Representations of Lie algebras of vector fields on algebraic varieties and supervarieties

Henrique de Oliveira Rocha

Thesis presented to the
Institute of Mathematics and Statistics
of the University of São Paulo
and
Faculty of Graduate and Postdoctoral Affairs
of the Carleton University
in partial fulfillment
of the requirements for the degree of
Doctor of Science

Program:	Mathematics
Supervisor (University of São Paulo):	Prof. Dr. Vyacheslav Futorny
Supervisor (Carleton University):	Prof. Dr. Yuly Billig

During the development of this work, the author received
financial support from FAPESP – grant 2020/13811-0 and 2022/00184-3

São Paulo
April, 2024

Representations of Lie algebras of vector fields on algebraic varieties and supervarieties

Henrique de Oliveira Rocha

This version of the thesis includes the corrections and modifications suggested by the Examining Committee during the defense of the original version of the work, which took place on April 29, 2024.

A copy of the original version is available at the Institute of Mathematics and Statistics of the University of São Paulo.

Examining Committee:

Prof. Dr. Vyacheslav Futorny (cosupervisor) – IME-USP
Prof. Dr. Yuly Billig (cosupervisor) – Carleton University
Prof. Dr. Adriano Adrega de Moura – UNICAMP
Prof. Dr. Colin Ingalls – Carleton University
Prof. Dr. João Fernando Schwarz – UFABC
Prof. Dr. Lucia Satie Ikemoto Murakami – IME-USP

Resumo

Henrique de Oliveira Rocha. **Representações de álgebras de Lie de campos vetoriais em variedades e supervariedades algébricas**. Tese (Doutorado). Instituto de Matemática e Estatística, Universidade de São Paulo [e] Faculty of Graduate and Postdoctoral Affairs, Carleton University, São Paulo, 2024.

Esta tese é dedicada a um estudo sobre a estrutura e a teoria de representação de algumas álgebras de Lie e superálgebras de Lie de dimensão infinita.

A primeira família estudada é a álgebra de Lie de campos vetoriais em uma variedade algébrica afim suave. Após uma exposição sobre a estrutura dessas álgebras de Lie, consideramos representações que admitem uma ação compatível do anel de coordenadas da variedade algébrica e são geradas finitamente como módulos sobre essa álgebra comutativa. Provamos que essas representações podem ser associadas a um feixe coerente que admite uma ação compatível do feixe tangente. Também provamos que a ação do feixe tangente é dada por um operador diferencial.

A segunda família considerada é a versão em supergeometria da anterior. Após uma investigação sobre a suavidade de supervariedades algébricas, provamos que as seções globais do feixe tangente de uma supervariedade afim integral suave é uma superálgebra de Lie simples. Em seguida, consideramos as representações dessa superálgebra de Lie que admitem uma ação compatível das seções globais do feixe estrutural da supervariedade afim. De forma análoga ao caso não-super, mostramos que o feixe de módulos associado admite uma ação compatível do feixe tangente quando é coerente. Além disso, mostramos que essa ação é definida por um operador diferencial.

Por fim, estudamos módulos de peso com multiplicidades finitas sobre a superálgebra de aplicações associada a uma superálgebra de Lie básica. Provamos que essas representações são cuspidais ou parabólicas induzidas de um módulo cuspidal limitado sobre uma subálgebra da superálgebra de aplicações. Mostramos também que módulos cuspidal limitados são módulos de avaliação.

Palavras-chave: (Super)álgebras de Lie de campos vetoriais. Representações de álgebras de Lie. Feixes de operadores diferenciais.

Abstract

Henrique de Oliveira Rocha. **Representations of Lie algebras of vector fields on algebraic varieties and supervarieties**. Thesis (Doctorate). Institute of Mathematics and Statistics, University of São Paulo [and] Faculty of Graduate and Postdoctoral Affairs, Carleton University, São Paulo, 2024.

This thesis is devoted to a study of the structure and representation theory of some infinite-dimensional Lie algebras and Lie superalgebras.

The first family studied is the Lie algebras of vector fields on smooth affine algebraic varieties. After an exposition of the structure of such Lie algebras, we consider representations that admit a compatible action of the coordinate ring of the algebraic variety and are finitely generated as modules over this commutative algebra. We prove that these representations can be associated with a vector bundle that admits a compatible action of the tangent sheaf. We also prove that the action of the tangent sheaf is given by a differential operator. These results allow us to solve a conjecture made in the first papers of this theory.

The second family considered is a supergeometry version of the previous. After an investigation of the smoothness of algebraic supervarieties, we prove that the global sections of the tangent sheaf of a smooth integral affine supervariety form a simple Lie superalgebra. Subsequently, we consider representations of this Lie superalgebra that admit a compatible action of global sections of the structure sheaf of the affine supervariety. Analogously to the non-super case, we show that the associated sheaf of modules admits a compatible action of the tangent sheaf when it is coherent. We also prove that this action is defined by a differential operator.

Lastly, we study the weight modules with finite multiplicities over the map superalgebra associated with a basic Lie superalgebra. We prove that these representations are either cuspidal or parabolically induced from a cuspidal bounded module over a subalgebra of the map superalgebra. We also show that cuspidal bounded modules are evaluation modules.

Keywords: Lie (super)algebras of vector fields. Representations of Lie algebras. Sheaves of differential operators.

Contents

Introduction

Throughout the whole thesis, $\mathbb{k}$ is an algebraically closed field of characteristic 0. Unless otherwise stated, vector spaces, linear maps and algebras are assumed to be over $\mathbb{k}$.

While aiming to establish a method for solving differential equations analogous to the Galois theory of algebraic equations, Sophus Lie discovered that continuous groups of transformations related to differential equations could be better understood by analyzing their infinitesimal counterpart. These continuous groups of transformations and infinitesimal transformations are known today as Lie groups and Lie algebras, respectively. Although Lie was the first, Killing also discovered Lie algebras independently around ten years later and classified all simple finite-dimensional complex Lie algebras (see [Kil90]) in a paper considered to be *the greatest mathematical paper of all time* [Col89]. In his thesis, Élie Cartan reviewed Killing's paper, gave a better exposition of this classification, fixed some mistakes he found and moved the theory forward providing new and important results. Cartan is also broadly regarded as the creator of the modern Lie theory, who better formulated the necessary tools to study the structure, representations and applications of the theory [Haw96].

Some examples of infinite-dimensional Lie algebras were considered by Lie, but this subject was difficult to tackle not only because of the lack of a well-defined theory describing the structure of infinite-dimensional Lie algebras at the time but also because there were too many of them. Cartan started a series of papers answering questions about Lie algebras and groups, culminating with a description of families of simple Lie algebras that he called *primitive infinite groups with n variables* [Car09]. These are divided into four infinite classes denoted by W (general vectorial algebras), S (divergence-free algebras), H (Hamiltonian algebras), and K (contact algebras). About sixty years later, Guillemin, Singer and Sternberg constructed an algebraic framework for filtered and graded Lie algebras [GS64; SS65]. An algebraic proof for Cartan's classification was found by Weisfeiler using this framework [GQS66; Wei68]. Later, Kac considered a larger class of Lie algebras called *graded Lie algebras of polynomial growth*. He managed to classify them under certain conditions adding the *loop algebras* to the four classes W, S, H and K [Kac68]. However, it was Mathieu who finalized the classification of infinite-dimensional graded Lie algebras of polynomial growth in 1992 [Mat92b], adding the centerless Virasoro algebra as the last Lie algebra of this family.

Kac [Kac68] and Moody [Moo68] discovered another class of infinite-dimensional Lie algebras, which is called today *Kac-Moody algebras*. These algebras are defined by generators and relations through a generalized Cartan matrix, and they can be seen as generalizations of semisimple Lie algebras. The importance of Kac-Moody algebras was quickly recognized, as well as their applications to both mathematics and physics. The most important class of Kac-Moody algebras is the affine Lie algebras. These can be realized as the central extensions of loop algebras associated with simple finite-dimensional Lie algebras together with a degree derivation.

Different generalizations of affine Lie algebras were considered by many authors. One of them is the *map algebras* or *current algebras*. For a finitely generated commutative algebra S and a Lie algebra $\mathfrak{g}$, there is a natural way to define the structure of a Lie algebra on the tensor product $\mathcal{G} = \mathfrak{g} \otimes S$. Map algebras can be seen as Lie algebras defined on the set of regular maps from $X = \operatorname{Spec}(S)$ to $\mathfrak{g}$ (seen as an affine space). If there is a group Γ that acts on both $\mathfrak{g}$ and S by automorphisms, then it is even possible to consider the equivariant map algebra $\mathcal{G}^{\Gamma}$, where only regular maps equivariant for the action of Γ are included. Building on earlier works for some specific examples of equivariant map algebras, Neher, Savage and Senesi showed that irreducible finite-dimensional representations of equivariant map algebras are evaluation representations, see [NSS12] and references therein. If $\mathfrak{g}$ is a simple Lie algebra, a classification of weight modules with finite-dimensional weight spaces over $\mathcal{G}$ was achieved by Britten, Lemire and Lau in [BLL15; Lau18].

Savage classified finite-dimensional modules over map superalgebras when $\mathfrak{g}$ is a basic classical Lie superalgebra. Additionally, in joint work with Calixto and Futorny, we gave a classification for weight $\mathcal{G}$-modules with finite-dimensional weight spaces assuming that $\mathfrak{g}$ is a basic classical Lie algebra, generalizing the results of [BLL15; Lau18] to the super case. Chapter 4 of this thesis will be about this paper. After proving several results on the representation theory of $\mathcal{G}$-modules and introducing the needed machinery, we prove the parabolic induction theorem and establish a classification of simple bounded weight modules (weight modules with dimensions of weight spaces bounded by some number) in terms of simple $\mathfrak{g}$-modules and maximal ideals of S. The main result of Chapter 4 can be summarized in the following theorem.

Theorem. *Let $\mathfrak{g}$ be a basic classical Lie superalgebra, S a finitely generated commutative algebra, and V a weight $\mathcal{G} = \mathfrak{g} \otimes S$-module with finite-dimensional weight spaces.*

1. *V is either a cuspidal bounded $\mathcal{G}$-module or a parabolically induced module from a simple cuspidal bounded module over a subalgebra of $\mathcal{G}$.*
2. *If V is a cuspidal bounded $\mathcal{G}$-module, then V is isomorphic to an evaluation module.*

However, the main focus of this thesis will be on a different class of infinite-dimensional Lie algebras — Lie algebras of vector fields on smooth irreducible affine algebraic varieties. One family of them already appeared in Cartan's classification, the general vectorial algebra. Another example that we already mentioned is the graded Lie algebras of polynomial growth — the centerless Virasoro algebra. Although graded examples of these algebras were already well-studied, there was no general theory of the Lie algebras of vector fields on arbitrary smooth affine varieties. Thus, after classifying weight modules with finite-dimensional weight spaces over the Lie algebra of vector fields on the torus [BF14], Billig and Futorny started to study the structure of the Lie algebras of vector fields on any algebraic variety [BF18]. Jordan [Jor86] and Siebert [Sie96] proved that, for an irreducible affine algebraic variety X, the Lie algebra $\mathcal{V}_X = \operatorname{Der}(A_X)$ of derivations of the coordinate ring A_X of X is simple if and only if X is smooth.

Billig and Futorny initiated the study of the representation theory of these algebras. Traditional techniques of Lie algebra theory, such as root systems, Cartan subalgebras, or weight decomposition, are not applicable in general for Lie algebras of vector fields. Their structure can vary greatly depending on the underlying algebraic variety. Some of them do have a Cartan subalgebra which decomposes it in root spaces, while others do not even have semisimple nor nilpotent elements. It was necessary to consider a different kind of representation that could work in the general setting of these Lie algebras. Billig and Futorny focused on modules over both A_X and $\mathcal{V}_X$ that satisfy the Leibniz rule, called $A_X\mathcal{V}_X$-modules. These are equivalent to modules over the smash product $A_X\#U(\mathcal{V}_X)$. $A_X\mathcal{V}_X$-modules can be seen as a generalization of the well-researched D_X-modules over X, since every D_X-module is an $A_X\mathcal{V}_X$-module. However, it is not required for the associated representation of $\mathcal{V}_X$ to be A_X-linear. Hence, there are plenty of examples of $A_X\mathcal{V}_X$-modules that are not D_X-modules, e.g. $\mathcal{V}$ itself, as well as the space of differential k-forms Ω^k_X.

$A_X\mathcal{V}_X$-modules appeared in the classification of weight modules over the Lie algebra of vector fields on the n-dimensional torus and affine space. Therefore, their importance is already recognized. Billig, Futorny and Nilsson constructed two families of simple $A_X\mathcal{V}_X$-modules in [BFN19]: gauge modules and Rudakov modules. Rudakov modules are the generalizations of modules constructed by Rudakov for the affine space [Rud74]. On the other hand, gauge modules were inspired by tensor modules, which were crucial in the papers [BF14; GS22]. It was conjectured in [BFN19] that every $A_X\mathcal{V}_X$-module that is finitely generated as an A_X-module is a *gauge module*, hence results on the simplicity of gauge modules were obtained in [BNZ21] by Billig, Nilson and Zaidan. The conjecture was proven when $X = \mathbb{A}^n$ is the affine space by Billig, Ingalls and Nasr in [BIN23], but we will prove it for a general algebraic variety.

So far we talked about objects of a geometric nature, however we did not associate them with an algebraic geometric object. The affine algebraic variety X relates naturally to a scheme on Spec(A_X) and the structure sheaf $\mathcal{O}_X$, while the Lie algebra $\mathcal{V}_X$ associates with the tangent sheaf Θ_X on X. If M is an $A_X\mathcal{V}_X$-module, then localization gives a quasi-coherent sheaf $\tilde{M}$ of $\mathcal{O}_X$-modules. Then the question is whether it is possible to make $\tilde{M}$ into a sheaf of representations of Θ_X in a way that for each affine open subset $U \subset X$ we have that the sections $\Gamma(U, \tilde{M})$ is an $A_U\mathcal{V}_U$-module. Guided by their earlier paper [BIN23], Billig and Ingalls studied a quasi-coherent sheaf that would govern these representations [BI23]. Moreover, they expanded the notions studied so far to quasi-projective varieties. Motivated by their work, in a joint paper with Bouaziz, we showed in [BR23] that the quasi-coherent sheaf $\tilde{M}$ admits a compatible action of Θ_X if M is a finitely generated module over A (i.e. $\tilde{M}$ is coherent). We also showed that the associated representation $\mathcal{V}_X \to \mathrm{End}_{\Bbbk}(M)$ is a differential operator, which means that the map $\Theta_X \to \mathrm{End}_{\Bbbk}(\tilde{M})$ is also a differential operator. These results will be presented in Chapter 2, with the main accomplishment summarized in the following theorem.

Theorem. *Let X be a smooth affine algebraic variety with a coordinate ring A_X and Lie algebra of vector fields $\mathcal{V}_X = \mathrm{Der}(A_X)$. Denote by $\mathcal{O}_X$ the structure sheaf of X and Θ_X the tangent sheaf. If M is an $A_X\mathcal{V}_X$-module finitely generated as an A_X-module, then*

1. *The coherent sheaf $\tilde{M}$ of $\mathcal{O}_X$-modules is a vector bundle and $\Gamma(U, X)$ is a $B\mathrm{Der}(B)$-*

module for each affine open subset $U = \operatorname{Spec}(B) \subset X$.

2. *Both the representation* $\mathcal{V}_X \to \operatorname{End}_{\Bbbk}(M)$ *and* $\Theta_X \to \operatorname{End}_{\operatorname{Spec}(\Bbbk)}(\tilde{M})$ *are differential operators.*

The supergeometric version of the theory of Lie algebras of vector fields will be investigated in this thesis as well. The above theorem can be generalized in this context, and we will prove it in Chapter 3. The existence of odd elements will be the main difficulty to handle, however, proofs follow a similar path to the non-super case. We will also prove that if an affine supervariety is smooth then the Lie superalgebra of vector fields on it is simple. Finally, we will study the $A\mathcal{V}$-module theory for the supervariety with only non-zero finite odd dimensions. The interesting fact here is that the Lie superalgebra of vector fields is finite-dimensional and it is one of the Cartan-type simple Lie superalgebras that appears on the classification of simple finite-dimensional Lie superalgebras [Kac77]. For this case, we first prove an isomorphism of associative superalgebras similar to the one that holds for the affine space [BIN23], and we will show that there is an equivalence of categories between the category of $A_X\mathcal{V}_X$-modules and the category of vector fields that vanish at the single point of the supervariety.

This thesis is organized as follows. In Chapter 1, we will present the preliminary results, fix our notations and review the basics of the structure of affine varieties and the Lie algebra of vector fields associated with it. Then, we move to the representation theory and prove our results about $A_X\mathcal{V}_X$-modules in Chapter 2. In Chapter 3, we give the preliminaries on supergeometry, prove the simplicity of the Lie superalgebra of vector fields and study its $A_X\mathcal{V}_X$-module theory. Finally, we present our results on weight modules with finite-dimensional weight spaces over the map algebra associated with basic Lie superalgebras in Chapter 4.

Chapter 1

Algebraic varieties and the Lie algebra of vector fields

This chapter presents an overview of the background results needed for this thesis.

In Section 1.1 we define the basics of Lie algebra theory and their representations. Then we present the affine algebraic varieties in Section 1.2 and talk about their dimensions and tangent spaces in Section 1.3. The Lie algebra of vector fields is introduced in Section 1.4, we give examples and a couple of results about their structure. In Section 1.5 we introduce the module of Kähler differentials and relate it to the Lie algebra of vector fields. In Section 1.6, the local theory of the Lie algebra of vector fields is investigated. We finish this chapter by introducing power series for functions and derivations in Section 1.7.

1.1 Lie algebras and representations

Definition 1.1.1. A Lie algebra $\mathfrak{g}$ (over $\Bbbk$) is a vector space with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called *Lie bracket*, that satisfies the following conditions:

1. $[\cdot,\cdot]$ is bilinear,
2. $[x, x] = 0$ for every $x \in \mathfrak{g}$,
3. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for every $x, y, z \in \mathfrak{g}$ (Jacobi identity).

If $\mathfrak{g}_1, \mathfrak{g}_2$ are Lie algebras and $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a linear map, then φ is a *homomorphism of Lie algebras* if $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for every $x, y \in \mathfrak{g}_1$.

A subspace $\mathfrak{l}$ of Lie algebra $\mathfrak{g}$ is a *(Lie) subalgebra* if $[x, y] \in \mathfrak{l}$ for every $x, y \in \mathfrak{l}$. Similarly, a subspace $\mathfrak{l}$ of $\mathfrak{g}$ is an *ideal* if $[x, y] \in \mathfrak{l}$ for every $x \in \mathfrak{l}$ and $y \in \mathfrak{g}$. We say that Lie algebra $\mathfrak{g}$ is simple if $[\mathfrak{g}, \mathfrak{g}] \neq 0$ and the only ideals of $\mathfrak{g}$ are 0 and itself.

For the details on the structure theory of simple finite-dimensional Lie algebras and their representations, we refer to the book of Humphreys [Hum78].

Example 1.1.2. The *commutator* defines a Lie algebra structure on every associative algebra. Let A be an associative algebra, then the commutator

$$[x, y] = xy - yx, \quad x, y \in A$$

defines a Lie bracket on A.

Example 1.1.3. Let V be a vector space and consider the set $\mathrm{End}_{\Bbbk}(V)$ of linear endomorphisms of V. The composition of maps makes $\mathrm{End}_{\Bbbk}(V)$ an associative algebra. The Lie algebra defined by its commutator will be denoted by $\mathfrak{gl}_{\Bbbk}(V)$ or simply $\mathfrak{gl}(V)$.

Example 1.1.4. Let A be an algebra. Define the set of derivations as

$$\mathrm{Der}(A) = \{D \in \mathfrak{gl}(A) \mid D(ab) = D(a)b + aD(b) \text{ for each } a, b \in A\}.$$

Since

$$\begin{aligned}
&[D_1, D_2](ab) \\
=&D_1(D_2(a)b + aD_2(b)) - D_2(D_1(a)b + aD_1(b)) \\
=&[D_1, D_2](a)b + D_2(a)D_1(b) + D_1(a)D_2(b) + a[D_1, D_2](b) - D_1(a)D_2(b) - D_2(a)D_1(b) \\
=&[D_1, D_2](a)b + +a[D_1, D_2](b)
\end{aligned}$$

for each $a, b \in A$ and $D_1, D_2 \in \mathrm{Der}(A)$, we have that $\mathrm{Der}(A)$ is a Lie subalgebra of $\mathfrak{gl}(A)$.

The above example will be the most important one for us in this thesis.

The structure of finite-dimensional Lie algebras is very well understood. There is a complete classification of simple finite-dimensional Lie algebras, which is covered in almost every introduction book including the one we referenced above. On the other hand, the structure of infinite-dimensional Lie algebras can vary greatly depending on which type is being studied.

Regardless of which Lie algebra is in the spotlight, its representations theory is always an essential subject to investigate. It often gives insights on the structure of the Lie algebra, since this theory can be used to translate problems in the Lie algebra theory to problems in linear algebra. Representation theory also studies how the Lie algebra could interact with other spaces, thus it is a great source of applications. Therefore, representation theory is both a tool and subject of study necessary for a better understating of Lie algebra characteristics, applications and impact.

Definition 1.1.5. Let $\mathfrak{g}$ be a Lie algebra and V be a vector space. A *representation* of $\mathfrak{g}$ is a map $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ that is a homomorphism of Lie algebras. When V is equipped with a representation $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$, we say that V is a *module over* $\mathfrak{g}$ or a $\mathfrak{g}$-*module*.

If V is a $\mathfrak{g}$-module, then a subspace $W \subset V$ is a $\mathfrak{g}$-*submodule* if $xw \in W$ for every $x \in \mathfrak{g}$ and $w \in W$. We say that V is *irreducible* or *simple* if it only has two submodules: V and 0; the trivial ones. If V and W are two $\mathfrak{g}$-modules, a linear map $\psi : V \to W$ is a *homomorphism* of $\mathfrak{g}$-modules if $\psi(xv) = x\psi(v)$.

Example 1.1.6. The Lie bracket defines a representation of a Lie algebra $\mathfrak{g}$ over itself, called the *adjoint representation.* Let $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ be the map defined by

$$\mathrm{ad}(x)(y) = [x, y].$$

By the Jacobi identity, ad is a representation.

For each Lie algebra $\mathfrak{g}$, there is an unital associative algebra whose modules correspond exactly to the representations of the Lie algebra $\mathfrak{g}$. This is the *universal enveloping algebra.* An *enveloping algebra* of $\mathfrak{g}$ is a pair (U, φ) where U is an unital associative algebra and $\varphi : \mathfrak{g} \to U$ is a homomorphism of Lie algebras (the bracket on U is the commutator). The *universal enveloping algebra* $(U(\mathfrak{g}), \pi)$ is an enveloping algebra that satisfies the following universal property: for every enveloping algebra (U, φ), there exists a unique homomorphism of associative algebras $\tilde{\varphi} : U(\mathfrak{g}) \to U$ such that the following diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \\ U(\mathfrak{g}) \arrow[ur, dotted, "\tilde{\varphi}"'] & \end{tikzcd}$$

commutes. Below we will present a construction of $U(\mathfrak{g})$. The universal property above shows the universal enveloping algebra is unique up to an isomorphism.

The universal enveloping algebra of $\mathfrak{g}$ can be realized using the tensor algebra of $\mathfrak{g}$. Define

$$\begin{aligned} T^0(\mathfrak{g}) &= \Bbbk, \\ T^1(\mathfrak{g}) &= \mathfrak{g}, \\ T^m(\mathfrak{g}) &= \mathfrak{g} \otimes \cdots \otimes \mathfrak{g} \ (m \text{ times}),\ m > 1. \end{aligned}$$

The *tensor algebra* $\mathcal{T}(\mathfrak{g})$ of $\mathfrak{g}$ is the associative algebra

$$\mathcal{T}(\mathfrak{g}) = \bigoplus_{i=0}^{\infty} T^i(\mathfrak{g}),$$

where the product is given by concatenation

$$(v_1 \otimes \cdots \otimes v_r)(w_1 \otimes \cdots \otimes w_s) = v_1 \otimes \cdots \otimes v_r \otimes w_1 \otimes \cdots \otimes w_s.$$

Let I be the two-sided ideal of $\mathcal{T}(\mathfrak{g})$ generated by elements of the form

$$x \otimes y - y \otimes x - [x, y], x, y \in \mathfrak{g}.$$

Denote by $\tilde{U}(\mathfrak{g})$ the quotient $\mathcal{T}(\mathfrak{g})/I$.

Theorem 1.1.7 (Poincaré–Birkhoff–Witt theorem)**.** *Let $\mathfrak{g}$ be a Lie algebra, then the universal enveloping algebra $(U(\mathfrak{g}), \pi)$ exists and it is isomorphic to $\tilde{U}(\mathfrak{g})$. Furthermore, if $\{x_i \mid i \in J\}$ is a basis of $\mathfrak{g}$ indexed by an ordered set J, then the image of the set*

$$\{1\} \cup \left\{ x_{i_1}^{n_1} \otimes x_{i_2}^{n_2} \otimes \cdots \otimes x_{i_r}^{n_r} \mid r \geq 1,\ i_1 < i_2 < \cdots < i_r,\ n_1, \ldots, n_r \geq 1 \right\}$$

forms a basis of $\tilde{U}(\mathfrak{g})$*. In particular,* $\mathfrak{g}$ *is a Lie subalgebra of* $U(\mathfrak{g})$*.*

Proof. See [Hum78, Section 17.3]. □

A module V over the universal enveloping algebra $U(\mathfrak{g})$ defines a representation $\rho : \mathfrak{g} \to \mathrm{End}_{\Bbbk}(V)$ of $\mathfrak{g}$ by taking $\rho(x)v = xv$. Similarly, a representation $\rho : \mathfrak{g} \to \mathrm{End}_{\Bbbk}(V)$ of $\mathfrak{g}$ gives an enveloping algebra $(\mathrm{End}_{\Bbbk}(V), \rho)$, thus there exists a map $\tilde{\rho} : U(\mathfrak{g}) \to \mathrm{End}_{\Bbbk}(V)$ that defines an action of $U(\mathfrak{g})$ on V. Hence, the category of representations of $\mathfrak{g}$ is equivalent to the category of modules over $U(\mathfrak{g})$. This equivalence provides us with numerous additional tools derived from associative algebra theory and allows us to use the structure of $U(\mathfrak{g})$ to infer theorems of the representation theory of $\mathfrak{g}$.

1.2 Affine algebraic varieties

Recall that $\Bbbk$ is an algebraically closed field with characteristic 0. The *affine space* $\mathbb{A}^n_{\Bbbk}$ of dimension n over $\Bbbk$ is the set $\Bbbk^n$. When $\Bbbk$ is fixed, we will denote $\mathbb{A}^n_{\Bbbk}$ simply by $\mathbb{A}^n$. The affine space comes with a topology, called *Zariski topology*, where its closed subsets are given by the *affine algebraic sets*

$$Z(S) = \{(a_1, \dots, a_n) \in \Bbbk^n \mid f(a_1, \dots, a_n) = 0 \ \forall f \in S\}$$

with $S \subset \Bbbk[x_1, \dots, x_n]$ being a subset of the polynomial algebra $\Bbbk[x_1, \dots, x_n]$. Note that if $\mathfrak{a}$ is the ideal of $\Bbbk[x_1, \dots, x_n]$ generated by S, then $Z(\mathfrak{a}) = Z(S)$. Thus, every algebraic affine set is given by a finite set of equations, because ideals of $\Bbbk[x_1, \dots, x_n]$ are finitely generated. The basis of Zariski topology is given by the basic open sets

$$D(f) = \{(a_1, \dots, a_n) \in \mathbb{A}^n \mid f(a_1, \dots, a_n) \neq 0\} \text{ for } f \in \Bbbk[x_1, \dots, x_n].$$

The basic open set $D(f)$ is the complement of the affine algebraic set $Z(\{f\})$.

Let $X \subset \mathbb{A}^n$ be an affine algebraic set, then we define

$$I(X) = \{f \in \Bbbk[x_1, \dots, x_n] \mid f(a_1, \dots, a_n) = 0 \ \forall (a_1, \dots, a_n) \in X\}$$

to be the ideal of polynomial functions vanishing on X. The *coordinate ring* $A_X = \Bbbk[X]$ of X is defined as the quotient

$$A_X = \Bbbk[x_1, \dots, x_n]/I(X).$$

Let $p = (a_1, \dots, a_n) \in X$ be any point of X, then we define

$$\mathfrak{m}_p = I(p) = \{f \in \Bbbk[x_1, \dots, x_n] \mid f(p) = f(a_1, \dots, a_n) = 0\} = \langle x_1 - a_1, \dots, x_n - a_n \rangle.$$

The ideal $\mathfrak{m}_p$ is maximal and corresponds to a maximal ideal of A_X, which will be denoted by $\mathfrak{m}_p$ as well. Denote by $\operatorname{Specm} A_X$ the set of all maximal ideals of A_X. Hilbert's Nullstellensatz states that the map $X \to \operatorname{Specm} A_X$ defined by $p \mapsto \mathfrak{m}_p$ is a bijection [Har77, Theorem 1.3A].

The induced Zariski topology from $\mathbb{A}^n$ makes an affine algebraic set $X \subset \mathbb{A}^n$ a topological space as well. An *affine algebraic variety* is an irreducible affine algebraic set, i.e. an algebraic set $X \subset \mathbb{A}^n_{\mathbb{k}}$ is an algebraic variety if it is not a union of two proper closed subsets. An algebraic set X is an algebraic variety if and only if A_X is an integral domain, which is true if and only if $I(X)$ is a prime ideal of the polynomial algebra $\mathbb{k}[x_1, \dots, x_n]$. In particular, the affine space $\mathbb{A}^n$ is an affine algebraic variety.

For more on affine algebraic varieties, we refer to [Har77, Chapter I] and [Sha94a]. Appendix A of this thesis gives the basics of scheme theory. Affine algebraic varieties can be defined in the scheme theoretical setting as *integral separated affine schemes of finite type over* $\mathbb{k}$, see Example A.0.14. Later in this thesis, we will use this association.

1.3 Tangent space and dimension

In this section, we define the notion of dimension and smoothness of an algebraic variety. For more details, we recommend [Sha94a, Chapter 2]. If $f \in \mathbb{k}[x_1, \dots, x_n]$ and $p = (a_1, \dots, a_n) \in X$ is a point, then f has a Taylor series expansion

$$f(x) = f(p) + f^{(1)}(x) + \dots + f^{(k)}(x),$$

where $f^{(i)}$ are homogeneous polynomials of degree i in the variables $x_j - a_j$. The linear form $f^{(1)} : \mathbb{k}^n \to \mathbb{k}^n$ is called the *differential* of f at p, and is denoted by $d_p f$. Explicitly, we have

$$d_p f(x_1, \dots, x_n) = f^{(1)}(x_1, \dots, x_n) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_i}(p)(x_i - a_i).$$

By definition,

$$d_p(f + g) = d_p f + d_p g, \text{ and } d_p(fg) = f(p) d_p g + g(p) d_p f$$

for all $f, g \in A_X$. If the ideal $I(X)$ is generated by $f_1, \dots, f_k$, we define the *tangent space* $T_p X$ of p as the vector space

$$T_p X = \{(x_1, \dots, x_n) \in \mathbb{k}^n \mid d_p f_i(x_1, \dots, x_n) = 0, \text{ for each } i = 1, \dots, k\}.$$

If $F \in A_X$ is such that $F = f + I_X$, then we set the *differential of* F *at* p as $d_p F := d_p f|_{T_p X}$. It is possible to show that $d_p F$ is a well-defined linear form on $T_p X$ [Sha94a, Section 2.1.3].

We define the dimension of X as

$$\dim X = \max_{p \in X} \dim T_p X$$

the maximal dimension of the tangent vector spaces of X. This is one of the many equivalent ways of defining the dimension of an affine algebraic variety. A point $p \in X$ such that $\dim T_p X = \dim X$ is called *non-singular point*. We say that X is *smooth* if every point is non-singular.

Let $\mathfrak{m}_p$ be the maximal ideal of A_X corresponding to $p \in X$. We have that $d_p : A_X \to$

$(T_pX)^*$ is a well-defined linear map, and it inherits the relations

$$d_p(FG) = F(p)d_pG + G(p)d_pF, \quad \text{for } F, G \in A_X$$

Since $d_p(\alpha) = 0$ for all $\alpha \in \Bbbk$ and $A_X = \Bbbk \oplus \mathfrak{m}_p$ as a vector space, we only need to consider its restriction $d_p : \mathfrak{m}_p \to (T_pX)^*$.

Proposition 1.3.1 ([Sha94a, Theorem 2.1, Section 2.1.3])**.** *The map d_p is an isomorphism between $\mathfrak{m}_p/\mathfrak{m}_p^2$ and $(T_pX)^*$.*

Therefore, if p is non-singular, we get that $\dim \mathfrak{m}_p/\mathfrak{m}_p^2 = \dim X$.

1.4 The Lie algebra of vector fields

In this section, we present one of the protagonists of this thesis: the Lie algebra of vector fields on an algebraic variety. By Example 1.1.4, the set of all derivations

$$\mathcal{V}_X = \mathrm{Der}_{\Bbbk}(A_X) = \{D \in \mathrm{End}_{\Bbbk}(A_X) \mid D(ab) = D(a)b + aD(b) \text{ for all } a, b \in A_X\}$$

is a Lie subalgebra of $\mathfrak{gl}_{\Bbbk}(A_X)$, because the commutator of two derivations is a derivation.

Example 1.4.1. Let $X = \mathbb{A}^n$, then $A_{\mathbb{A}^n} = \Bbbk[x_1, \dots, x_n]$, and we will denote $\mathcal{V}_{\mathbb{A}^n} := V_n = \mathrm{Der}_{\Bbbk}(\Bbbk[x_1, \dots, x_n])$. Thus,

$$V_n = \bigoplus_{i=1}^{n} \Bbbk[x_1, \dots, x_n]\frac{\partial}{\partial x_i},$$

where $\frac{\partial}{\partial x_i} \in \mathcal{V}_X$ is the partial derivative with respect to x_i.

Example 1.4.2. Let

$$X = \mathbb{T}^n = \prod_{i=1}^{n} \mathbb{S}^1 = \left\{(x_1, y_1, \dots, x_n, y_n) \in \mathbb{A}^{2n} \mid x_i^2 + y_i^2 = 1,\ i = 1, \dots, n\right\},$$

then $I_{\mathbb{T}^n} = \left(x_i^2 + y_i^2 - 1,\ i = 1, \dots, n\right)$. The algebra homomorphism given by $t_j \mapsto y_j - ix_j + I_{\mathbb{T}^n}$ and $t_j^{-1} \mapsto y_j + ix_j + I_{\mathbb{T}^n}$ defines an isomorphism between $A_{\mathbb{T}^n}$ and the Laurent polynomial algebra in n variables $\Bbbk[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. For each $i = 1, \dots, n$, we define $d_i = t_i\frac{\partial}{\partial t_i}$, where $\frac{\partial}{\partial t_i} \in \mathcal{V}_X$ is the partial derivative with respect to t_i. Then,

$$[t^r d_a, t^s d_b] = s_a t^{r+s} d_b - r_b t^{r+s} d_a, \quad a, b = 1, \dots, n,\ r, s \in \mathbb{Z}^n,$$

where we set $t^r = t_1^{r_1} \cdots t_n^{r_n}$ for each $r = (r_1, \dots, r_n) \in \mathbb{Z}^n$. Hence, the adjoint action of $d_1, \dots, d_n$ defines a $\mathbb{Z}^n$-grading on $\mathcal{V}_{\mathbb{T}^n}$, and

$$\mathcal{V}_{\mathbb{T}^n} \cong \bigoplus_{i=1}^{n} \Bbbk[t_1^{\pm 1}, \dots, t_n^{\pm 1}]d_i.$$

The Lie algebra $\mathcal{V}_{\mathbb{T}^n}$ is called *generalized Witt algebra* [BF18]. When $n = 1$, then $\mathcal{V}_{\mathbb{S}^1}$ is the Witt algebra whose universal central extension is the famous *Virasoro algebra* [Mat92a].

Various properties of the Lie algebra $\mathcal{V}_X$ depend on the affine variety it is associated with, including simplicity as the following theorem shows.

Theorem 1.4.3 ([Jor86; Sie96]). *Let X be an affine algebraic variety with coordinate ring A_X, then the Lie algebra $\mathrm{Der}(A_X)$ is simple if and only if X is smooth.*

Therefore, this thesis focuses on the representation theory of an infinite family of simple Lie algebras. Their properties change a lot depending on which affine algebraic variety they are associated with.

In [BF18], this theorem was reviewed and many examples of these Lie algebras were investigated. Using the next proposition, we may create a relation between the Lie algebra $\mathcal{V}_X$ and a quotient of $V_n = \mathrm{Der}\,(\Bbbk[x_1, \dots, x_n])$.

Proposition 1.4.4 ([BF18, Proposition 3.1]). *There exists an isomorphism between $\mathcal{V}_X = \mathrm{Der}_{\Bbbk}(A_X)$ and $\{\mu \in V_n \mid \mu\,(I(X)) \subset I(X)\} \,/\, \{\mu \in V_n \mid \mu(\Bbbk[x_1, \dots, x_n]) \subset I(X)\}$.*

Remark 1.4.5. As explained in [BF18], the isomorphism presented in Proposition 1.4.4 allows us to represent derivations in $\mathrm{Der}_{\Bbbk}(A_X)$ as elements of the free A_X-module $\bigoplus_{i=1}^{n} A_X \dfrac{\partial}{\partial x_i}$. If $A_X = \Bbbk[x_1, \dots, x_n]/(f_1, \dots, f_s)$, then an element $\mu = g_1 \dfrac{\partial}{\partial x_1} + \cdots + g_n \dfrac{\partial}{\partial x_n} \in \bigoplus_{i=1}^{n} A_X \dfrac{\partial}{\partial x_i}$ is a derivation of A_X if and only if

$$\begin{cases} g_1 \frac{\partial f_1}{\partial x_1} + \cdots + g_n \frac{\partial f_1}{\partial x_n} = 0 \\ g_1 \frac{\partial f_2}{\partial x_1} + \cdots + g_n \frac{\partial f_2}{\partial x_n} = 0 \\ \vdots \\ g_1 \frac{\partial f_s}{\partial x_1} + \cdots + g_n \frac{\partial f_s}{\partial x_n} = 0 \end{cases}$$

holds in A_X.

In general, the Lie algebras $\mathcal{V}_X$ can have different properties depending on the variety X considered. For instance, the Lie algebra $\mathcal{V}_X$ is always a projective module over the coordinate ring A_X (since X is smooth), however, it is not always a free module as the next example shows.

Example 1.4.6. Let $X = \mathbb{S}^2 = \left\{(a_1, a_2, a_3) \in \mathbb{A}^3 \mid a_1^2 + a_2^2 + a_3^2 = 1\right\}$. We may use Remark 1.4.5 to find elements of $\mathcal{V}_{\mathbb{S}^2}$ inside $\bigoplus_{i=1}^{3} A_{\mathbb{S}^2} \dfrac{\partial}{\partial x_i}$. The derivations

$$\Delta_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \in \mathcal{V}_{\mathbb{S}^2}$$

with $i < j \in \{1, 2, 3\}$ generate $\mathcal{V}_{\mathbb{S}^2}$ as a module over $A_{\mathbb{S}^2}$ but not freely since

$$x_1 \Delta_{23} - x_2 \Delta_{13} + x_3 \Delta_{12} = 0.$$

Both $\mathcal{V}_{\mathbb{A}^n}$ and $\mathcal{V}_{T^n}$ are not only free as modules over the respective coordinate ring, but they have a Cartan subalgebra that decomposes them in weight spaces. In general, this will rarely occur. Some of the Lie algebras may not even have semisimple or nilpotent elements as the next example shows.

Example 1.4.7. Let $h \in \Bbbk[x]$ be a polynomial of odd degree equal to $2m + 1 \geq 3$. The hyperelliptic curve $\mathcal{H} = \{(x, y) \mid y^2 = 2h(x)\} \subset \mathbb{A}^2$ is smooth if and only if $\gcd(h, h') = 1$ [BF18, Proposition 5.1]. As a vector space, its coordinate ring $A_{\mathcal{H}}$ is $\Bbbk[x] \oplus y\Bbbk[x]$ and it has a filtration given by

$$\deg(x^k) = 2k \quad \text{and} \quad \deg(yx^k) = 2k + 2m + 1.$$

The Lie algebra $\mathcal{V}$ is a free module over $A_{\mathcal{H}}$ and it is generated by $\tau = y\frac{\partial}{\partial x} + h'(x)\frac{\partial}{\partial y}$. The filtration on $A_{\mathcal{H}}$ induces a filtration on $\mathcal{V}_{\mathcal{H}}$ by $\deg(g\tau) = \deg(g) + 2m - 1$ such that

$$\deg([\eta, \mu]) = \deg(\eta) + \deg(\mu) > \deg(\eta) \geq 1$$

if $\deg(\eta) \neq \deg(\mu)$. Hence, η is the only eigenvector of ad_η for every $\eta \in \mathcal{V}_{\mathcal{H}}$, i.e. $\mathcal{V}_{\mathcal{H}}$ has no semisimple elements. Similarly, $\mathcal{V}_{\mathcal{H}}$ has no nilpotent elements as well.

1.5 The module of Kähler differentials and the tangent sheaf

When it comes to the infinitesimal theory of algebraic varieties, the *module of Kähler differentials* appears as a substitute for the notion of differential forms. In this section, we introduce the notion of derivation of an algebra into a module and we define the module of Kähler differentials, then we show how they are related. We will use these notions to introduce one of the main objects of study of this thesis.

Let R be a commutative ring and A be an R-algebra. For any A-module, a *derivation of A into M* is an R-linear map $D : A \to M$ such that

$$D(ab)m = (D(a)b + aD(b))\, m$$

for all $a, b \in A$ and $m \in M$. We denote the space of all derivations of A into M by $\mathrm{Der}_R(A, M)$. Since the sum of two derivations is a derivation and aD is a derivation for each $a \in A$ and $D \in \mathrm{Der}_R(A, M)$, we have that $\mathrm{Der}_R(A, M)$ is an A-module.

The A-module of Kähler differentials $\Omega_{R/A}$ relative to R is the A-module generated by the symbols $\mathrm{d}a$ for all $a \in A$ subject to the relations

$$\mathrm{d}(ra + b) = r\mathrm{d}a + \mathrm{d}b, \quad d(ab) = a\mathrm{d}b + b\mathrm{d}a, \quad a, b \in A,\ r \in R.$$

Note that $\Omega_{R/A}$ comes with a linear map $\mathrm{d} : A \to \Omega_{R/A}$ given by $a \mapsto \mathrm{d}a$. If $D \in \mathrm{Der}_R(A, M)$, then define $\varphi_D : \Omega_{A/R} \to M$ as $\varphi(\mathrm{d}a) = D(a)$. Since D is a derivation of A into M, it follows that φ_D is a homomorphism of A-modules. By construction, φ_D is the only A-module homomorphism such that $\varphi_D \circ \mathrm{d} = D$. It follows that d is a universal homomorphism.

Proposition 1.5.1. *Let R be a commutative ring, A an R-algebra and $\Omega_{R/A}$ the A-module of*

Kähler differentials relative to R. Then the universal derivation $\mathrm{d} : A \to \Omega_{R/A}$ *is a derivation of* A *into* $\Omega_{R/A}$ *and satisfies the universal property: for any derivation* $D \in \mathrm{Der}_R(A, M)$*, there exists a unique homomorphism of* A*-modules* $\varphi : \Omega_{A/R} \to M$ *such that* $D = \varphi \circ \mathrm{d}$*. Therefore, the composition with* d *provides the isomorphism*

$$\mathrm{Hom}_A(\Omega_{A/R}, M) \cong \mathrm{Der}_R(A, M).$$

for every A*-module* M*.*

In the case that $M = A$, we will denote $\mathrm{Der}_R(A, M)$ by simply $\mathrm{Der}_R(A)$. The last proposition together with Proposition 1.4.4 implies the following corollary.

Corollary 1.5.2. *Let* X *be an algebraic variety. Then the following are isomorphic to each other:*

1. $\mathcal{V}_X = \mathrm{Der}_{\Bbbk}(A_X)$;
2. $\{\mu \in V_n \mid \mu(I(X)) \subset I(X)\} / \{\mu \in V_n \mid \mu(\Bbbk[x_1, \dots, x_n]) \subset I(X)\}$;
3. $\mathrm{Hom}_{A_X}(\Omega_{A_X, \Bbbk}, A_X)$.

Example 1.5.3. Let $A = \Bbbk[x_1, \dots, x_n]$, then any element of $\Omega_{A/\Bbbk}$ can be written as $\sum_{i=1}^n f_i dx_i$ with $f_1, \dots, f_n \in A$. The image of A-homomorphism $\alpha_i : \Omega_{A/\Bbbk} \to A$ that sends $\mathrm{d}x_j$ to δ_{ij} under the isomorphism $\mathrm{Hom}_A(\Omega_{A/\Bbbk}, A) \cong \mathrm{Der}_{\Bbbk}(A)$ is the derivation $\frac{\partial}{\partial t_i}$. Furthermore, $\Omega_{A/\Bbbk}$ has a basis of $\mathrm{d}x_1, \dots, \mathrm{d}x_n$ as an A-module.

Let A be a $\Bbbk$-algebra. If $h \in A$, we denote by $A_h = S^{-1}A$ where S is the multiplicative set $S = \{h^k \mid k \geq 0\}$. If $\mathfrak{p}$ is a prime ideal of A, we denote by $A_{\mathfrak{p}} = S^{-1}A$ where S is the multiplicative set $S = A \setminus \mathfrak{p}$. We may localize A-modules as well, and we will use the notations $M_f = A_f \otimes_A M$ and $M_{\mathfrak{p}} = A_{\mathfrak{p}} \otimes_A M$ for the localization of an A-module M by the multiplicative sets $\{h^k \mid k \geq 0\}$ and $A \setminus \mathfrak{p}$, respectively. For details on the localization of algebras and modules, we refer to [AM69, Chapter 3].

Lemma 1.5.4. *Let* R *be a commutative algebra and* A *an* R*-algebra, then*

1. $\Omega_{A'/R'} \cong \Omega_{A'/R} \otimes_R R'$*, where* R' *is another* R*-algebra and* $A' = A \otimes_R R'$*.*
2. $S^{-1}\Omega_{A/R} \cong \Omega_{S^{-1}A/R}$ *for any multiplicative set* $S \subset A$*.*

Proof. See [Har77, Proposition 16.4, Proposition 16.9]. □

The Lemma 1.5.4 shows that the module of Kähler differentials interacts nicely with localization. Combined with Proposition 1.5.1, we may use Lemma 1.5.4 to express the *tangent sheaf* associated with an affine algebraic variety using the localization of its coordinate ring. We refer to Appendix A for the basics of scheme theory and sheaf theory that we will use in this thesis.

Lemma 1.5.4 implies that there is a sheaf Ω_X on $X = \mathrm{Spec}\,(A)$ such that

$$\Gamma(D(h), \Omega_X) = \Omega_{A_h, \Bbbk}, \quad h \in A.$$

This sheaf is called *sheaf of differentials* of X over $\mathrm{Spec}\,(\Bbbk)$.

Let X be a smooth affine algebraic variety, $A = A_X$ be its coordinate ring and $\mathcal{V} = \mathcal{V}_X = \mathrm{Der}_{\Bbbk}(A)$ the Lie algebra of vector fields on X. We can now define a quasi-coherent sheaf $\Theta_X = \tilde{\mathcal{V}}_X$ on X (more precisely, on $\mathrm{Spec}\,(A)$), called *tangent sheaf*. Since A is Noetherian, $S^{-1}\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$ for every finitely generated A-module M, N by [Eis95, Proposition 2.10]. Therefore, by Proposition 1.5.1 and Lemma 1.5.4,

$$\begin{aligned} S^{-1}\mathrm{Der}_{\Bbbk}(A) &\cong S^{-1}\mathrm{Hom}_A(\Omega_{A,\Bbbk}, A) \cong \mathrm{Hom}_{S^{-1}A}(S^{-1}\Omega_{A,\Bbbk}, S^{-1}A) \\ &\cong \mathrm{Hom}_{S^{-1}A}(\Omega_{S^{-1}A,\Bbbk}, S^{-1}A) \cong \mathrm{Der}_{\Bbbk}(S^{-1}A). \end{aligned}$$

This means that $\Theta_{X,h} = \mathrm{Der}_{\Bbbk}(A_{X,h})$ for every $h \in A$ and $\Theta_{X,p} = \mathrm{Der}_{\Bbbk}\left(A_{\mathfrak{m}_p}\right)$ for every $p \in X$. Furthermore, the isomorphism on Proposition 1.5.1 implies an isomorphism of sheaves

$$\mathrm{Hom}_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X) \cong \Theta_X.$$

1.6 Local parameters and uniformizing parameters

Suppose that X is smooth, and $\dim X = r$. Let $p \in X$. We say that $u_1, \dots, u_r \in A_X$ are local parameters at $p \in X$ if each $u_i \in \mathfrak{m}_p$, and their images form a basis of the vector space $\mathfrak{m}/\mathfrak{m}_p^2$. Using the isomorphism $d_p : \mathfrak{m}_p/\mathfrak{m}_p^2 \to T_pX^*$, we see that $u_1, \dots, u_r$ are local parameters if and only if the linear forms $d_pu_1, \dots, d_pu_r$ are linearly independent. This is equivalent to saying that the system of equations

$$d_pu_1 = \cdots = d_pu_r = 0$$

has 0 as its only solution on T_pX.

We know the images of $x_1 - a_1, \dots, x_n - a_n$ generate $\mathfrak{m}_p/\mathfrak{m}_p^2$. We will show how to get local parameters at p using these functions. Let $I_X = (f_1, \dots, f_m)$, and define

$$\mathrm{J}(p) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_n}(p) \\ & \cdots & \\ \frac{\partial f_m}{\partial x_1}(p) & \cdots & \frac{\partial f_m}{\partial x_n}(p) \end{bmatrix}.$$

By definition of T_pX, $\dim T_pX = n - \mathrm{rank}\, J(p)$. Thus, there exists a nonzero $(n-r)$-minor of $J(p)$, since p is nonsingular. Suppose the principal $(n-r)$-minor $h(p)$ is nonzero, then

$$t_1 = x_{n-r+1} - a_{n-r+1},\ \dots,\ t_r = x_n - a_n$$

may be chosen as local parameters at $p = (a_1, \dots, a_n)$. Similarly, if the $(n-r)$-minor with row indices $\alpha = \{i_1, \dots, i_{n-r}\}$ and columns indices $\beta = \{j_1, \dots, j_{n-r}\}$ is nonzero, then $t_i = x_i - a_i$, $i \notin \beta$, can be chosen as local parameters at $p = (a_1, \dots, a_n)$ [BN19, Lemma 3].

For each $h \in A_X$, define $D(h) = \{q \in X \mid h(q) \neq 0\}$. Thus, if $p \in X$ and $h(p) \neq 0$, then $D(h)$ is an open neighborhood of p. Let $J = \left(\frac{\partial f_i}{\partial x_j}\right)$, and $\{h_j \mid j \in J\}$ be the set of all nonzero $(n-r)$-minors of J. Thus, $\{D(h_j) \mid j \in J\}$ is an open cover of X [BN19, Lemma 2], and hence it is an atlas for X, called *standard atlas* and its open sets $D(h_j)$ are called *standard* *charts.*

Definition 1.6.1. Let $\mathcal{O}$ be the structure sheaf of X and $U \subset X$ an affine open subset, then $t_1, \dots, t_s \in A$ are called *uniformizing parameters* on U if

1. $t_1, \dots, t_s$ are algebraically independent over $\Bbbk$, thus $\Bbbk[t_1, \dots, t_s] \subset A_X$;

2. every $f \in A_X$ is algebraic over $\Bbbk[t_1, \dots, t_s]$, that is, there exists a polynomial $p(x) \in \Bbbk[t_1, \dots, t_s][x]$ such that $p(f) = 0$;

3. $\frac{\partial}{\partial t_i}$ extends uniquely to a derivation of the localized algebra $A_U = \Gamma(U, \mathcal{O})$.

If U has uniformizing parameters $t_1, \dots, t_s$, then we say that U is an *ètale chart.*

If $t_1, \dots, t_s \in A_X$ are uniformizing parameters in the chart $D(h)$, then $s = r = \dim X$, each element of $f \in A_{D(h)}$ is algebraic over $\Bbbk[t_1, \dots, t_s]$, and $\mathrm{Der}_{\Bbbk}(A_{D(h)}) = \bigoplus_{i=1}^{r} A_{D(h)} \dfrac{\partial}{\partial t_i}$. Furthermore, if $\eta \in \mathcal{V}$ then there exist unique $g_1, \dots, g_r \in A_{D(h)}$ such that $\eta = \sum_{i=1}^{r} g_i \dfrac{\partial}{\partial t_i}$, since $\mathcal{V} \subset \mathrm{Der}_{\Bbbk}(A_{D(h)})$.

If $x_{i_1}, \dots, x_{i_r}$ are the variables such that the i_a-collumn is not part of the minor $h \in \{h_j \mid j \in J\}$, then $x_{i_1}, \dots, x_{i_r}$ are uniformizing parameters in $D(h)$ [BN19, Lemma 3], called *standard uniformizing parameters.* If $t_1, \dots, t_r$ are standard uniformizing parameters in the standard chart $D(h)$, then $h\frac{\partial}{\partial t_i} \in \mathcal{V}_X$ for each $i = 1, \dots, r$ [BF18, Section 3].

The isomorphism $\mathrm{Hom}_{A_{D(h)}}\left(\Omega_{A_{D(h)}/\Bbbk}, A_{D(h)}\right) \xrightarrow{\cong} \mathrm{Der}_{\Bbbk}(A_{D(h)}) = \bigoplus_{i=1}^{r} A_{D(h)} \dfrac{\partial}{\partial t_i}$ sends the map $\alpha_j : \Omega_{A_{D(h)}/\Bbbk} \to A_{D(h)}$ given by $g dt_i \mapsto \delta_{ij}$ to $\frac{\partial}{\partial t_j}$. Hence, the maps $\alpha_1, \dots, \alpha_r$ freely generate $\mathrm{Hom}_{A_{D(h)}}\left(\Omega_{A_{D(h)}/\Bbbk}, A_{D(h)}\right)$. Since $\alpha_1, \dots, \alpha_r$ are the dual of $\mathrm{d}t_1, \dots, \mathrm{d}t_r$ in $\mathrm{Hom}_{A_{D(h)}}\left(\Omega_{A_{D(h)}/\Bbbk}, A_{D(h)}\right)$, we have that $\Omega_{A_{D(h)}/\Bbbk}$ is also a free $A_{D(h)}$-module, and $\mathrm{d}t_1, \dots, \mathrm{d}t_r$ forms a basis of it.

Proposition 1.6.2 ([BFN19, Lemma 3])**.** *Let $h \in A_X$ and $t_1, \dots, t_r \in A_X$ uniformizing parameters of $D(h)$. Let $p \in D(h)$, and define $\bar{t}_i = t_i - t_i(p)$ for each $i = 1, \dots, r$. Then, $\bar{t}_1, \dots, \bar{t}_r$ are local parameters at p.*

Example 1.6.3. Let $X = \mathbb{S}^2 = \{(x, y, z) \in \Bbbk^3 \mid x^2 + y^2 + z^2 = 1\}$ and $A = A_{\mathbb{S}^2}$. Its Jacobian matrix is

$$\mathrm{Jac} = \begin{bmatrix} 2x & 2y & 2z \end{bmatrix}.$$

We may choose $h = z$ and $D(h) = X \setminus \{(x, y, z) \in X \mid z = 0\}$ as standard chart. Note that x, y are algebraically independent over $\Bbbk$. Let $p \in \Bbbk[x, y][u]$ given by $p(u) = u^2 + x^2 + y^2 - 1$, then $p(z) = 0$ and z is algebraic over $\Bbbk[x, y]$. Since x, y, z generates A as a ring, every element of A is algebraic over $\Bbbk[x, y]$. Note that $0 = \frac{\partial}{\partial x}(x^2 + y^2 + z^2 - 1) = 2x + 2z\frac{\partial z}{\partial x}$, so $\frac{\partial z}{\partial x} = -\frac{x}{z}$ which determines $\frac{\partial}{\partial x} \in \mathrm{Der}(A_h)$. By symmetry $\frac{\partial z}{\partial y} = -\frac{y}{z}$. Therefore, x, y are uniformizing parameters in $D(z)$.

Example 1.6.4. Let

$$X = \mathbb{T}^n = \prod_{i=1}^n \mathbb{S}^1 = \left\{ (x_1, y_1, x_2, y_2, \dots, x_n, y_n) \in \mathbb{A}^{2n} \mid x_i^2 + y_i^2 = 1 \; \forall i = 1, \dots, n \right\}$$

be the n-torus. Note that $A = \Bbbk[x_1, y_1, \dots, x_n, y_n]/I$, where $I = \langle x_j^2 + y_j^2 = 1 \mid j = 1, \dots, n \rangle$. Let $h = \prod_{j=1}^n y_j + I$, then $D(h) = X \setminus \{(x_1, y_1, \dots, 0, \pm 1, \dots, x_n, y_n) \in X\}$ is X without two $(n-1)$-tori. We will prove that $x_1 + I, \dots, x_n + I$ are uniformizing parameters for $D(h)$. They are algebraically independent. If $q_j(u) = x_j^2 + u^2 - 1 + I \in \Bbbk[x_1 + I, \dots, x_n + I][u]$, then $q_j(y_j + I) = 0$. Since A is generated by x_j, y_j as an algebra, each element of A is algebraic over $\Bbbk[x_1 + I, \dots, x_n + I]$. Since $0 = \frac{\partial}{\partial x_a}(x_b^2 + y_b^2) = 2\delta_{ab} x_b + 2y_b \frac{\partial y_b}{\partial x_a}$, we have that $\frac{\partial y_a}{\partial x_b} = -\delta_{ab} \frac{x_a}{y_a}$ and it determines $\frac{\partial}{\partial x_a} \in A_h$. Therefore, $x_1, \dots, x_n$ are uniformizing parameters for $D(h)$. In the other hand, $y_1, \dots, y_n$ are uniformizing parameters for $D(x_1 \cdots x_n)$.

If $D(h)$ is a standard chart of X, we may see $D(h)$ itself as a smooth affine algebraic variety with coordinate ring

$$A_h \cong \Bbbk[x_1, \dots, x_n, t]/(f_1, \dots, f_r, th - 1)$$

where $A = \Bbbk[x_1, \dots, x_n]/(f_1, \dots, f_r)$ is the coordinate ring of X. Without loss of generality, assume that $t_1 = x_1, \dots, t_r = x_r$ are standard uniformizing parameters in $D(h)$. We may apply Remark 1.4.5 and solve that system to obtain a basis

$$\tau_i = \frac{\partial}{\partial x_i} + \frac{1}{h} \sum_{j=n-r+1}^{n} q_{1j} \frac{\partial}{\partial x_j}, \quad i = 1, \dots, r$$

of $\Gamma(D(h), \mathcal{O}_X) = \operatorname{Der}(A_h)$ as an $\Gamma(D(h), \mathcal{O}_X) = A_h$-module. We have that $\tau_i(x_j) = \delta_{ij}$, and $\tau_1, \dots, \tau_r$ are exactly the derivatives that $\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_r}$ extends to. For every $p \in D(h)$, $\tau_1, \dots, \tau_r$ extend to derivations of $\mathcal{O}_{X,p} = A_{\mathfrak{m}_p}$ and

$$\Theta_{X,p} = \operatorname{Der}_{\Bbbk}\left(\mathcal{O}_{X,p}\right) = \bigoplus_{i=1}^r \mathcal{O}_{X,p} \tau_i.$$

Example 1.6.5. Let $X = \mathbb{S}^2 = \left\{ (a_1, a_2, a_3) \in \mathbb{A}^3_{\Bbbk} \mid a_1^2 + a_2^2 + a_3^2 = 1 \right\} \subset \mathbb{A}^3_{\Bbbk}$ be the sphere, and $A = \Bbbk[x_1, x_2, x_3]/(x_1^2 + x_2^2 + x_3^2 - 1)$ its coordinate ring. Take the standard chart $D(2x_3) = D(x_3)$, then $\mu = g_1 \frac{\partial}{\partial x_1} + g_2 \frac{\partial}{\partial x_2} + g_3 \frac{\partial}{\partial x_3} \in \operatorname{Der}_{\Bbbk}(A_{x_3})$ if and only if

$$x_1 g_1 + x_2 g_2 + x_3 g_3 = 0.$$

Thus, $g_3 = -\frac{x_1}{x_3} g_1 - \frac{x_2}{x_3} g_2$, and

$$\mu = g_1 \left(\frac{\partial}{\partial x_1} - \frac{x_1}{x_3} \frac{\partial}{\partial x_3} \right) + g_2 \left(\frac{\partial}{\partial x_2} - \frac{x_2}{x_3} \frac{\partial}{\partial x_3} \right).$$

We have that $t_1 = x_1,\ t_2 = x_2$ are standard uniformizing parameters in the chart $D(x_3)$ and

$$\left(\frac{\partial}{\partial x_i} - \frac{x_i}{x_3}\frac{\partial}{\partial x_3}\right)(x_j) = \delta_{ij} \quad \text{for } i, j = 1, 2.$$

Hence, the local sections $\tau_1 = \frac{\partial}{\partial x_1} - \frac{x_1}{x_3}\frac{\partial}{\partial x_3}$ and $\tau_2 = \frac{\partial}{\partial x_2} - \frac{x_2}{x_3}\frac{\partial}{\partial x_3}$ are the partial derivatives $\frac{\partial}{\partial t_1}$ and $\frac{\partial}{\partial t_2}$, respectively. Moreover,

$$\Gamma(D(x_3), \Theta_X) = \mathrm{Der}(A_{x_3}) = A_{x_3}\tau_1 \oplus A_{x_3}\tau_2.$$

Note that τ_1 and τ_2 are not global, but $x_3\tau_1$ and $x_3\tau_2$ are. This process can be done similarly with the other two standard charts $D(x_1)$ and $D(x_2)$.

1.7 Power series and filtrations

Let $p \in X$ be a nonsingular point of X with local parameters $t_1, \dots, t_r \in A$. For any $f \in A_{\mathfrak{m}_p}$ is possible to find $F_i \in \Bbbk[T_1, \dots, T_s]$ of degree i, $i \geq 0$, such that $f - \sum_{i=0}^{k} F_i(t_1, \dots, t_s) \in \mathfrak{m}_p^{k+1}$ [Sha94a, Section II.2.2]. Thus we can define the *formal power series* $\Psi = \sum_{i=0}^{\infty} F_i$ of f, called *Taylor series*.

Theorem 1.7.1 ([Sha94a, Section II.2.2]). *Every function $f \in A_{\mathfrak{m}_p}$ has at least one Taylor series. If p is nonsingular, then a function has a unique Taylor series.*

Therefore, we have a uniquely determined map $\pi : A \to R$ that takes each functions to its Taylor series, where $R = \Bbbk[[T_1, \dots, T_s]]$ such that $\pi(t_i) = T_i$. It is possible to show that π is a homomorphism of algebras. The kernel of π is equal to $\{f \in \mathcal{O}_P \mid f \in \mathfrak{m}_p^{k+1}\ \forall k \geq 0\}$, thus $f \in \ker \pi$ if and only if $f \in \bigcap_{k=0}^{\infty} \mathfrak{m}_P^k = (0)$. Thus π is injective, and every element of $\mathfrak{m}_p/\mathfrak{m}_P^2$ is uniquely determined by its Taylor series.

Example 1.7.2. Let $X = \mathbb{A}^1$ with coordinate t, and let $p = 0$. Then $\mathfrak{m}_p = (t)$, and one can associate a power series $\sum_{m=0}^{\infty} \alpha_m t^m$ with any rational function $f(t) = P(t)/Q(t)$ with $Q(0) \neq 0$ such that

$$\frac{P(t)}{Q(t)} - \sum_{m=0}^{k} \alpha_m t^m = 0 \mod t^{k+1}.$$

For example,

$$\frac{1}{1-t} = \sum_{m=0}^{\infty} t^m, \text{ because } \frac{1}{1-t} - \sum_{m=0}^{k} t^m = \frac{t^{k+1}}{1-t} = 0 \mod t^{k+1}.$$

Let $\mathfrak{m}_0$ be the ideal in R of power series without constant term. Consider descending chain in A and R:

$$\begin{aligned} &A \supset \mathfrak{m}_p \supset \mathfrak{m}_p^2 \supset \mathfrak{m}_p^3 \supset \dots \\ &R \supset \mathfrak{m}_0 \supset \mathfrak{m}_0^2 \supset \mathfrak{m}_0^3 \supset \dots \end{aligned} \tag{1.1}$$

We define topologies on A and R by taking (1.1) to be bases of open neighborhoods of 0. Since $\bigcap_{k=0}^{\infty} \mathfrak{m}_p^k = (0)$ and $\bigcap_{k=0}^{\infty} \mathfrak{m}_p^k = (0)$, these topologias are separable. In this topology, the closure of $\Bbbk[t_1, \dots, t_r]$ is A, hence $\Bbbk[t_1, \dots, t_r]$ is a dense subset of A. By construction, $\pi(\mathfrak{m}_p^j) \subset \mathfrak{m}_0^j$, hence the map π is continuous.

