, as the partition function, to be invariant under modular transformations. In particular for the torus, we expect that partition function $\mathcal{Z}(\tau,\bar{\tau})$ to be invariant by transformations (2.242). We calculate the partition function of torus in a similar way we did for the tube in the last section, but considering the torus as a cylinder with ends glued together with periodic boundary conditions, but now we calculate the bosonic and ghost part separately by making use of the Hilbert space of these systems.

Scalar Partition Function

For the scalar system on the cylinder $H = L_0 + \bar{L}_0 - \frac{1}{24}(c+\bar{c})$ and the Momentum $L_0 - \bar{L}_0$ (due to $L_{cyl} = L_0 - \frac{c}{24}$). The partition function is

$$\begin{aligned}
\mathcal{Z}(\tau) &= (q\bar{q})^{-\frac{c}{24}} Tr[q^{L_0}\bar{q}^{\bar{L}_0}] \\
&= (q\bar{q})^{-\frac{c}{24}} Tr\left[q^{\frac{\alpha}{4}k^2+\sum_{n>0}\alpha^{\mu}_{n}\alpha_{n\mu}}\right] Tr\left[\bar{q}^{\frac{\alpha}{4}k^2+\sum_{n>0}\bar{\alpha}^{\mu}_{n}\bar{\alpha}_{n\mu}}\right] \\
&= (q\bar{q})^{-\frac{c}{24}} Tr[e^{-\alpha\pi\tau\, k^2}\prod_{n,\mu} q^{nN}\bar{q}^{n\bar{N}}] \\
&= (q\bar{q})^{-\frac{c}{24}} \int \frac{d^{26}k}{(2\pi)^{26}} e^{-\alpha\pi\alpha\, k^2} Tr[\prod_{n,\mu} q^{nN}\bar{q}^{n\bar{N}}],
\end{aligned}$$

the trace has to be calculated over all possible states with all possible occupation number N for every oscillation mode α_n.

$$\begin{aligned}
\mathcal{Z}(\tau) &= (q\bar{q})^{-\frac{c}{24}}[\frac{1}{(4\alpha\pi^2 Im\tau)^{\frac{1}{2}}}]^{26}\prod_{n,\mu}(\sum_{N} q^{nN})(\sum_{\bar{N}}\bar{q}^{n\bar{N}}) \\
&= (q\bar{q})^{-\frac{c}{24}}\frac{1}{(4\alpha\pi^2 Im\tau)^{13}}\prod_{n,\mu}(q^n+q^{2n}+...)(\bar{q}^n+\bar{q}^{2n}) \\
&= (q\bar{q})^{-\frac{c}{24}}\frac{1}{(4\alpha\pi^2 Im\tau)^{13}}\prod_{n,\mu}\frac{1}{1-q^n}\frac{1}{1-\bar{q}^n} \\
&= i\frac{1}{(4\alpha\pi^2 Im\tau)^{13}}(\frac{1}{|\eta(\tau)|}^{2})^{26}
\end{aligned} \tag{2.244}$$

where we have defined the Dedekind function

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}(1-q^n) \tag{2.245}$$

The i come from the Wick rotation of k^0 as in (2.207). Modular invariance of the scalar partition function $Z(\tau) = Z(\tau+1) = Z(-\frac{1}{\tau})$ is manifest due to Dedekind function transformations:

$$\eta(\tau+1) = e^{\frac{i\pi}{12}}\eta(\tau), \qquad \eta(-\frac{1}{\tau}) = (i\tau)^{\frac{1}{2}}\eta \tag{2.246}$$

remarking that $Im(\tau)$ is invariant under T and transform under S as

$$S : Im(\tau) \to \frac{Im(\tau)}{|\tau|^2} \tag{2.247}$$

We expect the measure to be modular invariant also, in order to get amplitudes with good moduli behaviour.

Ghost partition function

We now calculate the contribution for the ghost part. This can be a little trickier and will be instructive also in order to get intuition about the manipulation of fermionic variables in the partition function. The essential property to be in mind is that occupation number for any state can only be zero or one. Additionally the trace has to be weighted by $(-1)^F$, that commutes with grassmann fields,due to antiperiodic boundary conditions in the time direction. In order to calculate the trace for the ghost system we describe their Hilbert space. Let $|\Omega\rangle \otimes |\bar{\Omega}\rangle$ be the vacuum of the bc system, from which we are acting to create all the spectre as

$$|n_1, n_2...\rangle \otimes |\bar{n}_1, \bar{n}_2...\rangle = b_{-1}^{n_1}...c_{-1}^{m_1}...\bar{b}_1^{\bar{n}}...c_{-1}^{\bar{m}_1}...left|\Omega\rangle \otimes |\bar{\Omega}\rangle \tag{2.248}$$

where dots mean infinite series in ghost-modes. Just to simplify calculation, we focus in the he trace of the holomorphic part,

$$tr(q^{L_0}) = \sum_{n_1}\sum_{n_2}...\sum_{m_1}\sum_{m_2}...(\langle n_1...m_1...|q^{L_0}|n_1...m_1...\rangle) \tag{2.249}$$

with $L_0 = -\sum_p p : b_p c_{-p} : -1$. Considering

$$-\sum_p p : b_p c_{-p} : |n_1, n_2...\rangle = -(\sum_{p=1}^{\infty} p b_p c_{-p}) b_{-1}^{n_1}...c_{-1}^{m_1}...|\Omega\rangle \tag{2.250}$$

$$= -\sum_{p=1}^{\infty} p b_{-1}^{n_1}...c_{-1}^{m_1}...(b_p c_{-p}) c_p^{m_p}...|\Omega\rangle \tag{2.251}$$

$$= -\sum_{p=1}^{\infty} p b_{-1}^{n_1}...c_{-1}^{m_1}...m_p c_p^{m_p}...|\Omega\rangle \tag{2.252}$$

On the other hand

$$\begin{aligned} q^{L_0}|n_1,n_2...\rangle &= (\sum_k \frac{1}{k!}(2i\pi\tau)^k L_0^k) b_{-1}^{n_1}...c_{-1}^{m_1}...|\Omega\rangle \\ &= q^{-1}(\sum_k \frac{1}{k!}(2i\pi\tau)^k [\sum_{p=1}^{\infty} pc_{-p}b_p]^k) b_{-1}^{n_1}...c_{-1}^{m_1}...|\Omega\rangle \\ &= q^{-1}(\sum_k \frac{1}{k!}(2i\pi\tau)^k [\sum_{p=1}^{\infty} pn_p]^k) b_{-1}^{n_1}...c_{-1}^{m_1}...|\Omega\rangle \\ &= q^{-1} q^{\sum_{p=1}^{\infty} pn_p} |n_1,n_2...\rangle \end{aligned} \quad (2.253)$$

Calculation of the trace is direct. Expanding (2.253) as wi did for the bosonic case and taking into account the result in (2.252) we obtain

$$tr(q^{L_0}) = q^{-1} \sum_{n_1}\sum_{n_2} ...(q^{n_1} q^{2n_2}...) \sum_{m_1}\sum_{m_2} ...(q^{m_1} q^{2m_2}...) \quad (2.254)$$

but $n_i = m_i = 0, 1$ and a single oscillator n gives $(1 + q^n)$. The final result for the trace, after adding the anti-holomorphic part is

$$Tr[e^{2i\pi\tau_1 P - 2\pi\tau_2 H}] = (q\bar{q})^{\frac{1}{12}} \prod_{n=1} |1 + q^n|^4 \quad (2.255)$$

This result just come from one of the vacuum of the system, and has to be weighted by $(-)^F$ defined even for $|\downarrow\rangle$ and odd for $|\uparrow\rangle$. Moreover $\mathcal{Z}_{gh}$ has to be saturated with insertions of ghost fields to give a non-zero result. Partition function with the less number of insertions come from $\langle c(w_1)b(w_2)\bar{c}(\bar{w}_1)\bar{b}(\bar{w}_2)\rangle$, further, as we have shown only zero modes make contribution. Operators $c_0 b_0 \bar{c}_0 \bar{b}_0$ annihilate all vacuum states but $|\downarrow\rangle\otimes|\downarrow\rangle$. Taking into account the factor $(-)^F$ the final result is

$$\langle c(w_1)b(w_2)\bar{c}(\bar{w}_1)\bar{b}(\bar{w}_2)\rangle_T = (q\bar{q})^{\frac{1}{12}} \prod_{n=1} |1 - q^n|^4 \quad (2.256)$$

Before to write the total partition function for one loop vacuum, putting together the result for the scalar and ghost parts, we need to consider the CKV group for genus 1 which have not been fixed (as it was made for the sphere by fixing 3 points) by hand. The CKV for the torus correspond to constant vectors producing translations. We obtain this by representing the torus as a parallelogram with periodicity 1, and putting the moduli dependence on the metric as $ds^2 = |d\sigma_1 + \tau\, d\sigma_2|^2$, then $Vol(CKV) = 2Im(\tau)$. The full partition function remains

$$i \int_{F_0} \frac{d^2\tau}{4Im(\tau)} (4\pi^2 \alpha\, Im(\tau))^{-13} |\eta(\tau)|^{-48} \quad (2.257)$$

Modular invariance is easily shown by realizing that

$$\frac{d\tau\, d\bar{\tau}}{(Im\tau)^2} \tag{2.258}$$

is obviously invariant by T, and because of (2.246) invariant under S too.

Chapter 3

BRST quantization of the fermionic string

3.1 Introduction

In this chapter we are interested in quantizing a string theory which includes a realistic spectra incorporating bosons as well as fermions. For this purpose we extend the Polyakov action $S_P[x^\mu, g_{\alpha\beta}]$ by adding two dimensional Majorana fermions described by a 2d Dirac system.

$$\mathcal{S}_1[g^{\alpha\beta}, x, \Psi] = -\frac{1}{2\pi}\int d^2\sigma\sqrt{g}\left[g^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x_\mu - i\bar{\Psi}^\mu\gamma^\rho\nabla_\rho\Psi_\mu\right]. \tag{3.1}$$

This action presents an extra symmetry, global supersymmetry between two dimensional bosons x^μ and spinors Ψ^μ

$$\delta_s x^\mu = \epsilon\Psi^\mu, \qquad \delta_s\Psi^\mu = i\gamma^\rho\partial_\rho x^\mu\epsilon\,. \tag{3.2}$$

The two-dimensional fermions enlarge the Hilbert space of the bosonic string, including unphysical degrees of freedom also, which we have to eliminate from the spectrum. For the bosonic string this was achieved by imposing the Virasoro constraints, which appear due to the gauge fixing the local symmetries of the theory (conformal gauge). We expect to proceed in a similar way for the fermionic string; starting from a total action with local supersymmetry [14] [15], we expect to obtain the supersymmetric extensions of the Virasoro constraints after fixing a superconformal gauge. We will not rigorously demonstrate the construction of this action but will motivate the procedure via the Noether's method by coupling the supersymmetric Majorana matter system (3.1) to supergravity (2d graviton-gravitino system). After going further in this construction, we remark two things about two dimensional supergravity.

- First, the Rarita-Schwinger action can not be constructed in 2d because it is impossible to construct an antisymmetric tensor of third order $\gamma^{\alpha\beta\sigma}$ due to 2d symmetries.
- Second, as for the bosonic string, the Einstein-Hilbert action in 2d is related to a topological invariant (Euler characteristic). It follows that the graviton-gravitino system does not have dynamic in 2d and can be thought as Lagrange multipliers (as $g^{\alpha\beta}$ for the bosonic string).

As it is usual, we use Noether's method. We start by promoting the global supersymmetry to local supersymmetry (letting ϵ in (3.2) be coordinate dependent). Varying the action for local transformations (3.2) and noting that $\delta_s\bar{\Psi}^\mu = \gamma^0\delta\Psi^{\mu\dagger} = i\bar{\epsilon}\gamma^\alpha\partial_\alpha x^\mu$ we have that

$$\begin{aligned}
\delta\mathcal{S}_1 &= -\frac{1}{\pi}\int d^2\sigma\sqrt{g}\big[\partial_\alpha\bar{\epsilon}\Psi^\mu\partial^\alpha x_\mu + \bar{\epsilon}\partial_\alpha\Psi^\mu\partial^\alpha x_\mu + \epsilon\Psi^\mu\partial_\alpha\partial^\alpha x_\mu + \frac{1}{2}\bar{\epsilon}\gamma^\alpha\gamma^\beta\partial_\alpha x^\mu\partial_\beta\Psi_\mu \\
&\quad -\frac{1}{2}\bar{\Psi}^\mu(\gamma^\alpha\gamma^\beta\partial_\alpha\partial_\beta x_\mu)\epsilon - \frac{1}{2}\bar{\Psi}^\mu\gamma^\alpha(\partial_\alpha\epsilon)\gamma^\beta\partial_\beta x_\mu\big] \\
&= -\frac{1}{\pi}\int d^2\sigma\sqrt{g}\big[\partial_\alpha(\bar{\epsilon}\Psi^\mu\partial^\alpha x_\mu) + \frac{1}{2}\bar{\epsilon}\gamma^\alpha\gamma^\beta\partial_\alpha x^\mu\partial_\beta\Psi_\mu - \frac{1}{2}\bar{\Psi}^\mu\gamma^\alpha(\partial_\alpha\epsilon)\gamma^\beta\partial_\beta x_\mu \\
&\quad -\frac{1}{2}\bar{\Psi}^\mu\frac{1}{2}(\gamma^\alpha\gamma^\beta\partial_\alpha\partial_\beta + \gamma^\beta\gamma^\alpha\partial_\beta\partial_\alpha x_\mu)\epsilon\big] \\
&= -\frac{1}{\pi}\int d^2\sigma\sqrt{g}\left[\frac{1}{2}\bar{\epsilon}\gamma^\alpha\gamma^\beta\partial_\alpha x^\mu\partial_\beta\Psi_\mu - \frac{1}{2}\bar{\Psi}^\mu\gamma^\alpha(\partial_\alpha\epsilon)\gamma^\beta\partial_\beta x_\mu - \frac{1}{2}\bar{\Psi}^\mu(\partial_\alpha\partial^\alpha x_\mu)\epsilon\right] \\
&= \frac{1}{\pi}\int d^2\sigma\sqrt{g}(\partial_\alpha\bar{\epsilon})\gamma^\alpha\Psi_\mu\gamma^\beta\partial_\beta x^\mu\,. \qquad (3.3)
\end{aligned}$$

In order to remove this variation we include an extra term of order κ

$$\mathcal{S}_2 = -\frac{\kappa}{2\pi}\int d^2\sigma\sqrt{g}\bar{\chi}_\alpha\gamma^\alpha\Psi_\mu\gamma^\beta\partial_\beta x^\mu\,, \qquad (3.4)$$

such that the 2d spinor field χ_α transform as

$$\delta_s\chi_\alpha = \frac{2}{\kappa}\nabla_\alpha\epsilon\,, \qquad (3.5)$$

χ_α is a spinor field of spin 3/2, the gravitino field. It is important to note that ∇_α take a 1/2 spinor to one of spin 3/2. The variation of action $\mathcal{S}_3 = \mathcal{S}_1 + \mathcal{S}_2$, produces three extra terms

$$\delta\mathcal{S}_3 = -\frac{\kappa}{2\pi}\int d^2\sigma\sqrt{g}\left[\bar{\chi}_\alpha\gamma^\beta\gamma^\alpha\Psi^\mu\bar{\Psi}_\mu(\partial_\beta\epsilon) + \bar{\chi}_\alpha\gamma^\beta\gamma^\alpha\Psi^\mu\epsilon\partial_\beta\Psi_\mu - i\bar{\chi}_\alpha\gamma^\beta\gamma^\alpha\gamma^\sigma\partial_\sigma x^\mu\partial_\beta x_\mu\epsilon\right]. \qquad (3.6)$$

The second term can be matched by adding the term $i\epsilon\gamma^\alpha\bar{\Psi}\chi_\alpha$. Third term can be taken out by considering the supersymmetric variation of the metric $g^{\alpha\beta}$. It is convenient to rewrite the metric in the tetrad formalism $g_{\alpha\beta} = e^a_\alpha \delta_{ab} e^b_\beta$. The first term remains and can be equated by adding a new term $\mathcal{S}_3$ of second order in κ of the form

$$\mathcal{S}_4 = -\frac{\kappa^2}{8\pi}\int d^2\sigma\, e\left[\bar{\chi}_\alpha\gamma^\beta\gamma^\alpha\Psi^\mu\Psi_\mu\chi_\beta\right]. \tag{3.7}$$

