REPRESENTATIONS OF INFINITE-DIMENSIONAL LIE ALGEBRAS

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Preface

This thesis chronicles three studies in the representation theory of infinite-dimensional Lie algebras. The first work concerns the imaginary highest-weight theory of affine sl(2). Futorny **[14]** describes the structure of all but one of the universal objects of the theory, the imaginary Verma modules. If reducible, an imaginary Verma module $\mathbf{V}(\lambda)$ possesses an infinite descending series of submodules such that the quotient of any submodule by its successor is isomorphic to a certain module $\mathbf{M}(\lambda)$ that depends only upon the highest weight λ. The infinitely recurrent factor $\mathbf{M}(\lambda)$ is reducible precisely when $\lambda = 0$. The structure of $\mathbf{M}(0)$ is apparently complicated, and not at all understood. The principal result of the first work is a classification of the irreducible quotients of the submodules of $\mathbf{M}(0)$. This classification complements the result of Futorny to provide a structural description of the imaginary Verma modules. One may define, for any function on the integers with values in the field, an irreducible module of level zero for affine sl(2):

$$\varphi : \mathbb{Z} \to \mathbb{C} \qquad \text{defines} \qquad \mathbf{N}(\varphi) \text{ irreducible module for affine sl(2).}$$

The irreducible quotients of the submodules of $\mathbf{M}(0)$ are precisely the modules $\mathbf{N}(\varphi)$ where φ is a linear combination of exponential functions with coefficients that are integral, even, and negative. The classification follows from the construction of a family of singular vectors, and from a description of $\mathbf{M}(0)$ in terms of the symmetric functions.

The class of so-called exponential-polynomial modules, which is the class of those modules $\mathbf{N}(\varphi)$ defined by an exponential-polynomial function φ, therefore contains all the irreducible quotients of the submodules of $\mathbf{M}(0)$. An exponential-polynomial function φ is a function that may be written as a sum in which each summand is a product of an exponential and a polynomial function. Equivalently, a function φ is exponential-polynomial precisely when the sequence of its values

$$\cdots \quad \varphi(-2), \quad \varphi(-1), \quad \varphi(0), \quad \varphi(1), \quad \varphi(2), \quad \cdots$$

solves some linear homogeneous recurrence relation with constant coefficients. The values of the function φ provide the structure constants of the module $\mathbf{N}(\varphi)$, and so the exponential-polynomial modules may be thought of as modules with structure constants of a limited complexity. The isomorphism classes of the exponential-polynomial modules parameterise the isomorphism classes of the modules $\mathbf{N}(\varphi)$ whose representatives have only finite-dimensional weight spaces [**3**] [**12**]. The open problem of describing the multiplicities of the weight spaces of an exponential-polynomial module may be resolved through a study of the highest-weight representations of truncations of the loop algebra.

Any proper quotient of a loop algebra is isomorphic to a truncation of the form

$$\breve{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[\mathrm{t}]/\mathrm{t}^{\mathrm{N}+1}\mathbb{C}[\mathrm{t}],$$

where N is some non-negative integer. In the second work a highest-weight theory for the truncation $\breve{\mathfrak{g}}$ is developed when the underlying Lie algebra $\mathfrak{g}$ possesses a triangular decomposition. The principal result is a reducibility criterion for the Verma modules of $\breve{\mathfrak{g}}$ when $\mathfrak{g}$ is a symmetrisable Kac-Moody Lie algebra, the Heisenberg algebra, or the Virasoro algebra. This is achieved through a study of the Shapovalov form.

The third work employs the highest-weight theory of truncations of the loop algebra to describe the multiplicities of the weight spaces of an exponential-polynomial module in the case where $\mathfrak{g} = \mathrm{sl}(2)$. An exponential-polynomial module $\mathbf{N}(\varphi)$ may be realized as an irreducible constituent of a loop module built from an irreducible highest-weight module $\mathsf{L}(\varphi)$ for a truncation of the loop algebra. This realization, which is due to Chari and Pressley [**9**], expresses the multiplicities of $\mathbf{N}(\varphi)$ in terms of the multiplicities of $\mathsf{L}(\varphi)$ via a certain action of a finite cyclic group. In the particular case of $\mathfrak{g} = \mathrm{sl}(2)$, an expression for the formal character[1] of a module $\mathsf{L}(\varphi)$ may be deduced from the aforementioned reducibility criterion for Verma modules of $\breve{\mathfrak{g}}$. The third work develops a theory of semi-invariants for finite cyclic groups and employs this expression to solve the multiplicity problem for exponential-polynomial modules.

I am indebted to my supervisors, Alexander Molev at the University of Sydney, and Vyacheslav Futorny at the Universidade de São Paulo, for their academic guidance. I would like to extend further thanks to Vyacheslav for facilitating my transition to Brazil, and for his friendship and support during a challenging period. I am very grateful to Yuly Billig, under whose direction the second work was completed, for his counsel and encouragement.

[1]That is, the generating function defined by the multiplicities of the weight spaces.

I am fortunate to be part of a wonderful family that has provided unwaivering support throughout this most extended of adventures, and I am profoundly thankful for their understanding. A whimsical undergraduate foray into mathematics has afforded my passion and profession! Finally, I would like to thank my friends, in whose company the triumphs and tribulations of this period were marked.

The content of this thesis is original work of which I am the sole author. The use of existing works is explicitly and duly acknowledged in the text.

Benjamin J. Wilson
São Paulo, October 11, 2007.

Contents

CHAPTER 1

Recapitulation

Section 1 presents the preliminary results on exponential-polynomial modules and on truncated current Lie algebras that are necessary to describe the results of the thesis in Sections 2 – 4. The material of Section 1 is principally derived from Billig and Zhao [**3**], Chari [**6**], and Chari and Pressley [**9**].

1. Exponential-Polynomial Modules and Truncated Current Lie Algebras

The loop-module realisation motivates a study of an exponential-polynomial module via an irreducible highest-weight representation of a truncated current Lie algebra. In this section, the exponential-polynomial modules and their corresponding truncated current Lie algebras are constructed, and the realisation is described. Let $\Bbbk$ denote a field of characteristic zero. Denote by $\mathfrak{g} = \mathrm{span}_{\Bbbk}\{\,\mathrm{e}, \mathrm{h}, \mathrm{f}\,\}$ the three-dimensional Lie algebra $\mathrm{sl}(2)$ with the commutation relations

$$[\,\mathrm{e}, \mathrm{f}\,] = \mathrm{h}, \qquad [\,\mathrm{h}, \mathrm{e}\,] = 2\mathrm{e}, \qquad [\,\mathrm{h}, \mathrm{f}\,] = -2\mathrm{f},$$

and triangular decomposition

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-, \qquad \mathrm{e} \in \mathfrak{g}_+, \quad \mathrm{h} \in \mathfrak{h}, \quad \mathrm{f} \in \mathfrak{g}_-. \tag{1.1}$$

For any Lie algebra $\mathfrak{a}$ over $\Bbbk$, denote by

$$\hat{\mathfrak{a}} = \mathfrak{a} \otimes \Bbbk[\mathrm{t}, \mathrm{t}^{-1}]$$

the $\mathbb{Z}$-graded *loop algebra* associated to $\mathfrak{a}$, with the Lie bracket

$$[\,x \otimes \mathrm{t}^i, y \otimes \mathrm{t}^j\,] = [\,x, y\,] \otimes \mathrm{t}^{i+j}, \qquad x, y \in \mathfrak{a}, \quad i, j \in \mathbb{Z}.$$

An $\hat{\mathfrak{a}}$-module M is $\mathbb{Z}$-graded if $M = \bigoplus_{n \in \mathbb{Z}} M_n$ and

$$\mathfrak{a} \otimes \mathrm{t}^m \cdot M_n \subset M_{m+n}, \qquad m, n \in \mathbb{Z}.$$

The decomposition (1.1) defines a decomposition of $\hat{\mathfrak{g}}$

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{g}}_-$$

as a direct sum of $\mathbb{Z}$-graded subalgebras. We consider $\hat{\mathfrak{g}}$-modules M that are $\mathfrak{h}$-diagonalisable, i.e.

$$M = \bigoplus_{\chi \in \mathfrak{h}^*} M^\chi \quad \text{where} \quad h|_{M^\chi} = \chi(h), \quad h \in \mathfrak{h}.$$

A $\mathbb{Z}$-graded $\hat{\mathfrak{g}}$-module M may be decomposed

$$M = \bigoplus_{(\chi,n) \in \mathrm{s}(M)} M_n^\chi, \qquad M_n^\chi = M^\chi \cap M_n,$$

as a direct sum of *homogeneous components* M_n^χ, where

$$\mathrm{s}(M) = \{\, (\chi, n) \in \mathfrak{h}^* \times \mathbb{Z} \mid M_n^\chi \neq 0 \,\}.$$

The functional $\alpha \in \mathfrak{h}^*$ given by $\alpha(\mathrm{h}) = 2$ is the positive root of $\mathfrak{g}$. The *category* $\tilde{\mathcal{O}}$, introduced by Chari [**6**], consists of those $\mathbb{Z}$-graded $\hat{\mathfrak{g}}$-modules M such that

$$\mathrm{s}(M) \subset \bigcup_{\lambda \in A} (\lambda - \mathbb{Z}_+\alpha) \times \mathbb{Z}$$

for some finite subset $A = A_M \subset \mathfrak{h}^*$. The morphisms of the category are the homomorphisms of $\mathbb{Z}$-graded $\hat{\mathfrak{g}}$-modules. For any Lie algebra $\mathfrak{a}$, a map $\theta : M \to N$ is a homomorphism of $\mathbb{Z}$-graded $\hat{\mathfrak{a}}$-modules M and N if θ is a $\hat{\mathfrak{a}}$-module homomorphism and if there exists $k \in \mathbb{Z}$ such that $\theta(M_n) \subset N_{n+k}$, for all $n \in \mathbb{Z}$.

1.1. Exponential-polynomial modules. Denote by $\mathcal{F}$ the vector space of functions $\varphi : \mathbb{Z} \to \Bbbk$. For any $\varphi \in \mathcal{F}$, the rule

$$\tilde{\varphi} : \mathrm{h} \otimes \mathrm{t}^m \mapsto \varphi(m)\mathrm{t}^m, \qquad m \in \mathbb{Z}, \tag{1.2}$$

defines a homomorphism $\tilde{\varphi} : \mathcal{U}(\hat{\mathfrak{h}}) \to \Bbbk[\mathrm{t}, \mathrm{t}^{-1}]$ of $\mathbb{Z}$-graded associative algebras, where the grading of $\mathcal{U}(\hat{\mathfrak{h}})$ is defined by that of $\hat{\mathfrak{h}}$. Write $\mathbf{H}(\varphi)$ for $\mathrm{im}\,\tilde{\varphi}$ considered as a $\mathbb{Z}$-graded $\hat{\mathfrak{h}}$-module via $\tilde{\varphi}$. Let

$$\mathcal{F}' = \{\, \varphi \in \mathcal{F} \mid \mathrm{im}\,\tilde{\varphi} = \Bbbk[\mathrm{t}^r, \mathrm{t}^{-r}] \ \text{ for some } \ r > 0 \,\}.$$

For $\varphi \in \mathcal{F}'$, write $\deg \varphi = r$ for the *degree* of φ, where $\mathrm{im}\,\tilde{\varphi} = \Bbbk[\mathrm{t}^r, \mathrm{t}^{-r}]$. The set $\mathcal{F}'$ consists of those functions whose support is not wholly contained in $\mathbb{N}$ or $-\mathbb{N}$. The module $\mathbf{H}(\varphi)$ is irreducible and of dimension greater than one precisely when $\varphi \in \mathcal{F}'$. For $\varphi \in \mathcal{F}$, consider $\mathbf{H}(\varphi)$ as a $(\hat{\mathfrak{h}} \oplus \hat{\mathfrak{g}}_+)$-module via $\hat{\mathfrak{g}}_+ \cdot \mathbf{H}(\varphi) = 0$. Whenever $\mathbf{H}(\varphi)$ is irreducible, the induced $\hat{\mathfrak{g}}$-module

$$\mathrm{Ind}_{\hat{\mathfrak{h}} \oplus \hat{\mathfrak{g}}_+}^{\hat{\mathfrak{g}}} \mathbf{H}(\varphi) \tag{1.3}$$

has a unique irreducible $\mathbb{Z}$-graded quotient, which we denote by $\mathbf{N}(\varphi)$. It is shown in [6] that, over an algebraically closed field, any irreducible object of the category $\tilde{\mathcal{O}}$ is of the form $\mathbf{N}(\varphi)$. For any $\lambda \in \Bbbk^\times$, define the exponential function

$$\textsc{exp}(\lambda) : \mathbb{Z} \to \Bbbk, \qquad \textsc{exp}(\lambda)(m) = \lambda^m, \quad m \in \mathbb{Z}.$$

A function $\varphi \in \mathcal{F}$ is *exponential polynomial* if it can be written as a finite sum of products of polynomial and exponential functions, i.e.

$$\sum_{\lambda \in \Bbbk^\times} \varphi_\lambda \, \textsc{exp}(\lambda), \tag{1.4}$$

for some polynomial functions $\varphi_\lambda \in \mathcal{F}$ and distinct scalars $\lambda \in \Bbbk^\times$. Write

$$\mathcal{E} = \{ \varphi \in \mathcal{F} \mid \varphi \text{ is exponential polynomial} \}.$$

Then $\mathcal{E} \setminus \{0\} \subset \mathcal{F}' \subset \mathcal{F}$. The exponential-polynomial functions are those whose successive values solve a homogeneous linear recurrence relation with constant coefficients.

A module $\mathbf{N}(\varphi)$ is *exponential polynomial* if $\varphi \in \mathcal{E}$. It is shown in [3] that the homogeneous components of an exponential-polynomial module are finite dimensional. Conversely, for $\varphi \in \mathcal{F}'$, the module $\mathbf{N}(\varphi)$ has finite-dimensional homogeneous components only if $\varphi \in \mathcal{E}$ (cf. [12]). In particular, if $\Bbbk$ is algebraically closed, then the exponential-polynomial modules $\{ \mathbf{N}(\varphi) \mid \varphi \in \mathcal{E} \}$ are precisely those irreducible objects of the category $\tilde{\mathcal{O}}$ for which all homogeneous components are finite dimensional.

1.2. Loop modules and truncated current Lie algebras. For $\varphi \in \mathcal{F}$, let $\Bbbk \mathrm{v}_\varphi$ be the one-dimensional $\hat{\mathfrak{h}}$-module defined by

$$\mathrm{h} \otimes \mathrm{t}^m \cdot \mathrm{v}_\varphi = \varphi(m)\mathrm{v}_\varphi, \qquad m \in \mathbb{Z}.$$

Let $\hat{\mathfrak{g}}_+ \cdot \mathrm{v}_\varphi = 0$, and denote by

$$\mathsf{V}(\varphi) = \mathrm{Ind}_{\hat{\mathfrak{h}} \oplus \hat{\mathfrak{g}}_+}^{\hat{\mathfrak{g}}} \Bbbk \mathrm{v}_\varphi \tag{1.5}$$

the induced $\hat{\mathfrak{g}}$-module. This module has a unique irreducible quotient $\mathsf{L}(\varphi)$. The modules $\mathsf{V}(\varphi)$ and $\mathsf{L}(\varphi)$ are not $\mathbb{Z}$-graded. Suppose that $\varphi \in \mathcal{E}$ is an exponential-polynomial function, and write $\mathrm{c}_\varphi \in \Bbbk[\mathrm{t}]$ for the characteristic polynomial[1] of the minimal-order linear homogeneous recurrence relation that is solved by the values of φ. It can be shown that the ideal

$$\mathfrak{g} \otimes \mathrm{c}_\varphi \Bbbk[\mathrm{t}, \mathrm{t}^{-1}] \subset \hat{\mathfrak{g}}$$

[1]For example, if $\varphi = \textsc{exp}(\lambda)$, then $\mathrm{c}_\varphi = (\mathrm{t} - \lambda)$.

acts trivially on the $\hat{\mathfrak{g}}$-module $\mathsf{L}(\varphi)$. Hence $\mathsf{L}(\varphi)$ is a module for the "truncation"

$$\mathfrak{g}(\varphi) := \hat{\mathfrak{g}}/\mathfrak{g} \otimes c_\varphi \Bbbk[t, t^{-1}] \cong \mathfrak{g} \otimes \Bbbk[t, t^{-1}]/c_\varphi \Bbbk[t, t^{-1}] \tag{1.6}$$

of the loop algebra $\hat{\mathfrak{g}}$. A Lie algebra (1.6) is called a *truncated current Lie algebra.*

Let M be a $\hat{\mathfrak{g}}$-module. The vector space $\widehat{M} = M \bigotimes_{\Bbbk} \Bbbk[t, t^{-1}]$ is a $\mathbb{Z}$-graded $\hat{\mathfrak{g}}$-module (called a *loop module*) via

$$x \otimes a \cdot u \otimes b = (x \otimes a \cdot u) \otimes ab, \qquad x \in \mathfrak{g}, \quad u \in M, \quad a, b \in \Bbbk[t, t^{-1}].$$

The $\mathbb{Z}$-grading is defined by degree in the indeterminate $t \in \Bbbk[t, t^{-1}]$. The loop-module realisation, due to Chari and Pressley [**9**], relates the exponential-polynomial modules to the irreducible highest-weight representations of a truncated current Lie algebra. Chari and Pressley show that if $\varphi \in \mathcal{E}$ is non-zero and $r = \deg \varphi$, then the $\mathbb{Z}$-graded $\hat{\mathfrak{g}}$-module $\widehat{\mathsf{L}(\varphi)}$ has precisely r irreducible constituents, all of which are isomorphic to the exponential-polynomial module $\mathbf{N}(\varphi)$. Moreover, the weight spaces of $\mathbf{N}(\varphi)$ are described in terms of the semi-invariants of an action of the cyclic group $\mathbb{Z}_r$ on $\mathsf{L}(\varphi)$. Thus the exponential-polynomial modules, and in particular their weight-space multiplicities, may be studied via highest-weight representations of truncated current Lie algebras.

The Chinese Remainder Theorem implies that if φ is written in the form (1.4), then

$$\mathfrak{g}(\varphi) \cong \bigoplus_{\lambda \in \Bbbk^\times} \mathfrak{g}(\varphi_\lambda \mathrm{EXP}(\lambda))$$

is an isomorphism of Lie algebras. Remarkably, if $\Bbbk$ is algebraically closed, then it follows also that

$$\mathsf{L}(\varphi) \cong \bigotimes_{\lambda \in \Bbbk^\times} \mathsf{L}(\varphi_\lambda \mathrm{EXP}(\lambda))$$

is an isomorphism of $\hat{\mathfrak{g}}$-modules (cf. Chapter 5). Therefore, in order to study highest-weight representations of truncated current Lie algebras, it is sufficient to consider only truncated current Lie algebras of the form $\mathfrak{g}(\varphi)$, where $\varphi = a\mathrm{EXP}(\lambda)$ and $a \in \mathcal{F}$ is a polynomial function. In any such case,

$$\mathfrak{g}(\varphi) \cong \mathfrak{g} \otimes \Bbbk[t]/t^{N+1}\Bbbk[t].$$

where $N = \deg a$.

2. Imaginary Highest-Weight Representation Theory

Affine Lie algebras admit non-classical highest-weight theories through alternative partitions of the root system. Although significant inroads have been made, much of the classical machinery is inapplicable in this broader context, and some fundamental questions remain unanswered. In particular, the structure of the reducible objects in non-classical theories has not yet been fully understood. This question is addressed in this thesis for affine sl(2), which has a unique non-classical highest-weight theory, termed *imaginary*. The reducible Verma modules in the imaginary theory possess an infinite descending series, with all factors isomorphic to a certain canonically associated module, the structure of which depends upon the highest weight. If the highest weight is non-zero, then this factor module is irreducible, and conversely. Chapter 2 examines the degeneracy of the factor module of highest-weight zero. The intricate structure of this module is understood via a realisation in terms of the symmetric functions. The realisation permits the description of a family of singular vectors, and the classification of the irreducible subquotients as a certain subclass of the exponential-polynomial modules.

Let $\mathbb{k}$ denote a field of characteristic zero, and let $\tilde{\mathfrak{g}}$ denote an affine Kac-Moody Lie algebra over $\mathbb{k}$. Write $\tilde{\mathfrak{h}} \subset \tilde{\mathfrak{g}}$ for the Cartan subalgebra and $\Delta \subset \tilde{\mathfrak{h}}^*$ for the root system. Denote by $\tilde{\mathfrak{g}}^\phi$ the root space associated to any root $\phi \in \Delta$. So

$$\tilde{\mathfrak{g}} = (\textstyle\bigoplus_{\phi \in \Delta} \tilde{\mathfrak{g}}^\phi) \oplus \tilde{\mathfrak{h}}, \qquad \operatorname{ad} h|_{\tilde{\mathfrak{g}}^\phi} = \phi(h), \quad h \in \tilde{\mathfrak{h}}, \quad \phi \in \Delta.$$

A $\tilde{\mathfrak{g}}$-module V is called *weight* if the action of $\tilde{\mathfrak{h}}$ upon V is diagonalisable. That is,

$$V = \textstyle\bigoplus_{\lambda \in \tilde{\mathfrak{h}}^*} V_\lambda, \qquad h|_{V_\lambda} = \lambda(h), \qquad h \in \tilde{\mathfrak{h}}, \quad \lambda \in \tilde{\mathfrak{h}}^*.$$

2.1. Partitions and highest-weight theories. The notion of a highest-weight module for $\tilde{\mathfrak{g}}$ depends upon the partition of the root system. A subset $\mathbf{P} \subset \Delta$ is called a *partition* of the root system if both

i. $\mathbf{P}$ is closed under root space addition, i.e. if $\phi, \psi \in \mathbf{P}$ and $\phi + \psi \in \Delta$, then $\phi + \psi \in \mathbf{P}$;
ii. $\mathbf{P} \cap -\mathbf{P} = \emptyset$ and $\mathbf{P} \cup -\mathbf{P} = \Delta$.

If $\Delta_+(\pi)$ denotes the set of positive roots with respect to some basis $\pi \subset \Delta$ of the root system, then $\mathbf{P} = \Delta_+(\pi)$ is an example of a partition. A partition $\mathbf{P}$ defines a decomposition of the Lie algebra $\tilde{\mathfrak{g}}$ as a direct sum of subalgebras

$$\tilde{\mathfrak{g}} = \mathfrak{N}_- \oplus \tilde{\mathfrak{h}} \oplus \mathfrak{N}_+, \quad \text{where} \quad \mathfrak{N}_+ = \oplus_{\phi \in \mathbf{P}} \tilde{\mathfrak{g}}^{\phi}, \quad \mathfrak{N}_- = \oplus_{\phi \in \mathbf{P}} \tilde{\mathfrak{g}}^{-\phi}.$$

A weight $\tilde{\mathfrak{g}}$-module V is of *highest-weight* $\lambda \in \tilde{\mathfrak{h}}^*$ with respect to the partition $\mathbf{P}$ if there exists $v \in V_\lambda$ such that

$$\mathcal{U}(\tilde{\mathfrak{g}}) \cdot v = V, \quad \text{and} \quad \mathfrak{N}_+ \cdot v = 0.$$

Thus the choice of partition $\mathbf{P}$ defines a theory of highest-weight modules. A highest-weight theory defined by the set of all positive roots $\mathbf{P} = \Delta_+(\pi)$ with respect to some basis π of the root system is called *classical*. Two partitions are *equivalent* if they are conjugate under the action of $\mathrm{W} \times \{\pm 1\}$, where W denotes the Weyl group associated to the root system Δ. Equivalent partitions define similar highest-weight theories. All partitions of the root system of a finite-dimensional semisimple complex Lie algebra are classical, and hence equivalent. In contrast, it has been shown by Jakobsen and Kac [**19**], and by Futorny [**13**], that there are finitely many, but never one, inequivalent partitions of the root system of an affine Lie algebra. Thus any affine Lie algebra has multiple distinct highest-weight theories.