Remark 1.7.3. Suppose $X \subset A^n$. Denote by $\mathfrak{m}'_p$ the ideal of $\Bbbk[x_1, \dots, x_n]$ corresponding to p, then we have that

$$(T_pX)^* \cong \mathfrak{m}_p/\mathfrak{m}_p^2 \cong \mathfrak{m}'_p/(\mathfrak{m}'_p)^2 \cong \mathfrak{m}_0/\mathfrak{m}_0^2.$$

Fix a standard chart $D(h)$ with $p \in D(h)$ and suppose that $t_1, \dots, t_r \in A$ are standard uniformizing parameters. For $l \geq -1$ define $\mathcal{V}_X(l) = \left\{ \eta \in \mathcal{V}_X \mid \eta(A) \subset \mathfrak{m}_p^{l-1} \right\}$. If $l + k \geq 1$, then $[\mathcal{V}_X(l), \mathcal{V}_X(k)] \subset \mathcal{V}_X(l + k)$, thus $\mathcal{V}_X(0)$ is an ideal of $\mathcal{V}_X$. Furthermore, we have a filtration

$$\mathcal{V}_X = \mathcal{V}_X(-1) \supset \mathcal{V}_X(0) \supset \mathcal{V}_X(1) \supset \mathcal{V}_X(2) \supset \dots,$$

and $\mathcal{V}_X(l) = \mathfrak{m}_p^{l+1}\mathcal{V}_X$ for every $l \geq -1$ [BFN19, Lemma 5]. Similarly, we construct a filtration on $\widehat{\mathcal{L}} = \mathrm{Der}_{\Bbbk}(R) = \bigoplus_{i=1}^{r} R\frac{\partial}{\partial T_i}$ given by

$$\widehat{\mathcal{L}} = \widehat{\mathcal{L}}(-1) \supset \widehat{\mathcal{L}}(0) \supset \widehat{\mathcal{L}}(1) \supset \widehat{\mathcal{L}}(2) \supset \dots,$$

with $\widehat{\mathcal{L}}(l) = \mathfrak{m}_0^l\widehat{\mathcal{L}}$.

Proposition 1.7.4 ([BF18, Proposition 3.2])**.** *With the notation we fixed above, there exists a unique embedding*

$$\widehat{\pi} : \mathcal{V}_X \to \widehat{\mathcal{L}},$$

such that the following diagram is commutative

$$\begin{tikzcd} \mathcal{V}_X \times A \arrow[r] \arrow[d, "\widehat{\pi}\times\pi"'] & A \arrow[d, "\pi"] \\ \widehat{\mathcal{L}} \times R \arrow[r] & R \end{tikzcd}$$

where the horizontal arrows are the actions of a Lie algebra by derivations, i.e. $(\mu, f) \in \mathcal{V}_X \times A \mapsto \mu(f) \in A$ *and* $(d, \Psi) \in \widehat{\mathcal{L}} \times R \mapsto d(\Psi) \in R$.

Chapter 2

Sheafification of $A\mathcal{V}$-modules

We move to the representation theory of the Lie algebras of vector fields on an affine algebraic variety. Because of the lack of a common structure on these Lie algebras that would allow us to use popular Lie theory techniques, we will consider representations that admit a compatible action of the ring of functions associated with the variety. This allows us to use techniques from commutative algebra and algebraic geometry to infer interesting properties of these modules.

Suppose that A_X is the coordinate ring of the smooth affine algebraic variety X and $\mathcal{V}_X = \mathrm{Der}(A_X)$ is its Lie algebra of derivations. In this chapter, we study the modules called $A_X\mathcal{V}_X$-modules. These modules were instrumental in the classification of weight modules with finite multiplicities over both $\mathcal{V}_{T^n}$ and $\mathcal{V}_{\mathbb{A}^n}$, see [BF16] and [GS22].

There exist several examples of $A_X\mathcal{V}_X$-modules that have a fundamental role in various mathematical theories. For instance, both A_X and $\mathcal{V}_X$ are $A_X\mathcal{V}_X$-modules as well as the modules Ω^k_X of k-differential forms on X. Furthermore, every module over the algebra of differential operators on X is an $A_X\mathcal{V}_X$-module, which evolves into a theory that finds valuable applications across various mathematical domains.

The main objective of this chapter is to introduce a geometric object related to these modules. In other words, for an $A_X\mathcal{V}_X$-module M that is finitely generated as A_X-module, we prove there is an action of the tangent sheaf Θ_X on the coherent sheaf $\tilde{M}$ that is compatible with its structure of module over the structure sheaf $\mathcal{O}_X$ of X. We called these sheaves infinitesimally equivariant sheaves. Our approach to demonstrate this is fairly algebraic and any reader familiar with the basics of commutative algebra theory should be able to follow the proofs. We also prove that the associated representation $\mathcal{V}_X \to \mathrm{End}_{\Bbbk}(M)$ is a differential operator and finalize the chapter proving the main conjecture of [BFN19], which states that every finite $A\mathcal{V}$-module is a gauge module. These results can be found in the paper published by Bouaziz and the author [BR23], and we summarize them in the following theorem.

Theorem. *Let X be smooth irreducible affine algebraic variety, A be its coordinate ring and $\mathcal{V} = \mathrm{Der}(A)$. Then,*

1. *Every finite $A\mathcal{V}$-module is a gauge module.*

2. *There is an equivalence between the category of finite $A\mathcal{V}$-modules and the category of infinitesimally equivariant bundles on X.*
3. *If M is a finite $A\mathcal{V}$-module with associated $\mathcal{V}$-representation $\rho : \mathcal{V} \to \mathfrak{gl}_{\Bbbk}(M)$, then ρ is a differential transformation with order less or equal to a number that depends solely on the rank of M as an A-module.*

For a more explicit claim of these results, we refer to the conclusion Section 2.8 of this chapter.

In Section 2.1, we start the chapter with definitions and preliminary results. We give a precise definition of an $A_X\mathcal{V}_X$-module as well as a way to construct new $A_X\mathcal{V}_X$-modules using existing ones. After giving some examples, we prove the first fundamental result of the section, Theorem 2.1.8, that states that any $A_X\mathcal{V}_X$-module that is finitely generated as A_X-module is a projective A_X-module. This implies that the corresponding coherent sheaf is a vector bundle.

Afterward, we study an associative algebra that governs $A_X\mathcal{V}_X$-modules in Section 2.2, which has a similar role to the universal enveloping algebra of a Lie algebra. This associative algebra is the smash product $A_X\#U(\mathcal{V}_X)$ of A_X and the universal enveloping algebra of $\mathcal{V}_X$, which is an associative algebra defined on the tensor product $A \otimes U(\mathcal{V})$ using the coproduct of $U(\mathcal{V})$. We will prove several identities in this algebra, especially for its Lie subalgebra $A \otimes \mathcal{V}$. These identities will be used in the following sections to prove the main results of this chapter.

Section 2.3 is dedicated to finding certain annihilators in $A_X\#U(\mathcal{V}_X)$. We will use the identities proven in the previous section for certain elements of $A_X \otimes \mathcal{V}_X \subset A_X\#U(\mathcal{V}_X)$ to construct elements in the annihilator. These elements measure a degree A_X-nonlinearity of the representation of $\mathcal{V}_X$ associated with an $A_X\mathcal{V}_X$-module.

We define the action of the tangent sheaf on the coherent sheaf associated with an $A_X\mathcal{V}_X$-module finitely generated as an A_X-module in Section 2.4. We will use the annihilators we found in the previous section and a formula provided in [BI23] to define the action of the local section of $\mathcal{V}_X$ associated with affine basic open sets. After proving this is well-defined and is exactly the action we needed, we establish the main theorem of this chapter Theorem 2.4.5, building on the work done in previous sections.

We define *infinitesimally equivariant sheaves* in Section 2.5. They are the sheaf-theoretically version of $A\mathcal{V}$-modules. This was a term coined by Emile Bouaziz when we started working on this problem together with Yuly Billig and Collin Ingalls. In this section, we introduce this notion and relate it to the *Atiyah algebra* of a sheaf of modules. We finish the section explaining that for smooth irreducible algebraic varieties, the category of infinitesimally equivariant bundles is equivalent to the category of $A\mathcal{V}$-modules.

In Section 2.6, an analysis of the representation associated with an $A\mathcal{V}$-module or the *Lie map* associated with an infinitesimally equivariant sheaf is presented. We define the set of differential operators between two sheaves of modules (or modules over a commutative algebra). Additionally, we use the annihilators studied before to show the representation $\mathcal{V}_X \to \mathfrak{gl}_{\Bbbk}(M)$ associated with an $A_X\mathcal{V}_X$-module M is a differential operator. Since being a differential operator is a local property, we infer that the Lie map of infinitesimally equivariant sheaves on schemes that are covered by smooth affine algebraic varieties is a differential operator as well.

Before we give a summary of all results of this chapter in Section 2.8, we prove the main conjecture of the paper [BFN19] in Section 2.7, which states that every finite $A\mathcal{V}$-module is a gauge module. We combine the results of this chapter with the structure theorems of [BI23] to prove the conjecture.

2.1 Preliminaries

Let X be a smooth affine variety, $A = A_X$ its coordinate ring and $\mathcal{V} = \mathcal{V}_X = \mathrm{Der}(A_X)$. We say that M is an $A\mathcal{V}$-module if M is both a $\mathcal{V}$-module and an A-module such that

$$\eta \cdot (f \cdot m) = \eta(f) \cdot m + f \cdot (\eta \cdot m) \tag{2.1}$$

for all $\eta \in \mathcal{V}$, $f \in A$, and $m \in M$. The formula (2.1) is called the *Leibniz rule.*

Since A is a commutative algebra, we may consider the tensor product $M \otimes_A N$ of the $A\mathcal{V}$-modules M and N, which is an A-module. This vector space is also a $\mathcal{V}$-module, where the action is given by

$$\eta(m \otimes n) = (\eta m) \otimes n + m \otimes (\eta n), \text{ for all } m \in M, \; n \in N, \text{ and } \eta \in \mathcal{V}.$$

Because

$$\begin{aligned} \eta(f(m \otimes n)) &= \eta((fm) \otimes n) = \eta(fm) \otimes n + (fm) \otimes (\eta n) \\ &= (\eta(f)m) \otimes n + (f(\eta m)) \otimes n + (fm) \otimes (\eta n) \end{aligned}$$

for all $m \in M$, $n \in N$, $\eta \in \mathcal{V}$, and $f \in A$, we have that $M \otimes_A N$ is an $A\mathcal{V}$-module.

Consider the pth tensor product $M^{\otimes_A p}$ of M over A. This is an $A\mathcal{V}$-module as we just discussed in the previous paragraph. The permutation group $\mathfrak{S}_p$ acts on $M^{\otimes_A p}$ by

$$\sigma(v_1 \otimes_A \cdots \otimes_A v_p) = v_{\sigma(1)} \otimes_A \cdots \otimes_A v_{\sigma(p)}$$

for each tensor $v_1 \otimes_A \cdots \otimes_A v_p \in M^{\otimes_A p}$. We say that $v \in M^{\otimes_A p}$ is an *alternating p-tensor* if $\sigma(v) = \mathrm{sgn}(\sigma)v$. The set $\Lambda^p_A M$ of all alternating p-tensors is an $A\mathcal{V}$-submodule of $M^{\otimes_A p}$. There is a correspondence between $\Lambda^p_A M$ and the *p-exterior power* of M, which is the vector subspace of the exterior algebra

$$\Lambda^\bullet_A(M) = T_A(M)/(v \otimes_A v \mid v \in M)$$

generated by tensors $v_1 \wedge \cdots \wedge v_p = v_1 \otimes_A \cdots \otimes_A v_p + (v \otimes_A v \mid v \in M)$, $v_1, \ldots, v_p \in M$. Both $\Lambda^p_A(M)$ and $\Lambda^\bullet_A(M)$ are $A\mathcal{V}$-modules.

Another way to construct $A\mathcal{V}$-modules using existing ones is considering the *dual* of a module. For an $A\mathcal{V}$-module M the full dual space

$$M^* = \mathrm{Hom}_{\Bbbk}(M, \Bbbk)$$

is both an A-module and $\mathcal{V}$-module with $(f\cdot\varphi)(m) := \varphi(f\cdot m)$ and $(\eta\cdot\varphi)(m) = -\varphi(\eta\cdot m)$ for all $f\in A$, $\eta\in\mathcal{V}$, $m\in M$. These actions are compatible because

$$(\eta(f\varphi))(m) = (f\varphi)(-\eta m) = \varphi(-f(\eta m)) = \varphi(\eta(f)m - \eta(fm)) = (\eta(f)\varphi + f(\eta\varphi))(m).$$

We can also consider the dual of M as a module over the algebra A. The vector space

$$M^\circ = \mathrm{Hom}_A(M, A)$$

is naturally an A-module with $(f\cdot\varphi)(m) = \varphi(f\cdot m)$. However, the action of $\mathcal{V}$ is given by

$$(\eta\cdot\varphi)(m) = -\varphi(\eta\cdot m) + \eta\cdot(\varphi(m)).$$

Considering that

$$\begin{aligned}(\eta(f\varphi))(m) =& -(f\varphi)(\eta m) + \eta((f\varphi)(m)) = \varphi(-f(\eta m)) + \eta(\varphi(fm))\\ =&\varphi(\eta(f)m - f(\eta m)) + \eta(\varphi(fm)) = (\eta(f)\varphi)(m) - \varphi(\eta(fm)) + \eta((\varphi)(fm))\\ =&(\eta(f)\varphi + f(\eta\varphi))(m), \qquad f\in A,\ \eta\in\mathcal{V},\ m\in M,\end{aligned}$$

we see that M° is an $A\mathcal{V}$-module.

Example 2.1.1. The algebra A is an $A\mathcal{V}$-module naturally and it is a simple $A\mathcal{V}$-module when X is smooth [BF18, Theorem 4.1].

Example 2.1.2. The Lie algebra $\mathcal{V}$ is an $A\mathcal{V}$-module as well. For all $\eta,\mu\in\mathcal{V}$ and $f\in A$,

$$[\eta, f\mu] = \eta(f)\mu + f[\eta,\mu].$$

Thus, the adjoint representation and the natural action of A on $\mathcal{V}$ make $\mathcal{V}$ an $A\mathcal{V}$-module.

Example 2.1.3. Another example is the A-module Ω^1_A of Käller differentials. The action of $\mathcal{V}$ on Ω^1_A is given by

$$\eta(a\mathrm{d}b) = \eta(a)\mathrm{d}b + a\mathrm{d}(\eta b),$$

which is compatible with the action of A. Since exterior powers of an $A\mathcal{V}$-module is an $A\mathcal{V}$-module, we have that $\Omega^k_A = \Lambda^k_A(\Omega^1_A)$ is an $A\mathcal{V}$-module as well.

Example 2.1.4. The algebra $\mathcal{D} = \mathcal{D}_X$ of differential operators is the associative subalgebra of $\mathrm{End}_{\Bbbk}(A)$ generated by the subspaces $A\,\mathrm{id}$ and $\mathcal{V}$, where $\mathrm{id} : A\to A$ denotes the identity morphism. Therefore, a $\mathcal{D}$-module is both an A-module and $\mathcal{V}$-module. Since

$$\eta\circ f\mathrm{id} = \eta(f)\mathrm{id} + (f\mathrm{id})\circ\eta, \quad \text{for all } f\in A,\ \eta\in\mathcal{V},$$

we have that any $\mathcal{D}$-module is an $A\mathcal{V}$-module. However, a $\mathcal{D}$-module M also satisfies $(f\eta)m = f\cdot(\eta\cdot m)$. In fact, an $A\mathcal{V}$-module M is a $\mathcal{D}$-module if and only if $(f\eta)m = f\cdot(\eta\cdot m)$ for all $f\in A$, $\eta\in\mathcal{V}$, and $m\in M$ [Gro67]. The formal definition of the sheaf of differential operators on a scheme is different from the one presented here for a smooth affine variety. If $Y = (|Y|, \mathcal{O}_Y)$ is a scheme, then $\mathcal{D}_Y = \mathrm{Diff}_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{O}_Y)$, where $\mathrm{Diff}_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{N})$ is defined in Section 2.6 for two sheaves of $\mathcal{O}_Y$-modules $\mathcal{M}$ and $\mathcal{N}$.

Definition 2.1.5. An $A\mathcal{V}$-module M is called finite if M is finitely generated as an A-

module.

For an A-module M and prime ideal $\mathfrak{p} \in \operatorname{Spec}(A)$, we define

$$\operatorname{rank}_{\mathfrak{p}}(M) = \dim_{\Bbbk(\mathfrak{p})}(\Bbbk(\mathfrak{p}) \otimes_A M),$$

where $\Bbbk(\mathfrak{p}) = A_{\mathfrak{p}}/(\mathfrak{p})$ is the residue field of the local algebra $A_{\mathfrak{p}}$. Note that $\operatorname{rank}_{(0)}(M) = \dim_{\operatorname{Frac}(A)} \operatorname{Frac}(A) \otimes_A M$. By Nakayama's Lemma, $\operatorname{rank}_{(0)}(M) \leq \operatorname{rank}_{\mathfrak{p}}(M)$ for each $\mathfrak{p} \in \operatorname{Spec}(A)$.

Lemma 2.1.6. *Let M be an A-module with $r = \operatorname{rank}_{(0)}(M) < \infty$. Then M is a projective A-module if and only if $\Lambda^k_A(M) = 0$ for all $k > r$.*

Proof. If B is a local algebra and N is a B-module, then one of the consequences of Nakayama's Lemma is that $\operatorname{rank}(N) = 0$ implies $N = 0$. We have that $\operatorname{rank}_{\mathfrak{p}}(\Lambda^k(M)) = \binom{\operatorname{rank}_{\mathfrak{p}}(M)}{k}$. Therefore, $\operatorname{rank}_{\mathfrak{p}}\Lambda^k(M) = 0$ if $k > \operatorname{rank}_{\mathfrak{p}}(M)$, so the module $\left(\Lambda^k_A(M)\right)_{\mathfrak{p}} = \Lambda^k_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ over the local algebra $A_{\mathfrak{p}}$ is zero if $k > \operatorname{rank}_{\mathfrak{p}}(M)$.

If M is projective, then $\operatorname{rank}_{\mathfrak{p}}(M) = \operatorname{rank}_{(0)}(M) = r$ for every prime ideal $\mathfrak{p} \in \operatorname{Spec}(A)$. Hence, $(\Lambda^k_A(M))_{\mathfrak{p}} = 0$ for each $\mathfrak{p} \in \operatorname{Spec}(A)$, and $k > r$. Since its support is $\operatorname{Spec}(A)$, $\Lambda^k_A(M) = 0$.

On the other hand, suppose $\Lambda^k_A(M) = 0$ for all $k > r$. If $k > r$, then $\operatorname{rank}(\Lambda^k_A(M)) = 0$. So $\operatorname{rank}_{\mathfrak{p}}(\Lambda^k_A(M)) = 0$ for each $\mathfrak{p} \in \operatorname{Spec}(A)$. Therefore, $\operatorname{rank}_{\mathfrak{p}}(M) \leq \operatorname{rank}_{(0)}(M)$. We conclude $\operatorname{rank}_{\mathfrak{p}}(M) = \operatorname{rank}_{(0)}(M)$. Since A is a Notherian intregral domain, $\operatorname{rank}_{\mathfrak{p}}(M) = \operatorname{rank}_{(0)}(M)$ for every $\mathfrak{p} \in \operatorname{Spec}(A)$ is equivalent to M being projective [Eis95, Exercise 20.13]. □

If M is an A-module, we define the *A-annihilator of M* to be

$$\operatorname{Ann}_A(M) = \{f \in A \mid \forall m \in M\ fm = 0\}.$$

The A-annihilator is an ideal of A. When M is an $A\mathcal{V}$-module, its A-annihilator is trivial as the next lemma shows.

Lemma 2.1.7. *Let M be an $A\mathcal{V}$-module. Then*

$$\operatorname{Ann}_A(M) = \{f \in A \mid \forall m \in M\ fm = 0\} = 0.$$

Proof. We will show the ideal $\operatorname{Ann}_A(M)$ is an $A\mathcal{V}$-submodule of A. For every $f \in \operatorname{Ann}_A(M)$ and $\eta \in \mathcal{V}$,

$$\eta(f)m = \eta(fm) - f(\eta m) = 0, \qquad \text{for each } m \in M.$$

Therefore, $\eta(f) \in \operatorname{Ann}_A(M)$. Hence, $\operatorname{Ann}_A(M)$ is an $A\mathcal{V}$-submodule of A. Since A is a simple $A\mathcal{V}$-module (see [BF18, Theorem 4.1]) and $\operatorname{Ann}_A(M)$ is an ideal, $\operatorname{Ann}_A(M) = 0$. □

For an A-module M, we denote by

$$\operatorname{Supp}(M) = \left\{\mathfrak{p} \in \operatorname{Spec}(A) \mid M_{\mathfrak{p}} = 0\right\}$$

and it is called the *support of M*. It is known that $\operatorname{Supp}(M) = V(\operatorname{Ann}_A(M))$, where $V(P) := \{\mathfrak{p} \in \operatorname{Spec}(A) \mid I \subset \mathfrak{p}\}$ is the set of all prime ideals that contains the subset $P \subset A$.

Theorem 2.1.8. *Let M be a finite $A\mathcal{V}$-module, then M is a projective A-module.*

Proof. Let $r = \mathrm{rank}_{(0)}(M)$, hence $\Lambda^r_A(M)$ is nonzero. Suppose $\Lambda^p_A(M) \neq 0$ for some $p > r$. We know that localization commutes with exterior power, thus $\Lambda^p_{\mathrm{Frac}(A)}(\mathrm{Frac}(A) \otimes_A M) = 0$. Then there exists a prime ideal $\mathfrak{p}$ of A such that $\left(\Lambda^p_A(M)\right)_{\mathfrak{p}} = \Lambda^p_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$. Therefore,

$$\mathfrak{p} \in \mathrm{Supp}(\Lambda^p_A(M)) = V(\mathrm{Ann}_A(\Lambda^p_A(M))).$$

Hence, $\mathrm{Ann}_A(\Lambda^p_A(M)) \neq 0$, which is a contradiction by Lemma 2.1.7 and the fact that $\Lambda^p_A(M)$ is an $A\mathcal{V}$-module. Thus, $\Lambda^p_A M$ must be trivial for each $p > r$. We conclude that M is a projective A-module using Lemma 2.1.6. □

The previous theorem implies the sheaf $\tilde{M}$ on X associated with any finite $A\mathcal{V}$-module M is a vector bundle.

It was proved in [BIN23, Lemma 4.2] that the *torsion* of any finite $A\mathcal{V}$-module is trivial, i.e.

$$\mathrm{Tor}_A(M) = \{m \in M \mid \exists f \in A,\ f \neq 0,\ \text{such that } fm = 0\} = 0$$

for every finite $A\mathcal{V}$-module M. The above theorem can be seen as a generalization of this statement since every projective module over A is torsion-free.

2.2 The smash product $A\#U(\mathcal{V})$ and its Lie subalgebra $A\#\mathcal{V}$

Vector fields have a natural geometric origin, thus it is reasonable to ask whether $A\mathcal{V}$-modules have an algebraic geometric object related to it as well. This translates into the question of the possibility to construct a $\Theta_X = \tilde{\mathcal{V}}$-module structure on the coherent sheaf $\tilde{M}$ of $\mathcal{O}_X$-modules that satisfies the Leibniz rule for each affine subset of X, where $\tilde{M}$ is the sheaf associated to a finite $A\mathcal{V}$-module M. We give a positive answer to this question in Theorem 2.4.5. The second main problem is related to the structure of the map $\Theta \to \mathrm{End}_{\Bbbk}(\mathcal{M})$ associated with the representation $\mathcal{V} \to \mathrm{End}_{\Bbbk}(M)$. We wish to prove it is a differential operator. To prove these results, we will investigate certain associative algebra related to $A\mathcal{V}$-modules and prove some identities in it.

The universal enveloping algebra $U(\mathcal{V})$ of $\mathcal{V}$ is a Hopf Algebra with the coproduct given by

$$\Delta(\eta) = \eta \otimes 1 + 1 \otimes \eta, \qquad \text{for all } \eta \in \mathcal{V}.$$

The coproduct Δ is extended to $U(\mathcal{V})$ as an algebra homomorphism $\Delta : U(\mathcal{V}) \to U(\mathcal{V}) \otimes U(\mathcal{V})$. The usual notation for the coproduct of an arbitrary element $u \in U(\mathcal{V})$ is given by $\Delta(u) = \sum_{(u)} u_{(1)} \otimes u_{(2)}$. The action of $\mathcal{V}$ on the algebra A allows us to define the *smash product* $A\#U(\mathcal{V})$ which is an associative algebra. In fact, $A\#U(\mathcal{V})$ is a Hopf algebra, but we will only use its algebra structure. As a vector space, $A\#U(\mathcal{V})$ coincides with $A \otimes U(\mathcal{V})$, and the product is defined as

$$(f\#u)(g\#v) = \sum_{(u)} f(u_{(1)}g)\#u_{(2)}v \tag{2.2}$$

for all $f, g \in A$, $u, v \in U(\mathcal{V})$. In particular,

$$(f\#\eta)(g\#\mu) = f\eta(g)\#\mu + fg\#\eta\mu, \tag{2.3}$$

for each $f, g \in A$, $\eta, \mu \in \mathcal{V}$. Notice that $(f\#1)(g\#\mu) = fg\#\mu$. For details on Hopf algebras and smash products, we refer to [DNR01].

Due to the fact that $A \cong A\#\Bbbk \subset A\#U(\mathcal{V})$ and $U(\mathcal{V}) \cong \Bbbk\#U(\mathcal{V}) \subset A\#U(\mathcal{V})$, we see that every module over $A\#U(\mathcal{V})$ is both an A-module and $\mathcal{V}$-module. Taking $f = 1$ and $v = 1$ in the equation (2.2), we conclude that these actions satisfy (2.1). Therefore, every module over $A\#U(\mathcal{V})$ is an $A\mathcal{V}$-module. On the other hand, an $A\mathcal{V}$-module M is an $A\#U(\mathcal{V})$-module if we define $(f\#\mu)\cdot m := f\cdot(\mu\cdot m)$ for each $f \in A$, $\mu \in \mathcal{V}$ and $m \in M$. Consequently, there exists an equivalence of the categories of $A\mathcal{V}$-modules and modules over $A\#U(\mathcal{V})$.

From now on, results in this section do not depend on X. Therefore, we may assume that A is an integral domain and $\mathcal{V} = \mathrm{Der}(A)$.

The smash product $A\#U(\mathcal{V})$ is an associative algebra, thus the commutator defines a Lie algebra structure on it. By (2.3),

$$[f\#\eta, g\#\mu] = fg\#[\eta,\mu] + f\eta(g)\#\mu - g\mu(f)\#\eta$$

for all $f, g \in A$ and $\eta, \mu \in \mathcal{V}$. It follows that $A\#\mathcal{V}$ is a Lie subalgebra of $A\#U(\mathcal{V})$.

The vector space $A\#\mathcal{V} = A \otimes_{\Bbbk} \mathcal{V}$ is an (A, A)-bimodule. Explicitly, $(a, b)(f\#\eta) = af\#b\eta$ for all $a, b, f \in A$ and $\eta \in \mathcal{V}$. We may write these actions using the tensor product

$$(a \otimes b)(f\#\eta) = af\#b\eta.$$

The vector space $A\otimes_{\Bbbk} A$ is an algebra with product given by $(a\otimes b)(c\otimes d) = (ac)\otimes(bd)$. The multiplication map $\mathrm{m} : A \otimes A \to A$, $\mathrm{m}(a \otimes b) = ab$, is a homomorphism of commutative algebras, and its kernel is the ideal of $A \otimes A$ generated by

$$f \otimes 1 - 1 \otimes f, \quad f \in A.$$

Consider the linear map $\delta : A \to A \otimes A$ given by $\delta(f) = f \otimes 1 - 1 \otimes f$, then $\delta(f) \in \ker \mathrm{m}$, and $\delta(f_1)\cdots\delta(f_k) \in (\ker \mathrm{m})^k$ for all $f, f_1, \dots, f_k \in A$.

The algebra $A \otimes_{\Bbbk} A$ is an (A, A)-bimodule. We will write $f(a \otimes b) = (fa) \otimes b$, and $(a \otimes b)f = a \otimes (bf)$ for each $f, a, b \in A$. With this notation, we have that $\delta(fg) = f\delta(g) + \delta(f)g$ for all $f, g \in A$, hence the following lemma may be proved inductively.

Lemma 2.2.1. *For all $f, g \in A$ and $p \geq 1$,*

1. $(\delta(fg))^p = \displaystyle\sum_{k=0}^{p} \binom{p}{k} f^k \delta(g)^k \delta(f)^{p-k} g^{p-k}$;

2. $(\delta(f+g))^p = \sum_{k=0}^{p} \binom{p}{k} \delta(f)^{p-k}\delta(g)^k.$

The previous lemma shows how δ interacts with the structure of $A \otimes_{\mathbb{k}} A$ as both an algebra and (A, A)-bimodule.

For each $f \in A$, $\eta \in \mathcal{V}$ and $p \geq 1$ consider the following element of $A\#\mathcal{V}$

$$\Omega_p(f,\eta) = \delta(f)^p(1\#\eta) = \sum_{k=0}^{p}(-1)^k\binom{p}{k} f^{p-k}\#f^k\eta \in A\#U(\mathcal{V}). \tag{2.4}$$

We will prove identities for these elements inside the Lie algebra $A\#\mathcal{V}$. Their importance will be seen in the next sections of this thesis.

The first interesting property of these elements is that they commute with $A \cong A\#1 \subset A\#U(\mathcal{V})$. Therefore, their action on an $A\mathcal{V}$-module can be seen to be by operators in $\mathrm{End}_A(M)$.

Lemma 2.2.2. *For all $f, g \in A$, $\eta \in \mathcal{V}$, and $p \geq 1$,*

$$\Omega_p(f,\eta)(g\#1) = (g\#1)\Omega_p(f,\eta).$$

Proof. It follows from the Binomial Theorem that $0 = (1-1)^p = \sum_{k=0}^{p}(-1)^k\binom{p}{k}$. Thus,

$$\begin{aligned}
&\Omega_p(f,\eta)(g\#1) - (g\#1)\Omega_p(f,\eta)\\
&= \sum_{k=0}^{p}(-1)^k\binom{p}{k}\left((f^{p-k}\#f^k\eta)(g\#1) - (g\#1)(f^{p-k}\#f^k\eta)\right)\\
&= \sum_{k=0}^{p}(-1)^k\binom{p}{k}\left((f^p\eta(g)\#1 + f^{p-k}g\#f^k\eta - gf^{p-k}\#f^k\eta\right) = 0
\end{aligned}$$

for each $f, g \in A$, and $\eta \in \mathcal{V}$. □

The following lemmas give identities that will be used in other sections. The goal now is to investigate the bracket between the elements (2.4).

Lemma 2.2.3. *For every $f, g \in A$, $\eta, \mu \in \mathcal{V}$, and $p, q \geq 1$,*

$$\left[\Omega_p(f,\eta), \Omega_q(g,\mu)\right] = \sum_{k=0}^{p}\sum_{l=0}^{q}(-1)^{k+l}\binom{p}{k}\binom{q}{l} f^{p-k}g^{q-l}\#[f^k\eta, g^l\mu]$$

Proof. As we saw in the proof of the last lemma, $\sum_{k=0}^{r}(-1)^k\binom{r}{k} = 0$ for all $r \geq 1$. Hence,

$$\begin{aligned}
&\left[\Omega_p(f,\eta), \Omega_q(g,\mu)\right]\\
&= \sum_{k=0}^{p}\sum_{l=0}^{q}(-1)^{k+l}\binom{p}{k}\binom{q}{l}[f^{p-k}\#f^k\eta, g^{q-l}\#g^l\mu]
\end{aligned}$$

$$\begin{aligned}&= \sum_{k=0}^{p}\sum_{l=0}^{q}(-1)^{k+l}\binom{p}{k}\binom{q}{l}\left(f^p\eta(g^{q-l})\#g^l\mu - g^q\mu(f^{p-k})\#f^k\eta + f^{p-k}g^{q-l}\#[f^k\eta, g^l\mu]\right)\\ &= \sum_{k=0}^{p}\sum_{l=0}^{q}(-1)^{k+l}\binom{p}{k}\binom{q}{l}f^{p-k}g^{q-l}\#[f^k\eta, g^l\mu].\end{aligned}$$

□

The following lemma will be used frequently when we calculate other brackets inside $A\#\mathcal{V}$.

Lemma 2.2.4. *For every $f \in A$, $\eta, \mu \in \mathcal{V}$, and $p, q \geq 1$,*

$$\left[\Omega_p(f,\eta), \Omega_q(f,\mu)\right] = \Omega_{p+q}(f,[\eta,\mu]) + p\Omega_{p+q-1}(f,\mu(f)\eta) - q\Omega_{p+q-1}(f,\eta(f)\mu).$$

Proof. By the previous lemma,

$$\begin{aligned}&\left[\Omega_p(f,\eta), \Omega_q(f,\mu)\right]\\ &= \sum_{k=0}^{p}\sum_{l=0}^{q}(-1)^{k+l}\binom{p}{k}\binom{q}{l}f^{p+q-k-l}\#[f^k\eta, f^l\mu]\\ &= \sum_{k=0}^{p}\sum_{l=0}^{q}(-1)^{k+l}\binom{p}{k}\binom{q}{l}f^{p+q-k-l}\#\left(lf^{k+l-1}\eta(f)\mu - kf^{k+l-1}\mu(f)\eta + f^{k+l}[\eta,\mu]\right)\end{aligned}$$

Let us separate this into three sums. Set $u = k+l$. Thus, the coefficient at $f^{p+q-u}\#f^u[\eta,\mu]$ is the same as at y^u in

$$\begin{aligned}&\sum_{k=0}^{p}\sum_{l=0}^{q}(-1)^{k+l}\binom{p}{k}\binom{q}{l}y^{k+l}\\ &= \sum_{k=0}^{p}(-1)^k\binom{p}{k}y^k\left(\sum_{l=0}^{q}(-1)^{k+l}\binom{q}{l}y^l\right)\\ &=(1-y)^p(1-y)^q = (1-y)^{p+q}\\ &= \sum_{u=0}^{p+q}(-1)^u\binom{p+q}{u}y^u\end{aligned}$$

Therefore,

$$\sum_{k=0}^{p}\sum_{l=0}^{q}(-1)^{k+l}\binom{p}{k}\binom{q}{l}f^{p+q-k-l}\#f^{k+l}[\eta,\mu] = \Omega_{p+q}(f,[\eta,\mu]).$$

The coefficient at $f^{p+q-u}\#f^{u-1}\eta(f)\mu$ is the same as in y^u in

$$\sum_{k=0}^{p}\sum_{l=0}^{q}(-1)^{k+l}\binom{p}{k}\binom{q}{l}ly^{k+l}$$

$$
\begin{aligned}
&=\left(\sum_{k=0}^{p}(-1)^k\binom{p}{k}y^k\right)\left(\sum_{l=0}^{q}(-1)^l\binom{q}{l}ly^l\right)\\
&=\left(\sum_{k=0}^{p}(-1)^k\binom{p}{k}y^k\right)\left(y\frac{d}{dy}\right)\left(\sum_{l=0}^{q}(-1)^l\binom{q}{l}y^l\right)\\
&=(1-y)^p\left(y\frac{d}{dy}(1-y)^q\right)=-qy(1-y)^{p+q-1}\\
&=-q\sum_{u=0}^{p+q-1}(-1)^u\binom{p+q-1}{u}y^{u+1}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\sum_{k=0}^{p}\sum_{l=0}^{q}(-1)^{k+l}\binom{p}{k}\binom{q}{l}lf^{p+q-k-l}\# f^{k+l-1}\eta(f)\mu\\
&=-q\sum_{u=0}^{p+q-1}(-1)^u\binom{p+q-1}{u}f^{p+q-u-1}\# f^u\eta(f)\mu\\
&=-q\Omega_{p+q-1}(f,\eta(f)\mu).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&-\sum_{k=0}^{p}\sum_{l=0}^{q}(-1)^{k+l}\binom{p}{k}\binom{q}{l}kf^{p+q-k-l}\# f^{k+l-1}\eta(f)\mu\\
&=p\sum_{u=0}^{p+q-1}(-1)^u\binom{p+q-1}{u}f^{p+q-u-1}\# f^u\mu(f)\eta\\
&=p\Omega_{p+q-1}(f,\mu(f)\eta)
\end{aligned}
$$

We conclude that

$$
\left[\Omega_p(f,\eta),\Omega_q(f,\mu)\right]=\Omega_{p+q}(f,[\eta,\mu])+p\Omega_{p+q-1}(f,\mu(f)\eta)-q\Omega_{p+q-1}(f,\eta(f)\mu)
$$

□

Lemma 2.2.5. *For every $f,g,h\in A$, $\eta,\mu\in\mathcal{V}$, $p,q\geq 1$,*

1. $[\Omega_p(f,\eta),\Omega_q(f,g\mu)]-[\Omega_p(f,g\eta),\Omega_q(f,\mu)]=\Omega_{p+q}(f,\eta(g)\mu+\mu(g)\eta)$.
2. $[\Omega_p(f,g\eta),\Omega_q(f,h\eta)]-[\Omega_p(f,\eta),\Omega_q(f,gh\eta)]=2\Omega_{p+q}(f,h\eta(g)\eta)$
3. $[\Omega_p(f,\eta),\Omega_q(f,g\eta)]-[\Omega_p(f,g\eta),\Omega_q(f,\eta)]=2\Omega_{p+q}(f,\eta(g)\eta)$;
4. $[\Omega_p(f,g\eta),\Omega_q(f,\eta(h)\eta)]-[\Omega_p(f,\eta),\Omega_q(f,g\eta(h)\eta)]=2\Omega_{p+q}(f,\eta(g)\eta(h)\eta)$;

Proof. For part (1), we have that $[\eta,g\mu]-[g\eta,\mu]=\eta(g)\mu+\mu(g)\eta$. Therefore,

$$
[\Omega_p(f,\eta),\Omega_q(f,g\mu)]-[\Omega_p(f,g\eta),\Omega_q(f,\mu)]
$$

$$\begin{aligned}
=&\Omega_{p+q}(f,[\eta,g\mu])+p\Omega_{p+q-1}(f,g\mu(f)\eta)-q\Omega_{p+q-1}(f,\eta(f)g\mu)\\
&-\Omega_{p+q}(f,[g\eta,\mu])-p\Omega_{p+q-1}(f,\mu(f)g\eta)+q\Omega_{p+q-1}(f,g\eta(f)\mu)\\
=&\Omega_{p+q}(f,[\eta,g\mu]-[g\eta,\mu])=\Omega_{p+q}(f,\eta(g)\mu+\mu(g)\eta).
\end{aligned}$$

Part (2) follows from part (1) by substituting η by $h\eta$ and μ by η. All other parts follow from (1) and (2). □

Lemma 2.2.6. *For all $f \in A$, $\eta,\mu \in \mathcal{V}$, and $p \geq 1$,*

$$[\Omega_p(f,\eta),1\#\mu]=\Omega_p(f,[\eta,\mu])+p\Omega_{p-1}(f,\mu(f)\eta)-p\mu(f)\Omega_{p-1}(f,\eta).$$

Proof. For every $\eta,\mu \in \mathcal{V}$,

$$\begin{aligned}
&[\Omega_p(f,\eta),1\#\mu]\\
=&\sum_{k=0}^{p}(-1)^k\binom{p}{k}[f^{p-k}\#f^k\eta,1\#\mu]\\
=&\sum_{k=0}^{p}(-1)^k\binom{p}{k}\left(f^{p-k}\#[f^k\eta,\mu]-(p-k)\mu(f)f^{p-k-1}\#f^k\eta\right)\\
=&\sum_{k=0}^{p}(-1)^k\binom{p}{k}\left(f^{p-k}\#f^k[\eta,\mu]-kf^{p-k}\#f^{k-1}\mu(f)\eta-(p-k)\mu(f)f^{p-k-1}\#f^k\eta\right)\\
=&\Omega_p(f,[\eta,\mu])+p\Omega_{p-1}(f,\mu(f)\eta)-\mu(f)\sum_{k=0}^{p}(-1)^k\binom{p}{k}(p-k)f^{p-k-1}\#f^k\eta.
\end{aligned}$$

The coefficient at $f^{p-1-u}\#f^u\eta$ is the same as in y^u in

$$\begin{aligned}
&\sum_{k=0}^{p}(-1)^k\binom{p}{k}(p-k)y^k\\
=&p\sum_{k=0}^{p}(-1)^k\binom{p}{k}y^k-y\frac{d}{dy}\sum_{k=0}^{p}(-1)^k\binom{p}{k}y^k\\
=&p(1-y)^p-y\frac{d}{dy}(1-y)^p=p(1-y)^p+py(1-y)^{p-1}\\
=&p(1-y)^{p-1}(1-y+y)=p(1-y)^{p-1}=p\sum_{u=0}^{p-1}(-1)^u\binom{p-1}{u}y^u.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{k=0}^{p}(-1)^k\binom{p}{k}(p-k)f^{p-k-1}\#f^k\eta=&p\sum_{k=0}^{p-1}(-1)^k\binom{p-1}{k}f^{p-k-1}\#f^k\eta\\
=&p\Omega_{p-1}(f,\eta).
\end{aligned}$$

Therefore,

$$[\Omega_p(f,\eta), 1\#\mu] = \Omega_p(f,[\eta,\mu]) + p\Omega_{p-1}(f,\mu(f)\eta) - p\mu(f)\Omega_{p-1}(f,\eta).$$

$\square$

The last lemma of this section shows a way to define $\Omega_p(f,\eta)$ recursively.

Lemma 2.2.7. *For each $f \in A$, $p \geq 0$, and $\eta \in \mathcal{V}_f$,*

$$\Omega_p(f, f\eta) = f\Omega_p(f,\eta) - \Omega_{p+1}(f,\eta).$$

Proof. Using the well-known recurrence relation $\binom{p+1}{k} - \binom{p}{k} = \binom{p}{k-1}$, we get

$$\begin{aligned}
&f\Omega_p(f,\eta) - \Omega_{p+1}(f,\eta)\\
&=\sum_{k=0}^{p}(-1)^k\binom{p}{k}f^{p-k+1}\#f^k\eta - (-1)^k\binom{p+1}{k}f^{p+1-k}\#f^k\eta - (-1)^p 1\#f^{p+1}\eta\\
&=\sum_{k=0}^{p}(-1)^k\left(\binom{p}{k} - \binom{p+1}{k}\right)f^{p+1-k}\#f^k\eta - (-1)^p 1\#f^{p+1}\eta\\
&=\sum_{k=1}^{p+1}(-1)^{k+1}\binom{p}{k-1}f^{p+1-k}\#f^k\eta\\
&=\sum_{l=0}^{p}(-1)^l\binom{p}{l}f^{p-l}\#f^{l+1}\eta = \Omega_p(f,\eta).
\end{aligned}$$

$\square$

2.3 Annihilators of finite $A\mathcal{V}$-modules

Recall that X is a smooth irreducible algebraic variety, A is its coordinate ring and $\mathcal{V} = \mathrm{Der}(A)$ is its Lie algebra of polynomial vector fields. For an $A\mathcal{V}$-module M, we define the *annihilator* $\mathrm{Ann}(M)$ by

$$\mathrm{Ann}(M) = \{x \in A\#U(\mathcal{V}) \mid xm = 0 \text{ for all } m \in M\}.$$

Example 2.3.1. If M is an $A\mathcal{V}$-module, then $\Omega_1(f,\eta) \in \mathrm{Ann}(M)$ for every $f \in A$ and $\eta \in \mathcal{V}$ if and only if M is a $\mathcal{D}$-module by example 2.1.4. If we take $M = \mathcal{V}$, then $\Omega_2(f,\eta) \in \mathrm{Ann}(\mathcal{V})$ for every $f \in A$, and $\eta \in \mathcal{V}$, because

$$\begin{aligned}
&f^2[\eta,\mu] - 2f[f\eta,\mu] + [f^2\eta,\mu]\\
=&f^2[\eta,\mu] + 2f\mu(f)\eta - 2f^2[\eta,\mu] - 2f\mu(f)\eta + f^2[\eta,\mu] = 0.
\end{aligned}$$

We wish to prove that for every finite $A\mathcal{V}$-module M and for all $f \in A$, there exists $N > 0$ such that $\Omega_p(f,\eta) \in \mathrm{Ann}(M)$ for each $p > N$ and $\eta \in \mathcal{V}$. We will use the identities we provided earlier to show that.

Lemma 2.3.2. *Let M be a finite $A\mathcal{V}$-module with $r = \operatorname{rank}_A(M)$, $f \in A$ and $\eta \in \mathcal{V}$. For any $p > r^2$, there exist $a_1, \dots, a_{r^2} \in A$ and $b \in A \setminus \{0\}$ such that $b\Omega_p(f,\eta) + \sum_{i=1}^{r^2} a_i\Omega_i(f,\eta) \in \operatorname{Ann}(M)$.*

Proof. Since M is finitely generated with rank r, we have that $\operatorname{End}_A(M)$ is a finitely generated A-module with rank at most r^2. By Lemma 2.2.2, the action of $\Omega_i(f,\eta)$ commutes with the action of A in M for each $i \in \{1, 2, \dots, r^2, p\}$, hence $\{\Omega_k(f,\eta) \mid k = 1, \dots, r^2\} \cup \{\Omega_p(f,\eta)\}$ defines a family of endomorphisms in $\operatorname{End}_A(M)$. Therefore, it must be A-linearly dependent. Thus, there exists $a_1, \dots, a_N \in A$ and $b \in A$, $b \neq 0$, such that $b\Omega_p(f,\eta) + \sum_{i=1}^{r^2} a_i\Omega_i(f,\eta) \in \operatorname{Ann}(M)$. □

Lemma 2.3.3. *Let M be a finite $A\mathcal{V}$-module with rank r, $f \in A$ and $\eta \in \mathcal{V}$ such that $\eta(f) \neq 0$. Then exists N that depends on r such that $\Omega_p(f, \eta(f)^{r^2}\eta) \in \operatorname{Ann}(M)$ for all $p \geq N$.*

Proof. Let $m > r^2$, then by Lemma 2.3.2 there exists $a_1, \dots, a_{r^2} \in A$ not all zero and $a_{r^2+1} \in A \setminus \{0\}$ such that $\sum_{i=1}^{r^2+1} a_i\Omega_{m_i}(f,\eta) \in \operatorname{Ann}(M)$ where $m_i = i$ if $i \leq r^2$ and $m_i = p$ if $i = r^2+1$. Thus, for every $m \in M$,

$$\begin{aligned}
0 &= \Omega_{m_1}(f,\eta)\left(\sum_{i=1}^{r^2+1} a_i\Omega_{m_i}(f,\eta)\right)m \\
&= \left(\sum_{i=1}^{r^2+1} a_i\Omega_{m_i}(f,\eta)\right)\Omega_{m_1}(f,\eta)m + \left(\sum_{i=1}^{r^2+1} a_i\left[\Omega_{m_1}(f,\eta), \Omega_{m_i}(f,\eta)\right]\right)m \\
&= \left(\sum_{i=2}^{r^2+1} a_i(m_1 - m_i)\Omega_{m_1+m_i-1}(f, \eta(f)\eta)\right)m.
\end{aligned}$$

Therefore, $\sum_{i=2}^{r^2+1} a_i(m_1 - m_i)\Omega_{m_1+m_i-1}(f,\eta(f)\eta) \in \operatorname{Ann}(M)$. For all $m \in M$

$$\begin{aligned}
0 &= \Omega_{m_1+m_2-1}(f,\eta(f)\eta)\left(\sum_{i=2}^{r^2+1} a_i(m_1 - m_i)\Omega_{m_1+m_i-1}(f,\eta(f)\eta)\right)m \\
&= \left(\sum_{i=2}^{r^2+1} a_i(m_1 - m_i)\left[\Omega_{m_1+m_2-1}(f,\eta(f)\eta), \Omega_{m_1+m_i-1}(f,\eta(f)\eta)\right]\right)m \\
&= \left(\sum_{i=3}^{r^2+1} a_i(m_1 - m_i)(m_2 - m_i)\Omega_{2m_1+m_2+m_i-2}(f, \eta(f)^2\eta)\right)m.
\end{aligned}$$

We may do this process r^2 times to conclude that $b\Omega_{p+N}(f, \eta(f)^{r^2}\eta) \in \operatorname{Ann}(M)$ for some $0 \neq b \in A$ and $N \geq 0$ that depends on r. By Theorem 2.1.8, $\operatorname{Tor}_A(M) = 0$, hence

$\Omega_{m+N}(f,\eta(f)^{r^2}\eta) \in \mathrm{Ann}(M)$. Since this holds for every $p > r^2$, $\Omega_{k+r^2+N}(f,\eta(f)^{r^2}\eta) \in \mathrm{Ann}(M)$ for every $k \geq 1$. □

Proposition 2.3.4. *Let M be a finite $A\mathcal{V}$-module and $f \in A$. If there exists $\mu \in \mathcal{V}$ such that $\mu(f) \neq 0$, then there exist $\eta \in \mathcal{V}$ with $\eta(f) \neq 0$ and N that depends on the rank of M such that $\Omega_p(f,\eta) \in \mathrm{Ann}(M)$ for every $p > N$.*

Lemma 2.3.5. *Let M be a finite $A\mathcal{V}$-module, $f \in A$, and $\eta \in \mathcal{V}$. Suppose $\Omega_p(f,\eta) \in \mathrm{Ann}(M)$ for each $p > N$ for some $N > 0$. Then for all $g, h \in A$*

1. *$\Omega_{2N+1+k}(f,\eta(g)\eta) \in \mathrm{Ann}(M)$ for all $k \geq 1$;*
2. *$\Omega_{3N+2+k}(f,\eta(g)\eta(h)\eta) \in \mathrm{Ann}(M)$ for all $k \geq 1$;*
3. *$\Omega_{3N+2+k}(f,g\eta(\eta(h))\eta) \in \mathrm{Ann}(M)$ for all $k \geq 1$.*

Proof. The first and second claims follow from parts (3) and (4) of Lemma 2.2.5, respectively. Since

$$\Omega_p(f,g\eta(\eta(h))\eta) = \Omega_p(f,\eta(g\eta(h))\eta) - \Omega_p(f,\eta(g)\eta(h)\eta),$$

we get that $\Omega_p(f,g\eta(\eta(h))\eta)$ for $p > 3N+2$. □

The following proposition is key for this Section's main result.

Proposition 2.3.6. *Let M be a finite $A\mathcal{V}$-module, and $f \in A$. Let $\eta \in \mathcal{V}$ with $\eta(f) \neq 0$ and $N > 0$ such that $\Omega_p(f,\eta) \in \mathrm{Ann}(M)$ for every $p > N$. Let $g, h \in A$ and $I_{g,h,f,\eta}$ be the principal ideal of A generated by $\eta(g)\eta(\eta(h))$. Then for every $q \in I_{g,h,f,\eta}$, $\tau \in \mathcal{V}$ and $p > 3N+4$, $\Omega_p(f,q\tau) \in \mathrm{Ann}(M)$.*

Proof. The ideal $I_{g,h,f,\eta}$ is generated as a vector space by elements with the form $q = x\eta(g)\eta(\eta(h))$ with $x \in A$. By Lemma 2.3.5,

$$\begin{aligned}
&[\Omega_{3N+2+k}(f,x\eta(\eta(h))\eta),\Omega_l(f,g\tau)] - [\Omega_{3N+2+k}(f,gx\eta(\eta(h))\eta),\Omega_l(f,\tau)] \\
&\quad - \Omega_{3N+2+k+l}(f,\tau(g)x\eta(\eta(h))\eta) \\
&=\Omega_{3N+2+k+l}(f,\tau(g)x\eta(\eta(h))\eta + x\eta(g)\eta(\eta(h))\tau) - \Omega_{3N+2+k+l}(f,\tau(g)x\eta(\eta(h))\eta) \\
&=\Omega_{3N+2+k+l}(f,q\tau) \in \mathrm{Ann}(M)
\end{aligned}$$

for each $k, l \geq 1$, and $\tau \in \mathcal{V}$. Therefore, $\Omega_{3N+4+k}(f,q\tau) \in \mathrm{Ann}(M)$ for every $k \geq 1$, $q \in I_{g,h,f,\eta}$, $\tau \in \mathcal{V}$. □

Definition 2.3.7. For an ideal I of A, define $I^{(0)} = I$ and $I^{(k)}$ to be the ideal of A generated by $\{g, \mu(g) \mid g \in I^{(k-1)},\ \mu \in \mathcal{V}\}$.

Lemma 2.3.8. *Let M be an $A\mathcal{V}$-module, and $f \in A$. Suppose that I is an ideal of A such that $\Omega_p(f,q\tau) \in \mathrm{Ann}(M)$ for every $p > N$ for some $N > 0$. Then for each $p > N+k$, $\Omega_p(f,g\tau) \in \mathrm{Ann}(M)$ for all $g \in I^{(k)}$ and $\tau \in \mathcal{V}$.*

Proof. By Lemma 2.2.6,

$$0 = [\Omega_{p+1}(f,g\tau), 1\#\mu]v$$

$$
\begin{aligned}
&= \Omega_{p+1}(f,[g\tau,\mu])v + (p+1)\Omega_p(f,\mu(f)g\tau)v - (p+1)\mu(f)\Omega_p(f,g\tau)v \\
&= -\Omega_{p+1}(f,\mu(g)\tau)v + \Omega_{p+1}(f,g[\tau,\mu])v \\
&= -\Omega_{p+1}(f,\mu(g)\tau)v
\end{aligned}
$$

for every $g \in I$, $\mu, \tau \in \mathcal{V}$, $p > N$ and $v \in M$. Thus, $\Omega_{p+1}(f,\mu(g)\tau) \in \mathrm{Ann}(M)$ for every $g \in I$ and $\mu \in \mathcal{V}$.

Furthermore, for every $g \in I$ and $h \in A$, we have that $gh \in I$ and

$$\Omega_p(f,h\mu(g)\tau) = \Omega_p(f,\mu(gh)\tau) - \Omega_p(f,g\mu(h)\tau) \in \mathrm{Ann}(M).$$

Hence, for every $g \in I^{(1)}$, we have that $\Omega_p(f,g\tau) \in \mathrm{Ann}(M)$ for every $p > N+1$. Since $I^{(k)} = (I^{(k-1)})^{(1)}$, we conclude by induction that $\Omega_p(f,g\tau) \in \mathrm{Ann}(M)$ for every $p > N+k$, $\tau \in \mathcal{V}$ and $g \in I^{(k)}$. $\square$

Lemma 2.3.9. *For every $f \in A$ with $f \notin \Bbbk$, and $p \in X$, there exist $\mu_1,\dots,\mu_l \in \mathcal{V}$ for some $l \geq 1$ such that $(\mu_1 \circ \mu_2 \circ \cdots \circ \mu_l)(f)(p) \neq 0$.*

Proof. The proof is similar to [BF18, Proposition 3.3]. Let $p \in X$, $t_1,\dots,t_s$ be local parameters centered at p, $\tau_1,\dots,\tau_s \in \mathcal{V}$ with $\tau_i(t_j) = h\delta_{ij}$ for some $h \in A$ with $h(p) \neq 0$. Write

$$\sum_\alpha f_\alpha t^\alpha \in \Bbbk[[t_1,\dots,t_s]]$$

the Taylor series at p of f. Choose $\beta = (\beta_1,\dots,\beta_s)$ such that $f_\beta \neq 0$ and $|\beta|$ is minimal in $\{|\alpha| \mid f_\alpha \neq 0\}$. If $|\beta| = 0$, then $f(p) = f_\beta \neq 0$. If $|\beta| > 0$, then

$$\left(\frac{\partial}{\partial t_i}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial t_i}\right)^{\beta_s} f_\beta$$

is a nonzero multiple of 1. Therefore, the Taylor series at p of

$$\tau_1^{\beta_1} \circ \cdots \circ \tau_s^{\beta_s}(f),$$

has a nonzero constant term. Therefore, $\tau_1^{\beta_1} \circ \cdots \circ \tau_s^{\beta_s}(f)(p) \neq 0$. $\square$

Corollary 2.3.10. *For all $f \in A$, $f \notin \Bbbk$, there exists $\eta \in \mathcal{V}$ with $\eta(f) \neq 0$.*

Proof. By Lemma 2.3.9, there exist $\mu_1,\dots,\mu_l \in \mathcal{V}$ for some $l \geq 1$ such that $(\mu_1 \circ \mu_2 \circ \cdots \circ \mu_l)(f)(p) \neq 0$. In particular, $\mu_l(f) \neq 0$. $\square$

The main result of this Section is the following theorem.

Theorem 2.3.11. *Let M be a finite $A\mathcal{V}$-module, and $f \in A$. Then, there exists N_f, that depends on f, such that $\Omega_p(f,\eta) \in \mathrm{Ann}(M)$ for each $p > N_f$, and $\eta \in \mathcal{V}$.*

Proof. If $f \in \Bbbk 1$, then $\Omega_p(f,\eta) = 0$ for all $p \geq 1$. Suppose $f \notin \Bbbk$. By Corollary 2.3.10, there exists $\mu \in \mathcal{V}$ such that $\mu(f) \neq 0$. By Proposition 2.3.4, there exists $\eta \in \mathcal{V}$ with $\eta(f) \neq 0$ and $N > 0$ such that $\Omega_p(f,\eta) \in \mathrm{Ann}(M)$ for every $p > N$. Since $\eta(f) \neq 0$, there exists

$g \in \{f, f^2\}$ such that $\eta(f)\eta(\eta(g)) \neq 0$. By Proposition 2.3.6, there exists a nonzero ideal I of A such that $\Omega_p(f, q\tau) \in \mathrm{Ann}(M)$ for every $q \in I$, $\tau \in \mathcal{V}$ and $p > 3N + 4$. Since A is Noetherian and

$$I \subset I^{(1)} \subset I^{(2)} \subset \cdots$$

is an ascending chain of ideals of A, we have that $I^{(k)} = I^{(l)}$ for every $l \geq k$ for some $k \geq 1$. Let $p \in X$. If there exists $g \in I$ such that $g(p) \neq 0$, then we are done. Otherwise, by Lemma 2.3.9 there exists $g \in I^{(l)}$ for some l such that $g(p) \neq 0$. Since $I^{(l)} \subset I^{(k)}$ or $I^{(l)} = I^{(k)}$, we have that $g \in I^{(k)}$. Therefore, for every $p \in X$, there exists $g \in I^{(k)}$ such that $g(p) \neq 0$. By Hilbert's Nullstellensatz, $I^{(k)} = A$. By Lemma 2.3.8, for every $g \in I^{(k)} = A$ and $p > 3N + 4 + k$, $\Omega_p(f, g\tau) \in \mathrm{Ann}(M)$ for every $\tau \in \mathcal{V}$. In particular, $\Omega_p(f, \tau) \in \mathrm{Ann}(M)$ for each $p > N_f$ where $N_f = 3N + 4 + k$. □

2.4 Localizing $A\mathcal{V}$-modules

Let M be a finite $A\mathcal{V}$-module, and $f \in A$, $f \neq 0$. Then

$$M_f = A_f \otimes_A M$$

is an A_f-module. Since the Lie algebra $\mathcal{V}$ is an A-module, we may consider the localization $\mathcal{V}_f = \mathrm{Der}(A)_f \cong \mathrm{Der}(A_f)$ as well. The open set $D(f) = \{p \in X \mid f(p) \neq 0\} \subset X$ is an irreducible smooth affine variety, and

$$A_{D(f)} = A_f = \left\{ \frac{g}{f^k} \mid g \in A,\ k \geq 0 \right\}.$$

Therefore, we may consider the question whether M_f is an $A_f\mathcal{V}_f$-module or not.

We wish to define an action of $\mathcal{V}_f$ in such a way that M_f is a module over $A_f\#U(\mathcal{V}_f)$. If $\eta \in \mathcal{V} \subset \mathcal{V}_f$, then its action on M_f must be defined as

$$(1\#\eta)\left(\frac{m}{f^k}\right) = -k\frac{\eta(f)m}{f^{k+1}} + \frac{1}{f^k}(\eta m)$$

for each $m \in M$.

Denote $\Omega_0(f, \eta) = 1\#\eta$. By Theorem 2.3.11, there exists N_f such that $\Omega_p(f, \eta) \in \mathrm{Ann}(M)$ for every $p > N_f$. Hence, the sum

$$\sum_{p=0}^{\infty} \frac{1}{f^{p+1}} \Omega_p(f, \eta)m$$

is finite for all $\eta \in \mathcal{V}$, and $m \in M_f$. Inspired by [BI23], we will show that

$$\left(1\#\frac{\eta}{f}\right)m = \sum_{p=0}^{\infty} \frac{1}{f^{p+1}} \Omega_p(f, \eta)m = \sum_{p=0}^{N_f} \frac{1}{f^{p+1}} \Omega_p(f, \eta)m \tag{2.5}$$

is well-defined for all $\eta \in \mathcal{V}$, $m \in M_f$. Assuming it is well-defined, we may use this formula to define the action of $1\#\frac{\eta}{f^k}$ for every $k \geq 0$, and $\eta \in \mathcal{V}$ recursively.

Lemma 2.4.1. *The action of $\mathcal{V}_f$ given by 2.5 is well-defined.*

Proof. To show the action given by (2.5) is well-defined, we first must prove the action of $\frac{f\eta}{f^2}$ and $\frac{\eta}{f}$ coincides for each $\eta \in \mathcal{V}$. By Lemma 2.2.7,

$$\begin{aligned}
&\sum_{p=0}^{\infty} \frac{1}{f^{p+1}}\Omega_p(f,\eta) = \sum_{p=0}^{\infty} \frac{p+1}{f^{p+1}}\Omega_p(f,\eta) - \sum_{p=1}^{\infty} \frac{p}{f^{p+1}}\Omega_p(f,\eta) \\
&= \sum_{p=0}^{\infty} \frac{p+1}{f^{p+1}}\Omega_p(f,\eta) - \sum_{q=0}^{\infty} \frac{q+1}{f^{q+2}}\Omega_{q+1}(f,\eta) = \sum_{p=0}^{\infty} \frac{p+1}{f^{p+2}}\left(\Omega_p(f,\eta) - \Omega_{p+1}(f,\eta)\right) \\
&= \sum_{p=0}^{\infty} \frac{p+1}{f^{p+2}}\Omega_p(f,f\eta) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{1}{f^{p+q+2}}\Omega_{p+q}(f,f\eta) \\
&= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{1}{f^{p+q+2}} \sum_{k=0}^{p}\sum_{l=0}^{q}(-1)^{k+l}\binom{p}{k}\binom{q}{l} f^{p+q-k-l}\# f^{k+l+1}\eta \\
&= \sum_{p=0}^{\infty} \frac{1}{f^{p+1}} \sum_{k=0}^{p}(-1)^{k}\binom{p}{k} f^{p-k}\frac{f^{p-k}}{f^{p-k}} \sum_{q=0}^{\infty} \frac{1}{f^{q+1}} \sum_{l=0}^{q}(-1)^{l}\binom{q}{l} f^{q-l}\# f^{l+k+1}\eta \\
&= \sum_{p=0}^{\infty} \frac{1}{f^{p+1}} \sum_{k=0}^{p}(-1)^{k}\binom{p}{k} f^{p-k} \sum_{q=0}^{\infty}\sum_{l=0}^{q}(-1)^{l}\binom{q}{l} \frac{f^{p-k}}{f^{p-k+l+1}}\# f^{l+k+1}\eta \\
&= \sum_{p=0}^{\infty} \frac{1}{f^{p+1}} \sum_{k=0}^{p}(-1)^{k}\binom{p}{k} f^{p-k} \left(\frac{f^{p-k}}{f^{p+1}}\# f^{2k+1}\eta\right) \\
&= \sum_{p=0}^{\infty} \frac{1}{f^{2(p+1)}} \sum_{k=0}^{p}(-1)^{k}\binom{p}{k} f^{p-k}\# f^{2k+1}\eta = \sum_{p=0}^{\infty} \frac{1}{f^{2(p+1)}}\Omega_p(f^2,f\eta)
\end{aligned}$$

It remains to prove the action of η and $\frac{f\eta}{f}$ coincides for each $\eta \in \mathcal{V}_f$. By Lemma 2.2.7,

$$\begin{aligned}
\left(1\#\frac{f\eta}{f}\right)m &= \sum_{p=0}^{N_f} \frac{1}{f^{p+1}}\Omega_p(f,f\eta)m \\
&= \sum_{p=0}^{N_f} \frac{1}{f^{p+1}}\left(f\Omega_p(f,\eta) - \Omega_{p+1}(f,\eta)\right)m \\
&= \left(1\#\eta + \sum_{p=1}^{N_f} \frac{1}{f^{p}}\Omega_p(f,\eta) - \sum_{p=0}^{N_f-1} \frac{1}{f^{p+1}}\Omega_{p+1}(f,\eta)\right)m \\
&= (1\#\eta)\,m
\end{aligned}$$

for each $m \in M_f$. Therefore, the action (2.5) is well-defined. $\square$

In order to define a module over $A_f \# U(\mathcal{V}_f)$, we need to impose that

$$(g\#1)((h\#\mu)m) = ((g\#1)(h\#\mu))m \quad \text{and} \quad (h\#\mu)((g\#1)m) = ((h\#\mu)(g\#1))m$$

for every $\mu \in \mathcal{V}_f \cup \Bbbk$, and $g, h \in A_f$. We will use this to prove (2.5) satisfies the Leibniz rule.

Lemma 2.4.2. *For all $\eta \in \mathcal{V}_f$, $g \in A_f$, $m \in M$,*

$$\sum_{p=0}^{N_f} \left(\frac{1}{f^{p+1}}\#1\right) \Omega_p(f,\eta)(g\#1) = \frac{\eta(g)}{f}\#1 + (g\#1) \sum_{p=0}^{N_f} \left(\frac{1}{f^{p+1}}\#1\right) \Omega_p(f,\eta)$$

Proof. Let $\eta \in \mathcal{V}_f$, $g \in A_f$. By Proposition 2.2.2,

$$(g\#1)\Omega_p(f,\eta) = \Omega_p(f,\eta)(g\#1)$$

for all $p > 0$. Therefore,

$$\begin{aligned}
\sum_{p=0}^{N_f} \frac{1}{f^{p+1}} \Omega_p(f,\eta)(g\#1) &= \left(\frac{1}{f}\#1\right)(1\#\eta)(g\#1) + \sum_{p=1}^{N_f} \left(\frac{1}{f^{p+1}}\#1\right) \Omega_p(f,\eta) \\
&= \left(\frac{1}{f}\#1\right)(\eta(g)\#1 + g\#\eta) + (g\#1) \sum_{p=1}^{N_f} \left(\frac{1}{f^{p+1}}\#1\right) \Omega_p(f,\eta) \\
&= \frac{\eta(g)}{f}\#1 + (g\#1) \sum_{p=0}^{N_f} \left(\frac{1}{f^{p+1}}\#1\right) \Omega_p(f,\eta)
\end{aligned}$$

□

By the last lemma and the comment above,

$$(1\#\mu)\left((g\#1)m\right) = \left((1\#\mu)(g\#1)\right)m = (\mu(g)\#1)\,m + (g\#1)\left((1\#\mu)m\right)$$

for every $m \in M_f$, $\mu \in \mathcal{V}_f$, $g \in A_f$. That is, the action (2.5) satisfies the Leibniz rule.