The total action is

$$\begin{aligned}\mathcal{S}[g^{\alpha\beta}, x, \Psi, \chi] \;=\; -\frac{1}{2\pi}\int d^2\sigma\, e\Big[& e^\alpha_a e^{a\beta}\partial_\alpha x^\mu\partial_\beta x_\mu - i\bar{\Psi}^\mu e^\alpha_a\gamma^a\partial_\alpha\Psi_\mu + \kappa\bar{\chi}_\alpha\gamma^\alpha\Psi_\mu\gamma^\beta\partial_\beta x^\mu + \\ & -\frac{\kappa^2}{4}\bar{\chi}_\alpha\gamma^\beta\gamma^\alpha\Psi^\mu\Psi_\mu\chi_\beta\Big],\end{aligned} \tag{3.8}$$

which is invariant under local supersymmetry transformations

$$\delta_s x^\mu = \epsilon\Psi^\mu, \qquad \delta_s\Psi^\mu = i\gamma^\alpha\gamma_\alpha(x^\mu - \bar{\Psi}^\mu\chi_\alpha)\epsilon\,, \tag{3.9}$$

$$\delta_s\chi_\alpha = \frac{2}{\kappa}\nabla_\alpha\epsilon\,, \qquad \delta_s e^a_\alpha = \kappa\gamma^a\chi_\alpha\,. \tag{3.10}$$

3.1.1 Fixing the gauge and $\beta\gamma$ system

The fixing of the zweibein e^a_α is analogous to the gauge fixing of the metric for the bosonic case. Due to symmetry under diffeomorphism, we can choose the conformal gauge $g_{\alpha\beta} = e^{2\phi(\sigma)}\delta_{\alpha\beta}$, and the zweibein stays as $\hat{e}^a_\alpha = e^{\phi(\sigma)}\delta^a_\alpha$. In order to fix the degrees of freedom of the gravitino χ_α we have to note that $\mathcal{S}$ is invariant under transformations of the form

$$\delta\chi_\alpha = \gamma_\alpha\varphi\,, \tag{3.11}$$

called local S-supersymmetry, where φ is a Majorana spinor. In general, we can split χ_α in two parts, a γ_α-trace and a γ_α-traceless part as

$$\chi_\alpha = \gamma_\alpha\eta + \phi, \qquad \gamma^\alpha\phi_\alpha = 0\,. \tag{3.12}$$

Also,Taking into account the above mentioned, we note that (3.12) holds for $\phi_\alpha = \gamma^\beta\gamma_\alpha\nabla_\beta\epsilon$. Thus,

$$\begin{aligned}\chi_\alpha &= \gamma_\alpha\eta + \gamma^\beta\gamma_\alpha\nabla_\beta\epsilon \\ &= \gamma_\alpha\eta + 2\delta^\beta_\alpha\nabla_\beta\epsilon - \gamma_\alpha\gamma^\beta\nabla_\beta\epsilon \\ &= 2\nabla_\alpha\epsilon + \gamma_\alpha(\eta - \gamma^\beta\nabla_\beta\epsilon)\,.\end{aligned} \tag{3.13}$$

The gravitino has 4 degrees of freedom in 2d. Classically, we have enough symmetries (supersymmetry and S-supersymmetry) to fix χ_α to zero. Quantum mechanically this gauge fixing produce the ghost system due to local supersymmetries. Fixing the gauge to $\hat{\chi} = \gamma_\alpha \varphi$, the usual Faddeev-Popov prescription,

$$1 = \int D\chi \delta(\chi - \hat{\chi}) \to \triangle_{FP_\chi} \int D\epsilon \, D\eta \delta(2\nabla_\alpha \epsilon) \, , \tag{3.14}$$

then,

$$\triangle_{FP_\chi}^{-1} = \int D\varsigma \, D\epsilon \exp\left(2i\pi \langle \varsigma^\alpha \, , 2\nabla_\alpha \epsilon \rangle\right) \, . \tag{3.15}$$

Because of Faddeev-Popov prescription we interchange $\epsilon \, , \varsigma$ Grassmann variables for $\gamma \, , \beta$ commuting variables. With convenient normalizations of the fields, we can write the action for the $\beta\gamma$-system in conformal gauge as

$$\mathcal{S}_{\beta\gamma} = \frac{1}{2\pi} \int d^2 z \beta \partial_{\bar{z}} \gamma \, , \tag{3.16}$$

which gives the equations of motion

$$\partial_{\bar{z}} \gamma = \partial_{\bar{z}} \beta = 0 \, , \tag{3.17}$$

γ and β are holomorphic fields. We noted that the operator ∇_α take 1/2 spin field to a spinor field of spin 3/2, then we expect that γ and β have conformal weight 1/2 and $-3/2$ respectively. In general we consider β and γ fields as commutating holomorphic fields of weight $\lambda - 2$ and $3/2 - \lambda$ respectively, with infinitesimal transformations

$$\delta\beta = (\lambda - 1/2)\beta\partial_z \epsilon + \epsilon \partial_z \beta \qquad \delta\gamma = (3/2 - \lambda)\gamma \partial_z \epsilon + \epsilon \partial_z \gamma \, . \tag{3.18}$$

The energy-momentum tensor can be deduced

$$\begin{aligned}
\delta \mathcal{S}_{\beta\gamma} &= \frac{1}{2\pi} \int d^2 z (\lambda - 1/2)\beta\partial_z \epsilon + \epsilon \partial_z \beta \partial_{\bar{z}} ((3/2 - \lambda)\gamma \partial_z \epsilon + \epsilon \partial_z \gamma) \\
&= \frac{1}{2\pi} \int d^2 z ((\lambda - 1/2)\beta \partial_{\bar{z}} \gamma + (3/2 - \lambda)\beta \partial_z \gamma)\partial_z \epsilon + (\partial_z \beta \partial_{\bar{z}} \gamma + \beta \partial_{\bar{z}} \partial_z \gamma)\epsilon \\
&\quad + (3/2 - \lambda)\beta\gamma \partial_{\bar{z}} \partial_z \epsilon + \beta \partial_z \gamma \partial_{\bar{z}} \epsilon \\
&= \frac{1}{2\pi} \int d^2 z \left[\partial_z (\beta \partial_{\bar{z}} \gamma \epsilon) + (3/2 - \lambda)(\partial_z (\beta\gamma \partial_{\bar{z}} \epsilon) - \partial_z (\beta\gamma)\partial_{\bar{z}} \epsilon) + \beta \partial_z \gamma \partial_{\bar{z}} \epsilon\right] \\
&= -\frac{1}{2\pi} \int d^2 z \left[(3/2 - \lambda)\partial_z (\beta\gamma) - \beta \partial_z \gamma\right] \partial_{\bar{z}} \epsilon \, ,
\end{aligned} \tag{3.19}$$

and we can write the energy-momentum tensor for the $\beta\gamma$ system as

$$T_{\beta\gamma} = -\frac{2\lambda - 1}{2} \partial_z (\beta\gamma) + \partial_z \beta \gamma \, . \tag{3.20}$$

The correlator is obtained as we did in the first chapter for the matter and the *bc* system

$$\langle \beta(z)\,\gamma(w)\rangle = -\,\langle \gamma(z)\,\beta(w)\rangle = \frac{1}{z-w}\,. \tag{3.21}$$

Unlike the *bc* propagator, the minus is beacuse of the commutativity between β and γ. Taking into account all these details we can calculate the OPE

$$T_{\beta\gamma}(z)T_{\beta\gamma}(w) = \frac{1/2(3(2\lambda-2)^2-1)}{(z-w)^4} + \frac{2T_{\beta\gamma}(w)}{(z-w)^2} + \frac{\partial_2 T_{\beta\gamma}(w)}{z-w}\,. \tag{3.22}$$

Now, we will focus on getting the constraints for the fermionic string. To achieve this, we work in a similar way we obtained the Virasoro constraints for the bosonic string; solving the equations of motion for e_a^α and χ_α and then fixing the gauge. For the gravitino, we define

$$\begin{aligned} T_F &= \frac{2\pi}{ie}\frac{\delta \mathcal{S}}{\delta \bar{\chi}^\alpha} \\ &= \gamma^\beta\gamma^\alpha\Psi_\mu\partial_\beta x^\mu + \frac{\kappa}{4i}\gamma^\beta\gamma^\alpha\Psi^\mu\Psi_\mu\chi_\beta = 0\,, \end{aligned} \tag{3.23}$$

after fixing the gauge $\chi_\alpha = \gamma_\alpha\varsigma$ the *superconformal* constraint is

$$T_F^\alpha = \gamma^\beta\gamma^\alpha\Psi_\mu\partial_\beta x^\mu\,. \tag{3.24}$$

The energy-momentum tensor can be obtained in a similar manner.

$$T_{\alpha\beta} = -\frac{1}{2}\partial_\alpha x^\mu\partial_\beta x_\mu + \frac{1}{2}g^{\alpha\beta}\partial_\alpha x^\mu\partial^\alpha x_\mu - \frac{i}{2}\bar{\Psi}^\mu\gamma_\alpha\partial_\beta\Psi_\mu\,. \tag{3.25}$$

Let us focus in the fermionic part of the action

$$\mathcal{S}_F = \frac{i}{2\pi}\int d^2\sigma\sqrt{g}\Psi\gamma^\alpha\partial_\alpha\Psi\,. \tag{3.26}$$

Representing the gamma matrices in a real basis, components of the Majorana spinor $\Psi = \begin{pmatrix}\psi\\ \tilde{\psi}\end{pmatrix}$ are real. Then $\bar{\Psi} = \gamma^2\Psi^\dagger$. We work in this manner but performing a Wick rotation on $\sigma^1 \to -i\sigma^1$ because we are concerned with a 2d euclidean space in which the Clifford algebra,

$$\{\gamma^\alpha, \gamma^\beta\} = 2g^{\alpha\beta}\mathbf{1}\,, \tag{3.27}$$

holds for the gamma matrices

$$\gamma^0 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \gamma^1 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\,. \tag{3.28}$$

Now it is easy to see

$$\gamma^\alpha \partial_\alpha = \begin{pmatrix} 0 & \partial_0 \\ \partial_0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\partial_1 \\ i\partial_1 & 0 \end{pmatrix} = 2 \begin{pmatrix} 0 & \partial_z \\ \partial_{\bar{z}} & 0 \end{pmatrix}, \quad (3.29)$$

then,

$$\gamma^0 \gamma^\alpha \partial_\alpha = 2 \begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\bar{z}} \end{pmatrix}. \quad (3.30)$$

Rewriting everything in complex coordinates, $g^{1/2} = 1/2$, and the non-zero components of gamma matrices are $\gamma^z_{++} = \gamma^{\bar{z}}_{--} = 1$. It is easy to see that equation (3.26) can be split in their components and can be written as

$$\mathcal{S}_F = -\frac{1}{2\pi} \int d^2 z (\psi \partial_{\bar{z}} \psi + \tilde{\psi} \partial_z \tilde{\psi}). \quad (3.31)$$

Looking at the equations of motion, we note that ψ^μ and $\bar{\psi}^\mu$ are holomorphic and anti-holomorphic fields respectively

$$\partial_{\bar{z}} \psi^\mu = \partial_z \bar{\psi}^\mu = 0. \quad (3.32)$$

The propagator and the energy-momentum tensor can be obtained in a similar way it was computed for the bc system

$$\langle \psi^\mu(z)\, \psi^\nu(w) \rangle = \frac{g^{\mu\nu}}{z - w}, \quad (3.33)$$

$$T_\psi(z) = -\frac{1}{2} \psi^\mu(z) \partial_z \psi_\mu(z). \quad (3.34)$$

And we can obtain the OPE

$$T_\psi(z) T_\psi(w) = \frac{D/4}{(z-w)^4} + 2\frac{T_\psi(w)}{(z-w)^2} + \frac{\partial_w T_\psi(w)}{z - w}. \quad (3.35)$$

As it was mentioned above, as in the bosonic case comformal symmetry were still presents after fixing the local symmetries, the action (3.97) has a *superconformal* symmetry remaining [16][17][18]. In order to exploit this symmetry we focus the next section to explore it.

3.2 Superconformal field theory

As we saw in our study of bosonic strings, the natural arena to be defined a two-dimensional conformal field theory is over a one dimensional complex manifold endowed with a metric (Riemann surfaces).In the same way, superconformal field theories are better interpreted in one dimensional complex supermanifolds locally described by complex supercoordinates $\mathbf{z} = (z, \theta)$ (z, is the ordinary complex coordinate and θ an anticommuting complex coordinate).

In complex basis the superderivatives take the form

$$D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}, \qquad \bar{D} = \frac{\partial}{\partial\bar{\theta}} + \bar{\theta}\frac{\partial}{\partial\bar{z}}\,, \tag{3.36}$$

which can be seen as the square root of the ordinary derivative

$$D^2 = \partial_z, \qquad D = \partial_\theta + \theta\partial_z\,. \tag{3.37}$$

A function is called super analytic if $\bar{D}f = 0$, thus $f = f(z, \theta)$. From this it follows that under a super analytic transformation $\mathbf{z} \to \mathbf{z}'$ the superderivative transforms as

$$\begin{aligned} D &= \partial_\theta\theta'\partial_{\theta'} + \partial_\theta z'\partial_{z'} + \theta(\partial_z\theta'\partial_{\theta'} + \partial_z z'\partial_{z'}) \\ &= (\partial_\theta\theta' + \theta\partial_z\theta')\partial_{\theta'} + (\partial_\theta z'\partial_{z'} + \theta\partial_z z'\partial_{z'})\partial_{z'} \\ &= (D\theta')(\partial_{\theta'} + \theta\partial_{z'}) - (D\theta')\theta'\partial_{z'} + (Dz')\partial_{z'} \\ D &= (D\theta')D' + \left[Dz' - \theta' D\theta'\right] D'^2\,. \end{aligned} \tag{3.38}$$

Super analytic transformations that takes superderivatives into a multiple of itself are called superconformal

$$D = (D\theta')D' \qquad Dz' = \theta' D\theta'\,. \tag{3.39}$$

We define the superconformal tensor fields $\phi(\mathbf{z})$ of weight $(h, 0)$ by the condition that transform as

$$\phi(\mathbf{z}) = (D\theta')^{2h}\phi'(\mathbf{z})\,. \tag{3.40}$$

Defining the displacements

$$z_{12} = z_1 - z_2 - \theta_1\theta_2, \qquad \theta_{12} = \theta_1 - \theta_2\,. \tag{3.41}$$

The Taylor expansion for a super analytic function around $\mathbf{z}_2 = (z_2, \theta_2)$ is

$$\sum_{n=0}^{\infty} \frac{1}{n!} z_{12}^n \left[f(\mathbf{z}_2) + \theta_{12} D_2 f(\mathbf{z}_2)\right]\,. \tag{3.42}$$

Using this we can obtain the analogues for Cauchy's theorem. Noting that

$$\begin{aligned} \frac{1}{2i\pi}\int dz_1 d\theta_1 \frac{f(z_1,\theta_1)(\theta_{12})}{z_{12}^{n+1}} &= \frac{1}{2i\pi}\int dz_1 d\theta_1 \frac{f(z_1,\theta_1)(\theta_{12})}{(z_1 - z_2)^{n+1}} \left(1 - \frac{\theta_1\theta_2}{z_1 - z_2}\right)^{-(n+1)} \\ &= \frac{1}{2i\pi}\int dz_1 d\theta_1 \frac{f(z_1,\theta_1)(\theta_1 - \theta_2)}{(z_1 - z_2)^{n+1}} \\ &= \frac{1}{n!}\partial_{z_2}^n f(z_2, \theta_2)\,, \end{aligned} \tag{3.43}$$

noting that $(\theta_1-\theta_2)$ works as Dirac delta function. The same for

$$
\begin{aligned}
\frac{1}{2i\pi}\int dz_1 d\theta_1 \frac{f(z_1,\theta_1)}{z_{12}^{n+1}} &= \frac{1}{2i\pi}\int dz_1 d\theta_1 \frac{f(z_1,\theta_1)}{(z_1-z_2)^{n+1}}\left(1-\frac{\theta_1\theta_2}{z_1-z_2}\right)^{-(n+1)} \\
&= \frac{1}{2i\pi}\int dz_1 d\theta_1 \left[\frac{f(z_1,\theta_1)}{(z_1-z_2)^{n+1}}+(n+1)\frac{\theta_1\theta_2 f(z_1,\theta_1)}{(z_1-z_2)^{n+2}}\right] \\
&= \frac{1}{2i\pi}\int dz_1 \left[\frac{\partial_{\theta_2} f(z_1,\theta_2)}{(z_1-z_2)^{n+1}}+(n+1)\frac{\theta_2 f(z_1,\theta_1)}{(z_1-z_2)^{n+2}}\right] \\
&= \frac{\partial_z^n}{n!}\left[D_2 f(z_2,\theta_2)\right] . \qquad (3.44)
\end{aligned}
$$