2.2. Imaginary highest-weight theory for affine sl(2). Up to equivalence, there is precisely one non-classical partition, the *imaginary partition*, of the root system of affine sl(2). The associated *imaginary highest-weight theory* has been pioneered by Futorny in [**14**], [**15**]. These works provide an almost complete understanding of the universal objects of the theory, the *imaginary Verma modules*. Let $\tilde{\mathfrak{g}}$ denote the affinisation of $\mathfrak{g}$:

$$\tilde{\mathfrak{g}} = \hat{\mathfrak{g}} \oplus \Bbbk \mathrm{c} \oplus \Bbbk \mathrm{d},$$

with Lie bracket relations:

$$\begin{aligned} [\, x \otimes \mathrm{t}^k, y \otimes \mathrm{t}^l \,] &= [\, x, y \,] \otimes \mathrm{t}^{k+l} + k\delta_{k,-l}(\, x \,|\, y \,)\mathrm{c}, \qquad [\, \mathrm{c}, \tilde{\mathfrak{g}} \,] = 0, \\ [\, \mathrm{d}, x \otimes \mathrm{t}^k \,] &= kx \otimes \mathrm{t}^k, \qquad x, y \in \mathfrak{g}, \quad k, l \in \mathbb{Z}, \end{aligned}$$

where $(\,\cdot\,|\,\cdot\,)$ denotes the Killing form of $\mathfrak{g}$. The Cartan subalgebra $\tilde{\mathfrak{h}} \subset \tilde{\mathfrak{g}}$ is given by

$$\tilde{\mathfrak{h}} = \mathrm{span}\,\{\, \mathrm{h} \otimes \mathrm{t}^0, \mathrm{c}, \mathrm{d} \,\}\,.$$

Let $\alpha, \delta \in \tilde{\mathfrak{h}}^*$ be such that

$$\begin{aligned} \alpha(\mathrm{h} \otimes \mathrm{t}^0) &= 2, \qquad \alpha(\mathrm{c}) = 0, \qquad \alpha(\mathrm{d}) = 0, \\ \delta(\mathrm{h} \otimes \mathrm{t}^0) &= 0, \qquad \delta(\mathrm{c}) = 0, \qquad \delta(\mathrm{d}) = 1. \end{aligned}$$

Then $\Delta = \{\pm\alpha + i\delta \mid i \in \mathbb{Z}\} \cup \{i\delta \mid i \in \mathbb{Z},\ i \neq 0\}$. The imaginary partition $\mathbf{P} \subset \Delta$ is given by

$$\mathbf{P} = \{\alpha + i\delta \mid i \in \mathbb{Z}\} \cup \{i\delta \mid i \in \mathbb{Z},\ i > 0\}.$$

Thus the associated subalgebras $\mathfrak{N}_+, \mathfrak{N}_-$ of $\tilde{\mathfrak{g}}$ are given by

$$\mathfrak{N}_+ = \hat{\mathfrak{g}}_+ \oplus [\textstyle\bigoplus_{j>0} \mathfrak{h} \otimes \mathrm{t}^j], \qquad \mathfrak{N}_- = \hat{\mathfrak{g}}_- \oplus [\textstyle\bigoplus_{j<0} \mathfrak{h} \otimes \mathrm{t}^j].$$

Let $\lambda \in \tilde{\mathfrak{h}}^*$, and consider the one-dimensional vector space $\Bbbk \mathrm{v}_\lambda$ as an $(\tilde{\mathfrak{h}} \oplus \mathfrak{N}_+)$-module via

$$\mathfrak{N}_+ \cdot \mathrm{v}_\lambda = 0, \qquad h \cdot \mathrm{v}_\lambda = \lambda(h)\mathrm{v}_\lambda, \quad h \in \tilde{\mathfrak{h}}.$$

Let $\mathbf{V}(\lambda)$ denote the induced $\tilde{\mathfrak{g}}$-module:

$$\mathbf{V}(\lambda) = \mathrm{Ind}_{\tilde{\mathfrak{h}} \oplus \mathfrak{N}_+}^{\tilde{\mathfrak{g}}} \Bbbk \mathrm{v}_\lambda.$$

The $\tilde{\mathfrak{g}}$-module $\mathbf{V}(\lambda)$ is the universal highest-weight $\tilde{\mathfrak{g}}$-module of highest-weight λ, and so is called an *imaginary Verma module.*

Theorem. [14] Let $\lambda \in \tilde{\mathfrak{h}}^*$. Then:

i. If $\lambda(\mathrm{c}) \neq 0$, then $\mathbf{V}(\lambda)$ is irreducible.
ii. Suppose that $\lambda(\mathrm{c}) = 0$. Then $\mathbf{V}(\lambda)$ has an infinite descending series of submodules

$$\mathbf{V}(\lambda) = V^0 \supset V^1 \supset V^2 \supset \cdots$$

such that any factor V^i/V^{i+1}, $i \geqslant 0$, is isomorphic to the quotient of $\tilde{\mathfrak{g}}$-modules

$$\mathbf{M}(\lambda) = \mathbf{V}(\lambda) \;/\; \langle \mathrm{h} \otimes \mathrm{t}^j \cdot \mathrm{v}_\lambda \mid j < 0 \rangle,$$

up to a shift in the δ weight-decomposition. Moreover, if $\lambda(\mathrm{h} \otimes \mathrm{t}^0) \neq 0$, then $\mathbf{M}(\lambda)$ is irreducible.

The value $\lambda(\mathrm{d})$ of the action of d on the generating vector is immaterial to the structure of the imaginary Verma module $\mathbf{V}(\lambda)$. Hence the theorem above provides an almost complete description of the structure of the imaginary Verma modules for $\tilde{\mathfrak{g}}$, lacking only a statement about the imaginary Verma module of highest-weight zero. Part (ii) of the Theorem motivates a study of $\mathbf{V}(0)$ through its canonically associated and infinitely occurrent quotient $\mathbf{M}(0)$. Chapter 2 is an extensive study of the degeneracy of $\mathbf{M}(0)$.

The central element c acts trivially on the module $\mathbf{M}(0)$. Hence $\mathbf{M}(0)$ may be studied as a $\mathbb{Z}$-graded module for the loop algebra $\hat{\mathfrak{g}}$. The $\hat{\mathfrak{g}}$-module $\mathbf{M}(0)$ may be defined by

$$\mathbf{M}(0) = \mathrm{Ind}_{\hat{\mathfrak{h}} \oplus \hat{\mathfrak{g}}_+}^{\hat{\mathfrak{g}}} \Bbbk \mathrm{u}_0, \tag{2.1}$$

where $\Bbbk u_0$ is the trivial one-dimensional $(\hat{\mathfrak{h}} \oplus \hat{\mathfrak{g}}_+)$-module. The module $\mathbf{M}(0)$ is $\mathbb{Z}$-graded by construction, and

$$\mathrm{s}(\mathbf{M}(0)) \subset (-\mathbb{Z}_+\alpha) \times \mathbb{Z}.$$

Thus $\mathbf{M}(0) \in \tilde{\mathcal{O}}$. In fact, if $\lambda \in \tilde{\mathfrak{h}}^*$ is such that $\lambda(\mathrm{c}) = 0$, and $\varphi \in \mathcal{F}$ is given by

$$\varphi(m) = \lambda(\mathrm{h} \otimes \mathrm{t}^0)\delta_{m,0}, \qquad m \in \mathbb{Z},$$

then the induced module defined by (1.3) and the canonical quotient $\mathbf{M}(\lambda)$ are isomorphic as $\mathbb{Z}$-graded $\hat{\mathfrak{g}}$-modules.

Throughout the remainder of this section, all modules are $\mathbb{Z}$-graded.

2.3. Symmetric functions and singular vectors. It is apparent from the construction (2.1) that

$$\mathbf{M}(0) = \bigoplus_{n \in \mathbb{Z}_+} \mathbf{M}^{(n)} \quad \text{where} \quad \mathbf{M}^{(n)} = \mathbf{M}(0)^{-n\alpha}, \quad n \geqslant 0,$$

is a decomposition of $\mathbf{M}(0)$ as a direct sum of $\mathbb{Z}$-graded $\hat{\mathfrak{h}}$-modules. The $\hat{\mathfrak{h}}$-modules $\mathbf{M}^{(n)}$ have remarkable realisations in terms of the symmetric functions, and play a large part in the structural description of the $\hat{\mathfrak{g}}$-module $\mathbf{M}(0)$. For any positive integer n, let

$$\mathbf{A}_n = \Bbbk[\mathrm{z}_1, \mathrm{z}_1^{-1}, \ldots, \mathrm{z}_n, \mathrm{z}_n^{-1}]^{\mathrm{Sym}(n)}$$

denote the $\Bbbk$-algebra of symmetric Laurent polynomials in the n indeterminates $\mathrm{z}_1, \ldots, \mathrm{z}_n$. The module $\mathbf{M}^{(n)}$ may be considered as a graded $\mathbf{A}_n$-module in such a way that the action of $\hat{\mathfrak{h}}$ upon $\mathbf{M}^{(n)}$ factors through an epimorphism of algebras $\mathcal{U}(\hat{\mathfrak{h}}) \twoheadrightarrow \mathbf{A}_n$. Therefore, it is sufficient to consider the module $\mathbf{M}^{(n)}$ as a graded $\mathbf{A}_n$-module. In fact, it may be shown that as a graded $\mathbf{A}_n$-module, $\mathbf{M}^{(n)}$ is isomorphic to the graded regular module for $\mathbf{A}_n$. Thus, in particular, the $\hat{\mathfrak{h}}$-module structure of $\mathbf{M}^{(n)}$ may be understood via the graded algebra $\mathbf{A}_n$. The classification of the irreducible quotients of the algebra $\mathbf{A}_n$, and the construction of singular vectors of $\mathbf{M}(0)$, together permit the classification of the irreducible subquotients of the $\hat{\mathfrak{g}}$-module $\mathbf{M}(0)$.

An element $v \in \mathbf{M}(0)$ is *singular* if $\hat{\mathfrak{g}}_+ \cdot v = 0$. Non-trivial singular vectors generate proper submodules. For any positive integer n, let

$$\Omega_n = \prod_{1 \leqslant i < j \leqslant n} (\mathrm{z}_i - \mathrm{z}_j)^2 \qquad \in \mathbf{A}_n.$$

In the theory of symmetric functions, the function Ω_n appears both as the square of the Vandermonde determinant, and as the discriminant function for degree-n polynomials. Let $\mathrm{sgn}(\sigma) = \pm 1$ denote the sign of a permutation σ.

Theorem. Let n be a positive integer. Then

i. All elements of the $\hat{\mathfrak{h}}$-submodule $\Omega_n \cdot \mathbf{M}^{(n)}$ are singular.
ii. The space $\Omega_n \cdot \mathbf{M}^{(n)}$ is spanned by singular vectors of the form

$$\mathbf{w}(\chi) = \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sgn}(\sigma) \prod_{1 \leqslant i \leqslant n} \mathrm{f} \otimes \mathrm{t}^{\chi_i + \sigma(i)} \mathrm{u}_0,$$

where $\chi \in \mathbb{Z}^n$ and u_0 denotes the generator of $\mathbf{M}(0)$.

Conjecture. Let n be a positive integer, and suppose that $v \in \mathbf{M}^{(n)}$ is singular. Then $v \in \Omega_n \cdot \mathbf{M}^{(n)}$.

2.4. Structure of the canonical quotient $\mathbf{M}(0)$. A $\hat{\mathfrak{g}}$-module Q is a *subquotient* of a $\hat{\mathfrak{g}}$-module M if there exists a chain of $\hat{\mathfrak{g}}$-modules

$$M \supset N \supset P,$$

such that $N/P \cong Q$. The preceding results may be employed to classify the irreducible subquotients of the $\hat{\mathfrak{g}}$-module $\mathbf{M}(0)$. Let

$$\mathcal{E}^{(-)} = \{ \varphi \in \mathcal{E} \mid \varphi_\lambda \in -2\mathbb{Z}_+ \ \text{ for all } \ \lambda \in \mathbb{k}^\times \},$$

in the notation of (1.4).

Theorem. For any $\varphi \in \mathcal{E}^{(-)}$, the $\hat{\mathfrak{g}}$-module $\mathbf{N}(\varphi)$ is an irreducible subquotient of $\mathbf{M}(0)$. Moreover, if $\mathbb{k}$ is algebraically closed, then any irreducible subquotient of $\mathbf{M}(0)$ is of the form $\mathbf{N}(\varphi)$ for some $\varphi \in \mathcal{E}^{(-)}$.

3. Highest-Weight Theory for Truncated Current Lie Algebras

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{k}$ of characteristic zero, and a fix positive integer N. The Lie algebra

$$\check{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{k}} \mathbb{k}[\mathrm{t}]/\mathrm{t}^{\mathrm{N}+1}\mathbb{k}[\mathrm{t}] \tag{3.1}$$

over $\mathbb{k}$, with the Lie bracket

$$[\, x \otimes \mathrm{t}^i, y \otimes \mathrm{t}^j \,] = [\, x, y \,] \otimes \mathrm{t}^{i+j}, \qquad x, y \in \mathfrak{g}, \quad i, j \geqslant 0, \tag{3.2}$$

is a truncated current Lie algebra. In Chapter 4, a highest-weight theory for $\breve{\mathfrak{g}}$ is developed when the underlying Lie algebra $\mathfrak{g}$ possesses a triangular decomposition. The principal result is the reducibility criterion for the Verma modules of $\breve{\mathfrak{g}}$ for a wide class of Lie algebras $\mathfrak{g}$, including the symmetrisable Kac-Moody Lie algebras, the Heisenberg algebra, and the Virasoro algebra. This is achieved through a study of the Shapovalov form. In the particular case of $\mathfrak{g} = \mathrm{sl}(2)$, an expression for the formal character of the irreducible module $\mathsf{L}(\varphi)$ may be deduced from the reducibility criterion.

3.1. Truncated current Lie algebras. There have been various studies of truncated current Lie algebras and their representation theory in the particular case where $\mathfrak{g}$ is a semisimple finite-dimensional Lie algebra. There are applications in the theory of soliton equations [**5**] [**23**] [**26**], and in this context the Lie algebra $\breve{\mathfrak{g}}$ is called a *polynomial Lie algebra*. The paper [**4**] describes a construction of $\breve{\mathfrak{g}}$ via the Wigner contraction. Takiff considered this case with $\mathrm{N} = 1$ in [**31**], and that work was extended in [**28**], [**16**], [**17**] without the restriction on N. As such, when $\mathfrak{g}$ is a semisimple finite-dimensional Lie algebra, the Lie algebra $\breve{\mathfrak{g}}$ is often called a *generalised Takiff algebra*. The category of modules for a truncated current Lie algebra is examined in [**22**].

3.2. Highest-weight modules. We assume that the Lie algebra $\mathfrak{g}$ is equipped with a *triangular decomposition*

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+, \qquad \mathfrak{g}_\pm = \textstyle\bigoplus_{\alpha\in\Delta_+} \mathfrak{g}^{\pm\alpha}, \qquad \Delta_+ \subset \mathfrak{h}^*. \tag{3.3}$$

The fundamental definitions and results concerning Lie algebras with triangular decompositions and their highest-weight representation theory are the subject of Chapter 3 (here, $\mathfrak{h}_0 = \mathfrak{h}$). The exposition follows that of Moody and Pianzola [**24**], modified in accordance with our definitions. The triangular decomposition (3.3) of $\mathfrak{g}$ naturally defines a triangular decomposition of $\breve{\mathfrak{g}}$,

$$\breve{\mathfrak{g}} = \breve{\mathfrak{g}}_- \oplus \breve{\mathfrak{h}} \oplus \breve{\mathfrak{g}}_+, \qquad \breve{\mathfrak{g}}_\pm = \textstyle\bigoplus_{\alpha\in\Delta_+} \breve{\mathfrak{g}}^{\pm\alpha},$$

where the subalgebra $\breve{\mathfrak{h}}$ and the subspaces $\breve{\mathfrak{g}}^\alpha$ are defined in the manner of (3.1), and $\mathfrak{h} \subset \breve{\mathfrak{h}}$ is the diagonal subalgebra. Hence a $\breve{\mathfrak{g}}$-module M is *weight* if the action of $\mathfrak{h}$ on M is diagonalisable. A weight $\breve{\mathfrak{g}}$-module is of *highest weight* if there exists a non-zero vector $v \in M$, and a functional $\Lambda \in \breve{\mathfrak{h}}^*$ such that

$$\breve{\mathfrak{g}}_+ \cdot v = 0; \qquad \mathcal{U}(\breve{\mathfrak{g}}) \cdot v = M; \qquad h \cdot v = \Lambda(h)v, \quad \text{for all } \ h \in \breve{\mathfrak{h}}.$$

The unique functional $\Lambda \in \breve{\mathfrak{h}}^*$ is the *highest weight* of the highest-weight module M. Notice that the support of a weight module is a subset of $\mathfrak{h}^*$, while a highest-weight is an element of $\check{\mathfrak{h}}^*$. A highest-weight $\Lambda \in \check{\mathfrak{h}}^*$ may be thought of as a tuple of functionals on $\mathfrak{h}$,

$$\Lambda = (\Lambda_0, \Lambda_1, \ldots, \Lambda_\mathrm{N}), \quad \text{where} \quad \langle \Lambda_i, h \rangle = \langle \Lambda, h \otimes \mathrm{t}^i \rangle, \quad h \in \mathfrak{h}, \quad i \geqslant 0. \tag{3.4}$$

All $\check{\mathfrak{g}}$-modules of highest weight $\Lambda \in \check{\mathfrak{h}}^*$ are homomorphic images of a certain universal $\check{\mathfrak{g}}$-module of highest weight Λ, denoted by $\mathfrak{V}(\Lambda)$. These universal modules $\mathfrak{V}(\Lambda)$ are the *Verma modules* of the highest-weight theory.

3.3. Reducibility of Verma modules. A single hypothesis suffices for the derivation of a criterion for the reducibility of a Verma module $\mathfrak{V}(\Lambda)$ for $\check{\mathfrak{g}}$ in terms of the functional $\Lambda \in \check{\mathfrak{h}}^*$. We assume that the triangular decomposition of $\mathfrak{g}$ is *non-degenerately paired*, i.e. that for each $\alpha \in \Delta_+$, a non-degenerate bilinear form

$$(\,\cdot\,|\,\cdot\,)_\alpha : \mathfrak{g}^\alpha \times \mathfrak{g}^{-\alpha} \to \mathbb{k},$$

and a non-zero element $\mathbf{h}(\alpha) \in \mathfrak{h}$ are given, such that

$$[x, y] = (\,x\,|\,y\,)_\alpha\, \mathbf{h}(\alpha),$$

for all $x \in \mathfrak{g}^\alpha$ and $y \in \mathfrak{g}^{-\alpha}$. The symmetrisable Kac-Moody Lie algebras, the Virasoro algebra, and the Heisenberg algebra all possess triangular decompositions that are non-degenerately paired. The reducibility criterion is the following.

Theorem. The Verma module $\mathfrak{V}(\Lambda)$ for $\check{\mathfrak{g}}$ is reducible if and only if

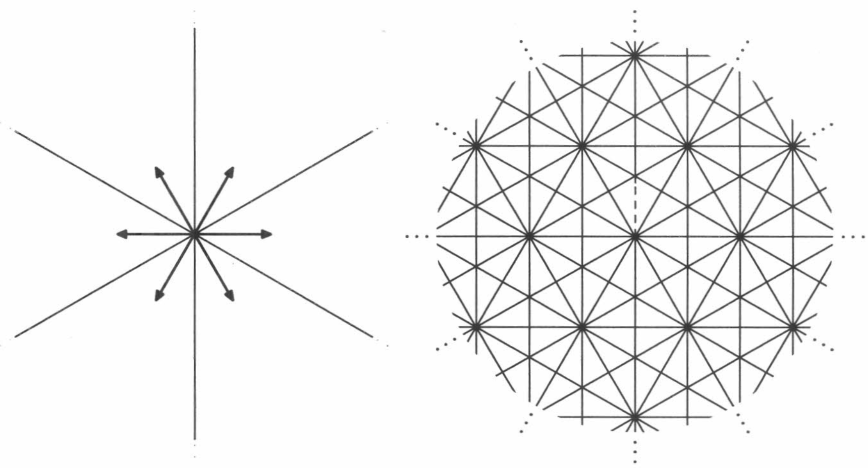
$$\langle \Lambda, \mathbf{h}(\alpha) \otimes \mathrm{t}^\mathrm{N} \rangle = 0,$$

for some positive root $\alpha \in \Delta_+$ of $\mathfrak{g}$.

Notice that the reducibility of $\mathfrak{V}(\Lambda)$ depends only upon $\Lambda_\mathrm{N} \in \mathfrak{h}^*$, the last component of the tuple (3.4).

3.4. Applications of the Theorem. The criterion described by the Theorem has many disguises, depending upon the underlying Lie algebra $\mathfrak{g}$.

Example 3.5. Let $\mathfrak{g} = \mathrm{sl}(3)$ be the Lie algebra of type A_2. The diagonal subalgebra $\mathfrak{h}$ is two-dimensional, and the root system $\Delta \subset \mathfrak{h}^*$ carries the geometry defined by the Killing form. A Verma module $\mathfrak{V}(\Lambda)$ for $\check{\mathfrak{g}}$ is reducible if and only if Λ_N is orthogonal to a root. This is precisely when Λ_N belongs to one of the three hyperplanes in $\mathfrak{h}^*$ illustrated in Figure 1(a). The arrows describe the root system.



(a) $\mathfrak{g}$ of type A_2 (b) $\mathfrak{g}$ of type $G_2^{(1)}$

Figure 1: Reducibility criterion for the Verma modules of $\breve{\mathfrak{g}}$, where $\mathfrak{g}$ is of type A_2 or $G_2^{(1)}$

Example 3.6. Let $\dot{\mathfrak{g}}$ denote the fourteen-dimensional simple Lie algebra of type G_2, and let

$$\mathfrak{g} = \dot{\mathfrak{g}} \otimes \Bbbk[s, s^{-1}] \oplus \Bbbk c \oplus \Bbbk d,$$

denote the affinisation of $\dot{\mathfrak{g}}$ with central extension c and degree derivation d. Then $\mathfrak{g}$ is the Kac-Moody Lie algebra of type $G_2^{(1)}$. The diagonal subalgebra

$$\mathfrak{h} = \dot{\mathfrak{h}} \oplus \Bbbk c \oplus \Bbbk d, \tag{3.7}$$

is obtained from the diagonal subalgebra $\dot{\mathfrak{h}}$ of $\dot{\mathfrak{g}}$. The Lie algebra $\mathfrak{g}$ has a triangular decomposition defined by a choice of simple roots. For any $\Gamma \in \mathfrak{h}^*$, denote by $\widetilde{\Gamma}$ the restriction of Γ to $\dot{\mathfrak{h}}$ defined by (3.7). A Verma module $\mathfrak{V}(\Lambda)$ for $\breve{\mathfrak{g}}$ is reducible precisely when $\widetilde{\Lambda_N}$ belongs to the infinite union of hyperplanes described in Figure 1(b), where the dashed line segment has length $|\langle \Lambda_N, c\rangle|$.

4. Characters of Exponential-Polynomial Modules

Let $\mathfrak{g}$ denote the Lie algebra sl(2) over an algebraically closed field $\Bbbk$ of characteristic zero, and adopt the notations of Section 1. An expression for the formal character of an exponential-polynomial module $\mathsf{L}(\varphi)$ is derived in Chapter 5. As described in Subsection 1.2, the loop-module realisation reduces this task to the study of the semi-invariants of a finite cyclic group acting on the irreducible highest-weight module $\mathsf{L}(\varphi)$ for the truncated current Lie algebra $\mathfrak{g}(\varphi)$ defined by (1.6). A certain generalisation of Molien's Theorem in the case of a cyclic group describes the multiplicities of these semi-invariants. The character of the exponential-polynomial module $\mathbf{N}(\varphi)$ is then expressed in terms of the character of $\mathsf{L}(\varphi)$, obtained in Chapter 4.