It remains to prove that (2.5) defines a representation of $\mathcal{V}_f$. To prove it, we will need the following lemma.

Lemma 2.4.3. *For every $\eta \in \mathcal{V}_f$, and $m \in M_f$,*

$$\left(1\#\frac{\eta}{f^k}\right) m = \sum_{p=0}^{\infty} \binom{p+k-1}{p} \frac{1}{f^{p+k}} \Omega_p(f,\eta)m.$$

Proof. We will prove the statement by induction on $k \geq 1$. For $k = 1$, it follows by Lemma 2.4.1. Suppose by induction that

$$\left(1\#\frac{\eta}{f^{k-1}}\right) m = \sum_{p=0}^{\infty} \binom{p+k-2}{p} \frac{1}{f^{p+k-1}} \Omega_p(f,\eta)m$$

for every $m \in M$ and $k > 1$. Hence, if $k > 1$, then

$$
\begin{aligned}
&\left(1\#\frac{\eta}{f^k}\right) m \\
&= \sum_{p=0}^{N_f} \frac{1}{f^{p+1}} \Omega_p\left(f, \frac{\eta}{f^{a-1}}\right) m \\
&= \sum_{p=0}^{N_f} \frac{1}{f^{p+1}} \sum_{a=0}^{p} (-1)^a \binom{p}{a} f^{p-a} \left(1\#\frac{f^a\eta}{f^{k-1}}\right) m \\
&= \sum_{p=0}^{N_f} \frac{1}{f^{p+1}} \sum_{a=0}^{p} (-1)^a \binom{p}{a} f^{p-a} \sum_{q=0}^{N_f} \binom{q+k-2}{q} \frac{1}{f^{q+k-1}} \sum_{b=0}^{q} (-1)^b \binom{q}{b} f^{q-b} \# f^{a+b} \eta m \\
&= \sum_{p=0}^{N_f} \sum_{q=0}^{N_f} \binom{q+k-2}{q} \frac{1}{f^{p+q+k}} \sum_{a=0}^{p} \sum_{b=0}^{q} (-1)^{a+b} \binom{p}{a} \binom{q}{b} f^{p+q-a-b} \# f^{a+b} \eta m \\
&= \sum_{p=0}^{N_f} \sum_{q=0}^{N_f} \binom{q+k-2}{q} \frac{1}{f^{p+q+k}} \Omega_{p+q}(f, \eta) m = \sum_{u=0}^{N_f} \binom{u+k}{u} \frac{u+1}{f^{u+2}} \Omega_u(f, \eta) m.
\end{aligned}
$$

□

Proposition 2.4.4. *M_f is a $\mathcal{V}_f$-module with the action given by* (2.5).

Proof. Fix $m \in M_f$. Since (2.5) is well-defined for all elements of V_f, we only need to prove that

$$
\left[\frac{\eta}{f}, \frac{\mu}{f}\right] m = -\frac{\eta(f)}{f^3}\mu m + \frac{\mu(f)}{f^3}\eta m + \frac{[\eta, \mu]}{f^2} m = \left[\sum_{p=0}^{N_f} \frac{1}{f^{p+1}} \Omega_p(f, \eta), \sum_{p=0}^{N_f} \frac{1}{f^{p+1}} \Omega_p(f, \mu)\right] m.
$$

By Lemma 2.2.4 and Lemma 2.4.3,

$$
\begin{aligned}
&\left[\sum_{p=0}^{N_f} \frac{1}{f^{p+1}} \Omega_p(f, \eta), \sum_{p=0}^{N_f} \frac{1}{f^{p+1}} \Omega_p(f, \mu)\right] m \\
&= \sum_{k=0}^{N_f} \sum_{l=0}^{N_f} \frac{1}{f^{k+l+2}} \left[\Omega_k(f, \eta), \Omega_l(f, \mu)\right] m \\
&= \sum_{k=0}^{N_f} \sum_{l=0}^{N_f} \frac{1}{f^{k+l+2}} \left(k\Omega_{k+l-1}(f, \mu(f)\eta) - l\Omega_{k+l-1}(f, \eta(f)\mu) + \Omega_{k+l}(f, [\eta, \mu])\right) m \\
&= \sum_{u=0}^{2N_f} \left(\binom{u+2}{u} \frac{1}{f^{u+3}} \Omega_u(f, \mu(f)\eta) - \binom{u+2}{u} \frac{1}{f^{u+3}} \Omega_u(f, \eta(f)\mu) + \frac{u+1}{f^{u+2}} \Omega_u(f, [\eta, \mu])\right) m \\
&= \sum_{u=0}^{N_f} \left(\binom{u+2}{u} \frac{1}{f^{u+3}} \Omega_u(f, \mu(f)\eta) - \binom{u+2}{u} \frac{1}{f^{u+3}} \Omega_u(f, \eta(f)\mu) + \frac{u+1}{f^{u+2}} \Omega_u(f, [\eta, \mu])\right) m
\end{aligned}
$$

$$=\frac{\mu(f)}{f^3}\eta m - \frac{\eta(f)}{f^3}\mu m + \frac{[\eta,\mu]}{f^2}m,$$

where $u = k + l - 1$ in the above formula.

□

Since the number of generators of M_f as an A_f-module is less or equal to the number of generators of M as an A-module, we have that M_f is a finitely generated as an A_f-module.

Theorem 2.4.5. *If M is a finite $A\mathcal{V}$-module and $f \in A$, $f \neq 0$, then $M_f = A_f \otimes_A M$ is a finite $A_f\mathcal{V}_f$-module, where the action of A_f is given by left side multiplication and*

$$\left(\frac{\eta}{f^k}\right)m = \sum_{p=0}^{\infty} \frac{1}{f^{k(p+1)}}\Omega_p(f^k,\eta)m = \sum_{p=0}^{\infty}\binom{p+k-1}{p}\frac{1}{f^{p+k}}\Omega_p(f,\eta)m$$

for each $\eta \in \mathcal{V}_f$.

Proof. We have that M_f is an A_f-module. By Lemma 2.4.1 and Proposition 2.4.4, the formula above gives a well-defined representation of the Lie algebra $\mathcal{V}_f$. By Lemma 2.4.2, this action satisfies the Leibniz rule. Therefore, M_f is an $A_f\mathcal{V}_f$-module. □

Corollary 2.4.6. *Let M be a finite $A\mathcal{V}$-module and $g, h \in A$ be nonzero elements. If $\eta \in \mathcal{V}_g$ and $\mu \in \mathcal{V}_h$ are such that $\eta = \mu$ as elements of $\mathcal{V}_{gh}$, then $\eta m = \mu m$ for all $m \in M_{gh}$.*

Proof. Let $\mu, \eta \in \mathcal{V}$ such that $\frac{h^l\eta}{a} = \frac{h^k\mu}{a}$ in $\mathcal{V}_a$, where $g, h \in A$, $g, h \neq 0$, $k, l \geq 0$, and $a = g^k h^l$. This means that $\frac{\eta}{g^k} = \frac{\mu}{h^l}$ in Der (Frac(A)). Hence, $a^i(ah^l\eta - ag^k\mu) = 0$ for some $i > 0$. Since A is a domain and $\mathcal{V}$ is torsion-free, $h^l\eta = g^k\mu$. Therefore,

$$\sum_{p=0}^{\infty}\frac{1}{a^{p+1}}\Omega_p(a, h^l\eta) = \sum_{p=0}^{\infty}\frac{1}{a^{p+1}}\Omega_p(a, g^k\mu)$$

as operators of M_a. □

Remark 2.4.7. Theorem 2.4.5 and Corollary 2.4.6 imply that the coherent sheaf $\tilde{M}$ on X associated to an $A\mathcal{V}$-module M is both a sheaf of $\mathcal{O}_X$-modules and a sheaf of $\Theta_X = \tilde{\mathcal{V}}$-modules that satisfies

$$\eta(fm) = \eta(f)m + f(\eta m) \quad \text{for each } \eta \in \Gamma(U,\Theta_X),\ f \in \Gamma(U,\mathcal{O}_X), \text{ and } m \in \Gamma(U,\tilde{M})$$

for each affine open set $U \subset X$. Note that we needed the assumption that X is smooth and irreducible to prove the results of this section.

2.5 Infinitesimally equivariant sheaves

In this section, we will extend the theory of $A\mathcal{V}$-modules to scheme theory. Appendix A reviews the basics of sheaves, schemes and quasi-coherent sheaves.

Let Y be a separated scheme with structure sheaf $\mathcal{O}$. Consider the *tangent sheaf* Θ over Y given by $\Gamma(U, \Theta) = \mathrm{Der}(B)$ for each affine open set $U = \mathrm{Spec}\,(B) \subset Y$.

Definition 2.5.1. A sheaf $\mathcal{M}$ over Y of $\mathcal{O}$-modules is called *infinitesimally equivariant*, or *infeq* for short, if for each affine open set $U = \mathrm{Spec}\,(U) \subset Y$, $\Gamma(U, \mathcal{M})$ is a $\Gamma(U, \Theta)$-module that satisfies the *Leibniz rule*:

$$\eta \cdot (f \cdot m) = \eta(f) \cdot m + f \cdot (\eta \cdot m) \text{ for each } \eta \in \Gamma(U, \Theta),\ f \in B,\ m \in \Gamma(U, \mathcal{M}). \tag{2.6}$$

If $\mathcal{M}$ is a vector bundle, we call $\mathcal{M}$ an *infeq bundle*.

Example 2.5.2. Both $\mathcal{O}$ and Θ are infeq sheaves, as well as each $\mathcal{D}$-module over Y. However, infeq sheaves need not be $\mathcal{D}$-modules.

Let $\mathcal{M}$ be a vector bundle on Y and $U \subset Y$ an affine open set. If $\eta \in \Gamma(U, \Theta)$, then a linear map $\alpha : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{M})$ is called a *η-derivation* of $\Gamma(U, \mathcal{M})$ if α satisfies the following "Leibniz rule": $\alpha(fm) = f\alpha(m) + \eta(f)m$ for each $f \in \Gamma(U, \mathcal{O})$ and $m \in \Gamma(U, \mathcal{M})$. A linear map $\alpha : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{M})$ is called *derivation* of $\Gamma(U, \mathcal{M})$ if there exists $\eta \in \Gamma(U, \Theta)$ such that α is an η-derivation.

If α_1 is an η_1-derivation and α_2 is an η_2-derivation, then $\alpha_1 + \lambda\alpha_2$ is a $(\eta_1 + \lambda\eta_2)$-derivation for every $\lambda \in \Bbbk$. Thus, the set of all derivations of $\Gamma(U, \mathcal{M})$ is a vector subspace of $\Gamma(U, \mathrm{End}_{\Bbbk}(\mathcal{M}))$.

Definition 2.5.3. The *Atiyah algebra* $\mathrm{At}(\mathcal{M})$ of a vector bundle $\mathcal{M}$ on Y is the sheaf defined by

$$\Gamma\,(U, \mathrm{At}(\mathcal{M})) = \{\alpha \in \Gamma\,(U, \mathrm{End}_{\Bbbk}(\mathcal{M})) \mid \alpha \text{ is a derivation of } \Gamma(U, \mathcal{M})\}$$

for each affine open subset $U \subset Y$.

For a detailed discussion of the Atiyah algebra definition and its related constructions, we refer the reader to [BS88].

Remark. In this section, we will abuse the language commonly used in algebraic geometry. For a sheaf S on an algebraic scheme Y, we say that a statement is true for each $\alpha \in S$ if it is true for each $\alpha \in \Gamma(U, S)$ in every affine open set $U \subset Y$. For instance, Definition 2.5.1 would be written as follows: an infeq bundle is a vector bundle $\mathcal{M}$ that is a Θ-module with $\eta \cdot (f \cdot m) = \eta(f) \cdot m + f \cdot (\eta \cdot m)$ for each $\eta \in \Theta$, $f \in \mathcal{O}$, and $m \in \mathcal{M}$.

Lemma 2.5.4. *The Atiyah algebra is both an $\mathcal{O}$-submodule and a Lie subalgebra of $\mathfrak{gl}_{\Bbbk}(\mathcal{M})$.*

Proof. Let $\alpha, \beta \in \mathrm{At}(\mathcal{M})$ be η, μ-derivations, respectively. For $f \in \mathcal{O}$ and $m \in \mathcal{M}$, we define $(f\alpha)(m) = f\alpha(m)$. Then $(f\alpha)(m) = f\alpha(gm) = fg\alpha(m) + f\eta(g)(m)$ for each $g \in \mathcal{O}$. Therefore, $f\alpha$ is a $(f\eta)$-derivation, and $\mathrm{At}(\mathcal{M})$ is a $\mathcal{O}$-module.

We define the bracket $[\alpha, \beta]$ by the commutator, then

$$\begin{aligned}[\alpha,\beta](fm) =&\alpha(\beta(fm)) - \beta(\alpha(fm))\\ =&\alpha(f\beta(m) + \mu(f)m) - \beta(f\alpha(m) + \eta(f)m)\\ =&f\alpha(\beta(m)) + \eta(f)\beta(m) + \mu(f)\alpha(m) + \eta(\mu(f))m\\ &- f\beta(\alpha(m)) - \mu(f)\alpha(m) - \eta(f)\beta(m) - \mu(\eta(f))m\\ =&f[\alpha,\beta](m) + [\eta,\mu](f)m.\end{aligned}$$

Hence, $[\alpha, \beta]$ is a $[\eta, \mu]$-derivation. We conclude $\mathrm{At}(\mathcal{M})$ is a Lie subalgebra of $\mathfrak{gl}_{\Bbbk}(\mathcal{M})$. □

We are assuming that $\mathcal{M}$ is a vector bundle, hence it is torsion-free. If $\alpha \in \mathrm{At}(\mathcal{M})$ is both an η-derivation and a μ-derivation, then $\eta(f)m = \alpha(fm) - f\alpha(m) = \mu(f)m$ for every $f \in \mathcal{O}$, $m \in \mathcal{M}$. Therefore, $\eta = \mu$ since $\mathcal{M}$ is torsion free. In other words, the map $\sigma : \mathrm{At}(\mathcal{M}) \to \Theta$ that assigns an η-derivation $\alpha \in \mathrm{At}(\mathcal{M})$ to η is well-defined. This map is often called *symbol.*

Definition 2.5.5. We call a Lie algebra homomorphism $L : \Theta \to \mathrm{At}(\mathcal{M})$ a *Lie map* if it is a $\Bbbk$-linear splitting of the symbol σ, i.e., $\sigma(L(\eta)) = \eta$ for every $\eta \in \Theta$.

Proposition 2.5.6. *A vector bundle $\mathcal{M}$ is an infeq bundle if and only if there exists a Lie map $L : \Theta \to \mathrm{At}(M)$.*

Proof. Suppose $\mathcal{M}$ is an infeq bundle, then define $L(\eta)m = \eta \cdot m$ for each $\eta \in \Theta$, and $m \in \mathcal{M}$. By definition of infeq sheaf, $L(\eta)(fm) = \eta(f)m + fL(\eta)(m)$. Therefore, the image of L lies in $\mathrm{At}(\mathcal{M})$, and L is a $\Bbbk$-linear splitting of σ. Furthermore, L is the representation of Θ in $\mathcal{M}$, so it is a Lie algebra homomorphism.

On the other hand, if $\mathcal{M}$ is a vector bundle equipped with a Lie map $L : \Theta \to \mathrm{At}(\mathcal{M})$, then $L : \Theta \to \mathfrak{gl}_{\Bbbk}(\mathcal{M})$ is a representation of Θ that satisfies the Leibniz rule (2.6). Therefore, $\mathcal{M}$ is an infeq bundle. □

Remark 2.5.7. In other words, Definition 2.5.1 could be rephrased as follows: a vector bundle $\mathcal{M}$ is said to be an infeq bundle if it is equipped with a linear splitting $L : \Theta \to \mathrm{At}(\mathcal{M})$ of the symbol $\sigma : \mathrm{At}(\mathcal{M}) \to \Theta$.

Theorem 2.5.8. *Let X be an affine algebraic variety and M be a finite $A_X\mathcal{V}_X$-module. Then $\tilde{M}$ is an infeq bundle on $X \cong \mathrm{Spec}\,(A)$.*

Proof. Let $\mathcal{O}$ the structure sheaf of A on $\mathrm{Spec}\,(A)$. Consider the quasi-coherent sheaf $\tilde{M}$ given by $\Gamma(D(f), \tilde{M}) = M_f$ for each $f \in A$. By Theorem 2.1.8, M is a projective module. This is equivalent to saying that $\tilde{M}$ is a locally free $\mathcal{O}$-module. Therefore, $\tilde{M}$ is a vector bundle.

By Theorem 2.4.5 and Corollary 2.4.6, $\Gamma(D(f), \tilde{M})$ is a $\Gamma(D(f), \mathcal{O})\#U(\Gamma(D(f), \tilde{\mathcal{V}}))$-module for each $f \in A$, actions agree on intersections and restrictions behaves nicely with respect to the $\mathcal{O}$-module structure. Therefore, $\tilde{M}$ is an infeq bundle. □

On the other hand, if $\mathcal{M}$ is an infeq bundle on X, then $\Gamma(X, \mathcal{M})$ is an $A\mathcal{V}$-module by definition. This, together with the previous theorem, shows that we have an equivalence of categories.

Corollary 2.5.9. *The category of infeq bundles on a smooth irreducible affine algebraic variety $X = \operatorname{Spec}(A)$ is equivalent to the category of finite $A_X\mathcal{V}_X$-modules.*

2.6 The Lie map as a differential operator

Let $\mathcal{M}$ and $\mathcal{N}$ be two vector bundles on a scheme Y with structure sheaf $\mathcal{O}_Y$. Following Grothendieck [Gro67, Section 16.8], we define the sheaf of differential operators $\operatorname{Diff}(\mathcal{M}, \mathcal{N})$ as a subsheaf of $\operatorname{Hom}_{\Bbbk}(\mathcal{M}, \mathcal{N})$ constructed inductively as follows. Let

$$\operatorname{Diff}^0_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{N}) = \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{N})$$

be the set of homomorphisms of $\mathcal{O}$-modules. For $n \geq 0$, define

$$\operatorname{Diff}^{n+1}_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{N}) = \left\{ D \in \operatorname{Hom}_{\Bbbk}(\mathcal{M}, \mathcal{N}) \mid [D, f] \in \operatorname{Diff}^n_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{N}) \; \forall f \in \mathcal{O}_Y \right\}.$$

Then the sheaf of *differential operators* from $\mathcal{M}$ to $\mathcal{N}$ is

$$\operatorname{Diff}_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{N}) = \bigcup_{n=0}^{\infty} \operatorname{Diff}^n_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{N}).$$

A sheaf $\mathcal{F} \in \operatorname{Diff}^n_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{N})$ is called a *differential operator of order less or equal than n*. All these notions can be defined similarly for two modules M and N over a commutative algebra B.

We record the following proposition which will be used in the main theorem of this section.

Proposition 2.6.1 ([Gro67, Proposition 16.8.8])**.** *Let $\mathcal{M}$, $\mathcal{N}$ be sheaves on Y of $\mathcal{O}_Y$-modules, and $D \in \operatorname{Hom}_{\Bbbk}(\mathcal{M}, \mathcal{N})$. Then, D is a differential operator of order less or equal than n if and only if for each $a_1, \dots, a_{n+1} \in \mathcal{O}_Y$ and $t \in \mathcal{M}$,*

$$\sum_{H \subset \{1,\dots,n+1\}} (-1)^{|H|} \left(\prod_{i \in H} a_i \right) D \left(\left(\prod_{i \notin H} a_i \right) t \right) = 0$$

where $|H|$ is the cardinality of H.

For a differential operator $D \in \operatorname{Diff}^1_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{N})$ of order less or equal to 1, we define

$$\varepsilon(D)(f) = [D, f] \in \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{N}) \quad \forall f \in \mathcal{O}_Y.$$

Using the natural isomorphism $\Theta \otimes \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{N}) \cong \operatorname{Der}_{\Bbbk}\left(\mathcal{O}_Y, \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{N})\right)$ given by $\eta \otimes T \mapsto (f \mapsto \eta(f)T)$, we have a map

$$\varepsilon : \operatorname{Diff}^1_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{N}) \to \Theta \otimes \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{N}),$$

which is also called *symbol.* In the case where $\mathcal{M} = \mathcal{N}$, the pre-image $\varepsilon^{-1}(\Theta \otimes \mathrm{id})$ is exactly the set of maps $D : \mathcal{M} \to \mathcal{M}$ such that $\varepsilon(D)(f) = [D, f] = \eta(f)\mathrm{id}$ for some $\eta \in \Theta$. That is, $\mathrm{At}(\mathcal{M}) = \varepsilon^{-1}(\Theta \otimes \mathrm{id})$. In other words,

$$\mathrm{At}(\mathcal{M}) = \left\{ D \in \mathrm{Diff}^1_{\mathcal{O}_Y}(\mathcal{M}) \mid \varepsilon(D) \in \Theta \otimes \mathrm{id} \right\},$$

and the symbol $\sigma : \mathrm{At}(\mathcal{M}) \to \Theta$ we defined earlier is exactly the restriction: $\varepsilon|_{\mathrm{At}(\mathcal{M})} = \sigma$.

We wish to show that the Lie map associated with an infeq bundle is a differential operator. We will prove this in the affine setting first. Let X be a smooth irreducible variety, A its coordinate ring and $\mathcal{V}$ the Lie algebra of vector fields as before.

Theorem 2.6.2. *If M is a finite $A\mathcal{V}$-module with rank r, then $\Omega_p(f, \tau) \in \mathrm{Ann}(M)$ for every $f \in A$, $\tau \in \mathcal{V}$, and $p > 3\,r^2 + 4$.*

Proof. Let $f \in A$. If $f \in \Bbbk$, then $\Omega_1(f, \eta) = 0$ for every $\eta \in \mathcal{V}$. Assume $f \notin \Bbbk$, then there exists $\mu \in \mathcal{V}$ such that $\mu(f) \neq 0$. Set $\eta = \frac{\mu}{\mu(f)} \in \mathcal{V}_{\mu(f)}$, then $\eta(f) = 1$. By Lemma 2.2.4,

$$[\Omega_1(f, \eta), \Omega_p(f, \eta)] = (1 - p)\Omega_p(f, \eta).$$

Consider $F = \mathrm{Frac}(A)$ the field of fractions of A, and $\overline{M} = F \otimes_A M$, then we may see each $\Omega_p(f, \eta)$ as an element of the vector space $\mathrm{End}_F(\overline{M})$ of F-linear endomorphisms of $\overline{M}$. We have that $\Omega_1(f, \eta)$ acts on $\mathrm{End}_F(\overline{M})$ by commutation $\Omega_1(f, \eta) \cdot T = [\Omega_1(f, \eta), T]$ for each $T \in \mathrm{End}_F(\overline{M})$. The elements of the set $\left\{ \Omega_p(f, \eta) \mid p = 1, \dots, \mathrm{rank}(M)^2 + 1 \right\} \subset \mathrm{End}_F(\overline{M})$ are eigenvectors of $\Omega_1(f, \eta)$ with distinct eigenvalues. Therefore, there exists $p \in \{1, \dots, \mathrm{rank}(M)^2 + 1\}$ such that $\Omega_p(f, \eta) = 0$. However, by Lemma 2.2.4

$$[\Omega_a(f, \eta), \Omega_b(f, \eta)] = (a - b)\Omega_{a+b-1}(f, \eta)$$

thus $\Omega_p(f, \eta) = 0$ implies $\Omega_q(f, \eta) = 0$ for every $q > p$. In particular, for all $q > \mathrm{rank}(M)^2$ we have $\Omega_q(f, \eta) \in \mathrm{Ann}(\overline{M})$. By Proposition 2.3.6, $\eta(f)\eta(\eta(f^2)) = 2$ implies that $\Omega_q(f, \tau) \in \mathrm{Ann}(\overline{M})$ for every $\tau \in \mathrm{Der}_{\Bbbk}(F)$, $q > 3\,\mathrm{rank}(M)^2 + 4$. Since $\mathcal{V}$ injects itself in $\mathrm{Der}_{\Bbbk}(F)$ and M is a torsion-free A-module, we have that $\Omega_p(f, \tau) \in \mathrm{Ann}(M)$ for every $p > 3\,\mathrm{rank}(M)^2 + 4$, and $\tau \in \mathcal{V}$. $\square$

In the beginning of Section 2.2, we use the linear map $\delta : A \to A \otimes A$ given $\delta(f) = f \otimes 1 - 1 \otimes f$ to define $\Omega_p(f, \eta)$. For each $f_1, \dots, f_p \in A$ and $\eta \in \mathcal{V}$, define

$$\Omega((f_1, \dots, f_p), \eta) = \delta(f_1) \cdots \delta(f_p)(1\#\eta).$$

These elements are a generalization of the Ω_p defined previously. If $f_i = f_j$ for each $i = 1, \dots, p$, then $\Omega((f_1, \dots, f_p), \eta) = \Omega_p(f_1, \eta)$. Similar to $\Omega_p(f, \eta)$, $\Omega((f_1, \dots, f_p), \eta)$ also commutes with $A\#1$.

Lemma 2.6.3. *For every $f_1, \dots, f_p, g \in A$ and $\eta \in \mathcal{V}$,*

$$\Omega((f_1, \dots, f_p), \eta)(g\#1) = (g\#1)\Omega((f_1, \dots, f_p), \eta).$$

Proof. Let $P = \{1, \dots, p\}$. For each $H \subset P$, let $|H|$ be the cardinality of H. Hence,

$$\begin{aligned}\Omega((f_1, \dots, f_p), \eta) = \delta(f_1) \cdots \delta(f_p)(1\#\eta) &= \prod_{i=1}^{p}(f_i \otimes 1 - 1 \otimes f_i)(1\#\eta) \\ &= -\sum_{H \subset P}(-1)^{|H|}\prod_{i\in H} f_i \# \prod_{i \notin H} f_i \eta.\end{aligned}$$

Thus,

$$\begin{aligned}&-\Omega((f_1, \dots, f_p), \eta)(g\#1) + (g\#1)\Omega((f_1, \dots, f_p), \eta) \\ &= \left(\sum_{H\subset P}(-1)^{|H|}\prod_{i\in H} f_i\#\prod_{i\notin H} f_i\eta\right)(g\#1) - (g\#1)\left(\sum_{H\subset P}(-1)^{|H|}\prod_{i\in H} f_i\#\prod_{i\notin H} f_i\eta\right) \\ &= \sum_{H\subset P}(-1)^{|H|}\prod_{i\in P} f_i\eta(g)\#1 + \sum_{H\subset P}\left((-1)^{|H|}\prod_{i\in H} f_i g\#\prod_{i\notin H} f_i\eta - (-1)^{|H|}g\prod_{i\in H} f_i\#\prod_{i\notin H} f_i\eta\right) \\ &= \eta(g)\prod_{i=1}^{p}(f_i - f_i)\#1 = 0.\end{aligned}$$

□

We want to prove $\Omega((f_1, \dots, f_p), \eta) \in \mathrm{Ann}(M)$ for large p, but first we need to prove the following lemma.

Lemma 2.6.4. *Let V be a vector space and let $F(z_1, \dots, z_p) \in \Bbbk[z_1, \dots, z_p] \otimes V$, $F = \sum_{\alpha\in\mathbb{Z}_+^p} z^\alpha v_\alpha$. If $F(a_1, \dots, a_p) = 0$ for all $a_1, \dots, a_p \in \Bbbk$, then $v_\alpha = 0$ for all $\alpha \in \mathbb{Z}_+^p$.*

Proof. Fix $k \geq 1$. For $\alpha = (a_1, \dots, a_p), \beta = (b_1, \dots, b_p) \in \mathbb{Z}_+^p$, denote $|\alpha| := a_1 + \cdots + a_p$ and $\alpha^\beta := a_1^{b_1} \cdots a_p^{b_p}$. Consider the set

$$S = \left\{\alpha \in \mathbb{Z}_+^p \mid |\alpha| \leq k\right\} \subset \mathbb{Z}_+^p.$$

Enumerate $S = \{\alpha_1, \dots, \alpha_l\}$, then the determinant $L_{k,p}$ of the matrix $\left(\alpha_i^{\alpha_j}\right)_{i,j=1}^n$ is nonzero. The value $L_{k,p}$ is equal to $\left(\prod_{i=1}^{k} i^{\binom{k-i+p}{p}}\right)^p \neq 0$.

If k is the maximum degree of α such that $v_\alpha \neq 0$, then $F(\alpha_i) = 0$, $i = 1, \dots, l$, implies that the associated homogeneous linear system in V has variables v_{α_i}, $i = 1, \dots, l$, and its associated matrix $\left(\alpha_i^{\alpha_j}\right)_{i,j=1}^n$ has determinant $L_{k,p} \neq 0$. Hence, this system has a unique solution. Consequently, $v_{\alpha_i} = 0$ for every $i = 1, \dots, l$. □

Lemma 2.6.5. *For each $p > 3\,\mathrm{rank}(M)^2 + 4$, we have that $\Omega((f_1, \dots, f_p), \eta) \in \mathrm{Ann}(M)$.*

Proof. Since each $\Omega_p(f_i, \eta) \in \mathrm{Ann}(M)$, we have that

$$\Omega_p\left(\sum_{i=1}^{p} a_i f_i, \eta\right) = \sum_{l_1+\cdots+l_p=p} \binom{p}{l_1, \dots, l_p} a_1^{l_1} \cdots a_p^{l_p} \prod_{i=1}^{p} \delta(f_i)^{l_i}(1\#\eta) \in \mathrm{Ann}(M) \tag{2.7}$$

for all $a_1, \dots, a_p \in \Bbbk$. We may see (2.7) as the evaluation $F(a_1, \dots, a_p)$ of

$$F = \sum_{l_1+\cdots+l_p=p} \binom{p}{l_1, \dots, l_p} z_1^{l_1} \cdots z_p^{l_p} \otimes \prod_{i=1}^{p} \delta(f_i)^{l_i}(1\#\eta) \in \Bbbk[z_1, \dots, z_p] \otimes \mathrm{End}_{\Bbbk}(M)$$

By Lemma 2.6.4, $\Omega((f_{i_1}, \dots, f_{i_p}), \eta) \in \mathrm{Ann}(M)$ for every $1 \leq i_1, \dots, i_p \leq p$.

□

The previous lemma and Proposition 2.6.1 imply the main result of this section.

Theorem 2.6.6. *Let $\mathcal{M}$ be an infeq bundle on a separated scheme Y that admits a finite open cover of smooth irreducible affine algebraic varieties. Denote by $\mathcal{O}_Y$ the structure sheaf of Y and Θ_Y its tangent sheaf. Then, the associated Lie map $L : \Theta_Y \to \mathrm{At}(\mathcal{M})$ is a differential operator of order less or equal to* $3\,\mathrm{rank}(\mathcal{M})^2 + 4$.

Proof. Since being a differential operator is a local property, we may assume that Y is affine without loss of generality. Suppose that Y is a smooth irreducible affine algebraic variety with coordinate ring A and Lie algebra of derivations $\mathcal{V}$. Set $p = 3\mathrm{rank}_A(\Gamma(X, \mathcal{M}))^2 + 4$. By Lemma 2.6.5,

$$\Omega((f_1, \dots, f_p), \eta)m = \sum_{H \subset \{1,\dots,p\}} (-1)^{|H|} \left(\prod_{i \in H} f_i\right) L\left(\left(\prod_{i \in H} f_i\right)\eta\right) m = 0$$

for each $m \in \mathcal{M}$, $\eta \in \mathcal{V}$, $f_1, \dots, f_p \in A$. By Proposition 2.6.1, L is a differential operator of order less than p. □

2.7 Gauge modules

In this section, we wish to give a more explicit description of the local structure of an $A\mathcal{V}$-module utilizing the results we proved in this thesis and the structure theorems proved in [BI23]. We will prove the conjecture from [BFN19] that states that every finite $A\mathcal{V}$-module is a *gauge module*.

For $k = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$, we denote

$$x^k = x_1^{k_1} \cdots x_n^{k_n},$$

where $x_1, \dots, x_n$ are variables.

Recall that X is a smooth affine variety with dimension $\dim X = r$, $\mathcal{O}$ is its structure sheaf with $\Gamma(X, \mathcal{O}) = A$, and Θ is its tangent sheaf with $\mathcal{V} = \Gamma(X, \Theta) = \mathrm{Der}(A)$. Let

$U \subset X$ be an ètale chart with uniformizing parameters $t_1, \dots, t_r \in A$. Hence, the sections $t_1, \dots, t_s \in B = \Gamma(U, \mathcal{O})$ define partial derivatives such that

$$\mathcal{W} := \Gamma(U, \Theta) = \bigoplus_{i=1}^{r} B \frac{\partial}{\partial t_i}.$$

We denote by $\mathcal{L}_{r,+}$ the Lie subalgebra of vector fields on the affine space $\mathbb{A}^n$ that vanishes at the origin. Explicitly,

$$\mathcal{L}_{r,+} = \bigoplus_{i=1}^{r} (T_1, \dots, T_r) \frac{\partial}{\partial T_i} \subset \bigoplus_{i=1}^{r} \Bbbk[T_1, \dots, T_r] \frac{\partial}{\partial T_i}.$$

Let V be a finite-dimensional vector space and $\alpha : \mathcal{L}_{r,+} \to \mathfrak{gl}(B \otimes V)$ be a representation of $\mathcal{L}_{r,+}$ such that $\alpha(x)(b \otimes v) = b\alpha(x)(1 \otimes v)$ for every $b \in B$, $x \in \mathcal{L}_{r,+}$ and $v \in V$. Then, B-linear functions $B_i : B \otimes V \to B \otimes V$, $i = 1, \dots, r$, are called *gauge fields* if

$$[B_i, \alpha(\mathcal{L}_{r,+})] = 0$$

and the operators

$$\left[\frac{\partial}{\partial t_i} \otimes 1 + B_i, \frac{\partial}{\partial t_j} \otimes 1 + B_j \right] = 0$$

for all $i, j = 1, \dots, r$. If $\{B_1, \dots, B_r\}$ is a set of gauge fields, then the space $B \otimes V$ is a $\mathcal{W}$-module with the following action

$$\left(f \frac{\partial}{\partial t_i} \right) (g \otimes v) = f \frac{\partial g}{\partial t_i} \otimes v + f g B_i (1 \otimes v) + g \sum_{k \in \mathbb{Z}^r \setminus \{0\}} \frac{1}{k!} \frac{\partial^k f}{\partial t^k} \alpha \left(t^k \frac{\partial}{\partial t_i} \right) (1 \otimes v)$$

where $f, g \in B$, $v \in V$ and $i = 1, \dots, r$. In this case, $B \otimes V$ is a $B \# U(\mathcal{W})$-module, and it is called a *local gauge module* on U. We say that a finite $A\mathcal{V}$-module M is a *gauge module* if there exists an affine open cover $\{U_i \mid i \in Y\}$ of X such that $\Gamma(U_i, \tilde{M})$ is a local gauge module on U_i for each $i \in Y$. The definition given here is a slight variation of the one given in [BFN19] to accommodate the now-known fact that $A\mathcal{V}$-modules sheafify.

Let M be a finite $A\mathcal{V}$-module with associated representation $\rho : \mathcal{V} \to \mathfrak{gl}(M)$ of $\mathcal{V}$. By Theorem 2.4.5, M is a projective A-module, thus the coherent sheaf $\tilde{M}$ is locally free. Let us assume that both $M' = \Gamma(U, \tilde{M})$ and $\mathcal{W} = \Gamma(U, \Theta)$ are free $B = \Gamma(U, \mathcal{O})$-modules.

For each $k \in \mathbb{Z}^r$, denote by $\Omega_{k,i} \in \mathfrak{gl}_{\Bbbk}(M')$ the endomorphism of M' that gives the action of

$$\delta(t)^k \frac{\partial}{\partial t_i} := \delta(t_1)^{k_1} \cdots \delta(t_s)^{k_s} \left(1 \# \frac{\partial}{\partial t_i} \right).$$

By Theorem 2.6.6, ρ is a differential transformation with finite order. If N is the order of differential transformation ρ, then $\Omega_{k,i} = 0$ for every $k \in \mathbb{Z}_+^s$ with $|k| > N$. Note that the family $\left\{ \Omega_{k,i} \mid k \in \mathbb{Z}_+^s \setminus \{0\},\ i = 1, \dots, s \right\} \subset \mathfrak{gl}_B(M')$ because the action of B commutes with these endomorphisms by Lemma 2.6.3.

Denote by $\Delta^p = (\ker \mathrm{m})^p$ the p-th power of the kernel of the multiplication map m :

$B \otimes B \to B$. We have that $\Delta^p \otimes_B \mathcal{W} \subset B\#\mathcal{W}$ is generated by $\Omega((f_1, \dots, f_p), \eta)$, $f_1, \dots, f_p \in B$, $\eta \in \mathcal{W}$. Let $\widehat{J}$ denotes the limit

$$\widehat{J} = \varprojlim (B\#\mathcal{W}) / (\Delta^p \otimes_B \mathcal{W}).$$

and $\widehat{\mathcal{L}}_{r,+}$ denote the Lie algebra

$$\widehat{\mathcal{L}}_{r,+} = \bigoplus_{i=1}^{r} (T_1, \dots, T_r) \frac{\partial}{\partial T_i} \subset \bigoplus_{i=1}^{r} \Bbbk[[T_1, \dots, T_r]] \frac{\partial}{\partial T_i}$$

of derivations of the power series in r variables that vanishes at the origin. The vector space $B \otimes \widehat{\mathcal{L}}_{r,+}$ is a Lie algebra with Lie bracket given by $[f \otimes \eta, g \otimes \mu] = fg \otimes [f, g]$ for every $f, g \in B$ and $\eta, \mu \in \widehat{\mathcal{L}}_{r,+}$. Define the semi-direct product $\mathcal{W} \ltimes (B \otimes \widehat{\mathcal{L}}_{r,+})$ using the bracket

$$[\eta, f \otimes \mu] = \eta(f) \otimes \mu, \ \eta \in \mathcal{W}, \ f \in B, \ \mu \in \widehat{\mathcal{L}}_{r,+}.$$

The main theorem of [BI23] states there is an isomorphism between $\widehat{J}$ and $\mathcal{W} \ltimes (B \otimes \widehat{\mathcal{L}}_{r,+})$. This isomorphism is given by two maps $\varphi : \widehat{J} \to \mathcal{W} \ltimes (B \otimes \widehat{\mathcal{L}}_{r,+})$ and $\psi : \mathcal{W} \ltimes (B \otimes \widehat{\mathcal{L}}_{r,+}) \to \widehat{J}$ defined by

$$\varphi\left(g\#f\frac{\partial}{\partial t_i}\right) = gf\frac{\partial}{\partial t_i} + \sum_{k \in \mathbb{Z}^r \setminus \{0\}} \frac{1}{k!} g \frac{\partial^k f}{\partial t^k} \otimes T^k \frac{\partial}{\partial T_i},$$

$$\psi\left(g\frac{\partial}{\partial t_i}\right) = g \otimes \frac{\partial}{\partial t_i},$$

$$\psi\left(g \otimes T^k \frac{\partial}{\partial T_i}\right) = (-1)^k (g\#1) \delta(t_1)^{k_1} \cdots \delta(t_r)^{k_r} \left(1\#\frac{\partial}{\partial t_i}\right).$$

This theorem allows us to give an explicit formula for the action of $B\#\mathcal{W}$ in M' in terms of $\Omega_{k,i}$. If $L : B\#\mathcal{W} \to \mathfrak{gl}_{\Bbbk}(M')$ is the representation of $B\#\mathcal{W}$ associated to M', then we define a representation $T : \mathcal{W} \ltimes (B \otimes \widehat{\mathcal{L}}_{r,+}) \to \mathfrak{gl}_{\Bbbk}(M')$ by $T = L \circ \psi$. Explicitly,

$$T\left(g \otimes T^k \frac{\partial}{\partial T_i}\right) m = (g\#1) \left(\delta(t_1)^{k_1} \cdots \delta(t_s)^{k_s} \left(1\#\frac{\partial}{\partial t_i}\right)\right) m = (g\#1)\Omega_{k,i} m,$$

$$T\left(f\frac{\partial}{\partial t_i}\right) m = \left(f\#\frac{\partial}{\partial t_i}\right) m,$$

for every $f, g \in B$, $i \in \{1, \dots, r\}$, $m \in M'$. This is well-defined because $\Omega_{k,i} = 0$ for $|k| > N$. Hence, calculating $T \circ \varphi$, we get that

$$\left(g\#f\frac{\partial}{\partial t_i}\right) m = \left(gf\#\frac{\partial}{\partial t_i}\right) m + \sum_{k \in \mathbb{Z}^r \setminus \{0\}} \frac{1}{k!} g \frac{\partial^k f}{\partial t^k} \Omega_{k,i} m$$

for every $m \in M'$, $f, g \in B$ and $i = 1, \dots, r$.

The isomorphisms above hint that the Lie subalgebra generated by

$$\left\{\Omega\left(\left(t_{i_1},\dots,t_{i_p}\right),\frac{\partial}{\partial t_a}\right) \mid p>0,\ i_1,\dots,i_p,a\subset\{1,\dots,r\}\right\}$$

in $B\#\mathcal{V}$ is isomorphic to the Lie algebra $\mathcal{L}_{r,+}$. The following lemma confirms that.

Lemma 2.7.1. *For each $k,l\in\mathbb{Z}_+^r\setminus\{0\}$ and $a,b\in\{1,\dots,r\}$,*

$$\left[\delta(t)^k\frac{\partial}{\partial t_a},\delta(t)^l\frac{\partial}{\partial t_b}\right]=l_a\delta(t)^{k+l-\epsilon_a}\frac{\partial}{\partial t_b}-k_b\delta(t)^{k+l-\epsilon_b}\frac{\partial}{\partial t_a}$$

Proof. We have that

$$\delta(t)^k\frac{\partial}{\partial t_a}=\sum_{0\le c\le k}(-1)^{k-c}\binom{k}{c}t^{k-c}\#t^c\frac{\partial}{\partial t_a}$$

where $\binom{k}{c}$ denotes the multinomial coefficient and $(-1)^c=(-1)^{c_1+c_2+\dots+c_r}$ for each $c\in\mathbb{Z}_+^r$. Thus,

$$\begin{aligned}
&\left[\delta(t)^k\frac{\partial}{\partial t_a},\delta(t)^l\frac{\partial}{\partial t_b}\right]\\
&=\left(\delta(t)^k\frac{\partial}{\partial t_a}\delta(t)^l\frac{\partial}{\partial t_b}-\delta(t)^l\frac{\partial}{\partial t_a}\delta(t)^k\frac{\partial}{\partial t_b}\right)\\
&=\sum_{0\le c\le k}\sum_{0\le d\le l}(-1)^{k+l-c-d}\binom{k}{c}\binom{l}{d}\left(t^{k-c}\#t^k\frac{\partial}{\partial t_a}\right)\left(t^{l-d}\#t^d\frac{\partial}{\partial t_b}\right)\\
&\quad-\sum_{0\le c\le k}\sum_{0\le d\le l}(-1)^{k+l-c-d}\binom{k}{c}\binom{l}{d}\left(t^{l-d}\#t^d\frac{\partial}{\partial t_b}\right)\left(t^{k-c}\#t^k\frac{\partial}{\partial t_a}\right)\\
&=\sum_{0\le c\le k}\sum_{0\le d\le l}(-1)^{k+l-c-d}\binom{k}{c}\binom{l}{d}\left(t^k\frac{\partial}{\partial t_a}\left(t^{l-d}\right)\#t^d\frac{\partial}{\partial t_b}-t^l\frac{\partial}{\partial t_b}\left(t^{k-c}\right)\#t^c\frac{\partial}{\partial t_a}\right)\\
&\quad+\sum_{0\le c\le k}\sum_{0\le d\le l}(-1)^{k+l-c-d}\binom{k}{c}\binom{l}{d}t^{k+l-c-d}\#\left[t^c\frac{\partial}{\partial t_a},t^d\frac{\partial}{\partial t_b}\right]\\
&=\left(\sum_{0\le c\le k}(-1)^{k-c}\binom{k}{c}(t^k\#1)\right)\sum_{0\le d\le l}(-1)^{l-d}\binom{l}{d}\frac{\partial}{\partial t_a}\left(t^{l-d}\right)\#t^d\frac{\partial}{\partial t_b}\\
&\quad-\left(\sum_{0\le d\le l}(-1)^{l-d}\binom{l}{d}(t^l\#1)\right)\sum_{0\le c\le k}(-1)^{k-c}\binom{k}{c}\frac{\partial}{\partial t_b}\left(t^{k-c}\right)\#t^c\frac{\partial}{\partial t_a}\\
&\quad+\sum_{0\le c\le k}\sum_{0\le d\le l}(-1)^{k+l-c-d}\binom{k}{c}\binom{l}{d}t^{k+l-c-d}\#\left(t^c\frac{\partial}{\partial t_a}\left(t^d\right)\frac{\partial}{\partial t_b}-t^d\frac{\partial}{\partial t_b}\left(t^c\right)\frac{\partial}{\partial t_a}\right)\\
&=\sum_{0\le c\le k}\sum_{0\le d\le l}(-1)^{k+l-c-d}d_a\binom{k}{c}\binom{l}{d}t^{k+l-c-d}\#t^{c+d-\epsilon_a}\frac{\partial}{\partial t_b}\\
&\quad-\sum_{0\le c\le k}\sum_{0\le d\le l}(-1)^{k+l-c-d}c_b\binom{k}{c}\binom{l}{d}t^{k+l-c-d}\#t^{c+d-\epsilon_b}\frac{\partial}{\partial t_a}
\end{aligned}$$

$$=l_a\delta(t)^{k+l-\epsilon_a}\frac{\partial}{\partial t_b}-k_b\delta(t)^{k+l-\epsilon_b}\frac{\partial}{\partial t_a}$$

The fifth equality follows from the fact that $\sum_{0\leq c\leq k}(-1)^{k-c}\binom{k}{c}x=0$ and the last equality follows from [BIN23, Lemma 3.2 (c)], which states that

$$\sum_{0<m\leq k}\sum_{0<j\leq l}(-1)^{k+l-m-j}\binom{k}{m}\binom{l}{j}j_p x^{k+l-m-j}y^{m+j-\epsilon_p}$$
$$=l_p\sum_{0<\leq k_l-\epsilon_p}(-1)^{k+l-\epsilon_p}x^{k+l-j-\epsilon_p}y^j$$

where x,y denote multi-variables. □

Let s be the rank of M' and $v_1,\dots,v_s\in M'$ be a basis of M' as a B-module, then $M'=BV\cong B\otimes_{\Bbbk}V$ is a module over $\mathcal{L}_{r,+}$ under the isomorphism given by the last lemma, where $V=\operatorname{span}_{\Bbbk}\{v_1,\dots,v_s\}$. If $\alpha:\mathcal{L}_{r,+}\to\mathfrak{gl}(BV)$ is the associated representation, then $\alpha(x)(bv)=b\alpha(x)v$ for every $b\in B, v\in V$ and $x\in\mathcal{L}_{r,+}$ because the action of $\Omega_{k,i}$ commutes with the action of B. For each $v\in V$, we have

$$\begin{aligned}\left(g\#f\frac{\partial}{\partial t_i}\right)(hv)&=\left(g\#f\frac{\partial}{\partial t_i}\right)(hv)+g\sum_{k\in\mathbb{Z}^r\setminus\{0\}}\frac{1}{k!}\frac{\partial^k f}{\partial t^k}\Omega_{k,i}(hv)\\&=\left(gf\frac{\partial}{\partial t_i}(h)\right)v+gfh\left(\frac{\partial}{\partial t_i}v\right)+gh\sum_{k\in\mathbb{Z}^r\setminus\{0\}}\frac{1}{k!}\frac{\partial^k f}{\partial t^k}\Omega_{k,i}v.\end{aligned}$$

For each $i=1,\dots,r$, consider the map

$$\begin{aligned}V&\to M'\\v&\mapsto\left(1\#\frac{\partial}{\partial t_i}\right)v.\end{aligned}$$

Let $B_i:B\otimes V\to M'$ be the B-linear extension of this map. Then, $B_1,\dots,B_r$ are gauge fields, because they are B-linear maps that commute with the action of $\mathcal{L}_{+,r}$ given by the $\Omega_{k,j}$ and

$$\begin{aligned}&\left[\frac{\partial}{\partial t_i}\otimes 1+B_i,\frac{\partial}{\partial t_j}\otimes 1+B_j\right]\\=&\frac{\partial}{\partial t_i}\left(\frac{\partial}{\partial t_j}(h)\right)v+hB_i(B_j(1\otimes v))-\frac{\partial}{\partial t_j}\left(\frac{\partial}{\partial t_i}(h)\right)v+hB_j(B_i(1\otimes v))\\=&h\left(B_j\circ B_i-B_i\circ B_j)(1\otimes v)\right)=h\left[1\#\frac{\partial}{\partial t_i},1\#\frac{\partial}{\partial t_j}\right]v=0.\end{aligned}$$

Therefore, the action of $B\#\mathcal{W}$ can be rewritten as

$$\left(g\#f\frac{\partial}{\partial t_i}\right)(hv) = \left(gf\frac{\partial}{\partial t_i}(h)\right)v + gfhB_i(1\otimes v) + gh\sum_{k\in\mathbb{Z}^r\setminus\{0\}}\frac{1}{k!}\frac{\partial^k f}{\partial t^k}\Omega_{k,i}v.$$

Thus, M' is a local gauge module on U. This solves the conjecture in [BFN19].

Theorem 2.7.2. *Every finite $A\mathcal{V}$-module is a gauge module.*

Proof. Let M be a finite $A\mathcal{V}$-module and $p \in X$, then $p \in D(h)$ is an element of a standard chart and there exists an open neighborhood $U \subset X$ of p such that both $\Gamma(U, \tilde{M})$ is a free $\Gamma(U, \mathcal{O})$-module. Take $U_p = D(h) \cap U$, hence $\Gamma(U_p, \tilde{M})$ is a local gauge module on U_p by the arguments given above. Doing this for each $p \in X$, we have that $\{U_p \mid p \in X\}$ is an open cover of X such that $\Gamma(U_p, \tilde{M})$ is a local gauge module on U_p for each $p \in X$. Therefore, M is a gauge module. □

Remark 2.7.3. Since the localization maps $A \hookrightarrow B$, $\mathcal{V} \hookrightarrow \mathcal{W}$ and $M \hookrightarrow M'$ are injective, M is a gauge $A\mathcal{V}$-module in the sense of the definition given in [BFN19].

2.8 Summary of results

In this chapter, we developed further the theory of $A\mathcal{V}$-modules and proved the main conjecture of the paper [BFN19] that made the foundations of this theory, which states that every finite $A\mathcal{V}$-module is a gauge module.

This was only possible because we proved that the coherent sheaf associated with a finite $A\mathcal{V}$-module is a vector bundle such that local sections admit a compatible action of the tangent sheaf.

Theorem (Theorem 2.4.5, Theorem 2.1.8)**.** *Let X be a smooth irreducible affine algebraic variety, A be its coordinate ring and $\mathcal{V} = \mathrm{Der}(A)$. Denote by $\mathcal{O}$ the structure sheaf of X and by Θ its tangent sheaf. Let M be a finite $A\mathcal{V}$-module. Then, the coherent sheaf $\tilde{M}$ of $\mathcal{O}$-modules is an infinitesimally equivariant bundle. That is, for every affine open set $U \subset X$, the $\Gamma(U, \mathcal{O})$-module $\Gamma(U, \tilde{M})$ is a $\Gamma(U, \mathcal{O})\Gamma(U, \Theta)$-module. In particular, if $U = D(f)$ for $f \in A$, then*

$$\sum_{p=0}^{\infty}\sum_{l=0}^{p}(-1)^l\binom{p+k}{p}\binom{p}{l}\frac{1}{f^{k+l}}\#f^l\eta$$

is a finite sum as an operator on M_f and express the action of $\dfrac{\eta}{f^k}$ for each $\eta \in \mathcal{V}$.

This theorem allowed us to show that the representation $\rho : \mathcal{V} \to \mathfrak{gl}_{\mathbb{k}}(M)$ associated with the $A\mathcal{V}$-module is a differential operator.

Theorem (Theorem 2.6.6)**.** *Let X be a smooth irreducible affine algebraic variety, A be its coordinate ring and $\mathcal{V} = \mathrm{Der}(A)$. If M is an $A\mathcal{V}$-module with $\mathcal{V}$-representation $\rho : \mathcal{V} \to \mathfrak{gl}_{\mathbb{k}}(M)$, then ρ is a differential operator with order less or equal to $N = \mathrm{rank}(M)^2 + 4$. In* *particular, for every $p > N$,*

$$\sum_{H \subset P} (-1)^{\mathrm{card}(H)} \prod_{i \in H} f_i \rho \left(\prod_{i \notin H} f_i \right) = 0$$

where $f_1, \dots, f_p \in A$, $\eta \in \mathcal{V}$ and $P = \{1, \dots, p\}$.

The theory of $A\mathcal{V}$-modules that started algebraically has now a compatible geometric counterpart, which we call Infinitesimally equivariant sheaves. The algebraic-geometric version of these results can be summarized in the following theorem.

Theorem (Theorem 2.5.9, Theorem 2.6.6)**.** *1. Let X be a smooth irreducible affine algebraic variety, A its coordinate ring and $\mathcal{V} = \mathrm{Der}(A)$. There is an equivalence of categories between the category of finite $A\mathcal{V}$-modules and the category of infinitesimally equivariant bundles.*

2. Let $\mathcal{M}$ be an infinitesimally equivariant bundle on a separated scheme Y that admits a finite open cover of smooth irreducible affine algebraic varieties. Denote by $\mathcal{O}_Y$ the structure sheaf of Y and Θ_Y its tangent sheaf. Then, the associated Lie map $L : \Theta_Y \to \mathrm{At}(\mathcal{M})$ is a differential operator of order less or equal to $3\,\mathrm{rank}(\mathcal{M})^2 + 4$.

These results are detailed in the paper authored by Bouaziz and the present author [BR23].

When combined with the structure theorems proved in [BI23], our results give us a proof for the conjecture in [BFN19] as we saw in Section 2.7. This also gives a picture of the local structure of $A\mathcal{V}$-modules.

Theorem (Theorem 2.7.2)**.** *Let X be a smooth irreducible affine algebraic variety with dimension $r = \dim(X)$ and structure sheaf $\mathcal{O}$. If $\mathcal{M}$ is an infinitesimally equivariant sheaf on X, then for every $p \in X$ there exists an ètale chart $U \ni p$ with uniformizing parameters $t_1, \dots, t_r \in \Gamma(U, \mathcal{O})$, a finite-dimensional subvector space $V \subset \Gamma(U, \mathcal{M})$ that admits representation $\rho : \mathcal{L}_{r,+} \to \mathfrak{gl}_{\Bbbk}(V)$ that commutes with the action of $\Gamma(U, \mathcal{O})$ and gauge fields $B_1, \dots, B_r : \Gamma(U, \mathcal{O}) \otimes_{\Bbbk} V \to \Gamma(U, \mathcal{M})$ such that $M \cong \Gamma(U, \mathcal{O}) \otimes_{\Bbbk} V$ and*

$$\left(g \# f \frac{\partial}{\partial t_i} \right)(hv) = \left(g f \frac{\partial}{\partial t_i}(h) \right) v + g f h B_i(1 \otimes v) + g h \sum_{k \in \mathbb{Z}^r \setminus \{0\}} \frac{1}{k!} \frac{\partial^k f}{\partial t^k} \rho \left(T^k \frac{\partial}{\partial T_i} \right) v$$

for every $v \in V$, $f, g, h \in \Gamma(U, \mathcal{O})$, and $i = 1, \dots, r$.

Chapter 3

Supervarieties, superalgebras and Lie superalgebras

In this chapter, we extend concepts and results from previous chapters to the supersymmetric case. This entails delving into super vector spaces, superalgebras, and supervarieties.

As we saw in Theorem 1.4.3, the Lie algebra of vector fields on a smooth irreducible affine algebraic variety is simple. We will prove the same result but now for the Lie superalgebra of vector fields on a smooth irreducible affine supervariety. We will also investigate the theory of $A\mathcal{V}$-modules, however, we will assume from the start we have an infinitesimally equivariant sheaf and prove that the associated Lie map is a differential operator. We will also study $A\mathcal{V}$-modules associated with the supervariety with purely odd n-dimensional superspace. In this case, the Lie superalgebra of vector fields is the simple finite-dimensional Lie superalgebra of Cartan type $\mathrm{W}(n)$ and the coordinate ring is the Grassmann algebra $\Lambda(n)$ in n variables. After showing an isomorphism analogous to the one proven in [BIN23], we prove there is an equivalence of categories between the category of finite $\mathrm{W}(n)\Lambda(n)$-modules and finite-dimensional modules over the Lie superalgebra of vector fields that vanishes in the unique point of the associated variety. All these results are explicitly displayed in Section 3.10 with the summary of our results.

We will start the chapter with the basics of superlinear algebra. In Section 3.1, we define super vector spaces, superalgebras and Lie superalgebras. To summarize, these can be seen as generalizations of the usual structures but with minor changes made to add a $\mathbb{Z}_2$-grading. Every vector space, algebra and Lie algebra are included in this context if we consider them as having a null odd part.

In the next section, we move to supergeometry. The main objective of Section 3.2 is to define supervarieties. To do it, we go through several concepts from supergeometry, introducing superschemes, morphisms of superschemes, functor of points and other related notions. Our definition of supervarieties is a direct generalization of the definition of algebraic varieties given in the scheme theoretical setting.

The infinitesimal theory of supervarierties is explored in Section 3.3. We will go through several related concepts, e.g. the tangent sheaf, the tangent space, the sheaf of differentials and smoothness. These concepts will be tied together in Theorem 3.3.11, which is part of a more general theorem proven in [She21, Theorem B.3].

Similar to what we have done in the first preliminary chapter of this thesis, we will define the system of parameters at a point in Section 3.4. We will give a more precise description of the tangent sheaf, especially of its local behavior. In Example 3.4.6, we show explicitly how to algebraically construct a local basis of the tangent sheaf seen as a sheaf of modules of the structure sheaf of a supervariety.

Before moving to our result on the simplicity of the Lie algebra of vector fields, we will need one more notion which will be introduced in Section 3.5. We will talk about topological algebras, completions and power series. These will give us a powerful tool to study the local structure of the tangent sheaf and the structure sheaf of a supervariety.

Section 3.6 contains one of our main theorems of this chapter. We will prove that if X is a smooth integral affine supervariety of dimension greater or equal to $1|0$, then global sections of the tangent sheaf form a simple Lie superalgebra. Our proof is algebraic and it mimics the one given in the non-super case by Billig and Futorny [BF18].

Infinitesimally equivariant sheaves on supervarieties are introduced in Section 3.7 as well as their module version. We show that every infeq finite module over S sheafifies into an infinitesimally equivariant sheaf if $X = \underline{\operatorname{Spec}}(S)$ is an integral smooth affine supervariety. In Section 3.8, we will prove that the Lie map associated with infinitesimally equivariant sheaves on a smooth integral supervariety is a differential operator.

In Section 3.9, we will study infinitesimally equivariant sheaves on the affine supervariety with only odd variables, i.e. the structure scheme of the Grassmann algebra. We will prove that the smash product $\Lambda(n)\#U(\mathrm{W}(n))$ of the Grassmann superalgebra $\Lambda(n)$ and the universal enveloping algebra of its Lie algebra $\mathrm{W}(n) = \operatorname{Der}(\Lambda(n))$ of derivations is isomorphic to the the tensor product of $\underline{\operatorname{End}}_{\Bbbk}(\Lambda(n))$ and the universal enveloping algebra of the Lie subalgebra $\mathrm{W}(n)_+$ of $\mathrm{W}(n)$ formed by the vector fields that vanish at the unique point of the associated supervariety. We finish Section 3.9 by proving the equivalence of categories between finite-dimensional modules over $\Lambda(n)\#U(\mathrm{W}(n))$ and finite-dimensional modules over $\mathrm{W}(n)_+$. Thus, such modules may be simultaneously viewed as Rudakov modules or tensor modules.

3.1 Super vector spaces and superalgebras

Definition 3.1.1. A *super vector space* V is a $\mathbb{Z}_2$-graded $\Bbbk$-vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$, where elements of $V_{\bar{0}}$ are called *even* and elements of $V_{\bar{1}}$ are called *odd*. An element $v \in V$ is called *homogeneous* if it is an element of either $V_{\bar{0}}$ or $V_{\bar{1}}$. For a homogeneous element $v \in V$ we define its *parity* as

$$|v| = \begin{cases} 0 & \text{if } v \in V_{\bar{0}} \\ 1 & \text{if } v \in V_{\bar{1}} \end{cases}$$

A *subspace* W of a super vector space is a vector super space $W = W_{\bar{0}} \oplus W_{\bar{1}} \subset V$ with $W_i \subset V_i$ for $i \in \mathbb{Z}_2$. If $\dim V_{\bar{0}} = m$ and $\dim V_{\bar{1}} = n$, the *superdimension* of V is the pair (m, n), which will be denoted by $\dim V = m|n$.

Remark 3.1.2. When we write $|v|$ for an element of a super vector space, we will implicitly assume that v is a homogeneous element.

Example 3.1.3. For $m, n \geq 0$, we define the super vector space $\mathbb{k}^{m|n}$ as the vector space $\mathbb{k}^{m+n}$ with the $\mathbb{Z}_2$-grading

$$(\mathbb{k}^{m|n})_{\bar{0}} = \mathbb{k}^m \times \{0\}, \quad (\mathbb{k}^{m|n})_{\bar{1}} = \{0\} \times \mathbb{k}^n.$$

Example 3.1.4. Let V, W be two super vector spaces. Denote by $\underline{\mathrm{Hom}}(V, W)$ the set of all linear maps from V to W. This vector space is a super vector space with the grading

$$\underline{\mathrm{Hom}}(V, W)_{\bar{0}} = \{T : V \to W \mid T(V_i) \subset W_i \text{ for all } i \in \mathbb{Z}_2\};$$
$$\underline{\mathrm{Hom}}(V, W)_{\bar{1}} = \{T : V \to W \mid T(V_i) \subset W_{i+\bar{1}} \text{ for all } i \in \mathbb{Z}_2\}.$$

Definition 3.1.5. A *morphism* from a super vector space V to a super vector space W is an element of $\underline{\mathrm{Hom}}(V, W)_{\bar{0}}$, which will be denoted by $\mathrm{Hom}(V, W)$

Thus we defined the category of super vector spaces that we denote by SVect. The category SVect admits tensor product. For super vector spaces V, W we give $V \otimes W$ the $\mathbb{Z}_2$-grading

$$(V \otimes W)_{\bar{0}} = (V_{\bar{0}} \otimes W_{\bar{0}}) \oplus (V_{\bar{1}} \oplus W_{\bar{1}}),$$
$$(V \otimes W)_{\bar{1}} = (V_{\bar{0}} \otimes W_{\bar{1}}) \oplus (V_{\bar{1}} \oplus W_{\bar{0}}).$$

This tensor product is associative, i.e. $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$. The object $\mathbb{k}^{1|0}$ is the unit element with respect to tensor multiplication. Furthermore, the linear map $c_{V_1,V_2} : V_1 \otimes V_2 \to V_2 \otimes V_1$ given by $c_{V_1,V_2}(v_1 \otimes v_2) = (-1)^{|v_1||v_2|} v_2 \otimes v_1$ is an isomorphism of super vector spaces.

Definition 3.1.6. For a super vector space V define the *parity shift* of V to be $\Pi V \in \mathrm{SVect}$ defined as $(\Pi V)_{\bar{0}} = V_{\bar{1}}$ and $(\Pi V)_{\bar{1}} = V_{\bar{0}}$.

A *superalgebra* is an object $S \in \mathrm{SVect}$ with a multiplication even map $\tau : S \otimes S \to S$, usually denoted by $\tau(a \otimes b) = ab$. It said to be *commutative* (or *supercommutative*) if $\tau = \tau \circ c_{S,S}$, that is, $ab = (-1)^{|a||b|} ba$. Similarly we say that S is *associative* if $\tau \circ (\tau \otimes \mathrm{id}) = \tau \circ (\mathrm{id} \otimes \tau)$ on $S \otimes S \otimes S$. In other words $a(bc) = (ab)c$. We say that S is *unital* if there exists an even element 1 so that $\tau(1 \otimes a) = \tau(a \otimes 1)a$ for all $a \in S$, i.e., $a \cdot 1 = 1 \cdot a = a$. The tensor product $S_1 \otimes S_2$ of two superalgebras S_1 and S_2 is a superalgebra as well, with the multiplication map defined by

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd.$$

For a superalgebra S, the *soul* J_S of S is the ideal of S generated by its odd part $S_{\bar{1}}$. The *body* of the superalgebra S is the quotient S/J_S.

Example 3.1.7. Let V be a super vector space and consider $\underline{\mathrm{End}}(V) = \underline{\mathrm{Hom}}(V, V)$ the set of linear endomorphisms of V. Example 3.1.4 shows that $\underline{\mathrm{End}}(V)$ is a super vector space. The composition of endomorphisms makes $\underline{\mathrm{End}}(V)$ an associative superalgebra.

Example 3.1.8. If V is a super vector space, then we may define the free associative algebra generated by V. Define $T^0V = \Bbbk$ and $T^kV = V^{\otimes k}$ for positive integer k. The *tensor superalgebra* $T(V)$ is the super vector space

$$T(V) = \bigoplus_{k\geq 0} T^k V$$

with multiplication map given by concatenation, that is,

$$(v_1 \otimes \cdots \otimes v_r) \cdot (w_1 \otimes \cdots \otimes w_s) = v_1 \otimes \cdots \otimes v_r \otimes w_1 \otimes \cdots \otimes w_s.$$

Thus, $T(V)$ is an unital associative superalgebra, but it is commutative if and only if V is even and one-dimensional.