Let $V(z,\theta)=v_0+\theta\, v_1$ be an infinitesimal super vector field that generates infinitesimal supercoordinate transformations.

$$
V(z,\theta)=v_0+\theta\, v_1 . \qquad (3.45)
$$

A superconformal transformations $\mathbf{z}\to\mathbf{z}'=\mathbf{z}+\delta\mathbf{z}$ (generated by an infinitesimal variation $\delta\mathbf{z}=\delta\, z+\theta\delta\theta$) must satisfy (3.39). It can be solved perturbatively,

$$
z'=z+\delta\, z \qquad \theta'=\theta+\delta\theta . \qquad (3.46)
$$

The most general analytic variations for the components

$$
\delta\, z=\epsilon(z)+\theta\eta(z) \qquad \delta\theta=\varphi+\theta\varsigma , \qquad (3.47)
$$

with ϵ and ς commuting, η and φ anticommuting. With this infinitesimal variations, condition (3.39) yields

$$
\theta(\partial_z\epsilon(z))+\eta(z)=\varphi(z)+2\theta\varsigma(z) \qquad (3.48)
$$

and by comparing both sides we see that $\varphi(z)=\eta(z)$ and $1/2\partial_z\epsilon(z)=\varsigma(z)$ and the variation of the components stays as $\delta\, z=\delta\, z=\epsilon(z)+\theta\eta(z)$ and $\delta\theta=\eta(z)+1/2\theta\partial_z\epsilon$. Setting this conformal variation in the form of (3.45), we obtain that

$$
\begin{aligned}
\delta\, z &= v_0+\frac{1}{2}\theta\, v_1 \qquad (3.49)\\
\delta\theta &= \frac{1}{2}(v_1+\theta\partial_z v_0)=\frac{1}{2}DV . \qquad (3.50)
\end{aligned}
$$

Now we can consider the infinitesimal form of (3.40)

$$
\begin{aligned}
\phi(\mathbf{z}) &= \phi'(\mathbf{z}+\delta\mathbf{z})\left[D(\theta+\delta\theta))\right] \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}z_{12}^n\left[\phi'(\mathbf{z})+\theta_{12}D_2\phi'(\mathbf{z})\right](1+2h\, D\delta\theta) \\
&= \left(\phi'(\mathbf{z})+\delta\theta\, D\phi'(\mathbf{z})+\delta\, z\partial_z\phi'(\mathbf{z})+\theta\delta\theta\partial_z\phi'(\mathbf{z})\right)(1+2h\, D(DV)) . \qquad (3.51)
\end{aligned}
$$

Reordering and considering (3.50) we obtain the infinitesimal variation for a superconformal field of weight h

$$\delta\phi(\mathbf{z}) = [V\partial_z + 1/2(DV)D + h\partial_z V]\,\phi(\mathbf{z})\,. \tag{3.52}$$

3.2.1 Super energy-momentum tensor

In the last subsection we reviewed how superconformal fields transform under superconformal transformations. These field transformations are generated by the chiral $(3/2, 0)$ field, the super energy-momentum Tensor $T(z,\theta)$

$$T(z,\theta) = T_F(z) + \theta\, T_B(z)\,, \tag{3.53}$$

where T_B is the usual energy-momentum tensor and T_F generates supersymmetric transformations. The action of $T(z,\theta)$ on $\mathcal{O}$, a superconformal field, can be written as a superspace contour integral

$$\delta_V\phi = \oint dz_1\, d\theta_1 V(z_1,\theta_1)T(z_1,\theta_1)\mathcal{O}(z_2,\theta_2)\,. \tag{3.54}$$

As we did for the bosonic string, the OPE $T(\mathbf{z}_1)\phi(\mathbf{z}_2)$ is obtained by comparing (3.54) with (3.52) and considering the results obtained for the contour integrals (3.118) and (3.119),

$$T(z_1,\theta_1)\phi(z_2,\theta_2) \sim h\frac{\theta_{12}}{z_{12}^2}\phi(z_2,\theta_2) + \frac{1/2}{z_{12}}D_2\phi(z_2,\theta_2) + \frac{\theta_{12}}{z_{12}}\partial_{z_2}\phi(z_2,\theta_2)\,. \tag{3.55}$$

One of the components of the super Energy-Momentum tensor is the ordinary Stress tensor which is an anomalous tensor field, so we expect that $T(z,\theta)$ be an anomalous superconfomal field which may transform as a conformal field (3.52) just for a small subgroup of conformal transformations. Due to T_B this anomalous term must only depend on derivatives of V and independent of T_B. The term $\partial_z^2 DV(z,\theta)$ has the correct weight and gives the expected anomalous term for T_B. From this heuristic argument we write the infinitesimal transformation for the super Energy-Momentum tensor as

$$\delta_V T(\mathbf{z}) = [V\partial_z + 1/2(DV)D + 3/2\partial_z V]\,T(\mathbf{z}) + \frac{\hat{c}}{8}\partial_z^2 DV\,, \tag{3.56}$$

where the factor $\frac{\hat{c}}{8}$ has been included for convenience. The OPE $T(z_1,\theta_1)T(z_2,\theta_2)$ can be reconstructed from (3.54)

$$T(z_1,\theta_1)T(z_2,\theta_2) \sim h\frac{\theta_{12}}{z_{12}^2}T(z_2,\theta_2) + \frac{1/2}{z_{12}}D_2T(z_2,\theta_2) + \frac{\theta_{12}}{z_{12}}\partial_{z_2}T(z_2,\theta_2) + \frac{\hat{c}}{4}\frac{1}{z_{12}^3}\,. \tag{3.57}$$

Exact solutions of $\partial_{z_1} D_1 V(z_1,\theta_1)$ represent globally defined superconformal vector fields on the sphere.

$$V(z_1,\theta_1) = a + bz_1 + cz_1^2 + \theta_1(m + nz_1)\,, \tag{3.58}$$

where, a, b, c, n, m, are complex parameters. The finite form of these transformations, obtained from successive infinitesimal transformations take the form

$$\theta_1' = \theta_1 + \frac{\theta_{12}}{z_{12}} \qquad z_1' = z_0 + \frac{\alpha + \theta_1\theta_0}{z_{12}}\,, \tag{3.59}$$

where the parameters are (z_0,θ_0), (z_2,θ_2) and α. These transformations yield the $\widehat{SL}_2$ group of transformation, the superconformal generalization of the $SL(2,\mathbb{C})$ group for ordinary conformal transformations. In order to know how the components of a super conformal field $\phi(z,\theta) = \phi_0 + \theta\phi_1$ vary under super conformal transformations, we expand (3.55) as

$$\begin{aligned}
T(z_1,\theta_1)T(z_2,\theta_2) &= (T_F + \theta\, T_B)(z_1)(\phi_0 + \theta\phi_1)(z_2) \\
&= h\frac{\theta_{12}}{z_{12}^2}\phi(z_2,\theta_2) + \frac{1/2}{z_{12}}D_2\phi(z_2,\theta_2) + \frac{\theta_{12}}{z_{12}}\partial_{z_2}\phi(z_2,\theta_2) \\
&= h\frac{\theta_{12}}{(z_1-z_2)^2}\left(1 + 2\frac{\theta_1\theta_2}{z_1-z_2}\right)\phi + \frac{1/2D_2\phi}{z_1-z_2}\left(1 + \frac{\theta_1\theta_2}{z_1-z_2}\right) \\
&\quad + \frac{\theta_{12}}{z_1-z_2}\left(1 + \frac{\theta_1\theta_2}{z_1-z_2}\right)\partial_{z_2}\phi \\
&= 1/2\frac{\phi_1}{z_1-z_2} + \theta_1\theta_2\left[(h+1/2)\frac{\phi_1}{(z_1-z_2)^2} + \frac{\partial_z\phi_1}{z_1-z_2}\right] \\
&+ \quad \theta_1\left[h\frac{\phi_0}{(z_1-z_2)^2} + \frac{\partial_z\phi}{z_1-z_2}\right] - \theta_2\left[h\frac{\phi_0}{(z_1-z_2)^2} + 1/2\frac{\partial_z\phi}{z_1-z_2}\right]\,,
\end{aligned}$$

where we have expanded ϕ and D_2 in their components. We obtain, after comparing term by term in θ

$$\begin{aligned}
T_B(z_1)\phi_0(z_2) &= h\frac{\phi_0}{(z_1-z_2)^2} + \frac{\partial_z\phi_0}{z_1-z_2}\,, \\
T_B(z_1)\phi_1(z_2) &= (h+1/2)\frac{\phi_1}{(z_1-z_2)^2} + \frac{\partial_z\phi_1}{z_1-z_2}\,, \\
T_F(z_1)\phi_0(z_2) &= 1/2\frac{\partial_z\phi_1}{z_1-z_2}\,, \\
T_F(z_1)\phi_1(z_2) &= h\frac{\phi_0}{(z_1-z_2)^2} + 1/2\frac{\partial_z\phi_0}{z_1-z_2}\,,
\end{aligned} \tag{3.60}$$

As it is expected $\epsilon(z)T_B(z)$ generates conformal transformations. We can see $\eta(z)T_F(z)$, for $\eta(z)$ a fermionic parameter, transforms the superpartner fields ϕ_0 and ϕ_1 into each other . We define the superconformal transformation for a conformal field $\mathcal{O}$ as

$$\delta_\eta \mathcal{O}(z) = \int \frac{dw}{2i\pi}\eta(w)T_F(w)\mathcal{O}(z)\,, \tag{3.61}$$

which produce the supersymmetric transformations

$$\delta_\eta\phi_0(z) = 1/2\eta\phi_1(z), \qquad \delta_\eta\phi_1(z) = h\partial_z\eta\phi_0(z) + 1/2\eta\partial_z\phi_0(z)\,, \tag{3.62}$$

Now we verify that the commutator of two superconformal transformations is a conformal transformation

$$\begin{aligned}
[\delta_{\eta_1}, \delta_{\eta_1}]\,\phi_0(z) &= (\eta_1\eta_2 - \eta_2\eta_1)\phi(z) \\
&= \delta_{\eta_1}(1/2\eta_2\partial_z\phi_1(z)) - \delta_{\eta_2}(1/2\eta_1\partial_z\phi_1(z)) \\
&= \eta_1(1/2\eta_2\partial_z\phi_1(z)) - \eta_2(1/2\eta_1\phi_1(z)) \\
&= 1/2\left[-\eta_2(h\partial_z\eta_1\phi_0(z) + 1/2\eta_1\partial_z\phi_0(z)) + \eta_1(h\partial_z\eta_2\phi_0(z) + 1/2\eta_2\partial_z\phi_0(z))\right] \\
&= 1/2\left[2\eta_1\eta_2\partial_z\phi_0(z) + h\phi_0((\eta_1)\partial_z\eta_2 + (\partial_z\eta_1)\eta_2\right] \\
[\delta_{\eta_1}, \delta_{\eta_1}]\,\phi_0(z) &= \eta_1\eta_2\partial_z\phi_0(z) + h\phi_0\partial_z(\eta_1\eta_2)\,.
\end{aligned} \tag{3.63}$$

And the commutator of a conformal and superconformal transformation produce a conformal transformation

$$\begin{aligned}
[\delta_\epsilon, \delta_\eta]\,\phi_0(z) &= \delta_\epsilon(1/2\eta\partial_z\phi_1(z)) - \delta_\eta(h\partial_z\epsilon\phi_0(z) + \epsilon\partial_z\phi_0(z)) \\
&= (h+1/2)\partial_z\epsilon\left(\frac{\eta}{2}\phi_1\right) + \epsilon\partial_z\left(\frac{\eta}{2}\phi_1\right) - \left(h\frac{\eta}{2}\partial_z\epsilon\phi_1\right) + \epsilon\partial_z\left(\frac{\eta}{2}\phi_1\right) \\
&= \left[(h+1/2)\frac{\eta}{2}\partial_z\epsilon + \frac{1}{2}\epsilon\partial_z\eta - h\frac{\eta}{2}\partial_z\epsilon - \frac{1}{2}\epsilon\partial_z\eta\right]\phi_1(z) \\
[\delta_\epsilon, \delta_\eta]\,\phi_0(z) &= \frac{1}{2}\eta\partial_z\epsilon\phi_1(z)\,.
\end{aligned} \tag{3.64}$$

The same algebra holds on ϕ_1. We can get the product expansion among the components of $T(z,\theta)$ from (3.57), in the same way we obtained (3.60).

$$T_B(z_1)T_B(z_2) = \frac{3\hat{c}/4}{(z_1-z_2)^4} + \frac{2T_B(z_2)}{(z_1-z_2)^2} + \frac{\partial_z T_B(z_2)}{z_1-z_2}\,, \tag{3.65}$$

$$T_B(z_1)T_F(z_2) = \frac{3/2T_F(z_2)}{(z_1-z_2)^2} + \frac{\partial_z T_F(z_2)}{z_1-z_2}\,, \tag{3.66}$$

$$T_F(z_1)T_F(z_2) = \frac{\hat{c}/4}{(z_1-z_2)^3} + \frac{1/2\partial_z T_F(z_2)}{z_1-z_2}\,, \tag{3.67}$$

The central charge of the superconformal system, $3/2\hat{c}$, comes from the contributions of $\hat{c}$ and $\hat{c}/2$ for the bosons and fermions respectively. As we can see from the OPE's with T_B, the fields ϕ_0, ϕ_1, T_F and T_B are holomorphic fieds of weights h, $h+1/2$, $3/2$, and 2 respectively, and expand in Laurent series as

$$T_B(z) = \sum_n \frac{L_n}{z^{n+2}} \qquad T_F(z) = \frac{1}{2} \sum_{n\in\mathbb{Z}+\nu} \frac{G_n}{z^{n+3/2}}\,, \tag{3.68}$$

$$\phi_0(z) = \sum_n \frac{\phi_{0,n}}{z^{n+h}} \qquad \phi_1(z) = \sum_n \frac{\phi_{1,n}}{z^{h+1/2}}\,. \tag{3.69}$$

And proceeding as we did in the last chapter for the Virasoro modes, we get the algebra for the superconformal generators

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{\hat{c}}{8}(m^3 - m)\delta_{m+n}\,, \\ \{G_m, G_n\} &= 2L_{m+n} + \frac{\hat{c}}{2}(m^2 - 1/4)\delta_{m+n}\,, \\ [L_m, G_n] &= (1/2m - n)G_{m+n}\,. \end{aligned} \tag{3.70}$$

Conformal fields with odd conformal weight (2d spinors) can be double-valued. We introduced ν in (3.68) to signalize two sectors. For $\nu = 1/2$ the modes are half-integers and T_F is single valued because of Laurent expansion in integer powers of z(Neveu-Schwarz sector). For $\nu = 0$ the modes of expansion are integers and T_F is doubled-value on the complex plane (Ramond sector) In general, for $\Theta(z)$, a conformal field with half-integer weight h, in the Ramond sector

$$\Theta(e^{i2\pi}z) = \sum_{n\in\mathbb{Z}} \theta_n z^{-(n+h)} e^{-2hi\pi} = -\Theta(z)\,. \tag{3.71}$$

Consequently, the algebra (3.70) split in two sectors as well the Hilbert space of the theory.