For any $\varphi \in \mathcal{E}$, write

$$\operatorname{char} \mathbf{N}(\varphi) = \sum_{k \geqslant 0} \sum_{n \in \mathbb{Z}} \dim \mathbf{N}(\varphi)_{k,n} \, \mathrm{X}^k \mathrm{Z}^n \qquad \in \mathbb{Z}_+[[\mathrm{X}, \mathrm{Z}, \mathrm{Z}^{-1}]],$$

for the formal character of $\mathbf{N}(\varphi)$, where

$$\mathbf{N}(\varphi)_{k,n} = \mathbf{N}(\varphi)_n^{(\varphi(0)-k)\alpha} \qquad k \geqslant 0, \quad n \in \mathbb{Z}.$$

For any positive integer r, define the function

$$\wp_r = \sum_{\zeta^r = 1} \mathrm{EXP}(\zeta) \qquad \in \mathcal{E},$$

where the sum is over all roots of unity ζ such that $\zeta^r = 1$. The function $\wp_r$ takes the constant value r on its support $r\mathbb{Z}$. If $\varphi \in \mathcal{E}$ is non-zero and $\deg \varphi = r$, then it can be shown that $r > 0$ and

$$\varphi = \wp_r \cdot \sum_i a_i \, \mathrm{EXP}(\lambda_i) \tag{4.1}$$

for some finite collection of polynomial functions $a_i \in \mathcal{F}$ and scalars $\lambda_i \in \Bbbk^\times$, such that if $(\lambda_i/\lambda_j)^r = 1$, then $i = j$. The formal character of the exponential-polynomial module $\mathbf{N}(\varphi)$ is described by the following theorem.

Theorem. Let $\varphi \in \mathcal{E}$ be non-zero, and write $r = \deg \varphi$. In the notation of (4.1),

$$\operatorname{char} \mathbf{N}(\varphi) = \frac{1}{r} \sum_{n \in \mathbb{Z}} \sum_{d \mid r} \mathrm{c}_d(n) \left(\mathrm{P}_\varphi(\mathrm{X}^d) \right)^{\frac{r}{d}} \mathrm{Z}^n, \tag{4.2}$$

where the inner sum is over the positive divisors d of r, the quantities $\mathrm{c}_d(n)$ are Ramanujan sums, and

$$\mathrm{P}_\varphi(\mathrm{X}) = \frac{\prod_{a_i \in \mathbb{Z}_+} (1 - \mathrm{X}^{a_i+1})}{(1 - \mathrm{X})^M},$$

where $M = \sum_i (\deg a_i + 1)$ and the product is over those indices i for which $a_i \in \mathbb{Z}_+$.

The *Ramanujan sum* $\mathrm{c}_d(n)$ is given by

$$\mathrm{c}_d(n) = \frac{\phi(d)\mu(d')}{\phi(d')}, \qquad d' = \frac{d}{\gcd(d, n)}, \tag{4.3}$$

where ϕ denotes Euler's totient function and μ denotes the Möbius function. The expression (4.2) is the Ramanujan-Fourier transform of $\operatorname{char}\mathbf{N}(\varphi)$.

It is apparent that $c_d(\cdot)$ is a function of period d, and thus it may be deduced from formula (4.2) that, for any $k \geqslant 0$, the multiplicity function

$$n \mapsto \dim \mathbf{N}(\varphi)_{k,n}, \qquad n \in \mathbb{Z},$$

has period $r = \deg \varphi$. Therefore the character of $\mathbf{N}(\varphi)$ is completely described by the array of weight-space multiplicities $[\dim \mathbf{N}(\varphi)_{k,n}]$ where $k \geqslant 0$ and $0 \leqslant n < r$. Examples of these arrays, such as those illustrated by Figures 2(a) – 2(d), may be computed in a straightforward manner using the formula (4.2). Columns are indexed left to right by n, where $0 \leqslant n < r$, while rows are indexed from top to bottom by $k \geqslant 0$.

Greenstein [**18**] (see also [**8**, Section 4.1]) has derived an explicit formula for the formal character of an integrable irreducible object of the category $\tilde{\mathcal{O}}$. These objects are precisely the exponential-polynomial modules $\mathbf{N}(\varphi)$ where φ is a linear combination of exponential functions with non-negative integral coefficients. Indeed, our result may alternatively be deduced by considering separately the case where $\mathbf{N}(\varphi)$ is integrable, employing the result of Greenstein, and the case where $\mathbf{N}(\varphi)$ is not integrable, using Molien's Theorem. Our approach, via a general study of finite cyclic-group actions, has the advantage of permitting a unified proof. Both approaches employ the explicit expression of the formal character of an irreducible highest-weight module for a truncated current Lie algebra described in Chapter 4.

1	0	0	0	0	0
1	1	1	1	1	1
3	2	3	2	3	2
4	3	3	4	3	3
3	2	3	2	3	2
1	1	1	1	1	1
1	0	0	0	0	0
0	0	0	0	0	0
	$\vdots$			$\vdots$	

(a) $\varphi = \wp_6$

1	0	0	0
2	2	2	2
10	8	10	8
30	30	30	30
86	80	84	80
198	198	198	198
434	424	434	424
858	858	858	858
	$\vdots$	$\vdots$	

(b) $\varphi = -\wp_4(\text{EXP}(\lambda) + \text{EXP}(\mu))$

1	0	0	0	0	0
1	1	1	1	1	1
4	3	4	3	4	3
10	9	9	10	9	9
22	20	22	20	22	20
42	42	42	42	42	42
80	75	78	76	78	75
132	132	132	132	132	132
217	212	217	212	217	212
335	333	333	335	333	333
	$\vdots$			$\vdots$	

(c) $\varphi = -\wp_6$

1	0
2	2
5	3
6	6
9	7
10	10
13	11
14	14
17	15
18	18
$\vdots$	$\vdots$

(d) $\varphi = \wp_2(\text{EXP}(\lambda) - \text{EXP}(\mu))$

Figure 2: Array of weight-space multiplicites of $\mathbf{N}(\varphi)$

CHAPTER 2

Imaginary Highest-Weight Representation Theory

Adopt the notation of Section 2 of Chapter 1. In particular, $\Bbbk$ denotes any field of characteristic zero, $\mathfrak{g}$ denotes the Lie algebra $\mathrm{sl}(2)$, and $\hat{\mathfrak{g}}$ denotes the $\mathbb{Z}$-graded loop algebra associated to $\mathfrak{g}$. In this chapter, all modules are $\mathbb{Z}$-graded, unless stated otherwise. For any $x \in \mathfrak{g}$ and $k \in \mathbb{Z}$, write $x(k) = x \otimes \mathrm{t}^k$.

1. The Canonical Quotient $\mathbf{M}(0)$

The following preliminary result provides a description of the action of $\hat{\mathfrak{g}}$ upon $\mathbf{M}(0)$. Here, and throughout, the use of a hat above a term in a sum or product indicates the omission of that term. The subalgebra $\hat{\mathfrak{g}}_-$ is abelian, and so $\mathcal{U}(\hat{\mathfrak{g}}_-)$ may be identified with the infinite-rank polynomial ring

$$\Bbbk[\mathrm{f}(j) \,|\, j \in \mathbb{Z}].$$

Proposition 1.1. The following hold:

i. The $\hat{\mathfrak{g}}$-module $\mathbf{M}(0)$ is generated by an element u_0 such that the action of $\mathcal{U}(\hat{\mathfrak{g}}_-)$ on u_0 is free, whilst the actions of $\mathcal{U}(\hat{\mathfrak{g}}_+)$ and $\mathcal{U}(\hat{\mathfrak{h}})$ are trivial.

ii. The $\hat{\mathfrak{g}}$-module $\mathbf{M}(0)$ has a basis

$$\bigcup_{n \geqslant 0} \{ \prod_{1 \leqslant i \leqslant n} \mathrm{f}(\gamma_i)\mathrm{u}_0 \mid \gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_n,\ \gamma \in \mathbb{Z}^n \} \tag{1.2}$$

iii. The action of $\hat{\mathfrak{g}}$ on $\mathbf{M}(0)$ is given by

$$\begin{aligned}
\mathrm{f}(k) \cdot \prod_{1 \leqslant i \leqslant n} \mathrm{f}(\gamma_i)\mathrm{u}_0 &= \mathrm{f}(k) \prod_{1 \leqslant i \leqslant n} \mathrm{f}(\gamma_i)\mathrm{u}_0, \\
\mathrm{h}(k) \cdot \prod_{1 \leqslant i \leqslant n} \mathrm{f}(\gamma_i)\mathrm{u}_0 &= -2 \sum_{1 \leqslant i \leqslant n} \mathrm{f}(\gamma_1) \cdots \widehat{\mathrm{f}(\gamma_i)} \cdots \mathrm{f}(\gamma_n)\mathrm{f}(\gamma_i + k)\,\mathrm{u}_0, \\
\mathrm{e}(k) \cdot \prod_{1 \leqslant i \leqslant n} \mathrm{f}(\gamma_i)\mathrm{u}_0 &= -2 \sum_{1 \leqslant i < j \leqslant n} \mathrm{f}(\gamma_1) \cdots \widehat{\mathrm{f}(\gamma_i)} \cdots \widehat{\mathrm{f}(\gamma_j)} \cdots \mathrm{f}(\gamma_n)\mathrm{f}(\gamma_i + \gamma_j + k)\,\mathrm{u}_0,
\end{aligned}$$

for all $\gamma \in \mathbb{Z}^n$, $n \geqslant 0$ and $k \in \mathbb{Z}$.

Proof. Part (i) is clear, and part (ii) follows from part (i). To prove part (iii), we firstly derive some commutation relations in $\mathcal{U}(\hat{\mathfrak{g}})$ before considering them in light of parts (i) and (ii). If $\mathcal{L}$ is any Lie algebra and $x \in \mathcal{L}$, then the adjoint map

$$\operatorname{ad} x : \mathcal{U}(\mathcal{L}) \to \mathcal{U}(\mathcal{L}), \quad \operatorname{ad} x : y \mapsto [x, y] = xy - yx, \quad y \in \mathcal{U}(\mathcal{L}),$$

is a derivation of the associative product of $\mathcal{U}(\mathcal{L})$. That is,

$$[x, \prod_{1 \leqslant i \leqslant n} y_i] = \sum_{1 \leqslant i \leqslant n} y_1 \cdots y_{i-1}[x, y_i] y_{i+1} \cdots y_n. \tag{1.3}$$

This formula yields immediately the commutation equation

$$[\mathrm{h}(k), \prod_{1 \leqslant i \leqslant n} \mathrm{f}(\gamma_i)] = -2 \sum_{1 \leqslant i \leqslant n} \mathrm{f}(\gamma_1) \cdots \widehat{\mathrm{f}(\gamma_i)} \cdots \mathrm{f}(\gamma_n) \mathrm{f}(\gamma_i + k)$$

for all $\gamma \in \mathbb{Z}^n$, $n \geqslant 0$ and $k \in \mathbb{Z}$. Using formula (1.3) and substituting the above commutation equation for $\mathrm{h}(k)$,

$$\begin{aligned}[\mathrm{e}(k), \prod_{1 \leqslant i \leqslant n} \mathrm{f}(\gamma_i)] &= -2 \sum_{1 \leqslant i < j \leqslant n} \mathrm{f}(\gamma_1) \cdots \widehat{\mathrm{f}(\gamma_i)} \cdots \widehat{\mathrm{f}(\gamma_j)} \cdots \mathrm{f}(\gamma_r) \mathrm{f}(\gamma_i + \gamma_j + k) \\ &\quad + \sum_{1 \leqslant i \leqslant n} \mathrm{f}(\gamma_1) \cdots \widehat{\mathrm{f}(\gamma_i)} \cdots \mathrm{f}(\gamma_r) \mathrm{h}(\gamma_i + k)\end{aligned}$$

for all $\gamma \in \mathbb{Z}^n$, $n \geqslant 0$ and $k \in \mathbb{Z}$. These formulae, in consideration of parts (i) and (ii), immediately imply the formulae of part (iii), and completely describe the action of $\hat{\mathfrak{g}}$ on $\mathbf{M}(0)$. $\square$

Corollary 1.4. The $\hat{\mathfrak{g}}$-module $\mathbf{M}(0)$ has a decomposition

$$\mathbf{M}(0) = \bigoplus_{n \in \mathbb{Z}_+} \mathbf{M}^{(n)},$$

as a direct sum of modules for $\hat{\mathfrak{h}}$, where

$$\mathbf{M}^{(n)} = \operatorname{span}\{ \prod_{1 \leqslant i \leqslant n} \mathrm{f}(\chi_i) \mathrm{u}_0 \mid \chi \in \mathbb{Z}^n \},$$

and $\mathbf{M}^{(n)} = \mathbf{M}(0)^{-n\alpha}$ for any $n \geqslant 0$.

Remark 1.5. It follows from Corollary 1.4 above, and Proposition 3.6 of [**20**], that the only integrable subquotient of $\mathbf{M}(0)$ is the trivial one-dimensional $\hat{\mathfrak{g}}$-module.

2. Symmetric Function Realisation

This section presents a realisation of the $\hat{\mathfrak{h}}$-module $\mathbf{M}^{(n)}$ as the graded regular module of the symmetric Laurent polynomials in n variables. The realisation allows the classification of the irreducible quotients of the $\hat{\mathfrak{h}}$-module $\mathbf{M}^{(n)}$ outlined in Section 4.

Fix a positive integer n. The $\mathbb{k}$-algebra $\mathbf{A}_n$ is $\mathbb{Z}$-graded by total degree. The elementary symmetric functions $\varepsilon_i \in \mathbf{A}_n$, $1 \leqslant i \leqslant n$, are defined by the polynomial equation

$$\prod_{i=1}^{n}(1+\mathrm{z}_i\mathrm{t}) = 1 + \sum_{i=1}^{n} \varepsilon_i(\mathrm{z}_1,\dots,\mathrm{z}_n)\mathrm{t}^i. \tag{2.1}$$

Notice that $\varepsilon_n = \mathrm{z}_1\cdots\mathrm{z}_n$ is invertible in $\mathbf{A}_n$. For any $k \in \mathbb{Z}$, let

$$\mathbf{p}(k) = \mathrm{z}_1^k + \cdots + \mathrm{z}_n^k \quad \in \mathbf{A}_n,$$

denote the sum of k-powers of the indeterminates, and for any $\gamma \in \mathbb{Z}^n$, write

$$\mathbf{m}(\gamma) = \frac{1}{n!}\sum_{\sigma\in\mathrm{Sym}(n)}\prod_{1\leqslant i\leqslant n} \mathrm{z}_{\sigma(i)}^{\gamma_i} \quad \in \mathbf{A}_n.$$

The symmetric polynomial $\mathbf{m}(\gamma)$ may alternatively be defined by

$$\mathbf{m}(\gamma) = \frac{1}{n!}\sum_{\sigma\in\mathrm{Sym}(n)}\prod_{1\leqslant i\leqslant n} \mathrm{z}_i^{\gamma_{\sigma(i)}}.$$

The set $\{\,\mathbf{m}(\gamma) \mid \gamma \in \mathbb{Z}^n\,\}$ spans the $\mathbb{k}$-algebra $\mathbf{A}_n$ of symmetric Laurent polynomials.

Lemma 2.2. Let $n > 0$ and $\gamma, \chi \in \mathbb{Z}^n$. Then

$$\mathbf{m}(\gamma)\cdot\mathbf{m}(\chi) = \frac{1}{n!}\sum_{\tau\in\mathrm{Sym}(n)} \mathbf{m}(\gamma+\chi_\tau),$$

where $\chi_\tau \in \mathbb{Z}^n$ is given by $(\chi_\tau)_i = \chi_{\tau(i)}$, for all $1 \leqslant i \leqslant n$ and $\tau \in \mathrm{Sym}(n)$.

Proof. Let $\gamma, \chi \in \mathbb{Z}^n$. Then

$$\begin{aligned}
\mathbf{m}(\gamma) \cdot \mathbf{m}(\chi) &= \left(\frac{1}{n!}\right)^2 \sum_{\sigma \in \mathsf{Sym}(n)} \sum_{\tau \in \mathsf{Sym}(n)} \prod_{1 \leqslant i \leqslant n} \mathrm{z}_i^{\gamma_{\sigma(i)} + \chi_{\tau(i)}} \\
&= \left(\frac{1}{n!}\right)^2 \sum_{\sigma \in \mathsf{Sym}(n)} \sum_{\tau \in \mathsf{Sym}(n)} \prod_{1 \leqslant i \leqslant n} \mathrm{z}_i^{\gamma_{\sigma(i)} + \chi_{(\tau \circ \sigma)(i)}} \\
&\quad \text{(substituting } \tau \circ \sigma \text{ for } \tau\text{)} \\
&= \left(\frac{1}{n!}\right)^2 \sum_{\sigma \in \mathsf{Sym}(n)} \sum_{\tau \in \mathsf{Sym}(n)} \prod_{1 \leqslant i \leqslant n} \mathrm{z}_i^{(\gamma + \chi_\tau)_{\sigma(i)}} \\
&\quad \text{(since } \chi_{\tau \circ \sigma} = (\chi_\tau)_\sigma\text{)} \\
&= \frac{1}{n!} \sum_{\tau \in \mathsf{Sym}(n)} \mathbf{m}(\gamma + \chi_\tau)
\end{aligned}$$

□

Proposition 2.3. For any positive integer n:

i. The $\Bbbk$-algebra $\mathbf{A}_n$ is generated by the set $\{ \varepsilon_i \mid 1 \leqslant i \leqslant n \} \cup \{ {\varepsilon_n}^{-1} \}$.
ii. The $\Bbbk$-algebra $\mathbf{A}_n$ is generated by the set of power sums $\{ \mathbf{p}(k) \mid k \in \mathbb{Z} \}$.

Proof. Let

$$\mathbf{A}_n^+ = \Bbbk[\mathrm{z}_1, \ldots, \mathrm{z}_n]^{\mathsf{Sym}(n)}, \qquad \mathbf{A}_n^- = \Bbbk[\mathrm{z}_1^{-1}, \ldots, \mathrm{z}_n^{-1}]^{\mathsf{Sym}(n)}.$$

Then

$$\mathbf{A}_n = \sum_{k \geqslant 0} \varepsilon_n^{-k} \cdot \mathbf{A}_n^+, \tag{2.4}$$

and, in particular,

$$\mathbf{A}_n = \mathbf{A}_n^- \cdot \mathbf{A}_n^+. \tag{2.5}$$

Part (i) follows from the Fundamental Theorem of Symmetric Functions and equation (2.4), while part (ii) follows from the Newton-Girard formulae and equation (2.5). □

Proposition 2.6. For any $n > 0$, $\mathbf{M}^{(n)}$ is a $\mathbb{Z}$-graded $\mathbf{A}_n$-module via linear extension of

$$\mathbf{m}(\gamma) \cdot \prod_{1 \leqslant i \leqslant n} \mathrm{f}(\chi_i)\,\mathrm{u}_0 = \frac{1}{n!} \sum_{\sigma \in \mathsf{Sym}(n)} \prod_{1 \leqslant i \leqslant n} \mathrm{f}(\chi_i + \gamma_{\sigma(i)})\,\mathrm{u}_0, \qquad \gamma, \chi \in \mathbb{Z}^n, \tag{2.7}$$

with the $\mathbb{Z}$-grading defined by

$$\deg \prod_{1 \leqslant i \leqslant n} \mathrm{f}(\chi_i)\,\mathrm{u}_0 = \sum_{1 \leqslant i \leqslant n} \chi_i, \qquad \chi \in \mathbb{Z}^n.$$

Proof. For any $\varepsilon, \gamma, \chi \in \mathbb{Z}^n$,

$$
\begin{aligned}
\mathbf{m}(\varepsilon) \cdot \big(\mathbf{m}(\gamma) \cdot \prod_{1\leqslant i\leqslant n} \mathrm{f}(\chi_i)\mathrm{u}_0\big) &= \big(\frac{1}{n!}\big)^2 \sum_{\sigma\in\mathrm{Sym}(n)} \sum_{\tau\in\mathrm{Sym}(n)} \prod_{1\leqslant i\leqslant n} \mathrm{f}(\chi_i + \varepsilon_{\sigma(i)} + \gamma_{\tau(i)})\mathrm{u}_0 \\
&= \frac{1}{n!} \sum_{\tau\in\mathrm{Sym}(n)} \frac{1}{n!} \sum_{\sigma\in\mathrm{Sym}(n)} \prod_{1\leqslant i\leqslant n} \mathrm{f}(\chi_i + (\varepsilon + \gamma_\tau)_{\sigma(i)})\mathrm{u}_0 \\
&\quad \text{(substituting } \tau \circ \sigma \text{ for } \tau) \\
&= \Big(\frac{1}{n!} \sum_{\tau\in\mathrm{Sym}(n)} \mathbf{m}(\varepsilon + \gamma_\tau)\Big) \cdot \prod_{1\leqslant i\leqslant n} \mathrm{f}(\chi_i)\mathrm{u}_0 \\
&= \big(\mathbf{m}(\varepsilon)\mathbf{m}(\gamma)\big) \cdot \prod_{1\leqslant i\leqslant n} \mathrm{f}(\chi_i)\mathrm{u}_0,
\end{aligned}
$$

by Lemma 2.2. As the polynomials $\mathbf{m}(\gamma)$ span $\mathbf{A}_n$, linear extension of (2.7) endows $\mathbf{M}^{(n)}$ with the structure of a $\mathbb{Z}$-graded $\mathbf{A}_n$-module. □

Theorem 2.8. Let $n > 0$. The action of $\mathcal{U}(\hat{\mathfrak{h}})$ on $\mathbf{M}^{(n)}$ factors through an epimorphism

$$\Psi : \mathcal{U}(\hat{\mathfrak{h}}) \twoheadrightarrow \mathbf{A}_n$$

of graded algebras defined by

$$\Psi : \mathrm{h}(k) \mapsto -2\mathbf{p}(k), \qquad k \in \mathbb{Z}.$$

That is, if ρ and ν denote the representations of $\mathcal{U}(\hat{\mathfrak{h}})$ and $\mathbf{A}_n$ on $\mathbf{M}^{(n)}$, respectively, then the following diagram commutes:

$$
\begin{tikzcd}
\mathcal{U}(\hat{\mathfrak{h}}) \arrow[r, two heads, "\Psi"] \arrow[dr, "\rho"'] & \mathbf{A}_n \arrow[d, "\nu"] \\
& \operatorname{End}\mathbf{M}^{(n)}
\end{tikzcd}
$$

Proof. The map Ψ is an algebra epimorphism by Proposition 2.3 part (ii). Let $k \in \mathbb{Z}$, and let $\iota_k = (k, 0, \dots, 0) \in \mathbb{Z}^n$. Then

$$\mathbf{p}(k) = n\mathbf{m}(\iota_k).$$

It follows that, for any $\chi \in \mathbb{Z}^n$,

$$\mathbf{p}(k) \cdot \prod_{1\leqslant i\leqslant n} \mathrm{f}(\chi_i)\mathrm{u}_0 = \sum_{1\leqslant i\leqslant n} \mathrm{f}(\chi_1)\cdots\widehat{\mathrm{f}(\chi_i)}\cdots\mathrm{f}(\chi_n)\mathrm{f}(\chi_i + k)\mathrm{u}_0$$

and so $\mathrm{h}(k)|_{\mathbf{M}^{(n)}} = \Psi(\mathrm{h}(k))|_{\mathbf{M}^{(n)}}$ by Proposition 1.1 part (iii). □

Therefore, for $n > 0$, it is sufficient to consider $\mathbf{M}^{(n)}$ as a $\mathbb{Z}$-graded $\mathbf{A}_n$-module. Write $\mathbf{A}_n^{\mathrm{reg}}$ for the regular $\mathbb{Z}$-graded $\mathbf{A}_n$-module, i.e. for $\mathbf{A}_n$ considered as a $\mathbb{Z}$-graded $\mathbf{A}_n$-module under multiplication.