If $\{x_1, \dots, x_m\}$ is a basis of $V_{\bar{0}}$ and $\{\theta_1, \dots, \theta_s\}$ is basis of $V_{\bar{1}}$, we denote $T(V)$ by

$$T(V) = \Bbbk\langle x_1, \dots, x_r \mid \theta_1, \dots, \theta_s\rangle.$$

Example 3.1.9. Let $S(m|n) = \Bbbk[x_1, \dots, x_m \mid \theta_1, \dots, \theta_n]$ denote the superalgebra given as the quotient of $\Bbbk\langle x_1, \dots, x_r \mid \theta_1, \dots, \theta_s\rangle$ by the ideal generated by

$$x_ix_j - x_jx_i, \quad \theta_k\theta_l + \theta_k\theta_l, \quad x_i\theta_k - \theta_kx_i$$

with $i, j = 1, \dots, m$, and $k, l = 1, \dots, n$. The superalgebra $S(m|n)$ is commutative and its $\mathbb{Z}_2$-graded structure is

$$\begin{aligned}
S(m|n)_{\bar{0}} &= \operatorname{span}_{\Bbbk} \left\{ x_1^{r_1} \cdots x_m^{r_m} \theta_{i_1} \cdots \theta_{i_{2k}} \mid r_1, \dots, r_m \geq 0,\ i_1 < i_2 < \cdots < i_{2k} \right\}, \\
S(m|n)_{\bar{1}} &= \operatorname{span}_{\Bbbk} \left\{ x_1^{r_1} \cdots x_m^{r_n} \theta_{i_1} \cdots \theta_{i_{2k+1}} \mid r_1, \dots, r_m \geq 0,\ i_1 < i_2 < \cdots < i_{2k+1} \right\}.
\end{aligned}$$

Note that $S(m|n)$ is isomorphic as a vector space to the tensor product of $\Bbbk[x_1, \dots, x_m]$ with the exterior algebra $\Lambda(\theta_1, \dots, \theta_n)$. Furthermore, the soul $J_{S(m|n)}$ of $S(m|n)$ is the ideal generated by $\theta_1, \dots, \theta_n$, and its body $S(m|n)/J_{S(m|n)}$ is isomorphic to $\Bbbk[x_1, \dots, x_m]$.

A *Lie superalgebra* is an object $\mathfrak{g} \in \mathrm{SVect}$ with linear even map $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, the Lie (super)bracket, such that $\tau = -\tau \circ c_{\mathfrak{g},\mathfrak{g}}$ and

$$\tau \circ (\mathrm{id} \otimes \tau) = \tau \circ (\tau \otimes \mathrm{id}) + \tau \circ (\mathrm{id} \otimes \tau) \circ (c_{\mathfrak{g},\mathfrak{g}} \otimes \mathrm{id}).$$

These two properties mean that if $\tau(x \otimes y) = [x, y]$ for each $x, y \in \mathfrak{g}$, then $[x, y] = -(-1)^{|x||y|}[y, x]$ and

$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$$

for every $x, y, z \in \mathfrak{g}$. A map $\alpha : \mathfrak{g}_1 \to \mathfrak{g}_2$ between two Lie superalgebras is a morphism of Lie superalgebras if $\alpha([x, y]) = [\alpha(x), \alpha(y)]$. Common notions, such as subalgebras and ideals, are defined in this setting accordingly. A Lie superalgebra $\mathfrak{g}$ is called *simple* if $[\mathfrak{g}, \mathfrak{g}] \neq 0$ and it has exactly two ideals, 0 and $\mathfrak{g}$.

Example 3.1.10. Let S be an associative superalgebra, then S is a Lie superalgebra with the bracket given by the supercommutator

$$[a, b] = ab - (-1)^{|a||b|} ba.$$

If V is a super vector space and $S = \underline{\mathrm{End}}(V)$, we denote by $\mathfrak{gl}(V)$ the associated Lie superalgebra.

Example 3.1.11. Let S be a an associative superalgebra. A *(super)derivation* of S is an element $D \in \mathfrak{gl}(S)$ such that $D(ab) = D(a)b + (-1)^{|D||a|} aD(b)$ for each $a, b \in D$. The set $\mathrm{Der}(S)$ of all derivations of S is a Lie subalgebra of $\mathfrak{gl}(S)$. If S is commutative, then $\mathrm{Der}(S)$ is also an S-module, and these structures are compatible following the rule

$$[f\eta, g\mu] = f\eta(g)\mu - (-1)^{(|f|+|\eta|)(|g|+|\mu|)} g\mu(f)\eta + (-1)^{|\eta||g|} fg[\eta, \mu] \tag{3.1}$$

where $f, g \in S$, and $\eta, \mu \in \mathrm{Der}(S)$.

3.2 Supervarieties

In this section, we assume that all superalgebras are associative, commutative and unital. We denote their category by **SAlg**. For an introduction to supergeometry, we refer to the book [CCF11]. For the basics of algebraic geometry, we refer to the Appendix A.

For $S \in$ **SAlg**, recall that $J_S = (S_{\bar{1}})$ is the ideal generated by the odd elements. Write S^r for the quotient S/J_S. We say that S is *reduced* or *super reduced* if S^r is commutative algebra with no nilpotents elements. Note that S may still contain some (even) nilpotents.

A *superspace* $X = (|X|, \mathcal{O}_X)$ is a topological space $|X|$ together with a sheaf of superalgebras $\mathcal{O}_X : |X| \to$ **SAlg** such that the stalk $\mathcal{O}_{X,x}$ is a local superalgebra for every point x in $|X|$, i.e., $\mathcal{O}_{X,x}$ has a unique maximal homogeneous ideal. We define the sheaves $\mathcal{O}_{X,\bar{0}}$ and $\mathcal{O}_{X,\bar{1}}$ by $\Gamma(U, \mathcal{O}_{X,\bar{0}}) = \Gamma(U, \mathcal{O}_X)_{\bar{0}}$ and $\Gamma(U, \mathcal{O}_{X,\bar{1}}) = \Gamma(U, \mathcal{O}_X)_{\bar{1}}$ for each open set $U \subset |X|$, respectively. Note that $\mathcal{O}_{X,\bar{0}} : |X| \to$ **SAlg** is a sheaf of commutative algebras. Furthermore, both $\mathcal{O}_X$ and $\mathcal{O}_{X,\bar{1}}$ are sheaves of $\mathcal{O}_{X,\bar{0}}$-modules.

A *superscheme* X is a superspace $(|X|, \mathcal{O}_X)$ such that $\mathcal{O}_{X,\bar{1}}$ is a quasi-coherent sheaf of $\mathcal{O}_{X,0}$-modules. $\mathcal{O}_X$ is called structure sheaf of X. Given a superscheme $X = (|X|, \mathcal{O}_X)$, let $\mathcal{O}_X^r$ denote the sheaf of algebras

$$\mathcal{O}_X^r(U) = (\mathcal{O}_X/J_X)(U)$$

where J_X is the ideal sheaf $U \mapsto J_{\mathcal{O}_X(U)}$. We will call X^r the *reduced space* associated to the superspace $X = (|X|, \mathcal{O}_X)$.

Example 3.2.1 (Affine Superscheme)**.** Let S be a superalgebra, then consider the affine scheme $\mathrm{Spec}\,(S_{\bar{0}})$ with structure sheaf $\mathcal{O}_{S_{\bar{0}}}$. There exists a correspondence between prime ideals of S and prime ideals of S^r that contains J_S. Since every element of J_S is nilpotent, $\mathrm{Spec}\,(S^r)$ and $\mathrm{Spec}\,(S_{\bar{0}})$ are essentially the same and homeomorphic as topological spaces.

The stalk of $\mathcal{O}_{S_{\bar{0}}}$ at a prime $\mathfrak{p}$ is given by localization $\mathcal{O}_{S_{\bar{0}},\mathfrak{p}} = \left(S_{\bar{0}}\right)_{\mathfrak{p}}$. The super vector space S is a module over $S_{\bar{0}}$, thus there exists a quasi-coherent sheaf $\widetilde{S}$ on $\operatorname{Spec}(S_{\bar{0}})$ associated with it. Explicitly, the stalk of $\widetilde{S}$ at prime ideal $\mathfrak{p}$ of $S_{\bar{0}}$ is the localization of the $\left(S_{\bar{0}}\right)_{\mathfrak{p}}$-module at $\mathfrak{p}$

$$\widetilde{S}_{\mathfrak{p}} = \left\{ \frac{f}{g} \mid f \in S,\ g \in S_{\bar{0}} \setminus \mathfrak{p} \right\}.$$

This is a sheaf of superalgebras and we will denote it by $\underline{\operatorname{Spec}}(S)$. In particular,

$$\Gamma\left(D(f), \underline{\operatorname{Spec}}(S)\right) = \left\{ \frac{g}{f^n} \mid g \in S,\ n \geq 0 \right\}.$$

if $D(f) = \{\mathfrak{p} \in \operatorname{Spec}(S_{\bar{0}}) \mid f \notin \mathfrak{p}\}$ is a basic open set of $\operatorname{Spec}(S_{\bar{0}})$ with $f \in S_{\bar{0}}$.

A superscheme X that is isomorphic to $\underline{\operatorname{Spec}}(S)$ for some commutative superalgebra S is called *affine superscheme.*

Example 3.2.2 (Affine superspace $\mathbb{A}^{m|n}$)**.** Consider the polynomial superalgebra

$$S(m|n) = \Bbbk[x_1, \dots, x_m \mid \theta_1, \dots, \theta_n]$$

over an algebraically closed field $\Bbbk$, where $x_1, \dots, x_m$ are even variables and $\theta_1, \dots, \theta_n$ are odd variables. We define $\mathbb{A}^{m|n} = \underline{\operatorname{Spec}}\ (S(m|n))$ to be the *affine superspace* of superdimension $m|n$ and denote it by $\mathbb{A}^{m|n}$.

The topological space underlying $\mathbb{A}^{m|n}$ is $\operatorname{Spec}(S(m|n)_{\bar{0}})$, which is essentially the same as $\operatorname{Spec}(\Bbbk[x_1, \dots, x_m])$. It consists of the even maximal ideals

$$(x_i - a_i,\ \theta_j\theta_k \mid i = 1, \dots, m,\ j, k = 1, \dots, n)$$

and the even prime ideals

$$(p_1, \dots, p_r,\ \theta_j\theta_k \mid j, k = 1, \dots, n)$$

where $(p_1, \dots, p_r)$ is a prime ideal in $\Bbbk[x_1, \dots, x_m]$.

Example 3.2.3 (Superscheme on the sphere S^2)**.** Consider the polynomial superalgebra $\Bbbk[x_1, x_2, x_3 \mid \theta_1, \theta_2, \theta_3, \xi_1, \xi_2, \xi_3]$ generated over an algebraically closed field $\Bbbk$ and the ideal

$$I = \left(x_1^2 + x_2^2 + x_3^2 - \xi_1\theta_1 - \xi_2\theta_2 - \xi_3\theta_3 - 1, x_1\theta_1 + x_2\theta_2 + x_3\theta_3\right).$$

Let $\Bbbk[X] = \Bbbk[x_1, x_2, x_3, \theta_1, \theta_2, \theta_3]/I$ and $X = \underline{\operatorname{Spec}}\,\Bbbk[X]$. Then X is a supervariety whose reduced variety X^r is the sphere S^2.

Example 3.2.4. Let X_0 be a scheme and $\mathcal{N}$ a coherent sheaf on X_0. Then $\Lambda^{\bullet}\mathcal{N}$ is a superscheme. Explicitly, if $U = \underline{\operatorname{Spec}}(B) \subset X_0$ is an affine open subset, then the exterior algebra of the B-module $N = \Gamma(U, \mathcal{N})$

$$\Gamma(U, \Lambda^{\bullet}\mathcal{N}) = \Lambda_B^{\bullet}N = B \oplus N \oplus \bigwedge_B^2 N \oplus \bigwedge_B^3 N \oplus \cdots$$

is a commutative superalgebra. The product of $x \in \bigwedge_B^k N$ with $y \in \bigwedge_B^l N$ is $x \wedge y \in \bigwedge_B^{k+l} N$.

Definition 3.2.5. A superscheme X is called *graded* if there exists a coherent sheaf $\mathcal{M}$ on X^r such that $X \cong (|X|, \Lambda^\bullet \mathcal{M})$. This isomorphism is called *grading*. If there exists a cover of X consisting of graded open subschemes, then we say that X is *locally graded.*

Remark 3.2.6. It was proved in [Kos94] that any connection on a supermanifold defines a grading on it, thus a graded superscheme may have many different gradings.

A superalgebra S is a superdomain if $S \setminus J_S$ does not have zero divisors. We say that a superscheme X is integral if the topological space $|X|$ is connected and $X = \cup X_i$ with $X_i = \underline{\operatorname{Spec}}(S_i)$ such that all S_i are superdomains. A *generic point* is a point $x \in |X|$ such that the closure of $\{x\}$ is X. If X is irreducible, then it admits an unique generic point. If X admits an unique generic point x, we denote by $\Bbbk(X) = \mathcal{O}_{X,x}$ the stalk of $\mathcal{O}_X$ at x. For any open affine subscheme $\underline{\operatorname{Spec}}(S) \subset X$, there exists a map $\Gamma(U, \mathcal{O}_X) \to \Bbbk(X)$. If $X = \underline{\operatorname{Spec}}(S)$ is an integral superscheme, then the soul J_S of S is the generic point of X.

A *morphism of superschemes* $f : X \to Y$ between the superschemes X and Y is a pair of maps $f = (|f|, f^\#)$ such that $|f| : |X| \to |Y|$ is a continuous map, $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_Y$ is a sheaf morphism and the map $f_p^\# : \mathcal{O}_{Y,|f|(p)} \to \mathcal{O}_{X,p}$ is a morphism of superalgebras such that the image of the maximal ideal of $\mathcal{O}_{Y,|f|(p)}$ is contained in the maximal ideal of $\mathcal{O}_{X,p}$. The sheaf $f_* \mathcal{O}_Y$ is the sheaf on $|Y|$ defined by $(f_* \mathcal{O}_X)(U) = \mathcal{O}_Y\left(|f|^{-1}(U)\right)$. Thus, the category **SSch** of superschemes is defined.

Definition 3.2.7. Let X be a superscheme and S is a commutative superalgebra. The set

$$h_X(S) := h_X(\underline{\operatorname{Spec}}(S)) = \operatorname{Hom}_{\mathbf{SSch}}(\operatorname{Spec}(S), X) \cong \operatorname{Hom}_{\mathbf{SAlg}}(\Gamma(X, \mathcal{O}_X), S)$$

is called the set of *S-points* of X.

There is a relation between points of the topological space $|X|$ and $\Bbbk$-points of X. Take a point $x \in |X|$ such that $\mathcal{O}_{X,x}/\mathsf{m}_{X,x} \cong \Bbbk$ where $\mathsf{m}_{X,x}$ is the maximal ideal of $\mathcal{O}_{X,x}$. Then, we define a morphism $f : \operatorname{Spec}(\Bbbk) \to X$ with $f^\# : \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/\mathsf{m}_{X,x}\Bbbk$ given by the canonical projection and $|f| : \operatorname{Spec}(\Bbbk) \to |X|$ that sends the unique point of $\operatorname{Spec}(\Bbbk)$ to x. Hence, $f = (|f|, f^*) \in h_X(\Bbbk)$. On the other hand, if $f = (|f|, f^\#) \in h_X(\Bbbk)$, then $f^\# : \mathcal{O}_{X,|f|(0)} \to \Bbbk$ defines an isomorphism $\mathcal{O}_{X,|f|(0)}/\mathsf{m}_{X,|f|(0)} \cong \Bbbk$ where $\mathsf{m}_{X,|f|(0)}$ is the maximal ideal of the local algebra $\mathcal{O}_{X,|f|(0)}$. Therefore, there is a correspondence between $\Bbbk$-points of X and points $x \in |X|$ with $\mathcal{O}_{X,x}/\mathsf{m}_{X,x} \cong \Bbbk$. This correspondence will be used freely in this text.

Definition 3.2.8. A $\Bbbk$-point x is *closed* if the corresponding prime ideal in $\operatorname{Spec}(S_{\bar{0}})$ is maximal for each affine open neighbourhood $\underline{\operatorname{Spec}}(S) \subset X$ of x.

Conversely, a $\Bbbk$-point $x \in h_X(\Bbbk)$ is closed if the corresponding point $x \in |X|$ is closed. If x is closed and $\underline{\operatorname{Spec}}(S)$ is an affine open neighborhood of x, then the quotient of S by the corresponding maximal ideal of x has finite dimension. Thus, it is a finite algebraic extension of $\Bbbk$, hence isomorphic to $\Bbbk$. In particular, $\mathcal{O}_{X,x}/\mathsf{m}_{X,x} \cong \Bbbk$. Therefore, every closed point corresponds to a $\Bbbk$-point.

Let X, Y, Z be superschemes. If $f : X \to Z$ and $g : Y \to Z$ are morphism of superschemes, then the *fibre product* is a superscheme $X \times_Z Y$ together with morphisms

$p_1 : X \times_Z Y \to X$ and $p_1 : X \times_Z Y \to Y$ such that the diagram

$$\begin{tikzcd}
X \times_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & Z
\end{tikzcd}$$

commutes and it is universal with that property. If $X = \underline{\mathrm{Spec}}\,(A)$, $Y = \underline{\mathrm{Spec}}\,(B)$ and $Z = \underline{\mathrm{Spec}}\,(S)$ are affine superschemes, then

$$X \times_Z Y = \underline{\mathrm{Spec}}\,(A \otimes_S B).$$

Consider the superscheme $X \times_Z X$. The diagonal map $\Delta : X \to X \times_Z X$ will play a role in a few definitions. This is a morphism of superschemes such that $\Delta \circ p_1 = \Delta \circ p_2 = 1_X$, where p_1 and p_2 are the projections maps that come with $X \times_Z X$.

There exists a unique map $X \to \mathrm{Spec}\,(\Bbbk)$. If $\underline{\mathrm{Spec}}\,(S) \subset X$ is an affine open subscheme, then this map comes from the unique morphism of (unital) commutative superalgebras $\Bbbk \to S$. We denote $X \times_{\mathrm{Spec}\,(\Bbbk)} X$ by simply $X \times X$. The superscheme X is called *separated* if $\Delta : X \to X \times X$ is a *closed embedding*, that is, $|\Delta| : |X| \to |X \times X|$ is a homeomorphism onto its image, the pullback morphism $\mathcal{O}_{X\times X} \to \Delta_* \mathcal{O}_X$ is surjective and its kernel is locally generated by its sections as a module over $\mathcal{O}_{X\times X}$. Similar to what happens in the usual algebraic geometry, any affine superscheme is separated. Furthermore, if $U, V \subset X$ are affine subschemes of a separated superscheme X, then $U \cap V$ is affine as well, see [Sha94b, Section V.4.3, Proposition 3]. This is the superscheme theoretical analog of a Hausdorff space in topology. For more on separated morphisms and schemes, see [Har77, Section II.4] or [Sha94b, Section V.4.3]. Schemes were originally required to be separated by Grothendick [Gro60] and the special class of superschemes studied by us will be separated as well.

We wish to consider a type of scheme that mimics the properties of algebraic varieties.

Definition 3.2.9. A *supervariety* X is an irreducible separated superscheme such that

1. X admits a finite open cover of affine superschemes of the form $\mathrm{Spec}\,(S)$, where S is a finitely generated superalgebra,
2. $\Gamma(U, \mathcal{O}_X) \to \Bbbk(X)$ is injective for any open subscheme $U \subset X$,
3. $\Bbbk(X)$ is an integral super domain.

If there exists a superalgebra S such that $X = \underline{\mathrm{Spec}}\,(S)$, then we say that X is an *affine supervariety.*

Remark 3.2.10.
1. An integral irreducible separated scheme of finite type over $\mathrm{Spec}\,(\Bbbk)$ is an algebraic variety, thus a supervariety X with structure sheaf $\mathcal{O}_X$ is an algebraic variety if $\mathcal{O}_{X,\bar{1}}$ is trivial.
2. Suppose that X_0 is a quasi-projective variety and $\mathcal{M}$ is a coherent sheaf of $\mathcal{O}_{X_0}$-modules, then $X = \Lambda^\bullet \mathcal{M}$ is a supervariety.

3. Our definition of a supervariety is the definition given in [She21]. By [She21, Remark 2.2], there exist supervarieties that are not integral. Furthermore, if X is a supervariety, then X^r is not necessarily a variety. However, if $X = \operatorname{Spec}(S)$ is an affine supervariety, then zero divisors of $S_{\overline{0}}$ are nilpotent elements.
4. If each S that appears in the open cover of the supervariety X is a super domain, then X is integral.

3.3 Infinitesimal theory

In this section, we study the infinitesimal theory of supervarieties. The main goal of this section is to introduce smooth supervarieties and give basic local properties of them.

Definition 3.3.1. Let X be a supervariety with dimension $r|s$ and $p \in X$ a closed point. We say that X is *smooth at* p if there exists an affine open neighbourhood $p \in U = \underline{\operatorname{Spec}}(S)$ such that X^r is smooth at p and $S \cong \Lambda^\bullet_{\overline{S}} \overline{S}^s$ where $\overline{S} = S/J_S$. We say that X is *smooth* if it is smooth at every closed point.

Remark 3.3.2. Every smooth supervariety is locally graded. By [She21, Remark 2.8], locally graded affine supervarieties are graded. Thus, if X is a smooth supervariety, there exists an open cover $\{U_i \mid i = 1, \dots, k\}$ of X such that $U_i = \underline{\operatorname{Spec}}(S_i)$ with $S_i \cong \Lambda^\bullet_{\overline{S_i}} M_i$, where M_i is a free module over the body algebra $\overline{S_i}$ and $\operatorname{Spec}(\overline{S_i})$ is a smooth variety. Furthermore, if $X = \underline{\operatorname{Spec}}(S)$ is a smooth affine supervariety, then $S \cong \Lambda^\bullet_{\overline{S}}(M)$ for some projective module over the body $\overline{S}$.

If $X = \underline{\operatorname{Spec}}(S)$ is affine, then the S-module $\mathcal{V} = \operatorname{Der}(S)$ defines a quasi-coherent sheaf $\widetilde{\mathcal{V}}$ on X. If S is finitely generated, this sheaf is coherent and

$$\Gamma(D(f), \widetilde{\mathcal{V}}) = \operatorname{Der}(S_f) \quad \text{for each } f \in S \setminus J_S.$$

This extends naturally to any supervariety.

Definition 3.3.3. Let X be a supervariety. The *tangent sheaf* Θ_X is the sheaf whose sections on an affine super subspace $U \subset X$ satisfies $\Gamma(U, \Theta_X) = \operatorname{Der}(\Gamma(U, \mathcal{O}_X))$.

The sheaf Θ_X is a coherent sheaf on X as we noted before. Moreover, $\mathcal{O}_X$ acts on Θ_X by left multiplication and Θ_X acts on $\mathcal{O}_X$ by derivations. If X is smooth, we know more about Θ_X. For instance, we will see that Θ_X is a locally free $\mathcal{O}_X$-module, and we will show how to construct an $\mathcal{O}_{X,x}$-basis for $\Theta_{X,x}$, and prove that $\Gamma(X, \Theta_X)$ is a simple Lie superalgebra if X is affine. These are super analogues of the results in [BF18].

Example 3.3.4. Let $X = \mathbb{A}^{m|n}$, then $X = \underline{\operatorname{Spec}}(S(m|n))$ where $S(m|n) = \Bbbk[x_1, \dots, x_m \mid \theta_1, \dots, \theta_n]$. The Lie algebra $W(m|n) = \operatorname{Der}(S(m|n))$ is a free $S(m|n)$-module generated by the partial derivations

$$\frac{\partial x_r}{\partial x_i} = \delta_{ir}, \quad \frac{\partial \theta_s}{\partial x_i} = 0$$

$$\frac{\partial x_r}{\partial \theta_j} = 0, \quad \frac{\partial \theta_s}{\partial \theta_j} = \delta_{sj}$$

for $r = 1, \ldots, m$ and $s = 1, \ldots, n$. Thus, $\Theta_X = \widetilde{W(m|n)}$ is a vector bundle over X.

Let $p \in X$ be a closed point of the supervariety X. The projection $\pi : \mathcal{O}_{X,x} \to \mathcal{O}_{X,p}/\mathsf{m}_{X,p} \cong \Bbbk$ is a map on $\Bbbk$. We define the *value* $f(p)$ *of* f *at* x to be $f(p) = \pi(p) \in \Bbbk$. Because $\pi(f - f(p)) = 0$, we have that $f - f(p) \in \mathsf{m}_{X,p} \subset \mathcal{O}_{X,p}$.

Definition 3.3.5. Let p be a $\Bbbk$-point of a supervariety X, then the *tangent space of* X *at* p is the super vector space

$$T_p X = \mathrm{Der}(\mathcal{O}_{X,p}, \Bbbk),$$

that is, $T_p X$ is the set of point derivations $\delta : \mathcal{O}_{X,p} \to \Bbbk$ such that $\delta(fg) = \delta(f)g(p) + f(p)\delta(g)$.

Lemma 3.3.6. *Let* X *be a supervariety and* $p \in X$ *a closed point, then*

$$T_p X \cong (\mathfrak{m}_p/\mathfrak{m}_p^2)^*.$$

Proof. Define the map $L : T_p X \to (\mathfrak{m}_p/\mathfrak{m}_p^2)^*$ by $L(\delta)(f) = \delta(f)$. If $f, g \in \mathfrak{m}_p$, then $L(\delta)(fg) = \delta(f)g(p) + f(p)\delta(g) = 0$. Thus, L is well-defined. Since p is a closed point, the exact sequence

$$0 \to \mathfrak{m}_p \to \mathcal{O}_{X,p} \to \Bbbk \to 0$$

splits, and $\mathcal{O}_{X,p} = \Bbbk \oplus \mathfrak{m}_p$. Take $\alpha \in (\mathfrak{m}_p/\mathfrak{m}_p^2)^*$, then we may see it as a map $\alpha : \mathfrak{m}_p \to \Bbbk$ with $\alpha(\mathfrak{m}_p^2) = 0$. Using that $\mathcal{O}_{X,p} = \Bbbk \oplus \mathfrak{m}_p$, we may extend α to $\mathcal{O}_{X,p}$ by setting $\alpha(k) = 0$ for each $k \in \Bbbk$. This defines the inverse map of L. $\square$

Remark 3.3.7. If $X = \mathrm{Spec}\,(S) \subset \mathbb{A}^{m|n}$ is an affine algebraic supervariety with

$$S \cong \Bbbk[x_1, \ldots, x_m \mid \theta_1, \ldots, \theta_n]/(f_1, \ldots, f_r, \phi_1, \ldots, \phi_s)$$

with $f_1, \ldots, f_r$ even and $\phi_1, \ldots, \phi_s$ odd. Let $p \in X$ be a closed point and consider the matrix

$$\mathrm{Jac}(p) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_m}(p) & \frac{\partial f_1}{\partial \theta_1}(p) & \cdots & \frac{\partial f_1}{\partial \theta_n}(p) \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial f_\alpha}{\partial x_1}(p) & \cdots & \frac{\partial f_\alpha}{\partial x_m}(p) & \frac{\partial f_\alpha}{\partial \theta_1}(p) & \cdots & \frac{\partial f_\alpha}{\partial \theta_n}(p) \\ \frac{\partial \phi_1}{\partial x_1}(p) & \cdots & \frac{\partial \phi_1}{\partial x_m}(p) & \frac{\partial \phi_1}{\partial \theta_1}(p) & \cdots & \frac{\partial \phi_1}{\partial \theta_n}(p) \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial \phi_\beta}{\partial x_1}(p) & \cdots & \frac{\partial \phi_\beta}{\partial x_m}(p) & \frac{\partial \phi_\beta}{\partial \theta_1}(p) & \cdots & \frac{\partial \phi_\beta}{\partial \theta_n}(p) \end{bmatrix}.$$

By [CCF11, Remark 10.6.16],

$$T_p X \cong \left\{ v \in \Bbbk^{m|n} \mid \mathrm{Jac}(p)v = 0 \right\}.$$

That is, $T_p X$ is isomorphic to the kernel of $\mathrm{Jac}(p)$ as a super vector space.

If $D : \mathcal{O}_{X,p} \to \mathcal{O}_{X,p}$ is a derivation of $\mathcal{O}_{X,p}$, then we may define an element of the tangent space $D_p : \mathcal{O}_{X,p} \to \Bbbk$ given by $D_p(f) = D(f)(p)$. This gives a linear map

$\Theta_{X,p} \to T_pX$ by $D \mapsto D_p$. This map is not always surjective.

Example 3.3.8. Let $X \subset \mathbb{A}^{2|2}$ be defined by the ideal $I = (x_1\theta_1 + x_2\theta_2) \subset \Bbbk[x_1, x_2|\theta_1, \theta_2]$. Assume $p = (0, 0)$, then the maps $\partial_i : \mathcal{O}_{X,p} \to \Bbbk$ defined by $\partial_i(j) = \delta_{ij}$, $i, j \in \{x_1, x_2, \theta_1, \theta_2\}$, are well-defined elements of $\mathrm{Der}(\mathcal{O}_{X,p}, \Bbbk)$, therefore $\dim T_pX = 2|2$. If $D \in \mathrm{Der}(\mathcal{O}_{X,x})_{\bar{1}}$ is an odd derivation such that $D(\theta_1) = 1 + f$ with $f(p) = 0$, then

$$0 = D(x_1\theta_1 + x_2\theta_2) \iff x_2D(x_2) = -x_1 - x_1f + D(x_1)\theta_1 + D(x_2)\theta_2.$$

Since D is odd, $x_2D(\theta_2) = -x_1 - fx_1 + g\theta_1\theta_2$ for some $g \in \mathcal{O}_{X,p}$. However, there is no even element $D(\theta_2) \in \mathcal{O}_{X,p}$ which makes this equality true. Therefore, there is no derivation $D \in \mathrm{Der}(\mathcal{O}_{X,p})$ such that $D_p = \partial_{\theta_1}$.

Consider the diagonal map $\Delta : X \to X \times X$ with $\Delta^* : \mathcal{O}_{X\times X} \to \Delta_*\mathcal{O}_X$. We have that $\Delta(X)$ is isomorphic to X which is a closed subscheme of $X \times X$ because X is separated. Let $\mathcal{I}$ be the sheaf of ideals of $\Delta(X)$. We define the *sheaf of differentials of* X (over $\mathrm{Spec}\,(\Bbbk)$) as the sheaf

$$\Omega_X = \Delta^*\left(\mathcal{I}/\mathcal{I}^2\right) = \Delta^{-1}\left(\mathcal{I}/\mathcal{I}^2\right) \otimes_{\Delta^{-1}\mathcal{O}_{X\times X}} \mathcal{O}_X.$$

Because $\mathcal{I}/\mathcal{I}^2$ is an $\mathcal{O}_{\Delta(X)}$-module and $\Delta : X \to \Delta(X)$ is an isomorphism, Ω_X is a sheaf of $\mathcal{O}_X$-module.

Remark 3.3.9. In the literature, one usually defines the sheaf of relative differentials of X over Y for a morphism $f : X \to Y$. For an explanation of sheaves of relatives differentials for the usual case, we refer to [Har77, Chapter II, Section 8].

Example 3.3.10. Suppose that $X = \underline{\mathrm{Spec}}\,(S)$ is an affine supervariety, then $X \times X$ is isomorphic to $\mathrm{Spec}\,(S \otimes_{\Bbbk} S)$, and $\Delta(X) \subset X \times X$ is the closed subscheme defined by the kernel I of the multiplication map

$$\begin{aligned} S \otimes S &\to S \\ f \otimes g &\mapsto fg. \end{aligned}$$

Thus, the super vector space $\Omega_S = I/I^2$ is an S-module by left multiplication. If $d : S \to \Omega_S$ is the map defined by $dg = 1 \otimes g - g \otimes 1 \in I$, then

$$\begin{aligned} d(fg) &= 1 \otimes fg - fg \otimes 1 = 1 \otimes fg - fg \otimes 1 - (1 \otimes f - f \otimes 1)(1 \otimes g - g \otimes 1) \\ &= 1 \otimes fg - fg \otimes 1 - 1 \otimes fg + (-1)^{|f||g|} g \otimes f + f \otimes g - fg \otimes 1 \\ &= f(1 \otimes g - g \otimes 1) + (-1)^{|f||g|} g(1 \otimes f - f \otimes 1) = fdg + (-1)^{|f||g|} gdf. \end{aligned}$$

Elements df, $f \in S$, generate Ω_S, and they satisfy $d(f + g) = df + dg$, $d(fg) = fdg + (-1)^{|f||g|}gdf$, $dcf = cdf$ for each $f, g \in S$ and $c \in \Bbbk$. We have that $\Omega_X \cong \widetilde{\Omega_S}$, thus the definition of the sheaf of differentials of X could have been done by covering X with affine open subvarieties $\underline{\mathrm{Spec}}\,(S_i)$ and gluing the corresponding $\widetilde{\Omega_{S_i}}$. In this case, the maps $d : S_i \to \Omega_{S_i}$ defines a map $d : \mathcal{O}_X \to \Omega_X$ of sheaves on X.

For any S-module M, $\underline{\mathrm{Hom}}_S(\Omega_S, M) \cong \mathrm{Der}_{\Bbbk}(S, M)$ where

$$\mathrm{Der}_{\Bbbk}(S, M) = \left\{ D : A \to M \mid D \text{ is linear and } D(fg) = D(f)g + (-1)^{|f||D|} fD(g)\ \forall f, g \in S \right\}.$$

Similarly, if $\mathcal{M}$ is a sheaf of $\mathcal{O}_X$-modules, then there is an isomorphism

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\Omega_X, \mathcal{M}) \cong \mathrm{Der}_{\mathrm{Spec}(\Bbbk)}(\mathcal{O}_X, \mathcal{M}),$$

where $\mathrm{Der}_{\mathrm{Spec}(\Bbbk)}(\mathcal{O}_X, \mathcal{M})$ is defined naturally.

Theorem 3.3.11. *Let X be a supervariety with $\dim X = m|n$ and $p \in X$ a closed point. The following are equivalent*

1. *There exists an affine open neighbourhood $U = \underline{\mathrm{Spec}(S)} \subset X$ of p such that X^r is smooth at $x \in \mathrm{Spec}(\overline{S})$ and $S \cong \Lambda^{\bullet}_{\overline{S}}\overline{S}^n$ where $\overline{S} = S/J_S$.*
2. *$\Theta_{X,p} \to T_pX$ is surjective.*
3. *$\Omega_{X,p}$ is a free $\mathcal{O}_{X,p}$-module.*

Proof. See [She21, Theorem B.3] for a more general version of this result. □

This theorem implies that a $\Bbbk$-point p of X is smooth if one item (hence all) of Theorem 3.3.11 is satisfied.

3.4 System of local parameters at a smooth point

Let X be a smooth integral supervariety. We start our chapter investigating further the local structure of X, analyzing the sheaves $\mathcal{O} = \mathcal{O}_X$ and $\Theta = \Theta_X$. For a closed point $p \in X$, we denote by $\mathfrak{m}_p$ the unique maximal ideal of the local algebra $\mathcal{O}_p$.

Definition 3.4.1. Let $U \subset X$ be an open affine subset and $p \in U$ be a smooth closed point. We say that $t_1, \dots, t_r, \xi_1, \dots, \xi_r \in \Gamma(U, \mathcal{O})$ form a *system of local parameters at p* if the sections $t_1, \dots, t_r, \xi_1, \dots, \xi_r \in \mathcal{O}_p$ form a basis of $\mathfrak{m}_p/\mathfrak{m}_p^2$.

Lemma 3.4.2. *Let p be a smooth closed point of X and $\dim T_pX = r|s$. Then, $t_1, \dots, t_r, \xi_1, \dots, \xi_s \in \Gamma(X, \mathcal{O})$ form a system of parameters at p if and only if $t_1, \dots, t_r, \xi_1, \dots, \xi_s$ generate $\mathfrak{m}_p$.*

Proof. If $t_1, \dots, t_r, \xi_1, \dots, \xi_s$ generate $\mathfrak{m}_p$, then the classes $t_1, \dots, t_r, \xi_1, \dots, \xi_s \in \mathfrak{m}_p/\mathfrak{m}_p^2$ generate the super vector space $\mathfrak{m}_p/\mathfrak{m}_p^2$ that has dimension $r|s$. Therefore, they form a basis for it. On the other hand, $\mathfrak{m}_p = \mathfrak{m}_p^2 + M$ if $M = (t_1, \dots, t_r, \xi_1, \dots, \xi_r) \subset \mathcal{O}_p$. Hence, $M = \mathfrak{m}_p$ by Nakayama's lemma (see [CCF11, Lemma B.3.3]). □

Since X is a smooth integral supervariety, there exists a projective $\mathcal{O}_p$-module M such that $\mathcal{O}_p \cong \Lambda^{\bullet}_{\overline{\mathcal{O}_p}}M$ by definition of smoothness. By Nakayama's lemma, M is free and has dimension s as an $\overline{\mathcal{O}_p}$-module, and, consequently, $\mathcal{O}_p \cong \Lambda^{\bullet}_{\overline{\mathcal{O}_p}}\overline{\mathcal{O}_p}^{\oplus s}$. Hence, there exists an injective homomorphism of superalgebras $\psi : \overline{\mathcal{O}_p} \to \mathcal{O}_p$ such that $\overline{\psi(f)} = f$ and odd elements $\xi_1, \dots, \xi_s \in \mathfrak{m}_p$ such that $\mathcal{O}_p = \psi(\overline{\mathcal{O}_p})[\xi_1, \dots, \xi_s]$. If $t_1, \dots, t_r$ is a system of parameters for $\overline{\mathcal{O}_p}$, then their images $\psi(t_1), \dots, \psi(t_r) \in \mathcal{O}_p$ are elements of $\mathfrak{m}_p$ and

$\psi(t_1), \dots, \psi(t_r), \theta_1, \dots, \theta_s$ is a system of parameters at p. From now on, we denote $\psi(f_0) \in \mathcal{O}_p$ by f_0 for each $f_0 \in \overline{\mathcal{O}_p}$. Any element $f \in \mathcal{O}_p$ may be written as

$$f = f_0 + \sum_{I \subset \{1,\dots,s\}} f_I \xi^I \tag{3.2}$$

where $f_0, f_I \in \overline{\mathcal{O}_p}$ and $\xi^I = \xi_{i_1}\xi_{i_2}\cdots\xi_{i_k}$ with $I = \{i_1, \dots, i_k\}$ and $i_1 < i_2 < \cdots < i_k$.

Since X^r is smooth at p, there exists derivations $\partial_{t_1}, \dots, \partial_{t_r} : \overline{\mathcal{O}_p} \to \overline{\mathcal{O}_p}$ such that $\partial_{t_i}(t_j) = \delta_{ij}$ and

$$\operatorname{Der}\left(\overline{\mathcal{O}_p}\right) = \bigoplus_{i=1}^{r} \overline{\mathcal{O}_p}\partial_{t_i}$$

is a free $\overline{\mathcal{O}_p}$-module generated by ∂_{t_i}. Using the expansion given on 3.2, we extend

$$\partial_{t_i}\left(f_0 + \sum_{I \subset \{1,\dots,s\}} f_I \xi^I\right) = \partial_{t_i}(f_0) + \sum_{I \subset \{1,\dots,s\}} \partial_{t_i}(f_I)\, \xi^I.$$

Similarly, we define the odd derivation $\partial_{\xi_i} : \mathcal{O}_p \to \mathcal{O}_p$ by $\partial_{\xi_i}(\xi_j) = \delta_{ij}$ and $\partial_{\xi_i}(f) = 0$ for each $f \in \overline{\mathcal{O}_p}$. Explicitly,

$$\partial_{\xi_i}\left(f_0 + \sum_{I \subset \{1,\dots,s\}} f_I \xi^I\right) = \sum_{I \subset \{1,\dots,s\}} f_I \partial_{t_i}\left(\xi^I\right).$$

Note that

$$\partial_{\xi_l}\left(\xi^I\right) = (-1)^{|I_1|}\xi^{I_1}\xi^{I_2}$$

if $I_1 = \{i_1, \dots, i_a\} \subset \{1, \dots, l-1\}$, $I_2 = \{i_1, \dots, i_b\} \subset \{l+1, \dots, s\}$ and $I = I_1 \cup I_2 \cup \{l\}$, and $\partial_{\xi_l}\left(\xi^I\right) = 0$ if $l \notin I$.

Lemma 3.4.3. *If $p \in X$ is a smooth closed point, then*

$$\Theta_p = \operatorname{Der}(\mathcal{O}_p) = \bigoplus_{i=1}^{r} \mathcal{O}_p\partial_{t_i} \oplus \bigoplus_{j=1}^{s} \mathcal{O}_p\partial_{\xi_j}.$$

Proof. It follows from $\mathcal{O}_p \cong \overline{\mathcal{O}_p}[\xi_1, \dots, \xi_s]$ and $\operatorname{Der}\left(\overline{\mathcal{O}_p}\right) = \bigoplus_{i=1}^{r} \overline{\mathcal{O}_p}\partial_{t_i}$. □

Remark 3.4.4. By Theorem 3.3.11, the map $\Theta_{X,p} \to T_pX$ is surjective if p is a smooth closed point of X. If $t_1, \dots, t_r, \xi_1, \dots, \xi_s$ is a system of parameters at p, then its dual basis on $T_pX \cong (\mathfrak{m}_p/\mathfrak{m}_p^2)^*$ is the image of the derivations $\partial t_1, \dots, \partial_{t_r}, \partial_{\xi_1}, \dots, \partial_{\xi_s}$.

Corollary 3.4.5. *There exists an affine open neighbourhood $U = \underline{\operatorname{Spec}}(B)$ of p and a system of parameters $t_1, \dots, t_r, \xi_1, \dots, \xi_s \in \Gamma(U, \mathcal{O})$ at p such that $\partial_{t_1}, \dots, \partial_{t_r}, \partial_{\xi_1}, \dots, \partial_{\xi_s} \in \Gamma(U, \Theta_X)$ are well-defined and form a basis of the $\Gamma(U, \mathcal{O})$-module $\Gamma(U, \Theta_X)$.*

Example 3.4.6. When $X \subset \mathbb{A}^{m|n}$, it is possible to explicitly construct a basis of Θ_p and also find an open neighborhood U of p such that $\Gamma(U, \Theta)$ is free as a module over $\Gamma(U, \mathcal{O})$.

Suppose that $X = \underline{\operatorname{Spec}} S$ is an affine integral smooth variety with

$$S = \Bbbk[x_1, \dots, x_m, \theta_1, \dots, \theta_n]/(f_1, \dots, f_\alpha, \phi_1, \dots, \phi_\beta).$$

Thus, $\Gamma(X, \Theta)$ is a projective module over S and can be seen as the submodule of the free S-module

$$\bigoplus_{i=1}^{m} S\frac{\partial}{\partial x_i} \oplus \bigoplus_{j=1}^{n} S\frac{\partial}{\partial \theta_j}$$

where $\eta = \sum_{i=1}^{m} g_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} \psi_j \frac{\partial}{\partial \theta_j} \in \Gamma(X, \Theta)$ if and only if

$$\sum_{i=1}^{m} g_i \frac{\partial f_1}{\partial x_i} + \sum_{j=1}^{n} \psi_j \frac{\partial f_k}{\partial \theta_j} = 0 \text{ and } \sum_{i=1}^{m} g_i \frac{\partial \phi_l}{\partial x_i} + \sum_{j=1}^{n} \psi_j \frac{\partial \phi_l}{\partial \theta_j} = 0 \tag{3.3}$$

for each $k = 1, \dots, \alpha$, $l = 1, \dots, \beta$. We may see this as a system of linear equations with coefficients on S. The associated matrix to this system is

$$\mathrm{Jac} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} & \frac{\partial f_1}{\partial \theta_1} & \cdots & \frac{\partial f_1}{\partial \theta_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial f_\alpha}{\partial x_1} & \cdots & \frac{\partial f_\alpha}{\partial x_m} & \frac{\partial f_\alpha}{\partial \theta_1} & \cdots & \frac{\partial f_\alpha}{\partial \theta_n} \\ \frac{\partial \phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_1}{\partial x_m} & \frac{\partial \phi_1}{\partial \theta_1} & \cdots & \frac{\partial \phi_1}{\partial \theta_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial \phi_\beta}{\partial x_1} & \cdots & \frac{\partial \phi_\beta}{\partial x_m} & \frac{\partial \phi_\beta}{\partial \theta_1} & \cdots & \frac{\partial \phi_\beta}{\partial \theta_n} \end{bmatrix}.$$

Take a closed point $p \in X$, then the following matrix (with coefficients in $\Bbbk \cong \mathcal{O}_p/\mathfrak{m}_p$)

$$\mathrm{Jac}(p) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_m}(p) & \frac{\partial f_1}{\partial \theta_1}(p) & \cdots & \frac{\partial f_1}{\partial \theta_n}(p) \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial f_\alpha}{\partial x_1}(p) & \cdots & \frac{\partial f_\alpha}{\partial x_m}(p) & \frac{\partial f_\alpha}{\partial \theta_1}(p) & \cdots & \frac{\partial f_\alpha}{\partial \theta_n}(p) \\ \frac{\partial \phi_1}{\partial x_1}(p) & \cdots & \frac{\partial \phi_1}{\partial x_m}(p) & \frac{\partial \phi_1}{\partial \theta_1}(p) & \cdots & \frac{\partial \phi_1}{\partial \theta_n}(p) \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial \phi_\beta}{\partial x_1}(p) & \cdots & \frac{\partial \phi_\beta}{\partial x_m}(p) & \frac{\partial \phi_\beta}{\partial \theta_1}(p) & \cdots & \frac{\partial \phi_\beta}{\partial \theta_n}(p) \end{bmatrix}$$

has rank $m|n - r|s$ by [Fio08, Lemma 3.5], where $\dim X = r|s = \dim(\mathfrak{m}_p/\mathfrak{m}_p^2) = \dim T_pX$. Assume that its principal submatrix has rank $(m - r|n - s)$.

Consider $\begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix}$, where

$$J_1 = \left(\frac{\partial f_i}{\partial x_j}\right)_{i,j=1}^{m-r}, \; J_2 = \left(\frac{\partial f_i}{\partial \theta_j}\right)_{i=1,j=1}^{m-r,n-s}, \; J_3 = \left(\frac{\partial \phi_i}{\partial x_j}\right)_{i=1,j=1}^{n-s,m-r}, \; J_4 = \left(\frac{\partial \phi_i}{\partial \theta_j}\right)_{i,j=1}^{n-s}.$$

Since that $J_1(p)$ and $J_4(p)$ are invertible, then $\det(J_1(p)), \det(J_4(p))$ are nonzero in $\mathcal{O}_p/\mathfrak{m}_p$, which implies $\det(J_1)$, $\det(J_4)$ are invertible in $\mathcal{O}_p$. Therefore, J_1 and J_4 are invertible, and it follows from [CCF11, Proposition 1.5.1] that $\begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix} \in \mathrm{GL}_{m-r|n-s}(\mathcal{O}_p)$. Furthermore, its inverse is

$$\begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix}^{-1} = \begin{bmatrix} (J_1 - J_2 J_4^{-1} J_3)^{-1} & -J_1^{-1} J_2 (J_4 - J_3 J_1^{-1} J_2)^{-1} \\ -J_4^{-1} J_3 (J_1 - J_2 J_4^{-1} J_3)^{-1} & (J_4 - J_3 J_1^{-1} J_2)^{-1} \end{bmatrix}$$

by [CCF11, Proposition 1.5.1, Proposition 1.5.9].

We can use this to solve the system (3.3) in $\mathcal{O}_p$, and we may write its solution space over $\mathcal{O}_p$ by choosing the first $m-r|n-s$ variables as leading and the last $r|s$ as free:

$$\left\{ \sum_{i=1}^{m-r} \frac{a_{ik}}{h} \frac{\partial}{\partial x_i} + \sum_{j=1}^{n-s} \frac{b_{jk}}{h} \frac{\partial}{\partial \theta_j} + \frac{\partial}{\partial x_{m-r+k}} \right\}_{k=1}^{r} \cup \left\{ \sum_{i=1}^{m-r} \frac{a'_{ik}}{h} \frac{\partial}{\partial x_i} + \sum_{j=1}^{n-s} \frac{b'_{jk}}{h} \frac{\partial}{\partial \theta_j} + \frac{\partial}{\partial \theta_{n-s+k}} \right\}_{k=1}^{s}$$

where $a_{ij}, b_{ij}, a'_{ij}, b'_{ij} \in S$ and $h \in \mathcal{O}_p$ is invertible. In particular, $h \notin \mathfrak{m}_p$ and hence $h \notin J_S$. Therefore,

$$p \in D(h) = \{\mathfrak{p} \in \operatorname{Spec}(S_{\bar{0}}) \mid h \notin \mathfrak{p}\} \subset X$$

is an open neighborhood of p. The global sections

$$t_1 = x_{m-r+1} - x_{m-r+1}(p), \ \dots, \ t_r = x_m - x_m(p), \ \xi_1 = \theta_{n-s+1}, \dots, \ \xi_s = \theta_s$$

form a system of parameters at p, and the derivatives

$$\partial_{t_k} = \sum_{i=1}^{m-r} \frac{a_{ik}}{h} \frac{\partial}{\partial x_i} + \sum_{j=1}^{n-s} \frac{b_{jk}}{h} \frac{\partial}{\partial \theta_j} + \frac{\partial}{\partial x_{m-r+k}}, \ k = 1, \dots, r,$$
$$\partial_{\xi_l} = \sum_{i=1}^{m-r} \frac{a'_{ik}}{h} \frac{\partial}{\partial x_i} + \sum_{j=1}^{n-s} \frac{b'_{jk}}{h} \frac{\partial}{\partial \theta_j} + \frac{\partial}{\partial \theta_{n-s+k}}, \ l = 1, \dots, s,$$

are a basis of $\Gamma(D(h), \Theta)$ as a $\Gamma(D(h), \mathcal{O})$-module. Note that these are also a basis of Θ_p as an $\mathcal{O}_p$-module.

Let us apply the technique of Example 3.4.6 in the following example.

Example 3.4.7. Consider $X = \underline{\operatorname{Spec}}(S) = \left(\operatorname{Spec}\left(\Bbbk\left[S^1\right]\right), \mathcal{O}\right)$ with

$$S = \Bbbk[x_1, x_2, \theta_1, \theta_2]/(x^2 + y^2 - 1 - \theta_1\theta_2, x_1\theta_1 + x_2\theta_2).$$

Set $f = x^2 + y^2 - 1 - \theta_1\theta_2$, $\psi = x_1\theta_1 + x_2\theta_2$ and $p = (x_1 - 1, x_2) \subset \overline{S} = \Bbbk[S^1]$. In this case, we have that

$$\operatorname{Jac} = \begin{bmatrix} 2x_1 & 2x_2 & \theta_1 & -\theta_2 \\ \theta_1 & \theta_2 & -x_1 & -x_2 \end{bmatrix} \text{ and } \operatorname{Jac}(p) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

The submatrix is $\begin{bmatrix} 2x_1 & \theta_1 \\ \theta_1 & -x_1 \end{bmatrix}$ is invertible over $\mathcal{O}_p$, and its inverse is $\begin{bmatrix} \frac{1}{2x_1} & \frac{\theta_1}{2x_1^2} \\ \frac{\theta_1}{2x_1^2} & -\frac{1}{x_1} \end{bmatrix}$. Hence, the derivations

$$\tau = \frac{\partial}{\partial x_1} - \left(\frac{x_2}{x_1} + \frac{\theta_1\theta_2}{2x_1^2} \right) \frac{\partial}{\partial x_1} + \left(\frac{\theta_2}{x_1} - \frac{x_2\theta_1}{x_1^2} \right) \frac{\partial}{\partial \theta_1}$$

$$\sigma = \frac{\partial}{\partial \theta_2} + \left(\frac{\theta_2}{2x_1} + \frac{x_2\theta_1}{2x_1^2} \right) \frac{\partial}{\partial x_1} + \left(\frac{\theta_1\theta_2}{2x_1^2} - \frac{x_2}{x_1} \right) \frac{\partial}{\partial \theta_1}$$

form a basis of $\mathrm{Der}\left(S_{x_1}\right)$ inside of the free module $S_{x_1}\frac{\partial}{\partial x_1} \oplus S_{x_1}\frac{\partial}{\partial x_2} \oplus S_{x_1}\frac{\partial}{\partial \theta_1} \oplus S_{x_1}\frac{\partial}{\partial \theta_2}$. Although σ and τ are not global sections of Θ, $x_1^2\tau,\ x_1^2\sigma \in \mathrm{Der}(S)$ are. If we set

$$t = x_2 - x_2(p) = x_2 \quad \text{and} \quad \xi = \theta_2,$$

then t, ξ form a system of parameters at p on $D(x_1)$. Moreover, τ and σ corresponds to ∂_t and ∂_ξ, respectively. Consequently, $\Theta_p = \mathcal{O}_p\tau \oplus \mathcal{O}_p\sigma$ and $\Gamma(D(x_1), \Theta) = \mathrm{Der}(S_{x_1}) = S_{x_1}\tau \oplus S_{x_1}\sigma$.

3.5 Completions and power series

A *topological* superalgebra is a ring that is a topological space such that the multiplication and addition maps are continuous. If I is an ideal of a commutative superalgebra S, then the filtration

$$S \supset I \supset I^2 \supset I^3 \supset \dots$$

defines a topology on S. In this case, the filtration gives a basis of open neighborhoods of $0 \in S$. Similarly, open neighborhoods of $f \in S$ are given by cosets $f + I^k$, $k \geq 0$. This topology is often called *I-adic topology* on S. The inverse limit

$$\widehat{S} = \varprojlim S/I^n$$

is a superalgebra and it is the *completion* of S with respect to this topology. There is a canonical map $S \to \widehat{S}$, and its kernel is the intersection of $\bigcap_{k\geq 0} I^k$. This topology is Hausdorff if $\bigcap_{k\geq 0} I^k = 0$. For more on topological rings and completions of rings, we refer to [AM69, Chapter 10].

Assume that $X = \underline{\mathrm{Spec}\,(S)}$ is an integral affine supervariety with $\dim X = r|s$. Let $p \in X$ be a smooth closed point of X. We consider the $\mathfrak{m}_p$-adic topology on $\mathcal{O}_p$ by its maximal ideal $\mathfrak{m}_p$, which defines the completion $\widehat{\mathcal{O}_p}$. We fix $t_1, \dots, t_r, \xi_1, \dots, \xi_s$ a system of parameters at p. Since X is affine, we may assume that this system is formed by global sections by Example 3.4.6.

Proposition 3.5.1 ([Fio08, Proposition 3.16]). *$\widehat{\mathcal{O}_p} \cong \Bbbk[[T_1, \dots, T_r|\Xi_1, \dots, \Xi_n]]$, where $t_i \mapsto T_i$ and $\xi_j \mapsto \Xi_j$.*

Since $S = \Gamma(X, \mathcal{O})$ is an integral superdomain, the localization map $S \to \mathcal{O}_p$ is an embedding. Thus, there is an embedding

$$\pi : S \hookrightarrow \Bbbk[[T_1, \dots, T_r|\Xi_1, \dots \Xi_s]]$$

such that $\pi(t_i) = T_i$ and $\pi(\xi_i) = \Xi_i$. Let $\widehat{W}(r,s) = \mathrm{Der}\,(\Bbbk[[T_1,\dots,T_r|\Xi_1,\dots,\Xi_s]])$, then

$$\frac{\partial}{\partial\Theta_1},\ \dots,\ \frac{\partial}{\partial\Theta_r},\ \frac{\partial}{\partial\Xi_1},\ \dots,\ \frac{\partial}{\partial\Xi_s}$$

form a basis of $\widehat{W}(r,s)$ as a module over $\widehat{S}(r,s) = \Bbbk[[T_1,\dots,T_r|\Xi_1,\dots,\Xi_s]]$. The superalgebra $\widehat{S}(r,s)$ is local, and its maximal ideal $\mathfrak{m}_0$ is the one generated by $T_1,\dots,T_r,\Xi_1,\dots,\Xi_s$, which is exactly the image of $\mathfrak{m}_p$ by the isomorphism $\widehat{\mathcal{O}_p}$ on Proposition 3.5.1. If $D \in \widehat{W}(r,s)$ is any derivation, then $D(\mathfrak{m}_0^k) \subset \mathfrak{m}_0^{k-1}$. Hence, any element of $\widehat{W}(r,s)$ is a continuous map under the $\mathfrak{m}_0$-adic topology. Similarly, if $\mathfrak{m}$ is the maximal ideal of S associated with p, any derivation of S is continuous on the $\mathfrak{m}$-adic topology. Finally, both topologies are Hausdorff because $\bigcap_{k\geq 0}\mathfrak{m}_0^k = 0$ and $\bigcap_{k\geq 0}\mathfrak{m}^k = 0$. These imply that if two continuous maps between S and $\widehat{S}(r,s)$ agree on a dense subset of S, they are identically equal.

Proposition 3.5.2. *Let $X = \underline{\mathrm{Spec}}\,(S)$ be an integral affine supervariety with structure sheaf $\mathcal{O}$ and tangent sheaf Θ. There exists an unique embedding of Lie algebras $\widehat{\pi} : \Gamma(X,\Theta) \to \widehat{W}(r,s)$ such that $\widehat{\pi}(\eta)\pi(f) = \pi(\eta(f))$ for all $f \in S$ and $\eta \in \Gamma(X,\Theta)$.*

Proof. There is an unique way to define $\widehat{\pi}$, which is

$$\widehat{\pi}(\eta) = \sum_{j=1}^{r}\pi(\eta(t_i))\frac{\partial}{\partial T_i} + \sum_{j=1}^{s}\pi(\eta(\xi_j))\frac{\partial}{\partial\Xi_j}$$

for each $\eta \in \mathrm{Der}(S)$. We have that $\Bbbk[t_1,\dots,t_r|\theta_1,\dots,\theta_s] \subset S$ is a dense subset of the Hausdorff space S. Since the continuous maps $\widehat{\pi}(\eta)\circ\pi$ and $\pi\circ\eta$ agree on $\Bbbk[t_1,\dots,t_r,\theta_1,\dots,\theta_s]$, they must be equal. Therefore, $\widehat{\pi}(\eta)(\pi(f)) = \pi(\eta(f))$ for each $\eta \in \mathrm{Der}(S)$ and $f \in S$. This identity implies that $\widehat{\pi}$ is a Lie algebra homomorphism, because

$$\left[\widehat{\pi}(\eta),\widehat{\pi}(\eta)\right](f) = \pi\left(\eta(\mu(f)) - (-1)^{|\mu||\eta|}\mu(\eta(f))\right) = \pi([\eta,\mu](f)) = \widehat{\pi}([\eta,\mu])(f).$$

Lastly, each derivation μ in the kernel of $\widehat{\pi}$ yields $\mu\left(\Bbbk[t_1,\dots,t_r|\xi_1,\dots,\xi_s]\right) = 0$, therefore μ is the trivial map because it is continuous in the $\mathfrak{m}$-adic topology. ☐

3.6 Simplicity of the Lie superalgebra of vector fields

In this section, we assume that $X = \underline{\mathrm{Spec}}\,(S)$ is an integral affine supervariety. Let $p \in X$ be a smooth closed point of X with a system of parameters $t_1,\dots,t_r,\xi_1,\dots,\xi_s$. To prove Lemma 3.4.3, we constructed partial derivatives $\partial_{t_1},\dots,\partial_{t_r},\partial_{\xi_1},\dots,\partial_{\xi_s}$ that form a basis of Θ_p. These may not be global, but there exists a basic open neighborhood $D(h)$ containing p such that $\partial_{t_1},\dots,\partial_{t_r},\partial_{\xi_1},\dots,\partial_{\xi_s} \in \Gamma(D(h),\Theta)$. If this is the case, then $h^k\partial_{t_1},\dots,h^k\partial_{t_r},h^k\partial_{\xi_1},\dots,h^k\partial_{\xi_s} \in \mathrm{Der}(S)$ for some $k \geq 0$. Take k' as the smallest integer such that this happens, set $h' = h^{k'}$, and define

$$\tau_i = h'\partial_{t_i}, \qquad \sigma_j = h'\partial_{\xi_j}$$

for $i = 1, \dots, r$, $j = 1, \dots, s$. With this notation, we have that

$$\widehat{\pi}(\tau_i) = \pi(h')\frac{\partial}{\partial T_i}, \quad \widehat{\pi}(\sigma_j) = \pi(h')\frac{\partial}{\partial \Xi_j}.$$

The Lie superalgebra $\widehat{W}(r, s)$ has a descending filtration

$$\widehat{W}(r, s) = \widehat{W}(r, s)_{-1} \supset \widehat{W}(r, s)_0 \supset \widehat{W}(r, s)_1 \supset \widehat{W}(r, s)_2 \supset \dots$$

induced by the one in $\widehat{S}(r, s)$ and it is defined by $\widehat{W}(r, s)_{k-1} = \mathfrak{m}_0^k \widehat{W}(r, s)$. Explicitly, a derivation $f\frac{\partial}{\partial Y} \in \widehat{W}(r, s)_{k-1}$ if the lowest degree of a monomial that occurs in $f \in \widehat{S}(r, s)$ is greater or equal to k, where $Y \in \{T_1, \dots, T_r, \Xi_1, \dots, \Xi_s\}$. This filtration satisfies $\left[\widehat{W}(r, s)_k, \widehat{W}(r, s)_l\right] \subset \widehat{W}(r, s)_{k+l}$ if $k + l \geq -1$. Moreover, there is an associated graded Lie algebra

$$\operatorname{Gr} \widehat{W}(r, s) = \widehat{W}(r, s)_{-1}/\widehat{W}(r, s)_0 \oplus \widehat{W}(r, s)_0/\widehat{W}(r, s)_1 \oplus \widehat{W}(r, s)_1/\widehat{W}(r, s)_2 \oplus \dots,$$

which is isomorphic to $W(r, s)$. Define $\omega : \widehat{W}(r, s) \to \operatorname{Gr} \widehat{W}(r, s)$ by $\omega(\mu) = \mu + \widehat{W}(r, s)_{j+1} \in \widehat{W}(r, s)_j/\widehat{W}(r, s)_{j+1}$ for $\mu \in \widehat{W}(r, s)_j/\widehat{W}(r, s)_{j+1}$ for each $\mu \in \widehat{W}(r, s)_j \setminus \widehat{W}(r, s)_{j+1}$. The map ω is not linear, however $[\omega(\eta), \omega(\mu)] = \omega([\eta, \mu])$ if $[\omega(\eta), \omega(\mu)] \neq 0$. Note that $\omega\left(\widehat{\pi}(\tau_i)\right)$ is a non-zero multiple of $\frac{\partial}{\partial T_i}$ and $\omega\left(\widehat{\pi}(\sigma_j)\right)$ is non-zero multiple of $\frac{\partial}{\partial \Xi_j}$.

Proposition 3.6.1. *Suppose $r|s \geq 1|0$. Let p be a smooth closed point of the affine supervariety X with* $\dim X = r|s$*, and $t_1, \dots, t_r, \xi_1, \dots, \xi_s$ be a system of parameters as before. If $\mathcal{J}$ is a nonzero $\mathbb{Z}_2$-graded ideal of* $\operatorname{Der}(S)$*, then*

1. *There exists $y \in \{t_1, \dots, t_r, \xi_1, \dots, \xi_s\}$ and $\mu \in \mathcal{J}_{|y|}$ such that $\mu(y)(p) \neq 0$.*
2. *For all $y \in \{t_1, \dots, t_r, \xi_1, \dots, \xi_s\}$, there exists $\mu \in \mathcal{J}_{|y|}$ such that $\mu(y)(p) \neq 0$. In particular, $\mathcal{J}_{\bar{0}} \neq 0$.*
3. *There exists $g \in A_{\bar{0}}$ and $\mu \in \mathcal{J}_{\bar{0}}$ such that $\mu(\mu(g))(p) \neq 0$.*

Proof. Let $\eta \in \mathcal{J}$ be a homogeneous element. Then, there exists $u_1, \dots, u_{r+s} \in \mathbb{k}[T_1, \dots, T_r | \Xi_1, \dots, \Xi_s]$ homogeneous polynomials with the same degree such that

$$\omega\left(\widehat{\pi}(\eta)\right) = \sum_{i=1}^{r} u_i \frac{\partial}{\partial T_i} + \sum_{j=1}^{r} u_{r+j} \frac{\partial}{\partial \Xi_j} \neq 0,$$

by the definition of ω. Note that $u_1, \dots, u_r$ have all the same parity, which is different from the parity of $u_{r+1}, \dots, u_{r+s}$. Choose a nonzero polynomial $u_{j_0} \in \{u_1, \dots, u_{r+s}\}$, and set $Y = T_{j_0}$ if $j_0 \leq r$ or $Y = \Xi_{j_0 - r}$ if $j_0 > r$. Take $T_1^{k_1} \cdots T_r^{k_r} \Xi_{i_1} \cdots \Xi_{i_l}$ a monomial that occurs in u_{j_0} so

$$\left(\frac{\partial}{\partial T_1}\right)^{k_1} \cdots \left(\frac{\partial}{\partial T_1}\right)^{k_s} \left(\frac{\partial}{\partial \Xi_{i_1}}\right) \cdots \left(\frac{\partial}{\partial \Xi_{i_l}}\right) u_{j_0} \in \mathbb{k} \setminus \{0\}.$$

Consider

$$\mu = \mathrm{ad}(\tau_1)^{k_1} \cdots \mathrm{ad}(\tau_r)^{k_r} \mathrm{ad}(\sigma_{i_1}) \cdots \mathrm{ad}(\sigma_{i_l})\eta \in \mathcal{J}.$$

Then, μ is homogeneous and it has the same parity of Y. Since ω preserves brackets, we see that

$$\omega(\widehat{\pi}(\mu)) = \mathrm{ad}(\omega(\widehat{\pi}(\tau_1)))^{k_1} \cdots \mathrm{ad}(\omega(\widehat{\pi}(\tau_s)))^{k_s} \mathrm{ad}(\omega(\widehat{\pi}(\sigma_{i_1}))) \cdots \mathrm{ad}(\omega(\widehat{\pi}(\sigma_{i_l})))\omega(\eta)$$

is an element of $\widehat{W}(r,s)_{-1} \setminus \widehat{W}(r,s)_0$ and it contains $\frac{\partial}{\partial Y}$ because $h'(p) \neq 0$. Hence, $\omega(\widehat{\pi}(\mu))Y \neq 0$, which implies that $\mu(y)(p) \neq 0$ where $y \in \{t_1, \dots, t_r, \xi_1, \dots, \xi_s\}$ is the parameter such that $\pi(y) = Y$.