3.2.2 Hilbert space

Let $|h\rangle$ be a state of weight h. From (3.70), commutation relations with L_0 produce the relations

$$\begin{aligned} [L_0, L_n] = -nL_n \qquad &\to \qquad L_n|h\rangle = (h-n)L_n|h\rangle\,, \\ [L_0, G_r] = -nG_n \qquad &\to \qquad G_r|h\rangle = (h-r)G_r|h\rangle\,. \end{aligned} \tag{3.72}$$

Now we can see that L_n and G_r works as annihilation operators for $n, r > 0$. Highest weight states $|h\rangle$ of the algebra (3.70) are defined as

$$\begin{aligned} L_0 |h\rangle &= h |h\rangle \,, \\ L_n |h\rangle &= 0 \qquad n > 0 \,, \\ G_r |h\rangle &= 0 \qquad r > 0 \,. \end{aligned} \tag{3.73}$$

Unitarity of superconformal quantum theory impose the following restrictions:
For $n > 0$,

$$\begin{aligned} \langle h|L_m L_{-m}|h\rangle &= \langle h| [L_m, L_{-m}] |h\rangle \\ &= \langle h|2mL_0 + \frac{\hat{c}}{8}(m^3 - m)|h\rangle \\ &= 2mh + \frac{\hat{c}}{8}(m^3 - m) \geq 0 \,. \end{aligned} \tag{3.74}$$

It follows from $m = 1$ that $h \geq 0$. Also, by taking n large, $\hat{c} \geq 0$ must hold.
For $r > 0$,

$$\begin{aligned} \langle h|G_r G_{-r}|h\rangle &= \langle h| [G_r, G_{-r}] |h\rangle \\ &= \langle h|2L_0 + \frac{\hat{c}}{2}(m^2 - 1/4)|h\rangle \\ &= 2h + \frac{\hat{c}}{2}(m^2 - 1/4) \geq 0 \,. \end{aligned} \tag{3.75}$$

This unitarity condition determines each sector. In the NS sector $h \geq 0$. On the other hand, in the R sector, $r \geq 0$, therefore $h \geq \hat{c}/16$. We will look at each sector by separately.

- Neveu-Schwarz Sector
  We observe that generators $L_{(-1,0,1)}$ and $G_{(-1/2,1/2)}$ belong to NS sector and form a finite sub-algebra free of the anomalous central charge; they are the generators of superconformal global transformations. Hence, the $\widehat{SL}_2$ invariant vacuum, $|0\rangle$ is found in this sector. Regularity of T_B and T_F requires that

$$\begin{aligned} L_n|0\rangle &= 0, \qquad n \geq -1 \,, \\ G_r|0\rangle &= 0, \qquad r \geq -1/2 \,. \end{aligned} \tag{3.76}$$

  It is interesting to note that

$$\begin{aligned} [\eta G_r, \phi_0(z)] &= 2\int \frac{dw}{2i\pi} w^{r+1/2} \eta \, T_F(w) \phi_0(z) \\ &= \int \frac{dw}{2i\pi} w^{r+1/2} \frac{\eta \partial_w \phi_1(z)}{w - z} \\ &= z^{r+1/2} \eta \phi_1(z) \,, \end{aligned} \tag{3.77}$$

and

$$\begin{aligned}
[\eta G_r, \phi_1(z)] &= 2\int \frac{dw}{2i\pi} w^{r+1/2} \eta\, T_F(w)\phi_1(z) \\
&= 2\int \frac{dw}{2i\pi} w^{r+1/2} \eta \left[h\frac{\phi_0}{(w-z)^2} + 1/2 \frac{\partial_z \phi_0}{w-z} \right] \\
&= \eta \left[2(m+1/2) h z^{m-1/2} \phi_0(z) + z^{m+1/2} \partial_z \phi_0(z) \right] . \qquad (3.78)
\end{aligned}$$

It follows that $G_{-1/2}$ is the generator of superconformal transformations (3.62). Let $|h\rangle = \phi(0)|0\rangle$ be a highest weight state. Then

$$\begin{aligned}
\phi_1(0)|0\rangle &= \left[G_{-1/2}, \phi_0(0) \right] |0\rangle \\
&= G_{-1/2}(\phi_0(0)|0\rangle) = G_{-1/2}|h\rangle\,, \qquad (3.79)
\end{aligned}$$

and

$$\begin{aligned}
L_0(G_{-1/2}|h\rangle) &= (\left[L_0, G_{-1/2} \right] + G_{-1/2}L_0)|h\rangle \\
&= (h+1/2)(G_{-1/2}|h\rangle) = (h+1/2)(\phi_1(0)|0\rangle)\,, \qquad (3.80)
\end{aligned}$$

so, $\phi_1(0)|0\rangle$ yields a highest weight state of weight $h+1/2$, as would be expected and we can conclude that in the NS sector, highest weight states correspond to super conformal fields. Moreover, hermiticity condition for T_F gives

$$G_r^\dagger = G_{-r}\,. \qquad (3.81)$$

This result combined with the algebra for the NS sector

$$\left\{ G_{1/2}, G_{-1/2} \right\} = 2L_{-1} = 2(G_{1/2}G_{1/2}^\dagger)\,, \qquad (3.82)$$

shows the superconformal transformation as the square root of translations on the complex plane, this is the operator representation for the global supersymmetric algebra

$$(G_{-1/2})^2 = L_{-1}\,. \qquad (3.83)$$

- Ramond Sector
  In Ramond sector L_0 commute with G_0 and Ramond ground state can be degenerated. From the superconformal algebra we see that

$$\{G_0, G_0\} = 2L_0 - \frac{\hat{c}}{8}\,. \qquad (3.84)$$

We have to note that $L_0 - \frac{\hat{c}}{16}$ is the generator of conformal transformation on the cylinder. We can back to the cylinder trough the inverse map $z = e^w$

$$T_F(w) = \frac{1}{2}\sum_n G_n e^{iwn}\,, \tag{3.85}$$

and G_0, in the cylinder, works as the generator of superconformal transformations

$$G_0 = 2\int d^2w T_F(w)\,. \tag{3.86}$$

If the Ramond ground state $|\Omega\rangle_R$ has conformal weight $\hat{c}/8$ then $G_0^2|\Omega\rangle_R = 0$. This ground state is not produced by the action of superconformal fields as the highest weight states of the NS sector. Ramond vacuum states are obtained by the action of Spinor fields on $\widehat{SL}_2$ invariant vacuum $|0\rangle$ and highest weight states come in orthogonal pairs

$$|\Omega\rangle_R = S^+(0)|0\rangle, \qquad G_0|\Omega\rangle_R = S^-(0)|0\rangle\,. \tag{3.87}$$

In operator form,

$$S^-(0)|0\rangle = G_0 S^+(0)|0\rangle = 2\int dz z^{1/2} T_F(z) S^+(0)|0\rangle\,, \tag{3.88}$$

therefore,

$$T_F(z)S^+(0) \sim \frac{1}{2}\frac{S^-(0)}{z^{3/2}}\,. \tag{3.89}$$

But this degeneracy is unacceptable if we want to maintain supersymmetry unbroken. On the other hand, if $|\Omega\rangle_R = S^+|0\rangle$ is a real vacuum, we easily note that $S^-|0\rangle$ is a null state and decouple since it is perpendicular to every descendent of $|\Omega\rangle_R$. So, we can set $G_0|\Omega\rangle_R = 0$ and stay with the Ramond ground state of weight $\hat{c}/16$ created by the action of a spinor field $S(0)$

$$|\Omega\rangle_R = S(0)|0\rangle\,. \tag{3.90}$$

Like the OPE (3.89), the operator product of Spin fields with the fermionic part of a superfield has an expansion in half integers power of z. In particular for a fermionic field $\phi_f(z)$ acting on $|\Omega\rangle_R$ is expanded in powers of half integers,

$$\phi_f|\Omega\rangle_R = \phi_f(z)S(0)|0\rangle\,, \tag{3.91}$$

but, $\phi_f(z)|0\rangle$ has an expansion in integer powers of z,therefore, $S(0)$ can be thought as the responsible of changing the boundary conditions on the cylinder.

We can conclude that the set of total states of the theory can be obtained by the action of the raising operators of the superconformal algebra on the ground states of each sector. Spin fields relate the ground state of the two sectors. All this can be schematized as

$$\begin{pmatrix} |NS'\rangle \\ |R'\rangle \end{pmatrix} = \begin{pmatrix} \phi & \\ & \phi \end{pmatrix} \begin{pmatrix} |NS\rangle \\ |R\rangle \end{pmatrix} . \tag{3.92}$$

Superconformal fields taking each sector into itself, and Spin fields interpolating different sectors

$$\begin{pmatrix} |NS'\rangle \\ |R'\rangle \end{pmatrix} = \begin{pmatrix} & S \\ S & \end{pmatrix} \begin{pmatrix} |NS\rangle \\ |R\rangle \end{pmatrix} . \tag{3.93}$$

Since the fermionic fields are double-valued, the total spectrum of the superconformal field theory is not local, i.e. the fermionic component of a superfield with a Spin field is expanded in half integers of z. In order to restore locality, we use the GSO projection [19] which eliminates half of the spectrum of each sector, projecting onto a local field theory. It is achieved by applying the projector $\Gamma = (-1)^F$ in the Ramond and Neveu-Schwarz sectors, where F is the world-sheet fermion number. In the NS sector, $\Gamma = 1$ selects even world-sheet fermion number operators and eliminates the tachyon. In the R sector, it acts as the chirality $\gamma^{11} = 2^{-5} \prod_{\mu=0}^{9} \gamma_0^\mu$, selecting left-handed space-time Majorana-Weyl spinors. The purpose of the last section of this chapter is to generate this projection from the requirement of modular invariance in loop diagrams.

3.3 Matter fields

The gauge-fixed action for the fermionic string is

$$\mathcal{S}_{matter} = \frac{1}{2\pi} \int d^2z (\partial_z x^\mu \partial_{\bar{z}} x_\mu - \psi \partial_{\bar{z}} \psi - \tilde{\psi} \partial_z \tilde{\psi}) . \tag{3.94}$$

We can combine free matter fields into a superfield

$$X^\mu(\mathbf{z}, \bar{z}) = X^\mu(\mathbf{z}) + X^\mu(\bar{\mathbf{z}}) , \tag{3.95}$$

where

$$X^\mu(z, \theta) = x^\mu(z) + \theta \psi^\mu(z) . \tag{3.96}$$

In these terms, the action (3.94) takes the form

$$\mathcal{S}_{matter} = \frac{1}{2\pi} \int d^2z d\theta\, d\bar{\theta} \bar{D} X^\mu D X_\mu . \tag{3.97}$$

The equation of motion for the Lagrangian is $\bar{D}DX^\mu = 0$, and $X^\mu(\ddagger, \bar{\ddagger})$ can be separated in holomorphic and anti-holomorphic components as in (3.95)

The two point function on the complex plane is

$$\begin{aligned}
\langle X^\mu(z_1,\theta_1)X^\mu(z,\theta_2)\rangle &= \langle x^\mu(z_1)x^\nu(z_2)\rangle + \theta_1\theta_2\langle\psi^\mu(z_1)x^\nu(z_2)\rangle \\
&= -g^{\mu\nu}\ln(z_1-z_2) + \theta_1\theta_2\frac{g^{\mu\nu}}{z_1-z_2} \\
&= -g^{\mu\nu}\ln\left[(z_1-z_2)\exp\left(\frac{\theta_1\theta_2}{z_1-z_2}\right)\right] \\
&= -g^{\mu\nu}\ln\left[(z_1-z_2)\left(1+\frac{\theta_1\theta_2}{z_1-z_2}\right)\right] \\
&= -g^{\mu\nu}\ln(z_{12})\,.
\end{aligned} \tag{3.98}$$

It is important to mention some useful computions

$$D_1\left(\frac{\theta_{12}}{z_{12}}\right) = D_2\left(\frac{\theta_{12}}{z_{12}}\right) = \frac{1}{z_{12}} \tag{3.99}$$

$$\partial_1\left(\frac{\theta_{12}}{z_{12}}\right) = -\partial_2\left(\frac{\theta_{12}}{z_{12}}\right) = -\frac{\theta_{12}}{z_{12}}\,. \tag{3.100}$$

We can obtain the super energy-momentum Tensor by considering a bilinear field combination of weight 3/2 of the superconformal field and their derivatives. The most general ansatz can be written as $T_{matter} = aDX(D^2X)$ and the factor can be obtained by requiring the correct field transformation

$$\begin{aligned}
(aDX\partial_z X)(\mathbf{z}_1)DX(\mathbf{z}_2) &= -a\left[DX(\mathbf{z}_1)\partial_1(\log\, z_{12}) + \partial_z X(\mathbf{z}_1)D_1(\log\, z_{12})\right] \\
&= -a\left[\frac{DX(\mathbf{z}_1)}{z_{12}} + \frac{\partial_z X(\mathbf{z}_1)}{z_{12}}\theta_{12}\right] \\
&= -a\left[\frac{1}{z_{12}}D(X(\mathbf{z}_2)\theta_{12}DX(\mathbf{z}_2) + \frac{\partial_z X(\mathbf{z}_1)}{z_{12}}\theta_{12}\right] \\
&= -2a\left[\frac{1/2D(X(\mathbf{z}_2)}{z_{12}} + \frac{\partial_z X(\mathbf{z}_1)}{z_{12}}\theta_{12}\right]\,.
\end{aligned} \tag{3.101}$$

We will use this procedure to obtain the correct values for operators in superconformal theory. We conclude that $a = -1/2$ in order to obtain the correct OPE for the superconformal field X of weight 0,

$$T_{matter}(\mathbf{z}_2) = -\frac{1}{2}DX\partial_z X(\mathbf{z}_2)\,, \tag{3.102}$$

or in components

$$T_{matter}(\mathbf{z}_2) = -\frac{1}{2}\psi\partial_z x^\mu + \theta(-\frac{1}{2}\partial_z x^\mu\partial_z x_\mu - \frac{1}{2}\partial_z\psi^\mu\psi_\mu)\,. \tag{3.103}$$

3.4 Superconformal ghosts

In adittion to the presence of the reparemetrization ghosts, the bc system, we obtained for the fermionic string the $\beta\gamma$ system (3.16) after fixing the local supersymmetry. In superconformal language, ghost fields c and γ are related to infinitesimal variations on the world-sheet. Meanwhile, b and β are associated with infinitesimal variations of the superconformal constraints. So, the ghost system for the superspace are

$$C = c + \theta\gamma, \qquad B = \beta + \theta\, b\,, \tag{3.104}$$

and the total ghost action take the form

$$S_{ghost} = \int d^2z\, d^2\theta\, B\bar{D}C\,, \tag{3.105}$$

whose equations of motion are

$$\bar{D}C = 0, \qquad \bar{D}B = 0\,, \tag{3.106}$$

proceeding as we did in (3.98), the propagator for the superconformal ghost system can be obtained

$$\langle B(z_1,\theta_1)C(z_2,\theta_2)\rangle = \frac{\theta}{z_{12}}\,. \tag{3.107}$$

The super energy-momentum tensor is achieved by the same argument we got (3.102), the most general bilinear superconformal field of weight $3/2$ is

$$a_1 C\partial_z B + a_2 DCDB + a_3\partial_z C\, B\,, \tag{3.108}$$

and requiring to produce the correct OPE with the ghost superfields. For C

$$\begin{aligned}
T^{gh}(\mathbf{z}_1)C(\mathbf{z}_2) &= (a_1 C\partial_z B + a_2 DCDB + a_3\partial_z C\, B)(\mathbf{z}_1)C(\mathbf{z}_2)\\
&= a_1 C(\mathbf{z}_1)\partial_1\langle B(\mathbf{z}_1)C(\mathbf{z}_2)\rangle + a_2 DC(\mathbf{z}_1)D_1\langle B(\mathbf{z}_1)C(\mathbf{z}_2)\rangle + a_1\partial_1 C(\mathbf{z}_1)\langle B(\mathbf{z}_1)C(\mathbf{z}_2)\rangle\\
&= a_1\theta_{12}\frac{C(\mathbf{z}_1)}{z_{12}^2} + a_2\frac{DC(\mathbf{z}_1)}{z_{12}} - a_3\theta_{12}\frac{\partial_1 C(\mathbf{z}_1)}{z_{12}}\\
&= \frac{a_1\theta_{12}}{z_{12}^2}\left[C(\mathbf{z}_2) + \theta_{12}DC(\mathbf{z}_2) + z_{12}\partial_2 C(\mathbf{z}_2) + \theta_{12}D_2C(\mathbf{z}_2)\right] + \frac{a_2}{z_{12}}\left[DC(\mathbf{z}_2) + \theta_{12}D^2C(\mathbf{z}_2)\right]\\
&- \frac{a_3\theta_{12}}{z_{12}}\partial_2 C(\mathbf{z}_2)\\
&= a_1\frac{\theta_{12}}{z_{12}^2}C(\mathbf{z}_2) + a_2\frac{DC(\mathbf{z}_2)}{z_{12}} + \frac{\theta_{12}}{z_{12}}\partial_2 C(\mathbf{z}_2)(a_1 + a_2 - a_3)\,,
\end{aligned}$$

for C a superconformald field of weight -1, the OPE is correct if $a_1 = -1$, $a_2 = 1/2$, $a_3 = -3/2$. Therefore the correct form for T^{gh} is

$$T^{gh}(\mathbf{z}) = -C\partial_z B + \frac{1}{2}DCDB - \frac{3}{2}\partial_z C\, B\,, \tag{3.109}$$

and this is expressed in components as follows

$$T^{gh}(z,\theta) = \left(-c\partial_z\beta + \frac{1}{2}\gamma\, b - \frac{3}{2}\partial_z c\beta\right) + \theta\left(c\partial_z b + 2(\partial\, c)b - \frac{1}{2}\gamma\partial_z\beta - \frac{3}{2}(\partial_z\gamma)\beta\right)\,. \tag{3.110}$$