Theorem 2.9. For any $n > 0$, the map

$$\Theta : \mathbf{M}^{(n)} \to \mathbf{A}_n^{\mathrm{reg}}$$

defined by linear extension of

$$\Theta : \prod_{1 \leqslant i \leqslant n} \mathrm{f}(\chi_i)\,\mathrm{u}_0 \mapsto \mathbf{m}(\chi), \qquad \chi \in \mathbb{Z}^n,$$

is an isomorphism of $\mathbb{Z}$-graded $\mathbf{A}_n$-modules.

Proof. The map Θ is a bijection, by Proposition 1.1 part (ii). Let $\gamma, \chi \in \mathbb{Z}^n$. Then

$$\begin{aligned}
\Theta\Big(\mathbf{m}(\gamma) \cdot \prod_{1 \leqslant i \leqslant n} \mathrm{f}(\chi_i)\,\mathrm{u}_0\Big) &= \Theta\Big(\frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}(n)} \prod_{1 \leqslant i \leqslant n} \mathrm{f}(\chi_i + \gamma_{\sigma(i)})\,\mathrm{u}_0\Big) \\
&= \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}(n)} \mathbf{m}(\chi + \gamma_\sigma) \\
&= \mathbf{m}(\gamma) \cdot \mathbf{m}(\chi) \quad \text{(by Lemma 2.2)} \\
&= \mathbf{m}(\gamma) \cdot \Theta\Big(\prod_{1 \leqslant i \leqslant n} \mathrm{f}(\chi_i)\,\mathrm{u}_0\Big).
\end{aligned}$$

Hence Θ is an isomorphism of $\mathbb{Z}$-graded $\mathbf{A}_n$-modules. □

Corollary 2.10. Let $n \geqslant 0$ and let $v \in \mathbf{M}^{(n)}$ be non-zero and homogeneous. Then $\mathcal{U}(\hat{\mathfrak{h}})v$ and $\mathbf{M}^{(n)}$ are isomorphic as $\hat{\mathfrak{h}}$-modules.

Proof. For $n = 0$, the statement is trivial. For $n > 0$, it is sufficient to employ Theorem 2.9, and observe that the ring $\mathbf{A}_n$ is an integral domain. □

3. Singular Vectors

The existence of non-zero singular vectors is related to the degeneracy of the module $\mathbf{M}(0)$:

Proposition 3.1. Let $n \geqslant 0$. Then

i. The set of all singular vectors in $\mathbf{M}^{(n)}$ forms an $\hat{\mathfrak{h}}$-submodule.

ii. If $v \in \mathbf{M}^{(n)}$ is non-zero and singular, and $V = \mathcal{U}(\hat{\mathfrak{g}})v$, then V has decomposition

$$V = \bigoplus_{m \geqslant n} V^{-m\alpha}, \qquad V^{-m\alpha} \subset \mathbf{M}^{(m)},$$

into non-trivial eigenspaces for $\mathrm{h}(0)$.

Proof. The set of all singular vectors in $\mathbf{M}^{(n)}$ clearly forms a vector space. Now suppose that $v \in \mathbf{M}^{(n)}$ is singular. Then

$$\mathrm{e}(k)(\mathrm{h}(l)v) = -2\mathrm{e}(k+l)v + \mathrm{h}(l)\mathrm{e}(k)v = 0,$$

for any $k, l \in \mathbb{Z}$. Thus if v is singular, then so is $\mathrm{h}(l)v$, for any $l \in \mathbb{Z}$, and so the set of singular vectors in $\mathbf{M}^{(n)}$ forms an $\hat{\mathfrak{h}}$-module, proving part (i). For part (ii), suppose again that $v \in \mathbf{M}^{(n)}$ is a non-zero singular vector, and let $V = \mathcal{U}(\hat{\mathfrak{g}})v$. Then

$$V = \mathcal{U}(\hat{\mathfrak{g}}) \cdot v = \mathcal{U}(\hat{\mathfrak{g}}_-) \otimes \mathcal{U}(\hat{\mathfrak{h}}) \cdot v \subset \bigoplus_{m \geqslant n} \mathbf{M}^{(m)},$$

since $\mathcal{U}(\hat{\mathfrak{g}}) = \mathcal{U}(\hat{\mathfrak{g}}_-) \otimes \mathcal{U}(\hat{\mathfrak{h}}) \otimes \mathcal{U}(\hat{\mathfrak{g}}+)$. □

Theorem 3.2. For any $n > 0$ and $\chi \in \mathbb{Z}^n$,

$$\mathbf{w}(\chi) = \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sgn}(\sigma) \prod_{1 \leqslant i \leqslant n} \mathrm{f}(\chi_i + \sigma(i))\, \mathrm{u}_0 \qquad \in \mathbf{M}^{(n)}$$

is a singular vector.

Proof. Singularity may be demonstrated directly by applying the formula for the action of $\mathrm{e}(k)$, $k \in \mathbb{Z}$, of Proposition 1.1 part (iii). □

Lemma 3.3. For any $n > 0$, the symmetric function Ω_n is equal to

$$\sum_{\sigma, \tau \in \mathrm{Sym}(n)} \mathrm{sgn}(\sigma \circ \tau) \prod_{1 \leqslant i \leqslant n} \mathrm{z}_i^{\sigma(i)+\tau(i)-2}$$

up to a change in sign.

Proof. Let $\Phi_n = \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sgn}(\sigma) \prod_{1 \leqslant i \leqslant n} \mathrm{z}_i^{\sigma(i)}$. It is not difficult to show that

$$\Phi_n|_{\mathrm{z}_i = \mathrm{z}_j} = 0, \qquad 1 \leqslant i < j \leqslant n,$$

and that $\Phi_n|_{\mathrm{z}_i = 0} = 0$ for all $1 \leqslant i \leqslant n$. Hence Φ_n is equal up to sign to

$$\prod_{1 \leqslant i \leqslant n} \mathrm{z}_i \cdot \prod_{1 \leqslant i < j \leqslant n} (\mathrm{z}_i - \mathrm{z}_j),$$

by degree considerations. Therefore Ω_n and

$$\prod_{1 \leqslant i \leqslant n} \mathrm{z}_i^{-2} \cdot \Phi_n^2$$

are equal up to sign, from which the result follows. □

Lemma 3.4. Suppose that $g \in \mathbf{A}_n$ and that $g|_{z_i=z_j} = 0$. Then $(z_i - z_j)^2$ divides g in $\Bbbk[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$.

Proof. Let σ_i^j denote the ring automorphism of $\Bbbk[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ that interchanges the variables z_i and z_j and leaves all other variables invariant. It is easy to convince oneself that

$$(\sigma_i^j h)|_{z_i=z_j} = h|_{z_i=z_j}, \qquad h \in \Bbbk[z_1^{\pm 1}, \dots, z_n^{\pm 1}].$$

Now $g|_{z_i=z_j} = 0$, and so $g = (z_i - z_j)h$, for some $h \in \Bbbk[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$. Moreover, $\sigma_i^j h = -h$, since $\sigma_i^j g = g$. Therefore

$$h|_{z_i=z_j} = (-\sigma_i^j h)|_{z_i=z_j} = -((\sigma_i^j h)|_{z_i=z_j}) = -(h|_{z_i=z_j}),$$

and so $h|_{z_i=z_j} = 0$. Hence $(z_i - z_j)$ divides h also. □

Theorem 3.5. For any $n > 0$,

$$\Omega_n \cdot \mathbf{M}^{(n)} = \operatorname{span}\{\, \mathbf{w}(\chi) \mid \chi \in \mathbb{Z}^n \,\},$$

and hence all elements of the non-zero $\hat{\mathfrak{h}}$-submodule $\Omega_n \cdot \mathbf{M}^{(n)}$ are singular.

Proof. Fix $n > 0$, and let

$$W_n = \operatorname{span}\{\, \mathbf{w}(\chi) \mid \chi \in \mathbb{Z}^n \,\}.$$

The symmetric function realisation of Theorem 2.9 may be used to demonstrate the inclusion $W_n \subset \Omega_n \cdot \mathbf{M}^{(n)}$. Let $\chi \in \mathbb{Z}^n$. Then

$$\begin{aligned}
\Theta(\mathbf{w}(\chi)) &= \frac{1}{n!} \sum_{\tau \in \mathrm{Sym}(n)} \sum_{\sigma \in \mathrm{Sym}(n)} \operatorname{sgn}(\sigma) \prod_{1 \leqslant i \leqslant n} z_i^{\chi_{\tau(i)} + \sigma(\tau(i))} \\
&= \frac{1}{n!} \sum_{\tau \in \mathrm{Sym}(n)} \operatorname{sgn}(\tau) \sum_{\sigma \in \mathrm{Sym}(n)} \operatorname{sgn}(\sigma) \prod_{1 \leqslant i \leqslant n} z_i^{\chi_{\tau(i)} + \sigma(i)} \\
&\quad \text{(substituting } \sigma \circ \tau^{-1} \text{ for } \sigma) \\
&= \frac{1}{n!} \sum_{\tau \in \mathrm{Sym}(n)} \operatorname{sgn}(\tau) F_\tau,
\end{aligned}$$

where, for any $\tau \in \mathrm{Sym}(n)$,

$$F_\tau = \sum_{\sigma \in \mathrm{Sym}(n)} \operatorname{sgn}(\sigma) \prod_{1 \leqslant i \leqslant n} z_i^{\chi_{\tau(i)} + \sigma(i)}.$$

It is not difficult to verify that

$$F_\tau|_{z_i=z_j} = 0, \qquad 1 \leqslant i < j \leqslant n, \quad \tau \in \mathrm{Sym}(n).$$

Therefore, for all $1 \leqslant i < j \leqslant n$, $\Theta(\mathbf{w}(\chi))$ is divisible by $(\mathrm{z}_i - \mathrm{z}_j)^2$ in $\Bbbk[\mathrm{z}_1^{\pm 1}, \ldots, \mathrm{z}_n^{\pm 1}]$, by Lemma 3.4. Thus

$$\Theta(\mathbf{w}(\chi)) = \Omega_n h \quad \text{for some} \quad h \in \Bbbk[\mathrm{z}_1^{\pm 1}, \ldots, \mathrm{z}_n^{\pm 1}]$$

since the factors $(\mathrm{z}_i - \mathrm{z}_j)^2$ are pairwise co-prime. In fact, $h \in \mathbf{A}_n$, since both $\Theta(\mathbf{w}(\chi))$ and Ω_n are symmetric. Therefore, $\Theta(\mathbf{w}(\chi)) \in \Omega_n \cdot \mathbf{A}_n$, and hence $W_n \subset \Omega_n \cdot \mathbf{M}^{(n)}$. Now let $\Lambda_n = (\prod_{1 \leqslant i \leqslant n} \mathrm{z}_i^2) \cdot \Omega_n$. The factor $\prod_{1 \leqslant i \leqslant n} \mathrm{z}_i^2$ is invertible in $\mathbf{A}_n$, and so

$$\Omega_n \cdot \mathbf{M}^{(n)} = \Lambda_n \cdot \mathbf{M}^{(n)},$$

by Theorem 2.9. In particular, by Corollary 1.4,

$$\Omega_n \cdot \mathbf{M}^{(n)} = \operatorname{span}\{\, \Lambda_n \cdot \prod_{1 \leqslant i \leqslant n} \mathrm{f}(\chi_i)\,\mathrm{u}_0 \mid \chi \in \mathbb{Z}^n \,\}.$$

By Lemma 3.3, the polynomial Λ_n is equal up to sign to

$$\sum_{\sigma, \tau \in \mathrm{Sym}(n)} \operatorname{sgn}(\sigma \circ \tau) \prod_{1 \leqslant i \leqslant n} \mathrm{z}_i^{\sigma(i) + \tau(i)}.$$

Therefore, for any $\chi \in \mathbb{Z}^n$,

$$\begin{aligned}
\Lambda_n \quad \cdot & \prod_{1 \leqslant i \leqslant n} \mathrm{f}(\chi_i)\,\mathrm{u}_0 \\
= \; & \frac{1}{n!} \sum_{\nu \in \mathrm{Sym}(n)} \sum_{\sigma, \tau \in \mathrm{Sym}(n)} \operatorname{sgn}(\sigma \circ \tau) \prod_{1 \leqslant i \leqslant n} \mathrm{f}(\chi_i + (\sigma \circ \nu)(i) + (\tau \circ \nu)(i))\,\mathrm{u}_0 \\
= \; & \frac{1}{n!} \sum_{\nu \in \mathrm{Sym}(n)} \sum_{\sigma, \tau \in \mathrm{Sym}(n)} \operatorname{sgn}(\sigma \circ \tau) \prod_{1 \leqslant i \leqslant n} \mathrm{f}(\chi_i + \sigma(i) + \tau(i))\,\mathrm{u}_0 \\
& (\text{substituting } \sigma \circ \nu^{-1} \text{ for } \sigma \text{ and } \tau \circ \nu^{-1} \text{ for } \tau) \\
= \; & \sum_{\sigma \in \mathrm{Sym}(n)} \operatorname{sgn}(\sigma) \sum_{\tau \in \mathrm{Sym}(n)} \operatorname{sgn}(\tau) \prod_{1 \leqslant i \leqslant n} \mathrm{f}(\chi_i + \sigma(i) + \tau(i))\,\mathrm{u}_0 \\
= \; & \sum_{\sigma \in \mathrm{Sym}(n)} \operatorname{sgn}(\sigma)\,\mathbf{w}(\chi(\sigma)),
\end{aligned}$$

where $\chi(\sigma) \in \mathbb{Z}^n$ is given by

$$\chi(\sigma)_i = \chi_i + \sigma(i), \qquad 1 \leqslant i \leqslant n,$$

for all $\sigma \in \mathrm{Sym}(n)$. Hence $\Omega_n \cdot \mathbf{M}^{(n)} \subset W_n$. □

Conjecture 3.6. Let $n > 0$, and suppose that $v \in \mathbf{M}^{(n)}$ is singular. Then $v \in \Omega_n \cdot \mathbf{M}^{(n)}$.

4. Irreducible Quotients of $\mathbf{M}^{(n)}$

For any $\mathbb{Z}$-graded $\Bbbk$-algebra B, write $B^{(k)}$, $k \in \mathbb{Z}$, for the graded components of B.

Proposition 4.1. Let n be a positive integer, and let B be a graded simple quotient of the graded algebra $\mathbf{A}_n$. Then $B = \mathbb{F}[\mathrm{t}^m, \mathrm{t}^{-m}]$ for some positive divisor m of n and finite algebraic field extension $\mathbb{F}$ of $\Bbbk$.

Proof. As $\varepsilon_n = \mathrm{z}_1 \cdots \mathrm{z}_n$ is invertible in $\mathbf{A}_n$, it must be that $B^{(n)} \neq 0$. Let m be the minimal positive integer such that $B^{(m)} \neq 0$, and let $u \in B^{(m)}$ be non-zero. Then u is invertible, since B is simple, and so multiplication by u^k is a vector-space automorphism of B such that

$$B^{(l)} \to B^{(km+l)}, \qquad l \in \mathbb{Z}.$$

In particular,

$$B^{(km)} = u^k \cdot B^{(0)}, \qquad k \in \mathbb{Z}. \tag{4.2}$$

Suppose that $B^{(l)} \neq 0$, for some $l \in \mathbb{Z}$, and let q, r be the unique integers such that

$$l = qm + r, \qquad 0 \leqslant r < m.$$

Then

$$0 \neq u^{-qm} \cdot B^{(l)} = B^{(r)},$$

and so $r = 0$ by the minimality of m. Hence

$$B = \bigoplus\nolimits_{k \in \mathbb{Z}} B^{(km)},$$

and in particular m must be a divisor of n. Moreover, by (4.2),

$$B \cong B^{(0)} \otimes_{\Bbbk} \Bbbk[\mathrm{t}^k, \mathrm{t}^{-k}]$$

via $u^k \mapsto \mathrm{t}^{km}$, $k \in \mathbb{Z}$. As $\mathbf{A}_n$ is finitely generated, by Proposition 2.3 part (i), so is the $\Bbbk$-algebra $\mathbf{A}_n^{(0)}$. Hence $B^{(0)}$ is a finite algebraic field extension of $\Bbbk$ (see, for example, [**1**], Proposition 7.9). $\square$

Proposition 4.3. Let n be a positive integer, and suppose that $\zeta : \mathbf{A}_n \to \Bbbk$ is a non-zero algebra homomorphism. Then there exist scalars $\alpha_1, \ldots, \alpha_n$, all non-zero and algebraic over $\Bbbk$, such that

$$\zeta(\mathbf{p}(k)) = \sum_{i=1}^{n} \alpha_i^k, \qquad k \in \mathbb{Z}.$$

Proof. Suppose that $\zeta : \mathbf{A}_n \to \Bbbk$ is a non-zero homomorphism, and let

$$g(\mathrm{t}) = 1 + \sum_{i=1}^{n} \zeta(\varepsilon_i)\mathrm{t}^i \quad \in \Bbbk[\mathrm{t}].$$

As $\varepsilon_n = \mathrm{z}_1 \cdots \mathrm{z}_n$ is invertible in $\mathbf{A}_n$, it must be that $\zeta(\varepsilon_n) \neq 0$. Let $\alpha_1, \ldots \alpha_n$ be some iteration of the scalars defined by

$$g(\mathrm{t}) = \prod_{i=1}^{n}(1 + \alpha_i \mathrm{t}).$$

Then by equation (2.1),

$$\zeta(\varepsilon_i) = \varepsilon_i(\alpha_1, \ldots, \alpha_n), \qquad 1 \leqslant i \leqslant n. \tag{4.4}$$

The α_i are necessarily non-zero since

$$\alpha_1 \cdots \alpha_n = \varepsilon_n(\alpha_1, \ldots, \alpha_n) = \zeta(\varepsilon_n) \neq 0.$$

By Proposition 2.3 part (i), there is a unique algebra homomorphism with the property (4.4), namely the restriction of the evaluation map

$$\mathrm{z}_i \mapsto \alpha_i, \qquad 1 \leqslant i \leqslant n.$$

In particular, $\zeta(\mathbf{p}(k)) = \sum_{i=1}^{n} \alpha_i^k$, for all $k \in \mathbb{Z}$. □

Recall that $\mathcal{E}^{(-)} \subset \mathcal{E}$ is given by

$$\mathcal{E}^{(-)} = \{ \varphi \in \mathcal{E} \mid \varphi_\lambda \in -2\mathbb{Z}_+ \;\text{ for all }\; \lambda \in \Bbbk^\times \}.$$

For any $n \geqslant 0$, let

$$\mathcal{E}^{(-n)} = \{ \varphi \in \mathcal{E}^{(-)} \mid \sum_{\lambda \in \Bbbk^\times} \varphi_\lambda = -2n \},$$

so that $\mathcal{E}^{(-)} = \bigsqcup_{n \geqslant 0} \mathcal{E}^{(-n)}$.

Theorem 4.5. For any $n \geqslant 0$ and $\varphi \in \mathcal{E}^{(-n)}$, the $\hat{\mathfrak{h}}$-module $\mathbf{H}(\varphi)$ is an irreducible quotient of the $\hat{\mathfrak{h}}$-module $\mathbf{M}^{(n)}$. Moreover, if $\Bbbk$ is algebraically closed, then any irreducible quotient of the $\hat{\mathfrak{h}}$-module $\mathbf{M}^{(n)}$ is of the form $\mathbf{H}(\varphi)$ for some $\varphi \in \mathcal{E}^{(-n)}$.

Proof. The statement is trivial for $n = 0$, so suppose that n is a positive integer. Let $\varphi \in \mathcal{E}^{(-n)}$, and let $\alpha_1, \ldots \alpha_n \in \Bbbk^\times$ be non-zero scalars such that

$$\varphi(k) = -2 \sum_{i=1}^{n} \alpha_i^k, \qquad k \in \mathbb{Z}.$$

Define

$$\eta = \eta_{\alpha_1, \ldots, \alpha_n} : \Bbbk[\mathrm{z}_1, \mathrm{z}_1^{-1}, \ldots, \mathrm{z}_n, \mathrm{z}_n^{-1}] \to \Bbbk[\mathrm{t}, \mathrm{t}^{-1}]$$

by extension of $\mathrm{z}_i \mapsto \alpha_i \mathrm{t}$, $1 \leqslant i \leqslant n$. Then

$$\eta|_{\mathbf{A}_n} : \mathbf{A}_n \to \operatorname{im} \eta|_{\mathbf{A}_n} \subset \Bbbk[\mathrm{t}, \mathrm{t}^{-1}]$$

is a homomorphism of graded algebras, and so $\operatorname{im} \eta|_{\mathbf{A}_n}$ may be considered as an $\mathbf{A}_n$-module, and as a quotient of the regular $\mathbf{A}_n$-module $\mathbf{A}_n^{\mathrm{reg}}$. Therefore $\operatorname{im} \eta|_{\mathbf{A}_n}$ is a quotient of the $\mathbf{A}_n$-module $\mathbf{M}^{(n)}$, by Theorem 2.9. Now $\operatorname{im} \eta|_{\mathbf{A}_n}$ is an $\hat{\mathfrak{h}}$-module via the map $\Psi : \mathcal{U}(\hat{\mathfrak{h}}) \to \mathbf{A}_n$ of Theorem 2.8. For any $k \in \mathbb{Z}$,

$$\begin{aligned}(\eta|_{\mathbf{A}_n} \circ \Psi)(\mathrm{h}(k)) &= -2\eta|_{\mathbf{A}_n}(\mathbf{p}(k)) \\ &= -2\sum_{i=1}^{n} \alpha_i^k \mathrm{t}^k \\ &= \tilde{\varphi}(\mathrm{h}(k)),\end{aligned}$$

where $\tilde{\varphi} : \mathcal{U}(\hat{\mathfrak{h}}) \to \Bbbk[\mathrm{t}, \mathrm{t}^{-1}]$ is defined by (1.2), page 2. Hence $\operatorname{im} \eta|_{\mathbf{A}_n} \cong \mathbf{H}(\varphi)$ as $\hat{\mathfrak{h}}$-modules, and so $\mathbf{H}(\varphi)$ is a quotient of the $\hat{\mathfrak{h}}$-module $\mathbf{M}^{(n)}$. It is not too difficult to verify that $\mathbf{H}(\varphi)$ is an irreducible $\hat{\mathfrak{h}}$-module for any $\varphi \in \mathcal{E}^{(-)}$.