If $i \in \{1, \dots, r\}$, then $\omega(\widehat{\pi}([y\tau_i, \mu])) = [\omega(\widehat{\pi}(y\tau_i)), \omega(\widehat{\pi}(\mu))]$ contains $\frac{\partial}{\partial T_i}$. Hence, $\omega(\widehat{\pi}([y\tau_i, \mu]))T_i \neq 0$, thus $[y\tau_i, \mu](t_i)(p) \neq 0$. By the same argument $[y\sigma_j, \mu](\xi_j)(p) \neq 0$.

Take $\mu' = [y\tau_i, \mu] \in \mathcal{J}_{\bar{0}}$. If $\mu'(\mu'(t_i))(p) \neq 0$, it is done. If $\mu'(\mu'(t_i))(p) = 0$, then $t_i^2 \neq 0$ and

$$\mu'(\mu'(t_i^2))(p) = 2\mu'(t_i\mu'(t_i))(p) = 2\mu'(t_i)(p)\mu'(t_i)(p) + 2t_i\mu'(\mu'(t_i')) = 2(\mu'(t_i)(p))^2 \neq 0.$$

□

Corollary 3.6.2. *If $\mathcal{J}$ is a nonzero ideal of* $\mathrm{Der}(S)$ *and p is a smooth closed point, then the map $\mathcal{J} \to T_pX$ given by $D \mapsto (f \mapsto D(f)(p))$ is surjective.*

Proof. The image of the vector fields $[y\tau_i, \mu]$, $[y\sigma_j, \mu] \in \mathcal{J}$, $i = 1, \dots, r$, $j = 1, \dots, s$, span T_pX. □

Corollary 3.6.3. *Suppose that X is smooth and integral. If $f \in S_{\bar{0}} \setminus (\Bbbk + J_S)$, there exists $\mu \in \mathrm{Der}(S)_{\bar{0}}$ such that $\mu(f) \notin J_S$.*

Proof. There exists a closed point $p \in X$ such that $f(p) \neq 0$. Since f is even and it is not in J_S, then f is not a zero divisor. Hence, there exists a neighborhood U of p such that

$$f = g + \sum_{\emptyset \neq \beta \subset \{1 \dots, s\}} f_\beta \xi^\beta \in \Gamma(U, \mathcal{O}),$$

where the Taylor series expansion of g does not have odd variables and $g \notin \Bbbk$. Therefore, there exists $i \in \{1, \dots, r\}$ such that $\tau_i(g) \notin J_{\Gamma(U,\mathcal{O})}$. In particular,

$$\tau_i(f) = \tau_i(g) + \sum_{\emptyset \neq \beta \subset \{1 \dots, s\}} \tau_i(f_\beta)\xi^\beta,$$

thus $\tau_i(f) \notin J_S$. □

Lemma 3.6.4. *Let $\mathcal{J}$ be an nonzero ideal of* $\mathrm{Der}(S)$, *then*

1. *If $\mu \in \mathcal{J}_{\bar{0}}$, then $\mu(f)\mu \in \mathcal{J}$ for all $f \in S$.*
2. *If $g \in S_{\bar{0}}$ and $\mu \in \mathcal{J}_{\bar{0}}$, then $\mu(f)\mu(g)\mu \in \mathcal{J}$ for all $f \in S$.*

3. *If* $g \in S_{\bar{0}}$ *and* $\mu \in \mathcal{J}_{\bar{0}}$, *then* $f\mu(\mu(g))\mu \in \mathcal{J}$ *for all* $f \in S$.

Proof. Let $\mu \in \mathcal{J}_{\bar{0}}$ and $f \in S$, then $[\mu, f\mu] = \mu(f)\mu + (-1)^{|f||\mu|}f[\mu,\mu] = \mu(f)\mu \in \mathcal{J}$ and item (1) follows. For each $g \in A_{\bar{0}}$, $[\mu, f\mu(g)\mu] - [f\mu, \mu(g)\mu] = 2\mu(g)\mu(f) \in \mathcal{J}$, hence (1) is proved. By (1), $\mu(f\mu(g))\mu \in \mathcal{J}$. By (2), $\mu(f)\mu(g)\mu \in \mathcal{J}$. Since $f\mu(\mu(g)) = \mu(f\mu(g)) - \mu(g)\mu(f)$, we have that $f\mu(\mu(g))\mu \in \mathcal{J}$.

□

Lemma 3.6.5. *Let* $f, g \in S_{\bar{0}}$, $\mu \in \mathcal{J}_{\bar{0}}$. *Let* $I_{f,g,\mu}$ *be the principal ideal of* S *generated by* $\mu(f)\mu(\mu(g))$. *Then for every* $a \in I_{f,g,\mu}$ *and every* $\tau \in \mathrm{Der}(S)$, $a\tau \in \mathcal{J}$.

Proof. If $a = b\mu(f)\mu(\mu(g))$, then $b\mu(\mu(g))\mu$, $fb\mu(\mu(g))\mu$, $\mu(f)b\mu(\mu(g))\mu \in \mathcal{J}$ by Lemma 3.6.4. Therefore, if $h = b\mu(\mu(g))$

$$\begin{aligned}
&[b\mu(\mu(g))\mu, f\tau] - [fb\mu(\mu(g))\mu, \tau] - (-1)^{|\tau||b|}\tau(f)b\mu(\mu(g))\mu \\
=&h\mu(f)\tau - (-1)^{|b||\tau|}f\tau(h)\mu + hf[\mu,\tau] + (-1)^{|\tau||b|}\tau(fh)\mu - fh[\mu,\tau] - (-1)^{|\tau||b|}\tau(f)h\mu \\
=&h\mu(f)\tau - (-1)^{|\tau||b|}f\tau(h)\mu + (-1)^{|\tau||b|}\tau(f)h\mu + (-1)^{|\tau||b|}f\tau(h)\mu - (-1)^{|\tau||b|}\tau(f)h\mu \\
=&h\mu(f)\tau = b\mu(f)\mu(\mu(g))\tau = a\tau \in \mathcal{J}
\end{aligned}$$

□

With all technical results proved, we may prove the main theorem of this section.

Theorem 3.6.6. *If* $X = \underline{\mathrm{Spec}}\,(S)$ *is a smooth integral affine supervariety and* $\dim X = r|s \geq 1|0$, *then* $\mathrm{Der}(S) = \Gamma(X, \Theta_X)$ *is a simple Lie superalgebra.*

Proof. Let $\mathcal{J}$ be a $\mathbb{Z}_2$-graded ideal of $\mathrm{Der}(S)$ and $J = \{a \in S \mid a\mathrm{Der}(S) \subset \mathcal{J}\}$. For each closed point $p \in X$ there exists $\mu \in \mathcal{J}_{\bar{0}}$ and $f, g \in S_{\bar{0}}$ such that $\mu(f)(p) \neq 0$ and $\mu(\mu(g))(p) \neq 0$ by Proposition 3.6.1. Thus, $\mu(f)\mu(\mu(g)) \neq 0$ and the principal ideal $I_{f,g,\mu}$ of S generated by $\mu(f)\mu(\mu(g))$ is nonzero. By Lemma 3.6.5, $I_{f,g,\mu} \subset J$. Thus, J is a nonzero $\mathrm{Der}(S)$-ideal of S. Furthermore, for each maximal ideal $\mathfrak{m}$ of S, there exists an $h \in J$ such that $h(p) \neq 0$ where $p \in X$ is the corresponding closed point. Hence, $h + \mathfrak{m} \neq 0$, and J is not contained in $\mathfrak{m}$. Since every proper ideal of S is contained in a maximal ideal, we conclude that $J = S$. Thus, $\mathrm{Der}(S) = S\mathrm{Der}(S) = J\mathrm{Der}(S) \subset \mathcal{J} \subset \mathrm{Der}(S)$, which implies that $\mathcal{J} = \mathrm{Der}(S)$. □

Corollary 3.6.7. *Let* $X = \underline{\mathrm{Spec}}\,(S)$ *be a smooth integral affine supervariety and* $\dim X = r|s \geq 1|0$ *and* $I \neq 0$ *be a nonzero ideal of* S. *If* I *is a* $\mathrm{Der}(S)$*-submodule of* S, *then* $I = S$.

Proof. If $f \in I$ and $\eta, \mu \in \mathrm{Der}(S)$, then

$$[\eta, f\mu] = \eta(f)\mu + (-1)^{|\eta||f|}f[\eta,\mu] \in I\mathrm{Der}(S).$$

Hence, $I\mathrm{Der}(S)$ is a nonzero ideal of $\mathrm{Der}(S)$. By Theorem 3.6.6, $I\mathrm{Der}(S) = \mathrm{Der}(S)$. However, this is only possible if $I = S$. □

3.7 Infinitesimally equivariant sheaves on supervarieties

From now on, $X = \underline{\mathrm{Spec}}(S)$ is a smooth integral affine supervariety with structure sheaf $\mathcal{O}$ and tangent sheaf Θ. An *Infinitesimally Equivariant* sheaf, or *infeq* sheaf for short, $\mathcal{M}$ over X is a sheaf on X of $\mathcal{O}$-modules and Θ-modules that satisfies the *Leibniz rule*

$$\eta \cdot (fm) = \eta(f)m + (-1)^{|\eta||f|} f(\eta \cdot m) \text{ for each } \eta \in \Gamma(U, \Theta),\ f \in B,\ m \in \Gamma(U, M)$$

for each affine open set $U = \underline{\mathrm{Spec}}(B) \subset X$. Any infeq sheaf $\mathcal{M}$ comes with a map $L : \Theta \to \mathrm{End}_{\Bbbk}(\mathcal{M})$ given by the action of Θ and it will be called the *Lie map*.

Suppose that X is affine. Let S denote the finitely generated integral commutative superalgebra such that $X = \underline{\mathrm{Spec}}(S)$. An *infinitesimally equivariant S-module*, or infeq S-module for short, M is a S-module M which is also a module over the Lie superalgebra $\mathrm{Der}(S)$ and satisfies the Leibniz rule. An S-infeq module is finite if it is a finitely generated S-module. We say that $V \subset M$ is an *infeq S-submodule* of M if V is a both a S-submodule of M and $\mathrm{Der}(S)$-submodule of M.

The main objective of this section is to prove that if M is a finite infinitesimally equivariant S-module, then the coherent sheaf $\tilde{M}$ on $X = \underline{\mathrm{Spec}}(S)$ is an infinitesimally equivariant sheaf.

We want to extend the main results of Chapter 2 to the super setting and we will use the calculations done there to achieve this.

Example 3.7.1. The module of Kahler differentials Ω^1_S is an infeq S-module as well as $\mathrm{Der}(S)$ and S. By Corollary 3.6.7, S is a simple infeq S-module, in the sense that if I is an ideal of S such that I is a $\mathrm{Der}(S)$-module, then $I = S$ or $I = 0$.

Example 3.7.2. If M is an infeq S-module, then the nth tensor product $\otimes^n_S M$ of M, the tensor algebra

$$\mathrm{T}_A(M) = A \oplus \bigoplus_{n=1}^{\infty} \otimes^n_S M,$$

$M^* = \mathrm{Hom}_{\Bbbk}(M, \Bbbk)$ and $M^\circ = \mathrm{Hom}_S(M, S)$ are infeq S-modules. The exterior n-power $\Lambda^n_S M$ and the exterior algebra

$$\Lambda^\bullet_S(M) = S \oplus \bigoplus_{n=1}^{\infty} \bigwedge_S^n M$$

are infeq S-modules as well. The exterior n-power $\Lambda^n_S M$ is the tensor n-power $\otimes^n_S M$ quotient by the S-submodule generated by

$$v_1 \otimes \cdots \otimes v_n - v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}, \quad v_1, \ldots, v_n \in M \text{ and } \sigma \in \mathfrak{S}_n,$$

where $\mathfrak{S}_n$ denotes the permutation group of $\{1, \ldots, n\}$.

Similarly to the non-super case, infeq S-modules are equivalent to modules over the smash product $S\#U(\mathrm{Der}(S))$.

Remark 3.7.3. The associative superalgebra $S\#U(\mathrm{Der}(S))$ is a superalgebra defined on the super space $S\otimes U(\mathrm{Der}(S))$ by the coproduct of $U(\mathrm{Der}(S))$. Explicitly, if $\Delta(u) = \sum_{(u)} u_{(1)}\otimes u_{(2)}$, then

$$(f \otimes u)(g \otimes v) = \sum_{(u)}(-1)^{|u_{(2)}||g|} f u_{(1)}(g) \otimes u_{(2)}v.$$

The commutator makes $S\#U(\mathrm{Der}(S))$ a Lie superalgebra, and the subspace $S\#\mathrm{Der}(S)$ is a Lie subalgebra of $S\#U(\mathrm{Der}(S))$ with the Lie bracket given by

$$[f\#\eta, g\#\mu] = f\eta(g)\#\mu - (-1)^{(|f|+|\eta|)(|g|+|\mu|)} g\eta(f)\#\mu + (-1)^{|\eta||g|} fg\#[\eta, \mu].$$

Lemma 3.7.4. *Let M be a finite infeq S-module. If $f \in S \setminus J_S$ and $m \in M$, then $fm = 0$ implies $f = 0$.*

Proof. The argument is similar to [BIN23, Lemma 4.2]. Denote by $\rho : \mathrm{Der}(S) \to \mathfrak{gl}_{\Bbbk}(M)$ the $\mathrm{Der}(S)$-representation associated to M. We will use the same notation for the associated map $U(\mathrm{Der}(S)) \to \mathrm{End}_{\Bbbk}(M)$. Let $V = \{m \in| \exists f \in S \setminus J_S$ such that $fm = 0\}$. Take $m \in M$ and $f \notin J_S$ such that $fm = 0$. We have that $f^2 \notin J_S$ and

$$f^2\rho(\eta)(m) = -\eta(f^2)m + \rho(\eta)(f^2 m) = -2\eta(f)fm = 0$$

for every $\eta \in \mathrm{Der}(S)$. Thus, $\rho(\eta)(m) \in V$ for every $\eta \in M$.

Since M is finitely generated, V is finitely generated. Suppose V is generated by $v_1, \dots, v_l$ and $f_1, \dots, f_l \in S \setminus J_S$ satisfy $f_i v_i = 0$ for each $i = 1, \dots, l$. Take $f = f_1 \cdots f_l$, then $fV = 0$. As we saw in Example 3.7.1, S is a simple infeq S-module, thus there exists $\sum_{i=1}^{k} g_i\#u_i \in S\#U(\mathrm{Der}(S))$ such that $\left(\sum_{i=1}^{k} g_i\#u_i\right) f = 1$. Therefore, for every $v \in V$,

$$0 = \sum_{i=1}^{k} g_i\rho(u_i)(fv) = \sum_{i=1}^{k} (gu_i(f))\, v + (-1)^{|u_i||f|} g_i f\rho(u_i)(v) = \left(\sum_{i=1}^{k} gu_i(f)\right) v = v$$

$\square$

Note that $S \otimes S$ is a superalgebra as well. Let $\delta : S \to S \otimes S$ be the map defined $\delta(f) = 1 \otimes f - f \otimes 1$ for each $f \in S$. For each $f \in S$, we define

$$\Omega_p(f, \eta) = \delta(f)^p \eta \in S\#U\,(\mathrm{Der}(S))$$

In particular, if $f \in S \setminus J_S$, then

$$\Omega_p(f, \eta) = \sum_{k=0}^{p}(-1)^l \binom{p}{k} f^{p-k}\#f^k\eta \in S\#\mathrm{Der}(S) \subset S\#U\,(\mathrm{Der}(S))\,.$$

We want to show that there exists N such that $\Omega_p(f, \eta) = 0$ for every $p > N$ as an endomorphism of a finite infeq S-module. For each infeq S-module, we denote the annihilator of M as the set

$$\mathrm{Ann}(M) = \{u \in S\#U(\mathrm{Der}(S)) \mid uM = 0\}.$$

Lemma 3.7.5. *Let $f \in S_{\bar{0}}$, $\eta, \mu \in \mathrm{Der}(S)$, then*

$$[\Omega_p(f,\eta), \Omega_q(f,\mu)] = \Omega_{p+q}(f,[\eta,\mu]) + (-1)^{|\eta||\mu|} p\Omega_{p+q-1}(f,\mu(f)\eta) - q\Omega_{p+q-1}(f,\eta(f)\mu).$$

Additionally, $\Omega_p(f,\eta)(g\#1) = (-1)^{|g||\eta|}(g\#1)\Omega_p(f,\eta)$ for every $g \in S$. Furthermore, if η is even and $g, h \in S$ then

1. $\left[\Omega_p(f,\eta), \Omega_q(f,g\mu)\right] - \left[\Omega_p(f,g\eta), \Omega_q(f,\mu)\right] = \Omega_{p+q}\left(f, \eta(g)\mu + (-1)^{|\mu||g|}\mu(g)\eta\right);$
2. $\left[\Omega_p(f,\eta), \Omega_q(f,gh\eta)\right] - \left[\Omega_p(f,g\eta), \Omega_q(f,h\eta)\right] = \Omega_{p+q}(f,\eta(g)h\eta);$

Proof. The proof follows the same steps as in Lemma 2.2.2, Lemma 2.2.4 and Lemma 2.2.5, □

Proposition 3.7.6. *Let M be a finite infeq S-module and $f \in S \setminus (\Bbbk + J_S)$, then there exists $\eta \in \mathrm{Der}(S)_{\bar{0}}$ with $\eta(f) \neq 0$ and $N > 0$ such that $\Omega_p(f,\eta) \in \mathrm{Ann}(M)$ for all $p > N$.*

Proof. Let r be the rank of M. By Lemma 3.6.3, there exists $\mu \in \mathrm{Der}(S)_{\bar{0}}$ such that $\mu(f)^k \neq 0$ for all $k > 0$. By Lemma 3.7.5, we may see each $\Omega_k(\eta,\mu)$ as an element of $\mathfrak{gl}_S(M)$. Since M is finitely generated as an A-module, there exists $a_1, \dots, a_{r^2}, a_{r^2+1} \in S$ with $a_{r^2+1} \neq 0$ such that

$$\sum_{i=1}^{r^2+1} a_i \Omega_{p_i}(f,\mu) \in \mathrm{Ann}(M),$$

where $p_i = i$ if $i \leq r^2$ and $p_i = p > r^2 + 1$. Since both μ and f are even, we may apply the proof of Lemma 2.3.3 to get that $a_{p_{r^2+1}}\Omega_p(f,\mu(f)^{r^2}\mu) \in \mathrm{Ann}(M)$ for every $p > N$, where N depends solely on the rank of M. Since $\mu(f)^{r^2}(f) \neq 0$, we have that $\Omega_p(f,\mu(f)^{r^2}\mu) \in \mathrm{Ann}(M)$ by Lemma 3.7.4. □

Proposition 3.7.7. *Let M be a finite infeq S-module, $f \in S \setminus (\Bbbk + J_S)$. Let $\eta \in \mathrm{Der}(S)_{\bar{0}}$ with $\eta(f) \neq 0$ and $N > 0$ such that $\Omega_p(f,\eta) \in \mathrm{Ann}(M)$ for all $p > N$. If $g, h \in S$, then $\Omega_k(f,q\tau) \in \mathrm{Ann}(M)$ for every $\tau \in \mathcal{V}$, $k > 3N+4$ and $q \in (\eta(g)\eta(\eta(h)))$.*

Proof. Note that η is even. By Lemma 3.7.5,

$$[\Omega_{N+1}(f,\eta), \Omega_{N+k}(f,g\eta)] - [\Omega_{N+1}(f,g\eta), \Omega_{N+k}(f,\eta)] = \Omega_{2N+k+1}(f,\eta(g)\eta) \in \mathrm{Ann}(M)$$

for all $k > 1$ and $g \in S$. Therefore, by Lemma 3.7.5 again,

$$\begin{aligned}&[\Omega_{N+1}(f,\eta), \Omega_{2N+k+1}(f,g\eta(h)\eta)] - [\Omega_{N+1}(f,g\eta), \Omega_{2N+k+1}(f,\eta(h)\eta)]\\ =&\Omega_{3N+k+2}(f,\eta(g)\eta(h)\eta) \in \mathrm{Ann}(M)\end{aligned}$$

for all $k \geq 1$ and $g, h \in S$. We conclude that

$$\Omega_{3N+k+2}(f,\eta(g\eta(h))\eta) - \Omega_{3N+k+2}(f,\eta(g\eta(h))\eta) = \Omega_{3N+k+2}(f,\eta(g\eta(h))\eta) \in \mathrm{Ann}(M)$$

for all $k \geq 1$.

Let $p \in S$ and set $q = p\eta(g)\eta(\eta(h))$, then

$$\begin{aligned}
&[\Omega_{3N+2+k}(f, p\eta(\eta(h))\eta), \Omega_l(f, g\tau)] - [\Omega_{3N+2+k}(f, gp\eta(\eta(h))\eta), \Omega_l(f, \tau)] \\
&\quad - (-1)^{|g||\tau|}\Omega_{3N+2+k+l}(f, \tau(g)p\eta(\eta(h))\eta) \\
=&\Omega_{3N+2+k+l}(f, p\eta(\eta(h))\eta(f)\tau + (-1)^{|g||\tau|}\tau(g)p\eta(g)\eta(\eta(h))\tau) \\
&\quad - (-1)^{|g||\tau|}\Omega_{3N+2+k+l}(f, \tau(g)p\eta(\eta(h))\eta) \\
=&\Omega_{3N+2+k+l}(f, q\tau) \in \mathrm{Ann}(M).
\end{aligned}$$

We conclude that $\Omega_{3N+3+k}(f, q\tau) \in \mathrm{Ann}(M)$ for every q in the principal ideal generated by $\eta(g)\eta(\eta(h))$ and $k \geq 1$. □

Lemma 3.7.8. *Let M be a finite infeq S-module and $f \in S$. If $f \in \Bbbk + J_S$, then there exists N that depends on S such that $\Omega_p(f, \eta) \in \mathrm{Ann}_S(M)$ for each $p > N$.*

Proof. If $f \in J_S$, then f is nilpotent and its degree of nilpotency depends solely on the odd dimension of X, which is fixed. Thus, $\Omega_q(f, \tau) = 0$ for every $\tau \in \Theta$ and q greater than some number that depends on S.

On the other hand, assume $f \in \Bbbk + J_S$. Without loss of generality, we may assume that $f = 1 + g$ with $g \in J_S \cap S_{\bar{0}}$. Since g is nilpotent and

$$\delta(f)^k = \delta(1+g)^k = \sum_{l=0}^{k} \binom{k}{l} \delta(1)^l \delta(g)^{k-l},$$

thus $\Omega_q(f, \tau) = 0$ if q is greater than some number that depends on S for every $\tau \in \mathcal{V}$. □

Similar to what we have done for the non-super case, for an ideal I of S, we define $I^{(0)} = I$ and $I^{(k)}$ as the ideal of S generated by $\{g, \mu(g) \mid g \in I^{(k-1)},\ \mu \in \mathcal{V}\}$.

Lemma 3.7.9. *Let M be a infeq S-module, and $f \in S_{\bar{0}} \setminus (\Bbbk + J_S)$. Suppose that I is an ideal of A such that $\Omega_p(f, q\tau) \in \mathrm{Ann}(M)$ for every $p > N$ for some $N > 0$. Then for each $p > N + k$, $\Omega_p(f, g\tau) \in \mathrm{Ann}(M)$ for all $g \in I^{(k)}$ and $\tau \in \mathcal{V}$.*

Proof. By an argument close to Lemma 2.2.6, we have that

$$[\Omega_p(f, \eta), 1\#\mu] = \Omega_p(f, [\eta, \mu]) + p(-1)^{|\eta||\mu|}\Omega_{p-1}(f, \mu(f)\eta) - (-1)^{|\eta||\mu|}\mu(f)\Omega_{p-1}(f, \eta)$$

for every $\eta, \mu \in \mathrm{Der}(S)$. Therefore, as endomorphisms of M,

$$\begin{aligned}
0 &= [\Omega_{p+1}(f, g\tau), 1\#\mu] \\
&= \Omega_{p+1}(f, [g\tau, \mu]) + (-1)^{(|g|+|\tau|)|\mu|}(p+1)\Omega_p(f, \mu(f)g\tau) - (-1)^{|\eta||\mu|}(p+1)\mu(f)\Omega_p(f, g\tau) \\
&= -(-1)^{(|g|+|\tau|)|\mu|}\Omega_{p+1}(f, \mu(g)\tau) + \Omega_{p+1}(f, g[\tau, \mu]) \\
&= -(-1)^{(|g|+|\tau|)|\mu|}\Omega_{p+1}(f, \mu(g)\tau)
\end{aligned}$$

for every $g \in I$, $\mu, \tau \in \mathcal{V}$, and $p > N$. Thus, $\Omega_{p+1}(f, \mu(g)\tau) \in \operatorname{Ann}(M)$ for every $g \in I$ and $\mu \in \mathcal{V}$.

Furthermore, for every $g \in I$ and $h \in A$, we have that $gh \in I$ and

$$\Omega_p(f, h\mu(g)\tau) = \Omega_p(f, \mu(gh)\tau) - (-1)^{|\mu||h|}\Omega_p(f, g\mu(h)\tau) \in \operatorname{Ann}(M).$$

Hence, for every $g \in I^{(1)}$, we have that $\Omega_p(f, g\tau) \in \operatorname{Ann}(M)$ for every $p > N + 1$. Since $I^{(k)} = (I^{(k-1)})^{(1)}$, we conclude by induction that $\Omega_p(f, g\tau) \in \operatorname{Ann}(M)$ for every $p > N + k$, $\tau \in \mathcal{V}$ and $g \in I^{(k)}$. □

Theorem 3.7.10. *Let M be a finite infeq S-module, and $f \in S$. Then there exists N_f, that depends on f, such that $\Omega_p(f, \eta) \in \operatorname{Ann}(M)$ for each $p > N_f$, and $\eta \in \mathcal{V}$.*

Proof. If $f \in \Bbbk 1 + J_S$, then $\Omega_p(f, \eta) = 0$ for all p greater than a number that depends on the odd dimension of S by Lemma 3.7.8. Suppose $f \notin \Bbbk + J_S$. By Proposition 3.7.6, there exists $\eta \in \operatorname{Der}(S)_{\bar{0}}$ with $\eta(f) \neq 0$ and $N > 0$ such that $\Omega_p(f, \eta) \in \operatorname{Ann}(M)$ for all $p > N$. Since $\eta(f) \neq 0$, there exists $g \in \{f, f^2\}$ such that $\eta(f)\eta(\eta(g)) \neq 0$. By Proposition 3.7.7, $\Omega_p(f, q\tau) \in \operatorname{Ann}(M)$ for every $q \in I = (\eta(f)\eta(\eta(g))) \neq 0$, $\tau \in \mathcal{V}$ and $p > 3N + 4$. Since S is Noetherian and

$$I \subset I^{(1)} \subset I^{(2)} \subset \cdots$$

is an ascending chain of ideals of A, we have that $I^{(k)} = I^{(l)}$ for every $l \geq k$ for some $k \geq 1$. Hence, $I^{(k)}$ is an infeq S-submodule of S. But S is a simple infeq S-module, thus $I^{(k)} = S$ by Corollary 3.6.7. By Lemma 3.7.9, for every $g \in I^{(k)}$ and $p > 3N + 4 + k$, $\Omega_p(f, g\tau) \in \operatorname{Ann}(M)$ for every $\tau \in \mathcal{V}$. In particular, $\Omega_p(f, \tau) \in \operatorname{Ann}(M)$ for each $p > N_f$ where $N_f = 3N + 4 + k$. □

For each $\eta \in \operatorname{Der}(S)$, set $\Omega_0(f, \eta) = 1\#\eta$.

Corollary 3.7.11. *Let M be a finite infeq S-module, and $f \in S \setminus J_S$. Then,*

$$\sum_{i=0}^{\infty} \frac{1}{f^{k(p+1)}}\Omega_p(f, \eta) \tag{3.4}$$

is a well-defined endomorphism of M_f for every $\eta \in \operatorname{Der}(S)$ and $k \geq 1$.

Proof. As an endomorphism of M_f,

$$\sum_{i=0}^{\infty} \frac{1}{f^{p+1}}\Omega_p(f, \eta) = \sum_{i=0}^{N_f} \frac{1}{f^{p+1}}\Omega_p(f, \eta)$$

converges to a well-defined map of M_f, where $\Omega_0(f, \eta) = 1\#\eta$, $\eta \in \operatorname{Der}(S)$, $f \in S_{\bar{0}} \setminus J_S$ and N_f is given by Theorem 3.7.10. □

Proposition 3.7.12. *Let M be a finite infeq S-module with associated $\operatorname{Der}(S)$-representation $\rho : \operatorname{Der}(S) \to \mathfrak{gl}_{\Bbbk}(M)$, and $f \in S \setminus J_S$. Then, there exists a representation $\rho_f : \operatorname{Der}(S)_f \to$*

$\mathfrak{gl}_{\Bbbk}(M)$ *such that* $\rho_f|_{\mathrm{Der}(S)} = \rho$ *and*

$$\rho_f\left(\frac{\eta}{f^k}\right) = \sum_{i=0}^{\infty} \frac{1}{f^{k(p+1)}}\Omega_p(f,\eta) = \sum_{p=0}^{\infty}\sum_{l=0}^{p}(-1)^l\binom{p}{l}\frac{1}{f^{kl+k}}\rho(f^{kl}\eta)$$

for each $\eta \in \mathrm{Der}(S)$ *and* $k \geq 1$

Proof. We want to use the representation $\rho : \mathrm{Der}(S) \to \mathfrak{gl}_{\Bbbk}(M)$ and Corollary 3.7.11 to define a representation $\rho_f : \mathrm{Der}(S)_f \to \mathfrak{gl}_{\Bbbk}(M_f)$ of $\mathrm{Der}(S)_f \cong \mathrm{Der}(S_f)$. If $\eta \in \mathrm{Der}(S) \subset \mathrm{Der}(S)_f$, then we set

$$\rho_f(\eta)\left(\frac{m}{f^k}\right) = -k\frac{\eta(f)m}{f^{k-1}} + \frac{1}{f^k}(\rho(\eta)m)$$

for each $m \in M$. Because $f \in S_{\bar{0}} \setminus J_S$ is even, Lemma 2.4.2 is true in this context. Therefore the map ρ_f given by

$$\rho_f\left(\frac{\eta}{f^k}\right) = \sum_{i=0}^{\infty} \frac{1}{f^{k(p+1)}}\Omega_p(f,\eta) = \sum_{p=0}^{\infty}\sum_{l=0}^{p}(-1)^l\binom{p}{l}\frac{1}{f^{kl+k}}\rho(f^{kl}\eta)$$

is a well-defined map from $\mathrm{Der}(S)_f$.

It remains to show that $\left[\rho_f(\eta), \rho_f(\mu)\right] = \rho_f([\eta,\mu])$ for every $\eta, \mu \in \mathrm{Der}(S)_f$. This holds for elements of $\mathrm{Der}(S) \subset \mathrm{Der}(S)_f$, hence if we show that

$$\left[\rho_f\left(\frac{\eta}{f}\right), \rho_f\left(\frac{\mu}{f}\right)\right] = \rho_f\left(\left[\frac{\eta}{f}, \frac{\mu}{f}\right]\right)$$

for every $\eta, \mu \in \mathrm{Der}(S)_f$, then the result follows by recursion. Since f is even, Lemma 2.4.3 holds in this context as written. Thus, by this lemma and Lemma 3.7.5,

$$\begin{aligned}
\left[\rho_f\left(\frac{\eta}{f}\right), \rho_f\left(\frac{\mu}{f}\right)\right] &= \left[\sum_{p=0}^{\infty}\frac{1}{f^{p+1}}\Omega_p(f,\eta), \sum_{p=0}^{\infty}\frac{1}{f^{p+1}}\Omega_p(f,\mu)\right] \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{f^{k+l+2}}\left[\Omega_k(f,\eta), \Omega_p(f,\mu)\right] \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{f^{k+l+2}}\left((-1)^{|\eta||\mu|}k\Omega_{k+l-1}(f,\mu(f)\eta) - l\Omega_{k+l-1}(f,\eta(f)\mu) + \Omega_{k+l}(f,[\eta,\mu])\right) \\
&= \sum_{u=0}^{\infty}\left((-1)^{|\eta||\mu|}\frac{(u+1)(u+2)/2}{f^{u+3}}\Omega_u(f,\mu(f)\eta) - \frac{(u+1)(u+2)/2}{f^{u+3}}\Omega_u(f,\eta(f)\mu)\right) \\
&\quad + \sum_{u=0}^{\infty}\frac{u+1}{f^{u+2}}\Omega_u(f,[\eta,\mu]) \\
&= \sum_{u=0}^{\infty}\left((-1)^{|\eta||\mu|}\frac{(u+1)(u+2)/2}{f^{u+3}}\Omega_u(f,\mu(f)\eta) - \frac{(u+1)(u+2)/2}{f^{u+3}}\Omega_u(f,\eta(f)\mu)\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{[\eta, \mu]}{f^2} \\
=&(-1)^{|\eta||\mu|}\frac{\mu(f)}{f^3}\eta - \frac{\eta(f)}{f^3}\mu + \frac{[\eta,\mu]}{f^2} = \left[\frac{\eta}{f}, \frac{\mu}{f}\right]
\end{aligned}
$$

as endomorphisms of M_f. Therefore, ρ_f is indeed a representation of $\mathcal{V}_f$. □

Theorem 3.7.13. *If $X = \underline{\mathrm{Spec}}\,(S)$ is an affine smooth integral supervariety and M be a finite infeq S-module, then $\tilde{M}$ is an infinitesimally equivariant sheaf on X. In particular, for every basic open set $D(f) \subset X$, $f \in S \setminus J_S$,*

$$\left(1\#\frac{\eta}{f^k}\right) v = \sum_{p=0}^{\infty} \frac{1}{f^{k(p+1)}}\Omega_p(f,\eta)$$

for every $v \in M_f$ and $\eta \in \mathrm{Der}(S)_f$.

Proof. Let $f \in S \setminus J_S$. By Proposition 3.7.12, there is a presentation $\rho_f : \mathrm{Der}(S) \to \mathfrak{gl}_{\Bbbk}(M_f)$ such that $\rho_f|_{\mathrm{Der}(S)} = \mathrm{Der}(S)$ and

$$\rho_f\left(\frac{\eta}{f^k}\right) = \sum_{i=0}^{\infty} \frac{1}{f^{k(p+1)}}\Omega_p(f,\eta) = \sum_{p=0}^{\infty}\sum_{l=0}^{p}(-1)^l\binom{p}{l}\frac{1}{f^{kl+k}}\rho(f^{kl}\eta).$$

It remains to show that $\rho_f(\eta)(gv) = \eta(g)v + (-1)^{|\eta||g|}g\rho_f(\eta)(v)$ for every $\eta \in \mathrm{Der}(S)_f$, $v \in M_f$ and $g \in S$. For every $\eta \in \mathrm{Der}(S)_f$ and $g \in S$,

$$
\begin{aligned}
\rho_f\left(\frac{\eta}{f}\right)(gv) &= \sum_{p=0}^{\infty}\frac{1}{f^{p+1}}\Omega_p(f,\eta)(g\#1)v \\
&= \left(\left(\frac{1}{f}\#\eta\right)(g\#1) + \sum_{p=1}^{\infty}\frac{1}{f^{p+1}}\Omega_p(f,\eta)(g\#1)\right)v \\
&= \left(\frac{\eta(g)}{f}\#1 + (-1)^{|\eta||g|}(g\#1)\left(\frac{1}{f}\#\eta\right) + (-1)^{|\eta||g|}\sum_{p=1}^{\infty}\frac{1}{f^{p+1}}(g\#1)\Omega_p(f,\eta)\right)v \\
&= \frac{\eta(g)}{f}v + (-1)^{|\eta||g|}g\rho_f\left(\frac{\eta}{f}\right)(v),
\end{aligned}
$$

hence ρ_f turns M_f an infeq S-module.

Let $f, g \in S \setminus J_S$ and $\eta, \mu \in \mathrm{Der}(S)$ such that $\frac{\eta}{f^k} = \frac{\mu}{g^l}$. Take $h = f^k g^l \in S \setminus J_S$, then $\frac{g^l\eta}{h} = \frac{f^k\mu}{h}$ and $h^c(hf^k\mu - hg^l\eta) = 0$ for some $c > 0$. Since $h^{c+1} \in S_{\bar{0}} \setminus J_S$, we have that $g^l\eta = f^k\mu$ by Lemma 3.7.4. Therefore, as an endomorphism of M_h,

$$\rho_g\left(\frac{\mu}{g^l}\right) = \rho_h\left(\frac{f^k\mu}{h}\right) = \sum_{p=0}^{\infty}\frac{1}{h^{p+1}}\sum_{a=0}^{p}(-1)^a\binom{p}{a}h^{p-a}\#h^a f^k\mu$$

$$= \sum_{p=0}^{\infty} \frac{1}{h^{p+1}} \sum_{a=0}^{p} (-1)^a \binom{p}{a} h^{p-a} \# h^a g^l \eta$$

$$= \rho_h \left(\frac{g^l \eta}{h} \right) = \rho_f \left(\frac{\eta}{f^l} \right).$$

We conclude that $\rho : \mathrm{Der}(S) \to \mathfrak{gl}_{\Bbbk}(M)$ sheafifies to $\rho : \Theta_X \to \mathfrak{gl}_{\Bbbk}(\tilde{M})$ with $\rho_{D(f)} = \rho_f$. □

3.8 Infinitesimally equivariant sheaves are differential operators

We wish to prove that the Lie map $L : \Theta \to \mathrm{End}_{\Bbbk}(\mathcal{M})$ of an infinitesimally equivariant sheaf $\mathcal{M}$ is a differential operator. Let S be a commutative superalgebra. Recall that a map $M \to N$ between two S-modules is a differential operator if it is an element of

$$\mathrm{Diff}(M, N) = \bigcup_{n \geq 0} \mathrm{Diff}^n_S(M, N),$$

where $\mathrm{Diff}^0(M, N) = \mathrm{Hom}_S(M, N)$ and

$$\mathrm{Diff}^{n+1}(M, N) = \{D \in \mathrm{Hom}_{\Bbbk}(M, N) \mid [D, f] \in \mathrm{Diff}^n(M, N)\ \forall f \in S\}.$$

If $D \in \mathrm{Diff}^n(M, N)$, we say that D is a differential operator of order less or equal to n. In this case,

$$D(fm) - (-1)^{|f||D|} f(D(m))$$

is a differential operator of order less or equal to $n - 1$ for each $f \in S$. Note that each element of $\mathrm{Der}(S)$ is a differential operator $S \to S$ of order less or equal to 1 because

$$D(f) = D \circ f - (-1)^{|f||D|} f \circ D \quad \text{for each } f \in S,\ D \in \mathrm{Der}(S).$$

A map $\mathcal{M} \to \mathcal{N}$ between the vector bundles $\mathcal{M}$ and $\mathcal{N}$ on X is a differential operator of order less or equal to n if and only if sections $\Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N})$ are differential operators for each affine open set $U \subset X$.

Since being a differential operator is a local property, we may assume that $X = \underline{\mathrm{Spec}}\,(S)$ is affine.

For a linear map $T \in \underline{\mathrm{Hom}}_{\Bbbk}(\mathrm{Der}(S), \mathrm{End}(M))$, we define

$$\delta(f_1) \cdots \delta(f_k) T(\eta) m = \delta(f_1) \cdots \delta(f_k)(1 \# \eta) m$$

for each $m \in M$, $\eta \in \mathrm{Der}(S)$, $f_1, \ldots, f_k \in S$, where $\delta(f) = 1 \otimes f - f \otimes 1$. Note that $\delta(f) T(\eta) = T(f\eta) - (-1)^{|f||L|} f T(\eta)$.

Lemma 3.8.1. $L : \mathrm{Der}(S) \to \mathrm{End}(M)$ *is a differential operator of order less or equal to* k *if and only if* $\delta(f_1) \cdots \delta(f_{k+1}) L(\eta) = 0$ *for each* $f_1, \ldots f_{k+1} \in S$ *and* $\eta \in \mathrm{Der}(S)$.

Proof. We will prove this lemma by induction on the order of the differential operator. L

will be a differential operator of order less or equal to 0 if and only if

$$0 = L(f\eta)m - fL(\eta)m = (1\#f\eta - f\#\eta)m = \delta(f)(1\#\eta)m$$

for each $f \in S$, $\eta \in \mathrm{Der}(S)$, and $m \in M$. Assume that L is a differential operator of order k, then $L_f(\eta) = \delta(f)(\eta) = L(f\eta) - fL(\eta)$ defines a differential operator of order less or equal to $k-1$. By induction, L_f is a differential operator of order less or equal to $k-1$ if and only if there exist $f_1, \dots, f_k \in S$ such that

$$0 = \delta(f_1)\cdots\delta(f_k)L_f(\eta) = \delta(f_1)\cdots\delta(f_k)\delta(f)L(\eta)$$

for each $\eta \in \mathrm{Der}(S)$. Thus, the lemma follows. □

Example 3.8.2. Similar to Example 2.3.1, the adjoint representation $\mathrm{ad} : \mathrm{Der}(S) \to \mathrm{End}(\mathrm{Der}(S))$ is a differential operator of degree less or equal to 1 since

$$\begin{aligned}
&\delta(g)\delta(f)\mathrm{ad}(\eta)(\mu)\\
&= \left(\mathrm{ad}(gf\eta) - g\mathrm{ad}(f\eta) - f\mathrm{ad}(g\eta) + (-1)^{|f||g|}gf\mathrm{ad}(\eta)\right)(\mu)\\
&=[gf\eta,\mu] - g[f\eta,\mu] - f[g\eta,\mu] + (-1)^{|f||g|}gf[\eta,\mu]\\
&= -(-1)^{(|g|+|f|+|\eta|)|\mu|}\mu(gf)\eta + gf[\eta,\mu] + (-1)^{(|f|+|\eta|)|\mu|}g\mu(f)\eta - gf[\eta,\mu]\\
&\quad + (-1)^{(|g|+|\eta|)\mu}f\mu(g)\eta - fg[\eta,\mu] + (-1)^{|f||g|}gf[\eta,\mu]\\
&=(-1)^{(|g|+|f|+|\eta|)|\mu|}\left(-\mu(fg)\eta + \mu(f)g\eta + (-1)^{|g||\mu|}f\mu(g)\eta\right) = 0
\end{aligned}$$

for every $f, g \in S$ and $\eta, \mu \in \mathrm{Der}(S)$ by (3.1).

Lemma 3.8.3. *Let M be an infeq S-module and $f \in S \setminus J_S$. Assume that there exist $\eta \in \mathrm{Der}(S)$ with $\eta(f) = 1$. If there exists N such that $\Omega_k(f,\eta) \in \mathrm{Ann}(M)$ for every $k > N$, then $\Omega_k(f,\tau) \in \mathrm{Ann}(M)$ for every $k > 3N+4$ and $\tau \in \mathrm{Der}(S)$.*

Proof. The argument is analogous to the one given on Proposition 2.3.6. Note that we may assume η homogeneous. In this case, η must be an even element, since $\eta(f) = 1$ and $f \in S_{\overline{0}}$. Using Lemma 3.7.5 and the assumption that $\eta(f) = 1$, we get that $\Omega_k(f,\eta) \in \mathrm{Ann}(M)$ for every $k \geq N$.

We apply Lemma 3.7.5 again to get that $\Omega_p(f, \eta(r)\eta) \in \mathrm{Ann}(M)$ for $p > 2N+1$ and for every $r \in S$. Similarly, $\Omega_p(f, \eta(r)\eta(s)\eta) \in \mathrm{Ann}(M)$ for $p > 3N+2$ by taking $g = r$, $h = \eta(s)$ on Lemma 3.7.5. Since

$$\Omega_p(f, g\eta(\eta(h))\eta) = \Omega_p(f, \eta(g\eta(h))\eta) - \Omega_p(f, \eta(g)\eta(h)\eta),$$

we have that $\Omega_p(f, r\eta(\eta(s))\eta) \in \mathrm{Ann}(M)$ for every $p > 3N+2$ and $r \in S$. Hence, $\Omega_k(f,\tau) \in \mathrm{Ann}(M)$ for every $k > 3N+4$ because

$$\begin{aligned}
&[\Omega_p(f, \eta(\eta(f^2))\eta), \Omega_q(f, f\tau)] - [\Omega_p(f, f\eta(\eta(f^2))\eta), \Omega_q(f,\tau)]\\
&- (-1)^{|\tau||g|}\Omega_{p+q}(f, \tau(f)\eta(\eta(f^2))\eta)
\end{aligned}$$

$$=\Omega_{p+q}(f, \eta(f)\eta(\eta(f^2))\tau) = 2\Omega_{p+q}(f, \tau)$$

by Lemma 3.7.5 for $p > 3N + 2$ and $q \geq 1$. □

Lemma 3.8.4. *Let M be a finite infeq S-module and $f \in S \setminus J_S$. If there exist $\eta \in \mathrm{Der}(S)$ such that $\eta(f) = 1$, then there exists N that depends on the rank of M such that $\Omega_p(f, \tau) \in \mathrm{Ann}_S(M)$ for each $p > N$ and $\tau \in \mathrm{Der}(S)$.*

Proof. Let $f \in S_{\bar{0}} \setminus J_S$ with $f + J_S \notin \Bbbk + J_S$. Suppose $\mathrm{rank}_S(A) = a|b$ and take $\eta \in \mathrm{Der}(S)$ with $\eta(f) = 1$. Then,

$$[\Omega_1(f, \eta), \Omega_p(f, \eta)] = (1 - p)\Omega_p(f, \eta)$$

by Lemma 3.7.5. The element $\Omega_1(f, \eta)$ acts on $\mathrm{End}_S(M)$ by the superbracket of endomorphisms. With this action, $\{\Omega_p(f, \eta) \mid p = 1, \dots, a^2 + b^2 + 1\}$ is a set of even eigenvectors on $\mathrm{End}_S(M)$ for $\Omega_1(f, \eta)$ with distinct eigenvalues. Since the number of eigenvalues of an operator cannot exceed the dimension of the space, there exists $p \in \{1, \dots, a^2 + b^2 + 1\}$ such that $\Omega_p(f, \eta) \in \mathrm{Ann}(M)$, thus $\Omega_q(f, \eta) \in \mathrm{Ann}(M)$ for each $q > p$ since

$$[\Omega_k(f, \eta), \Omega_l(f, \eta)] = (l - k)\Omega_{k+l-1}(f, \eta).$$

By Lemma 3.8.3, $\Omega_q(f, \tau)$ for every $q > 3(a^2 + b^2) + 4$ and $\tau \in \mathrm{Der}(S)$. □

Lemma 3.8.5. *Let $N > 0$ be such that $\Omega_p(f, \tau) \in \mathrm{Ann}_S(M)$ for each $p > N$, $f \in S$ and $\tau \in \mathrm{Der}(S)$. Then,*

$$\Omega((f_1, \dots, f_p), \tau) \in \mathrm{Ann}(M)$$

for every $f_1, \dots, f_p \in S$ and for each $p > N + s$, where s is the odd dimension of S.

Proof. Suppose $\dim S = r|s$. Then the product of $s + 1$ homogeneous elements of J_S is zero, thus $\delta(f_1) \cdots \delta(f_{s+1})\eta = 0$ for each $f_1, \dots, f_{s+1} \in J_S$. Suppose $p \geq N$ and take $f_1, \dots, f_p \in S \setminus J_S$ then

$$\Omega_p\left(\sum_{i=1}^{p} a_i f_i, \tau\right) = \sum_{l_1+\cdots+l_p=p} \binom{p}{l_1, \dots, l_p} \delta(f_1)^{l_1} \cdots \delta(f_p)^{l_p}(1\#\tau) \in \mathrm{Ann}(M)$$

for every $a_1, \dots, a_p \in \Bbbk$. Similarly to Corollary 2.6.5, we may apply Lemma 2.6.4 to get that $\Omega((f_{i_1}, \dots, f_{i_p}), \tau)$ for every $1 \leq i_1, \dots, i_p \leq p$. Combining both cases, we get that $\Omega((g_1, \dots, g_p), \tau) \in \mathrm{Ann}(M)$ for every $p > N + s$ and $g_1, \dots, g_p \in S$. □

The main theorem of this section shows that, similarly to what was proved in the non-super case, infeq coherent sheaves are given by differential operators.

Theorem 3.8.6. *If $\mathcal{M}$ is an infeq coherent sheaf on a smooth supervariety X with Lie map $\rho : \Theta \to \mathrm{End}_{\Bbbk}(M)$, then ρ is a differential operator.*

Proof. Being a differential operator is a local property, so it is sufficient to prove the restriction $\rho|_U : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N})$ is a differential operator for each small enough affine open set $U = \underline{\mathrm{Spec}}(S)$ of X. By Lemma 3.8.1, we only need to prove there exists N such that $\Omega((f_1, \dots, f_p), \tau) \in \mathrm{Ann}(M)$ for every $f_1, \dots, f_p \in S$, $p > N$, and $\tau \in \Gamma(U, \Theta)$.

Take a closed point $x \in U$ and $f \in \Gamma(U, \mathcal{O})$ with $f(x) \neq 0$ and $f + J_S \notin \Bbbk + J_S$. Since X is smooth, there exist $\mu \in \mathrm{Der}(S)_{\bar{0}}$ such that $\mu(f)(x) \neq 0$. We may assume U is a small enough neighborhood of x to $\mu(f)$ to be invertible. Take $\eta = \frac{\mu}{\mu(f)} \in \mathrm{Der}(S)$ so $\eta(f) = 1$. Thus, the claim of the theorem follows by Lemma 3.8.4 and Lemma 3.8.5. □

3.9 Grassmann algebras and their infinitesimally equivariant modules

In the previous section, we studied infeq modules associated with affine supervarieties with dimensions greater than $1|0$. In this section, we investigate the case where the dimension is $0|n$. Take a finitely generated superalgebra S that is an integral superdomain with dimension $0|n$. The algebra S/J_S is both a finitely generated algebra and a field, thus S/J_S is an algebraic extension of the algebraically closed field $\Bbbk$. It follows from the weak Nullstellensatz theorem that $S/J_S \cong \Bbbk$, thus $X = \underline{\mathrm{Spec}}(S)$ is a point. Assuming that X is a smooth affine supervariety, there is a vector space V such that $S \cong \Lambda^{\bullet}_{\Bbbk} V$ by definition, i.e., S is a Grassmann algebra. Consequently, we may study solely the infinitesimally equivariant modules over a Grassmann algebra. The main objective of this section is to demonstrate that there is an equivalence of categories between infeq modules and modules over the Lie algebra of vector fields vanishing at the single point of X. To accomplish this, we will use an algebraic approach following the ideas presented in [BIN23].

Let $\Lambda(n)$ the Grassmann algebra in variables $\theta_1, \dots, \theta_n$. Explicitly, if $\Bbbk\langle\theta_1, \dots, \theta_n\rangle$ is the free associative algebra in the variables $\theta_1, \dots, \theta_n$, then

$$\Lambda(n) = \Bbbk\langle\theta_1, \dots, \theta_n\rangle / (\theta_i\theta_j + \theta_j\theta_i | i, j = 1, \dots, n).$$

The algebra $\Lambda(n)$ is a finite-dimensional associative algebra generated by the monomials of the form $\theta_1^{k_1} \cdots \theta_n^{k_n}$, $k_1, \dots, k_n \in \{0, 1\}$. It inherits a compatible $\mathbb{Z}$-grading from the free associative algebra given by the degree of the monomials, i.e.

$$\Lambda(n)_k = \mathrm{span}_{\Bbbk} \left\{ \theta_{i_1} \cdots \theta_{i_k} \mid i_1 < \dots i_k \right\}$$

for each $k \in \mathbb{Z}_+$. We will denote $\deg\left(\theta_1^{k_1} \cdots \theta_n^{k_n}\right) = k_1 + \cdots + k_n$. With this grading, $\Lambda(n)_k \Lambda(n)_l \subset \Lambda(n)_{k+l}$ for each $k, l \in \mathbb{Z}$, and $\Lambda(n)_r = 0$ for all $r > n$. Furthermore, $\dim_{\Bbbk} \Lambda(n)_k = \binom{n}{k}$ for each $k = 0, 1, \dots, n$, which implies that $\dim_{\Bbbk} \Lambda(n) = 2^n$. Its $\mathbb{Z}$-grading and its relations demonstrate that $\Lambda(n) = \Lambda(n)_{\bar{0}} \oplus \Lambda(n)_{\bar{1}}$ is a commutative superalgebra where

$$\Lambda(n)_{\bar{0}} = \mathrm{span}_{\Bbbk} \left\{ \theta_1^{k_1} \cdots \theta_n^{k_n} \mid k_i \in \{0, 1\},\ k_1 + k_2 + \cdots + k_n \in 2\mathbb{Z} \right\} = \bigoplus_{k \text{ is even}} \Lambda(n)_k,$$
$$\Lambda(n)_{\bar{1}} = \mathrm{span}_{\Bbbk} \left\{ \theta_1^{k_1} \cdots \theta_n^{k_n} \mid k_i \in \{0, 1\},\ k_1 + k_2 + \cdots + k_n \in 1 + 2\mathbb{Z} \right\} = \bigoplus_{k \text{ is odd}} \Lambda(n)_k.$$

We denote by $\mathrm{J}(n)$ the soul of $\Lambda(n)$, which is the ideal generated by its odd part $\Lambda(n)_{\bar{1}}$. We note that its body $\Lambda(n)/\mathrm{J}(n)$ is isomorphic to $\Bbbk$. Thus, $\mathrm{J}(n)$ is the only prime ideal on $\Lambda(n)$. Moreover, if $x \in \mathrm{J}(n)$, then x is nilpotent, and $1 - x$ is invertible. Therefore, every element of $\Lambda(n) \setminus \mathrm{J}(n)$ is invertible.

It is well-known that the Lie superalgebra of superderivations

$$\mathrm{W}(n) = \mathrm{Der}(\Lambda(n)) = \{D \in \mathrm{End}(\Lambda(n)) \mid D(ab) = D(a)b + (-1)^{|D||a|} aD(b)\}$$

is simple if $n \geq 2$ (see [Kac77]). Denote by $\dfrac{\partial}{\partial \theta_i} : \Lambda(n) \to \Lambda(n)$ the odd derivation of $\Lambda(n)$ given by

$$\frac{\partial}{\partial \theta_i}(\theta_j) = \delta_{ij}.$$

Then, $\mathrm{W}(n)$ can be realized as a free $\Lambda(n)$-module,

$$\mathrm{W}(n) = \bigoplus_{i=1}^{n} \Lambda(n) \frac{\partial}{\partial \theta_i}.$$

Therefore, $\dim(\mathrm{W}(n)) = n2^n$, and $\dim(\mathrm{W}(n)_{\bar{0}}) = \dim(\mathrm{W}(n)_{\bar{1}})$. The $\mathbb{Z}$-grading of $\Lambda(n)$ induces a $\mathbb{Z}$-grading on $\mathrm{W}(n)$ by

$$\mathrm{W}(n)_k = \left\{ \sum_{i=1}^{n} f_i \frac{\partial}{\partial \theta_i} \mid f_1, \dots, f_n \in \Lambda(n)_{k+1} \right\}.$$

This $\mathbb{Z}$-grading satisfies

$$\mathrm{W}(n) = \bigoplus_{k=-1}^{n-1} \mathrm{W}(n)_k, \quad [\mathrm{W}(n)_k, \mathrm{W}(n)_l] \subset W(n)_{k+l}.$$

The subspace $\mathrm{W}(n)_0$ is a subalgebra of $\mathrm{W}(n)$ and it is isomorphic to $\mathfrak{gl}_{\Bbbk}(n)$, with the isomorphism given by $\theta_i \dfrac{\partial}{\partial \theta_j} \mapsto E_{ij}$. Another important subalgebra of $\mathrm{W}(n)$ is

$$\mathrm{W}(n)_+ = \bigoplus_{k=0}^{n-1} \mathrm{W}(n)_k.$$

Alternatively, this subalgebra may be defined as $\mathrm{W}(n)_+ = \mathrm{J}(n)\mathrm{W}(n)$.

From now on, n is fixed, and we will denote $\Lambda(n)$, $\mathrm{W}(n)$, $\mathrm{W}(n)_+$, $\mathrm{J}(n)$ by Λ, W, W_+, and J, respectively. We denote by $\widehat{n}$ the set $\{1, \dots, n\}$.

3.9.1 Isomorphism of superalgebras

Consider the smash product $\Lambda \# U(\mathrm{W})$ of Λ with $U(\mathrm{W})$. It is an associative superalgebra, and its product comes from the coproduct of $U(\mathrm{W})$ and its action on Λ. As a vector space, $\Lambda \# U(\mathrm{W})$ is isomorphic to $\Lambda \otimes_{\Bbbk} U(\mathrm{W})$, and the following relation holds

$$(f \# \eta)(g \# \mu) = f\eta(g) \# \mu + (-1)^{|\eta||g|} fg \# \eta\mu \tag{3.5}$$

for each $f, g \in \Lambda$, and $\eta, \mu \in \mathrm{W}$. In this section, we will use the ideas introduced on [BIN23] to construct an isomorphism between $\Lambda \# U(\mathrm{W})$ and the tensor product of two other associative superalgebras.

The first associative superalgebra that will appear in the isomorphism is the algebra of differential operators. Consider $\mathrm{D} = \mathrm{D}(n)$, the associative subalgebra of $\mathrm{End}_{\Bbbk}(\Lambda(n))$ generated by $\Lambda(n)$ and $\mathrm{W}(n)$. The assignment $\theta_i \mapsto \xi_i$ and $\dfrac{\partial}{\partial \theta_j} \mapsto \partial_j$ defines an isomorphism of superalgebras

$$\mathrm{D}(n) \cong \Bbbk\langle \xi_1, \dots, \xi_n, \partial_1, \dots, \partial_n \rangle / \left(\xi_i \xi_j + \xi_j \xi_i,\ \partial_i \partial_j + \partial_j \partial_i,\ \partial_j \xi_i + \xi_i \partial_j - \delta_{ij} \right).$$

Thus, the set

$$\left\{ \theta_{i_1} \cdots \theta_{i_k} \frac{\partial}{\partial \theta_{j_1}} \cdots \frac{\partial}{\partial \theta_{j_l}} \mid i_1 < \cdots < i_k,\ j_1 < \cdots < j_k \right\}$$

is a basis of $\mathrm{D}(n)$, and its dimension is $2^{2n} = \dim \mathrm{End}(\Lambda)$. Therefore, $\mathrm{D} = \mathrm{End}_{\Bbbk}(\Lambda)$ as associative algebras, but we will keep its realization with the differential operators to define the isomorphism.

The second associative superalgebra will be the universal enveloping algebra of a Lie superalgebra isomorphic to $\mathrm{W}(n)_+$. Let $\mathrm{L} = \mathrm{L}(n)$ be the Lie superalgebra of superderivations of the Grassmann algebra in the variables $\Theta_1, \dots, \Theta_n$, and $\mathrm{L}_+ = \mathrm{L}(n)_+$ the subalgebra of derivations with non-negative degree. It is naturally isomorphic to W_+, but we will show that there is another Lie subalgebra of $\Lambda \# \mathrm{W}$ that is isomorphic to L_+.

To define the isomorphism, we will fix some notation. For a subset $P \subset \{i_1, \dots, i_k\} = I \subset \widehat{n}$, set $P = \{i_{r_1}, \dots, i_{r_a}\}$ with $r_1 < \cdots < r_a$, $P^c = I \setminus P = \{i_{s_1}, \dots, i_{s_{k-a}}\}$ with $s_1 < \cdots < s_{k-a}$. Moreover, we denote

$$s(P, I) = (-1)^{r_1 + \cdots + r_a - \frac{a(a+1)}{2}} = (-1)^{(k-a)k - s_1 - \cdots - s_{k-a} - \frac{(a-1)a}{2}},$$

$$\theta^P = \theta_{i_{r_1}} \cdots \theta_{i_{r_a}}, \quad \Theta^P = \Theta_{i_{r_1}} \cdots \Theta_{i_{r_k}}.$$

The definition above depends on the order of $i_1, \dots, i_k$. It may be useful to consider orders different from the natural order inherited from $\widehat{n}$. However, unless otherwise stated, we will take the inherited order $\widehat{n}$. With this notation,

$$\begin{aligned}
\Lambda(n) &= \mathrm{span}_{\Bbbk} \left\{ \theta^P \mid P \subset \widehat{n} \right\} \\
\mathrm{W}(n) &= \mathrm{span}_{\Bbbk} \left\{ \theta^P \frac{\partial}{\partial \theta_q} \mid q \in \widehat{n},\ P \subset \widehat{n} \right\} \\
\mathrm{L}(n)_+ &= \mathrm{span}_{\Bbbk} \left\{ \theta^P \frac{\partial}{\partial \theta_q} \mid q \in \widehat{n},\ \emptyset \neq P \subset \widehat{n} \right\}
\end{aligned}$$

If we assume that $P \subset I'$ for some $I' \subset I$, we define $P^c = I' \setminus P$.

Theorem 3.9.1. *The map*

$$\varphi : \Lambda \# U(\mathrm{W}) \to \mathrm{D} \otimes U(\mathrm{L}_+)$$

is an isomorphism of associative algebras, where $\varphi(f\#\eta) = (f \otimes 1)\varphi(\eta)$ *for every* $f \in \Lambda$, $\eta \in \mathrm{W}$, *and*

$$\varphi\left(1\#\theta^I \frac{\partial}{\partial\theta_p}\right) = \theta^I \frac{\partial}{\partial\theta_p} \otimes 1 + \sum_{P \subsetneq I} s(P,I)\theta^P \otimes \Theta^{P^c} \frac{\partial}{\partial\Theta_p}$$

for every $I \subset \{1,\dots,n\}$.

Explicitly, the definition of φ is given by

$$\begin{aligned}\varphi\left(\theta_{i_1}\cdots\theta_{i_k}\frac{\partial}{\partial\theta_j}\right) =& \theta_{i_1}\cdots\theta_{i_k}\frac{\partial}{\partial\theta_j} \\ &+ \sum_{\substack{0\le a<k \\ \{r_1,\dots,r_a,s_1,\dots,s_{k-a}\}=\{1,\dots,k\} \\ r_1<\cdots<r_l,\ s_1<\cdots<s_{k-l}}} (-1)^{r_1+\cdots+r_a+\frac{a(a+1)}{2}}\theta_{i_{r_1}}\cdots\theta_{i_{r_a}} \otimes \Theta_{i_{s_1}}\cdots\Theta_{i_{s_{k-a}}}\frac{\partial}{\partial\theta_j}\end{aligned}$$

This formula implies that

$$\begin{aligned}\varphi\left(\theta^I\theta^J\frac{\partial}{\partial\theta_q}\right) =& \theta^I\theta^J\frac{\partial}{\partial\theta_q}\otimes 1 + \sum_{Q\subsetneq J} s(Q,J)\theta^I\theta^Q \otimes \Theta^{Q^c}\frac{\partial}{\partial\Theta_q} \\ &+ \sum_{\substack{P\subsetneq I\\ Q\subset J}} (-1)^{|P^c||Q|} s(P,I)s(Q,J)\theta^P\theta^Q\otimes\Theta^{P^c}\Theta^{Q^c}\frac{\partial}{\partial\Theta_q}.\end{aligned}$$

for every $I, J \subset \{1,\dots,n\}$.

We will split the proof of Theorem 3.9.1 into several lemmas.