The matter and ghost energy-momentum tensor can be put together to give the total stress-tensor

$$T^{tot}(z,\theta) = T^{matter}(z,\theta) + T^{gh}(z,\theta)\,. \tag{3.111}$$

The free Weyl anomaly condition is imposed on the fermionic string, giving the critical dimension $D = 10$. This can be shown by taking into account that the OPE

$$T_B^{tot}(z_1)T_B^{tot}(z_2)\,, \tag{3.112}$$

can be obtained by the summation of their components (2.156),2.157, (3.22) and (3.35). The free anomaly condition reduces to

$$T_B^{tot}(z_1)T_B^{tot}(z_2) \sim \frac{1/2}{(z_1 - z_2)^4}\left\{(3D/2) + (9 - 12\lambda)\right\} + \ldots\,, \tag{3.113}$$

for $\lambda = 2$, we have the expected result

$$D = 10\,. \tag{3.114}$$

3.4.1 Mode expansion

The mode expansion for the holomorphic free fields are

$$\partial_z x^\mu(z) = \sum_n \frac{\alpha_n}{z^{n+1}} \qquad \psi(z) = \sum_n \frac{\psi_n}{z^{n+1/2}}\,, \tag{3.115}$$

$$c(z) = \sum_n \frac{c_n}{z^{n-1}} \qquad \gamma(z) = \sum_n \frac{\gamma_n}{z^{n-1/2}}\,, \tag{3.116}$$

$$b(z) == \sum_n \frac{b_n}{z^{n+2}} \qquad \beta(z) = \sum_n \frac{\beta_n}{z^{n++3/2}}\,. \tag{3.117}$$

By using the above mode expansions we can findd the form of the superconformal generators for the matter and ghost systems. Let us obtain the mode expansion for T_F^{gh} the superconfomal generator of the ghost system

$$T_F^{gh} = \sum_k \frac{G_k^{gh}}{z^{k+3/2}} . \tag{3.118}$$

It can be expanded as

$$\begin{aligned} T_F^{gh} &= \sum_{m,n} (n+5/2) \frac{c_m}{z^{m-1}} \frac{\beta_n}{z^{n+3/2}} + \frac{3}{2} \sum_{m,n} (m-1) \frac{c_m}{z^m} \frac{\beta_n}{z^{n+3/2}} + \frac{1}{2} \sum_{p,q} \frac{b_p}{z^{p+2}} \frac{\gamma_q}{z^{q-1/2}} \\ &= \sum_{m,n} \left[(n+3/2\, m) \frac{c_m \beta_n}{z^{m+n+3/2}} + \frac{1}{2} \frac{b_m}{z^{n+2}} \frac{\gamma_n}{z^{n-1/2}} \right] , \end{aligned} \tag{3.119}$$

renaming $m+n=k$, we get the G^{gh} expansion by comparing (3.118) and (3.119). In the same way we get the mode expansion if the Virasoro generators and analogous for the matter system

$$G_k^{gh} = \sum_n \left[(3k-n) c_{l-n} \beta_n + \frac{1}{2} b_n \gamma_{l-m} \right] , \tag{3.120}$$

$$L_k^{gh} = \sum_m (m-2k) : c_{l-m} b_m : + \sum_r (\frac{3}{2} k - r) : \gamma_{k-r} \beta_r : + a^g \delta_{k,0} , \tag{3.121}$$

$$G_k^{matter} = \sum_n -\psi^\mu_{k-n} \alpha^\mu_n , \tag{3.122}$$

$$L_k^{matter} = \sum_m -\frac{1}{2} : \alpha^\mu_{k-m} \alpha^\mu_m : + \frac{1}{2} \sum_r (\frac{1}{2} k - r) : \psi^\mu_{k-r} \psi_{r\mu} : + a^m \delta_{k,0} . \tag{3.123}$$

In the NS sector the mode expansion r is in half integers ,while in the Ramond sector all the indices are integers. We have added a normal ordering constant to the Virasoro modes due to the ordering ambiguity of the zero modes. The values of these constant will be obtained further in our discussion. The algebra of the modes are computed by contour integrals of the operator product expansion of the two fields. We enumerate some important relations

$$[\alpha^\mu_m, \alpha^\nu_n] = -m g^{\mu\nu} \delta_{m+n} , \qquad \{\psi^\mu_m, \psi^\nu_n\} = -g^{\mu\nu} \delta_{m+n} , \tag{3.124}$$

$$[\gamma_m, \beta_n] = \delta_{m+n} , \qquad \{c_m, b_n\} = \delta_{m+n} . \tag{3.125}$$

3.4.2 Ramond ground state and $SO(9,1)$ current

As we have seen the total Hilbert space of the theory can be split in two sectors. The full Hilbert space of the theory are generated from the ground states of these sectors by the action of raising operators α^{μ}_{-n} and ψ_{-n}, for $n>0$. In the NS sector there is a ground state $|k\rangle$ for each eigenvalue of $k^{\mu}=\alpha^{\mu}_0$ satisfying

$$\psi^{\mu}_n k^{\mu} = \alpha^{\mu}_0 = 0\,, \qquad n>0\,. \tag{3.126}$$

It follows that the NS sector contains only states which transform as Lorentz vectors.

Of particular interest is the relation in the Ramond sector

$$\{\psi^{\mu}_0, \psi^{\nu}_0\} = -g^{\mu\nu}\,, \tag{3.127}$$

which can be identified as the Dirac algebra and the fermionic zero modes are $\Gamma^{\mu}=\sqrt{2}\psi^{\mu}_0$. The R ground state is degenerate because $\{\psi^{\mu}_n, \psi^{\nu}_0\}=0$, so the fermionic zero modes take R ground states into R ground states. Spin fields, which create them acting on the NS vacuum, must be $SO(9,1)$ spinors of weight 5/8 (for $\hat{c}=D=10$). The R ground states can be labeled by the spinor indexes as

$$S_{\alpha}(0)|k\rangle = |k,\alpha\rangle\,. \tag{3.128}$$

The total spectrum is obtained by the action of vector operators which modify the spin by integers, then the Ramond sector only contains $SO(9,1)$ spinors, spacetime fermions. The computation of fermionic string amplitudes requires some correlation functions between Spin fields and fermionic fields. We will obtain some important results by making use of the $SO(9,1)$ current. Let us define

$$J^{\mu\nu}(w) =: \psi^{\mu}(w)\psi^{\mu}(w):\,, \tag{3.129}$$

which holds for the operator product expansion

$$J^{\mu\nu}(z)\psi^{\lambda}(w) \sim \frac{1}{z-w}\left[g^{\mu\lambda}\psi^{\nu}(w) - g^{\nu\lambda}\psi^{\mu}(w)\right]\,. \tag{3.130}$$

The space-time Lorentz symmetry of the fermionic string is produced by $J^{\mu\nu}(z)$ from which ψ^{μ} transform as a vector. This current is a representation of the generators of the $SO(9,1)$ current algebra. Although we use a particular representation for the current algebra, it is independent of the particular way we represent it in terms of conformal fields.

On the other hand, the spinorial nature of S_α means that it has to transform under a $SO(9,1)$ current as

$$J^{\mu\nu}(z)S_\alpha(w) \sim \frac{1/2}{z-w}(\gamma^{\mu\nu})_\alpha^\beta S_\beta(w)\,, \qquad (3.131)$$

where $\gamma^{\mu\nu} = \frac{1}{2}\gamma^{[\mu}\gamma^{\nu]}$ is the generator of Lorentz transformations for a ten dimensional spinor.

Moreover, we know that the insertion of S_α at $z=0$ flips the boundary conditions of the fields between periodic and anti-periodic. The branch cut produced on ϕ^μ by the action of S_α can be demand by requiring that the OPE between them expands in half-integer powers

$$\psi^\mu(z)S_\alpha(w) \sim \frac{\gamma^\mu_{\alpha\beta}S^\beta}{(z-w)^{1/2}}\,. \qquad (3.132)$$

In this way, we have used the $SO(9,1)$ current in order to obtain some important coefficients of operator product expansion. The advantage of this approach is its manifest Lorentz covariance. Some other important OPE's which can be constructed from those are

$$\begin{aligned}
J^{\mu\nu}(z)J^{\sigma\tau}(w) &= \frac{g^{\mu\tau}g^{\nu\sigma}-(\mu\longleftrightarrow\nu)}{(z-w)^2}+\frac{1}{z-w}\left[g^{\mu\sigma}J^{\nu\tau}(1-\mu\longleftrightarrow\nu)(\sigma\longleftrightarrow\tau)\right] \\
S^\alpha(z)S_\beta(w) &= \frac{\delta^\alpha_\beta}{(z-w)^{5/4}}+\frac{\frac{1}{2}(\gamma_\mu\gamma_\nu)^\alpha_\beta J^{\mu\nu(w)}}{(z-w)^{1/4}} \qquad (3.133) \\
S_\alpha(z)S_\beta(w) &= \frac{\gamma^\mu_{\alpha\beta}\psi_\mu}{(z-w)^{3/4}}\,. \qquad (3.134)
\end{aligned}$$

3.5 Super BRST current

Using the general prescription for constructing the BRST charge for a general constraint algebra, the super-BRST charge can be defined as the line integral of the current

$$J_{BRST} \sim CT^{matter} + \frac{1}{2}CT^{gh}\,. \qquad (3.135)$$

Using (3.109) it can be written as

$$\begin{aligned}
&-\frac{1}{2}CDX\partial_z X+\frac{1}{4}CDCDB-\frac{3}{4}C\,D(DC)B \\
&-\frac{1}{2}CDX\partial_z X+CDCDB-\frac{3}{4}C\,D(DC\,B)\,,
\end{aligned}$$

As we did for the bosonic case we can add a total derivative term $-\frac{3}{4}D(CDC\,B)$ that does not contribute to the $BRST$ charge, and the BRST current stays as

$$J_{BRST} = -\frac{1}{2}CDX\partial_z X + CDCDB - \frac{3}{4}DCDCB\,, \tag{3.136}$$

and the BRST charge is

$$\varepsilon\, Q_{BRST} = \int \frac{dz\, d\theta}{2i\pi}\varepsilon\, J_{BRST}(\mathbf{z})\,, \tag{3.137}$$

where ε is an anticommuting parameter. Now, it is simple to obtain the BRST transformation of the fields. For example for a matter field Φ with conformal weight h,

$$\begin{aligned}
[\varepsilon\, Q_{BRST}, \Phi(\mathbf{z}_2)] &= \int dz_1\, d\theta_1 \varepsilon(\mathbf{z}_1) T^{matter}(\mathbf{z}_1)\Phi(\mathbf{z}_2) \\
&= \int dz\, d\theta \varepsilon(\mathbf{z}_1)\left[h\frac{\theta_{12}}{z_{12}^2}\Phi(z_2,\theta_2) + \frac{1/2}{z_{12}}D_2\Phi(z_2,\theta_2) + \frac{\theta_{12}}{z_{12}}\partial_{z_2}\Phi(z_2,\theta_2)\right] \\
&= h\partial_z(\varepsilon\, C)\Phi + \frac{1}{2}D(\varepsilon\, C)D\Phi\, + \partial_z(\varepsilon\, C)\partial_z\Phi\,.
\end{aligned} \tag{3.138}$$

In the same way we can obtain the BRST transformations

$$[\varepsilon\, Q_{BRST}, C] = \varepsilon(C\partial_z C - \frac{1}{4}DCDC)\,, \tag{3.139}$$

$$[\varepsilon\, Q_{BRST}, B] = -\varepsilon\, T\,. \tag{3.140}$$

Possible BRST anomalies can be exclude if J_{BRST} transform covarientely on any world surface. This can be imposed by requiring J_{BRST} transform as a superconformal field of weight 1/2, free of anomaly terms

$$T(z_1,\theta_1)J_{BRST}(z_2,\theta_2) \sim \frac{1}{2}\frac{\theta_{12}}{z_{12}^2}J_{BRST}(z_2,\theta_2) + \frac{1/2}{z_{12}}D_2 J_{BRST}(z_2,\theta_2) + \frac{\theta_{12}}{z_{12}}\partial_{z_2}J_{BRST}(z_2,\theta_2)\,. \tag{3.141}$$

We can fulfil this condition by looking at the higher pole oh the expansion $T(\mathbf{z}_1)J_{BRST}(\mathbf{z}_2)$ and spliting the OPE as

$$\begin{aligned}
& T^{gh}(\mathbf{z}_1)J_{BRST}(\mathbf{z}_2) \\
= & \left[-C\partial_z B + \frac{1}{2}DCDB - \frac{3}{2}\partial_z C\, B\right](\mathbf{z}_1)\left[CDCDB - \frac{3}{4}DCDCB - \frac{1}{2}CDX\partial_z X\right](\mathbf{z}_2) \\
= & -\partial_1 D_2\langle B_1C_2\rangle\, D_2\langle C_1B_2\rangle C_2 + \partial_1\langle B_1C_2\rangle\, D_2\langle C_1B_2\rangle D_2C_2 + \frac{3}{4}.2\partial_1 D_2\langle B_1C_2\rangle\langle C_1B_2\rangle D_2C_2 \\
- & \frac{1}{2}D_1D_2\langle B_1C_2\rangle\, D_1D_2\langle C_1B_2\rangle C_2 D_1\langle B_1C_2\rangle\, D_1D_2\langle C_1B_2\rangle D_2C_2 - \frac{3}{8}.2D_1D_2\langle B_1C_2\rangle\, D_1\langle C_1B_2\rangle D_2C_2 \\
- & \frac{3}{2}D_2\langle B_1C_2\rangle\,\partial_1 D_2\langle C_1B_2\rangle C_2 - \langle B_1C_2\rangle\,\partial_1 D_2\langle C_1B_2\rangle D_2C_2 + \frac{9}{8}.2D_2\langle B_1C_2\rangle\,\partial_1\langle C_1B_2\rangle D_2C_2 + ... \\
= & \frac{5}{2}\frac{C_2}{(z_1-z_2)^3} + \frac{D_2C_2}{(z_1-z_2)^3}(\theta_{12})(1-3/2+1/2+3/4+3/2-9/4) + ... \\
= & \frac{5}{2}\frac{C(\mathbf{z}_2)}{(z_1-z_2)^3} + ...\,, \qquad (3.142)
\end{aligned}$$

here we have considered the results (3.107) and (3.100) and dots stand for less singular terms. The contribution to the OPE from matter part of the super energy-momentum tensor is given by

$$\begin{aligned}
T^{matter}(\mathbf{z}_1)J_{BRST}(\mathbf{z}_2) &= -\frac{C(\mathbf{z}_2)}{4}\left[\partial_1 D_2\langle X_{1\mu}X_2^\nu\rangle\, D_1\partial_2\langle X_1^\mu X_{2\nu}\rangle + D_1D_2\langle X_1^\mu X_2^\nu\rangle\,\partial_1\partial_2\langle X_{1\mu}X_{2\nu}\rangle\right] + ... \\
&= -C(\mathbf{z}_2)\frac{D/4}{(z_1-z_3)^3} + ...\,. \qquad (3.143)
\end{aligned}$$

Adding the two terms (3.142) and (3.143) which contribute to the anomaly, we confirm that it vanishes only if $D = 10$. By an analogous argument we used for the bosonic case we can show that the nilpotence of Q_{BRST} only holds if J_{BRST} transforms as a superconformal field. Of course the same statement holds for the anti-chiral piece of the algebra.