Now suppose that $\Bbbk$ is algebraically closed, and that Γ is an irreducible quotient of the $\hat{\mathfrak{h}}$-module $\mathbf{M}^{(n)}$. By Theorems 2.8 and 2.9, Γ is an $\mathbf{A}_n$-module, and a quotient of $\mathbf{A}_n^{\mathrm{reg}}$. Hence there exists a simple quotient B of the graded algebra $\mathbf{A}_n$

$$\eta : \mathbf{A}_n \twoheadrightarrow B,$$

such that $B \cong \Gamma$ as $\mathbf{A}_n$-modules, when B is considered as an $\mathbf{A}_n$-module via the algebra epimorphism η. Moreover, by Proposition 4.1, $B = \Bbbk[\mathrm{t}^m, \mathrm{t}^{-m}]$ for some positive divisor m of n. Let $\zeta : \mathbf{A}_n \to \Bbbk$ be given by

$$\eta(x) = \zeta(x)\, \mathrm{t}^{\deg x},$$

for all homogeneous $x \in \mathbf{A}_n$. Then ζ is a non-zero algebra homomorphism, and so by Proposition 4.3 there exist non-zero scalars $\alpha_1, \ldots, \alpha_n \in \Bbbk^\times$ such that

$$\eta(\mathbf{p}(k)) = \zeta(\mathbf{p}(k))\, \mathrm{t}^k = \sum_{i=1}^{n} \alpha_i^k \mathrm{t}^k, \qquad k \in \mathbb{Z}.$$

Therefore $\Gamma \cong \mathbf{H}(\varphi)$, where

$$\varphi : \mathbb{Z} \to \Bbbk, \qquad \varphi(k) = -2\sum_{i=1}^{n} \alpha_i^k, \qquad k \in \mathbb{Z},$$

by Theorem 2.8. □

5. Irreducible Subquotients of $\mathbf{M}(0)$

An $\hat{\mathfrak{h}}$-module Γ is *weight* if $\mathrm{h}(0)$ acts by a scalar $\mathrm{h}(0)|_\Gamma$ on Γ. For any weight $\hat{\mathfrak{h}}$-module Γ, write $\mathscr{V}(\Gamma)$ for the induced $\hat{\mathfrak{g}}$-module

$$\mathscr{V}(\Gamma) = \mathrm{Ind}_{\hat{\mathfrak{h}}\oplus\hat{\mathfrak{g}}_+}^{\hat{\mathfrak{g}}} \Gamma, \quad \text{where} \quad \hat{\mathfrak{g}}_+ \cdot \Gamma = 0.$$

Proposition 5.1. Let Γ be a weight $\hat{\mathfrak{h}}$-module. Then the $\hat{\mathfrak{g}}$-module $\mathscr{V}(\Gamma)$ has a unique maximal submodule that has trivial intersection with Γ.

Proof. Let $\lambda = \mathrm{h}(0)|_\Gamma$. Then

$$\mathscr{V}(\Gamma) = \bigoplus\nolimits_{n\geqslant 0} \mathscr{V}(\Gamma)^{(\lambda-n)\alpha},$$

and $\mathscr{V}(\Gamma)^\lambda = \Gamma$. If $N \subset \mathscr{V}(\Gamma)$ is a $\hat{\mathfrak{g}}$-submodule that has trivial intersection with Γ, then

$$N \subset \bigoplus\nolimits_{n>0} \mathscr{V}(\Gamma)^{(\lambda-n)\alpha}.$$

Hence the same is true of the sum of all such $\hat{\mathfrak{g}}$-submodules. This sum is itself a $\hat{\mathfrak{g}}$-submodule, and its maximality and uniqueness follow from construction. □

For any $\hat{\mathfrak{h}}$-module Γ, denote by $\mathscr{L}(\Gamma)$ the quotient of $\mathscr{V}(\Gamma)$ by its unique maximal submodule that has trivial intersection with Γ. Hence $\mathscr{L}(\Gamma)$ is an irreducible $\hat{\mathfrak{g}}$-module if Γ is an irreducible $\hat{\mathfrak{h}}$-module. In particular, if $\varphi \in \mathcal{F}'$, then the $\hat{\mathfrak{h}}$-module $\mathbf{H}(\varphi)$ is irreducible, and so $\mathscr{L}(\mathbf{H}(\varphi)) = \mathbf{N}(\varphi)$.

Theorem 5.2. For any $\varphi \in \mathcal{E}^{(-)}$, the $\hat{\mathfrak{g}}$-module $\mathbf{N}(\varphi)$ is an irreducible subquotient of $\mathbf{M}(0)$. Moreover, if $\Bbbk$ is algebraically closed, then any irreducible subquotient of $\mathbf{M}(0)$ is of the form $\mathbf{N}(\varphi)$ for some $\varphi \in \mathcal{E}^{(-)}$.

Proof. Let n be a positive integer, and let $\varphi \in \mathcal{E}^{(-n)}$. Theorem 3.5 guarantees the existence of a non-zero singular vector $v \in \mathbf{M}^{(n)}$. The $\hat{\mathfrak{h}}$-module $\mathcal{U}(\hat{\mathfrak{h}})v$ contains only singular vectors, and is isomorphic to $\mathbf{M}^{(n)}$, by Corollary 2.10 and Proposition 3.1. Let $P = \mathcal{U}(\hat{\mathfrak{g}})v$. By the Poincaré-Birkhoff-Witt Theorem,

$$\begin{aligned} \mathcal{U}(\hat{\mathfrak{g}}) \cdot v &= \mathcal{U}(\hat{\mathfrak{g}}_-) \otimes \mathcal{U}(\hat{\mathfrak{h}}) \otimes \mathcal{U}(\hat{\mathfrak{g}}_+) \cdot v \\ &= \mathcal{U}(\hat{\mathfrak{g}}_-) \otimes \mathcal{U}(\hat{\mathfrak{h}}) \cdot v. \end{aligned}$$

Therefore $P = \mathcal{U}(\hat{\mathfrak{g}}_-)P^{-n\alpha}$ where $P^{-n\alpha} = \mathcal{U}(\hat{\mathfrak{h}})v \cong \mathbf{M}^{(n)}$. Hence, by Theorem 4.5, there is an epimorphism of $\hat{\mathfrak{h}}$-modules

$$P^{-n\alpha} \twoheadrightarrow \mathbf{H}(\varphi),$$

which extends to an epimorphism of $\hat{\mathfrak{g}}$-modules

$$P \twoheadrightarrow \mathbf{N}(\varphi).$$

Thus $\mathbf{N}(\varphi)$ is an irreducible subquotient of $\mathbf{M}(0)$.

Now suppose that $\Bbbk$ is algebraically closed, and that N is an irreducible subquotient of $\mathbf{M}(0)$. The support of N is a subset of the support $-\mathbb{Z}_{+}\alpha$ of $\mathbf{M}(0)$. Let n denote the minimal non-negative integer such that $N^{-n\alpha} \neq 0$. Then $\hat{\mathfrak{g}}_{+} \cdot N^{-n\alpha} = 0$. Thus there is an epimorphism of $\hat{\mathfrak{g}}$-modules $N \twoheadrightarrow \mathscr{L}(N^{-n\alpha})$, and since N is irreducible, this map is an isomorphism. Therefore, the weight space $N^{-n\alpha}$ is an irreducible $\hat{\mathfrak{h}}$-module. Indeed, a proper $\hat{\mathfrak{h}}$-submodule of $N^{-n\alpha}$ generates a proper $\hat{\mathfrak{g}}$-submodule of $\mathscr{L}(N^{-n\alpha}) \cong N$. Now let $P' \subset P$ be $\hat{\mathfrak{g}}$-submodules of $\mathbf{M}(0)$ such that $N = P/P'$. Then $N^{-l\alpha} = P^{-l\alpha}/P'^{-l\alpha}$, for all $l \geqslant 0$, and in particular $N^{-n\alpha}$ is a subquotient of the $\hat{\mathfrak{h}}$-module $\mathbf{M}^{(n)}$. Therefore, by Corollary 2.10 and Theorem 4.5 there exists $\varphi \in \mathcal{E}^{(-n)}$ such that $N^{-n\alpha} \cong \mathbf{H}(\varphi)$ as $\hat{\mathfrak{h}}$-modules. Thus there is an isomorphism of $\hat{\mathfrak{g}}$-modules

$$N \cong \mathscr{L}(N^{-n\alpha}) \cong \mathscr{L}(\mathbf{H}(\varphi)) = \mathbf{N}(\varphi),$$

which completes the proof of the Theorem. □

The following Corollary is immediate from [**3**] and the inclusion $\mathcal{E}^{(-)} \subset \mathcal{E}$.

Corollary 5.3. Suppose that $\Bbbk$ is algebraically closed. Then the homogeneous components of any irreducible subquotient of $\mathbf{M}(0)$ have finite dimension.

CHAPTER 3

Lie Algebras with Triangular Decomposition

This chapter develops the technology necessary for the study of the highest-weight theory of truncated current Lie algebras undertaken in Chapter 4. The notion of a Lie algebra with triangular decomposition is introduced, and several examples are considered. Fundamental results in the highest-weight representation theory are then described, concluding with a proof of Shapovalov's Lemma. The content of this chapter is entirely derivative of the book of Moody and Pianzola [**24**]. Let $\Bbbk$ denote a field of characteristic zero.

1. Lie Algebras with Triangular Decomposition

Let $\mathfrak{g}$ be a Lie algebra over the field $\Bbbk$. A *triangular decomposition* of $\mathfrak{g}$ is specified by a pair of non-zero abelian subalgebras $\mathfrak{h}_0 \subset \mathfrak{h}$, a pair of distinguished non-zero subalgebras $\mathfrak{g}_+$, $\mathfrak{g}_-$, and an anti-involution (i.e. an anti-automorphism of order 2)

$$\omega : \mathfrak{g} \longrightarrow \mathfrak{g},$$

such that:

i. $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$;
ii. the subalgebra $\mathfrak{g}_+$ is a non-zero weight module for $\mathfrak{h}_0$ under the adjoint action, with weights Δ_+ all non-zero;
iii. $\omega|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}$ and $\omega(\mathfrak{g}_+) = \mathfrak{g}_-$;
iv. the semigroup with identity $\mathcal{Q}_+$, generated by Δ_+ under addition, is freely generated by a finite subset $\{\, \alpha_j \,\}_{j \in \mathrm{J}} \subset \mathcal{Q}_+$ consisting of linearly independent elements of $\mathfrak{h}_0^*$.

This definition is a modification of the definition of Moody and Pianzola [**24**]. There, the set J is not required to be finite, root spaces may be infinite-dimensional, and $\mathfrak{h}_0 = \mathfrak{h}$. We distinguish between $\mathfrak{h}_0$ and $\mathfrak{h}$ in order to include Example 1.6.

Write $\mathcal{Q} = \sum_{j \in J} \mathbb{Z}\alpha_j$. Call the weights Δ_+ of the $\mathfrak{h}_0$-module $\mathfrak{g}_+$ the *positive roots*, and the weight space $\mathfrak{g}^\alpha$ corresponding to $\alpha \in \Delta_+$ the *α-root space*, so that $\mathfrak{g}_+ = \oplus_{\alpha \in \Delta_+} \mathfrak{g}^\alpha$. The anti-involution ensures an analogous decomposition of $\mathfrak{g}_- = \oplus_{\alpha \in \Delta_-} \mathfrak{g}^\alpha$, where $\Delta_- = -\Delta_+$ (the *negative roots*) and $\mathfrak{g}^{-\alpha} = \omega(\mathfrak{g}^\alpha)$ for all $\alpha \in \Delta_+$. Write $\Delta = \Delta_+ \cup \Delta_-$ for the *roots* of $\mathfrak{g}$. Consider $\mathcal{Q}_+$ to be partially ordered in the usual manner, i.e. for $\gamma, \gamma' \in \mathcal{Q}_+$,

$$\gamma \leqslant_{\mathcal{Q}_+} \gamma' \quad \iff \quad (\gamma' - \gamma) \in \mathcal{Q}_+.$$

We assume that all root spaces are finite-dimensional, and that Δ_+ is a countable set. For clarity, a Lie algebra with triangular decomposition may be referred to as a five-tuple $(\mathfrak{g}, \mathfrak{h}_0, \mathfrak{h}, \mathfrak{g}_+, \omega)$.

Example 1.1. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$, with Cartan subalgebra $\mathfrak{h}$ and root system Δ. Then

$$\mathfrak{g} = \mathfrak{h} \oplus (\textstyle\bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha).$$

Let π be a basis for Δ, and let $\mathcal{Q}_+$ be the additive semigroup generated by π. Write $\Delta_+ = \Delta \cap \mathcal{Q}_+$, and let $\mathfrak{g}_+$, $\mathfrak{g}_-$ be given by

$$\mathfrak{g}_+ = \textstyle\bigoplus_{\alpha \in \Delta_+} \mathfrak{g}^\alpha, \qquad \mathfrak{g}_- = \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}^\alpha,$$

where $\Delta_- = -\Delta_+$. Then $\mathfrak{g}_+$ is a weight-module for $\mathfrak{h}_0 = \mathfrak{h}$ with weights Δ_+, and

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+.$$

All root spaces are one-dimensional. For any $\alpha \in \Delta_+$, choose non-zero elements

$$\mathrm{x}(\alpha) \in \mathfrak{g}^\alpha, \qquad \mathrm{y}(\alpha) \in \mathfrak{g}^{-\alpha}.$$

An anti-involution ω on $\mathfrak{g}$ is defined by extension of

$$\omega|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}, \qquad \omega(\mathrm{x}(\alpha)) = \mathrm{y}(\alpha), \quad \omega(\mathrm{y}(\alpha)) = \mathrm{x}(\alpha), \quad \alpha \in \pi.$$

Thus $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{h}, \mathfrak{h}, \omega)$ is a Lie algebra with triangular decomposition. The semisimple finite-dimensional Lie algebras over $\mathbb{C}$ are parameterised by Euclidean root systems, or equivalently by the Cartan matrices. The Serre relations permit the construction of any such Lie algebra from its Cartan matrix, and this construction works over an arbitrary field $\Bbbk$ of characteristic zero. The preceding assertions hold also for the Lie algebras over $\Bbbk$ constructed in this manner. Here and throughout, *semisimple finite-dimensional Lie algebra* means a Lie algebra over $\Bbbk$ defined by a Cartan matrix and the Serre relations.

Example 1.2. It shall be convenient to consider the following particular case of Example 1.1 in greater detail. Let $\mathfrak{g}$ denote sl(3), the finite-dimensional semisimple Lie algebra over $\Bbbk$ with root system A_2. Denote by α_1, α_2 the simple roots, by

$$\mathrm{x}(\alpha_1),\ \mathrm{x}(\alpha_2),\ \mathrm{y}(\alpha_1),\ \mathrm{y}(\alpha_2),\ \mathrm{h}(\alpha_1),\ \mathrm{h}(\alpha_2)$$

the Chevalley generators, and by $\mathfrak{h} = \Bbbk \mathrm{h}(\alpha_1) \oplus \Bbbk \mathrm{h}(\alpha_2)$ the Cartan subalgebra, so that

$$\alpha_1(\mathrm{h}(\alpha_1)) = \alpha_2(\mathrm{h}(\alpha_2)) = 2, \quad \alpha_1(\mathrm{h}(\alpha_2)) = \alpha_2(\mathrm{h}(\alpha_1)) = -1.$$

Then the root system is defined by $\Delta = \Delta_+ \cup \Delta_-$, where $\Delta_+ = \{\, \alpha_1, \alpha_2, \alpha_1 + \alpha_2 \,\}$ and $\Delta_- = -\Delta_+$. Write

$$\begin{aligned} &\mathrm{x}(\alpha_1+\alpha_2) = [\,\mathrm{x}(\alpha_1), \mathrm{x}(\alpha_2)\,], \quad \mathrm{y}(\alpha_1+\alpha_2) = [\,\mathrm{y}(\alpha_2), \mathrm{y}(\alpha_1)\,], \\ &\mathrm{h}(\alpha_1+\alpha_2) = \mathrm{h}(\alpha_1) + \mathrm{h}(\alpha_2). \end{aligned}$$

Then for each $\alpha \in \Delta_+$, the elements $\mathrm{x}(\alpha), \mathrm{y}(\alpha), \mathrm{h}(\alpha)$ span a subalgebra of $\mathfrak{g}$ isomorphic to $\mathrm{sl}(2)$. The anti-involution ω fixes $\mathfrak{h}$ point-wise, and interchanges $\mathrm{x}(\alpha)$ with $\mathrm{y}(\alpha)$ for every $\alpha \in \Delta_+$. Write

$$\mathfrak{g}^{\alpha} = \Bbbk \mathrm{x}(\alpha), \quad \mathfrak{g}^{-\alpha} = \Bbbk \mathrm{y}(\alpha), \quad \alpha \in \Delta_+,$$

and $\mathfrak{g}_{\pm} = \oplus_{\alpha \in \Delta_+} \mathfrak{g}^{\pm\alpha}$. Then $\mathfrak{g}_+$ is a weight-module for $\mathfrak{h}_0 = \mathfrak{h}$ with weights Δ_+, and

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+.$$

The semigroup $\mathcal{Q}_+$ is generated by $\pi = \{\, \alpha_1, \alpha_2 \,\}$. Note that the $\mathrm{h}(\alpha)$ defined here are only proportional to the elements $\mathbf{h}(\alpha)$ defined later on.

Example 1.3. Let $\mathfrak{g}$ be the Kac-Moody Lie algebra over $\Bbbk$ associated to an $n \times n$ generalised Cartan matrix (we paraphrase [**20**]). Let $\mathfrak{h}$ denote the Cartan subalgebra, and Δ the root system. Then

$$\mathfrak{g} = \mathfrak{h} \oplus (\textstyle\bigoplus_{\alpha \in \Delta} \mathfrak{g}^{\alpha}),$$

and all root spaces are finite-dimensional. The collection Π of simple roots is a linearly-independent subset of the finite-dimensional space $\mathfrak{h}^*$. Let $\mathcal{Q}_+$ denote the additive semigroup generated by Π, let $\Delta_+ = \Delta \cap \mathcal{Q}_+$, and write

$$\mathfrak{g}_+ = \textstyle\bigoplus_{\alpha \in \Delta_+} \mathfrak{g}^{\alpha}, \qquad \mathfrak{g}_- = \textstyle\bigoplus_{\alpha \in \Delta_-} \mathfrak{g}^{\alpha},$$

where $\Delta_- = -\Delta_+$. Then $\mathfrak{g}_+$ is a weight-module for $\mathfrak{h}_0 = \mathfrak{h}$ with weights Δ_+, and

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+.$$

If e_i, f_i, $1 \leqslant i \leqslant n$, denote the Chevalley generators of $\mathfrak{g}$, then $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are the subalgebras generated by the e_i and by the f_i, respectively. An anti-involution ω of $\mathfrak{g}$ is defined by extension of

$$\omega|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}, \qquad \omega(e_i) = f_i, \quad \omega(f_i) = e_i, \quad 1 \leqslant i \leqslant n,$$

(this ω differs from the ω of [**20**]). Thus $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{h}, \mathfrak{h}, \omega)$ is a Lie algebra with triangular decomposition.

Example 1.4. Let $\mathfrak{g}$ denote the $\Bbbk$-vector space with basis the symbols

$$\{\mathrm{L}_m \mid m \in \mathbb{Z}\} \cup \{\mathrm{c}\},$$

endowed with the Lie bracket given by

$$[\mathrm{c}, \mathfrak{g}] = 0, \quad [\mathrm{L}_m, \mathrm{L}_n] = (m-n)\mathrm{L}_{m+n} + \delta_{m,-n}\psi(m)\mathrm{c}, \quad m, n \in \mathbb{Z},$$

where $\psi : \mathbb{Z} \to \Bbbk$ is any function satisfying $\psi(-m) = -\psi(m)$ for $m \in \mathbb{Z}$, and

$$\psi(m+n) = \tfrac{2m+n}{n-m}\psi(n) + \tfrac{m+2n}{m-n}\psi(m), \qquad m, n \in \mathbb{Z}, \quad m \neq n.$$

If $\psi = 0$, then the symbols L_m span a copy of the *Witt algebra.* The *Virasoro algebra* is the only non-split one-dimensional central extension of the Witt algebra, up to isomorphism [**21**], and is typically defined with $\psi(m) = \frac{m^3-m}{12}$. Let

$$\mathfrak{g}_\pm = \bigoplus_{m>0} \Bbbk\mathrm{L}_{\pm m}, \quad \mathfrak{h}_0 = \mathfrak{h} = \Bbbk\mathrm{L}_0 \oplus \Bbbk\mathrm{c},$$

and let $\delta \in \mathfrak{h}^*$ be given by

$$\delta(\mathrm{L}_0) = -1, \quad \delta(\mathrm{c}) = 0.$$

Then $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$, and $\mathfrak{g}_+$ is a weight module for $\mathfrak{h}_0 = \mathfrak{h}$, with weights

$$\Delta_+ = \{m\delta \mid m > 0\}.$$

The semigroup $\mathcal{Q}_+$ is generated by δ. An anti-involution ω is given by

$$\omega(\mathrm{c}) = \mathrm{c}, \quad \omega(\mathrm{L}_m) = \mathrm{L}_{-m}, \quad m \in \mathbb{Z},$$

and in this notation $\mathfrak{g}$ is a Lie algebra with triangular decomposition.

Example 1.5. Let $\mathfrak{a}$ denote the $\Bbbk$-vector space with basis the symbols

$$\{\mathrm{a}_m \mid m \in \mathbb{Z}\} \cup \{\hbar, \mathrm{d}\},$$

endowed with the Lie bracket given by

$$[\mathrm{a}_m, \mathrm{a}_n] = m\delta_{m,-n}\hbar, \quad [\hbar, \mathfrak{a}] = 0, \quad [\mathrm{d}, \mathrm{a}_m] = m\mathrm{a}_m, \quad m, n \in \mathbb{Z}.$$

The Lie algebra $\mathfrak{a}$ is called the *extended Heisenberg* or *oscillator algebra.* Let

$$\mathfrak{a}_\pm = \bigoplus_{m>0} \Bbbk\mathrm{a}_{\pm m}, \quad \mathfrak{h} = \Bbbk\mathrm{a}_0 \oplus \Bbbk\hbar \oplus \Bbbk\mathrm{d},$$

and let $\delta \in \mathfrak{h}^*$ be given by

$$\delta(\mathrm{a}_0) = \delta(\hbar) = 0, \quad \delta(\mathrm{d}) = 1.$$

Then $\mathfrak{a} = \mathfrak{a}_- \oplus \mathfrak{h} \oplus \mathfrak{a}_+$, and $\mathfrak{a}_+$ is a weight module for $\mathfrak{h}_0 = \mathfrak{h}$, with weights

$$\Delta_+ = \{m\delta \mid m > 0\}.$$

The semigroup $\mathcal{Q}_+$ is generated by δ. An anti-involution ω is given by

$$\omega(\hbar) = \hbar, \quad \omega(\mathrm{d}) = \mathrm{d}, \quad \omega(\mathrm{a}_m) = \mathrm{a}_{-m}, \quad m \in \mathbb{Z},$$

and in this notation $\mathfrak{a}$ is a Lie algebra with triangular decomposition.

Example 1.6. Let $\mathfrak{g}$ be a $\Bbbk$-Lie algebra with triangular decomposition, denoted as above, and let R be a commutative, associative $\Bbbk$-algebra with 1 (e.g. $R = \Bbbk[\mathrm{t}]/\mathrm{t}^{\mathrm{N}+1}\Bbbk[\mathrm{t}]$, $\mathrm{N} > 0$). Write $\check{\mathfrak{g}} = \mathfrak{g} \otimes_{\Bbbk} R$, and similarly for the subalgebras of $\mathfrak{g}$. Then $\check{\mathfrak{g}}$ is a $\Bbbk$-Lie algebra with Lie bracket

$$[x \otimes r, y \otimes s] = [x, y] \otimes rs, \quad x, y \in \mathfrak{g}, \; r, s \in R,$$

and contains $\mathfrak{g}$ as a subalgebra via $x \mapsto x \otimes 1$. Moreover, $\check{\mathfrak{g}} = \check{\mathfrak{g}}_- \oplus \check{\mathfrak{h}} \oplus \check{\mathfrak{g}}_+$, and $\mathfrak{h}_0 \subset \check{\mathfrak{h}}$ are non-zero abelian subalgebras of $\check{\mathfrak{g}}$. The subalgebra $\check{\mathfrak{g}}_+$ is a weight module for $\mathfrak{h}_0$ with weights coincident with the weights Δ_+ of the $\mathfrak{h}_0$-module $\mathfrak{g}_+$, and $(\check{\mathfrak{g}}_+)^\alpha = (\mathfrak{g}_+^\alpha)\check{}$. So $\mathfrak{g}$ and $\check{\mathfrak{g}}$ share the same roots Δ and root lattices $\mathcal{Q}$, $\mathcal{Q}_+$. The anti-involution ω of $\check{\mathfrak{g}}$ is given by R-linear extension

$$\omega : x \otimes r \mapsto \omega(x) \otimes r, \quad x \in \mathfrak{g}, \; r \in R,$$

and fixes $\check{\mathfrak{h}}$ point-wise. Thus $(\check{\mathfrak{g}}, \mathfrak{h}_0, \check{\mathfrak{h}}, \check{\mathfrak{g}}_+, \omega)$ is a $\Bbbk$-Lie algebra with triangular decomposition.