Lemma 3.9.2. *Let* $I, J \subset \{1,\dots,n\}$, *then*

$$\begin{aligned}\left[\varphi\left(\theta^I\frac{\partial}{\partial\theta_p}\right),\varphi\left(\theta^J\frac{\partial}{\partial\theta_q}\right)\right] =& \left[\theta^I\frac{\partial}{\partial\theta_p},\theta^J\frac{\partial}{\partial\theta_q}\right]\otimes 1 + \sum_{Q\subsetneq J} s(Q,J)\left[\theta^I\frac{\partial}{\partial\theta_p},\theta^Q\right]\otimes\Theta^{Q^c}\frac{\partial}{\partial\Theta_q} \\ &- (-1)^{(|I|+1)(|J|+1)}\sum_{P\subsetneq I} s(P,I)\left[\theta^J\frac{\partial}{\partial\theta_q},\theta^P\right]\otimes\Theta^{P^c}\frac{\partial}{\partial\Theta_p} \\ &+ \sum_{\substack{P\subsetneq I\\ Q\subsetneq J}} s(P,I)s(Q,J)(-1)^{(|P^c|+1)|Q|}\theta^P\theta^Q\otimes\left[\Theta^{P^c}\frac{\partial}{\partial\Theta_p},\Theta^{Q^c}\frac{\partial}{\partial\Theta_q}\right]\end{aligned}$$

Proof. Using the definition of the supercommutator and the product of $\mathrm{D}\otimes U(\mathrm{L}_+)$, we have

$$\begin{aligned}&\left[\varphi\left(\theta^I\frac{\partial}{\partial\theta_p}\right),\varphi\left(\theta^J\frac{\partial}{\partial\theta_q}\right)\right] \\ =& \left[\theta^I\frac{\partial}{\partial\theta_p},\theta^J\frac{\partial}{\partial\theta_q}\right]\otimes 1 \\ &+ \sum_{Q\subsetneq J} s(Q,J)\left(\left(\theta^I\frac{\partial}{\partial\theta_p}\right)\theta^Q - (-1)^{(|I|+1)(|J|+1)+(|I|+1)(|Q^c|+1)}\theta^Q\theta^I\frac{\partial}{\partial\theta_p}\right)\otimes\Theta^{Q^c}\frac{\partial}{\partial\Theta_q}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{P\subsetneq I} s(P,I)\left((-1)^{(|J|+1)(|P^c|+1)}\theta^P\theta^J\frac{\partial}{\partial\theta_q}-(-1)^{(|I|+1)(|J|+1)}\left(\theta^J\frac{\partial}{\partial\theta_q}\right)\theta^P\right)\otimes\Theta^{P^c}\frac{\partial}{\partial\Theta_p}\\
&+\sum_{\substack{P\subsetneq I\\ Q\subsetneq J}} s(P,I)s(Q,J)\theta^P\theta^Q\otimes((-1)^{(|P^c|+1)|Q|}\left(\Theta^{P^c}\frac{\partial}{\partial\Theta_p}\right)\left(\Theta^{Q^c}\frac{\partial}{\partial\Theta_q}\right)\\
&\qquad-(-1)^{|P||Q|+(|I|+1)(|J|+1)+|P|(|Q^c|+1)}\left(\Theta^{Q^c}\frac{\partial}{\partial\Theta_q}\right)\left(\Theta^{P^c}\frac{\partial}{\partial\Theta_p}\right))\\
=&\left[\theta^I\frac{\partial}{\partial\theta_p},\theta^J\frac{\partial}{\partial\theta_q}\right]\otimes 1\\
&+\sum_{Q\subsetneq J} s(Q,J)\left(\left(\theta^I\frac{\partial}{\partial\theta_p}\right)\theta^Q-(-1)^{(|I|+1)|Q|}\theta^Q\theta^I\frac{\partial}{\partial\theta_p}\right)\otimes\Theta^{Q^c}\frac{\partial}{\partial\Theta_q}\\
&+\sum_{P\subsetneq I} s(P,I)(-1)^{(|J|+1)(|P^c|+1)}\left(\theta^P\theta^J\frac{\partial}{\partial\theta_q}-(-1)^{|P|(|J|+1)}\left(\theta^J\frac{\partial}{\partial\theta_q}\right)\theta^P\right)\otimes\Theta^{P^c}\frac{\partial}{\partial\Theta_p}\\
&+\sum_{\substack{P\subsetneq I\\ Q\subsetneq J}} s(P,I)s(Q,J)(-1)^{(|P^c|+1)|Q|}\theta^P\theta^Q\otimes(\left(\Theta^{P^c}\frac{\partial}{\partial\Theta_p}\right)\left(\Theta^{Q^c}\frac{\partial}{\partial\Theta_q}\right)\\
&\qquad-(-1)^{(|P^c|+1)(|Q^c|+1)}\left(\Theta^{Q^c}\frac{\partial}{\partial\Theta_q}\right)\left(\Theta^{P^c}\frac{\partial}{\partial\Theta_p}\right))\\
=&\left[\theta^I\frac{\partial}{\partial\theta_p},\theta^J\frac{\partial}{\partial\theta_q}\right]\otimes 1+\sum_{Q\subsetneq J} s(Q,J)\left[\theta^I\frac{\partial}{\partial\theta_p},\theta^Q\right]\otimes\Theta^{Q^c}\frac{\partial}{\partial\Theta_q}\\
&-(-1)^{(|I|+1)(|J|+1)}\sum_{P\subsetneq I} s(P,I)\left[\theta^J\frac{\partial}{\partial\theta_q},\theta^P\right]\otimes\Theta^{P^c}\frac{\partial}{\partial\Theta_p}\\
&+\sum_{\substack{P\subsetneq I\\ Q\subsetneq J}} s(P,I)s(Q,J)(-1)^{(|P^c|+1)|Q|}\theta^P\theta^Q\otimes\left[\Theta^{P^c}\frac{\partial}{\partial\Theta_p},\Theta^{Q^c}\frac{\partial}{\partial\Theta_q}\right].
\end{aligned}
$$

□

Lemma 3.9.3. *φ is a homomorphism of associative superalgebras.*

Proof. We want to prove that the formulas provided can be used to extend $\varphi|_{\mathrm{W}}$ to $U(\mathrm{W})$. Thus, it is sufficient to prove that $(\mathrm{D}\otimes U(\mathrm{L}_+),\varphi)$ is an enveloping algebra for W, i.e. $\varphi|_{\mathrm{W}}:\mathrm{W}\to\mathrm{D}\otimes U(\mathrm{L}_+)$ is a homomorphism of Lie algebras. We have that $\varphi|_{\mathrm{W}}$ is a homomorphism of Lie algebras if and only if

$$
\varphi\left(\left[\theta^I\frac{\partial}{\partial\theta_p},\theta^J\frac{\partial}{\partial\theta_q}\right]\right)=\left[\varphi\left(\theta^I\frac{\partial}{\partial\theta_p}\right),\varphi\left(\theta^J\frac{\partial}{\partial\theta_q}\right)\right]\tag{3.6}
$$

for every $p,q\in\{1,\dots,n\}$, $I,J\subset\{1,\dots,n\}$.

Let $p,q\in\{1,\dots,n\}$, $I,J\subset\{1,\dots,n\}$. If $q\notin I$, and $p\notin J$, then the bracket in the right-hand side of (3.6) is zero. The left-hand side is zero since all brackets on Lemma 3.9.2 are zero in this case.

Suppose $q \notin I$ and $p \in J$. Without loss of generality, we may assume that p is the first element of J. Thus, $\theta^J = \theta_p \theta^{J'}$, and

$$s(Q, J) = \begin{cases} s(Q \setminus \{p\}, J') & \text{if } p \in Q, \\ (-1)^{|Q|} s(Q \setminus \{p\}, J') & \textit{if } p \notin Q \end{cases}$$

where $J' = J \setminus \{p\}$. First, we have that $\left[\theta^I \frac{\partial}{\partial \theta_p}, \theta^J \frac{\partial}{\partial \theta_q}\right] = \theta^I \theta^{J'} \frac{\partial}{\partial q}$. Secondly,

$$\begin{aligned}
&\sum_{Q \subsetneq J} s(Q, J) \left[\theta^I \frac{\partial}{\partial \theta_p}, \theta^Q\right] \otimes \Theta^{Q^c} \frac{\partial}{\partial \Theta_q} \\
&= \sum_{Q \subset J'} (-1)^{|Q|} s(Q, J') \left[\theta^I \frac{\partial}{\partial \theta_p}, \theta^Q\right] \otimes \Theta_p \Theta^{Q^c} \frac{\partial}{\partial \Theta_q} + \sum_{Q \subsetneq J'} s(Q, J') \left[\theta^I \frac{\partial}{\partial \theta_p}, \theta_p \theta^Q\right] \otimes \Theta^{Q^c} \frac{\partial}{\partial \Theta_q} \\
&= \sum_{Q \subsetneq J'} s(Q, J') \theta^I \theta^Q \otimes \Theta^{Q^c} \frac{\partial}{\partial \Theta_q}
\end{aligned}$$

Moreover,

$$\sum_{P \subsetneq I} s(P, I) \left[\theta^J \frac{\partial}{\partial \theta_q}, \theta^P\right] \otimes \Theta^{P^c} \frac{\partial}{\partial \Theta_p} = 0,$$

because $q \notin P$. Finally,

$$\begin{aligned}
&\sum_{\substack{P \subsetneq I \\ Q \subsetneq J}} s(P, I) s(Q, J) (-1)^{(|P^c|+1)|Q|} \theta^P \theta^Q \otimes \left[\Theta^{P^c} \frac{\partial}{\partial \Theta_p}, \Theta^{Q^c} \frac{\partial}{\partial \Theta_q}\right] \\
&= \sum_{\substack{P \subsetneq I \\ Q \subset J'}} s(P, I) s(Q, J') (-1)^{(|P^c|+1)|Q|+|Q|} \theta^P \theta^Q \otimes \left[\Theta^{P^c} \frac{\partial}{\partial \Theta_p}, \Theta_p \Theta^{Q^c} \frac{\partial}{\partial \Theta_q}\right] \\
&\quad + \sum_{\substack{P \subsetneq I \\ Q \subsetneq J'}} s(P, I) s(Q, J') (-1)^{(|P^c|+1)(|Q|+1)} \theta^P \theta_p \theta^Q \otimes \left[\Theta^{P^c} \frac{\partial}{\partial \Theta_p}, \Theta^{Q^c} \frac{\partial}{\partial \Theta_q}\right] \\
&= \sum_{\substack{P \subsetneq I \\ Q \subset J'}} s(P, I) s(Q, J') (-1)^{|P^c||Q|} \theta^P \theta^Q \otimes \Theta^{P^c} \Theta^{Q^c} \frac{\partial}{\partial \Theta_q} \\
&= \sum_{\substack{P \subsetneq I \\ Q \subsetneq J'}} s(P, I) s(Q, J') (-1)^{|P^c||Q|} \theta^P \theta^Q \otimes \Theta^{P^c} \Theta^{Q^c} \frac{\partial}{\partial \Theta_q} \\
&\quad + \sum_{P \subsetneq I} s(P, I) (-1)^{|P^c||J'|} \theta^P \theta^{J'} \otimes \Theta^{P^c} \Theta^{Q^c} \frac{\partial}{\partial \Theta_q}
\end{aligned}$$

Therefore, the right hand side of equation 3.6 by Lemma 3.9.2 is

$$\left[\varphi\left(\theta^I \frac{\partial}{\partial \theta_p}\right), \varphi\left(\theta^J \frac{\partial}{\partial \theta_q}\right)\right] = \theta^I \theta^{J'} \frac{\partial}{\partial q} \otimes 1 + \sum_{Q \subsetneq J'} s(Q, J') \theta^I \theta^Q \otimes \Theta^{Q^c} \frac{\partial}{\partial \Theta_q}$$

$$+\sum_{\substack{P\subsetneq I\\ Q\subset J'}} s(P,I)s(Q,J')(-1)^{|P^c||Q|}\theta^P\theta^Q\otimes\Theta^{P^c}\Theta^{Q^c}\frac{\partial}{\partial\Theta_q}.$$

On the other hand, the left-hand side of equation 3.6

$$\begin{aligned}\varphi\left(\theta^I\theta^{J'}\frac{\partial}{\partial q}\right)=&\theta^I\theta^{J'}\frac{\partial}{\partial\theta_q}\otimes 1+\sum_{Q\subsetneq J'} s(Q,J')\theta^I\theta^Q\otimes\Theta^{Q^c}\frac{\partial}{\partial\Theta_q}\\ &+\sum_{P\subsetneq I}(-1)^{|P^c||J'|}s(P,I)\theta^P\theta^J\otimes\Theta^{P^c}\frac{\partial}{\partial\Theta_q}\\ &+\sum_{\substack{P\subsetneq I\\ Q\subsetneq J'}}(-1)^{|P^c||Q|}s(P,I)s(Q,J')\theta^P\theta^Q\otimes\Theta^{P^c}\Theta^{Q^c}\frac{\partial}{\partial\Theta_q}.\end{aligned}$$

We conclude (3.6) is satisfied if $q\notin I$ and $p\in J$. Analogously, equation (3.6) is true when $q\in I$ and $p\notin J$.

It remains to prove that equation (3.6) is true when $q\in I$, and $p\in J$. Suppose that q is the first element of I and p is the first element of J. Denote $I'=I\setminus\{q\}$, and $J'=J\setminus\{p\}$. Hence, $\theta^I=\theta_q\theta^{J'}$, $\theta^J=\theta_p\theta^{J'}$, and

$$s(P,I)=\begin{cases}s(P\setminus\{q\},I') & \text{if } q\in P,\\ (-1)^{|P|}s(P,I') & \textit{if } q\notin P;\end{cases}$$

$$s(Q,J)=\begin{cases}s(Q\setminus\{p\},J') & \text{if } p\in Q,\\ (-1)^{|Q|}s(Q,J') & \textit{if } p\notin Q.\end{cases}$$

Furthermore,

$$\left[\theta^I\frac{\partial}{\partial\theta_p},\theta^J\frac{\partial}{\partial\theta_q}\right]=\theta^I\theta^{J'}\frac{\partial}{\partial\theta_p}-(-1)^{(|I|+1)(|J|+1)}\theta^J\theta^{I'}\frac{\partial}{\partial\theta_p}.$$

With this notation, we use the calculations we did in the last case to infer that

$$\begin{aligned}\sum_{Q\subsetneq J} s(Q,J)\left[\theta^I\frac{\partial}{\partial\theta_p},\theta^Q\right]\otimes\Theta^{Q^c}\frac{\partial}{\partial\Theta_q}&=\sum_{Q\subsetneq J'} s(Q,J')\theta^I\theta^Q\otimes\Theta^{Q^c}\frac{\partial}{\partial\Theta_q},\\ \sum_{P\subsetneq I} s(P,I)\left[\theta^J\frac{\partial}{\partial\theta_q},\theta^P\right]\otimes\Theta^{P^c}\frac{\partial}{\partial\Theta_p}&=\sum_{P\subsetneq I'} s(P,I')(-1)^{|P||J|}\theta^P\theta^J\otimes\Theta^{P^c}\frac{\partial}{\partial\Theta_p}.\end{aligned}$$

The last sum given by Lemma 3.9.2 is

$$\begin{aligned}&\sum_{\substack{P\subsetneq I\\ Q\subsetneq J}} s(P,I)s(Q,J)(-1)^{(|P^c|+1)|Q|}\theta^P\theta^Q\otimes\left[\Theta^{P^c}\frac{\partial}{\partial\Theta_p},\Theta^{Q^c}\frac{\partial}{\partial\Theta_q}\right]\\ =&\sum_{\substack{P\subsetneq I\\ Q\subset J'}} s(P,I)s(Q,J')(-1)^{(|P^c|+1)|Q|+|Q|}\theta^P\theta^Q\otimes\left[\Theta^{P^c}\frac{\partial}{\partial\Theta_p},\Theta_p\Theta^{Q^c}\frac{\partial}{\partial\Theta_q}\right]\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\substack{P\subsetneq I\\ Q\subsetneq J'}} s(P,I)s(Q,J')(-1)^{(|P^c|+1)(|Q|+1)}\theta^P\theta_p\theta^Q \otimes \left[\Theta^{P^c}\frac{\partial}{\partial\Theta_p}, \Theta^{Q^c}\frac{\partial}{\partial\Theta_q}\right] \\
&= \sum_{\substack{P\subset I'\\ Q\subset J'}} s(P,I')s(Q,J')(-1)^{(|P^c|+1)|Q|+|Q|+|P|}\theta^P\theta^Q \otimes \left[\Theta_q\Theta^{P^c}\frac{\partial}{\partial\Theta_p}, \Theta_p\Theta^{Q^c}\frac{\partial}{\partial\Theta_q}\right] \\
&+ \sum_{\substack{P\subsetneq I'\\ Q\subset J'}} s(P,I')s(Q,J')(-1)^{(|P^c|+1)|Q|+|Q|}\theta_q\theta^P\theta^Q \otimes \left[\Theta^{P^c}\frac{\partial}{\partial\Theta_p}, \Theta_p\Theta^{Q^c}\frac{\partial}{\partial\Theta_q}\right] \\
&- \sum_{\substack{P\subset I'\\ Q\subsetneq J'}} s(P,I')s(Q,J')(-1)^{|P^c|(|Q|+1)+|P|+|P^c|(|Q^c|+1)}\theta^P\theta_p\theta^Q \otimes \left[\Theta^{Q^c}\frac{\partial}{\partial\Theta_q}, \Theta_q\Theta^{P^c}\frac{\partial}{\partial\Theta_p}\right] \\
&= \sum_{\substack{P\subset I'\\ Q\subset J'}} s(P,I')s(Q,J')(-1)^{|P^c||Q|+|P|}\theta^P\theta^Q \otimes \left(\Theta_q\Theta^{P^c}\Theta^{Q^c}\frac{\partial}{\partial\Theta_q} - (-1)^{|P^c||Q^c|}\Theta_p\Theta^{Q^c}\Theta^{P^c}\frac{\partial}{\partial\Theta_q}\right) \\
&+ \sum_{\substack{P\subsetneq I'\\ Q\subset J'}} s(P,I')s(Q,J')(-1)^{|P^c||Q|}\theta_q\theta^P\theta^Q \otimes \Theta^{P^c}\Theta^{Q^c}\frac{\partial}{\partial\Theta_q} \\
&- \sum_{\substack{P\subset I'\\ Q\subsetneq J'}} s(P,I)s(Q,J')(-1)^{|P^c|(|J|+1)+|P|}\theta^P\theta_p\theta^Q \otimes \Theta^{Q^c}\Theta^{P^c}\frac{\partial}{\partial\Theta_p} \\
&= \sum_{\substack{P\subsetneq I\\ Q\subset J'}} s(P,I)s(Q,J')(-1)^{|P^c||Q|}\theta^P\theta^Q \otimes \Theta^{P^c}\Theta^{Q^c}\frac{\partial}{\partial\Theta_q} \\
&- \sum_{\substack{P\subset I'\\ Q\subsetneq J}}(-1)^{|P^c|(|J|+1)+|P|}s(P,I')s(Q,J)\theta^P\theta^Q \otimes \Theta^{Q^c}\Theta^{P^c}\frac{\partial}{\partial\Theta_p}.
\end{aligned}
$$

With all in place, we have that

$$
\begin{aligned}
\left[\varphi\left(\theta^I\frac{\partial}{\partial\theta_p}\right), \varphi\left(\theta^J\frac{\partial}{\partial\theta_q}\right)\right] =&\varphi\left(\theta^I\theta^{J'}\frac{\partial}{\partial\theta_q}\right) - (-1)^{(|I|+1)(|J|+1)}\varphi\left(\theta^J\theta^{I'}\frac{\partial}{\partial\theta_p}\right) \\
&- (-1)^{(|I|+1)(|J|+1)}\sum_{P\subsetneq I'} s(P,I')\theta^J\theta^P \otimes \Theta^{P^c}\frac{\partial}{\partial\Theta_p} \\
&+ \sum_{\substack{P\subsetneq I\\ Q\subset J'}} s(P,I)s(Q,J')(-1)^{|P^c||Q|}\theta^P\theta^Q \otimes \Theta^{P^c}\Theta^{Q^c}\frac{\partial}{\partial\Theta_q} \\
&- \sum_{\substack{P\subset I'\\ Q\subsetneq J}}(-1)^{|P^c|(|J|+1)+|P|}s(P,I')s(Q,J)\theta^P\theta^Q \otimes \Theta^{Q^c}\Theta^{P^c}\frac{\partial}{\partial\Theta_p} \\
=&\varphi\left(\theta^I\theta^{J'}\frac{\partial}{\partial\theta_q}\right) - (-1)^{(|I|+1)(|J|+1)}\varphi\left(\theta^J\theta^{I'}\frac{\partial}{\partial\theta_p}\right) \\
=&\varphi\left(\left[\theta^I\frac{\partial}{\partial\theta_p}, \theta^J\frac{\partial}{\partial\theta_q}\right]\right),
\end{aligned}
$$

thus equation (3.6) is satisfied when $q \in I$, and $p \in J$. □

Lemma 3.9.4. *φ is an isomorphism.*

Proof. We start by proving that φ is surjective. Since φ is an algebra homomorphism, we only need to show that generators of D and $U(\mathrm{L}_+)$ are elements of its image. Let $J \subset \{1, \dots, n\}$ and $p, p_1, \dots, p_k \in \{1, \dots, n\}$, then

$$\varphi\left(\theta^J \# \left(\frac{\partial}{\partial\theta_{p_1}}\right) \cdots \left(\frac{\partial}{\partial\theta_{p_k}}\right)\right) = \theta^I \left(\frac{\partial}{\partial\theta_{p_1}}\right) \cdots \left(\frac{\partial}{\partial\theta_{p_k}}\right) \otimes 1$$

Furthermore,

$$\begin{aligned}
&\varphi\left(\sum_{I\subset J}(-1)^{|I|}s(I^c,J)\theta^{I^c}\#\theta^I\frac{\partial}{\partial\theta_p}\right)\\
&=\sum_{I\subset J}(-1)^{|I|}s(I^c,J)\theta^{I^c}\varphi\left(1\#\theta^I\frac{\partial}{\partial\theta_p}\right)\\
&=\sum_{I\subset J}(-1)^{|I|}s(I^c,J)\theta^{I^c}\theta^I\frac{\partial}{\partial\theta_p}\otimes 1+\sum_{I\subset J}\sum_{P\subsetneq I}(-1)^{|J|-|I^c|}s(I^c,J)s(P,I)\theta^{I^c}\theta^P\otimes\Theta^{P^c}\frac{\partial}{\partial\Theta_p}\\
&=\sum_{I\subset J}(-1)^{|I|}\theta^J\frac{\partial}{\partial\theta_p}\otimes 1\\
&\quad+\sum_{Q\subsetneq J}\left(\sum_{P\subset Q}(-1)^{|J|-|Q|+|P|}s(Q\setminus P,J)s(P,P\cup Q^c)s(P,Q)\right)\theta^Q\otimes\Theta^{Q^c}\frac{\partial}{\partial\Theta_p}\\
&=(-1)^{|J|}\otimes\Theta^J\frac{\partial}{\partial\Theta_p}.
\end{aligned}$$

Therefore, φ is surjective.

Let $\psi : \mathrm{D} \otimes U(\mathrm{L}_+) \to \Lambda\# U(\mathrm{W})$ be the map defined by

$$\begin{aligned}
\psi\left(\theta^J\frac{\partial}{\partial\theta_{p_1}}\cdots\frac{\partial}{\partial\theta_{p_k}}\otimes 1\right) &= \theta^J\#\frac{\partial}{\partial\theta_{p_1}}\cdots\frac{\partial}{\partial\theta_{p_k}},\\
\psi\left(1\otimes\Theta^I\frac{\partial}{\partial\Theta_p}\right) &= \sum_{I\subset J}(-1)^{|I|}s(I^c,J)\theta^{I^c}\#\theta^I\frac{\partial}{\partial\theta_p},
\end{aligned}$$

for each $I, J \subset \hat{n}$, $I \neq \emptyset$, and $p, p_1, \dots, p_k \in \hat{n}$. Note that we have shown that $\varphi \circ \psi$ is the identity of $\mathrm{D} \otimes U(\mathrm{L}_+)$. On the other hand,

$$\begin{aligned}
&\psi\left(\theta^I\frac{\partial}{\partial\theta_p}\otimes 1+\sum_{P\subsetneq I}s(P,I)\theta^P\otimes\Theta^{P^c}\frac{\partial}{\partial\Theta_p}\right)\\
&=\psi\left(\theta^I\frac{\partial}{\partial\theta_p}\otimes 1\right)+\sum_{P\subsetneq I}s(P,I)\psi\left(\theta^P\otimes\Theta^{P^c}\frac{\partial}{\partial\Theta_p}\right)\\
&=\theta^I\#\frac{\partial}{\partial\theta_p}+\sum_{P\subsetneq I}s(P,I)\sum_{Q\subset P}(-1)^{|Q^c|}s(Q^c,P^c)\theta^P\theta^{Q^c}\#\theta^Q\frac{\partial}{\partial\theta_p}
\end{aligned}$$

$$
\begin{aligned}
&=\sum_{P\subset I}\sum_{Q\subset P^c}(-1)^{|Q|}s(P,I)s(Q^c,P)s(P,P\cup Q^c)\theta^{P\cup Q^c}\#\theta^Q\frac{\partial}{\partial\theta_p}\\
&=\sum_{Q\subset I}s(Q^c,I)\left(\sum_{Q\subset P\subset I}(-1)^{|Q|}s(P\setminus Q,I\setminus Q)\right)\theta^{Q^c}\#\theta^Q\frac{\partial}{\partial\theta_p}\\
&=1\#\theta^I\frac{\partial}{\partial\theta_p},
\end{aligned}
$$

because $s(P,I)s(Q,P) = s(Q,I)$ if $Q \subset P \subset I$. Thus, $\psi\circ\varphi$ is the identity of $\Lambda\#U(\mathrm{W})$. Therefore, φ is an isomorphism, and its inverse is ψ. □

Therefore, φ is an isomorphism of associative superalgebras by Lemma 3.9.3, and Theorem 3.9.1 follows.

Corollary 3.9.5. *The map* $\psi : \mathrm{D}\otimes U(\mathrm{L}_+)\to\Lambda\#U(\mathrm{W})$ *defined on the previous lemma is a homomorphism of associative superalgebras and it is the inverse of* ϕ*.*

Remark 3.9.6. By PBW Theorem, there is a canonical linear isomorphism

$$\Lambda\#U(\mathrm{W})\cong\left(\Lambda(\theta_1,\dots,\theta_n)\#\Lambda\left(\frac{\partial}{\partial\theta_1},\dots,\frac{\partial}{\partial\theta_n}\right)\right)(1\#U(\mathrm{W}_+)).$$

The first term is isomorphic to D and the second to $U(\mathrm{L}_+)$. However,

$$\left[1\#\frac{\partial}{\partial\theta_p},\theta_l\frac{\partial}{\partial\theta_q}\right]=1\#\frac{\partial}{\partial\theta_p}(\theta_l)\frac{\partial}{\partial\theta_q}$$

is non-zero if $p\neq l$. Therefore, the isomorphism given by the PBW Theorem is not an isomorphism of associative superalgebras, since the subalgebras D and $U(\mathrm{L}_+)$ do not commute in this case.

3.9.2 Rudakov modules

Let U be a W_+-module. We can make U a Λ-module by evaluation:

$$\theta^P u=\begin{cases}0, & \text{if } \emptyset\neq P\subset\{1,\dots,n\}\\ u & \text{if } P=\emptyset\end{cases}$$

The action of $\Bbbk\subset\Lambda$ is the same of U as a vector space. Therefore, $\mathrm{J}U=0$, and

$$\theta^P\frac{\partial}{\partial\theta_q}(\theta^Q v)=\theta^P\frac{\partial}{\partial\theta_q}(\theta^Q)v+\theta^Q\left(\theta^P\frac{\partial}{\partial\theta_q}v\right)=\theta^Q\left(\theta^P\frac{\partial}{\partial\theta_q}v\right)$$

for each $P,Q\subset\{1,\dots,n\}$, $P\neq\emptyset$, $q\in\{1,\dots,n\}$. Therefore, U is a $\Lambda\#U(\mathrm{W}_+)$-module.

The *Rudakov module* associated to U is the infeq Λ-module

$$\mathcal{R}(U)=\Lambda\#U(\mathrm{W})\otimes_{\Lambda\#U(\mathrm{W}_+)}U.$$

If U is a finite-dimensional vector space, then $\mathcal{R}(U)$ is a finite-dimensional vector space, thus it is finitely generated as an Λ-module. This is a major difference between Rudakov modules over $\Lambda\#U(\mathrm{W})$ and other settings.

Let M be an infeq Λ-module, and define the *body module* $\overline{M} = M/\mathrm{J}M$. If $\emptyset \neq P, Q \subset \{1, \dots, n\}$, then

$$\theta^P \frac{\partial}{\partial \theta_p} \left(\theta^Q v\right) = \theta^P \frac{\partial}{\partial \theta_p} (\theta^Q) v + (-1)^{(|P|+1)|Q|} \theta^Q \left(\theta^P \frac{\partial}{\partial \theta_p} v \right) \in \mathrm{J}M$$

for each $p \in \{1, \dots, n\}$, and $v \in M$. Therefore, $\overline{M}$ is a module over W_+.

Define $\overline{M}^* = \mathrm{Hom}_{\Bbbk}(\overline{M}, \Bbbk)$ the dual space of $\overline{M}$. If $M = M_{\bar{0}} \oplus M_{\bar{1}}$ is $\mathbb{Z}_2$-graded, then $\overline{M}^*$ is $\mathbb{Z}_2$-graded by

$$\overline{M}^*_i = \left\{ \alpha \in \overline{M}^* \mid \alpha \left(\overline{M_{i+1}} \right) = 0 \right\}.$$

Moreover, it is a $\Lambda\#U(\mathrm{W}_+)$-module with the standard actions of Λ and W_+,

$$(f\alpha)(m) = (-1)^{|f||\alpha|} \alpha(fm),$$
$$(\eta\alpha)(m) = -(-1)^{|\eta||\alpha|} \alpha(\eta m)$$

for all $\eta \in \mathrm{W}_+$, $f \in \Lambda$, $\alpha \in \overline{M}^*$, and $m \in M$.

Let $\pi : M \to \overline{M}$ be the $\Lambda\#U(\mathrm{W}_+)$-homomorphism given by the canonical projection, then the pullback of π defines a homomorphism $\pi^* : \overline{M}^* \to M^*$ between the dual modules by

$$\pi^*(\alpha) = \alpha \circ \pi, \quad \alpha \in \overline{M}^*.$$

Proposition 3.9.7. *The canonical homomorphism $\pi : M \to \overline{M}$ extends uniquely to a $\Lambda\#U(W)$-homomorphism $\overline{\pi}^* : \mathcal{R}\left(\overline{M}^*\right) \to M^*$.*

Proof. The induction functor $\mathcal{R}(\_) : \Lambda\#U(\mathrm{W}_+) - \mathrm{Mod} \to \Lambda\#U(\mathrm{W})$ is left adjoint to the restriction functor $\mathrm{Res}_\iota : \Lambda\#U(\mathrm{W}) - \mathrm{Mod} \to \Lambda\#U(\mathrm{W}_+) - \mathrm{Mod}$ induced by the inclusion map $\iota : \Lambda\#U(\mathrm{W}_+) \hookrightarrow \Lambda\#U(\mathrm{W})$. Hence,

$$\begin{aligned} \mathrm{Hom}_{\Lambda\#U(\mathrm{W}_+)}\left(\overline{M}^*, M^*\right) &\cong \mathrm{Hom}_{\Lambda\#U(\mathrm{W}_+)}\left(\overline{M}^*, \mathrm{Res}_\iota\left(M^*\right)\right) \\ &\cong \mathrm{Hom}_{\Lambda\#U(\mathrm{W})}\left(\mathcal{R}(\overline{M}^*), M^*\right). \end{aligned}$$

Therefore, the canonical homomorphism $\pi \in \mathrm{Hom}_{\Lambda\#U(\mathrm{W}_+)}\left(\overline{M}^*, M^*\right)$ corresponds to a unique $\Lambda\#U(\mathrm{W})$-homomorphism $\overline{\pi}^* \in \mathrm{Hom}_{\Lambda\#U(\mathrm{W})}\left(\mathcal{R}(\overline{M}^*), M^*\right)$. Explicitly,

$$\overline{\pi}^*(f\#u \otimes \alpha)(m) = \alpha(\pi(f(um)))$$

for each $f\#u \otimes \alpha \in \Lambda\#U(\mathrm{W}) \otimes_{\Lambda\#U(\mathrm{W}_+)} \overline{M}^*$. □

The dual module $M^* = \mathrm{Hom}_{\Bbbk}(M, \Bbbk)$ is a super vector space if M is, and it is an infeq Λ-module with the standard actions of Λ and W

$$(f\alpha)(m) = (-1)^{|f||\alpha|} \alpha(fm), \quad (\eta\alpha)(m) = -(-1)^{|\eta||\alpha|} \alpha(\eta m)$$

for all $\eta \in \mathrm{W}$, $f \in \Lambda$, $\alpha \in M^*$, and $m \in M$. It is not always true that M and M^* are isomorphic as Λ-infeq modules, but this is the case when $M = \Lambda$.

Example 3.9.8. The superalgebra Λ is naturally an infeq Λ-module, thus its dual Λ^* is also an infeq Λ-module. For $P \subset \widehat{n}$, we denote $\theta^P(0) = 0$ if $P \neq \emptyset$, and $\theta^{\emptyset}(0) = 1$. A point derivation $D : \Lambda \to \Bbbk$ is a map such that $D(ab) = D(a)b(0) + (-1)^{|D||b|}a(0)D(b)$. For each $p \in \widehat{n}$, the map $\partial_p(\theta_q) = 1$ defines a point derivation $\partial_p : \Lambda \to \Bbbk$. We define the product $\partial_p\partial_q : \Lambda \to \Bbbk$ by

$$(\partial_p\partial_q)(\theta^P) = \partial_p\left(\frac{\partial}{\partial\theta_q}(\theta^P)\right)$$

Note that $\partial_p\partial_q = -\partial_q\partial_p$, and we define ∂^P for each $P \subset \widehat{n}$ accordingly (note that $\partial^{\emptyset}(1) = 1$ and it is zero otherwise). The set $\{\partial^P \mid P \subset \widehat{n}\}$ has 2^n linearly independent elements, hence it is a basis of Λ^*. The action of Λ on ∂^P is by derivation, i.e.

$$\theta_p(\partial_p\partial^Q) = \partial^Q, \quad p \in \widehat{n},\ Q \subset \widehat{n} \setminus \{p\}.$$

On the other hand, the action of W_{-1} is given by a product

$$\frac{\partial}{\partial\theta_p}\theta^P = -(-1)^{|P|}\partial^P\partial_p = -\partial_p\partial^P.$$

We have that 1 and $\partial^{\widehat{n}}$ are a basis of $\Lambda(n)$ and Λ^* as Λ-modules, respectively. Therefore, we may consider the unique Λ-module homomorphism $T : \Lambda \to \Lambda^*$ such that $T(1) = \partial^{\widehat{n}}$. Because it satisfies

$$T\left(\frac{\partial}{\partial\theta_p}(\theta^P)\right) = \frac{\partial}{\partial\theta_p}(\theta^P)\partial^{\widehat{n}} = \frac{\partial}{\partial\theta_p}(\theta^P\partial^{\widehat{n}}) - (-1)^{|P|}\theta^P\left(\frac{\partial}{\partial\theta_p}\partial^{\widehat{n}}\right) = \frac{\partial}{\partial\theta_p}T(\theta^P),$$

T is an isomorphism of $\Lambda\#U(\mathrm{W})$-modules.

The dual M^{**} of M^* is an infeq module as well. For each $m \in M$, we define $m^* \in M^{**}$ by $m^*(\alpha) = (-1)^{|m||\alpha|}\alpha(m)$ for each $\alpha \in M^*$. Since M is finite-dimensional, the map $\phi : M \to M^{**}$ defined by $\phi(m) = m^*$ gives an isomorphism between M and M^{**}. Furthermore,

$$(fm^*)(\alpha) = (fm)^*(\alpha), \quad (\eta m^*)(\alpha) = (\eta m)^*(\alpha), \quad f \in \Lambda,\ \eta \in \mathrm{W},\ \text{and } m \in M.$$

Therefore, ϕ is an isomorphism of $\Lambda\#U(\mathrm{W})$-modules.

Let $U = M^*/\mathrm{J}M^*$ the body module of M^*. As we know, U is a module over $\Lambda\#U(\mathrm{W}_+)$. By Proposition 3.9.7, the canonical homomorphism $\pi : M^* \to U$ extends uniquely to a $\Lambda\#U(\mathrm{W})$-homomorphism $\overline{\pi}^* : \mathcal{R}(U^*) \to M^{**}$. Thus, $\phi \circ \overline{\pi}^* : \mathcal{R}(U^*) \to M$ is an isomorphism. Therefore, we have proved the following proposition.

Proposition 3.9.9. *Every finite infeq Λ-module is a Rudakov module. Explictly, if M is a finite infeq Λ-module, then $M \cong \mathcal{R}\left((\overline{M^*})^*\right)$.*

Therefore, the functor $\mathcal{R}$ defined on the category W_+-mod of finite-dimensional W_+-modules to the category $\mathrm{InfEq}(\Lambda)$ of finite infeq Λ-modules is an essentially surjective functor, i.e. each object of $\mathrm{InfEq}(\Lambda)$ is isomorphic to an object of the form $\mathcal{R}(U)$ for some object U of W_+-mod. We want to prove that $\mathcal{R}$ is an equivalence of categories, thus it remains to prove that $\mathcal{R}$ is a *fully faithful functor.* A fully faithful functor $\mathcal{F} : C \to C'$ between the two categories C and C' is a functor such that the map

$$\mathrm{Hom}_C(x, y) \to \mathrm{Hom}_{C'}(Fx, FY)$$

induced by $\mathcal{F}$ is bijective for each $x, y \in \mathbb{C}$.

Lemma 3.9.10. *If V is a W_+-module and $v \in \mathcal{R}(V)$, then $\mathrm{J}v = 0$ if and only if $v \in V$.*

Proof. By the PBW Theorem

$$\mathcal{R}(V) \cong \left(\Lambda \otimes \Lambda\left(\frac{\partial}{\partial\theta_1}, \dots, \frac{\partial}{\partial\theta_n}\right)\right) U\left(\Lambda \# U(\mathrm{W}_+)\right) \otimes_{\Lambda\# U(\mathrm{W}_+)} V \cong \Lambda\left(\frac{\partial}{\partial\theta_1}, \dots, \frac{\partial}{\partial\theta_n}\right) \otimes V$$

as a vector space, where $\Lambda\left(\frac{\partial}{\partial\theta_1}, \dots, \frac{\partial}{\partial\theta_n}\right)$ denotes the exterior product in variables $\frac{\partial}{\partial\theta_1}, \dots, \frac{\partial}{\partial\theta_n}$. Using the notation introduced in the Example 3.9.8, there is an isomorphism of modules over $\Lambda\# U(\mathrm{W})$ between this exterior product and Λ^*, which is defined by $\frac{\partial}{\partial\theta_p} \mapsto \partial_p$. If $P \subset \widehat{n}$, then

$$\theta_p \partial^P v = (-1)^{|P|}\partial^P \theta_p v + c_{P,p}\partial^{P\setminus\{p\}} v = c_{P,p}\partial^{P\setminus\{p\}} v,$$

where $c_{P,p} = 0$ if $p \notin P$ and $c_{P,p} \in \{1, -1\}$ otherwise. Thus, $\theta_p \partial^P v = 0$ for every $p \in \widehat{n}$ if and only if $v \in \Bbbk \otimes V$. □

Proposition 3.9.11. *The functor $\mathcal{R} : \mathrm{W}_+\text{-mod} \to \mathrm{InfEq}(\Lambda)$ is a fully faithful functor.*

Proof. By definition, $\mathcal{R}$ is a fully faithful functor if the map

$$\mathrm{Hom}_{\mathrm{W}_+}(U, V) \to \mathrm{Hom}_{\Lambda\# U(\mathrm{W})}(\mathcal{R}(U), \mathcal{R}(V))$$

induced by the functor $\mathcal{R}$ is bijective for every object U, V of W_+-mod. Because the induction functor $\mathcal{R}$ is left adjoint to the restriction functor Res_ι, we have that

$$\mathrm{Hom}_{\Lambda\# U(\mathrm{W})}(\mathcal{R}(U), \mathcal{R}(V)) \cong \mathrm{Hom}_{\Lambda\# U(\mathrm{W}_+)}\left(U, \mathrm{Res}_\iota\left(\mathcal{R}(V)\right)\right).$$

Additionally, the action of Λ in both U and V is given by evaluation on J, thus if $\alpha \in \mathrm{Hom}_{\Lambda\# U(\mathrm{W}_+)}(U, \mathcal{R}(V))$ then $0 = \alpha(fu) = f\alpha(u)$ for each $f \in \mathrm{J}$ and $u \in U$. By Lemma 3.9.10, the image of α is a subset of $V \cong \Bbbk \otimes V \subset \mathcal{R}(V)$, therefore $\alpha : U \to V$ is a W_+-homomorphism. In other words,

$$\mathrm{Hom}_{\mathrm{W}_+}(U, V) \cong \mathrm{Hom}_{\Lambda\# U(\mathrm{W}_+)}\left(U, \mathcal{R}(V)\right).$$

We conclude that $\mathrm{Hom}_{\mathrm{W}_+}(U, V) \cong \mathrm{Hom}_{\Lambda\# U(\mathrm{W})}(\mathcal{R}(U), \mathcal{R}(V))$, thus $\mathcal{R}$ is a fully faithful functor. □

Theorem 3.9.12. *The functor $\mathcal{R} : \mathrm{W}_+\text{-mod} \to \mathrm{InfEq}(\Lambda)$ is an equivalence of categories.*

Proof. By Proposition 3.9.9 and Proposition 3.9.11, the functor $\mathcal{R}$ is essentially surjective, full and faithful. Thus, it is an equivalence of categories. $\square$

We can use this theorem and the previous discussions to describe simple finite infeq Λ-modules. Because the functor $\mathcal{R}$ is an equivalence of categories, a finite-dimensional W_+-module U is simple if and only if the finite infeq Λ-module $\mathcal{R}(U)$ is simple. Let us describe simple finite infeq Λ-modules in terms of irreducible representations of W_+. We start analyzing finite-dimensional simple W_+-modules by applying the following lemma.

Lemma 3.9.13 ([CK98, Lemma 1])**.** *Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra and let $\mathfrak{n}$ be a solvable ideal of $\mathfrak{g}$. Let $\mathfrak{a}$ be an even subalgebra of $\mathfrak{g}$ such that $\mathfrak{n}$ is a completely reducible $\operatorname{ad}\mathfrak{a}$-module with no trivial summand. Then $\mathfrak{n}$ acts trivially in any irreducible finite-dimensional $\mathfrak{g}$-module.*

Lemma 3.9.14. *Let V be a simple finite-dimensional W_+-module, then $\mathrm{W}_{+,i}V = 0$ for each $i > 1$. In particular, V is a simple module over $\mathrm{W}_+/\mathrm{J}\mathrm{W}_+ \cong \mathfrak{gl}_n$.*

Proof. Assume $\mathfrak{g} = \mathrm{W}_+$, $\mathfrak{n} = \mathrm{J}\mathrm{W}_+$ and $\mathfrak{a} = \mathrm{W}_{+,0} = \bigoplus_{i,j=1}^{n} \Bbbk\theta_i \frac{\partial}{\partial\theta_j}$. We have that $\mathfrak{a} \cong \mathfrak{gl}_n(\Bbbk)$ and the adjoint representation makes $\mathfrak{n}$ is a module over $\mathfrak{a}$. With this action, $\mathfrak{n}$ is a weight module over $\mathfrak{a}$ with non trivial weights because

$$\left[\theta_i \frac{\partial}{\partial\theta_i}, \theta_i\theta^I \frac{\partial}{\partial\theta_k}\right] = \theta^I \frac{\partial}{\partial\theta_k}$$

where $i, k \in \widehat{n}$ with $i \neq k$ and $\emptyset \neq I \subset \widehat{n}$ with $i \notin I$. Hence, $\mathfrak{g}$, $\mathfrak{n}$ and $\mathfrak{a}$ satisfy the hypothesis of Lemma 3.9.13. Therefore, $\mathrm{J}\mathrm{W}_+$ acts trivially in any irreducible finite-dimensional W_+-module. $\square$

Let $\rho : \mathrm{W}_+ \to \mathfrak{gl}_{\Bbbk}(U)$ be a representation of W_+ on the finite-dimensional super vector space U. As we saw in proof of Lemma 3.9.10,

$$\mathcal{R}(U) \cong \Lambda\left(\frac{\partial}{\partial\theta_1}, \dots, \frac{\partial}{\partial\theta_n}\right) \otimes U$$

as a vector space. If $\emptyset \neq \{i_1, \dots, i_k\} = I \subset \widehat{n}$ with $i_1 < i_2 < \cdots < i_k$, denote $\frac{\partial}{\partial\theta^I} = \frac{\partial}{\partial\theta_{i_1}} \cdots \frac{\partial}{\partial\theta_{i_k}}$. The action of Λ in $\frac{\partial}{\partial\theta^I} \otimes U$ is given by

$$\theta^Q\left(\frac{\partial}{\partial\theta^I} \otimes v\right) = \begin{cases} 0 & \text{if } Q \setminus I \neq \emptyset, \\ s(Q,I)\frac{\partial}{\partial\theta^{I\setminus Q}} & \text{if } Q \subset I, \end{cases}$$

for each $v \in U$. Passing the isomorphism $\Lambda \cong \Lambda^* \cong \Lambda\left(\frac{\partial}{\partial\theta_1}, \dots, \frac{\partial}{\partial\theta_n}\right)$ as Λ-modules constructed in Example 3.9.8, we have that $\mathcal{R}(U) \cong \Lambda \otimes U$ as Λ-modules.

$$\theta^Q\left(\theta^I \otimes v\right) = \left(\theta^Q\theta^I\right) \otimes v = s(Q,I)\theta^{Q\cup I} \otimes v$$

for each $v \in U$.

Assume that U is a simple W_+-module, then $\mathrm{J}\mathrm{W}_+ U = 0$ by Lemma 3.9.14. Therefore, U is simply a $\mathfrak{gl}_n$-module. Let $\rho : \mathfrak{gl}_n \to \mathfrak{gl}(U)$ be the associated representation, then the action of W on $\mathcal{R}(U) \cong \Lambda \otimes U$ in the above isomorphism is given by

$$\left(\theta^I \frac{\partial}{\partial \theta_q}\right)\left(\theta^P \otimes v\right) = \theta^P \frac{\partial \theta^Q}{\partial \theta_p} + \sum_{i=1}^{n} s(\{i\}, I)\theta^Q \frac{\partial \theta^I}{\partial \theta_i} \otimes \rho\left(E_{iq}\right) v \tag{3.7}$$

for each $I, P \subset \widehat{n}$, $q \in \widehat{n}$ and $v \in U$, where $\left\{E_{iq} \mid i, j \in \widehat{n}\right\} \subset \mathfrak{gl}_n$ is the canonical basis of $\mathfrak{gl}_n$.

A *tensor module* over Λ is an infeq Λ-module defined in the tensor product $T(U) = \Lambda \otimes U$, where U is a $\mathfrak{gl}_n$-module. The action of Λ is given by left-side multiplication and the action of W is defined by (3.7). In the above discussion, we proved the following theorem.

Theorem 3.9.15. *Every simple finite infeq Λ-module is a tensor module. That is, if M is a simple finite infeq Λ-module, then there exists an irreducible representation $\rho : \mathfrak{gl}_n \to \mathfrak{gl}(U)$ of $\mathfrak{gl}_n$ such that $M \cong \Lambda \otimes U$ as a vector space. The action of Λ is defined by left multiplication and the action of* W *is given by*

$$\left(\theta^I \frac{\partial}{\partial \theta_q}\right)\left(\theta^P \otimes v\right) = \theta^P \frac{\partial \theta^Q}{\partial \theta_p} + \sum_{i=1}^{n} s(\{i\}, I)\theta^Q \frac{\partial \theta^I}{\partial \theta_i} \otimes \rho\left(E_{iq}\right) v \tag{3.8}$$

for each $I, P \subset \widehat{n}$, $q \in \widehat{n}$ and $v \in U$.

Remark 3.9.16. A peculiarity about simple finite infeq modules over Λ is that Rudakov modules and tensor modules are isomorphic. This does not happen when the even dimension is positive, since Rudakov modules are not finitely generated as modules over the algebra of functions, see [BFN19].

3.10 Summary of results

In this chapter, we extended results about the Lie algebra of vector fields and its representations to supergeometry, including results given in Chapter 2.

After establishing the preliminary results, we proved that the Lie superalgebra of vector fields on a smooth affine supervariety is simple.

Theorem (Theorem 3.6.6). *Let $X = \underline{\mathrm{Spec}}\,(S)$ be a smooth integral affine supervariety with* $\dim X = r|s \neq 0|0$. *Then, the Lie superalgebra* $\mathrm{Der}(S) = \Gamma(X, \Theta_X)$ *is simple.*

This result gives us an infinite family of infinite-dimensional simple Lie superalgebras. We studied representations of Lie superalgebras in this family that admit a compatible action of the superalgebra of functions of the affine supervariety. When this representation is finitely generated as a module over the ring of functions, we proved that the coherent sheaf associated with it is an infinitesimally equivariant sheaf.

Theorem (Theorem 3.7.13). *Let $X = \underline{\mathrm{Spec}}(S)$ be a smooth integral affine supervariety and M a finite infinitesimally equivariant S-module with associated* $\mathrm{Der}(S)$*-representation* $\rho : \mathrm{Der}(S) \to \mathfrak{gl}(M)$*. Then, the coherent sheaf $\tilde{M}$ is an infinitesimally equivariant sheaf on X. In particular, its Lie map $L : \Theta_X \to \mathfrak{gl}_{\Bbbk}(\tilde{M})$ is given by*

$$L_{D(f)}\left(\frac{\eta}{f^k}\right) = \sum_{p=0}^{\infty}\sum_{l=0}^{p}(-1)^l\binom{p}{l}\frac{1}{f^{kl+k}}\rho(f^{kl}\eta) = \sum_{p=0}^{\infty}\sum_{l=0}^{p}(-1)^l\binom{p+k}{p}\binom{p}{l}\frac{1}{f^{k+l}}\rho\left(f^l\eta\right)$$

for every $f \in S \setminus J_S$ and $\eta \in \mathrm{Der}(S)$.

With this result, we proved that the associated Lie map of an infinitesimally equivariant sheaf on a smooth supervariety is a differential operator.

Theorem (Theorem 3.8.6). *Let X be a smooth integral supervariety and $\mathcal{M}$ an infinitesimally equivariant coherent sheaf with Lie map $L : \mathrm{Der}(S) \to \mathfrak{gl}(M)$. Then L is a differential operator of order bounded by a constant that depends on the rank of M.*

A special family of Lie superalgebras of vector fields is the family $\mathrm{W}(n)$ of vector fields on the exterior algebra $\Lambda(n)$ in n variables. We studied the associative algebra that governs infeq $\Lambda(n)$-modules and established the following isomorphism theorem.

Theorem (Theorem 3.9.1). *The associative superalgebra $\Lambda(n)\#U(\mathrm{W}(n))$ is isomorphic to the tensor product of associative superalgebras $\mathrm{End}_{\Bbbk}(\Lambda(n)) \otimes U(\mathrm{W}(n)_+)$, where $\mathrm{W}(n)_+$ is the subalgebra of $\mathrm{W}(n)$ of vector fields vanishing at the point of $\underline{\mathrm{Spec}}(\Lambda(n))$.*

Using this isomorphism, we studied finite infinitesimally equivariant $\Lambda(n)$-modules and constructed an equivalence of categories between the category $\mathrm{InfEq}(\Lambda(n))$ of finite infinitesimally equivariant $\Lambda(n)$-modules and the category $\mathrm{W}(n)_+ - \mathrm{mod}$ of $\mathrm{W}(n)_+$-modules.

Theorem (Theorem 3.9.12). *The induction functor $\mathcal{R} : \mathrm{W}(n)_+ - \mathrm{mod} \to \mathrm{InfEq}(\Lambda(n))$ is an equivalence of categories, where*

$$\mathcal{R}(U) = \Lambda(n)\#U(\mathrm{W}(n)) \otimes_{\Lambda(n)\#U(\mathrm{W}(n)_+)} U$$

for every $\mathrm{W}(n)_+$-module U.

We wrapped up this chapter by illustrating a unique aspect within this context: the isomorphism between the tensor modules and Rudakov modules.

Theorem (Theorem 3.9.15). *Every simple finite infeq $\Lambda(n)$-module is a tensor module. That is, if M is a simple finite infeq $\Lambda(n)$-module, then there exists an irreducible representation $\rho : \mathfrak{gl}_n \to \mathfrak{gl}(U)$ of $\mathfrak{gl}_n$ such that $M \cong \Lambda(n) \otimes U$ as a vector space. The action of $\Lambda(n)$ is defined by left multiplication and the action of $\mathrm{W}(n)$ is given by*

$$\left(\theta^I\frac{\partial}{\partial\theta_q}\right)\left(\theta^P \otimes v\right) = \theta^P\frac{\partial\theta^Q}{\partial\theta_p} + \sum_{i=1}^{n} s(\{i\}, I)\theta^Q\frac{\partial\theta^I}{\partial\theta_i} \otimes \rho\left(E_{iq}\right)v \tag{3.9}$$

for each $I, P \subset \hat{n}$, $q \in \hat{n}$ and $v \in U$.

Chapter 4

Finite weight modules

In this chapter, we will shift away from the theory of Lie algebras of vector fields on algebraic varieties and delve into the realm of the representation theory of another (important) infinite-dimensional Lie superalgebra: the map superalgebra. The map superalgebra is a Lie superalgebra defined on the tensor product $\mathcal{G} = \mathfrak{g} \otimes S$ of a Lie superalgebra $\mathfrak{g}$ and an unital commutative (super)algebra S. Most of the time, the representation theory of $\mathcal{G}$ depends heavily on the structure and representation theory of both $\mathfrak{g}$ and S. For instance, when $\mathfrak{g}$ is a simple Lie algebra and S is a finitely generated algebra, the classification of bounded weight modules over $\mathcal{G}$ was done in terms of evaluation modules in [BLL15], which are modules over $\mathcal{G}$ constructed using maximal ideals of S and simple modules over $\mathfrak{g}$.

We will assume that $\mathfrak{g}$ is a basic classical Lie superalgebra and use its structure to define and study weight modules with finite multiplicities over $\mathcal{G}$. Throughout the chapter, we will also need to put restrictions over S, since its characteristics influence the representation theory of $\mathcal{G}$ as well. This chapter is the fruit of a collaboration with Vyacheslav Futorny and Lucas Calixto [CFR23]. All results are stated explicitly in the last Section 4.10 of this chapter.

We will start the chapter with the basics of Lie superalgebras and introduce the basic Lie superalgebras that we will use throughout the chapter. This will be done in Section 4.1 and our main reference for it is the paper by Kac that gives the classification of all simple finite-dimensional Lie superalgebras [Kac77].

In Section 4.2, we will prove some basic results on the representation theory of Lie superalgebras, like Schur's lemma and the density theorem. However, the main result of this section is Proposition 4.2.5 which shows how the irreducible representations of the direct sum of two superalgebras will behave. This is also the section where we will define the *irreducible tensor product* of two irreducible representations.

We will also prove a few propositions on Lie superalgebras that admit a weight decomposition through the adjoint action of an abelian subalgebra. We will define in Section 4.3 weight representations for this kind of Lie superalgebras and show that a simple weight module with finite multiplicities over the direct sum of two of such Lie superalgebras is given by an irreducible tensor product of two irreducible representations of the involved

Lie superalgebras.

We will then focus on the representations of map superalgebra $\mathcal{G} = \mathfrak{g} \otimes S$ associated with a basic classical Lie superalgebra $\mathfrak{g}$ and a commutative finitely generated superalgebra S. In Section 4.4, we study the S-annihilator of a weight representation, which is the largest ideal I of S such that $\mathfrak{g} \otimes I$ annihilates a module. As it happens in the Lie algebra case, we will show this ideal of S has a finite codimension.

Afterward, we study the shadow of a module in Section 4.5. To summarize, this exposes the actions of $\mathcal{G}$ on the finite weight module associated with it, showing which parts of $\mathcal{G}$ act locally nilpotently or injectively on the representation. It also allows us to define a partition of the root system of $\mathfrak{g}$ that will be used in Section 4.6 on one of the main theorems of this chapter.

Section 4.6 dives into the relation of the structure of the weight representation and its shadow. For instance, we will show that if the set of injective roots is empty, then the representation is finite-dimensional. On the other hand, if the whole root system acts injectively, then the module is cuspidal and bounded (i.e. the dimension of its weight spaces is bounded by a fixed number). We will use the concepts analyzed so far to show that each simple weight $\mathcal{G}$-module with weight spaces with finite dimension is either a cuspidal bounded $\mathcal{G}$-module or parabolically induced from a simple cuspidal bounded module over a certain subalgebra of $\mathcal{G}$.

After talking briefly about evaluation modules in Section 4.7, we will classify in Section 4.8 cuspidial bounded modules over map superalgebras associated with a basic classical Lie superalgebra $\mathfrak{g}$ with even part $\mathfrak{g}_{\bar{0}}$ semisimple and a commutative algebra. We will show that in this case, cuspidal bounded modules are evaluation modules.

We will finish this chapter applying our results to affine Lie algebras, which is the central extension of the map algebra $\mathfrak{g} \otimes \mathbb{k}[t, t^{-1}]$.

4.1 Basic Lie superalgebras

This chapter begins with a concise overview of the theory surrounding finite-dimensional simple Lie superalgebras. However, our primary emphasis will be on delving into the theory of classical basic Lie superalgebras right from the outset. All results in this section are well-known and most of them may be found in the work of Kac [Kac77].

We recall that a *Lie superalgebra* is a super vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called *bracket*, that satisfies

1. $[x, y] + (-1)^{|x||y|}[y, x] = 0$ for each $x, y \in \mathfrak{g}$;
2. (Super Jacobi Identity) $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|x||y|}[z, [x, y]] = 0$.

Subalgebras and ideals of Lie superalgebras are defined as in other algebra structures with the added part that they need to be $\mathbb{Z}_2$-graded. Homomorphisms of two superalgebras need to preserve the $\mathbb{Z}_2$-gradation.

Lemma 4.1.1. *Let $\mathfrak{g}_{\bar{0}}$ be a Lie algebra with bracket $[\cdot,\cdot]_0$ and V a $\mathfrak{g}_{\bar{0}}$-module with associated representation ρ. Suppose we have a $\mathfrak{g}_{\bar{0}}$-homomorphism $\sigma : S^2(V) \to \mathfrak{g}_{\bar{0}}$. Consider the* *super vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus V$ with even part $\mathfrak{g}_{\bar{0}}$ and odd part V. Define a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by*

$$[x,y] = [x,y]_0, \quad [x,v] = -[v,x] = x \cdot v, \quad [v,u] = \sigma(v \otimes u)$$

for each $x, y \in \mathfrak{g}_{\bar{0}}$, $v, u \in V$. Then, $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus V$ is a Lie superalgebra if

$$\sigma(u \otimes v)w + \sigma(v \otimes w)u + \sigma(w \otimes u)v = 0 \text{ for } u, v, w \in V.$$

Conversely, if $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a Lie superalgebra, then

1. *$\mathfrak{g}_{\bar{0}}$ is a Lie algebra,*
2. *$\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$ induces by the adjoint action,*
3. *$u \otimes v \mapsto [u, v]$ defined a $\mathfrak{g}_{\bar{0}}$-homomorphism $S^2(\mathfrak{g}_{\bar{1}}) \to \mathfrak{g}_{\bar{0}}$,*
4. *the bracket satisfies the super Jacobi identity.*

From now on, suppose that $\mathfrak{g}$ is a finite-dimensional Lie superalgebra. We say that $\mathfrak{g}$ is *simple* if $\mathfrak{g}$ is not abelian and it has exactly two ideals, 0 and itself. A Lie superalgebra $\mathfrak{g}$ is called *classical* if it is simple and the adjoint action makes $\mathfrak{g}_{\bar{1}}$ a completely reducible module over the Lie algebra $\mathfrak{g}_{\bar{0}}$. A classical Lie superalgebra $\mathfrak{g}$ is called *basic* if it is classical and it admits a non-degenerate bilinear form similar to a Killing form of Lie algebras, that is, a non-degenerate bilinear form $\langle\cdot,\cdot\rangle : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that $\langle x, [y,z]\rangle = \langle [x,y], z\rangle$ for all $x, y, z \in \mathfrak{g}$. A bilinear form that satisfies this last condition is called *invariant*. If $\mathfrak{g}$ is a simple Lie superalgebra, then any invariant bilinear form on $\mathfrak{g}$ is *supersymmetric*, that is, $\langle x, y\rangle = (-1)^{|x||y|}\langle y, x\rangle$ for $x, y \in \mathfrak{g}$.

The classification of all simple finite-dimensional Lie superalgebras was done by Kac [Kac77]. It was shown that a simple Lie superalgebra need not be classical. The non-classical simple Lie superalgebras are called *Cartan type* Lie superalgebras. Not all classical Lie superalgebras are basic. In his classification, Kac showed there are two series of Lie superalgebras that do not admit a non-degenerate invariant bilinear form. They are divided into two infinite families called *strange series*. We will construct most of the basic classical Lie superalgebras, and we will give realizations of their *root systems*.

For a super vector space V, we denote by $\mathfrak{gl}(V)$ the super vector space of endomorphisms of V. If $V = \Bbbk^{m|n}$, then $\mathfrak{gl}(V)$ may be identified with the super vector space of $m|n \times m|n$ super matrices $\mathfrak{gl}(m|n)$. As a vector space, $\mathfrak{gl}(m|n)$ is the set $\mathrm{M}_{m+n}(\Bbbk)$ of square $(m+n) \times (m+n)$ matrices. For $X \in \mathfrak{gl}(m|n)$, write X in blocs of matrices

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A \in \mathrm{M}_m(\Bbbk)$, $D \in \mathrm{M}_n(\Bbbk)$, $B \in \mathrm{M}_{m\times n}(\Bbbk)$ e $C \in \mathrm{M}_{n\times m}(\Bbbk)$. Then the even part of $\mathfrak{gl}(m|n)$ consists of matrices X with $B = 0$ and $C = 0$, and the odd part of $\mathfrak{gl}(m|n)$ consists of matrices with $A = 0$ and $D = 0$. Therefore, $\dim \mathfrak{gl}(m|n) = m^2 + n^2 | 2mn$, and

$$\mathfrak{gl}(m|n)_{\bar{0}} = \left\{ \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \mid A \in \mathrm{M}_m(\Bbbk),\ D \in \mathrm{M}_n(\Bbbk) \right\}$$

$$\mathfrak{gl}(m|n)_{\bar{1}} = \left\{ \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \mid B \in \mathrm{M}_{m\times n}(\Bbbk),\ C \in \mathrm{M}_{n\times n}(\Bbbk) \right\}$$

The usual product of matrices makes $\mathfrak{gl}(m|n)$ an associative superalgebra, and it is a Lie superalgebra with the bracket

$$[X, Y] = XY - (-1)^{|X||Y|} YX, \quad X, Y \in \mathfrak{gl}(m|n).$$

For a supermatrix $X = (x_{ij}) \in \mathfrak{gl}(m|n)$, we define the *supertrace* $\mathrm{str} : \mathfrak{gl}(m|n) \to \Bbbk$ by

$$\mathrm{str}(X) = \sum_{i=1}^{m} x_{ii} - \sum_{j=1}^{n} x_{jj}.$$

Its kernel

$$\mathfrak{sl}(m|n) = \{X \in \mathfrak{gl}(m|n) \mid \mathrm{str}(X) = 0\}$$

is a subalgebra of $\mathfrak{gl}(m|n)$ called *special linear Lie superalgebra.* When $m = n$, the identitiy matrix $I_{2m} \in \mathfrak{sl}(m|m)$ and it generates an ideal of $\mathfrak{sl}(m|m)$. In this case, the quotient $\mathfrak{psl}(m|m) = \mathfrak{sl}(m|m)/\Bbbk I_{2m}$ is called *projective special linear Lie superalgebra.* Their even parts are

$$\mathfrak{sl}(m|m)_{\bar{0}} = \mathfrak{sl}_m \oplus \mathfrak{sl}_m \oplus \Bbbk \text{ and } \mathfrak{psl}(m|m)_{\bar{0}} = \mathfrak{sl}_m \oplus \mathfrak{sl}_m.$$

For $m, n \geq 1$, we define

$$\mathrm{A}(m, n) = \begin{cases} \mathfrak{sl}(m+1|n+1), & \text{if } m \neq n \\ \mathfrak{psl}(m+1|m+1). & \text{if } m = n. \end{cases}$$

For the analogs of orthogonal and sympletic Lie algebras, consider the bilinear form ϕ in $\Bbbk^{m|n}$ given by the matrices

$$\phi = \begin{bmatrix} 0 & I_k & 0 & 0 & 0 \\ I_k & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_k \\ 0 & 0 & 0 & -I_k & 0 \end{bmatrix} \qquad \text{if } m = 2k+1,\ n = 2k,$$

$$\phi = \begin{bmatrix} 0 & I_k & 0 & 0 & 0 \\ I_k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_k \\ 0 & 0 & 0 & -I_k & 0 \end{bmatrix} \qquad \text{if } m = 2k,\ n = 2k.$$

The *orthosymplectic Lie superalgebra* is defined by

$$\mathfrak{osp}(m|n) = \left\{ X \in \mathfrak{gl}(m|n) \mid X\phi + \phi X^{st} = 0 \right\},$$

where X^{st} is the *supertranspose* of $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ given by $X^{st} = \begin{bmatrix} A^t & C^t \\ -B^t & D^t \end{bmatrix}$. Similarly to what happens in the Lie algebra case, there are differences in the structure of the orthosymplectic Lie superalgebras depending on the dimension of the defining representation. For this reason, we define

$$\begin{aligned} \mathrm{B}(m,n) &= \mathfrak{osp}(2m+1|2n), && m \geq 0,\ n \geq 1,\\ \mathrm{C}(n) &= \mathfrak{osp}(2|2n-2), && n \geq 2,\\ \mathrm{D}(m,n) &= \mathfrak{osp}(2m|2n), && m \geq 2,\ n \geq 1. \end{aligned}$$

In this case, their even part is

$$\begin{aligned} \mathrm{B}(m,n)_{\bar{0}} &\cong B_m \oplus C_n,\\ \mathrm{C}(n)_{\bar{0}} &\cong \Bbbk \oplus C_{n-1},\\ \mathrm{D}(m,n)_{\bar{0}} &\cong D_m \oplus C_n, \end{aligned}$$

where B_m, C_n, D_m are the usual simple Lie algebras of type B, C, D.

There are a few exceptional Lie superalgebras. The first is the one parameter family $\mathrm{D}(2,1,\alpha)$. For each $i = 1,2,3$, let $\mathfrak{g}_i$ be the Lie algebra $\mathfrak{sl}_2$, and denote by V_i the standard $\mathfrak{sl}_2$-module associated to $\mathfrak{g}_i$. A $\mathfrak{g}_i$-module homomorphism $S^2(V_i) \to \mathfrak{g}_i$ will be an isomorphism, it is given by a non-zero scalar $a_i \in \Bbbk$ by Schur's Lemma. By Lemma 4.1.1, the super vector space $\mathfrak{g}(a_1,a_2,a_3) = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, with $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$ and $\mathfrak{g}_{\bar{1}} = V_1 \otimes V_2 \otimes V_3$, will be a Lie superalgebra if $a_1 + a_2 + a_3 = 0$. For any nonzero scalar $c \in \Bbbk$, $\mathfrak{g}(ca_1, ca_2, ca_3)$ and $\mathfrak{g}(a_1,a_2,a_3)$ are isomorphic. Furthermore, if we change the order of $\mathfrak{g}_i$ and V_i, we will have isomorphic superalgebras. Therefore, $\mathfrak{g}(a_1,a_2,a_3) \cong \mathfrak{g}\left(1, \frac{1}{a_1}, -\frac{1}{a_1} - 1\right)$. Thus, we have a one-parameter family of Lie superalgebras given by nonzero scalar $a \in \Bbbk$

$$\mathrm{D}(2,1,\alpha) = \mathfrak{g}\left(1, \frac{1}{\alpha}, -\frac{1}{\alpha} - 1\right)$$

The Lie superalgebra $\mathrm{D}(2,1,\alpha)$ is simple if $\alpha \neq 0,-1$. Note that its even part is the semisimple Lie algebra $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.

There are other two exceptional Lie superalgebras, F(4) and G(3). The even part of F(4) is the semisimple Lie algebra $\mathfrak{sl}_2 \oplus \mathfrak{so}_7$. On the other hand, the Lie superalgebra G(3) has an even part isomorphic to $\mathfrak{sl}_2 \oplus G_2$.