Now we proceed to obtain a complete description of the quantized fermionic string by making use of the BRST invariance. In order to compute amplitudes we need to construct BRST invariant vertex operators which describes the different states of both sectors, NS and R. For the NS sector, the highest weight fields are BSRT invariant and it follows from (3.138) that the superspace integral of a superconformal vector field of weight $(1/2, 1/2)$ is BRST invariant. Then, superconformal vertex operators of the NS sector are o f the form

$$V_{NS} = \int d^2z\, d^2\theta V_{h=1/2}(z,\theta)\bar{V}_{\bar{h}=1/2}(\bar{z},\bar{\theta})\,. \qquad (3.144)$$

For example, from the highest weight field $V_B = DX^\mu e^{ik.X}(z,\theta)$,

$$T^{matter}(\mathbf{z}_1)DX^\mu e^{ik.X}(\mathbf{z}_2) \sim \theta_{12}\frac{\frac{1}{2}(k^2+1)}{z_{12}^2})DX^\mu e^{ik.X}(\mathbf{z}_2) + \dots . \tag{3.145}$$

we can construct the BRST invariant massless vertex operator $k^2 = 0$.

$$V_{massless} = \xi_{\mu\nu}(k)\int d^2z\, d^2\theta DX^\mu \bar{D}X^\nu e^{ik.X} \tag{3.146}$$

These kind of vertex operators only represent space time bosons and if we want to obtain the complete picture it is necessary to get fermionic vertex operators which represent the states on the Ramond sector. BRST invariance gives us crucial information about the construction of the physical fermion vertex. After rewriting Q_{BRST} in their ordinary conformal fields components, we can separate this into three sectors, each of these labelled by their spinor ghost charge,

$$\begin{aligned} Q_{BRST} &= Q_0 + Q_1 + Q_2\,, \\ Q_0 &= \int \frac{dz}{2i\pi}(c\tilde{T}_B(x,\psi\,;\beta\,,\gamma) - bc\partial_z c)\,, \\ Q_1 &= -\frac{1}{2}\int \frac{dz}{2i\pi}\gamma\psi^\mu\partial_z x_\mu\,, \\ Q_2 &= \frac{1}{4}\int \frac{dz}{2i\pi}b\gamma^2\,, \end{aligned} \tag{3.147}$$

where $\tilde{T}_B(x,\psi\,;\beta\,,\gamma)$ can be thought as the Energy-Momentum tensor of the fields mentioned; treating γ and β as they were extra matter fields, Q_0 is the bosonic string BRST operator. For an integrated fermionic vertex operator

$$\mathcal{V}_R = \int d^2z V(z,\bar{z})\,, \tag{3.148}$$

Q_0 constraints $V(z,\bar{z})$ to be a primary operator of dimension (1,1). $\mathcal{V}_R$ must create fermionic fields from the NS vacuum, so it has to be constructed out of spin fields. But S_α only has conformal weight 5/8, so we must incorporate some extra fields Σ of weight 3/8 to accomplish this requirement. On the other hand, beacause of the OPE (3.132), Q_1 acts over S_α as a generalized Dirac operator. It is easy to see that a masless operator of the form

$$u^\alpha(k)\Sigma\, S_\alpha e^{ik.X}\,, \tag{3.149}$$

is invariant under Q_1 for $u^\alpha(k)$ the space-time wave function for the massless fermion ($\gamma^\mu k_\mu u^\alpha = 0$, $k^2 = 0$).

3.6 First order free fields

Let $\mathbf{b}$ and $\mathbf{c}$ be conjugate conformal fields of dimension λ and $1-\lambda$ respectively,

$$S = \frac{1}{2\pi}\int d^2z\mathbf{b}\partial_{\bar{z}}\mathbf{c}\,, \tag{3.150}$$

with equations of motion $\partial_{\bar{z}}\mathbf{b} = \partial_{\bar{z}}\mathbf{c} = 0$. Conformal ghost systems bc and $\gamma\beta$ of rhe fermionic string are particular cases for $\lambda = 2$ and $\lambda = 3/2$ respectively. The fields $\mathbf{b}$ $\mathbf{c}$ can be either Bose or Fermi fields. Considering ϵ to be equal 1 for Fermi statistics and -1 for Bose statistics,

$$\mathbf{c}(z)\mathbf{b}(w) \sim \frac{1}{z-w},, \qquad \mathbf{b}(z)\mathbf{c}(w) \sim \frac{\epsilon}{z-w}\,. \tag{3.151}$$

Basic facts to know are

$$\mathbf{c}(z) = \sum_n \frac{\mathbf{c}_n}{z^{n+(1-\lambda)}} \qquad \mathbf{b}(z) = \sum_n \frac{\mathbf{b}_n}{z^{n+\lambda}}\,,$$

$$\mathbf{c}_n^\dagger = \mathbf{c}_n \qquad \mathbf{b}_n^\dagger = \epsilon\mathbf{c}_n\,,$$

$$\mathbf{c}_m\mathbf{b}_n + \epsilon\mathbf{b}_n\mathbf{c}_m = \delta_{m+n}\,. \tag{3.152}$$

We have two sectors in the theory. Fields have expansions in modes $n \in \mathbb{Z} - \lambda + \nu$, $\mu = 0$ for NS sector and $\nu = 1/2$ for Ramond. Energy-momentum tensor satisfies

$$T = -\lambda\mathbf{b}\partial_z\mathbf{c} + (1-\lambda)\partial_z\mathbf{bc}\,,$$

$$T(z)T(w) = \frac{\frac{1}{2}\epsilon(1-3Q^2)}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}\,,$$

$$Q = \epsilon(1-2\lambda)\,. \tag{3.153}$$

3.6.1 U(1) current

The action of the system $\mathbf{bc}$ exhibit a chiral $U(1)$ symmetry whose current is

$$j(z) = -\mathbf{bc}(z) = \epsilon\mathbf{bc}(z)\,. \tag{3.154}$$

The mode expansion of $j(z)$

$$j_n = \sum_l \epsilon\mathbf{c}_{n-l}\mathbf{b}_l\,. \tag{3.155}$$

It will be useful to know the relation with the adjoint modes. By using (3.152) it follows

$$j_n^\dagger = -j_n\,. \tag{3.156}$$

Direct calculations show that

$$j(z)\mathbf{b}(w) \sim \frac{-\mathbf{b}}{z-w}\,, \qquad j(z)\mathbf{c}(w) \sim \frac{\mathbf{c}}{z-w}\,, \tag{3.157}$$

$$\begin{aligned} \mathbf{j}(z)\mathbf{j}(w) &\sim \frac{\epsilon}{(z-w)^2} \\ [j_m, j_n] &= \epsilon\, m\delta_{m+n}\,, \end{aligned} \tag{3.158}$$

$$[j_m, \mathbf{b}_n] = -\mathbf{b}_{m+n}\,, \qquad [j_m, \mathbf{c}_n] = \mathbf{c}_{m+n}\,. \tag{3.159}$$

We note that the contour integral of the current $U(1)$ is the charge operator j_0. As it would be expected from the discussion in the past chapter, the algebra of the chiral and the Energy-Momentum tensor is anomalous.

$$T(z)j(w) \sim \frac{Q}{(z-w)^3} + \frac{j(w)}{(z-w)^2} + \frac{\partial\, j(w)}{z-w}\,, \tag{3.160}$$

then,

$$\begin{aligned} \left[L_m^{gh}, j_n\right] &= \oint \frac{dz}{2i\pi} \oint \frac{dw}{2i\pi} z^{m+1} w^n \left[\frac{Q}{(z-w)^3} + \frac{j(w)}{(z-w)^2} + \frac{\partial\, j(w)}{z-w}\right] \\ &= 2i\pi \oint \frac{dw}{2i\pi}\left[\frac{Q}{2}m(m+1)w^{m-1} + (m+1)w^{m+n}j(w) + \partial\, j(w) w^{n+m+1}\right] \\ &= 2i\pi \oint \frac{dw}{2i\pi}\left[\frac{Q}{2}m(m+1)w^{m-1} + (m+1)w^{m+n}j(w) - (m+n+1)j(w)w^{n+m}\right] \\ &= \frac{Q}{2}m(m+1)\delta_{m+n} - n j_{n+m}\,, \end{aligned} \tag{3.161}$$

we see that $j(z)$ is not a quasi-prymary field because it is only covariant under translations and dilations (L_{-1}, L_0). From the algebra we take into account some important results

$$\begin{aligned} [L_{-1}, j_1] = -j_0\,, &\qquad [L_1, j_{-1}] = Q + j_0\,, \\ [L_1, j_{-1}]^\dagger = \left[L_{-1}, -j_{-1}^\dagger\right] &= [L_{-1}, j_1] = -j_0\,. \end{aligned} \tag{3.162}$$

It follows that $j_0^\dagger = -(j_0 + Q)$. We can see the consecuences of this charge asymmetry by considering the expectation value of an 0perator $\mathcal{O}$ with charge q

$$\begin{aligned} q\langle 0|\mathcal{O}|0\rangle &= \langle 0|\,[j_0\mathcal{O}|0]\rangle \\ &= \langle 0|(j_0^\dagger - Q)\mathcal{O}|0\rangle = -Q\langle 0|\mathcal{O}|0\rangle\,. \end{aligned} \tag{3.163}$$

We only obtain a non-zero amplitude if the operator insertion fulfil the background charge $-Q$. In particular $\langle 0|0\rangle = 0$. In general, for two arbitrary states with charge q and q' respectively

$$\begin{aligned} q\langle q'|q\rangle &= \langle q'|j_0|q\rangle \\ &= \langle q'|-Q-j_0^\dagger|q\rangle = (-q'-Q)\langle q'|q\rangle\,. \end{aligned} \tag{3.164}$$

Then $q = q' + Q$ must hold for $\langle q'|q\rangle$ be different from zero.
The current $U(1)$ is not conserved by itself. This anomaly is related to the difference between zero modes of the ghost fields, the exact solutions of their equations of motion, and this value is given by the Riemann-Roch theorem as we found in the last chapter.

3.6.2 Bosonization

Let us take into consideration the chiral scalar field $\phi(z)$ which satisfies the OPE

$$\phi(z)\phi(w) \sim \epsilon \ln(z-w)\,, \tag{3.165}$$

then, we can rewrite the $U(1)$ current algebra (3.158) by representing the chiral current $J(z)$ as

$$J(z) = \epsilon\partial_z\phi(z)\,. \tag{3.166}$$

We can obtain the OPE's

$$\begin{aligned} J(z)\phi(w) &\sim \epsilon\partial_z\langle\phi(z)\phi(w)\rangle\,, \\ &\sim \frac{1}{z-w} \end{aligned} \tag{3.167}$$

$$\begin{aligned} J(z)e^{\phi(z)} &= \epsilon\partial_z\phi(z)\sum_n \frac{(q\phi(w))^n}{n!} \\ &\sim nq\epsilon\partial_z\langle\phi(z)\phi(w)\rangle\sum_n \frac{(q\phi(w))^{n-1}}{n!} \\ &\sim \frac{q}{z-w}e^{q\phi(w)}\,, \end{aligned} \tag{3.168}$$

thus,

$$\begin{aligned} \left[j_0, e^{q\phi(w)}\right] &= \int\frac{dz}{2i\pi}J(z)e^{q\phi(w)} \\ &= \int\frac{dz}{2i\pi}\frac{q}{z-w}e^{q\phi(w)} \\ &= qe^{q\phi(w)}\,, \end{aligned} \tag{3.169}$$

this means that the field $e^{q\phi(z)}$ is the bosonized representation of a field with current charge $U(1)$ equal to q. We emphasize that the current charge is quantum mechanically anomalous and holds the relation (2.54) in our discussion of the bosonic string. We can construct an action whose equation of motion give us the expected value of this anomaly. It can be obtained from the action

$$S_Q = -\frac{1}{2\pi}\int d^2z\epsilon\partial_z\phi\partial_{\bar{z}}\phi + \frac{1}{4}Qg^{1/2}R\phi\,. \tag{3.170}$$

The energy-momentum tensor for this action is given by

$$T_J(z) = \frac{\epsilon}{2}J^2(z) - \frac{\epsilon\, Q}{2}\partial_z J(z)\,. \tag{3.171}$$

The essential point of this construction is that this Tensor reproduces the algebra (3.160). The central charge for T_J is obtained by the OPE

$$\begin{aligned} T_J(z)T_J(w) &= \frac{1}{4}\left[4J(z)J(w)\langle J(z)J(w)\rangle + 2\langle J(z)J(w)\rangle\langle J(z)J(w)\rangle\right] \\ &\quad \frac{Q}{4}\left[2J(z)\partial_w\langle J(z)J(w)\rangle + 2J(w)\partial_z\langle J(z)J(w)\rangle\right] + \frac{Q^2}{4}\partial_z\partial_w\langle J(z)J(w)\rangle \\ &= \frac{\frac{1}{2}(1-3\epsilon\, Q^2)}{(z-w)^4} + \frac{Q\epsilon}{(z-w)^3}(J(z)-J(w)) + \frac{\epsilon\, J(z)J(w)}{(z-w)^2} \\ &= \frac{\frac{1}{2}(1-3\epsilon\, Q^2)}{(z-w)^4} + \frac{\epsilon(J^2(w)+Q\partial\, J(w))}{(z-w)^2} + \frac{\epsilon\, J(w)\partial\, J(w) + \frac{\epsilon q}{2}\partial^2 J(w)}{z-w} \\ &= \frac{\frac{1}{2}(1-3\epsilon\, Q^2)}{(z-w)^4} + \frac{2T_J(w)}{(z-w)^2} + \frac{\partial\, T_J(w)}{z-w}\,, \end{aligned} \tag{3.172}$$

and the central charge c_j is $1-3Q^2$. Comparing this value with the central charge of the **bc** energy-momentum tensor (3.153) we see that for $\epsilon = 1$, this central charge is the same T_J given a correct representation of the bc algebra. However, for the bosonic ghost (ϵ=-1) c_j differs from the correct value by 2.

$$c_j = 1 - 3Q^2 = \begin{cases} c & \text{for} \quad \epsilon = 1 \\ c+2 & \text{for} \quad \epsilon = -1 \end{cases}\,. \tag{3.173}$$

Therefore,in order to give a complete description of the $\beta\gamma$ system we must include extra fields in order to match the correct central charge. We introduce the energy-momentum tensor with central charge $= -2\ T_{-2}$. This contribution has to come from a fermionic system ($\epsilon = 1$) with charge $Q = -1$. It follows that T_{-2} is the energy-momentum tensor of a fermionic system of fields, namely, η and ξ with conformal weights of 1 and 0 respectively.