2. Highest-Weight Representation Theory

Throughout this section, let $(\mathfrak{g}, \mathfrak{h}_0, \mathfrak{h}, \mathfrak{g}_+, \omega)$ denote a Lie algebra with triangular decomposition. The universal highest-weight modules of $\mathfrak{g}$, called *Verma modules*, exist and possess the usual properties. An extensive treatment of Verma modules and the Shapovalov form can be found in [**24**]; we present only the definitions and the most important properties.

2.1. Highest-weight modules. A $\mathfrak{g}$-module M is *weight* if the action of $\mathfrak{h}_0$ on M is diagonalisable, i.e.

$$M = \bigoplus\nolimits_{\chi \in \mathfrak{h}_0^*} M^\chi, \qquad h|_{M^\chi} = \chi(h) \text{ for all } h \in \mathfrak{h}_0, \chi \in \mathfrak{h}_0^*. \tag{2.1}$$

The decomposition (2.1) is called the *weight-space decomposition* of M; the components M^χ are called *weight spaces*. The *support* of a weight module M is the set

$$\{ \chi \in \mathfrak{h}_0^* \mid M^\chi \neq 0 \} \subset \mathfrak{h}_0^*.$$

For any $\chi \in \mathfrak{h}_0^*$, an element $v \in M^\chi$ is a *primitive vector* of M if the submodule $\mathcal{U}(\mathfrak{g}) \cdot v \subset M$ is proper. Clearly M is reducible if and only if M has a non-zero primitive vector. A non-zero vector $v \in M$ is a *highest-weight vector* if

i. $\mathfrak{g}_+ \cdot v = 0$;
ii. there exists $\Lambda \in \mathfrak{h}^*$ such that $h \cdot v = \Lambda(h)v$, for all $h \in \mathfrak{h}$.

The unique functional $\Lambda \in \mathfrak{h}^*$ is called the *highest weight* of the highest-weight vector v. A weight $\mathfrak{g}$-module M is called *highest weight* (of highest weight Λ) if there exists a highest-weight vector $v \in M$ (of highest weight Λ) that generates it.

Proposition 2.2. Suppose that M is a highest-weight $\mathfrak{g}$-module, generated by a highest-weight vector $v \in M$ of highest weight $\Lambda \in \mathfrak{h}^*$. Then

i. the support of M is contained in $\Lambda|_{\mathfrak{h}_0} - \mathcal{Q}_+$;
ii. $M^{\Lambda|_{\mathfrak{h}_0}} = \mathbb{k}v$, and all weight spaces of M are finite-dimensional;
iii. M is indecomposable, and has a unique maximal submodule;
iv. if $u \in M$ is a highest-weight vector of highest-weight $\Lambda' \in \mathfrak{h}^*$, and u generates M, then $\Lambda' = \Lambda$ and u is proportional to v.

Let $\Lambda \in \mathfrak{h}^*$, and consider the one-dimensional vector space $\mathbb{k}\mathrm{v}_\Lambda$ as an $(\mathfrak{h} \oplus \mathfrak{g}_+)$-module via

$$\mathfrak{g}_+ \cdot \mathrm{v}_\Lambda = 0; \quad h \cdot \mathrm{v}_\Lambda = \Lambda(h)\mathrm{v}_\Lambda, \ h \in \mathfrak{h}.$$

The induced module

$$\mathfrak{V}(\Lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h}\oplus\mathfrak{g}_+)} \mathbb{k}\mathrm{v}_\Lambda$$

is called the *Verma module* of highest-weight Λ.

Proposition 2.3. For any $\Lambda \in \mathfrak{h}^*$,

i. Up to scalar multiplication, there is a unique epimorphism from $\mathfrak{V}(\Lambda)$ to any highest-weight module of highest-weight Λ, i.e. $\mathfrak{V}(\Lambda)$ is the universal highest-weight module of highest-weight Λ;
ii. $\mathfrak{V}(\Lambda)$ is a free rank one $\mathcal{U}(\mathfrak{g}_-)$-module.

2.2. The Shapovalov Form. The Shapovalov form is a contragredient symmetric bilinear form on $\mathcal{U}(\mathfrak{g})$ with values in $\mathcal{U}(\mathfrak{h}) = \mathrm{S}(\mathfrak{h})$. The evaluation of the Shapovalov form at $\Lambda \in \mathfrak{h}^*$ is a $\mathbb{k}$-valued bilinear form, and is degenerate if and only if the Verma module

$\mathfrak{V}(\Lambda)$ is reducible. By the Leibniz rule, $\mathcal{U}(\mathfrak{g})$ is a weight $\mathfrak{g}$-module, with weight-space decomposition

$$\mathcal{U}(\mathfrak{g}) = \bigoplus_{\gamma \in \mathcal{Q}} \mathcal{U}(\mathfrak{g})^{\gamma}.$$

The anti-involution ω of $\mathfrak{g}$ extends uniquely to an anti-involution of $\mathcal{U}(\mathfrak{g})$ (denoted identically), and is such that

$$\omega : \mathcal{U}(\mathfrak{g})^{\gamma} \to \mathcal{U}(\mathfrak{g})^{-\gamma}, \quad \gamma \in \mathcal{Q}.$$

It follows from the Poincaré-Birkhoff-Witt (PBW) Theorem that $\mathcal{U}(\mathfrak{g})$ may be decomposed

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h}) \oplus \{\mathfrak{g}_{-}\mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{g}_{+}\}$$

as a direct sum of vector spaces. Further, both summands are two-sided $\mathcal{U}(\mathfrak{h})$-modules preserved by ω. Let $\mathbf{q} : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h})$ denote the projection onto the first summand parallel to the second; the restriction $\mathbf{q}|_{\mathcal{U}(\mathfrak{g})^0}$ is an algebra homomorphism. Define

$$\mathbf{F} : \mathcal{U}(\mathfrak{g}) \times \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h}) \quad \text{via} \quad \mathbf{F}(x, y) = \mathbf{q}(\omega(x)y), \quad x, y \in \mathcal{U}(\mathfrak{g}).$$

The bilinear form $\mathbf{F}$ is called the *Shapovalov form*; we consider its restriction

$$\mathbf{F} : \mathcal{U}(\mathfrak{g}_{-}) \times \mathcal{U}(\mathfrak{g}_{-}) \to \mathcal{U}(\mathfrak{h}).$$

Distinct $\mathfrak{h}_0$-weight spaces of $\mathcal{U}(\mathfrak{g}_{-})$ are orthogonal with respect to $\mathbf{F}$, and so the study of $\mathbf{F}$ on $\mathcal{U}(\mathfrak{g}_{-})$ reduces to the study of the restrictions

$$\mathbf{F}_{\chi} : \mathcal{U}(\mathfrak{g}_{-})^{-\chi} \times \mathcal{U}(\mathfrak{g}_{-})^{-\chi} \to \mathcal{U}(\mathfrak{h}), \quad \chi \in \mathcal{Q}_{+}.$$

Any $\Lambda \in \mathfrak{h}^*$ extends uniquely to a map $\mathcal{U}(\mathfrak{h}) \to \Bbbk$; write $\mathbf{F}_{\chi}(\Lambda)$ for the composition of $\mathbf{F}_{\chi}$ with this extension, and write $\mathrm{Rad}\,\mathbf{F}_{\chi}(\Lambda)$ for its radical. The importance of the Shapovalov form stems from the following fact.

Proposition 2.4. Let $\chi \in \mathcal{Q}_{+}$, $\Lambda \in \mathfrak{h}^*$. Then $\mathrm{Rad}\,\mathbf{F}_{\chi}(\Lambda) \subset \mathfrak{V}(\Lambda)^{\Lambda|_{\mathfrak{h}_0} - \chi}$ is the $\Lambda|_{\mathfrak{h}_0} - \chi$ weight space of the maximal submodule of the Verma module $\mathfrak{V}(\Lambda)$.

In particular, a Verma module $\mathfrak{V}(\Lambda)$ is irreducible if and only if the forms $\mathbf{F}_{\chi}(\Lambda)$ are non-degenerate for every $\chi \in \mathcal{Q}_{+}$. Thus an understanding of the forms $\mathbf{F}_{\chi}$, $\chi \in \mathcal{Q}_{+}$, is an understanding of the irreducibility criterion of the Verma modules of the highest-weight theory.

2.3. Partitions and the Poincaré-Birkhoff-Witt Monomials. Let $\mathcal{C}$ be a set parameterizing a root-basis (i.e. an $\mathfrak{h}_0$-weight basis) of $\mathfrak{g}_+$, via

$$\mathcal{C} \ni \quad \gamma \quad \leftrightarrow \quad \mathrm{x}(\gamma) \quad \in \mathfrak{g}_+.$$

Define $\Delta : \mathcal{C} \to \Delta_+$ by declaring $\mathrm{x}(\gamma) \in \mathfrak{g}_+^{\Delta(\gamma)}$, for all $\gamma \in \mathcal{C}$. A *partition* is a finite multiset with elements from $\mathcal{C}$; write $\mathcal{P}$ for the set of all partitions. Set notation is used for multisets throughout. The *length* $|\lambda|$ of a partition $\lambda \in \mathcal{P}$ is the number of elements of λ, counting all repetition. Fix some ordering of the basis $\{\,\mathrm{x}(\gamma) \mid \gamma \in \mathcal{C}\,\}$ of $\mathfrak{g}_+$; for any $\lambda \in \mathcal{P}$, let

$$\mathrm{x}(\lambda) = \mathrm{x}(\lambda_1)\cdots\mathrm{x}(\lambda_k) \quad \in \mathcal{U}(\mathfrak{g}_+) \tag{2.5}$$

where $k = |\lambda|$ and $(\lambda_i)_{1\leqslant i\leqslant k}$ is an enumeration of the entries of λ such that (2.5) is a PBW monomial with respect to the basis ordering. For any partition $\lambda \in \mathcal{P}$, write $\mathrm{y}(\lambda) = \omega(\mathrm{x}(\lambda))$. By the PBW Theorem, the spaces $\mathcal{U}(\mathfrak{g}_+)$, $\mathcal{U}(\mathfrak{g}_-)$ have bases

$$\{\,\mathrm{x}(\lambda) \mid \lambda \in \mathcal{P}\,\}, \qquad \{\,\mathrm{y}(\lambda) \mid \lambda \in \mathcal{P}\,\},$$

respectively. For any partition $\lambda \in \mathcal{P}$ and positive root $\alpha \in \Delta_+$, write

$$\Delta(\lambda) = \sum_{\gamma\in\lambda} \Delta(\gamma); \qquad \lambda^\alpha = \{\,\gamma \in \lambda \mid \Delta(\gamma) = \alpha\,\}.$$

2.4. Shapovalov's Lemma. The proof of the following useful lemma is elementary.

Lemma 2.6. Suppose that $\lambda \in \mathcal{P}$, that $|\lambda| = r$, that $(\lambda_i)_{1\leqslant i\leqslant r}$ is an enumeration of λ and that $\tau \in \mathrm{Sym}(r)$. Then

$$\mathrm{x}(\lambda_1)\cdots\mathrm{x}(\lambda_r) = \mathrm{x}(\lambda_{\tau(1)})\cdots\mathrm{x}(\lambda_{\tau(r)}) + R$$

where R is a linear combination of terms $\mathrm{x}(\phi_1)\cdots\mathrm{x}(\phi_s)$ where $\phi_i \in \mathcal{C}$ for $1 \leqslant i \leqslant s$ and $s < r$.

The following Lemma is due to Shapovalov [**30**]. Our proof follows that of an analogous statement in [**24**].

Lemma 2.7. Let $(\mathfrak{g}, \mathfrak{h}_0, \mathfrak{h}, \mathfrak{g}_+, \omega)$ be a Lie algebra with triangular decomposition. Suppose that $\lambda, \mu \in \mathcal{P}$, that $|\lambda| = r$ and $|\mu| = s$, and that $(\lambda_i)_{1\leqslant i\leqslant r}$ and $(\mu_i)_{1\leqslant i\leqslant s}$ are arbitrary enumerations of λ and μ, respectively. Let

$$Z = \mathrm{x}(\lambda_r)\cdots\mathrm{x}(\lambda_1)\mathrm{y}(\mu_1)\cdots\mathrm{y}(\mu_s).$$

Then

i. $\deg_{\mathfrak{h}} \mathbf{q}(Z) \leqslant r, s$;
ii. if $r = s$, but $|\lambda^\alpha| \neq |\mu^\alpha|$ for some $\alpha \in \Delta_+$, then $\deg_{\mathfrak{h}} \mathbf{q}(Z) < r = s$;
iii. if $r = s$ and $|\lambda^\alpha| = |\mu^\alpha| =: m_\alpha$ for all $\alpha \in \Delta_+$, then the degree $r = s$ term of $\mathbf{q}(Z)$ is

$$\prod_{\alpha \in \Delta_+} \sum_{\tau \in \mathrm{Sym}(m_\alpha)} \prod_{1 \leqslant j \leqslant m_\alpha} [\mathrm{x}(\lambda^\alpha_{\tau(j)}), \mathrm{y}(\mu^\alpha_j)],$$

where for each $\alpha \in \Delta_+$, $(\lambda^\alpha_j)_{1 \leqslant j \leqslant m_\alpha}$, $(\mu^\alpha_j)_{1 \leqslant j \leqslant m_\alpha}$ are any fixed enumerations of λ^α and μ^α respectively.

Proof. The proof is by induction on $|\lambda| + |\mu|$. It is straightforward to show that all three parts hold whenever $|\lambda| = 0$ or $|\mu| = 0$. Suppose then that all three parts hold for all $\lambda', \mu' \in \mathcal{P}$ such that $|\lambda'| + |\mu'| < |\lambda| + |\mu|$. Let $\varsigma \in \mathrm{Sym}(r), \tau \in \mathrm{Sym}(s)$, and write

$$Z' = \mathrm{x}(\lambda_{\varsigma(r)}) \cdots \mathrm{x}(\lambda_{\varsigma(1)}) \mathrm{y}(\mu_{\tau(1)}) \cdots \mathrm{y}(\mu_{\tau(s)}).$$

By Lemma 2.6, $Z = Z' + R$, where R is a linear combination of terms

$$\mathrm{x}(\phi_{r'}) \cdots \mathrm{x}(\phi_1) \mathrm{y}(\psi_1) \cdots \mathrm{y}(\psi_{s'})$$

with $r' < r$ or $s' < s$. Therefore, by inductive hypothesis, the Lemma holds for arbitrary enumerations of λ and μ, if it holds for any particular pair of enumerations. Consider Δ_+ to carry some linearisation of its usual partial order, and choose any enumerations of λ, μ such that

$$\Delta(\lambda_1) \leqslant \cdots \leqslant \Delta(\lambda_r) \quad \text{and} \quad \Delta(\mu_1) \leqslant \cdots \leqslant \Delta(\mu_s).$$

Moreover, as

$$\begin{aligned} \mathbf{q}(\mathrm{x}(\lambda_r) \cdots \mathrm{x}(\lambda_1) \mathrm{y}(\mu_1) \cdots \mathrm{y}(\mu_s)) &= \mathbf{q}(\omega(\mathrm{x}(\lambda_r) \cdots \mathrm{x}(\lambda_1) \mathrm{y}(\mu_1) \cdots \mathrm{y}(\mu_s))) \\ &= \mathbf{q}(\mathrm{x}(\mu_s) \cdots \mathrm{x}(\mu_1) \mathrm{y}(\lambda_1) \cdots \mathrm{y}(\lambda_r)), \end{aligned}$$

it may supposed without loss of generality that $\Delta(\mu_1) \leqslant \Delta(\lambda_1)$. Now

$$\begin{aligned} \mathbf{q}(Z) &= \mathbf{q}(\mathrm{x}(\lambda_r) \cdots \mathrm{x}(\lambda_1) \mathrm{y}(\mu_1) \cdots \mathrm{y}(\mu_s)) \\ &= \mathbf{q}([\mathrm{x}(\lambda_r) \cdots \mathrm{x}(\lambda_1), \mathrm{y}(\mu_1)] \mathrm{y}(\mu_2) \cdots \mathrm{y}(\mu_s)) \\ &= \sum_{i=1}^{r} \mathbf{q}(A_i), \end{aligned}$$

where, by the Leibniz rule,

$$A_i = \mathrm{x}(\lambda_r) \cdots \mathrm{x}(\lambda_{i+1}) [\mathrm{x}(\lambda_i), \mathrm{y}(\mu_1)] \mathrm{x}(\lambda_{i-1}) \cdots \mathrm{x}(\lambda_1) \mathrm{y}(\mu_2) \cdots \mathrm{y}(\mu_s),$$

for $1 \leqslant i \leqslant r$. Let $0 \leqslant k \leqslant r$ be maximal such that $\Delta(\lambda_i) = \Delta(\mu_1)$ for all $1 \leqslant i \leqslant k$; the terms A_i with $1 \leqslant i \leqslant k$ and $k < i \leqslant r$ are to be considered separately. If $1 \leqslant i \leqslant k$, then $[\,\mathrm{x}(\lambda_i), \mathrm{y}(\mu_1)\,] \in \mathfrak{h}$. Therefore, by the Leibniz rule,

$$A_i = Z_i[\,\mathrm{x}(\lambda_i), \mathrm{y}(\mu_1)\,] + R_i,$$

where $Z_i = \mathrm{x}(\lambda_r) \cdots \mathrm{x}(\lambda_{i+1})\mathrm{x}(\lambda_{i-1}) \cdots \mathrm{x}(\lambda_1)\mathrm{y}(\mu_2) \cdots \mathrm{y}(\mu_s)$ and R_i is a linear combination of terms

$$\mathrm{x}(\phi_{r-1}) \cdots \mathrm{x}(\phi_1)\mathrm{y}(\psi_1) \cdots \mathrm{y}(\psi_{s-1}), \quad \phi_1, \ldots, \phi_{r-1}, \psi_1, \ldots, \psi_{s-1} \in \mathcal{C}.$$

If instead $k < i \leqslant r$, then A_i is a linear combination of terms

$$\mathrm{x}(\lambda_r) \cdots \mathrm{x}(\lambda_{i+1})\mathrm{x}(\gamma)\mathrm{x}(\lambda_{i-1}) \cdots \mathrm{x}(\lambda_1)\mathrm{y}(\mu_2) \cdots \mathrm{y}(\mu_s),$$

where $\Delta(\gamma) = \Delta(\lambda_i) - \Delta(\mu_1) \in \Delta_+$, since $\Delta(\mu_1) \leqslant \Delta(\lambda_1) \leqslant \Delta(\lambda_i)$.

Note that $\deg_{\mathfrak{h}}\mathbf{q}(Z) \leqslant \max\{\,\deg_{\mathfrak{h}}\mathbf{q}(A_i)\,\}$. Consider now each of the three parts of the claim.

Part (i). For $1 \leqslant i \leqslant k$,

$$\mathbf{q}(A_i) = \mathbf{q}(Z_i)[\,\mathrm{x}(\lambda_i), \mathrm{y}(\mu_1)\,] + \mathbf{q}(R_i), \tag{2.8}$$

since $\mathbf{q}|_{\mathcal{U}(\mathfrak{g})^0}$ is an algebra homomorphism. By part (i) of the inductive hypothesis,

$$\deg_{\mathfrak{h}}\mathbf{q}(Z_i),\ \deg_{\mathfrak{h}}\mathbf{q}(R_i) \leqslant r-1,\ s-1,$$

and so $\deg_{\mathfrak{h}}\mathbf{q}(A_i) \leqslant r, s$. For $k < i \leqslant r$, again by part (i) of the inductive hypothesis, $\deg_{\mathfrak{h}}\mathbf{q}(A_i) \leqslant r,\ s-1$. Hence $\deg_{\mathfrak{h}}\mathbf{q}(Z) \leqslant r, s$, and so part (i) holds.

Part (ii). Suppose that $r = s$, and let $\alpha \in \Delta_+$ be such that $|\lambda^\alpha| \neq |\mu^\alpha|$. For $1 \leqslant i \leqslant k$, consider $\mathbf{q}(A_i)$ by equation (2.8). By part (i) of the inductive hypothesis,

$$\deg_{\mathfrak{h}}\mathbf{q}(R_i) \leqslant r-1 < r,$$

and so it remains only to consider $\mathbf{q}(Z_i)$. Write λ' (respectively, μ') for the partition consisting of the components of λ (respectively, μ) except for λ_i (respectively, μ_1). Then $|\lambda'| = |\mu'|$ and $|\lambda'^\alpha| \neq |\mu'^\alpha|$. Therefore, by part (ii) of the inductive hypothesis, $\deg_{\mathfrak{h}}\mathbf{q}(Z_i) < r-1$; hence $\deg_{\mathfrak{h}}\mathbf{q}(A_i) < r$. For $k < i \leqslant r$, part (i) of the inductive hypothesis implies that $\deg_{\mathfrak{h}}\mathbf{q}(A_i) \leqslant s-1 < r$. Therefore $\deg_{\mathfrak{h}}\mathbf{q}(Z) < r$, as required.

Part (iii). Suppose that $r = s$ and that $|\lambda^\alpha| = |\mu^\alpha|$ for all $\alpha \in \Delta_+$. Observe that for $k < i \leqslant r$, part (i) of the inductive hypothesis implies that

$$\deg_{\mathfrak{h}}\mathbf{q}(A_i) \leqslant s-1 < r;$$

and that for $1 \leqslant i \leqslant k$, by the same,

$$\deg_{\mathfrak{h}} \mathbf{q}(R_i) \leqslant r - 1 < r.$$

Therefore, the terms $\mathbf{q}(A_i)$ for $k < i \leqslant r$ and the terms $\mathbf{q}(R_i)$ for $1 \leq i \leq k$ can not contribute to the degree-r component of $\mathbf{q}(Z)$; thus the degree-r component of $\mathbf{q}(Z)$ is the degree-r component of

$$\sum_{i=1}^{k} \mathbf{q}(Z_i)[\mathrm{x}(\lambda_i), \mathrm{y}(\mu_1)].$$

As Z_i satisfies the conditions of part (iii) of the inductive hypothesis, for $1 \leqslant i \leqslant k$, the formula follows. $\square$

CHAPTER 4

Highest-Weight Theory for Truncated Current Lie Algebras

In this chapter, the highest-weight theory for truncated current Lie algebras is extensively studied, culminating in a reducibility criterion for the Verma modules. References to material from Chapter 3 are distinguished by the specification of a page number in parentheses. The notations of that chapter are used throughout. In particular, $\Bbbk$ is any field of characteristic zero.