If $\mathfrak{g}$ is a classical basic simple Lie superalgebra, then $\mathfrak{g}$ is either a simple Lie algebra or isomorphic to one of the following algebras

$$\mathrm{A}(m,n) \text{ with } m > n \geq 0,\ \mathrm{A}(n,n) \text{ with } n > 0,$$

$$\mathrm{B}(m,n) \text{ with } m \geq 0,\ n > 0,\ \mathrm{C}(n) \text{ with } n \geq 2,$$

$$\mathrm{D}(m,n) \text{ with } m \geq 2,\ n \geq 1,\ \mathrm{D}(2,1,\alpha) \text{ with } \alpha \neq 0,-1,$$

$$\mathrm{F}(3),\ \mathrm{G}(4).$$

Some of these Lie superalgebras are isomorphic to each other. For instance, $D(2,1) \cong D(2,1,1)$ and $\mathrm{A}(2,1) \cong \mathrm{C}(2)$. Furthermore, $\mathrm{D}(2,1,\alpha) \cong \mathrm{D}(2,1,\frac{1}{\alpha}) \cong \mathrm{D}(2,1,-1-\alpha)$. Using the last number of each 3-uple in these isomorphisms, we may define an action of the permutation group S_3 in $\mathbb{C} \setminus \{0,1\}$. This action gives the isomorphism classes of the various $\mathrm{D}(2,1,\alpha)$. There are no further isomorphisms between the above superalgebras.

From now on, we suppose that $\mathfrak{g}$ is a classical basic simple Lie superalgebra. It is known the representation of $\mathfrak{g}_{\bar 0}$ on $\mathfrak{g}_{\bar 1}$ is either irreducible or a direct sum of two irreducible representations. We say $\mathfrak{g}$ is of *type II* if $\mathfrak{g}_{\bar 1}$ is a simple $\mathfrak{g}_{\bar 0}$-module. Otherwise, we say that $\mathfrak{g}$ is of *type I*. There exists a *distinguished $\mathbb{Z}$-grading* $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ such that $\mathfrak{g}_{\bar 0} = \bigoplus_{n\in 2\mathbb{Z}} \mathfrak{g}_n$, $\mathfrak{g}_{\bar 1} = \bigoplus_{n\notin 2\mathbb{Z}} \mathfrak{g}_n$, and it satisfies:

1. $\mathfrak{g}_{\bar 0} = \mathfrak{g}_0$ and $\mathfrak{g}_{\bar 1} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ if $\mathfrak{g}$ is of type I, and
2. $\mathfrak{g}_{\bar 0} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$ and $\mathfrak{g}_{\bar 1} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ if $\mathfrak{g}$ is of type II.

We summarize in Table 4.1 the list of all basic classical Lie superalgebras, their type, their even part and dimensions.

$\mathfrak{g}$	$\mathfrak{g}_{\bar 0}$	Type	Dimension
$\mathrm{A}(m,n)$, $m > n \geq 0$	$A_m \oplus A_n \oplus \Bbbk$	I	$m^2 + n^2 - 1 \vert 2mn$
$\mathrm{A}(n,n)$, $n \geq 1$	$A_n \oplus A_n$	I	$2n^2 - 2 \vert 2n^2$
$\mathrm{C}(n+1)$, $n \geq 1$	$C_n \oplus \Bbbk$	I	$2n^2 + n + 1 \vert 4n$
$\mathrm{B}(m,n)$, $m \geq 0$, $n \geq 1$	$B_m \oplus C_n$	II	$2m^2 + m + 2n^2 + n \vert 4mn + 2n$
$\mathrm{D}(m,n)$, $m \geq 2$, $n \geq 1$	$D_m \oplus C_n$	II	$2m^2 - m + 2n^2 + n \vert 4mn$
$\mathrm{F}(4)$	$A_1 \oplus B_3$	II	$24 \vert 16$
$\mathrm{G}(3)$	$A_1 \oplus G_2$	II	$17 \vert 14$
$\mathrm{D}(2,1,\alpha)$, $\alpha \neq 0, -1$	$A_1 \oplus A_1 \oplus A_1$	II	$9 \vert 8$

Table 4.1: *Basic classical Lie superalgebras that are not Lie algebras, their even part and their type*

A *Cartan subalgebra* of a basic Lie superalgebra $\mathfrak{g}$ is a Cartan subalgebra of the reductive Lie algebra $\mathfrak{g}_{\bar 0}$. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then $\mathfrak{g}$ is a weight module over $\mathfrak{h}$ with $\mathfrak{g}_0 = \mathfrak{h}$. Denote by Δ the set of nonzero weights of $\mathfrak{g}$ as a $\mathfrak{h}$-module. The set Δ is called the *root system* of $\mathfrak{g}$. Denote by $\Delta_{\bar 0}$ the weights of $\mathfrak{g}_{\bar 0}$ and $\Delta_{\bar 1}$ the weights of $\mathfrak{g}_{\bar 1}$.

We finish this section with a theorem that shows how similar basic classical Lie superalgebras are to semisimple Lie algebras.

Theorem 4.1.2. *Let $\mathfrak{g}$ be a basic Lie superalgebra with a Cartan subalgebra $\mathfrak{h}$.*

1. *We have a root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}^\alpha \quad \textit{and} \quad \mathfrak{g}_0 = \mathfrak{h}$$

2. $\dim \mathfrak{g}^\alpha = 1$ *for* $\alpha \in \Delta$.
3. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ *for each* $\alpha, \beta \in \Delta$ *such that* $\alpha + \beta \in \Delta$.
4. *There exists a non-degenerate even invariant supersymmetric bilinear form* $\langle \cdot, \cdot \rangle$ *on* $\mathfrak{g}$.

5. $\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta \rangle = 0$ *unless* $\alpha = -\beta \in \Delta$.

6. *The restriction of the bilinear form* $\langle \cdot, \cdot \rangle$ *on* $\mathfrak{h} \times \mathfrak{h}$ *is non-degenerate.*

7. $\Delta = -\Delta$, $\Delta_{\bar{0}} = -\Delta_{\bar{0}}$, *and* $\Delta_{\bar{1}} = -\Delta_{\bar{1}}$.

8. *Let* $\alpha \in \Delta$. *Then* $k\alpha \in \Delta$ *for* $k \neq \pm 1$ *if and only if* $\alpha \in \Delta_{\bar{1}}$ *and* $\langle \alpha, \alpha \rangle \neq 0$*; in this case* $k = \pm 2$.

9. *There exists* $x_\alpha \in \mathfrak{g}^\alpha$ *such that* $[x_\alpha, x_{-\alpha}] = \langle x_\alpha, x_{-\alpha} \rangle h_\alpha$ *where* h_α *is the cooroot determined by* $\langle h_\alpha, h \rangle = \alpha(h)$ *for* $h \in \mathfrak{h}$.

From now on, if $\mathfrak{g}$ is a basic Lie superalgebra, we define $x_\alpha \in \mathfrak{g}^\alpha$, $h_\alpha \in \mathfrak{h}$, elements such that $[x_\alpha, x_{-\alpha}] = h_\alpha$ for each $\alpha \in \Delta$. An odd root $\alpha \in \Delta$ is called *isotropic* if $\langle \alpha, \alpha \rangle = 0$.

4.2 Tensor product theorem

In this section, we will show a few general results about the representation theory of Lie superalgebras. Unless otherwise stated, we will not make any assumptions about $\mathfrak{g}$ other than being over $\Bbbk$.

For a vector space V, we denote ι_V as the identity map of V.

Lemma 4.2.1 (Schur's Lemma)**.** *Let* $\mathfrak{g}$ *be a Lie superalgebra and* V *be an irreducible* $\mathfrak{g}$*-module. Then either* $\mathrm{End}_{\mathfrak{g}}(V) = \mathrm{End}_{\mathfrak{g}}(V)_{\bar{0}} = \Bbbk\iota_V$ *or* $V_{\bar{0}} \cong V_{\bar{1}}$ *and* $\mathrm{End}_{\mathfrak{g}}(V) = \Bbbk\iota_V \oplus \Bbbk\sigma$*, where* $\sigma^2 = \iota_V$ *is a parity reversing map that permutes* $V_{\bar{0}}$ *and* $V_{\bar{1}}$.

Proof. First, note that every element of $\mathrm{End}_{\mathfrak{g}}(V)$ is an isomorphism, because V is simple, and kernels and images of $\mathfrak{g}$-endomorphisms of V are submodules of it. Since composition of $\mathfrak{g}$-modules isomorphisms is an isomorphism and $\iota_V \in \mathrm{End}_{\mathfrak{g}}(V)_{\bar{0}}$, the vector space $\mathrm{End}_{\mathfrak{g}}(V)_{\bar{0}}$ is a division ring. Hence, $\mathrm{End}_{\mathfrak{g}}(V)_{\bar{0}} = \Bbbk\iota_V$ by the argument of the Schur's lemma (due to Diximier) for Lie algebras. If $\mathrm{End}_{\mathfrak{g}}(V)_{\bar{1}} = 0$, then $\mathrm{End}_{\mathfrak{g}}(V) = \mathrm{End}_{\mathfrak{g}}(V)_{\bar{0}} = \Bbbk\iota_V$. Suppose $\mathrm{End}_{\mathfrak{g}}(V)_{\bar{1}} \neq 0$. Take a non-zero element $\sigma \in \mathrm{End}_{\mathfrak{g}}(V)_{\bar{1}}$, then $\sigma^2 \in \mathrm{End}_{\mathfrak{g}}(V)_{\bar{1}}$ is a non-zero even element. So $\sigma^2 = c\iota_V$ for some non-zero $c \in \Bbbk$. We may assume that $c = 1$. The restriction $\sigma|_{V_{\bar{0}}}$ is a $\mathfrak{g}_{\bar{0}}$-isomorphism between $V_{\bar{1}}$ and $V_{\bar{0}}$ with inverse $\sigma|_{V_{\bar{1}}}$, thus σ is a parity reversing map. If $\tau \in \mathrm{End}_{\mathfrak{g}}(V)_{\bar{1}}$ is any other non-zero element, then $\tau^2 = a\iota_V$ for some non-zero scalar $a \in \Bbbk$, and $\tau = b\sigma$ for some $b \in \Bbbk$ with $b^2 = a$. Therefore, $\mathrm{End}_{\mathfrak{g}}(V)_{\bar{1}} = \Bbbk\sigma$. □

Remark 4.2.2. Note that $\sigma : V_{\bar{0}} \to V_{\bar{1}}$ is an isomorphism of $\mathfrak{g}_{\bar{0}}$-modules. We used that $\Bbbk$ is an uncountable algebraically closed field of characteristic zero. If we assume that $\mathfrak{g}$ is finite-dimensional, we may drop the assumption that $\Bbbk$ is uncountable using Quillen's argument, see [Qui69].

Lemma 4.2.3 (Density Theorem)**.** *Let* $\mathfrak{g}$ *be a Lie superalgebra, and* V *be a simple* $\mathfrak{g}$*-module*

1. *Assume* $\mathrm{End}_{\mathfrak{g}}(V)_{\bar{1}} = 0$. *If* $v_1, \dots, v_n \in V$ *are linearly independent and* $w_1, \dots, w_n \in V$*, then there exists* $u \in U(\mathfrak{g})$ *such that* $uv_i = w_i$ *for each* $i = 1, \dots, n$.

2. *Assume* $\mathrm{End}_{\mathfrak{g}}(V)_{\bar{1}} \neq 0$. *If* $v_1, \dots, v_n \in V_i$ *are linearly independent homogeneous elements and* $w_1, \dots, w_n \in V_i$ *with* $i \in \{\bar{0}, \bar{1}\}$, *then there exists an even element* $u \in U(\mathfrak{g})_{\bar{0}}$ *such that* $uv_j = w_j$ *for each* $j = 1, \dots, n$.

Proof. Using that $\mathrm{End}_{\mathfrak{g}}(V)_{\bar{0}} = \Bbbk\iota_V$, both statements follow from the Jacobson Density Theorem (see [Isa09, Theorem 14.15]) and Schur's Lemma (Lemma 4.2.1). □

Lemma 4.2.4. *Let* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *be Lie superalgebras, and* V_1, V_2 *be simple modules over* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$, *respectively. If* $\mathrm{End}_{\mathfrak{g}_1}(V_1) \cong \Bbbk$, *then* $V_1 \otimes V_2$ *is a simple* $\mathfrak{g}_1 \oplus \mathfrak{g}_2$*-module.*

Proof. We only need to show that each non-zero homogeneous element of $V_1 \otimes V_2$ generates the whole $\mathfrak{g}_1 \oplus \mathfrak{g}_2$-module. Let $v \in V_1 \otimes V_2$ be an arbitrary non-zero homogeneous element, and write $v = \sum_{j=1}^{n} v_j^1 \otimes v_j^2$. We may assume that $\{v_1^1, \dots, v_n^1\}$ is a linearly independent set of homogeneous elements, and $v_1^2 \neq 0$. By Lemma 4.2.3 (1), there exists $u \in U(\mathfrak{g})$ such that $uv_j^1 = \delta_{j1}v_1^1$. Therefore,

$$uv = \sum_{j=1}^{n} (uv_j^1) \otimes v_j^2 = v_1^1 \otimes v_1^2 \neq 0.$$

Take $w_1 \in V_1$ and $w_2 \in V_2$. Since both V_1 and V_2 are simple, there exist $u_1 \in U(\mathfrak{g}_1)$ and $u_2 \in U(\mathfrak{g}_2)$ such that $u_1 v_1^1 = w_1$ and $u_2 v_1^2 = w_2$. Hence,

$$u_2(u_1(uv)) = (-1)^{|u_2||v_1^1|} w_1 \otimes w_2.$$

We conclude that any simple tensor is an element of $U(\mathfrak{g}_1 \oplus \mathfrak{g}_2)v$. Therefore, $U(\mathfrak{g}_1 \otimes \mathfrak{g}_2)v = V_1 \otimes V_2$. □

Proposition 4.2.5. *Let* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *be Lie superalgebras, and* V_1, V_2 *be simple modules over* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$, *respectively. Then* $V_1 \otimes V_2$ *is either a simple* $\mathfrak{g}_1 \oplus \mathfrak{g}_2$*-module, or it is isomorphic to* $V \oplus V$, *where* V *is a simple* $\mathfrak{g}_1 \oplus \mathfrak{g}_2$*-module.*

Proof. If $\mathrm{End}_{\mathfrak{g}_1}(V_1) \cong \Bbbk$ or $\mathrm{End}_{\mathfrak{g}_2}(V_2) \cong \Bbbk$, then $V_1 \otimes V_2$ is a simple $\mathfrak{g}_1 \oplus \mathfrak{g}_2$-module by Lemma 4.2.4. By Lemma 4.2.1, we assume that $\mathrm{End}_{\mathfrak{g}_1}(V_1) = \Bbbk I_{V_1} \oplus \Bbbk\sigma_1$ and $\mathrm{End}_{\mathfrak{g}_2}(V_2) \cong \Bbbk I_{V_2} \oplus \Bbbk\sigma_2$, where $\sigma_1^2 = I_{V_1}$ and $\sigma_2^2 = -I_{V_2}$.

The endomorphism $\sigma : V_1 \otimes V_2 \to V_1 \otimes V_2$ given by $\sigma(v_1 \otimes v_2) = (-1)^{|v_1|}\sigma_1(v_1) \otimes \sigma_2(v_2)$ is a $\mathfrak{g}_1 \oplus \mathfrak{g}_2$-module isomophism. Therefore, $\sigma^2 = id_{V_1 \otimes V_2}$.

For each $x \in V_1 \otimes V_2$

$$x = \frac{x + \sigma(x)}{2} + \frac{x - \sigma(x)}{2}.$$

Thus, $V_1 \otimes V_2 = V \oplus V'$ with

$$V = \{x \in V_1 \otimes V_2 \mid \sigma(x) = x\} \quad \text{and} \quad V' = \{x \in V_1 \otimes V_2 \mid \sigma(x) = -x\}.$$

Since σ is a $\mathfrak{g}_1 \oplus \mathfrak{g}_2$-module homomorphism, both V and V' are $\mathfrak{g}_1 \oplus \mathfrak{g}_2$-submodules of $V_1 \otimes V_2$.

The map σ_2 gives an isomorphism between the even and the odd part of V_2, so if $\{w_i \in V_2 \mid i \in I\}$ is a basis of the even part of V_2, then $\{w_i, \sigma_2(w_i) \mid i \in I\}$ is a basis of V_2. $V_1 \otimes V_2$ is generated, as a vector space, by the set $\{v \otimes w_j \pm \sigma(v \otimes w_j) \mid v \in V_1,\ j \in I\}$. Since $\{v \otimes w_j + \sigma(v \otimes w_j) \mid v \in V_1,\ j \in I\} \subset V$ and $\{v \otimes w_j - \sigma(v \otimes w_j) \mid v \in V_1,\ j \in I\} \subset V'$, they generate V and V' as vector spaces, respectively. For each $j \in I$ and $v \in V$, we have

$$\begin{aligned}\sigma\left(v \otimes w_j + (-1)^{|v|}\sigma_1(v) \otimes \sigma_2(w_j)\right) &= (-1)^{|v|}\sigma_1(v) \otimes \sigma_2(w_j) - \sigma_1^2(v) \otimes \sigma_2^2(w_j) \\ &= -v \otimes w_j + (-1)^{|v|}\sigma_1(v) \otimes \sigma_2(w_j).\end{aligned}$$

Similarly, $\sigma\left(v \otimes w_j - (-1)^{|v|}\sigma_1(v) \otimes \sigma_2(w_j)\right) = v \otimes w_j + (-1)^{|v|}\sigma_1(v) \otimes \sigma_2(w_j)$. Therefore, σ sends V to V' and V' to V. Thus, V and V' are isomorphic.

It remains to prove that V is a simple $\mathfrak{g}_1 \oplus \mathfrak{g}_2$-module. Let $v \in V$ be a non-zero homogeneous element of V. Then there exists homogeneous elements $v_1, \dots, v_n$ such that

$$v = \sum_{j=1}^{n} v_j \otimes w_{i_j} + \sigma(v_j \otimes w_{i_j})$$

where $i_j \neq i_l$ if $j \neq l$, and $v_1 \neq 0$. By Lemma 4.2.3, there exists $u \in U(\mathfrak{g}_2)$ such that $uw_{i_j} = \delta_{1j} w_{i_1}$. Hence, $uv = v_1 \otimes w_{i_1} + \sigma(v_j \otimes w_{i_1})$. Take a non-zero homogeneous element $v_0 \in V_1$ and $k \in I$, then there exists $a \in U(\mathfrak{g}_1)$ such that $av_1 = v_0$ and $b w_{i_1} = w_j$ because V_1 and V_2 are simple. Thus,

$$a(b(uv)) = v_0 \otimes w_k + \sigma(v_0 \otimes w_k).$$

We conclude that the generating set $\{v \otimes w_j + \sigma(v \otimes w_j) \mid v \in V_1,\ j \in I\}$ of V is a subset of $U(\mathfrak{g}_1 \oplus \mathfrak{g}_2)v$ for every non-zero homogeneous element $v \in V$. Therefore, V is a simple $\mathfrak{g}_1 \oplus \mathfrak{g}_2$-module. □

Definition 4.2.6. Using the notation of Proposition 4.2.5, we define the *irreducible tensor product* of V_1 and V_2 as

$$V_1 \hat{\otimes} V_2 = \begin{cases} V_1 \otimes V_2, & \text{if } V_1 \otimes V_2 \text{ is simple,} \\ V, & \text{if } V_1 \otimes V_2 \text{ is not simple,} \end{cases}$$

where V is a simple submodule of $V_1 \otimes V_2$, obtained in the proof of Proposition 4.2.5, for which we have an isomorphism of $(\mathfrak{g}_1 \oplus \mathfrak{g}_2)$-modules $V_1 \otimes V_2 \cong V \oplus V$.

4.3 Weight modules

Let $\mathcal{G}$ be a Lie superalgebra and $\mathfrak{h} \subset \mathcal{G}$ be an abelian subalgebra. An $\mathfrak{h}$-module V is said to be a *weight* module if

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V^\lambda,$$

where $V^\lambda = \{v \in V \mid hv = \lambda(h)v\}$. The set $\mathrm{Supp}(V) = \{\lambda \in \mathfrak{h}^* \mid V^\lambda \neq 0\}$ is called the *support* of V, and an element $\lambda \in \mathrm{Supp}(V)$ is called a *weight* of V. We say that V^λ is a *weight space* and its nonzero elements are called *weight vectors*. It is known that every submodule of a weight module is a weight module.

If the adjoint representation makes $\mathcal{G}$ a weight module over $\mathfrak{h}$ and $\mathfrak{h}$ is finite-dimensional, we say that $(\mathcal{G}, \mathfrak{h})$ is a *splitting pair*. If $(\mathcal{G}, \mathfrak{h})$ is a splitting pair, a module over $\mathcal{G}$ is *weight $\mathcal{G}$-module* if it is a weight module as a module over $\mathfrak{h}$. A weight $\mathcal{G}$-module is called *finite* if the dimension of its weight spaces is finite. A weight $\mathcal{G}$-module is called *bounded* if the set of the dimensions of all its weight spaces is bounded by some positive number.

Example 4.3.1. If $(\mathfrak{g}, \mathfrak{h})$ is splitting pair and A is commutative superalgebra, then $\mathcal{G} = \mathfrak{g}\otimes A$ is a Lie superalgebra with bracket given by

$$[x \otimes f, y \otimes g] = (-1)^{|f||y|}[x, y] \otimes fg, \quad x, y \in \mathfrak{g},\ f, g \in S.$$

Therefore, the adjoint representation makes $\mathcal{G}$ a weight module over $\mathfrak{h} \cong \mathfrak{h} \otimes \Bbbk \subset \mathcal{G}$. Hence, $(\mathcal{G}, \mathfrak{h} \otimes \Bbbk)$ is a splitting pair.

Example 4.3.2. If $(\mathcal{G}_1, \mathfrak{h}_1)$ and $(\mathcal{G}_2, \mathfrak{h}_2)$ are splitting pairs, then $(\mathcal{G}_1\oplus\mathcal{G}_2, \mathfrak{h}_1\oplus\mathfrak{h}_2)$ is a splitting pair.

Lemma 4.3.3. *Let $(\mathfrak{g}, \mathfrak{h})$ be a splitting pair and V be a finite weight $\mathfrak{g}$-module. If there is $\lambda \in \mathfrak{h}^*$ such that $\{w \in W \mid hw = \lambda(h)w$ for all $h \in \mathfrak{h}\}$ is nonzero for all submodule $W \subset V$, then V contains a simple $\mathfrak{g}$-module.*

Proof. It follows from the same argument given in the Lie algebra case, see [BLL15, Lemma 3.3].

□

Proposition 4.3.4. *Let $(\mathcal{G}_1, \mathfrak{h}_1)$ and $(\mathcal{G}_2, \mathfrak{h}_2)$ be splitting pairs and V a simple finite weight module over $\mathcal{G}_1 \oplus \mathcal{G}_2$. Then, there exist finite weight modules V_1 and V_2 over $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively, such that $V \cong V_1\widehat{\otimes}V_2$.*

Proof. This proof is heavily based on the argument given for Lie algebras, see [BLL15, Proposition 3.4]. Let $v \in V^{(\lambda,\mu)}$ be a non-zero vector of weight $(\lambda, \mu) \in \mathfrak{h}_1^* \times \mathfrak{h}_2^*$. For each $u \in U(\mathcal{G}_2)$ and $h_1 \in \mathfrak{h}_1^*$, $h_1 uv = \lambda(h_1)uv$. Therefore, $(U(\mathcal{G}_2)v)^\eta \subset V^{(\lambda,\eta)}$, and $W = U(\mathcal{G}_2)v \subset V$ is a finite weight module over $\mathcal{G}_2$.

Let N be any non-zero $\mathcal{G}_2$-submodule of W. Consider the subspace H^N of $\mathrm{Hom}_\Bbbk(N, V)$ of all generated by all homogeneous elements $\varphi \in \mathrm{Hom}_\Bbbk(N, V)$ such that $y\varphi(w) = (-1)^{|\varphi||y|}\varphi(yw)$ for all $y \in \mathcal{G}_2$, $w \in N$. Then H^N is a $\mathcal{G}_1$-module with action given by $(x\varphi)(w) = x\varphi(w)$ for each $x \in \mathcal{G}_1$, $\varphi \in H^N$, and $w \in W$. For a $\mathcal{G}_1$-submodule $M \subset H^N$, the map

$$\begin{aligned} \Psi_{M,N} : M \otimes N &\to V \\ \varphi \otimes w &\mapsto \varphi(w) \end{aligned}$$

is a $\mathcal{G}_1 \oplus \mathcal{G}_2$-module homomorphism. Suppose M is non-zero, then $\Psi_{M,N}$ is surjective because V is simple. Therefore, the image of $(M \otimes N)^{(\lambda,\mu)} = M^\lambda \otimes N^\mu$ under $\Psi_{M,N}$ is exactly $V^{(\lambda,\mu)}$. Thus, both M^λ and N^μ are non-zero.

For every non-zero submodule $N \subset W$, we have that $N^\mu \neq 0$. By Lemma 4.3.3, W contains a simple $\mathcal{G}_2$-module Q. By the simplicity of Q and the finiteness of the weight spaces of V, it is possible to show that H^Q is a finite weight module over $\mathcal{G}_1$. Since for every weight module $M \subset H^Q$ we have that $M^\lambda \neq 0$, there exists a simple $\mathcal{G}_1$-submodule P of H^Q.

By Proposition 4.2.5, $P \otimes Q$ is either simple or there exists a simple $\mathcal{G}_1 \oplus \mathcal{G}_2$-module L such that $P \otimes Q \cong L \oplus L$. If $P \otimes Q$ is simple, then $\Psi_{P,Q}$ is an isomorphism. On the other hand, if $P \otimes Q \cong L \oplus L$, then every non-trivial proper submodule and quotient of $P \otimes Q$ is isomorphic to L. Therefore, $\Psi_{P,Q}$ induces a isomorphism between L and V. We conclude that $V \cong P \hat{\otimes} Q$. □

4.4 The S-annihilator of a representation

From now on, let $\mathfrak{g}$ be a basic classical Lie superalgebra, S an unital finitely generated commutative superalgebra and $\mathcal{G} = \mathfrak{g} \otimes S$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then, $(\mathfrak{g}, \mathfrak{h})$ is a splitting pair. By example 4.3.1, $(\mathcal{G}, \mathfrak{h} \otimes \Bbbk)$ is a splitting pair as well. We aim to give a classification of finite weight $\mathcal{G}$-modules in terms of maximal ideals of S and finite weight $\mathfrak{g}$-modules. To do it, we need to define and study *S-annihilator* of a $\mathcal{G}$-representation.

Definition 4.4.1. If V is a $\mathcal{G}$-module, then we define the *S-annihilator* $\mathrm{Ann}_S(V)$ of V as the largest ideal I of S with the property $(\mathfrak{g} \otimes I)V = 0$.

Lemma 4.4.2. *If V is a finite $\mathcal{G}$-module, then*

$$\mathrm{Ann}_S(V) = \{f \in S \mid (\mathfrak{g} \otimes f)V = 0\}.$$

Proof. The set $I = \{f \in S \mid (\mathfrak{g} \otimes f)V = 0\}$ is the largest set with the property $(\mathfrak{g} \otimes I)V = 0$. We have that $(\mathfrak{g} \otimes fg)V = [\mathfrak{g} \otimes f, \mathfrak{g} \otimes g]V$ for every $f \in I$ and $g \in S$ because $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Thus, I is an ideal. □

Proposition 4.4.3. *Let $\mathfrak{g}$ be a basic classical Lie superalgebra, and S a commutative superalgebra. If $\rho : \mathcal{G} \to \mathrm{End}(V)$ is a representation of $\mathcal{G}$, then* $\ker(\rho) = \mathfrak{g} \otimes \mathrm{Ann}_S(V)$.

Proof. Set $\mathcal{I} = \ker(\rho)$. Fix a homogeneous element $v \in \mathcal{I}$, then there exists $a_\alpha \in S$, $\alpha \in \Delta$, $w \in \mathfrak{h} \otimes S$ such that

$$v = \sum_{\alpha \in \Delta} x_\alpha \otimes a_\alpha + w.$$

Since $\mathcal{I}$ is a submodule of the weight $\mathcal{G}$-module $\mathcal{G}$, we have that $\mathcal{I}$ is a weight module, and its weight spaces are $(\mathfrak{g}^\alpha \otimes S) \cap \mathcal{I}$. Therefore, we have that $x_\alpha \otimes a_\alpha \in \mathcal{I}$ for each $\alpha \in \Delta$ because $\dim \mathfrak{g}^\alpha = 1$. If $x \otimes a \in \mathcal{I}$, we may consider the $\mathfrak{g}$-module generated by $x \otimes a$ inside of $\mathcal{I}$. We have that $U(\mathfrak{g})(x \otimes a) = (U(\mathfrak{g})x) \otimes a = \mathfrak{g} \otimes a$, because $\mathfrak{g}$ is simple. Therefore, $\mathfrak{g} \otimes a_\alpha \in \mathcal{I}$, so $a_\alpha \in \mathrm{Ann}_S(V)$.

Now, consider $w \in \mathcal{I} \cap (\mathfrak{h} \otimes S)$. Since $\mathfrak{g}$ is a basic classical Lie superalgebra, there exists an even nondegenerated $\mathfrak{g}$-invariant bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ such that $[x_\alpha, x_\alpha] = \langle x_\alpha, x_{-\alpha} \rangle h_\alpha$ where h_α is such that $\langle h_\alpha, h \rangle = \alpha(h)$. This form is still nondegenerate when restricted to $\mathfrak{h}$.

Consider a simple root system $\alpha_1, \dots, \alpha_r$ of Δ, and write

$$w = \sum_{i=1}^{r} h_{\alpha_i} \otimes a_i.$$

Hence,

$$\left[x_{\alpha_j} \otimes 1, \sum_{i=1}^{r} h_{\alpha_i} \otimes a_i \right] = x_{\alpha_j} \otimes \left(-(-1)^{|x_{\alpha_j}|} \sum_{i=1}^{r} \alpha_j(h_{\alpha_i}) a_i \right) \in \mathcal{I}$$

for each $j = 1, \dots, r$. By a similar argument we gave before, we have that $\sum_{i=1}^{r} \alpha_j(h_{\alpha_i}) a_i \in \operatorname{Ann}_S(V)$. Since $\langle \cdot, \cdot \rangle$ is non-degenrated, the matrix $(\langle \alpha_i, \alpha_j \rangle)_{i,j=1}^{r} = (\alpha_i(h_{\alpha_j}))_{i,j=1}^{r}$ is invertible. Hence, the linear system $\sum_{i=1}^{r} \alpha_j(h_{\alpha_i}) a_i$, $i = 1, \dots, r$, has a solution, and this implies that $a_i \in \operatorname{Ann}_S(V)$.

□

Proposition 4.4.4. *If V is a simple finite weight $\mathcal{G}$-module, then* $\dim S/\operatorname{Ann}_S(V) < \infty$.

Proof. To prove that $S/\operatorname{Ann}_S(V)$ is a finite-dimensional algebra, we will use the same argument given in [BLL15]. We will leave it here for completion. Since $\operatorname{Ann}_S(V)$ is an ideal, we only need to prove the vector space $S/\operatorname{Ann}_S(V)$ has finite dimension. For each $\lambda \in \operatorname{Supp}(V)$, define

$$J(\lambda) = \left\{ a \in S \mid (x \otimes a)V^{\lambda} = 0 \ \forall x \in \mathfrak{g} \right\},$$

and $J(\lambda, \Delta) = \bigcap_{\mu \in \lambda + \Delta \cup \{0\}} J(\mu)$. Let B_α be a basis of $\mathfrak{g}^{\alpha}$ and B^{μ} is a basis for V^{μ} for each $\alpha \in \Delta \cup \{0\}$, $\mu \in \operatorname{Supp}(V)$.

Claim 1: $\dim S/J(\lambda, \Delta) < \infty$.

For each $\alpha \in \Delta \cup \{0\}$, $x \in \mathfrak{g}$, and $v \in V^{\mu}$ where $\mu \in \lambda + \Delta$, the map $\eta_{x,v} : S \to V^{\mu+\alpha}$ given by $\eta(a) = (x \otimes a)v$ induces an injection from $S/\ker \eta_{x,v}$ to the finite-dimensional vector space $V^{\mu+\alpha}$. Therefore, the vector space $S/\ker \eta_{x,v}$ has finite dimension, and

$$J(\lambda, \Delta) = \bigcap_{\mu \in \lambda + \Delta \cup \{0\}} \bigcap_{v \in B^{\mu}} \bigcap_{\alpha \in \Phi} \bigcap_{x \in B_\alpha} \ker_{x,v}$$

is the intersection of finitely many vector spaces with finite codimension. Hence, the map

$$\begin{aligned} S/J(\lambda, \Delta) \to & \bigoplus_{\substack{\alpha, \beta \in \Delta \cup \{0\} \\ x \in B_\alpha, \ v \in B^{\lambda+\beta}}} S/\ker \eta_{x,v} \\ a + J(\lambda, \Delta) \mapsto & \ (a + \ker \eta_{x,v}) \end{aligned}$$

is a monomorphism on a finite-dimensional vector space. Thus, $S/J(\lambda, \Delta)$ has finite dimension.

Claim 2: $J(\lambda,\Delta)S \subset J(\lambda)$.

Let $a \in J(\lambda,\Delta)S$, $b \in S$, $v \in V^\lambda$. If $\alpha,\beta \in \Delta \cup \{0\}$, $x \in B_\alpha$ and $y \in B_\beta$, then

$$\begin{aligned}((-1)^{|b||y|}[x,y] \otimes ba)v =&[x \otimes b, y \otimes a]v \\ =&(x \otimes b)(y \otimes a)v - (-1)^{(|x|+|b|)(|y|+|b|)}(y \otimes a)(x \otimes b)v \\ =&(x \otimes b)(y \otimes a)v \in (x \otimes r)V^{\lambda+\beta} = \{0\}.\end{aligned}$$

Since $\mathfrak{g}$ is simple, $([\mathfrak{g},\mathfrak{g}] \otimes ba)v = (\mathfrak{g} \otimes ba)v$ for all $v \in V^\lambda$. Thus, $J(\lambda,\Delta)S \subset J(\lambda)$.

Claim 3: $S/\mathrm{Ann}_S(V)$ has finite dimension.

Let $v \in V^\lambda$ be a nonzero weight vector. Since V is simple, every element of V is equal to a linear combination of elements of the form $(x_1 \otimes a_1)\cdots(x_k \otimes a_k)v$, where $x_1,\dots,x_k \in$, and $a_1,\dots,a_k \in S$. We will use induction on $k \geq 0$ to prove that

$$(x \otimes ra)(x_1 \otimes a_1)\cdots(x_k \otimes a_k)v = 0$$

for every $x \in \mathfrak{g}$, $r \in J(\lambda,\Delta)$, $a \in S$. If $k = 0$, then $(x \otimes ra)v = 0$ by Claim 2. Assume $k > 0$. Thus,

$$\begin{aligned}(x \otimes ra)(x_1 \otimes a_1)\cdots(x_k \otimes a_k)v =&(-1)^{(|x_1|+|a_1|)(|x|+|ra|)}(x_1 \otimes a_1)(x \otimes ra)\cdots(x_k \otimes a_k)v \\ &+ (-1)^{|ra||x_1|}([x,x_1] \otimes raa_1)\cdots(x_k \otimes a_k)v\end{aligned}$$

By induction hypotheses, both summands on the right-hand side of the previous equation are zero. If we take $a = 1$, we conclude that $J(\lambda,\Delta) \subset \mathrm{Ann}_S(V)$. By Claim 1, $\dim(S/\mathrm{Ann}_S(V)) \leq \dim(S/J(\lambda,\Delta)) < \infty$. □

4.5 The shadow of a module

Recall that $\mathfrak{h}$ is a Cartan subalgebra of the basic classical Lie superalgebra $\mathfrak{g}$. The following proposition is a simple generalization of a well-known result on weight modules over Lie algebras.

Proposition 4.5.1. *Let V be a simple finite weight $\mathcal{G}$-module. If α is an even root and $a \in S_{\bar{0}}$, then $x_\alpha \otimes a$ acts either injectively or nilpotently on V. Furthermore, the following conditions are equivalent:*

1. *For each $\lambda \in \mathrm{Supp}(V)$, the weight space $V^{\lambda+n\alpha}$ is zero for all but finitely many $n > 0$.*
2. *There exists $\lambda \in \mathrm{Supp}(V)$ such that the weight space $V^{\lambda+n\alpha}$ is zero for all but finitely many $n > 0$.*
3. *The element $x_\alpha \otimes s$ acts locally nilpotently on V for all $s \in S_{\bar{0}}$.*
4. *The element $x_\alpha \otimes 1$ acts locally nilpotently on V.*

Proof. It follows from the same arguments given for Lie algebras, see [Lau18, Lemma 2.1, Proposition 2.2]. □

Definition 4.5.2. The set $\operatorname{inj}(V) \subset \Delta_{\bar{0}}$ is defined as the set of all $\alpha \in \Delta_{\bar{0}}$ such that $x_\alpha \otimes 1$ acts injectively on V.

Lemma 4.5.3. *If V is a simple finite weight $\mathcal{G}$-module, then* $\operatorname{inj}(V)$ *is closed, i.e., if $\alpha, \beta \in$* $\operatorname{inj}(V)$ *with $\alpha + \beta \in \Delta$, then $\alpha + \beta \in \operatorname{inj}(V)$.*

Proof. Let $\alpha, \beta \in \Delta_{\bar{0}}$ such that $\alpha + \beta \in \Delta$. By Proposition 4.5.1, $V^{x+n\alpha+n\beta} = V^{\alpha+n(\alpha+\beta)}$ is non-zero for infinite many $n > 0$ and $\lambda \in \operatorname{Supp}(V)$. By the same proposition, $x_{\alpha+\beta} \otimes 1$ acts injectively on V, so $\alpha + \beta \in \operatorname{inj}(V)$. □

Let V be a simple finite weight $\mathcal{G}$-module. We want to extend the definition of $\operatorname{inj}(V)$ for odd roots as well. However, elements like $x_\alpha \otimes 1$ will always act nilpotently on V if α is an isotropic root because $2(x_\alpha \otimes 1)^2 = [x_\alpha \otimes 1, x_\alpha \otimes 1] = 0$. Therefore, we will use the ideas presented in [DMP00] to extend our definition.

A *cone* C is a finitely generated submonoid of Q. The *saturation* $\overline{C}$ of a cone C is the set

$$\overline{C} = \{\alpha \in Q \mid m\alpha \in C \text{ for some integer } m > 0\}.$$

For a weight module V, define C_V^1 as the cone generated by $\operatorname{inj}(V)$ and C_V^2 as the cone generated by all $\alpha \in Q$ such that $\alpha + \operatorname{Supp}(V) \subset \operatorname{Supp}(V)$.

Definition 4.5.4. A simple finite weight $\mathcal{G}$-module is called *compatible* if there exists a finite set $\Theta \subset \operatorname{Supp}(V)$ such that $\operatorname{Supp}(V) = \Theta + C_V^1$.

Proposition 4.5.5. *Let V be a simple finite weight $\mathcal{G}$-module. If $S_{\bar{1}} = 0$, then V is compatible. In particular, $\overline{C_V^1} = \overline{C_V^2}$.*

Proof. Since $S_{\bar{1}} = 0$,

$$U(\mathcal{G}) \cong U(\mathfrak{g}^{\alpha_t} \otimes S) \cdots U(\mathfrak{g}^{\alpha_1}) U(\mathfrak{h} \otimes S) \bigwedge (\mathfrak{g}_{\bar{1}} \otimes S)$$

as a vector space, where $\Delta_{\bar{0}} = \{\alpha_1, \ldots, \alpha_t\}$. Let $\lambda \in \operatorname{Supp}(V)$, and define

$$W_0(\lambda) = \left(U(\mathfrak{h} \otimes S) \bigwedge (\mathfrak{g}_{\bar{1}} \otimes S)\right) V^\lambda,$$
$$W_i(\lambda) = U(\mathfrak{g}^{\alpha_i} \otimes S) \cdots U(\mathfrak{g}^{\alpha_1} \otimes S) W_0(\lambda).$$

for each $i = 1, \ldots, t$. Each $W_i(\lambda)$ is a weight module over $\mathfrak{h}$. Define $S_i(\lambda)$ as the set of weights of $W_i(\lambda)$.

Because both V^λ and $S/\operatorname{Ann}_S(V)$ are finite-dimensional super vector spaces and the exterior algebra of a finite-dimensional vector space is finite-dimensional, we have that $(\bigwedge \mathfrak{g}_{\bar{1}} \otimes S/\operatorname{Ann}_S(V))\, V^\lambda$ is finite-dimensional. Thus, the subspace

$$W_0(\lambda) = \left(U(\mathfrak{h} \otimes S/\operatorname{Ann}_S(V)) \bigwedge (\mathfrak{g}_{\bar{1}} \otimes (S/\operatorname{Ann}_S(V)))\right) V^\lambda$$

has finite dimension as well, since it has the same weights as $(\bigwedge \mathfrak{g}_{\bar{1}} \otimes S/\operatorname{Ann}_S(V))\, V^\lambda$. Therefore, $S_0(\lambda)$ is a finite set. Assume that $\Delta_{\bar{0}} \setminus \operatorname{inj}(V) = \{\alpha_1, \ldots, \alpha_a\}$ and $\operatorname{inj}(V) = \{\alpha_{a+1}, \ldots, \alpha_t\}$. By Proposition 4.5.1, the set $\operatorname{Supp} V \cap \{\gamma + n\alpha_k \mid n \geq 0\}$ is finite for each

$\gamma \in \operatorname{Supp} V$. Therefore, for each $0 < k \leq a$,

$$S_k(\lambda) = \bigcup_{\gamma \in S_{k-1}(\lambda)} \operatorname{Supp} V \cap \{\gamma + n\alpha_k \mid n \geq 0\}$$

is a finite union of finite sets. We conclude that $S_a(\lambda)$ is a finite set. Set $\Theta = S_a(\lambda)$. Since each root α_{a+i} is injective, $\gamma + n\alpha_{a+i} \in \operatorname{Supp}(V)$ for every $\gamma \in \Theta$. Therefore, $S_t(\lambda) = \Theta + C_V^1$. Since V is simple, $S_t(\lambda) = \operatorname{Supp}(V)$. We conclude that $\operatorname{Supp}(V) = \Theta + C_V^1$. □

Remark 4.5.6. We note that the same proof works for the case where $\mathfrak{g}$ is a simple Lie algebra and S is a commutative superalgebra.

For a simple compatible finite weight $\mathcal{G}$-module V, we denote by C_V the saturation of the cone C_V^1 and C_V^2.

Proposition 4.5.7. *Let V a simple compatible finite weight $\mathcal{G}$-module, and $\alpha \in \Delta_{\bar{1}}$. Then $V^{\lambda+n\alpha}$ is non-zero for infinitely many $n > 0$ and any $\lambda \in \operatorname{Supp}(V)$ if and only if $\alpha \in C_V$.*

Proof. Suppose $V^{\lambda+n\alpha}$ is non-zero for infinitely many $n > 0$ and any $\lambda \in \operatorname{Supp}(V)$. Let $\Theta \subset \operatorname{Supp}(V)$ be a finite set given by Proposition 4.5.5 such that $\operatorname{Supp}(V) = C_V^1 + \Theta$. There exists a sequence $n_1 < n_2 < n_3 < \cdots$ of positive integers and $\gamma \in \Theta$ such that $\gamma + n_k\alpha \in \operatorname{Supp}(M) \setminus \Theta$ for every $k > 0$. Since the set $\{n_k \mid k > 0\}$ is infinite and every $\gamma + n_k\alpha$ is in the same coset of $\Theta + C_V^1$, there exists $\lambda \in \Theta$ and $0 < p < q$ such that

$$\gamma + n_p\alpha = \lambda + \beta_1 \quad \text{and} \quad \gamma + n_q\alpha = \lambda + \beta_2$$

for some $\beta_1, \beta_2 \in C_V^1$. Note that we can assume n_q is big enough in such a way that $\beta_2 - \beta_1 \in C_V^1$. Therefore, we have that $(n_q - n_p)\alpha \in C_V^1$, and we conclude $\alpha \in \overline{C_V^1} = C_V$.

On the other hand, assume that $\alpha \in C_V$. Thus there exists $\alpha_1, \ldots, \alpha_r \in \operatorname{inj}(V)$ and positive integers $a_1, \ldots, a_r, m$ such that $m\alpha = \sum_{i=1}^r a_i\alpha_i$. If $\lambda \in \operatorname{Supp}(M)$, we see that $V^{\lambda+mn\alpha}$ is non-zero infinitely many $n > 0$ by Proposition 4.5.1. □

Suppose V is compatible. Elements of $C_V \cap \Delta$ are called *injective* roots, and $\alpha \in \Delta$ is said to be *locally finite* if α is not injective. Decompose Δ in the four disjoint sets Δ_V^i, Δ_V^f, Δ_V^+, Δ_V^- given by

$$\begin{aligned}
\Delta_V^i &= \{\alpha \in \Delta \mid \pm\alpha \in C_V\}, \\
\Delta_V^f &= \{\alpha \in \Delta \mid \pm\alpha \notin C_V\}, \\
\Delta_V^+ &= \{\alpha \in \Delta \mid -\alpha \in C_V,\ \alpha \notin C_V\}, \\
\Delta_V^- &= \{\alpha \in \Delta \mid \alpha \in C_V,\ -\alpha \notin C_V\}.
\end{aligned}$$

Call the decomposition $\Delta = \Delta_V^i \sqcup \Delta_V^f \sqcup \Delta_V^+ \sqcup \Delta_V^-$ as the *V-decomposition* of Δ.

For a root $\alpha \in \Delta$ and $\lambda \in \operatorname{Supp}(V)$, define the *$\alpha$-string through λ* as the set $\{x \in \mathbb{Q} \mid \lambda + x\alpha \in \operatorname{Supp}(V)\}$. It may be bounded, unbounded from bellow or above, or unbounded in both directions as a subset of $\mathbb{Q}$. The following is a corollary of Proposition 4.5.1 and Proposition 4.5.7.

Corollary 4.5.8. *Let V be a simple compatible finite weight $\mathcal{G}$-module, $\alpha \in \Delta$, and $\lambda \in \mathrm{Supp}(V)$. Then,*

1. *$\alpha \in \Delta_V^i$ if and only if the α-string through any $\lambda \in \mathrm{Supp}(V)$ is unbounded in both directions.*
2. *$\alpha \in \Delta_V^f$ if and only if the α-string through any $\lambda \in \mathrm{Supp}(V)$ is bounded.*
3. *$\alpha \in \Delta_V^+$ if and only if the α-string through any $\lambda \in \mathrm{Supp}(V)$ is bounded from above only.*
4. *$\alpha \in \Delta_V^-$ if and only if the α-string through any $\lambda \in \mathrm{Supp}(V)$ is bounded from bellow only.*

Therefore, it is clear that Δ_V^i, Δ_V^f , Δ_V^+, Δ_V^- are closed subsets of Δ. Thus, the subspaces

$$\mathfrak{g}_V^+ = \bigoplus_{\alpha\in\Delta_V^+} \mathfrak{g}^\alpha, \quad \mathfrak{g}_V^i = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta_V^i} \mathfrak{g}^\alpha, \quad \mathfrak{g}_V^- = \bigoplus_{\alpha\in\Delta_V^-} \mathfrak{g}^\alpha.$$

of $\mathfrak{g}$ are subalgebras. The triple $(\mathfrak{g}_V^+, \mathfrak{g}_V^i, \mathfrak{g}_V^-)$ will be called $\mathfrak{g}$-*shadow* of V. The $\mathcal{G}$-*shadow* of V is defined as the triple $\left(\mathcal{G}_V^+, \mathcal{G}_V^i, \mathcal{G}_V^-\right)$, where $\mathcal{G}_V^+ = \mathfrak{g}_V^+ \otimes S$, $\mathcal{G}_V^i = \mathfrak{g}_V^i \otimes S$, and $\mathcal{G}_V^- = \mathfrak{g}_V^- \otimes S$.

Lemma 4.5.9. *Let V be a simple compatible finite weight $\mathcal{G}$-module. Then, the monoid $\mathcal{Q}_V^i$ generated by all even roots in Δ_V^i is a group, and for every odd root $\alpha \in \Delta_V^i$ there is $m > 0$ such that $m\alpha \in \mathcal{Q}_V^i$.*

Proof. By Lemma 4.5.3, $\mathrm{inj}(V) \subset \Delta_{\bar{0}}$ is closed, thus the group generated by $\mathrm{inj}(V) \cap \Delta_V^i = \mathrm{inj}(V) \cap -\mathrm{inj}(V)$ is $\mathcal{Q}_V^i = C_V^1 \cap (-C_V^1)$. If $\alpha \in \Delta_V^i$ is odd, then $\alpha \in \overline{\mathcal{Q}_V^i}$ by Corollary 4.5.8. □

Remark 4.5.10. The previous lemma was proved for weight modules over a simple Lie superalgebra in [DMP00, Theorem 3.6].

4.6 Parabolic induction theorem

We start this section by providing two results about some extreme cases. The first is $\Delta_V^f = \Delta$, and the second case is $\Delta_V^i = \Delta$. Then we define what is a triangular decomposition in our setting, which leads to a construction similar to Verma modules. We prove that every finite $\mathcal{G}$-module is either cuspidal or parabolically induced module from a cuspidal module over a certain subalgebra of $\mathcal{G}$. For some of these statements, we will assume that V is compatible.

Proposition 4.6.1. *Let V be a simple finite weight $\mathcal{G}$-module. Then $\Delta_V^f = \Delta$ if and only if V has finite dimension.*

Proof. If V has finite dimension, then every even root is locally finite by Proposition 4.5.1. Therefore, $C_V = \{0\}$, and $\Delta_V^f = \Delta$.

Assume that $\Delta_V^f = \Delta$. We use ideas presented in the proof of Proposition 4.5.5 to show that V has finite dimension. By PBW theorem,

$$V \cong \left(\bigotimes_{\alpha\in\Delta_{\bar 1}} U(\mathfrak{g}^\alpha \otimes S_{\bar 1})\right)\left(\bigotimes_{\alpha\in\Delta_{\bar 0}} U(\mathfrak{g}^\alpha \otimes S_{\bar 0})\right) U(\mathfrak{h}\otimes S) \bigwedge(\mathfrak{g}_{\bar 0}\otimes S_{\bar 1}) \bigwedge(\mathfrak{g}_{\bar 1}\otimes S_{\bar 0}) V^\lambda$$

as a vector space for every $\lambda \in \operatorname{Supp}(V)$. With this in mind, fix $\lambda \in \operatorname{Supp}(V)$, and define

$$W_0(\lambda) = U(\mathfrak{h}\otimes S) \bigwedge(\mathfrak{g}_{\bar 0}\otimes S_{\bar 1}) \bigwedge(\mathfrak{g}_{\bar 1}\otimes S_{\bar 0}) V^\lambda$$

Furthermore, enumerate $\Delta_{\bar 0} = \{\alpha_1, \dots, \alpha_a\}$ and $\Delta_{\bar 1} = \{\alpha_{a+1}, \dots, \alpha_{a+b}\}$. Finally, define

$$W_i(\lambda) = U(\mathfrak{g}^{\alpha_i}\otimes S_{\bar 0}) \cdots U(\mathfrak{g}^{\alpha_1}\otimes S_{\bar 0}) W_0(\lambda)$$

for $i = 1, \dots, a$, and

$$W_i(\lambda) = U(\mathfrak{g}^{\alpha_i}\otimes S_{\bar 1}) \cdots U(\mathfrak{g}^{\alpha_{a+1}}\otimes S_{\bar 1}) W_a(\lambda).$$

for $i = a+1, \dots, a+b$.

Note that $W_i(\lambda)$ is a weight $\mathfrak{h}\otimes\mathbb{k}$-module. Thus, we can consider $S_i(\lambda)$ the set of its weights. Since V is simple, $W_{a+b}(\lambda) = V$ and $S_{a+b}(\lambda) = \operatorname{Supp}(V)$ by the PBW Theorem. We will prove by induction on k that $S_k(\lambda)$ is finite. The weights of

$$W_0(\lambda) = U(\mathfrak{h}\otimes S/\operatorname{Ann}_S(V)) \bigwedge(\mathfrak{g}_{\bar 0}\otimes S_{\bar 1}/\operatorname{Ann}_S(V)) \bigwedge(\mathfrak{g}_{\bar 1}\otimes (S/\operatorname{Ann}_S(V)))\, V^\lambda$$

contains the weights of $\bigwedge(\mathfrak{g}_{\bar 0}\otimes S_{\bar 1}/\operatorname{Ann}_S(V)) \bigwedge(\mathfrak{g}_{\bar 1}\otimes (S/\operatorname{Ann}_S(V)))\, V^\lambda$, which is finite-dimensional because both V^λ and $S/\operatorname{Ann}_S(V)$ are finite-dimensional. Thus, we have that $W_0(\lambda)$ is finite-dimensional and the exterior algebra of a finite-dimensional vector space is finite-dimensional. Therefore, $S_0(\lambda)$ is a finite set. Suppose $k > 0$, then

$$S_k(\lambda) \subset \bigcup_{\gamma\in S_{k-1}(\lambda)} \operatorname{Supp} V \cap \{\gamma + n\alpha_k \mid n \geq 0\}.$$

By Corollary 4.5.8 and the assumption that $\Delta^f = \Delta$, the set $\operatorname{Supp} V \cap \{\gamma + n\alpha_k \mid n \geq 0\}$ is finite for every element of the finite set $S_{k-1}(\lambda)$. Hence, $S_k(\lambda)$ is finite because it is a finite union of finite sets. We conclude that $S_{a+b}(\lambda) = \operatorname{Supp}(V)$ is finite, thus V is a finite-dimensional vector space because it has finite support and its weight spaces have finite dimension.

□

Proposition 4.6.2. *Let V be a simple compatible finite weight $\mathcal{G}$-module. If $\Delta_V^i = \Delta$ (equivalently, $\operatorname{inj}(V) = \Delta_{\bar 0}$), then V is bounded. Furthermore, there exists a finite set $\Theta \subset \operatorname{Supp}(V)$ such that $\Theta + Q_{\bar 0} = \operatorname{Supp}(V)$, and $\dim V^\lambda = \dim V^\mu$ if $\lambda, \mu \in \gamma + Q_{\bar 0}$ for some $\gamma \in \Theta$.*

Proof. Since V is compatible, there exists a finite set $\Theta \subset \operatorname{Supp}(V)$ such that $\operatorname{Supp}(V) = \Theta + C_V^1$. Since every root is injective, $C_V^1 = Q_{\bar 0}$. Therefore, $\operatorname{Supp}(V) = \Theta + Q_{\bar 0}$.

Let $\alpha \in \Delta_{\bar{0}}$ and $\gamma \in \Theta$. Then, $\alpha, -\alpha \in \operatorname{inj} V$ and $V^{\gamma+n\alpha} \neq 0$ for every $n \in \mathbb{Z}$ by Corollary 4.5.8. Since $x_\alpha \otimes 1$ acts injectively, the linear map

$$\begin{aligned} V^\lambda &\to V^{\lambda+\alpha} \\ v &\mapsto (x_\alpha \otimes 1)v \end{aligned}$$

is injective. Thus, $\dim V^\lambda \leq \dim V^{\lambda+\alpha}$. Likewise, the map $v \mapsto (x_{-\alpha} \otimes 1)v$ is injective, hence $\dim V^\lambda \geq \dim V^{\lambda+\alpha}$. We conclude that $\dim V^\lambda = \dim V^{\lambda+n\alpha}$ for every $n \in \mathbb{Z}$, therefore $\dim V^\lambda = \dim V^{\lambda+\beta}$ for every $\beta \in \mathcal{Q}_{\bar{0}}$. □

Definition 4.6.3 (Triangular decomposition)**.** A *triangular decomposition* T of $\mathfrak{g}$ is a decomposition $\mathfrak{g} = \mathfrak{g}_T^+ \oplus \mathfrak{g}_T^0 \oplus \mathfrak{g}_T^-$ and a linear map $l : \mathcal{Q} \to \mathbb{Z}$ for which

$$\mathfrak{g}_T^+ = \bigoplus_{l(\alpha)>0} \mathfrak{g}^\alpha, \quad \mathfrak{g}_T^0 = \bigoplus_{l(\alpha)=0} \mathfrak{g}^\alpha \quad \text{and} \quad \mathfrak{g}_T^- = \bigoplus_{l(\alpha)<0} \mathfrak{g}^\alpha.$$

Similarly, a triangular decomposition T of $\mathcal{G}$ is a decomposition of the form $\mathcal{G} = \mathcal{G}_T^+ \oplus \mathcal{G}_T^0 \oplus \mathcal{G}_T^-$, where $\mathcal{G}_T^\bullet = \mathfrak{g}_T^\bullet \otimes S$ and T is a triangular decomposition of $\mathfrak{g}$. A triangular decomposition is *proper* if $\mathfrak{g}_T^0 \neq \mathfrak{g}$. Finally, we set $\Delta_T^+ = \{\alpha \in \Delta \mid l(\alpha) > 0\}$, $\Delta_T^- = \{\alpha \in \Delta \mid l(\alpha) < 0\}$ and $\Delta_T^0 = \{\alpha \in \Delta \mid l(\alpha) = 0\}$.

Lemma 4.6.4. *There is a triangular decomposition T of $\mathcal{G}$ such that $\Delta_V^i = \Delta_T^0$, $\Delta_V^+ \subset \Delta_T^+$, and $\Delta_V^- \subset \Delta_T^-$.*

Proof. The statement follows from Corollary 4.5.8, since it will be possible to construct a $\mathcal{G}$-shadow consisting of Lie subalgebras of $\mathcal{G}$. □

For a triangular decomposition T of $\mathcal{G}$ and a weight $\mathcal{G}_T^0$-module W, we define the induced module

$$M_T(W) = U(\mathcal{G}) \otimes_{U(\mathcal{G}_T^0 \oplus \mathcal{G}_T^+)} W,$$

where the action of $\mathcal{G}_T^+$ on W is trivial. This can be seen as a generalization of a Verma module. In fact, if $\mathcal{G}_T^0 = \mathfrak{h} \otimes S$, then $M_T(\mathbb{C}_\lambda)$ is the Verma module with highest weight λ for some $\lambda \in (\mathfrak{h} \otimes S)^*$.

Proposition 4.6.5. *Let W be a weight $\mathcal{G}_T^0 = \mathfrak{g}_T^0 \otimes S$-module whose support is included in a single $\mathcal{Q}^T$-coset, where $\mathcal{Q}^T$ is the root lattice of $\mathfrak{g}_T^0$.*

1. *$M_T(W)$ has a unique submodule $N_T(W)$ which is maximal among all submodules of $M_T(W)$ with trivial intersection with W.*

2. *$N_T(W)$ is maximal if and only if W is simple. In particular, $L_T(W) = M_T(W)/N_T(W)$ is a simple $\mathcal{G}$-module if and only if W is a simple $\mathcal{G}_T^0$-module.*

3. *If W is simple, the space*

$$L_T(W)^{\mathcal{G}_T^+} = \left\{ v \in L_T(W) \mid xv = 0 \text{ for all } x \in \mathcal{G}_T^+ \right\}$$

of $\mathcal{G}_T^+$-invariants is equal to W.

Proof. This proof is standard and follows the same idea as Verma modules. See [DMP00, Lemma 2.3, Corollary 2.4] for the proof of this proposition on the finite-dimensional simple Lie superalgebra case. □

Theorem 4.6.6. *Let V be a simple compatible finite weight $\mathcal{G}$-module, then there is a triangular decomposition T such that $\Delta_V^i = \Delta_T^0$, $\Delta_V^+ \subset \Delta_T^+$, $\Delta_V^- \subset \Delta_T^-$, the vector space of $\mathcal{G}_T^+$-invariants $V^{\mathcal{G}_T^+}$ is a simple bounded $\mathcal{G}_V^i$-module, and $V \cong L_T(V^{\mathcal{G}_T^+})$.*

Proof. Let $v \in V^\lambda$ be a nonzero weight vector with weight $\lambda \in \mathrm{Supp}(V)$. By Lemma 4.5.9, we may fix a triangular decomposition T of $\mathcal{G}$ such that $\Delta_V^i = \Delta_T^0$, $\Delta_V^+ \subset \Delta_T^+$, and $\Delta_V^- \subset \Delta_T^-$. Then, $U(\mathcal{G}_T^+)v$ is a finite-dimensional $\mathcal{G}_T^+$-module by a similar argument to the given on Proposition 4.6.1. Therefore, $U(\mathcal{G}_T^+)v \cap V^{\mathcal{G}_T^+} \neq 0$, and $V^{\mathcal{G}_T^+}$ is a non-zero vector space.

Let $w \in V^{\mathcal{G}_T^+}$ be a non-zero weight vector, and $W = U(\mathcal{G}_v^i)w$. The vector space $\mathcal{G}_V^i \oplus \mathcal{G}_T^+$ is a Lie subalgebra of $\mathcal{G}$, and $U(\mathcal{G}_V^i \oplus \mathcal{G}_T^+)w = U(\mathcal{G}_V^i)U(\mathcal{G}_T^+)w = U(\mathcal{G}_V^i)w = W$ by PBW theorem. Hence, W is a $\mathcal{G}_V^i \oplus \mathcal{G}_T^+$-module. The linear map

$$\begin{aligned} \varphi : M_T(W) &\to V \\ u \otimes_{U(\mathcal{G}_V^i \oplus \mathcal{G}_T^+)} v &\mapsto uv \end{aligned}$$

is a well-defined $\mathcal{G}$-module homomorphism. Its image contains W, so φ is a non-zero $\mathcal{G}$-module homomorphism on the simple $\mathcal{G}$-module V. Therefore, it is surjective, $V \cong M_T(W)/\ker(\varphi)$, and $\ker(\varphi)$ is a maximal $\mathcal{G}$-module of $M_T(W)$. As a vector space, $M_T(W)$ is isomorphic to $U(\mathcal{G}_T^-)\mathcal{G}_T^- \otimes W \oplus \Bbbk \otimes W$ thus $\varphi(1 \otimes v) = v$ for every $v \in W$. The restriction of φ to $\Bbbk \otimes W \cong W$ is a $\mathcal{G}_V^i$-module monomorphism, thus $W \cap \ker(\varphi) = 0$. Being $\ker(\varphi)$ a maximal submodule with trivial intersection with W, we apply Proposition 4.6.5 to conclude that $\ker(\varphi) = N_T(W)$, $V \cong L_T(W)$, and $W = V^{\mathcal{G}_T^+}$ is a simple $\mathcal{G}_V^i$-module.

It remains to show that W is a bounded $\mathcal{G}_V^i$-module. We use Lemma 4.5.9 to conclude that $\mathfrak{g}_V^i$ is a *good Levi subalgebra*. For a complete list of all such subalgebras of $\mathfrak{g}$, we refer to [DMP00; DMP04]. The important fact for our proof is that $\mathfrak{g}_V^i \cong \mathfrak{z} \oplus \mathfrak{l}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}_V^i$ and a subalgebra of Cartan subalgebra $\mathfrak{h}$, and $\mathfrak{l} \cong \bigoplus_{r=1}^{k} \mathfrak{l}^r$ is a direct sum of certain simple finite-dimensional basic Lie superalgebras where at most one $\mathfrak{l}^i$ has nontrivial odd part. W is a simple module over $\mathcal{G}_V^i = \mathfrak{g}_V^i \otimes S$ if and only if it is a simple module over $\mathfrak{l} \otimes S$. Since all but one $\mathfrak{l}^r$ are simple Lie algebras, Proposition 4.3.4 says that

$$W \cong \bigotimes_{r=1}^{k} W_r$$

as a module over $\mathfrak{l}^r \otimes S$, where W_r is a simple finite weight module over $\mathfrak{l}^r \otimes S$ for each $r \in \{1, \dots, r\}$. We wish to use Proposition 4.6.2 to conclude that W_r is a bounded module over $\mathfrak{l}^r \otimes S$. First, we note that $\mathrm{inj}(W_r) \subset \Delta_V^i$ as a module over $\mathfrak{l}^r \otimes S$ is exactly the set of even roots of $\mathfrak{l}^r$. Secondly, since $\mathcal{V}$ is compatible, each W is compatible as well a $\mathfrak{l}^r \otimes S$-module, because $\mathrm{Supp}(W) \subset \mathrm{Supp}(V) = C_V^1$ and the root lattice of $\mathfrak{l}^r$ is equal to $C_{W_r}^1 \subset C_V^1$. Therefore, each W_r is a bounded $\mathfrak{l}^r \otimes S$-module by Proposition 4.6.2. We conclude that W is bounded as a module over $\mathfrak{g}_V^i \otimes S$ because $\mathfrak{l}^r$'s are orthogonal among each other. □

We use Theorem 4.6.6 to define what is a cuspidal module in our context.

Definition 4.6.7. Let V be a simple compatible finite weight $\mathcal{G}$-module. If there exists a proper triangular decomposition T of $\mathcal{G}$ and a simple $\mathcal{G}_T^0$-module W such that $V \cong L_T(W)$, we say that V is a *parabolically induced* module. We call V a *cuspidal* module if it is not parabolically induced.

Theorem 4.6.6 states that every simple compatible finite weight $\mathcal{G}$-module is either cuspidal or parabolically induced. We emphasize that there exist parabolically induced modules that are not finite weight $\mathcal{G}$-modules as the next example demonstrates.

Example 4.6.8. Suppose $\mathfrak{g}$ is an arbitrary basic classical Lie superalgebra and $S = \Bbbk[t]$ the polynomial algebra on the variable t. A choice of simple roots defines a triangular decomposition T of $\Delta = \Delta^+ \cup \Delta^-$. In this case, $\mathcal{G}_T^0 = \mathfrak{h} \otimes \Bbbk[t]$. Take $h \in \mathfrak{h}$ a non-zero element of the Cartan subalgebra of $\mathfrak{g}$, and a linear functional $\Lambda : \mathfrak{h} \otimes \Bbbk[t]$ such that $\Lambda(h \otimes t^k) = \frac{1}{k+1}$ for every $k \geq 0$. The 1-dimensional $\mathcal{G}_T^0$-module $\Bbbk v_\Lambda$ defined by $xv_\Lambda = \Lambda(x)v_\Lambda$ is a simple bounded finite weight $\mathcal{G}_T^0$-module, but the simple module $L_T(\Bbbk v_\Lambda)$ has infinite-dimensional weight spaces by [Sav14, Theorem 4.16].