Furthermore, by using (3.168) we see the exponential field $e^{q\phi(z)}$ transforms as a conformal field of weight $\epsilon\, q(q+Q)$

$$T_j(z)e^{q\phi(w)} = \frac{\frac{\epsilon}{2}q(q+Q)}{(z-w)^2} + \frac{\partial\, e^{q\phi(w)}}{z-w}\,. \tag{3.174}$$

In this way we obtain the bosonization of the b and c fields for $Q=-3$

$$\begin{aligned} b(z) &= e^{\phi(z)}\,, \qquad q=1\,, \qquad h=2\,, \\ c(z) &= e^{-\phi(z)}\,, \qquad q=-1\,, \qquad h=1\,. \end{aligned}$$

For the $\eta\xi$ system we can work in the same way. Let us define the $U(1)$ chiral current $J_{\eta\xi}(z)$ which can be bosonized as $\partial\chi(z)$, and η and χ can be bosonized as

$$\eta(z) = e^{-\chi(z)}\,, \qquad \xi(z) = e^{\chi(z)}\,, \tag{3.175}$$

The exponentials $e^{\phi(z)}$ and $e^{-\phi(z)}$, for $\epsilon=-1$ and $Q=2$, represent fields of weight $-3/2$ and $1/2$ respectively. In terms of these variables, we can see that

$$\gamma(z) = e^{\phi(z)}\eta(z)\,, \beta(z) = e^{-\phi(z)}\partial_z\xi(z)\,, \tag{3.176}$$

are the correct bosonization of the $\gamma\beta$ system which gives the correct conformal weights of the fields and reproduce the OPE (3.21)

$$\begin{aligned} \beta(z)\gamma(w) &\sim \langle e^{\phi(z)}e^{-\phi(w)}\rangle\partial_w\langle\eta(z)\xi(w)\rangle \\ &\sim (z-w)^{-\epsilon}\frac{1}{(z-w)^2} = \frac{1}{z-w}\,. \end{aligned} \tag{3.177}$$

It is essential to note that an irreducible representation of the $\beta\gamma$ algebra is generated for ϕ, η and $\partial_z\xi$ and the zero mode of ξ never appears in this algebra. The inclusion of ξ_0 makes the representation reducible with a twofold degenerate ground state. To conclude this section we look at possible ambiguities in the definition of the vacuum state. Let us define a different vacuum $|q\rangle$ with charge q. These states can be generated by the action of the field $e^{q\phi(z)}$ at the origin. Then, analyticity at $z=0$ implies

$$\begin{aligned} b_n|q\rangle &= 0, \qquad n > \epsilon\, q - \lambda\,, \\ c_n|q\rangle &= 0, \qquad n \geq \epsilon\, q + \lambda\,, \end{aligned} \tag{3.178}$$

where $q \in \mathbb{Z}+1/2$ for the NS sector and $q\in\mathbb{Z}$ for the R sector.
In this way we have constructed an infinite number of vacua $|q\rangle$ which can be interpreted

as a Fermi/Bose sea. The only non-zero inner product is $\langle -q-Q|q\rangle$ then, the adjoint vacuum is $\langle -q-Q|$. We can obtain the following relation

$$\begin{aligned}\left[L_n, e^{q\phi(w)}\right] &= \int \frac{dz}{2i\pi} z^{n+1}\frac{\frac{\epsilon}{2}q(q+Q)}{(z-w)^2} + \frac{\partial\, e^{q\phi(w)}}{z-w} \\ &= \frac{\epsilon}{2}q(q+Q)(n+1)w^n e^{q\phi(w)} + w^{n+1}\partial_w e^{q\phi(0)}\,, \end{aligned} \tag{3.179}$$

which tell us that both $|0\rangle$ and $|-Q\rangle$ are invariant under L_0. However, only $|0\rangle$ is annihilated by L_{-1}, at $w=0$, hence the real $\hat{SL}_2$ vacuum.

3.7 The covariant fermion vertex

After obtaining the necessary information on ghost systems, we can return to our original problem, as it was stated at the end of section 5, to construct BRST invariant vertex operators for space-time fermions. We claimed that the remaining factor Σ of weight $3/8$ should come from the ghost system. This condition is fulfilled for the exponential $e^{-\phi/2}$, where $\phi(z)$ is a scalar field obtained through bosonizating the $\beta\gamma$ system. The field $e^{-\phi/2}$ modify the ghost charge of the vacuum by a half unit interpolating between the NS and R sectors, working as a spin field, as we expected to be Σ. Taking into account of all these factors, we propose the fermionic vertex operator of ghost charge $-1/2$

$$\mathcal{V}_{-1/2}(z) = u_\alpha S^\alpha e^{-1/2\phi} e^{ik\cdot X}(z)\,. \tag{3.180}$$

We test now the BRST invariance of $\mathcal{V}_{1/2}$. In order to prove this we split the BRST charge as in (3.147), then

$$\begin{aligned}\left[Q_0, \mathcal{V}_{1/2}(0)\right] &= \int \frac{dz}{2i\pi}\left(\frac{c\mathcal{V}_{1/2}(0)}{z^2} + \frac{\partial_w \mathcal{V}_{1/2}(0)}{z}\right) \\ &= \partial(c\mathcal{V}_{1/2}(0))\,, \end{aligned} \tag{3.181}$$

$$\begin{aligned}\left[Q_1, \mathcal{V}_{1/2}(0)\right] &= \frac{1}{2}\int \frac{dz}{2i\pi} e^{-\chi(z)} u_\alpha \langle e^{\phi(z)} e^{-1/2\phi(0)}\rangle\langle\psi^\mu(z) S^\alpha(0)\rangle\langle\partial_z x_\mu(z) e^{ik\cdot X(0)}\rangle \\ &= \frac{i/2}{z-w} e^{-\chi-\phi/2}(\gamma^\mu k_\mu u)^\alpha_\beta S^\beta(0) e^{ik\cdot X(0)}\,, \end{aligned} \tag{3.182}$$

$$\begin{aligned}\left[Q_2, \mathcal{V}_{1/2}(0)\right] &= \frac{1}{4}\int \frac{dz}{2i\pi}\langle\, e^{2\phi(z)} e^{-1/2\phi}\rangle e^{-2\chi(z)} b(z) u_\alpha S^\alpha e^{ik\cdot X}(0) \\ &= \frac{1}{4}\int \frac{dz}{2i\pi} z e^{3\phi/2-2\chi(z)} b\, u_\alpha S^\alpha e^{ik\cdot X}(0) = 0\,. \end{aligned} \tag{3.183}$$

The second terms gives zero contribution by demanding u_α satisfies the on-shell condition for massless fermions $\gamma^\mu k_\mu u = 0$. The first term is a total derivative and vanishes when integrated. However, $\mathcal{V}_{1/2}$ can not provide a complete description of the fermionic scattering amplitudes because it possess spinor ghost charge $-1/2$ and only four point amplitude would fulfil the background charge $Q_{\beta\gamma} = 2$. This difficulty can be avoided by obtaining a second version of the fermion vertex with opposite charge. From our exposition of the bosonic string we know that fields of the form $[Q_{BRST}, O]$ are $BRST$ invariant. Normally, vertex operators created in this way are all spurious and decouple. However, this reasoning does not apply to the vertex operator

$$\mathcal{V}_{1/2} = 2\left[Q_{BRST}, \xi\mathcal{V}_{-1/2}\right], \tag{3.184}$$

because ξ is not part of the irreducible algebra of the $\beta\gamma$ system. We obtain $\mathcal{V}_{1/2}$ by spliting Q_{BRST}. It is easily to see that Q_0 contributes with a total derivative

$$2\left[Q_0, \xi\mathcal{V}_{-1/2}(0)\right] = 2\partial_z(c\xi\mathcal{V}_{-1/2}), \tag{3.185}$$

that vanishes upon integration over z. The contribution from Q_1 is a little trickier to get

$$\begin{aligned} 2\left[Q_1, \xi\mathcal{V}_{-1/2}(0)\right] &= \int \frac{dz}{2i\pi} u^\alpha \langle e^{-\chi(z)} e^{\chi(0)} \langle e^{\phi(z)} e^{-\phi/2}\rangle [-\frac{ik_\mu\psi^\mu}{z} S_\alpha e^{ik\cdot X(0)} + \frac{\partial X_\mu}{z^{1/2}} e^{ik\cdot X(0)} (\gamma^\mu)_\alpha^\beta S_\beta \\ &\quad - \frac{i(k_\mu\gamma^\mu)^\beta\alpha}{z^{3/2}} S_\beta e^{ik\cdot X(0)}] \\ &= \int \frac{dz}{2i\pi} e^{\phi/2} u^\alpha \left[-\frac{i(k\dot{\psi})S_\alpha}{z^{3/2}} + \partial X_\mu \gamma_\alpha^{\mu\beta} S_\beta\right] e^{ik\cdot X(0)}. \end{aligned} \tag{3.186}$$

We need to put the first term in integer powers of z to compute the integral. In order to accomplish this requirement we obtain some sub leading terms of the OPE's (3.132).

$$\psi^\mu(z) S_\alpha(w) = \frac{\gamma_{\alpha\beta} S^\beta}{(z-w)^{1/2}} + z^{1/2}\psi^\mu_{-1} S_\alpha + \ldots \tag{3.187}$$

and of the OPE

$$\begin{aligned} \psi^\mu\psi^\nu(z) S_\alpha(0) &\sim \psi^\mu \frac{\gamma^\nu_{\alpha\beta} S^\beta}{z^{1/2}} - \psi^\nu \frac{\gamma^\mu_{\alpha\beta} S^\beta}{z^{1/2}} \\ &\sim \sum_n \left[\frac{\psi^\mu_n \gamma^\nu_{\alpha\beta} - \psi^\nu_n \gamma^\mu_{\alpha\beta}}{z^{n+1}}\right] S^\beta \\ &\sim \sum_n \frac{(\psi_n^{[\mu}\gamma^{\nu]})_\alpha^\beta}{z^{n+1}} S_\beta(0). \end{aligned} \tag{3.188}$$

After multiplying it by the term $u^\alpha k^\mu \gamma^\nu_{\alpha\gamma}$ and impose the on-shell condition for the wave-function u^α we get the identity

$$u^\alpha k^\mu \gamma^{\nu\beta}\alpha \psi^\mu \psi^\nu S_\beta = \frac{1-d/2}{\sqrt{2}} u_\alpha k^\mu \psi^\nu_{-1} S_\alpha\,, \tag{3.189}$$

and finally by taking into consideration (3.187) we obtain the relation

$$-i\frac{u^\alpha k_\mu \psi^\mu S_\alpha}{z^{3/2}} = i\frac{\sqrt{2}}{4}\frac{u^\alpha k_\mu \gamma^\beta_{\nu\alpha}\psi^\mu\psi^\nu S_\beta}{z}\,, \tag{3.190}$$

and the contribution to $\mathcal{V}_{1/2}$ is

$$\left[Q_1, 2\xi\mathcal{V}_{-1/2}(0)\right] = \frac{e^{\phi/2}u^\alpha}{\sqrt{2}}\left[\frac{i}{4}k_\mu\gamma^\beta_{\nu\alpha}\psi^\mu\psi^\nu S_\beta + \partial X_\mu \gamma^{\mu\beta}_\alpha S_\beta\right] e^{ik\cdot X(0)}\,. \tag{3.191}$$

Lastly, the contribution from Q_2 is

$$\begin{aligned}\left[Q_2, 2\xi\mathcal{V}_{-1/2}(0)\right] &= -\frac{1}{2}\int \frac{dz}{2i\pi} b(z) u^\alpha \langle e^{2\phi(z)}e^{-\phi(0)/2}\rangle\langle e^{-2\chi(z)}e^{\chi(0)}e^{ik\cdot X(0)}\rangle S_\alpha(0)\\ &= -\frac{1}{2}\int \frac{dz}{2i\pi} b(z) u^\alpha e^{3\phi(z)/2} z\frac{e^{-\chi(z)}}{z^2} e^{ik\cdot X(0)}\rangle S_\alpha(0)\\ &= -\frac{1}{2} b\, u^\alpha e^{3\phi/2-\chi}e^{ik\cdot X}S_\alpha(0)\,.\end{aligned} \tag{3.192}$$

Joining all the pieces, the fermionic vertex operator of ghost charge 1/2 is

$$\mathcal{V}_{1/2} = \frac{e^{\phi/2}u^\alpha}{\sqrt{2}}\left[\frac{i}{4}k_\mu\gamma^\beta_{\nu\alpha}\psi^\mu\psi^\nu + \partial X_\mu\gamma^{\mu\beta}_\alpha\right] S_\beta e^{ik\cdot X} - -\frac{1}{2} b\, u^\alpha e^{3\phi/2-\chi}e^{ik\cdot X}S_\alpha\,. \tag{3.193}$$

The last term of the above equation does not contribute to correlation functions, at least at tree level, because neither $\mathcal{V}_{-1/2}$ nor $\mathcal{V}_{1/2}$ include $c(z)$.

An important fact is that the zero mode of ξ does not contribute in the vertex $\mathcal{V}_{1/2}$ and its ghost components are also in the $\beta\gamma$ algebra. Having obtained these two vertices we note that the fermion amplitudes factorize on the NS amplitudes. For example, the operator product $\mathcal{V}_{-1/2}\mathcal{V}_{-1/2}$ factorize the NS vertex operator $\mathcal{V}_{-1}$ as

$$\begin{aligned}\mathcal{V}_{-1/2}(z)\mathcal{V}_{-1/2}(0) &= u^\alpha\langle S_\alpha(z)S_\beta(0)\rangle\langle e^{-1/2\phi(z)}e^{-1/2\phi(0)}\rangle\langle e^{ik\cdot X(z)}e^{ik\cdot X(0)}\rangle u^\beta\\ &= u^\alpha\left(z^{-3/4}\gamma^\mu_{\alpha\beta}\psi_\mu\right)\left(z^{-1/4}e^{-\phi(0)}\right)\left(z^{k_1\cdot k_2}e^{i(k_1+k_2)\cdot X(0)}\right)u^\beta\\ &= z^{-1+k_1\cdot k_2}\left(u^\alpha\gamma^\mu_{\alpha\beta}u^\beta\right)\left(e^{-\phi}\psi_\mu e^{i(k_1+k_2)\cdot X}(z_2)\right)\,,\end{aligned} \tag{3.194}$$

and the integral over z_1 near z_2 gives a pole at $(k_1 + k_2)^2 = 0$ whose residue is the NS massless vertex operator.

$$\mathcal{V}_{-1}(z)\left(\xi^{\mu\nu} = u^{\alpha}\gamma^{\mu}_{\alpha\beta}u^{\beta}; k = k_1 + k_2\right)\right) \xi^{\mu}\psi_{\mu}e^{i(k)\cdot X(0)} . \tag{3.195}$$

We remark that this is a bosonic vertex operator with ghost charge -1, however, it is not the only possible vertex for the NS sector. Let us look at the product $\mathcal{V}_{-1/2}\mathcal{V}_{1/2}$

$$\mathcal{V}_{-1/2}(z_1)\mathcal{V}_{1/2}(z_2) = \mathcal{V}_{-1/2}(z_2)\left[Q_{BRST}, 2\xi\mathcal{V}_{-1/2}(z_2)\right] . \tag{3.196}$$

By using the relation

$$\mathcal{V}_{-1/2}(z_1)\left[Q_{BRST}, \xi\mathcal{V}_{-1/2}(z_2)\right] = -\left\{Q_{BRST}, \mathcal{V}_{-1/2}(z_1)\xi\mathcal{V}_{-1/2}(z_2)\right\} ,$$

the above vertex product takes the form

$$\begin{aligned}
\mathcal{V}_{-1/2}(z_1)\mathcal{V}_{1/2}(z_2) &= 2\left\{Q_{BRST}, \xi\mathcal{V}_{-1/2}(z_1)\xi\mathcal{V}_{-1/2}(z_2)\right\} \\
&= z^{-1+k_1\cdot k_2}u^{\alpha}\gamma^{\mu}_{\alpha\beta}u^{\beta}\oint_{C(z_2)}\frac{dz}{2i\pi}\left(e^{-\phi-\chi}\psi_{\mu}\partial_z x^{\mu}(z)\right)\left(e^{\chi-\phi}\psi_{\mu}e^{i(k_1+k_2)\cdot X}(0)\right) \\
&= z^{-1+k_1\cdot k_2}\left(u^{\alpha}\gamma^{\mu}_{\alpha\beta}u^{\beta}\right)\left(\frac{g_{\mu\nu}}{z-z_2}\partial_z x^{\mu} + i\frac{(k_1+k_2)^{\mu}}{z-z_2}\psi_{\mu}\psi_{\nu}\right)e^{i(k_1+k_2)\cdot X}(z_2) \\
&= z^{-1+k_1\cdot k_2}\left(u^{\alpha}\gamma^{\mu}_{\alpha\beta}u^{\beta}\right)\left[\partial x_{\mu} + i\psi_{\mu}(\psi\dot{(}k_1+k_2))\right]e^{i(k_1+k_2)\cdot X}(z_2) ,
\end{aligned}$$

and factorize the NS vertex of ghost charge zero.

$$\mathcal{V}_{-1/2}(z_1)\mathcal{V}_{1/2}(z_2) \cong \mathcal{V}_0(z_1)(\xi^{\mu} = u^{\alpha}\gamma^{\mu}_{\alpha\beta}u^{\beta}; k = k_1 + k_2) . \tag{3.197}$$

We define the operator P_1 which by action on a vertex operator of picture n, $\mathcal{V}_n$ increases the picture by one. From our above discussion this operator takes the form

$$P_1(z) = \{Q_{BRST}, \xi(z)\} . \tag{3.198}$$

This is the picture-changing-operator and produces equivalent operators by mixing the pictures as

$$V_{n+1}(0) = \lim_{x\to 0}\left[P_1(z)\mathcal{V}_n(0)\right] . \tag{3.199}$$

From the above discussion, we might have realized that exactly as we derived $\mathcal{V}_{1/2}$ from $\mathcal{V}_{-1/2}$ by manipulation of ξ, we may construct an infinite number of fermion vertex operators as

$$\begin{aligned}
\mathcal{V}_{3/2} &= \left[Q_{BRST}, 2\xi\mathcal{V}_{1/2}\right] \\
\mathcal{V}_{5/2} &= \left[Q_{BRST}, 2\xi\mathcal{V}_{3/2}\right] \\
&\vdots ,
\end{aligned} \tag{3.200}$$

and similarly for the NS vector operators. However, it can be shown that for scattering amplitudes, all these vertices are equivalent to each other. This redundancy of different operators, expressing the same physical states come from the infinite number of inequivalent vacua of the $\beta\gamma$ system. The distinct copies, each of these with a different ghost number, are said to be in different ghost picture.