1. Truncated Current Lie Algebras

Let $(\mathfrak{g}, \mathfrak{h}_0, \mathfrak{h}, \mathfrak{g}_+, \omega)$ be a Lie algebra with triangular decomposition, and let $\mathcal{C}$ denote a set parameterizing a root-basis for $\mathfrak{g}_+$. Fix a positive integer N, and let

$$\breve{\mathfrak{g}} = \mathfrak{g} \otimes \Bbbk[\mathrm{t}]/\mathrm{t}^{\mathrm{N}+1}\Bbbk[\mathrm{t}]$$

denote the associated truncated current Lie algebra with the triangular decomposition of Example 1.6 (page 35). The integer N is the *nilpotency index* of $\breve{\mathfrak{g}}$. Let $\hat{\mathcal{C}} = \mathcal{C} \times \{0, \ldots, \mathrm{N}\}$. Then $\hat{\mathcal{C}}$ parameterises a basis for $\breve{\mathfrak{g}}_+$ consisting of $\mathfrak{h}_0$-weight vectors of homogeneous degree in t, via

$$\hat{\mathcal{C}} \ni \quad \gamma \quad \leftrightarrow \quad \mathrm{x}(\gamma) \quad \in \breve{\mathfrak{g}}_+,$$

where $\mathrm{x}(\gamma) = \mathrm{x}(\tau) \otimes \mathrm{t}^d$ if $\gamma = (\tau, d) \in \hat{\mathcal{C}}$. Define

$$\Delta : \hat{\mathcal{C}} \to \Delta_+, \qquad \deg_{\mathrm{t}} : \hat{\mathcal{C}} \to \{0, \ldots, \mathrm{N}\}$$

via $\mathrm{x}(\gamma) \in \mathfrak{g}^{\Delta(\gamma)} \otimes \mathrm{t}^{\deg_{\mathrm{t}}(\gamma)}$ for all $\gamma \in \hat{\mathcal{C}}$. Order the basis $\{\mathrm{x}(\gamma) \mid \gamma \in \hat{\mathcal{C}}\}$ of $\breve{\mathfrak{g}}_+$ by fixing an arbitrary linearisation of the partial order by increasing homogeneous degree in t, i.e. so that

$$\deg_{\mathrm{t}}(\gamma) < \deg_{\mathrm{t}}(\gamma') \quad \Rightarrow \quad \mathrm{x}(\gamma) < \mathrm{x}(\gamma'), \qquad \gamma, \gamma' \in \hat{\mathcal{C}}.$$

As per Subsection 2.3 (page 38), the PBW basis monomials of $\mathcal{U}(\breve{\mathfrak{g}}_+)$ with respect to this ordered basis are parameterised by a collection $\mathcal{P}$ of partitions. Partitions here are (finite) multisets with elements from $\hat{\mathcal{C}}$. For any $\chi \in \mathcal{Q}_+$, let

$$\mathcal{P}_\chi = \{\, \lambda \in \mathcal{P} \mid \Delta(\lambda) = \chi \,\}.$$

For any $0 \leqslant d \leqslant \mathrm{N}$ and $\lambda \in \mathcal{P}$, define

$$\lambda^d = \{\, \gamma \in \lambda \mid \deg_{\mathrm{t}} \gamma = d \,\};$$

λ is homogeneous of degree-d in t if $\lambda = \lambda^d$. The ordering of the basis of $\breve{\mathfrak{g}}_+$ is such that for all $\lambda \in \mathcal{P}$,

$$\mathrm{x}(\lambda) = \mathrm{x}(\lambda^0)\mathrm{x}(\lambda^1)\cdots\mathrm{x}(\lambda^{\mathrm{N}}), \qquad \mathrm{y}(\lambda) = \mathrm{y}(\lambda^{\mathrm{N}})\cdots\mathrm{y}(\lambda^1)\mathrm{y}(\lambda^0).$$

For any $\Lambda \in \breve{\mathfrak{h}}^*$ and $0 \leqslant d \leqslant \mathrm{N}$, let $\Lambda_d \in \mathfrak{h}^*$ be given by

$$\langle \Lambda_d, h \rangle = \langle \Lambda, h \otimes \mathrm{t}^d \rangle, \qquad h \in \mathfrak{h}.$$

1.1. The Shapovalov form. As in Section 2 (page 35), there is a decomposition

$$\mathcal{U}(\breve{\mathfrak{g}}) = \mathcal{U}(\breve{\mathfrak{h}}) \oplus \{\breve{\mathfrak{g}}_- \mathcal{U}(\breve{\mathfrak{g}}) + \mathcal{U}(\breve{\mathfrak{g}})\breve{\mathfrak{g}}_+\},$$

as a direct sum of two-sided $\mathcal{U}(\breve{\mathfrak{g}})^0$-modules. Denote by $\mathbf{q} : \mathcal{U}(\breve{\mathfrak{g}}) \to \mathcal{U}(\breve{\mathfrak{h}})$ the projection onto the first summand, parallel to the second. Let

$$\mathbf{F} : \mathcal{U}(\breve{\mathfrak{g}}) \times \mathcal{U}(\breve{\mathfrak{g}}) \to \mathcal{U}(\breve{\mathfrak{h}})$$

denote the Shapovalov form, and write $\mathbf{F}_\chi$ for the restriction of $\mathbf{F}$ to the subspace $\mathcal{U}(\breve{\mathfrak{g}}_-)^{-\chi}$, $\chi \in \mathcal{Q}_+$.

The algebra $\mathcal{U}(\breve{\mathfrak{g}}) = \bigoplus_{m \geqslant 0} \mathcal{U}(\breve{\mathfrak{g}})_m$ is graded by total degree in the indeterminate t,

$$\mathcal{U}(\breve{\mathfrak{g}})_m = \operatorname{span}\{\, (x_1 \otimes \mathrm{t}^{d_1}) \cdots (x_k \otimes \mathrm{t}^{d_k}) \mid \sum_{i=1}^{k} d_i = m, \quad k \geqslant 0 \,\}.$$

For any subspace $V \subset \mathcal{U}(\breve{\mathfrak{g}})$, let

$$V_m = \mathcal{U}(\breve{\mathfrak{g}})_m \cap V, \qquad m \geqslant 0,$$

and call V *graded* in t if $V = \bigoplus_{m \geqslant 0} V_m$. The subalgebras $\mathcal{U}(\breve{\mathfrak{g}}_+)$, $\mathcal{U}(\breve{\mathfrak{g}}_-)$, and $\mathcal{U}(\breve{\mathfrak{h}})$ are graded in t.

Lemma 1.1. For any $m \geqslant 0$, $\mathbf{q}(\mathcal{U}(\breve{\mathfrak{g}})_m) \subset \mathcal{U}(\breve{\mathfrak{h}})_m$.

Proof. The spaces $\breve{\mathfrak{g}}_- \mathcal{U}(\breve{\mathfrak{g}})$ and $\mathcal{U}(\breve{\mathfrak{g}})\breve{\mathfrak{g}}_+$ are graded in t; hence so is the sum

$$\{\breve{\mathfrak{g}}_- \mathcal{U}(\breve{\mathfrak{g}}) + \mathcal{U}(\breve{\mathfrak{g}})\breve{\mathfrak{g}}_+\}.$$

Therefore,

$$\mathcal{U}(\breve{\mathfrak{g}})_m = \mathcal{U}(\breve{\mathfrak{h}})_m \oplus \{\breve{\mathfrak{g}}_- \mathcal{U}(\breve{\mathfrak{g}}) + \mathcal{U}(\breve{\mathfrak{g}})\breve{\mathfrak{g}}_+\}_m,$$

for any $m \geqslant 0$. □

Example 1.2. Let $\mathfrak{g} = \mathrm{sl}(3)$, and recall the notation of Example 1.2 (page 32). Let $\mathrm{N} = 1$, so that $\breve{\mathfrak{g}} = \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathrm{t})$. Write $\mathcal{C} = \Delta_+$ and $\hat{\mathcal{C}} = \mathcal{C} \times \{0, 1\}$. Then $\breve{\mathfrak{g}}_+$ has a basis parameterised by $\hat{\mathcal{C}}$:

$$\hat{\mathcal{C}} \ni \quad (\alpha, d) \quad \leftrightarrow \quad \mathrm{x}(\alpha) \otimes \mathrm{t}^d \quad \in \breve{\mathfrak{g}}_+.$$

Let $\chi = \alpha_1 + \alpha_2$. Then $\mathcal{P}_\chi$ consists of the six partitions

$$\begin{array}{lll} \{ (\alpha_1, 0), (\alpha_2, 0) \}, & \{ (\alpha_1 + \alpha_2, 0) \}, & \{ (\alpha_1, 0), (\alpha_2, 1) \}, \\ \{ (\alpha_1, 1), (\alpha_2, 0) \}, & \{ (\alpha_1 + \alpha_2, 1) \}, & \{ (\alpha_1, 1), (\alpha_2, 1) \}. \end{array} \tag{1.3}$$

Order the set $\{ \mathrm{x}(\gamma) \mid \gamma \in \hat{\mathcal{C}} \}$ by the enumeration

$$\mathrm{x}(\alpha_1) \otimes \mathrm{t}^0, \quad \mathrm{x}(\alpha_1 + \alpha_2) \otimes \mathrm{t}^0, \quad \mathrm{x}(\alpha_2) \otimes \mathrm{t}^0, \quad \mathrm{x}(\alpha_1) \otimes \mathrm{t}^1, \quad \mathrm{x}(\alpha_1 + \alpha_2) \otimes \mathrm{t}^1, \quad \mathrm{x}(\alpha_2) \otimes \mathrm{t}^1.$$

Then the PBW basis monomials of $\mathcal{U}(\breve{\mathfrak{g}}_-)^{-\chi}$ corresponding to the partitions (1.3) are, respectively,

$$\begin{array}{lll} \mathrm{y}(\alpha_2) \otimes \mathrm{t}^0 \cdot \mathrm{y}(\alpha_1) \otimes \mathrm{t}^0, & \mathrm{y}(\alpha_1 + \alpha_2) \otimes \mathrm{t}^0, & \mathrm{y}(\alpha_2) \otimes \mathrm{t}^1 \cdot \mathrm{y}(\alpha_1) \otimes \mathrm{t}^0, \\ \mathrm{y}(\alpha_1) \otimes \mathrm{t}^1 \cdot \mathrm{y}(\alpha_2) \otimes \mathrm{t}^0, & \mathrm{y}(\alpha_1 + \alpha_2) \otimes \mathrm{t}^1, & \mathrm{y}(\alpha_2) \otimes \mathrm{t}^1 \cdot \mathrm{y}(\alpha_1) \otimes \mathrm{t}^1. \end{array} \tag{1.4}$$

For notational convenience, write $\mathrm{h}_{\alpha_i,j} = \mathrm{h}(\alpha_i) \otimes \mathrm{t}^j$, for $i = 1, 2$ and $j = 0, 1$. The restriction $\mathbf{F}_\chi$ of the Shapovalov form, expressed as a matrix with respect to the ordered basis (1.4), appears below.

$$\begin{pmatrix} \mathrm{h}_{\alpha_1+\alpha_2,0} + \mathrm{h}_{\alpha_1,0} & \mathrm{h}_{\alpha_1,0} & \mathrm{h}_{\alpha_1,0}\mathrm{h}_{\alpha_2,1} + \mathrm{h}_{\alpha_1,1} & \mathrm{h}_{\alpha_1,1}(\mathrm{h}_{\alpha_2,0} + 2) & \mathrm{h}_{\alpha_1,1} & \mathrm{h}_{\alpha_1,1}\mathrm{h}_{\alpha_2,1} \\ \mathrm{h}_{\alpha_1,0} & \mathrm{h}_{\alpha_1+\alpha_2,0} & \mathrm{h}_{\alpha_1,1} & -\mathrm{h}_{\alpha_2,1} & \mathrm{h}_{\alpha_1+\alpha_2,1} & 0 \\ \mathrm{h}_{\alpha_1,0}\mathrm{h}_{\alpha_2,1} + \mathrm{h}_{\alpha_1,1} & \mathrm{h}_{\alpha_1,1} & 0 & \mathrm{h}_{\alpha_1,1}\mathrm{h}_{\alpha_2,1} & 0 & 0 \\ \mathrm{h}_{\alpha_1,1}(\mathrm{h}_{\alpha_2,0} + 2) & -\mathrm{h}_{\alpha_2,1} & \mathrm{h}_{\alpha_1,1}\mathrm{h}_{\alpha_2,1} & 0 & 0 & 0 \\ \mathrm{h}_{\alpha_1,1} & \mathrm{h}_{\alpha_1+\alpha_2,1} & 0 & 0 & 0 & 0 \\ \mathrm{h}_{\alpha_1,1}\mathrm{h}_{\alpha_2,1} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This is an elementary calculation using the commutation relations. Observe that this matrix is triangular, and that in particular the determinant (the Shapovalov determinant at χ) must be the product of the diagonal entries, viz.,

$$(\mathrm{h}(\alpha_1) \otimes \mathrm{t}^1)^4 (\mathrm{h}(\alpha_2) \otimes \mathrm{t}^1)^4 (\mathrm{h}(\alpha_1 + \alpha_2) \otimes \mathrm{t}^1)^2, \tag{1.5}$$

up to sign. This provides a criterion for the existence of primitive vectors in the weight space $\Lambda|_{\mathfrak{h}_0} - \chi$ of a Verma module $\mathfrak{V}(\Lambda)$, $\Lambda \in \breve{\mathfrak{h}}^*$. We shall prove that the Shapovalov determinant always lies in $\mathrm{S}(\mathfrak{h} \otimes \mathrm{t}^{\mathrm{N}})$, and that for $\mathfrak{g}$ a semisimple finite-dimensional Lie algebra, the factors of the Shapovalov determinant are the analogues of those of (1.5).

Example 1.6. Let $\mathfrak{g}$ be the Virasoro/Witt algebra, and adopt the notation of Example 1.4 (page 1.4). Let $\mathrm{N} = 1$, and $\chi = 2\delta$. Write $\mathcal{C} = \Delta_+$ and $\hat{\mathcal{C}} = \mathcal{C} \times \{0, 1\}$. Then $\mathcal{P}_\chi$ consists of the five partitions

$$\{(\delta,0),(\delta,0)\},\quad \{(2\delta,0)\},\quad \{(\delta,0),(\delta,1)\},\quad \{(2\delta,1)\},\quad \{(\delta,1),(\delta,1)\}. \tag{1.7}$$

Order the basis

$$\{\mathrm{x}(\gamma) \mid \gamma \in \hat{\mathcal{C}}\} = \{\mathrm{L}_m \otimes \mathrm{t}^d \mid m > 0,\quad d = 0, 1\}$$

for $\breve{\mathfrak{g}}_+$ firstly by increasing degree d, and secondly by increasing index m. Then the PBW basis monomials of $\mathcal{U}(\breve{\mathfrak{g}}_-)^{-\chi}$ corresponding to (1.7) are, respectively,

$$(\mathrm{L}_{-1} \otimes \mathrm{t}^0)^2,\quad \mathrm{L}_{-2} \otimes \mathrm{t}^0,\quad \mathrm{L}_{-1} \otimes \mathrm{t}^1 \cdot \mathrm{L}_{-1} \otimes \mathrm{t}^0,\quad \mathrm{L}_{-2} \otimes \mathrm{t}^1,\quad (\mathrm{L}_{-1} \otimes \mathrm{t}^1)^2. \tag{1.8}$$

Write $\Omega_{m,i} = (2m\mathrm{L}_0 + \psi(m)\mathrm{c}) \otimes \mathrm{t}^i$, for $m, i \geqslant 0$. The matrix of $\mathbf{F}_\chi$, expressed with respect to the ordered basis (1.8), appears below:

$$\begin{pmatrix} 2\Omega_{1,0}(\Omega_{1,0}+1) & 3\Omega_{1,0} & 2\Omega_{1,1}(\Omega_{1,0}+1) & 3\Omega_{1,1} & 2(\Omega_{1,1})^2 \\ 3\Omega_{1,0} & \Omega_{2,0} & 3\Omega_{1,1} & \Omega_{2,1} & 0 \\ 2\Omega_{1,1}(\Omega_{1,0}+1) & 3\Omega_{1,1} & \Omega_{1,1}^2 & 0 & 0 \\ 3\Omega_{1,1} & \Omega_{2,1} & 0 & 0 & 0 \\ 2(\Omega_{1,1})^2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence the Shapovalov determinant at χ is given by $\det \mathbf{F}_\chi = 4\Omega_{1,1}^6 \Omega_{2,1}^2$.

Example 1.9. Let $\mathfrak{g}$ be the Virasoro/Witt algebra, and adopt the notation of Example 1.4 (page 1.4). Let $\mathrm{N} = 2$, and $\chi = 2\delta$. Write $\mathcal{C} = \Delta_+$ and $\hat{\mathcal{C}} = \mathcal{C} \times \{0, 1, 2\}$. Then $\mathcal{P}_\chi$ consists of the nine partitions

$$\begin{array}{lll} \{(\delta,0),(\delta,0)\}, & \{(2\delta,0)\}, & \{(\delta,0),(\delta,1)\}, \\ \{(\delta,0),(\delta,2)\}, & \{(2\delta,1)\}, & \{(\delta,1),(\delta,1)\}, \\ \{(\delta,1),(\delta,2)\}, & \{(2\delta,2)\}, & \{(\delta,2),(\delta,2)\}. \end{array} \tag{1.10}$$

Order the basis $\{\mathrm{x}(\gamma) \mid \gamma \in \hat{\mathcal{C}}\}$ as per Example 1.6. Then the PBW basis monomials of $\mathcal{U}(\breve{\mathfrak{g}}_-)^{-\chi}$ corresponding to (1.10) are, respectively,

$$\begin{array}{lll} (\mathrm{L}_{-1} \otimes \mathrm{t}^0)^2, & \mathrm{L}_{-2} \otimes \mathrm{t}^0, & \mathrm{L}_{-1} \otimes \mathrm{t}^1 \cdot \mathrm{L}_{-1} \otimes \mathrm{t}^0, \\ \mathrm{L}_{-1} \otimes \mathrm{t}^2 \cdot \mathrm{L}_{-1} \otimes \mathrm{t}^0, & \mathrm{L}_{-2} \otimes \mathrm{t}^1, & (\mathrm{L}_{-1} \otimes \mathrm{t}^1)^2, \\ \mathrm{L}_{-1} \otimes \mathrm{t}^2 \cdot \mathrm{L}_{-1} \otimes \mathrm{t}^1, & \mathrm{L}_{-2} \otimes \mathrm{t}^2, & (\mathrm{L}_{-1} \otimes \mathrm{t}^2)^2. \end{array} \tag{1.11}$$

The matrix of $\mathbf{F}_\chi$ with respect to the ordered basis (1.11) appears on page 48. Notice that the matrix has seven non-zero entries on the diagonal. Hence there is no reordering of the basis (1.11) that will render the matrix triangular.

1.2. A modification of the Shapovalov Form. As observed in Example 1.9, it is not always the case that the matrix for $\mathbf{F}_\chi$, $\chi \in \mathcal{Q}_+$, can be made triangular by an ordering of the chosen PBW monomial basis for $\mathcal{U}(\breve{\mathfrak{g}}_-)^{-\chi}$. A further permutation of columns is necessary; this is performed by an involution * on the partitions, and encapsulated in a modification $\mathbf{B}$ of the Shapovalov form $\mathbf{F}$. For any $\gamma = (\tau, d) \in \hat{\mathcal{C}}$, write $\gamma^\star = (\tau, \mathrm{N} - d) \in \hat{\mathcal{C}}$, and for any $\lambda \in \mathcal{P}$, write

$$\lambda^\star = \{\, \gamma^\star \mid \gamma \in \lambda \,\}.$$

So $(\lambda^d)^\star = (\lambda^\star)^{\mathrm{N}-d}$ for all $\lambda \in \mathcal{P}$ and all degrees d. For any $\chi \in \mathcal{Q}_+$, let

$$\mathbf{B}_\chi : \mathcal{U}(\breve{\mathfrak{g}}_-)^{-\chi} \times \mathcal{U}(\breve{\mathfrak{g}}_-)^{-\chi} \to \mathcal{U}(\breve{\mathfrak{h}})$$

be the bilinear form defined by

$$\mathbf{B}_\chi(\mathrm{y}(\lambda), \mathrm{y}(\mu)) = \mathbf{F}_\chi(\mathrm{y}(\lambda), \mathrm{y}(\mu^\star)), \qquad \lambda, \mu \in \mathcal{P}_\chi.$$

Relative to any linear order of the basis $\{\, \mathrm{y}(\lambda) \mid \lambda \in \mathcal{P}_\chi \}$ of $\mathcal{U}(\breve{\mathfrak{g}}_-)^{-\chi}$, the matrices of $\mathbf{B}_\chi$ and $\mathbf{F}_\chi$ are equal after a reordering of columns determined by the involution *. In particular, the determinants $\det \mathbf{B}_\chi$ and $\det \mathbf{F}_\chi$ are equal up to sign.

$$\begin{pmatrix}
2\Omega_{1,0}(\Omega_{1,0}+1) & 3\Omega_{1,0} & 2\Omega_{1,1}(\Omega_{1,0}+1) & 2\Omega_{1,2}(\Omega_{1,0}+1) & 3\Omega_{1,1} & 2(\Omega_{1,1}^2+\Omega_{1,2}) & 2\Omega_{1,1}\Omega_{1,2} & 3\Omega_{1,2} & 2\Omega_{1,2}^2 \\
3\Omega_{1,0} & \Omega_{2,0} & 3\Omega_{1,1} & 3\Omega_{1,2} & \Omega_{2,1} & 3\Omega_{1,2} & 0 & \Omega_{2,2} & 0 \\
2\Omega_{1,1}(\Omega_{1,0}+1) & 3\Omega_{1,1} & \Omega_{1,2}(\Omega_{1,0}+2)+\Omega_{1,1}^2 & \Omega_{1,1}\Omega_{1,2} & 3\Omega_{1,2} & 2\Omega_{1,1}\Omega_{1,2} & \Omega_{1,2}^2 & 0 & 0 \\
2\Omega_{1,2}(\Omega_{1,0}+1) & 3\Omega_{1,2} & \Omega_{1,1}\Omega_{1,2} & \Omega_{1,2}^2 & 0 & 0 & 0 & 0 & 0 \\
3\Omega_{1,1} & \Omega_{2,1} & 3\Omega_{1,2} & 0 & \Omega_{2,2} & 0 & 0 & 0 & 0 \\
2(\Omega_{1,1}^2+\Omega_{1,2}) & 3\Omega_{1,2} & 2\Omega_{1,1}\Omega_{1,2} & 0 & 0 & 2\Omega_{1,2}^2 & 0 & 0 & 0 \\
2\Omega_{1,1}\Omega_{1,2} & 0 & \Omega_{1,2}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
3\Omega_{1,2} & \Omega_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\Omega_{1,2}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Matrix of the Shapovalov form $\mathbf{F}_\chi$ for the Virasoro/Witt truncated current Lie algebra.

$$\chi = 2\delta, \quad \mathrm{N} = 2, \quad \Omega_{m,i} := (2m\mathrm{L}_0 + \psi(m)\mathrm{c}) \otimes \mathrm{t}^i, \quad m, i \geqslant 0.$$

2. Decomposition of the Shapovalov Form

Throughout this section, let $(\mathfrak{g}, \mathfrak{h}_0, \mathfrak{h}, \mathfrak{g}_+, \omega)$ denote a Lie algebra with triangular decomposition, and let $\breve{\mathfrak{g}}$ denote the associated truncated current Lie algebra of nilpotency index N. Let $\mathcal{L}$ denote the collection of all two-dimensional arrays of non-negative integers with rows indexed by Δ_+ and columns indexed by $\{0, \ldots, \mathrm{N}\}$, with only a finite number of non-zero entries. For any $\chi \in \mathcal{Q}_+$, let

$$\mathcal{L}_\chi = \{ L \in \mathcal{L} \mid \chi = \sum_{\alpha \in \Delta_+} \sum_{0 \leqslant d \leqslant \mathrm{N}} L_{\alpha,d}\, \alpha \}.$$

The entries of an array in $\mathcal{L}_\chi$ specify the multiplicity of each positive root in each homogeneous degree component of a partition of χ, i.e.

$$\lambda \in \mathcal{P}_\chi \quad \Longleftrightarrow \quad (\, |\lambda^{\alpha,d}|\,)_{\substack{\alpha \in \Delta_+, \\ 0 \leqslant d \leqslant \mathrm{N}}} \in \mathcal{L}_\chi.$$

Let

$$\mathcal{P}_L = \{ \lambda \in \mathcal{P} \mid |\lambda^{\alpha,d}| = L_{\alpha,d}, \text{ for all } \alpha \in \Delta_+,\ 0 \leqslant d \leqslant \mathrm{N} \}.$$

Then for any $L \in \mathcal{L}$, $\mathcal{P}_L$ is a non-empty finite set; if the root spaces of $\mathfrak{g}$ are one-dimensional, then $\mathcal{P}_L$ is a singleton. The set $\mathcal{L}_\chi$ parameterises a disjoint union decomposition of the set $\mathcal{P}_\chi$:

$$\mathcal{P}_\chi = \textstyle\bigsqcup_{L \in \mathcal{L}_\chi} \mathcal{P}_L. \tag{2.1}$$

For any $S \subset \mathcal{P}$, let

$$\operatorname{span}(S) = \operatorname{span}_{\Bbbk} \{ \mathrm{y}(\lambda) \mid \lambda \in S \},$$

so that, for example, $\operatorname{span}(\mathcal{P}) = \mathcal{U}(\breve{\mathfrak{g}}_-)$ and $\operatorname{span}(\mathcal{P}_\chi) = \mathcal{U}(\breve{\mathfrak{g}}_-)^{-\chi}$ for any $\chi \in \mathcal{Q}_+$. For any $\chi \in \mathcal{Q}_+$, the decomposition (2.1) of $\mathcal{P}_\chi$ defines a decomposition of $\mathcal{U}(\breve{\mathfrak{g}}_-)^{-\chi} = \operatorname{span}(\mathcal{P}_\chi)$:

$$\mathcal{U}(\breve{\mathfrak{g}}_-)^{-\chi} = \bigoplus_{L \in \mathcal{L}_\chi} \operatorname{span}(\mathcal{P}_L).$$

We construct an ordering of the set $\mathcal{L}_\chi$ and show that, relative to this ordering, any matrix expression of the modified Shapovalov form $\mathbf{B}_\chi$ for $\breve{\mathfrak{g}}$ is block-upper-triangular (cf. Theorem 2.14). The following Corollary, immediate from Theorem 2.14, provides a multiplicative decomposition of the Shapovalov determinant, and is the most important result of this section.