4.7 Evaluation modules and finite-dimensional $\mathcal{G}$-modules

Let A be a commutative finitely generated algebra. Assume that $\mathsf{m}_1, \ldots, \mathsf{m}_r$ are distinct maximal ideals of A and $n_1, \ldots, n_r$ are non-negative integers, define the *generalized evaluation map*

$$\mathrm{ev}_{\mathsf{m}_1^{n_1}, \ldots, \mathsf{m}_r^{n_r}} : \mathfrak{g} \otimes A \to \mathfrak{g} \otimes A/\mathsf{m}_1^{n_1} \oplus \cdots \oplus \mathfrak{g} \otimes A/\mathsf{m}_r^{n_r}$$

by $\mathrm{ev}_{\mathsf{m}_1^{n_1}, \ldots, \mathsf{m}_r^{n_r}}(x \otimes a) = (x \otimes a + \mathsf{m}^{n_1}, \ldots, x \otimes a + \mathsf{m}_r^{n_r})$ for each $x \in \mathfrak{g}$ and $a \in A$. When $n_1 = \cdots = n_r = 1$, we call $\mathrm{ev}_{\mathsf{m}_1, \ldots, \mathsf{m}_r}$ *evaluation map*. In this case, we note that $\mathrm{ev}_{\mathsf{m}_1^{n_1}, \ldots, \mathsf{m}_r^{n_r}} : \mathfrak{g} \otimes A \to \mathfrak{g}^{\oplus r}$, because $A/\mathsf{m}_i \cong \Bbbk$ since $\Bbbk$ is algebraically closed.

Let V_i be simple finite weight $\mathfrak{g}_i \otimes A/\mathsf{m}_i^{n_i}$-module for each $i = 1, \ldots, r$, then $V_1 \otimes \cdots \otimes V_r$ is a $\bigoplus_{i=1}^{r} \mathfrak{g}_i \otimes A/\mathsf{m}_i^{n_i}$-module with the action given by

$$(x_1, \ldots, x_r)v_1 \otimes \cdots \otimes v_r = \sum_{i=1}^{r} v_1 \otimes \cdots \otimes x_i v_i \otimes \cdots \otimes v_r$$

for each $v_i \in V_i$, $i = 1, \ldots, r$, and $x_i \in \mathfrak{g} \otimes A/\mathsf{m}_i^{n_i}$. In other words, if $\rho_i : \mathfrak{g} \otimes A/\mathsf{m}_i^{n_i} \to \mathfrak{gl}(V_i)$ is the representation associated to V_i, then

$$\bigoplus_{i=1}^{r} \mathfrak{g}_i \otimes A/\mathsf{m}_i^{n_i} \xrightarrow{\rho_1 \otimes \cdots \otimes \rho_r} \mathfrak{gl}(V_1 \otimes \cdots \otimes V_r)$$

is a representation for $\bigoplus_{i=1}^{r} \mathfrak{g}_i \otimes A/\mathsf{m}_i^{n_i}$.

With this notation, we define the *generalized evaluation module* $\mathrm{ev}_{\mathfrak{m}_1^{n_1},\dots,\mathfrak{m}_r^{n_r}}(V_1,\dots,V_r)$, which is the $\mathcal{G}$-module $V_1 \otimes \dots V_r$ given by the composition

$$\mathcal{G} \xrightarrow{\mathrm{ev}_{\mathfrak{m}_1^{n_1},\dots,\mathfrak{m}_r^{n_r}}} \bigoplus_{i=1}^{r} \mathfrak{g}_i \otimes A/\mathfrak{m}_i^{n_i} \xrightarrow{\rho_1\otimes\cdots\otimes\rho_r} \mathfrak{gl}\left(\bigoplus_{i=1}^{r} V_i\right)$$

When $n_1 = \dots = n_r = 1$, we call $\mathrm{ev}_{\mathfrak{m}_1,\dots,\mathfrak{m}_r}(V_1,\dots,V_r)$ *evaluation representation* and we may denote it by $\bigotimes_{i=1}^{r} V_i^{\mathfrak{m}_i}$.

Theorem 4.7.1. *[Sav14] If V is a finite-dimensional weight $\mathcal{G}$-module, then V is a generalized evaluation module. In particular, if $\mathfrak{g}_{\bar{0}}$ is a semisimple Lie algebra, then V is an evaluation module.*

4.8 Cuspidal $\mathcal{G}$-modules

In this section, we prove that every bounded cuspidal $\mathcal{G}$-module is an evaluation module. However, we will need the assumption that $S_{\bar{1}} = 0$, that is, S is a commutative algebra instead of a commutative superalgebra. To make a distinction, we will use the symbol A for S instead.

Let V be a simple finite weight $\mathcal{G}$-module. Suppose that Δ_V^i is not empty. By Theorem 4.6.6, the vector space of $\mathcal{G}_T^+$-invariants $V^{\mathcal{G}_T^+}$ is a simple bounded $\mathcal{G}_V^i$-module. All roots of $\mathcal{G}_V^i$ acts injectively on $V^{\mathcal{G}_T^+}$, hence $V^{\mathcal{G}_T^+}$ is a cuspidal $\mathcal{G}_V^i$-module. We may regard $V^{\mathcal{G}_T^+}$ as a module over $\mathfrak{g}_V^i$. Since $\mathfrak{g}_V^i$ and $\mathfrak{g}$ have the same Cartan subalgebra, $V^{\mathcal{G}_T^+}$ is a bounded finite weight module over $\mathfrak{g}_V^i$. By [DMP00], it has finite length over $\mathfrak{g}_V^i$. Therefore, $V^{\mathcal{G}_T^+}$ has a cuspidal $\mathfrak{g}_V^i$-submodule. Thus, $\mathfrak{g}_V^i$ admits cuspidal modules and is a *cuspidal Levi subalgebra* of $\mathfrak{g}$.

All cuspidal Levi subalgebras were classified in [DMP00; DMP04]. It was shown that a Levi subalgebra that admits cuspidal modules have components isomorphic to $\mathrm{A}(n,n)$, $\mathfrak{osp}(n|2m)$ with $n \neq 2$ and $n \leq 6$, $\mathrm{D}(2,1,\alpha)$, a proper Levi subalgebra of the simple Lie algebra G_2, or a reductive Lie algebra with simple components of type A or C. We point out that the notion of weight and cuspidal modules from [DMP00] are different from ours. Our definition is aligned with that of [EF09]. Therefore, the list of cuspidal Levi superalgebras presented in [EF09] is precisely the one that might appear in our context, which is $\mathfrak{osp}(1|2)$, $\mathfrak{osp}(1|2) \oplus \mathfrak{sl}_2$, $\mathfrak{osp}(n|2n)$ with $2 < n \leq 6$, $D(2,1;a)$, or a reductive Lie algebra with irreducible components of type A and C. Either way, simple components will all be either a reductive Lie algebra or a basic classical Lie superalgebra with $\mathfrak{g}_{\bar{0}}$ isomorphic to a semisimple Lie algebra. Not only that, at most one simple component is a Lie superalgebra. For an extended study of bounded cuspidal modules over $\mathfrak{g} \otimes S$ where $\mathfrak{g}$ is simple Lie algebra and S is a commutative Lie algebra, we refer to [BLL15]. Therefore, with these arguments and Proposition 4.3.4 in mind, we only need to classify cuspidal bounded weight $\mathcal{G}$-modules when $\mathfrak{g}$ is a basic classical Lie superalgebra with $\mathfrak{g}_{\bar{0}}$ semisimple.

Proposition 4.8.1. *Suppose that $\mathfrak{g}$ is a basic classical Lie superalgebra such that $\mathfrak{g}_{\bar{0}}$ is semisimple, and A is a commutative algebra. If V is a simple bounded weight $\mathfrak{g} \otimes A$-module,* *then the ideal* $\mathrm{Ann}_A(V)$ *is radical.*

Proof. The proof is quite similar to the Lie algebra case done on [BLL15]. We will explicitly prove only the parts that differ from that setting.

Denote $\mathcal{G} = \mathfrak{g} \otimes A$, $I = \mathrm{Ann}_A(V)$, and $\rho : \mathcal{G} \to \mathrm{End}(V)$ the associated representation of the $\mathcal{G}$-module V. By Proposition 4.4.3 and Proposition 4.4.4, A/I is a finite-dimensional algebra, and V is a faithful representation of $\mathcal{G}/\ker(\rho) = \mathfrak{g} \otimes A/I$. The commutative algebra A/I is Artinian since it is finite-dimensional. Thus, its Jacobson ideal $J = \sqrt{I}/I$ is nilpotent. If $J = 0$, then I is radical and the proof is complete. Suppose $J \neq 0$. Let m be the smallest integer such that $J^m = 0$ but $J^{m-1} \neq 0$, and m' be the smallest integer greater or equal to $m/2$. Denote $N = J^{m'}$, thus $N \neq 0$, and $N^2 = 0$.

Claim 1: $\mathfrak{g}^\alpha \otimes N$ *acts nilpotently on* V *for every* $\alpha \in \Delta$*.*

Let $\alpha \in \Delta_{\bar{1}}$ be an odd root, $x \in \mathfrak{g}^\alpha$, and $a \in N$. Hence, $x \otimes a$ is an odd element of $\mathcal{G}$, and for each $v \in V$

$$0 = ([x, x] \otimes a^2)v = [x \otimes a, x \otimes a]v = 2(x \otimes a)v$$

because $a^2 \in N^2 = 0$. Therefore, $\mathfrak{g}^\alpha \otimes N$ acts nilpotently on V for every odd root α. If $\alpha \in \Delta_{\bar{0}}$ is an even root, then the proof that $\mathfrak{g}^\alpha \otimes N_{\bar{0}}$ acts nilpotently follows from Step 1 and 2 of [BLL15, Proposition 4.4].

Claim 2: *There exists a nonzero weight vector* $w \in V$ *such that* $(\mathfrak{g} \otimes N)\, w = 0$.

Let $\lambda \in \mathrm{Supp}(V)$ be a weight of V. Then, $(\mathfrak{h} \otimes N)V^\lambda \subset V^\lambda$, because $\mathfrak{h} \otimes N$ is an abelian algebra that commutes with $\mathfrak{h} \otimes 1$. Since $\mathfrak{h} \otimes N$ is an abelian Lie algebra (so solvable) and V^λ has finite dimension, there exists a non-zero weight vector $v_0 \in V^\lambda$ that is an eigenvector for all elements of $(\mathfrak{h} \otimes N)$ by Lie's Theorem.

Let $\Delta = \Delta^- \cup \Delta^+$ be a choice of positive roots for Δ, and denote $\mathfrak{n}^\pm = \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}^\alpha \otimes N$. Since $N^2 = 0$, $[\mathfrak{g}^\alpha \otimes N, \mathfrak{g} \otimes N] \subset \mathfrak{g} \otimes N^2 = 0$. By Claim 1, $U(\mathfrak{n}^+ \otimes N)v_0$ is a finite-dimensional $\mathfrak{n}^+ \otimes N$-module. Since $\mathfrak{n}^+ \otimes N$ acts nilpotently on the finite-dimensional vector space $U(\mathfrak{n}^+ \otimes N)v_0$, and $\mathfrak{n}^+ \otimes N$ form a family of commutative operators on $U(\mathfrak{n}^+ \otimes N)v_0$, then $\rho(\mathfrak{n}^+ \otimes N)$ is family of simultaneously diagonalizable endomophisms of $U(\mathfrak{n}^+ \otimes N)v_0$. Hence, there exists a weight vector $u \in U(\mathfrak{n}^+ \otimes N)v_0$ such that $(\mathfrak{n}^+ \otimes N)u = 0$. Similarly, there exists a weight vector $w \in U(\mathfrak{n}^- \otimes N)u$ such that $(\mathfrak{n}^- \otimes N)w = 0$. Since $\mathfrak{n}^- \otimes N$ and $\mathfrak{n}^+ \otimes N$ commutes, $(\mathfrak{n}^+ \otimes N)w = 0$ as well.

The weight vector w is a eigenvector for every element of $\mathfrak{h} \otimes N$, because $\mathfrak{h} \otimes N$ commutes with both $\mathfrak{n}^+ \otimes N$ and $\mathfrak{n}^- \otimes N$, $w \in U(\mathfrak{n}^- \otimes N \oplus \mathfrak{n}^+ \otimes N)v_0$, and $(\mathfrak{h} \otimes N)v_0 \subset \Bbbk v_0$. By Step 5 of [BLL15, Proposition 4.4], $(h_\alpha \otimes N)w = 0$ for every $\alpha \in \Delta_{\bar{0}}$. The Cartan algebra of $\mathfrak{g}$ is the Cartan algebra of $\mathfrak{g}_{\bar{0}}$, which is semisimple by hypothesis. Therefore, the set $\{h_\alpha \mid \alpha \in \Delta_{\bar{0}}\}$ generates $\mathfrak{h}$ as a vector space. Thus, $(\mathfrak{h} \otimes N)w = 0$.

Claim 3: $\mathrm{Ann}_A(V)$ is a radical ideal.

We wish to conclude that N must be equal to 0. Set $W = \{w \in V \mid (\mathfrak{g} \otimes N)w = 0\}$. By Claim 2, W is a non-zero vector space. For each $x \otimes a \in \mathfrak{g} \otimes A/I$, and $y \otimes b \in \mathfrak{g} \otimes N$,

$$(y \otimes b)(x \otimes a) = (-1)^{|y||x|}(x \otimes a)(y \otimes b)w + ([x, y] \otimes ba)w = 0$$

because $ba \in N$. Therefore, W is a $(\mathfrak{g} \otimes A/I)$-submodule of V. But V is simple, thus $V = W$. We conclude that $0 \neq N \subset \mathrm{Ann}_{A/I}(V)$, which is a contradiction with the fact that V is a faithful representation of $\mathfrak{g} \otimes A/I$. Hence, $N = 0$, thus $J = \sqrt{I}/I = 0$ as well, which allows us to conclude that $I = \mathrm{Ann}_A(V)$ is a radical ideal. □

Lemma 4.8.2. *Let $r \geq 2$, and $V_1, \dots, V_r$ be bounded weight $\mathfrak{g}$-modules such that $\Delta = \cup_{i=1}^r R_{V_i}$. Then the $\mathfrak{g}$-module $V := \bigotimes_{i=1}^r V_i$ is bounded if and only if* $\dim V_i = \infty$ *for at most one $i = 1, \dots, r$.*

Proof. Suppose $\dim V_i = \infty$ for at most one $i \in \{1, \dots, r\}$. Without loss of generality, we assume that $\dim(V_1)$ may be infinity. Let $\lambda \in \mathrm{Supp}(V)$, and $\lambda_i \in \mathrm{Supp}(V_i)$ such that $\lambda = \lambda_1 + \cdots + \lambda_r$. Then,

$$V^\lambda = \bigoplus_{\substack{\alpha_1,\dots,\alpha_r \in Q \\ \alpha_1+\cdots+\alpha_r=0}} V_1^{\lambda_1+\alpha_1} \otimes \cdots \otimes V_r^{\lambda_r+\alpha_r}.$$

This sum is finite, and the maximum number of non-trivial summands is the sum $N = |\mathrm{Supp}(V_2)| + \cdots + |\mathrm{Supp}(V_r)|$ of numbers of weights of $V_2, \dots, V_r$. Therefore, the dimension of V^λ is bounded by $N \cdot L_1 \cdot \dim(V_2) \cdot \dim(V_3) \cdots \dim(V_r)$, where $L_1 \in \mathbb{Z}$ satisfy $\dim(V_1^\gamma) < L_1$ for every $\gamma \in \mathrm{Supp}(V_1)$.

We wish to prove that if $V = \bigotimes_{i=1}^{r} V_i$ is bounded, then at most one V_i has infinite dimension. Assume that $\dim(V_1) = \dim(V_2) = \infty$. Since each R_{V_i} is closed, the subspace

$$\mathfrak{g}^i := \bigoplus_{\alpha \in R_{V_i}} \Bbbk h_\alpha \oplus \bigoplus_{\alpha \in R_{V_i}} \mathfrak{g}^\alpha$$

is a subalgebra of $\mathfrak{g}$.

Consider the set $R = R_{V_1} + \bigcup_{i=2}^{r} R_{V_i} \cap \Delta$. If $R = \emptyset$, then $\mathfrak{g}^1$ commutes with the algebra generated by $\mathfrak{g}^2 + \mathfrak{g}^3 + \cdots + \mathfrak{g}^r$. Therefore, $\mathfrak{g}^1$ is an ideal of $\mathfrak{g}$, because $\mathfrak{g} = \mathfrak{g}^1 + \cdots + \mathfrak{g}^r$ and $[\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_1$. Similarly, if $R \subset R_{V_1}$, then the commutator of an element of $\mathfrak{g}^1$ with an element of the algebra generated by $\mathfrak{g}^2 + \mathfrak{g}^3 + \cdots + \mathfrak{g}^r$ is an element of $\mathfrak{g}^1$. Thus, $\mathfrak{g}^1$ will also be an ideal of $\mathfrak{g}$. Being $\mathfrak{g}$ simple, we have that $\mathfrak{g} = \mathfrak{g}^1$ if $R = \emptyset$ or $R \subset R_{V_1}$. Since $\dim(V_2) = \infty$, there exists $\alpha \in \mathrm{inj}(V_2)$, and $m > 0$ such that $\tilde{\alpha} = m\alpha \in C_{V_1}^1 \cap C_{V_2}^1$. Take $\lambda_i \in \mathrm{Supp}(V_i)$ for each $i = 1, \dots, r$. Then,

$$0 \neq V^{\lambda_1+l\tilde{\alpha}} \otimes V^{\lambda_2+(n-l)\alpha_2} \otimes V^{\lambda_3} \otimes \cdots \otimes V^{\lambda_r} \subset V^{\lambda_1+\cdots+\lambda_r+n\tilde{\alpha}}$$

for each $n > 0$, and $l = 0, \dots, n$. Therefore, the dimension of $V^{\lambda_1+\cdots+\lambda_r+n\tilde{\alpha}}$ grows as n gets larger. We conclude that $R \neq \emptyset$ and $R \not\subset R_{V_1}$.

Hence, we may suppose that there exists $\alpha \in R_{V_1}$ and $\beta \in R_{V_2}$ such that $\alpha + \beta \in \bigcup_{i=2}^{r} R_{V_i}$. If $\alpha + \beta \in R_{V_2}$, then there exists $m \in \mathbb{Z}_+$ such that $\tilde{\alpha} = m\alpha \in C_{V_1}^1$, $\tilde{\beta} = m\beta \in C_{V_2}^1$, and

$\tilde{\alpha} + \tilde{\beta} = m(\alpha + \beta) \in C^1_{V_2}$. In this case, we have that

$$\bigoplus_{l=0}^{n} V^{\lambda_1 + l\tilde{\alpha}} \otimes V^{\lambda_2 + l\tilde{\beta} + (n-l)(\tilde{\alpha}+\tilde{\beta})} \otimes V^{\lambda_3} \otimes \cdots \otimes V^{\lambda_r} \subset V^{\lambda_1 + \cdots + \lambda_r + n(\tilde{\alpha}+\tilde{\beta})}.$$

Therefore, $\dim V^{\lambda_1 + \cdots + \lambda_r + n(\tilde{\alpha}+\tilde{\beta})} > n$, and we would have that V is not bounded, a contradiction. Thus, it remains to consider the case on which $\alpha + \beta \in R_{V_i}$ for some $i \in \{3, \dots, r\}$. Without loss of generality, we assume $\alpha + \beta \in R_{V_3}$. Similarly to the previous case, there exists m such that $\tilde{\alpha} = m\alpha \in C^1_{V_2}$, $\tilde{\beta} = m\beta \in C^1_{V_2}$, and $\tilde{\alpha} + \tilde{\beta} = m(\alpha+\beta) \in C^1_{V_3}$. Therefore,

$$\bigoplus_{l=0}^{n} V^{\lambda_1 + l\tilde{\alpha}} \otimes V^{\lambda_2 + l\tilde{\beta}} \otimes V_3^{\lambda_3 + (n-l)(\tilde{\alpha}+\tilde{\beta})} \otimes V_4^{\lambda_4} \otimes \cdots \otimes V_r^{\lambda_r}.$$

Hence, $\dim V^{\lambda_1 + \cdots + \lambda_r + n(\tilde{\alpha}+\tilde{\beta})} > n$, which contradicts the assumption that V is bounded.

We conclude that at most one V_i has infinite dimension. □

Theorem 4.8.3. *Suppose that $\mathcal{G}$ has cuspidal modules. If V is a cuspidal bounded $\mathcal{G}$-module, then V is isomorphic to an evaluation module.*

Proof. We have that $\mathfrak{g}_{\bar{0}}$ is a semisimple Lie algebra because $\mathcal{G}$ is a Lie algebra. By Proposition 4.8.1, $\operatorname{Ann}_A(V)$ is a radical ideal of A, so there exists pairwise distinct maximal ideals $\mathsf{m}_1, \dots, \mathsf{m}_r$ of A such that $\operatorname{Ann}_A(V) = \mathsf{m}_1 \cap \cdots \cap \mathsf{m}_r$. By the Chinese Remainder Theorem and the assumption that A is a finitely generated algebra over an algebraically closed field,

$$\mathcal{G}/\left(\mathfrak{g} \otimes \bigcap_{i=1}^{r} \mathsf{m}_i\right) \cong \mathfrak{g} \oplus \left(A/\bigcap_{i=1}^{r} \mathsf{m}_i\right) \cong \mathfrak{g} \otimes (A/\mathsf{m}_1 \oplus \cdots \oplus A/\mathsf{m}_r) \cong \bigoplus_{i=1}^{r} \mathfrak{g} \otimes (A/\mathsf{m}_i) \cong \mathfrak{g}^r.$$

Hence, V is a simple $\mathfrak{g}^r$-module. By Proposition 4.3.4, there exist simple weight $\mathfrak{g}$-modules $V_1, \dots, V_r$ such that V is a simple $\mathfrak{g}^r$-submodule of $\tilde{V} = V_1 \otimes \cdots \otimes V_r$. Furthermore, there exists N such that $\tilde{V} \cong \bigoplus_{i=1}^{N} V$, i.e. $\tilde{V}$ is isomorphic to N copies of V. We wish to prove that $N = 1$.

The $\mathcal{G}$-module V is bounded if and only if $\tilde{V}$ is bounded as a $\mathfrak{g}$-module. Moreover, $\Delta = \Delta^i_V = R_V = C_V \cap \Delta$, because V is a cuspidal $\mathcal{G}$-module. Note that $V^\lambda \subset \tilde{V}^\lambda = \bigotimes_{\lambda_1 + \dots \lambda_r = \lambda} V_i^{\lambda_i}$. Thus, if $\alpha \in R_V$, then there exists $i \in \{1, \dots, r\}$ such that $V_i^{\lambda_i + n\alpha} \neq 0$ for infinite many $n > 0$, which implies that $\alpha \in R_{V_i}$. On the other hand, take $\alpha \in R_{V_l} \cap \Delta_{\bar{0}}$ and $a_1, \dots, a_r \in A$ with $a_i + \mathsf{m}_j = \delta_{ij} + \mathsf{m}_j$. Then, $x_\alpha \otimes a_i$ acts injectively. By Proposition 4.5.1, $\alpha \in R_V$. Since all injective even roots of V_l are elements of R_V, we conclude that $R_{V_l} \subset R_V$ by Proposition 4.5.5. Hence, $R_V = \bigcup_{i=1}^{r} R_{V_i}$. By Lemma 4.8.2, at most one V_i has infinite dimension. We may assume without loss of generality that V_1 is infinite-dimensional. In particular, for all $i > 1$, the $\mathfrak{g}$-modules V_i are highest weight modules, which implies that $\operatorname{End}_{\mathfrak{g}} V_i \cong \Bbbk$ (see [Sav14, Lemma 4.7]). By Proposition 4.3.4,

$$V \cong V_1 \otimes \cdots \otimes V_r.$$

We conclude that the associated representation $\rho : \mathcal{G} \to \mathfrak{gl}(V)$ of V factors through

$$\mathcal{G} \twoheadrightarrow \mathfrak{g} \otimes (A/\mathrm{Ann}_A(V)) \xrightarrow{\mathrm{ev}_{\mathsf{m}_1,\dots,\mathsf{m}_n}} \mathfrak{g}^{\oplus r} \xrightarrow{\rho_1 \otimes \cdots \otimes \rho_r} \mathfrak{gl}\left(V_1 \otimes \cdots \otimes V_r\right),$$

where $\rho_i : \mathfrak{g} \to \mathfrak{gl}(V_i)$ is the associated $\mathfrak{g}$-representation of V_i. Hence. V is isomorphic to the evaluation module $\bigotimes_{i=1}^{r} V_i^{\mathsf{m}_i}$.

□

Proposition 4.8.4. *Let $V_1, \dots, V_r$ be simple finite weight $\mathfrak{g}$-modules and $\mathsf{m}_1, \dots, \mathsf{m}_r$ pairwise distinct maximal ideals of A. Then the evaluation $\mathcal{G}$-module $\bigotimes_{i=1}^{r} V_i^{\mathsf{m}_i}$ is a cuspidal bounded module only if $\dim V_i = \infty$ for precisely one V_i, in which case V_i is a cuspidal bounded $\mathfrak{g}$-module. In particular, if $\bigotimes_{i=1}^{r} V_i^{\mathsf{m}_i}$ is a cuspidal bounded $\mathcal{G}$-module, then it is simple.*

Proof. Set $V = \bigotimes_{i=1}^{r} V_i^{\mathsf{m}_i}$. If $V_1, \dots, V_r$ are all bounded and no more than one of them is infinite-dimensional, then the dimension of weight spaces of V is bounded by the argument given on Lemma 4.8.2. On the other hand, if V is bounded and $N > 0$ is such that $\dim V^\lambda \leq N$ for each $\lambda \in \mathrm{Supp}\, V$, then the dimension of the weight spaces of each V_i has to be less or equal to N as well. Thus each V_i is bounded, and, by Lemma 4.8.2, no more than one V_i can be infinite-dimensional. This proves the first statement.

The second statement follows from Lemma 4.2.4 along with the fact that $\mathrm{End}_{\mathfrak{g}}(V_i) \cong \mathbb{k}$ for all, except at most one $i = 1, \dots, r$. □

4.9 Affine Lie superalgebras

The main example of map superalgebra is the Affine Lie superalgebra. Recall that $\mathfrak{g}$ is a basic classical Lie superalgebra. The associated *loop superalgebra* is the map algebra $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. It has a universal central extension $\mathcal{A}(\mathfrak{g})$ called *Affine Lie superalgebra* associated to $\mathfrak{g}$. It is possible to construct $\mathcal{A}(\mathfrak{g})$ explicitly. Let $\langle \cdot, \cdot \rangle$ be an even invariant supersymmetric bilinear form on $\mathfrak{g}$. As a vector space, $\mathcal{A}(\mathfrak{g}) = L(\mathfrak{g}) \oplus \mathbb{C}c$, where c is an even element. The bracket is given by

$$[c, \mathcal{A}(\mathfrak{g})] = 0, \quad [x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + n(x, y)\delta_{n,-m} c,$$

for any $x, y \in \mathfrak{g}$, $m, n \in \mathbb{Z}$.

If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c$ is a Cartan subalgebra of $\mathcal{A}(\mathfrak{g})$. Any simple $L(\mathfrak{g})$-module V is a simple $\mathcal{A}(\mathfrak{g})$-module if we assume that c acts trivially. On the other hand, if we take a $\mathcal{A}(\mathfrak{g})$-module such that c acts trivially, then V is a module over $L(\mathfrak{g}) \cong \mathcal{A}(\mathfrak{g})/\mathbb{C}c$.

Proposition 4.9.1. *Let V be a finite weight $\mathcal{A}(\mathfrak{g})$-module, then c acts trivially on V. In other words, there is a bijection between simple finite weight $L(\mathfrak{g})$-modules and simple finite weight $\mathcal{A}(\mathfrak{g})$-modules.*

Proof. By Schur's Lemma (Lemma 4.2.1), c must act as scalar on V, i.e. $cv = kv$ for every $v \in V$ for some fixed $k \in \mathbb{C}$. Let $\lambda \in \operatorname{Supp}(V)$, and $h \in \mathfrak{h}$ with $\langle h, h\rangle \neq 0$. The trace of the operator

$$[h \otimes t, h \otimes t^{-1}] = (h \otimes t)(h \otimes t^{-1}) - (h \otimes t^{-1})(h \otimes t) \in \operatorname{End}_{\Bbbk}(V^{\lambda})$$

is zero. However,

$$[h \otimes t, h \otimes t^{-1}] = \langle h, h\rangle c$$

implies that the trace of $[h \otimes t, h \otimes t^{-1}]$ is $\dim(V^{\lambda})\langle h, h\rangle c$. Since both $\langle h, h\rangle$ and $\dim(V^{\lambda})$ are non-zero, we have that c acts trivially. □

The argument given in the proof of the previous result should work on the universal central extension of other map superalgebras if they exist. This result was proved for the case of Lie algebras in [BLL15, Theorem 2.2].

4.10 Summary of results

The main goal of this chapter was to classify simple weight modules with finite-dimensional weight spaces over the map superalgebra $\mathcal{G} = \mathfrak{g} \otimes A$, where $\mathfrak{g}$ is a basic classical Lie superalgebra and A is a finitely generated commutative algebra. In some results in this chapter, A can be assumed to be a finitely generated commutative superalgebra.

To attain this goal, we established numerous theorems concerning the representation of Lie superalgebras. These encompassed theorems addressing simple modules over the direct sum of Lie superalgebras (Proposition 4.2.5 and Proposition 4.3.4).

We studied the action of the space $\mathfrak{g}_\alpha \otimes S$, where α is an element of the root system Δ of $\mathfrak{g}$, and used the shadow of the weight module to prove a parabolic induction theorem.

Theorem (Theorem 4.6.6)**.** *Let V be a simple a finite weight $\mathcal{G}$-module, then there is a triangular decomposition T such that $\Delta_V^i = \Delta_T^0$, $\Delta_V^+ \subset \Delta_T^+$, $\Delta_V^- \subset \Delta_T^-$, the vector space of $\mathcal{G}_T^+$-invariants $V^{\mathcal{G}_T^+}$ is a simple bounded $\mathcal{G}_V^i$-module, and $V \cong L_T(V^{\mathcal{G}_T^+})$.*

We proceed to the classification of modules that are not parabolically induced, the cuspidal modules. We gave a complete description of these modules in terms of simple weight $\mathfrak{g}$-modules with finite-dimensional weight spaces and maximal ideals of A.

Theorem (Theorem 4.8.3, Preposition 4.8.4)**.** *Suppose that $\mathcal{G}$ has cuspidal modules. If V is a cuspidal bounded $\mathcal{G}$-module, then V is isomorphic to an evaluation module. On the other hand, let $V_1, \dots, V_r$ be simple finite weight $\mathfrak{g}$-modules and $\mathsf{m}_1, \dots, \mathsf{m}_r$ pairwise distinct maximal ideals of A. Then the evaluation $\mathcal{G}$-module $\bigotimes_{i=1}^{r} V_i^{\mathsf{m}_i}$ is a cuspidal bounded module* *only if* $\dim V_i = \infty$ *for precisely one* V_i*, in which case* V_i *is a cuspidal bounded* $\mathfrak{g}$*-module. In particular, if* $\bigotimes_{i=1}^{r} V_i^{\mathsf{m}_i}$ *is a cuspidal bounded* $\mathcal{G}$*-module, then it is simple.*

These findings will be presented in a paper co-authored with Vyacheslav Futorny and Lucas Calixto [CFR23].

Appendix A

Sheaves, ringed spaces and schemes

Let X be any topological space and $\tau(X)$ the set of all open sets of X. We construct the category $\mathbf{top}(X)$ using this data. The objects of this category are the open sets of X, i.e. the elements of $\tau(X)$. If $U, V \in \tau(X)$, then the morphisms in this category are defined by inclusions

$$\mathrm{Hom}_{\mathbf{top}(X)}(U, V) = \begin{cases} \{U \to V\} & \text{if } U \subset V, \\ \emptyset & \text{otherwise.} \end{cases}$$

Definition A.0.1. Let X be a topological space and consider $\mathbf{top}(X)$ the category of its open sets. A *presheaf* of commutative algebebras on X is a contravariant functor $\mathcal{F} : top(M)^{op} \to \mathrm{CAlg}$. Explicitly,

1. for each open set U of X, $\mathcal{F}(U)$ is a commutative algebra;
2. If $U \subset V$ are two open sets of X, there exists a morphism $r_{U,V} : \mathcal{F}(V) \to \mathcal{F}$, called the *restritiction morphism* (often denoted by $r_{V,U}(f) = f|_U$), such that
   (a) $r_{U,U} = id$,
   (b) $r_{U,V} = r_{U,W} \circ r_{w,V}$ if W is an open set of X such that $U \subset W \subset V$.

A presheaf $\mathcal{F}$ is called a sheaf if given an open covering $\{U\}_{i\in I}$ of U and a family $\{f_i\}_{i\in I}$, $f_i \in \mathcal{F}(U_i)$, such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, there exists a unique $f \in \mathcal{F}(U)$ with $f|_{U_i} = f_i$.

If $\mathcal{F}, \mathcal{G}$ are two presheaves on X, then a *morphism of presheaves* $\varphi : \mathcal{F} \to \mathcal{G}$ if for each open set $U \subset X$ there exists a morphism of algebras $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that if $V \subset U$ are open sets then the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ \big\downarrow r^{\mathcal{F}}_{U,V} & & \big\downarrow r^{\mathcal{G}}_{U,V} \\ \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{F}(V) \end{array}$$

commutes. A morphism of sheaves is a morphism of the presheaves associated with it.

We defined sheaves and presheaves of commutative algebras in the previous definition. The definition of sheaves of abelian groups, algebras, modules and other algebraic structures can be easily generalized from it.

Notation A.0.2. If $\mathcal{F}$ is a sheaf of commutative algebras on X, we will often denote $\mathcal{F}(U)$ by $\Gamma(U, \mathcal{F})$, where $U \subset X$ is an open set. Elements of $\Gamma(U, \mathcal{F})$ are often called *sections* of $\mathcal{F}$ over the open set U.

Definition A.0.3. Let $\mathcal{F}$ be a sheaf on the topological space X and let $x \in X$. We define the *stalk* $\mathcal{F}_x$ of $\mathcal{F}$ at the point x as the direct limit

$$\mathcal{F}_x = \operatorname{inj} \lim_{x \in U} \mathcal{F}(U).$$

Explicitly, $\mathcal{F}_x$ consists of the disjoint union of all pairs (U, s) with U open in X, $x \in U$, and $s \in \mathcal{F}(U)$, module the equivalence relation: $(U, s) \cong (V, s)$ if and only if there exists a neighborhood W of x, $W \subset U \cap V$ such that $s|_W = t|_W$.

Let $f : X \to Y$ be a continuous function between two topological spaces and $\mathcal{F}$ be a presheaf of commutative algebras on X. We define $f_* \mathcal{F}$ to be the presheaf on Y given by $(f_*)(U) = \mathcal{F}(f^{-1}(U))$ for each open subset $U \subset Y$. Its restriction morphisms are exactly the ones from $\mathcal{F}$. If $\mathcal{F}$ is a sheaf on X, then $f_* \mathcal{F}$ is a sheaf on Y. The sheaf $f_* \mathcal{F}$ is called *direct image* and it defines a functor from the category of (pre)sheaves on Y to the categories of (pre)sheaves on X called direct image functor.

Definition A.0.4. A *ringed space* is a pair $X = (|X|, \mathcal{F})$ consisting of a topological space $|X|$ and a sheaf of commutative rings $\mathcal{F}$ on $|X|$. We say that a ringed space $(|X|, \mathcal{F})$ is a *locally ringed space* if the stalk $\mathcal{F}_x$ is a local ring for all $x \in |X|$. A *morphism of locally ringed spaces* $f : X \to Y$ between the locally ringed spaces $X = (X, \mathcal{F})$ and $Y = (Y, \mathcal{G})$ is a pair of maps $f = (|f|, f^\#)$ such that $|f| : |X| \to |Y|$ is a continuous map, $f^\# : \mathcal{O}_Y \to |f|_* \mathcal{O}_X$ is a sheaf morphism and $f^\#_p : \mathcal{O}_{Y,p} \to |f|_* \mathcal{O}_{X,p}$ is a homomorphism of local algebras.

Remark A.0.5. For a locally ringed space $(X, \mathcal{O}_X)$ and an open subset $|U| \subset |X|$, we will often write $U \subset X$ for the locally ringed space $(|U|, \mathcal{O}_X|_{|U|})$. When the distinction is not necessary, we might refer to X as a topological space instead of $|X|$.

Example A.0.6 (Structure sheaf)**.** Let A be a commutative algebra. Remember that for each $h \in A$

$$D(h) = \{\mathfrak{p} \in \operatorname{Spec} A \mid h \notin \mathfrak{p}\}$$

is an open set on $\operatorname{Spec} A$, and $\{D(h) \mid h \in A\}$ is an open basis of the Zariski topology of $\operatorname{Spec} A$. Then for each $h \in A$, define

$$\mathcal{O}_A(D(h)) = A_h,$$

then this assignment extends uniquely to a sheaf of commutative rings on $\operatorname{Spec} A$ (see [Har77, Section II.2]), called structure sheaf and denoted by $\mathcal{O}_A$. The stalk at a point $\mathfrak{p} \in \operatorname{Spec} A$, $\mathcal{O}_{A,\mathfrak{p}}$ is the localization $A_\mathfrak{p}$ of the ring A at the prime $\mathfrak{p}$. The pair $(\operatorname{Spec} A, \mathcal{O}_A)$ is a locally ringed space.

Definition A.0.7. A locally ringed space X is called *affine scheme* over $\Bbbk$ if $X \cong (\operatorname{Spec} A, \mathcal{O}_A)$ for some commutative $\Bbbk$-algebra A. A locally ringed space $(X, \mathcal{O}_X)$ is a *scheme* over $\Bbbk$ if there is an open cover $\{U_i\}_{i\in I}$ of X and commutative algebras A_i, $i \in I$, such that

$$(U_i, \mathcal{O}_X|_{U_i}) \cong (\operatorname{Spec} A_i, O_{A_i}) \quad \forall i \in I.$$

In this case, we say $\{U_i = \operatorname{Spec} A_i\}_{i\in I}$ is an *affine cover* of X. A morphism of schemes $X \to Y$ is simply a morphism of the locally ringed spaces X and Y.

The category of affine schemes $\mathbf{Sch}(\Bbbk)$ over $\Bbbk$ is closely related to the category of commutative $\Bbbk$-algebras $\Bbbk - \mathbf{CAlg}$. The functor $\Gamma : \mathbf{Sch}(\Bbbk) \to \Bbbk - \mathbf{CAlg}^{op}$ that associates each affine scheme $X = (|X|, \mathcal{O}_X) = \operatorname{Spec}(A)$ to the global sections $\Gamma(X) = \Gamma(X, \mathcal{O}_X) = A$ gives this equivalence. This follows from the definition and the following proposition.

Proposition A.0.8. *Let X be any locally ringed space and Y a scheme, then the map*

$$\operatorname{Hom}_{\mathbf{Sch}(\Bbbk)}(X, Y) \to \operatorname{Hom}_{\Bbbk-\mathbf{CAlg}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$$

that sends $f = (|f|, f^\#) : X \to Y$ to $f^\#_{|X|} : \mathcal{O}_Y(|Y|) \to f_*\mathcal{O}_X(|X|)$ is a bijection.

Proof. See [Gro61, Errata (Liste I), Proposition 1.8.1]. □

Example A.0.9. Let X be an algebraic variety with coordinate ring $A = A_X$. For each basic open set $D(h) \subset X$, we define $\mathcal{O}(D(h)) = A_h$ the localization of A at h. This defines a sheaf $\mathcal{O}_X$ on X of commutative algebras, and since $\mathcal{O}_{X,p} = A_{\mathfrak{m}_p}$ for every $p \in X$, we have that $(X, \mathcal{O}_X)$ is a locally ringed space. Thus, $(X, \mathcal{O}_X)$ is a locally ringed space. However, $\mathcal{O}_X$ is not the structure sheaf $\mathcal{O}_A$ of A, because $\operatorname{Spec}(A)$ and X are not homeomorphic, hence $(X, \mathcal{O}_X)$ is not a scheme. The map $f : X \to \operatorname{Spec}(A)$ defined by $f(p) = \mathfrak{m}_p$ defines a bijection between X and the set of all maximal ideals of A, but it is not surjective. Nevertheless, the map $f : X \to \operatorname{Spec}(A)$ is still a continuous map between X and $\operatorname{Spec}(A)$, and the assignment $U \mapsto f^{-1}(U)$ defines a bijection between open sets of X and open sets of $\operatorname{Spec}(A)$. Since $\mathcal{O}_X(U) = \mathcal{O}_A(f^{-1}(U)) = f_*\mathcal{O}_A(U)$, the morphism of sheaves $\mathcal{O}_X \to f_*\mathcal{O}_A$ is an isomorphism. Therefore, the assignment $X \mapsto \operatorname{Spec}(A_X)$ associates a scheme to the affine algebraic variety X. This assignment defines a functor from the category of affine varieties over $\Bbbk$ to the category of schemes over $\Bbbk$ and this functor is a natural fully faithful functor [Har77, Chapter II, Proposition 2.6]. In this thesis, we will often intertwine both structures X and $\operatorname{Spec}(A_X)$, using scheme theoretically concepts on X even though we are in reality examining $\operatorname{Spec}(A_X)$.

Example A.0.10. It is possible to *glue* two schemes together creating a new scheme provided the existence of *gluing maps*. Let $X_1 = (|X_1|, \mathcal{O}_1)$ and $X_2 = (|X_2|, \mathcal{O}_2)$ be two schemes, $U_1 \subset |X_1|$ and $U_2 \subset |X_2|$ be open subsets, and suppose there exists an isomorphism $\varphi : (U_1, \mathcal{O}_1|_{U_1}) \xrightarrow{\cong} (U_2, \mathcal{O}_2|_{U_2})$ of locally ringed spaces. Consider the relation $\sim$ in the disjoint union $|X_1| \sqcup |X_2|$ defined by

$$x_1 \sim x_2 \iff x_1 \in U_1,\ x_2 \in U_2,\ x_2 = \varphi(x_1).$$

We wish to define a scheme $X = (|X|, \mathcal{O}_X)$, where the topological space $|X| = |X_1| \sqcup |X_2| / \sim$ is the quotient of the disjoint union $|X_1| \sqcup |X_2|$ by the equivalence relation $\sim$ with projection maps $\iota_1 : |X_1| \to |X|$ and $\iota_2 : |X_2| \to |X|$. For an open subset $U \subset |X|$, the section $\mathcal{O}_X(U)$ is defined by

$$\mathcal{O}_X(U) = \left\{ (s_1, s_2) \in \mathcal{O}_1\left(\iota_1^{-1}(U)\right) \times \mathcal{O}_2\left(\iota_2^{-1}(U)\right) \mid s_2|_{U_2 \cap \iota_1^{-1}(U)} = \varphi\left(s_1|_{U_1 \cap \iota_2^{-1}(U)}\right) \right\}.$$

Then $\mathcal{O}_X$ is a sheaf of commutative algebras and $X = (|X|, \mathcal{O}_X)$ is a locally ringed space. Since each X_1 and X_2 are schemes, the locally ringed space X is a scheme as well.

Example A.0.11. Let $X_1 = \operatorname{Spec}(\Bbbk[x])$ and $X_2 = \operatorname{Spec}(\Bbbk[y])$. We will use Example A.0.10 to glue X_1 and X_2 through the open subsets $U_1 = D(x)$ and $U_2 = D(y)$. Note that $\Gamma(U_1, \mathcal{O}_{X_1}) = \Bbbk[x, x^{-1}]$ and $\Gamma(U_2, \mathcal{O}_{X_1}) = \Bbbk[y, y^{-1}]$. Let $\varphi : U_1 \to U_2$ be the scheme isomorphism defined by the isomorphism of algebras $y \mapsto x^{-1}$. The scheme obtained by gluing X_1 and X_2 through the gluing map φ is called *projective line* and it is denoted by $\mathbb{P}^1_{\Bbbk}$. Note that if $(s_1, s_2) \in \mathcal{O}_{X_1}(X_1) \times \mathcal{O}_{X_2}(X_2) = \Bbbk[x] \times \Bbbk[y]$ satisfies $\varphi^{\#}_{U_1}\left(s_2|_{U_2}\right) = s_1|_{U_2}$, then $s_2(\frac{1}{x}) = s_1\left(\frac{1}{x}\right)$ as rational functions in $\Bbbk[x, x^{-1}]$. Since $s_2 \in \Bbbk[y] \cong \Bbbk[x^{-1}]$ is polynomial, we get that $s_2(x) = s_1(x) \in \Bbbk \subset \Bbbk[x, x^{-1}]$. Thus, $\mathcal{O}_{\mathbb{P}^1_{\Bbbk}}(|X|) = \Bbbk$. In particular, $\mathbb{P}^1_{\Bbbk}$ is not affine, because $\operatorname{Spec}(\Bbbk)$ has a single point while $\mathcal{O}_{\mathbb{P}^1_{\Bbbk}}(\mathbb{P}^1_{\Bbbk})$ has infinite many.

Example A.0.12. Let $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$. Consider $T = \operatorname{Spec}(A \otimes_{\Bbbk} B)$, then there exists projections morphisms $p_1 : T \to X$ and $p_2 : T \to Y$ that are defined by the homomorphism of algebras $A \to A \otimes B$ and $B \to A \otimes B$ given by $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$. By Proposition A.0.8

$$\begin{aligned}
\operatorname{Hom}_{\mathbf{Sch}(\Bbbk)}(Z, T) &= \operatorname{Hom}_{\Bbbk-\mathbf{CAlg}}(A \otimes B, \mathcal{O}_Z(Z)) \\
&= \operatorname{Hom}_{\Bbbk-\mathbf{CAlg}}(A, \mathcal{O}_Z(Z)) \times \operatorname{Hom}_{\Bbbk-\mathbf{CAlg}}(B, \mathcal{O}_Z(Z)) \\
&= \operatorname{Hom}_{\mathbf{Sch}(\Bbbk)}(Z, X) \times \operatorname{Hom}_{\mathbf{Sch}(\Bbbk)}(Z, Y).
\end{aligned}$$

Therefore, T satisfies the universal property that for every morphism of schemes $f_1 : Z \to X$ and $f_2 : Z \to$ there exists a unique morphism $f : Z \to T$ such that the diagram

$$\begin{tikzcd}
Z \arrow[drr, bend left, "f_1"] \arrow[dr, dashed, "f"] \arrow[ddr, bend right, "f_2"'] & & \\
& T \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d] \\
& Y \arrow[r] & \operatorname{Spec}(\Bbbk)
\end{tikzcd}$$

commutes. The affine scheme T is usually denoted by $X \times Y$ or $X \times_{\operatorname{Spec}(\Bbbk)} Y$, and it is called *fibre product*. For general schemes X and Y, the scheme $X \times Y$ is defined by the universal property above and it comes with projection maps $X \times Y \to X$ and $X \times Y \to Y$.

Definition A.0.13. Let $X = (|X|, \mathcal{O}_X)$ be a scheme. We say that X is

1. *irreducible* if the topological space $|X|$ is irreducible.
2. *reduced* if $\mathcal{O}_{X,p}$ is a reduced algebra for every $p \in X$.
3. *integral* if X is both reduced and irreducible.

4. of *finite type* if for every affine open subset $U = \operatorname{Spec}(B) \subset X$ we have that B is a finitely generated $\Bbbk$-algebra.

5. *separated* if there exists a closed subset $F \subset Y$ such that $|f| : |X| \to F$ is a homeomorphism and $f_p^* : \mathcal{O}_{X\times X, f(p)} \to \mathcal{O}_{X,p}$ is surjective for every $p \in X$, where $f : X \to X \times X$ is the diagonal morphism $p_1 \circ f = p_2 \circ f = \iota_X$ and $p_1, p_2 : X \times X \to X$ are the projections.

Example A.0.14. If X is an affine algebraic variety with coordinate ring A_X, then $\operatorname{Spec}(A_X)$ is an integral separated affine scheme of finite type over $\Bbbk$. Reciprocally, if $Y = \operatorname{Spec}(B)$ is an integral separated affine scheme of finite type over $\Bbbk$, then B is a finitely generated $\Bbbk$-algebra because Y is of finite type. Thus, $B = \Bbbk[x_1, \dots, x_n]/I$ for some $n \geq 1$ and ideal I of $\Bbbk[x_1, \dots, x_n]$. Additionally, B is a reduced algebra because $B_\mathfrak{p}$ is reduced for every prime ideal $\mathfrak{p} \in \operatorname{Spec}(B)$ (see [AM69, Exercise 3.5]). Since Y is irreducible, the intersection of two nonempty open subsets must be nonempty. For every $f, g \in B$ with $f \neq 0$ and $g \neq 0$, $D(f) \cap D(g) = D(fg) \neq \emptyset$, thus fg is not nilpotent. In particular, $fg \neq 0$ for every $f, g \in B$ with $f \neq 0$ and $g \neq 0$. We conclude that B is an integral domain. Therefore, the ideal I is a prime ideal of $\Bbbk[x_1, \dots, x_n]$, thus $Z(I) \subset \mathbb{A}^n_\Bbbk$ is an affine algebraic variety, $B = \Bbbk[Z(I)] = A_{Z(I)}$ and $Y = \operatorname{Spec}(A_{Z(I)})$. In other words, in the scheme theoretically setting, affine algebraic varieties can be defined as integral separated affine schemes of finite type over $\Bbbk$.

Definition A.0.15. Let $X = (X, \mathcal{O}_X)$ be a scheme and $\mathcal{F}$ be a sheaf on X of abelian groups. We say that $\mathcal{F}$ is a *sheaf of $\mathcal{O}_X$-modules* if, for each open set $U \subset X$, the abelian group $\Gamma(U, \mathcal{F})$ is a $\Gamma(U, \mathcal{O}_X)$-module, and the restriction morphisms $\Gamma(V, \mathcal{F}) \to \Gamma(V, \mathcal{F})$ is compatible with the module structure via the algebra homomorphism $\Gamma(U, \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_X)$ for each inclusion of open sets $U \subset V$.

Example A.0.16. Let A be a commutative ring and M an A-module. For each $h \in A$, consider the assignment $D(h) \mapsto M_h$, where M_h is the A_h-module given by the localization of M at h. This assignment defines a sheaf on $\operatorname{Spec} A$ denoted by $\tilde{M}$. For all open set $U \subset \operatorname{Spec}(A)$, $\tilde{M}(U)$ is an $\mathcal{O}_A(U)$-module, therefore $\tilde{M}$ is a sheaf of $\operatorname{Spec}(A)$-modules. Furthermore, $(\tilde{M})_\mathfrak{p} = M_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} A$, and $\tilde{M}(\operatorname{Spec} A) = M$.

Example A.0.17. Let A be a commutative ring and M an A-module. Denote $\mathrm{T}^0(M) = A$ and $\mathrm{T}^n(M) = M \otimes_A M \otimes \cdots \otimes_A M$ the tensor product of M with itself n-times, then the concatenation of tensors makes $\mathrm{T}(M) = \bigoplus_{n\geq 0} \mathrm{T}^n(M)$ an unital associative algebra, which is called *tensor algebra of M*. Denote by $\mathrm{S}(M) = \bigoplus_{n\geq 0}$ the *symmetric algebra of M*, which is the quotient of $\mathrm{T}(M)$ by the ideal generated by $x \otimes_A y - y \otimes_A x$, $x, y \in M$.

Let $X = (X, \mathcal{O}_X)$ be a scheme and $\mathcal{F}$ be a sheaf on $\mathcal{O}_X$-modules. We can use the above construction to define the *tensor algebra* $\mathrm{T}(\mathcal{F})$ and the *symmetric algebra* $\mathrm{S}(\mathcal{F})$ of $\mathcal{F}$, which $\Gamma(U, \mathrm{T}(\mathcal{F})) = \mathrm{T}(\Gamma(U, \mathrm{T}))$ and $\Gamma(U, \mathrm{S}(\mathcal{F})) = \mathrm{S}(\Gamma(U, \mathrm{S}))$ for each open set $U \subset X$. Both $\mathrm{T}(\mathcal{F})$ and $\mathrm{S}(\mathcal{F})$ are $\mathcal{O}_X$-algebras and each degree component is an $\mathcal{O}_X$-module.

Definition A.0.18. Let $X = (X, \mathcal{O}_X)$ be a scheme, and $\mathcal{F}$ a sheaf on X of $\mathcal{O}_X$-modules, i.e. $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module for all U open in X and the restriction morphism behave nicely with respect to the $\mathcal{O}_X$-module structure. We say $\mathcal{F}$ is *quasi-coherent*, if there exists an open affine cover $\{U_i = \operatorname{Spec} A_i\}_{i\in I}$ of X such that $\mathcal{F}|_{U_i} \cong \tilde{M}_i$ (see Example A.0.16), where $\tilde{M}_i$ is the sheaf on $\operatorname{Spec} A_i$ defined for a suitable A_i-module M_i. If X is a locally Noetherian scheme, $\mathcal{F}$ is called *coherent* if the affine cover can be chosen so that the M_i's are finitely generated A_i-modules.

Example A.0.19. Let B be an associative algebra and $A \subset B$ a commutative subalgebra. Then, B is an A-module and $\tilde{B}$ is a quasi-coherent sheaf over $\operatorname{Spec}(A)$. If B is finitely generated as A-module, then $\tilde{B}$ is coherent.

Definition A.0.20. Let $X = (X, \mathcal{O}_X)$ be a scheme and $\mathcal{F}$ be quasi-coherent sheaf on X. We say that $\mathcal{F}$ is *locally free sheaf* of $\mathcal{O}_X$-modules if X can be covered by open sets $\{U_i \mid i \in I\}$ such that $\Gamma(U_i, \mathcal{F})$ is free as an $\Gamma(U_i, \mathcal{O}_X)$-module. If X is irreducible, then we define the *rank of* $\mathcal{F}$ as the size of a basis of $\Gamma(U_i, \mathcal{F})$ as an $\Gamma(U_i, \mathcal{O}_X)$-module, where $i \in I$.

Example A.0.21. Let A be a Noetherian integral domain and $X = \operatorname{Spec}(A)$, then $\tilde{M}$ is a locally free sheaf of $\mathcal{O}_X$-modules if and only if M is a finitely generated projective A-module [Eis95, Theorem A3.2].

Example A.0.22. Let $X = (X, \mathcal{O}_X)$ be a scheme and $\mathcal{F}$ be a locally free quasi-coherent sheaf on X with rank n. Denote by $Y = \operatorname{Spec}(\mathrm{S}(\mathcal{F}))$ be the spectrum of symmetric algebra of $\mathcal{F}$, then Y comes with a projection morphism $p : Y \to X$. Take $U \subset X$ such that $M = \Gamma(U, \mathcal{F})$ is free as $A = \Gamma(U, \mathcal{O})$-module. Choose a basis of M. This choice induces an isomorphism $\mathrm{S}(M) \cong A[x_1, \dots, x_n]$, which induces an isomorphism $p^{-1}(U) \to \mathcal{O}_U^n$. Thus, for each $x \in X$, there exists an open affine neighborhood $x \in U = \operatorname{Spec}(A)$ of x such that $p^{-1}(U)$ is isomorphic to

$$\begin{aligned} p^{-1}(U) &\cong \operatorname{Spec}(\mathcal{O}_U^n) \cong \operatorname{Spec}(A[x_1, \dots, x_n]) \cong \operatorname{Spec}(A \otimes_{\Bbbk} \Bbbk[x_1, \dots, x_n]) \\ &\cong \operatorname{Spec}(A) \times \operatorname{Spec}(\Bbbk[x_1, \dots, x_n]) = U \times \mathbb{A}_{\Bbbk}^n. \end{aligned}$$

Definition A.0.23. Let $X = (X, \mathcal{O}_X)$ be a scheme. A *vector bundle* Y of rank n over X is a scheme Y together with a morphism $p : Y \to X$ such that for each $p \in X$ there exists an open neighborhood $U \subset X$ of p such that $\Gamma(Y, \mathcal{O}_Y)$ is isomorphic to $U \times \mathbb{A}_{\Bbbk}^n$.

Example A.0.24. By Example A.0.22, every locally free sheaf of $\mathcal{O}_X$-modules can be associated uniquely (up to isomorphism) to a vector bundle. This is a one-to-one correspondence (up to isomorphism) between vector bundles and locally free sheaves of finite rank. [Har77, Exercise II.5.18].

Several other concepts arise in the scheme theory that we will not cover or explore further, because we will only need basic definitions in this text since we will be working with affine algebraic varieties. If the reader is looking for a reference on these concepts and results related to them, we recommend the books [Har77] and [Mum99].

Bibliography

[AM69] Michael F. Atiyah and Ian G. Macdonald. Introduction to commutative algebra. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969, ix+128.

[BF14] Yuly Billig and Vyacheslav Futorny. Representations of the Lie algebra of vector fields on a torus and the chiral de Rham complex. *Trans. Amer. Math. Soc.* 366.9 (2014), 4697–4731.

[BF16] Yuly Billig and Vyacheslav Futorny. Classification of irreducible representations of Lie algebra of vector fields on a torus. *J. Reine Angew. Math.* 720 (2016), 199–216.

[BF18] Yuly Billig and Vyacheslav Futorny. Lie algebras of vector fields on smooth affine varieties. *Comm. Algebra* 46.8 (2018), 3413–3429.

[BFN19] Yuly Billig, Vyacheslav Futorny, and Jonathan Nilsson. Representations of Lie algebras of vector fields on affine varieties. *Israel J. Math.* 233.1 (2019), 379–399.

[BI23] Yuly Billig and Colin Ingalls. A universal sheaf of algebras governing representations of vector fields on quasi-projective varieties. 2023. arXiv: 2302.07918 [math.RT].

[BIN23] Yuly Billig, Colin Ingalls, and Amir Nasr. $\mathcal{AV}$ modules of finite type on affine space. *J. Algebra* 623 (2023), 481–495.

[BLL15] Daniel Britten, Michael Lau, and Frank Lemire. Weight modules for current algebras. *J. Algebra* 440 (2015), 245–263.

[BN19] Yuly Billig and Jonathan Nilsson. Representations of the Lie algebra of vector fields on a sphere. *J. Pure Appl. Algebra* 223.8 (2019), 3581–3593.

[BNZ21] Yuly Billig, Jonathan Nilsson, and André Zaidan. Gauge Modules for the Lie Algebras of Vector Fields on Affine Varieties. *Algebr. Represent. Theory* 24.5 (2021), 1141–1153.

[BR23] Emile Bouaziz and Henrique Rocha. Annihilators of $\mathcal{AV}$-modules and differential operators. *J. Algebra* 636 (2023), 869–887.

[BS88] Alexander A. Beĭlinson and Vadim V. Schechtman. Determinant bundles and Virasoro algebras. *Comm. Math. Phys.* 118.4 (1988), 651–701.

[Car09] Elie Cartan. Les groupes de transformations continus, infinis, simples. *Ann. Sci. École Norm. Sup. (3)* 26 (1909), 93–161.

[CCF11] Claudio Carmeli, Lauren Caston, and Rita Fioresi. Mathematical foundations of supersymmetry. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2011, xiv+287.

[CFR23] Lucas Calixto, Vyacheslav Futorny, and Henrique Rocha. Harish-Chandra modules for map and affine Lie superalgebras. 2023. arXiv: 2104.07517 [math.RT].

[CK98] Shun-Jen Cheng and Victor G. Kac. Erratum: "Conformal modules" [Asian J. Math. **1** (1997), no. 1, 181–193; MR1480993 (98j:17026)]. *Asian J. Math.* 2.1 (1998), 153–156.

[Col89] Albert J. Coleman. The greatest mathematical paper of all time. *Math. Intelligencer* 11.3 (1989), 29–38.

[DMP00] Ivan Dimitrov, Olivier Mathieu, and Ivan Penkov. On the structure of weight modules. *Trans. Amer. Math. Soc.* 352.6 (2000), 2857–2869.

[DMP04] Ivan Dimitrov, Olivier Mathieu, and Ivan Penkov. Errata to: "On the structure of weight modules" [Trans. Amer. Math. Soc. **352** (2000), no. 6, 2857–2869; MR1624174]. *Trans. Amer. Math. Soc.* 356.8 (2004), 3403–3404.

[DNR01] Sorin Dăscălescu, Constantin Năstăsescu, and Şerban Raianu. Hopf algebras. Vol. 235. Monographs and Textbooks in Pure and Applied Mathematics. An introduction. Marcel Dekker, Inc., New York, 2001, x+401.

[EF09] Senapathi Eswara Rao and Vyacheslav Futorny. Integrable modules for affine Lie superalgebras. *Trans. Amer. Math. Soc.* 361.10 (2009), 5435–5455.

[Eis95] David Eisenbud. Commutative algebra. Vol. 150. Graduate Texts in Mathematics. With a view toward algebraic geometry. Springer-Verlag, New York, 1995, xvi+785.

[Fio08] Rita Fioresi. Smoothness of algebraic supervarieties and supergroups. *Pacific J. Math.* 234.2 (2008), 295–310.

[GQS66] Victor Guillemin, Daniel Quillen, and Shlomo Sternberg. The classification of the complex primitive infinite pseudogroups. *Proc. Nat. Acad. Sci. U.S.A.* 55 (1966), 687–690.

[Gro60] Alexander Grothendieck. Éléments de géométrie algébrique. I. Le langage des schémas. *Inst. Hautes Études Sci. Publ. Math.* 4 (1960), 228.

[Gro61] Alexander Grothendieck. Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes. *Inst. Hautes Études Sci. Publ. Math.* 8 (1961), 222.

[Gro67] Alexander Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.* 32 (1967), 361.

[GS22] Dimitar Grantcharov and Vera Serganova. Simple weight modules with finite weight multiplicities over the Lie algebra of polynomial vector fields. *J. Reine Angew. Math.* 792 (2022), 93–114.

[GS64] Victor W. Guillemin and Shlomo Sternberg. An algebraic model of transitive differential geometry. *Bull. Amer. Math. Soc.* 70 (1964), 16–47.

[Har77] Robin Hartshorne. Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977, xvi+496.

[Haw96] Thomas Hawkins. Élie Cartan (1869–1951).By M. A. Akivis and B. A. Rosenfeld. Translated from a Russian manuscript by V. V. Goldberg. Providence (American Mathematical Society). *Historia Mathematica* 23.1 (1996), 92–95.

[Hum78] James E. Humphreys. Introduction to Lie algebras and representation theory. Vol. 9. Graduate Texts in Mathematics. Second printing, revised. Springer-Verlag, New York-Berlin, 1978, xii+171.

[Isa09] Irving Martin Isaacs. Algebra: a graduate course. Vol. 100. Graduate Studies in Mathematics. Reprint of the 1994 original. American Mathematical Society, Providence, RI, 2009, xii+516.

[Jor86] David A. Jordan. On the ideals of a Lie algebra of derivations. *J. London Math. Soc. (2)* 33.1 (1986), 33–39.

[Kac68] Victor G. Kac. Simple irreducible graded Lie algebras of finite growth. *Izv. Akad. Nauk SSSR Ser. Mat.* 32 (1968), 1323–1367.

[Kac77] Victor G. Kac. Lie superalgebras. *Advances in Math.* 26.1 (1977), 8–96.

[Kil90] Wilhelm Killing. Die Zusammensetzung der stetigen endlichen Transformationsgruppen. *Math. Ann.* 36.2 (1890), 161–189.

[Kos94] Jean-Louis Koszul. Connections and splittings of supermanifolds. *Differential Geom. Appl.* 4.2 (1994), 151–161.

[Lau18] Michael Lau. Classification of Harish-Chandra modules for current algebras. *Proc. Amer. Math. Soc.* 146.3 (2018), 1015–1029.

[Mat92a] Olivier Mathieu. Classification of Harish-Chandra modules over the Virasoro Lie algebra. *Invent. Math.* 107.2 (1992), 225–234.

[Mat92b] Olivier Mathieu. Classification of simple graded Lie algebras of finite growth. *Invent. Math.* 108.3 (1992), 455–519.

[Moo68] Robert V. Moody. A new class of Lie algebras. *J. Algebra* 10 (1968), 211–230.

[Mum99] David Mumford. The red book of varieties and schemes. expanded. Vol. 1358. Lecture Notes in Mathematics. Includes the Michigan lectures (1974) on curves and their Jacobians, With contributions by Enrico Arbarello. Springer-Verlag, Berlin, 1999, x+306.

[NSS12] Erhard Neher, Alistair Savage, and Prasad Senesi. Irreducible finite-dimensional representations of equivariant map algebras. *Trans. Amer. Math. Soc.* 364.5 (2012), 2619–2646.

[Qui69] Daniel Quillen. On the endomorphism ring of a simple module over an enveloping algebra. *Proc. Amer. Math. Soc.* 21 (1969), 171–172.

[Rud74] Alexei N. Rudakov. Irreducible representations of infinite-dimensional Lie algebras of Cartan type. *Izv. Akad. Nauk SSSR Ser. Mat.* 38 (1974), 835–866.

[Sav14] Alistair Savage. Equivariant map superalgebras. *Math. Z.* 277.1-2 (2014), 373–399.

[Sha94a] Igor R. Shafarevich. Basic algebraic geometry. 1. Second. Varieties in projective space, Translated from the 1988 Russian edition and with notes by Miles Reid. Springer-Verlag, Berlin, 1994, xx+303.

[Sha94b] Igor R. Shafarevich. Basic algebraic geometry. 2. Second. Schemes and complex manifolds, Translated from the 1988 Russian edition by Miles Reid. Springer-Verlag, Berlin, 1994, xiv+269.

[She21] Alexander Sherman. Spherical supervarieties. *Ann. Inst. Fourier (Grenoble)* 71.4 (2021), 1449–1492.

[Sie96] Thomas Siebert. Lie algebras of derivations and affine algebraic geometry over fields of characteristic 0. *Math. Ann.* 305.2 (1996), 271–286.

[SS65] Isadore M. Singer and Shlomo Sternberg. The infinite groups of Lie and Cartan. I. The transitive groups. *J. Analyse Math.* 15 (1965), 1–114.

[Wei68] Boris Weisfeiler. Infinite dimensional filtered Lie algebras and their connection with graded Lie algebras. *Funkcional. Anal. i Priložen* 2.1 (1968), 94–95.