We saw in the last chapter for the bc system that although $|0\rangle$ is a highest weight state of the Virasoro-algebra,it is not a highest weight state for the bc algebra, namely, $|0\rangle$ is not annihilated by c_1. Also, for the $\beta\gamma$ algebra, the vacuum $|0\rangle$ is not annihilated by $\gamma_{1/2}$. This situation can be remedied by changing the Bose/Fermi level, and the highest weight vacuum of the combined ghost system is

$$|0\rangle_{NS} = c(0)e^{-\phi(0)}|0\rangle \,, \tag{3.201}$$

with energy $-1 + 1/2 = -1/2$. This is the canonical vacuum. From this discussion and considering the energy of the Ramond vacua of the matter system and ghost system as produced by the spin fields of weight $5/8$ and $3/8$ respectively (S_α, Σ), we easily can obtain the normal-ordering constants in (3.123)

$$\begin{array}{lll} \text{matter} & R : a^m = 5/8, & NS : a^m = 0 \\ \text{ghost} & R : a^g = -5/8, & NS : a^g = -1/2 \,. \end{array} \tag{3.202}$$

3.7.1 The rearrangement lemma

The tree level fermion scattering amplitude can be obtained now by using the fermionic vertex operators $\mathcal{V}_{-1/2}$ and $\mathcal{V}_{1/2}$

$$A(1 \dots n) = \int dz_1 \dots dz_n \langle \mathcal{V}_{q_1} \dots \mathcal{V}_{q_1} \rangle \,. \tag{3.203}$$

At this point, we would like to show that the above scattering amplitude is independent of the distribution of ghost charges among them. In particular, for the amplitude

$$\langle ... \mathcal{V}_{-1/2}(z_1) ... \mathcal{V}_{1/2}(z_2) ... \rangle \,, \tag{3.204}$$

since the vertex operators does not depend on ξ_0 we insert the in the amplitude $\int d\xi_0 \xi_0$, then

$$\langle \int d\xi_0 \xi_0 ... \mathcal{V}_{-1/2}(z_1) ... \mathcal{V}_{1/2}(z_2) ... \rangle \,, \tag{3.205}$$

since ξ_0 is a Grassmann variable we can replace it $\xi(z)$ for an arbitrary z that can be we chosen $z = z_1$. Thus,

$$\begin{aligned} &\langle \int d\xi_0 ... \xi(z_1)\mathcal{V}_{-1/2}(z_1) ... \left[Q_{BRST}, \xi(z_n)\mathcal{V}_{-1/2}(z_n)\right] ...\rangle \\ = \;& \int \frac{dz}{2i\pi} \langle \int d\xi_0 ... \xi(z_1)\mathcal{V}_{-1/2}(z_1) ... J_{BRST}(z)\xi(z_2)\mathcal{V}_{-1/2}(z_2) ...\rangle \,, \end{aligned}$$

By deforming the contour integration by pulling it off the back of the sphere, the BRST current passes through all intermediate vertex operator until the contour encircle the point z_1,

$$\begin{aligned} &\langle \textstyle\int d\xi_0 ... \left[Q_{BRST}, \xi(z_1)\mathcal{V}_{-1/2}(z_1)\right] ... \xi(z_2)\mathcal{V}_{-1/2}(z_2) ...\rangle \\ &\qquad = \langle ... \mathcal{V}_{1/2}(z_1) ... \mathcal{V}_{-1/2}(z_2) ...\rangle \qquad . \end{aligned} \tag{3.206}$$

This means that, although there is an infinite number of fermion vertices, they will all yield the same matrix element.

3.8 One loop partition function

In this section we are concerned to obtain the one-loop partition function for the femionic string. In order to get it we will use our knowledge obtained in the last chapter for the fermionic case. First of all, we parametrize the torus by using the complex coordinates $w = \sigma_1 + \tau\sigma_2$, fixing the period of the real coordinates σ_1 and σ_2 to 2π and let the moduli τ be a free parameter. For the bosonic fields we required the periodic boundary conditions

$$x(\sigma_1, \sigma_2) = x(\sigma_1 + 2\pi, \sigma_2) = x(\sigma_1, \sigma_2 + 2\phi)\,, \tag{3.207}$$

however, for fermions we can choose two periodic or antiperiodic boundary conditions in each direction. Therefore we have four possible boundary conditions, namely, four different spin structures: $(++)$, $(+-)$, $(-+)$ and $(--)$, where $+$ and $-$ stand for periodic or antiperiodic boundary conditions. We know from our study of the torus that global diffeomorphisms in the torus are generated by the modular group $SL(2,\mathbb{Z})$, i.e., T and S modular transformations. The moduli inversion pruduced by $S : -\frac{1}{\tau}$ are generated by the change of coordinates $(\sigma_1, \sigma_2) \to (\sigma_2, -\sigma_1)$, then the action of S on $\psi(\sigma_1, \sigma_2)$ produces the changes on the boundary conditions

$$\begin{aligned} (++) &\to (++)\,, \\ (+-) &\to (-+)\,, \\ (-+) &\to (+-)\,, \\ (--) &\to (--)\,. \end{aligned} \tag{3.208}$$

In a similar way, under T transformations, $(\sigma_1, \sigma_2) \to (\sigma_1 + \sigma_2, \sigma_2)$, and the boundary conditions change as

$$\begin{aligned}
(++) &\to (++)\,, \\
(+-) &\to (+-)\,, \\
(-+) &\to (--)\,, \\
(--) &\to (-+)\,.
\end{aligned} \tag{3.209}$$

We can see that the spin structure $(++)$ is invariant under the action of the modular group, then the one-loop partition function of the spin structure is modular invariant by itself. For the other cases, modular transformations change the spin structure. In order to obtain a modular invariant partition function we would have to sum all the contributions from the different spin structures

For $D = 10$, we can assume that we have 8 transverse degrees of freedom for the fermonic string and treat them as 8 two dimensional fermions. We obtain a similar result for the bosonic case. The path integrals can be written for each sector as

$$A((--), \tau) = \eta_{--} \mathrm{Tr} \exp(i\tau\, H_{NS})\,, \tag{3.210}$$

$$A((+-), \tau) = \eta_{+-} \mathrm{Tr} \exp(i\tau\, H_{R})\,, \tag{3.211}$$

$$A((-+), \tau) = \eta_{-+} \mathrm{Tr} \exp(i\tau\, H_{NS})(-1)^F\,, \tag{3.212}$$

$$A((++), \tau) = \eta_{++} \mathrm{Tr} \exp(i\tau\, H_{R})(-1)^F\,, \tag{3.213}$$

where the η's are phases constants to be determined. The choose for Hamiltonians to be expanded in R or NS modes comes from the boundary conditions on the direction of σ_1, and the factor $(-1)^F$ for antiperiodic boundary conditions in the σ_2 direction. The world-sheet Hamiltonian of the total systemcomes from the value $L_0^{gh} + L_0^m$ with the correct normal-ordering factors (3.202). We are only concern with the fermionic part, so we substract the bosonic contribution $-\frac{10-2}{24}$.

$$H_{NS} = \sum_{r=1/2} r\psi^i_{-r}\psi^i_r - \frac{1}{6}\,, \tag{3.214}$$

$$H_{NS} = \sum_{r=1} r\psi^i_{-r}\psi^i_r + \frac{1}{3}\,, \tag{3.215}$$

where i labels $i = 1...8$ transverse space coordinates.
In the last section of the chapter 2 we showed explicitly how to calculate these partition

functions. By similar computations we get

$$A((--),\tau) = \eta_{--} q^{-1/6} \prod_{n=1} (1+q^{n-1/2})^8 , \tag{3.216}$$

$$A((+-),\tau) = \eta_{+-} 16 q^{1/3} \prod_{n=1} (1+q^{n})^8 , \tag{3.217}$$

$$A((-+),\tau) = \eta_{-+} q^{-1/6} \prod_{n=1} (1-q^{n-1/2})^8 , \tag{3.218}$$

$$A((++),\tau) = 0 , \tag{3.219}$$

the vanishing of $A((++),\tau)$ is due to the zero mode of the Dirac operator. In order to exploit the modular invariance we can rewrite the last results in form of Theta functions defined as

$$\Theta_{\alpha\beta}(0|\tau) = \eta(\tau) e^{2i\pi\alpha\beta} q^{\frac{\alpha^2}{2}-\frac{1}{24}} \prod_{n=1}^{\infty} \left(1+q^{n+\alpha-1/2} e^{2i\pi\beta}\right) \left(1+q^{n-\alpha-1/2} e^{-2i\pi\beta}\right) , \tag{3.220}$$

where $\eta(\tau)$ is the usual Dedekind function. In this language, partition functions (3.219) take the form of

$$A((--),\tau) = \eta_{--} \frac{\Theta_3^4(0|\tau)}{\eta^4(\tau)} , \tag{3.221}$$

$$A((+-),\tau) = \eta_{+-} \frac{\Theta_2^4(0|\tau)}{\eta^4(\tau)} , \tag{3.222}$$

$$A((-+),\tau) = \eta_{-+} \frac{\Theta_4^4(0|\tau)}{\eta^4(\tau)} , \tag{3.223}$$

$$A((++),\tau) = \eta_{++} \frac{\Theta_1^4(0|\tau)}{\eta^4(\tau)} , \tag{3.224}$$

where we have simplified the notation by the identification

$$\Theta_3 = \Theta_{0\,0} , \qquad \Theta_4 = \Theta_{0\,\frac{1}{2}} , \qquad \Theta_2 = \Theta_{\frac{1}{2}\,0} , \qquad \Theta_1 = \Theta_{\frac{1}{2},\frac{1}{2}} , \tag{3.225}$$

The modular transformations of these Theta functions are well known. Some important relations are

$$\begin{aligned} \Theta_2(0|\tau+1) &= e^{i\pi/4}\Theta_2(0|\tau) , \\ \Theta_3(0|\tau+1) &= \Theta_4(0|\tau) , \\ \Theta_4(0|\tau+1) &= \Theta_3(0|\tau) , \\ \eta(0|\tau+1) &= e^{i\pi/12}\eta(\tau) \end{aligned} \tag{3.226}$$

$$
\begin{aligned}
\Theta_2(0|\frac{-1}{\tau}) &= \sqrt{-i\tau}\Theta_4(0|\tau)\,, \\
\Theta_3(0|\frac{-1}{\tau}) &= \sqrt{-i\tau}\Theta_3(0|\tau)\,, \\
\Theta_4(0|\frac{-1}{\tau}) &= \sqrt{-i\tau}\Theta_2(0|\tau)\,, \\
\eta(0|\frac{-1}{\tau}) &= \sqrt{-i\tau}\eta(\tau)\,.
\end{aligned}
\tag{3.227}
$$

These are the desired modular transformations which relate different partition functions of the spin structures. We arbitrarily fix $\eta_{--} = 1$, it is not an inconvenient beacause we are interested in relative phases. The total amplitude for each sector need to be multiplied by the contribution of 8 right moving degrees of freedom which was obtained in the last chapter($\eta^{-8}(\tau)$).

By $T : \tau \to \tau'$ transformations, $(--)$ spin structure remains as $(-+)$. By using the transformations (3.227) we find that

$$
\begin{aligned}
\eta_{-+}\frac{\Theta_4^4(0|\tau)}{\eta^{12}(\tau)} &= \frac{\Theta_3^4(0|\tau')}{\eta^{12}(\tau')} \\
&= e^{-i\pi/}\frac{\Theta_4^4(0|\tau)}{\eta^4(\tau)}\,,
\end{aligned}
\tag{3.228}
$$

therefore, modular invariance imposes that $\eta_{-+} = -1$. Similarly, we can obtain η_{+-} by using $STS : \tau \to \tau'$ we produce the change in the spin structure of $STS : (+-) \to (--)$

$$
STS : \eta_{+-}\frac{\Theta_2^4(0|\tau)}{\eta^{12}(\tau)} = -\eta_{+-}\frac{\Theta_3^4(0|\tau)}{\eta^{12}(\tau)}\,,
\tag{3.229}
$$

and $\eta_{+-} = -1$.

Although the value of η_{++} can not be fixed by modular invariance, it only can takes values of $0, 1, -1$. In fact, the one loop partition function only can be interpretated as sum of $\exp(i\tau\, H)$ over states for these values. The total sum over spins structure give us

$$
A = \text{Tr}e^{i\tau\, H_{NS}}\frac{1}{2}(1 + (-1)^{F+1}) - \text{Tr}e^{i\tau\, H_{NS}}\frac{1}{2}(1 - (-1)^F)\,.
\tag{3.230}
$$

Further discussions, for 2-loops partition function show that η_{++} can not be zero in order to preserve unitiraty [20]. This result is highly interesting, we see that modular invariance force us to sum over the spin structures. This is equivalent to performing the

GSO projection, and the partition function is obtained by summing only the contribution of the supersymmetric spectrum. Moreover, due to the identity for Theta functions

$$\Theta_2^4(01\tau) - \Theta_3^4(01\tau) + \Theta_4^4(01\tau) = 0\,, \tag{3.231}$$

the one loop partition function vanishes as expected for a supersymmetric spectrum.

Chapter 4

Conclusions

We have implemented a covariant quantization of bosonic and fermionic string theories by exploiting the tools provided by conformal and BRST methods. After having reviewed the basics of two dimensional conformal field theories, we quantized the bosonic string by using the Polyakov's path integral. By fixing the gauge and using the Faddeev-Popov method we found the correct measure for the functional integral which was represented by the world-sheet reparametrization ghost fields. The complete gauge-fixed action, including the matter field as well as the ghost system, presents BRST invariance. Vertex operators are obtained by requiring BRST invariance of the physical states.

The study of the closed string amplitude for four tachyons at tree level shows us divergence of the amplitude due to resonances corresponding to propagation of the intermediate string states over long spacetime distances. By factorizing the bosonic amplitude we find out that this divergence is due to a string propagating for a long proper time, the infrared region, $l \to \infty$, describing a long tube where one of the string states goes on-shell. These are the only potential divergences in string theory, which arises at the boundary of the moduli space ($q \to 0$) associated with potential BRST anomalies. The treatment of this anomalies is beyond the scope of this work (for more details [21] and references therein). The roots of the statement that there is no ultraviolet region is modular invariance. This symmetry of the theory always gives us an alternative description in which the proper time parameters are safely away from zero. We illustrated this feature for the particular case of the torus by studying the fundamental region of its moduli space.

The quantization of the RNS string was obtained by exploiting its superconformal structure. The use of BRST method was crucial to construct the correct massless fermion vertex operators.

The algebra of the $\beta\gamma$ system allow us to have an infinite number of inequivalent vacua and hence to construct an infinite number of fermion vertex operators representing the same state for each vacuum.

Finally, we have seen that modular invariance imposes the sum over the spin structures. This requirement forces us to perform the GSO projections. The states in the Ramond and in the Neveu-Schwarz sector satisfy opposite statistics, therefore the states in the Ramond sector are indeed space-time fermions.

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