Corollary 2.2. Let $\chi \in \mathcal{Q}_+$. Then

$$\det \mathbf{B}_\chi = \prod_{L \in \mathcal{L}_\chi} \det \mathbf{B}_\chi|_{\operatorname{span}(\mathcal{P}_L)}.$$

2.1. An order on $\mathcal{L}_\chi$. Fix an arbitrary linearisation of the partial order on $\mathcal{Q}_+$. If X is a set with a linear order, write $X^\dagger$ for the set X with the reverse order, i.e. $x \leqslant y$ in $X^\dagger$ if and only if $x \geqslant y$ in X. For example, the order on $\mathbb{Z}_+{}^\dagger$ is such that 0 is maximal. Suppose that $(X_i)_{i\geqslant 1}$ is a sequence of linearly ordered sets, and let

$$X = X_1 \times X_2 \times X_3 \times \cdots$$

denote the ordered Cartesian product. The set X carries an order $<_X$ defined by declaring, for all tuples $(x_i), (y_i) \in X$, that $(x_i) <_X (y_i)$ if and only if there exists some $m \geqslant 1$ such that $x_i = y_i$ for all $1 \leqslant i < m$, and $x_m < y_m$. This order on X is linear, and is called the *lexicographic order* (or *dictionary order*). Fix an arbitrary enumeration of the countable set $\Delta_+ \times \{0, 1, \ldots, \mathrm{N}\}$. Consider $\mathcal{L}$ as a subset of the ordered Cartesian product of copies of the set $\mathbb{Z}_+$ indexed by this enumeration. Write $\mathcal{L}(\leqslant)$ for the set $\mathcal{L}$ with the associated lexicographic order. For any $L \in \mathcal{L}$, write

$$\begin{aligned}
\Delta(L) &= \Big(\sum_{\alpha\in\Delta_+} L_{\alpha,0}\alpha, \sum_{\alpha\in\Delta_+} L_{\alpha,1}\alpha, \ldots, \sum_{\alpha\in\Delta_+} L_{\alpha,\mathrm{N}}\alpha\Big) \quad \in (\mathcal{Q}_+{}^\dagger)^{\mathrm{N}+1},\\
|L| &= \Big(\sum_{\alpha\in\Delta_+} L_{\alpha,0}, \sum_{\alpha\in\Delta_+} L_{\alpha,1}, \ldots, \sum_{\alpha\in\Delta_+} L_{\alpha,\mathrm{N}}\Big) \quad \in \mathbb{Z}_+{}^\dagger \times \mathbb{Z}_+^{\mathrm{N}}.
\end{aligned}$$

For any $\chi \in \mathcal{Q}_+$, define a map

$$\theta_\chi : \mathcal{L}_\chi \to (\mathcal{Q}_+{}^\dagger)^{\mathrm{N}+1} \times (\mathbb{Z}_+{}^\dagger \times \mathbb{Z}_+^{\mathrm{N}}) \times \mathcal{L}(\leqslant)$$

by

$$\theta_\chi(L) = (\Delta(L), |L|, L), \qquad L \in \mathcal{L}_\chi.$$

Consider the sets $(\mathcal{Q}_+{}^\dagger)^{\mathrm{N}+1}$ and $\mathbb{Z}_+{}^\dagger \times \mathbb{Z}_+^{\mathrm{N}}$ to both carry lexicographic orders. Thus the Cartesian product

$$(\mathcal{Q}_+{}^\dagger)^{\mathrm{N}+1} \times (\mathbb{Z}_+{}^\dagger \times \mathbb{Z}_+^{\mathrm{N}}) \times \mathcal{L}(\leqslant) \tag{2.3}$$

carries a lexicographic order, and this order is linear. For any $\chi \in \mathcal{Q}_+$, we consider the set $\mathcal{L}_\chi$ to carry the linear order defined by the injective map θ_χ and the linearly ordered set (2.3).

2.2. Decomposition of the Shapovalov form.

Lemma 2.4. For any partitions $\lambda, \mu \in \mathcal{P}$,

$$\mathrm{x}(\lambda)\mathrm{y}(\mu) = \mathrm{x}(\lambda^0)\mathrm{y}(\mu^{\mathrm{N}})\mathrm{x}(\lambda^1)\mathrm{y}(\mu^{\mathrm{N}-1})\cdots\mathrm{x}(\lambda^{\mathrm{N}})\mathrm{y}(\mu^0).$$

Proof. By choice of order for the basis $\{\, \mathrm{x}(\gamma) \mid \gamma \in \hat{\mathcal{C}} \,\}$ of $\breve{\mathfrak{g}}_+$, and since $\mathrm{y}(\mu) = \omega(\mathrm{x}(\mu))$, by definition,

$$\mathrm{x}(\lambda)\mathrm{y}(\mu) = \mathrm{x}(\lambda^0) \cdots \mathrm{x}(\lambda^{\mathrm{N}})\mathrm{y}(\mu^{\mathrm{N}}) \cdots \mathrm{y}(\mu^0).$$

As $[\,\mathrm{x}(\lambda^i), \mathrm{y}(\mu^j)\,] = 0$ if $i + j > \mathrm{N}$, the claim follows. □

Proposition 2.5. Suppose that $\lambda, \mu \in \mathcal{P}_\chi$, $\chi \in \mathcal{Q}_+$, and further that $\Delta(\lambda^d) = \Delta(\mu^d)$, for all $0 \leqslant d \leqslant k$, for some $0 \leqslant k \leqslant \mathrm{N}$. Then

$$\mathbf{B}\big(\mathrm{y}(\lambda), \mathrm{y}(\mu)\big) = \prod_{0 \leqslant d \leqslant k} \mathbf{B}\big(\mathrm{y}(\lambda^d), \mathrm{y}(\mu^d)\big) \cdot \mathbf{B}\big(\mathrm{y}(\lambda'), \mathrm{y}(\mu')\big),$$

where $\lambda' = \bigcup_{k<d\leqslant \mathrm{N}} \lambda^d$ and $\mu' = \bigcup_{k<d\leqslant \mathrm{N}} \mu^d$.

Proof. Under the hypotheses of the claim,

$$\begin{aligned}
\mathbf{B}\big(\mathrm{y}(\lambda), \mathrm{y}(\mu)\big) &= \mathbf{q}(\mathrm{x}(\lambda)\mathrm{y}(\mu^\star)) \\
&= \mathbf{q}(\mathrm{x}(\lambda^0)\mathrm{y}((\mu^\star)^{\mathrm{N}})\mathrm{x}(\lambda^1)\mathrm{y}((\mu^\star)^{\mathrm{N}-1}) \cdots \mathrm{x}(\lambda^{\mathrm{N}})\mathrm{y}((\mu^\star)^0)) \\
&\qquad \text{(by Lemma 2.4)} \\
&= \mathbf{q}(\mathrm{x}(\lambda^0)\mathrm{y}((\mu^0)^\star)\mathrm{x}(\lambda^1)\mathrm{y}((\mu^1)^\star) \cdots \mathrm{x}(\lambda^{\mathrm{N}})\mathrm{y}((\mu^{\mathrm{N}})^\star)) \\
&= \prod_{0\leqslant d\leqslant k} \mathbf{q}(\mathrm{x}(\lambda^d)\mathrm{y}(\mu^{d\star})) \cdot \mathbf{q}(\mathrm{x}(\lambda^{k+1})\mathrm{y}((\mu^{k+1})^\star) \cdots \mathrm{x}(\lambda^{\mathrm{N}})\mathrm{y}((\mu^{\mathrm{N}})^\star)) \\
&\qquad \text{(since } \mathbf{q}|_{\mathcal{U}(\breve{\mathfrak{g}})^0} \text{ is an algebra homomorphism)} \\
&= \prod_{0\leqslant d\leqslant k} \mathbf{q}(\mathrm{x}(\lambda^d)\mathrm{y}(\mu^{d\star})) \cdot \mathbf{B}\big(\mathrm{y}(\lambda'), \mathrm{y}(\mu')\big) \\
&\qquad \text{(by Lemma 2.4)} \\
&= \Big(\prod_{0\leqslant d\leqslant k} \mathbf{B}\big(\mathrm{y}(\lambda^d), \mathrm{y}(\mu^d)\big)\Big) \cdot \mathbf{B}\big(\mathrm{y}(\lambda'), \mathrm{y}(\mu')\big).
\end{aligned}$$

□

Lemma 2.6. Suppose that $\lambda, \mu \in \mathcal{P}$ are partitions of homogeneous degree d.

i. If $d = 0$ and $|\lambda| < |\mu|$, or if $d > 0$ and $|\lambda| > |\mu|$, then $\mathbf{B}\big(\mathrm{y}(\lambda), \mathrm{y}(\mu)\big) = 0$.
ii. If $|\lambda| = |\mu|$ and $|\lambda^\alpha| \neq |\mu^\alpha|$ for some $\alpha \in \Delta_+$, then $\mathbf{B}\big(\mathrm{y}(\lambda), \mathrm{y}(\mu)\big) = 0$.

Proof. This Lemma follows essentially from Lemma 2.7 (page 38), applied to the Lie algebra with triangular decomposition $(\breve{\mathfrak{g}}, \mathfrak{h}_0, \check{\mathfrak{h}}, \breve{\mathfrak{g}}_+, \omega)$. Let $\lambda, \mu \in \mathcal{P}$ be partitions of homogeneous degree d. Since

$$\mathrm{x}(\lambda)\mathrm{y}(\mu^\star) \in \mathcal{U}(\breve{\mathfrak{g}})_{|\lambda|d+|\mu|(\mathrm{N}-d)},$$

it follows from Lemma 1.1 that

$$\mathbf{B}\big(\mathrm{y}(\lambda), \mathrm{y}(\mu)\big) \in \mathcal{U}(\check{\mathfrak{h}})_{|\lambda|d+|\mu|(\mathrm{N}-d)}. \tag{2.7}$$

On the other hand,

$$\deg_{\check{\mathfrak{h}}} \mathbf{B}\big(\mathrm{y}(\lambda), \mathrm{y}(\mu)\big) \leqslant |\lambda|,\ |\mu| \tag{2.8}$$

by Lemma 2.7 (page 38). Therefore, if $\mathbf{B}\big(\mathrm{y}(\lambda), \mathrm{y}(\mu)\big) \neq 0$, and

$$\mathbf{B}\big(\mathrm{y}(\lambda), \mathrm{y}(\mu)\big) \in \mathcal{U}(\check{\mathfrak{h}})_m,$$

it must be that

$$m \leqslant |\lambda|\mathrm{N} \quad \text{and} \quad m \leqslant |\mu|\mathrm{N}, \tag{2.9}$$

since the degree of $h \in \check{\mathfrak{h}}$ in t is at most N. Combining (2.7) and (2.9), it follows that if $\mathbf{B}\big(\mathrm{y}(\lambda), \mathrm{y}(\mu)\big) \neq 0$, then

$$|\lambda|d + |\mu|(\mathrm{N} - d) \leqslant |\lambda|\mathrm{N}, \tag{2.10}$$

and

$$|\lambda|d + |\mu|(\mathrm{N} - d) \leqslant |\mu|\mathrm{N}. \tag{2.11}$$

If $d = 0$, then inequality (2.10) becomes $|\mu| \leqslant |\lambda|$. Hence, if $d = 0$, and $|\lambda| < |\mu|$, then $\mathbf{B}\big(\mathrm{y}(\lambda), \mathrm{y}(\mu)\big) = 0$. If $d > 0$, then inequality (2.11) yields $|\lambda| \leqslant |\mu|$. Hence, if $d > 0$ and $|\lambda| > |\mu|$, it must be that $\mathbf{B}\big(\mathrm{y}(\lambda), \mathrm{y}(\mu)\big) = 0$. This proves part (i).

Suppose now that $|\lambda| = |\mu| = r$, and that $|\lambda^\alpha| \neq |\mu^\alpha|$ for some $\alpha \in \Delta_+$. Then, by Lemma 2.7 (page 38), the inequality (2.8) becomes strict. Hence, if $\mathbf{B}\big(\mathrm{y}(\lambda), \mathrm{y}(\mu)\big) \neq 0$, then the inequalities (2.10) and (2.11) are also strict. These both yield $r\mathrm{N} < r\mathrm{N}$, which is absurd. Hence it must be that $\mathbf{B}\big(\mathrm{y}(\lambda), \mathrm{y}(\mu)\big) = 0$, and part (ii) is proven. □

Lemma 2.12. Suppose that $\nu \in \mathcal{Q}$ and $\nu \notin \mathcal{Q}_+$. Then $\mathcal{U}(\check{\mathfrak{g}})^\nu \subset \check{\mathfrak{g}}_- \mathcal{U}(\check{\mathfrak{g}})$.

Proof. Because $\check{\mathfrak{g}} = \check{\mathfrak{g}}_- \oplus (\check{\mathfrak{h}} \oplus \check{\mathfrak{g}}_+)$, we have $\mathcal{U}(\check{\mathfrak{g}}) = \mathcal{U}(\check{\mathfrak{g}}_-) \otimes \mathcal{U}(\check{\mathfrak{h}} \oplus \check{\mathfrak{g}}_+)$ by the PBW Theorem. The set of all weights of the $\mathfrak{h}_0$-module $\mathcal{U}(\check{\mathfrak{h}} \oplus \check{\mathfrak{g}}_+)$ is precisely $\mathcal{Q}_+$, and so, for any $\nu \in \mathcal{Q}$,

$$\mathcal{U}(\check{\mathfrak{g}})^\nu = \textstyle\sum_{\chi \in \mathcal{Q}_+} \mathcal{U}(\check{\mathfrak{g}}_-)^{\nu-\chi} \otimes \mathcal{U}(\check{\mathfrak{h}} \oplus \check{\mathfrak{g}}_+)^\chi. \tag{2.13}$$

Suppose that $\nu \in \mathcal{Q}$ and $\nu \notin \mathcal{Q}_+$. Then, in particular, $\nu - \chi \neq 0$, for any $\chi \in \mathcal{Q}_+$, and so

$$\mathcal{U}(\check{\mathfrak{g}}_-)^{\nu-\chi} \subset \check{\mathfrak{g}}_- \mathcal{U}(\check{\mathfrak{g}}).$$

Hence $\mathcal{U}(\check{\mathfrak{g}})^\nu \subset \check{\mathfrak{g}}_- \mathcal{U}(\check{\mathfrak{g}})$ by equation (2.13). □

Theorem 2.14. Suppose that $\chi \in \mathcal{Q}_+$, and that $L, M \in \mathcal{L}_\chi$. If $L > M$, then

$$\mathbf{B}\big(\mathrm{y}(\lambda), \mathrm{y}(\mu)\big) = 0,$$

for all $\lambda \in \mathcal{P}_L$ and $\mu \in \mathcal{P}_M$.

Proof. Suppose that $L, M \in \mathcal{L}_\chi$ and that $L > M$. Then one of the following hold:

- $\Delta(L) > \Delta(M)$; or
- $\Delta(L) = \Delta(M)$ and $|L| > |M|$; or
- $\Delta(L) = \Delta(M)$, $|L| = |M|$ and $L > M$ in $\mathcal{L}(\leqslant)$.

Let $\lambda \in \mathcal{P}_L$ and let $\mu \in \mathcal{P}_M$.

Suppose that $\Delta(L) > \Delta(M)$. Then there exists $0 \leqslant l \leqslant \mathrm{N}$ such that $\Delta(\lambda^d) = \Delta(\mu^d)$ for all $0 \leqslant d < l$, and $\Delta(\lambda^l) > \Delta(\mu^l)$ in ${\mathcal{Q}_+}^\dagger$, i.e. $\Delta(\lambda^l) < \Delta(\mu^l)$ in $\mathcal{Q}_+$. If $l > 0$, then Proposition 2.5 with $k = l - 1$ gives that

$$\mathbf{B}\big(\mathrm{y}(\lambda), \mathrm{y}(\mu)\big) = \theta \cdot \mathbf{B}\big(\mathrm{y}(\lambda'), \mathrm{y}(\mu')\big) \tag{2.15}$$

for some $\theta \in \mathrm{S}(\check{\mathfrak{h}})$, where $\lambda' = \bigcup_{l \leqslant d \leqslant \mathrm{N}} \lambda^d$ and $\mu' = \bigcup_{l \leqslant d \leqslant \mathrm{N}} \mu^d$. In the remaining case where $l = 0$, equation (2.15) holds with $\lambda = \lambda'$, $\mu = \mu'$ and $\theta = 1$. By Lemma 2.4,

$$\mathbf{B}\big(\mathrm{y}(\lambda'), \mathrm{y}(\mu')\big) = \mathbf{q}(\mathrm{x}(\lambda^l)\mathrm{y}((\mu^\star)^{\mathrm{N}-l}) \cdots \mathrm{x}(\lambda^{\mathrm{N}})\mathrm{y}((\mu^\star)^0)). \tag{2.16}$$

Since $\Delta((\mu^\star)^{\mathrm{N}-l}) = \Delta(\mu^l)$, the monomial $\mathrm{x}(\lambda^l)\mathrm{y}((\mu^\star)^{\mathrm{N}-l})$ has weight $\nu = \Delta(\lambda^l) - \Delta(\mu^l)$. Now $\nu \notin \mathcal{Q}_+$, since $\Delta(\lambda^l) < \Delta(\mu^l)$ in $\mathcal{Q}_+$, and so

$$\mathrm{x}(\lambda^l)\mathrm{y}((\mu^\star)^{\mathrm{N}-l}) \in \check{\mathfrak{g}}_-\mathcal{U}(\check{\mathfrak{g}})$$

by Lemma 2.12. Therefore

$$\mathbf{B}\big(\mathrm{y}(\lambda'), \mathrm{y}(\mu')\big) = 0,$$

by equation (2.16) and the definition of the projection $\mathbf{q}$. Hence $\mathbf{B}\big(\mathrm{y}(\lambda), \mathrm{y}(\mu)\big) = 0$ by equation (2.15).

Suppose instead that $\Delta(L) = \Delta(M)$. Then by Proposition 2.5,

$$\mathbf{B}\big(\mathrm{y}(\lambda), \mathrm{y}(\mu)\big) = \prod_{0 \leqslant d \leqslant \mathrm{N}} \mathbf{B}\big(\mathrm{y}(\lambda^d), \mathrm{y}(\mu^d)\big).$$

Suppose that $|L| > |M|$. Then either $|\lambda^d| < |\mu^d|$, with $d = 0$, or $|\lambda^d| > |\mu^d|$ for some $0 < d \leqslant \mathrm{N}$. In either case,

$$\mathbf{B}\big(\mathrm{y}(\lambda^d), \mathrm{y}(\mu^d)\big) = 0,$$

by Lemma 2.6 part (i), applied to the partitions λ^d, μ^d. Suppose that $|L| = |M|$ and that $L > M$ in $\mathcal{L}(\leqslant)$. Then

$$L_{\alpha,d} \neq M_{\alpha,d} \quad \text{for some} \quad \alpha \in \Delta_+,\ 0 \leqslant d \leqslant \mathrm{N},$$

so that $|(\lambda^d)^\alpha| \neq |(\mu^d)^\alpha|$. Therefore, Lemma 2.6 part (ii), applied to the partitions λ^d, μ^d, implies that $\mathbf{B}(\mathrm{y}(\lambda^d), \mathrm{y}(\mu^d)) = 0$. Hence $\mathbf{B}(\mathrm{y}(\lambda), \mathrm{y}(\mu)) = 0$. □

3. Values of the Shapovalov Form

Throughout this section, let $(\mathfrak{g}, \mathfrak{h}_0, \mathfrak{h}, \mathfrak{g}_+, \omega)$ denote a Lie algebra with triangular decomposition, and let $\check{\mathfrak{g}}$ denote the truncated current Lie algebra of nilpotency index N associated to $\mathfrak{g}$. In Section 2, the space $\mathcal{U}(\check{\mathfrak{g}}_-)^{-\chi}$ is decomposed,

$$\mathcal{U}(\check{\mathfrak{g}}_-)^{-\chi} = \bigoplus_{L \in \mathcal{L}_\chi} \operatorname{span}(\mathcal{P}_L)$$

and it is demonstrated that the determinant of the (modified) Shapovalov form $\mathbf{B}_\chi$ on $\mathcal{U}(\check{\mathfrak{g}}_-)^{-\chi}$ is the product of the determinants of the restrictions of $\mathbf{B}_\chi$ to the spaces $\operatorname{span}(\mathcal{P}_L)$, $L \in \mathcal{L}_\chi$. In this section, the restrictions $\mathbf{B}|_{\operatorname{span}(\mathcal{P}_L)}$ are studied. Firstly, the values of $\mathbf{B}|_{\operatorname{span}(\mathcal{P}_L)}$ with respect to the basis $\mathrm{y}(\lambda)$, $\lambda \in \mathcal{P}_L$, are calculated (cf. Proposition 3.3). This permits the recognition of $\mathbf{B}|_{\operatorname{span}(\mathcal{P}_L)}$, in the case where $\mathfrak{g}$ carries a non-degenerate pairing, as an $\mathrm{S}(\check{\mathfrak{h}})$-multiple of a non-degenerate bilinear form on $\operatorname{span}(\mathcal{P}_L)$ (cf. Theorem 3.20). The form on $\operatorname{span}(\mathcal{P}_L)$ is constructed as a symmetric tensor power of the non-degenerate form on $\mathfrak{g}$.

3.1. Values of the restrictions $\mathbf{B}|_{\operatorname{span}(\mathcal{P}_L)}$. Whenever $\lambda, \mu \in \mathcal{P}$ and $|\lambda| = |\mu| = n$, let

$$\mathrm{S}(\lambda, \mu) = \sum_{\tau \in \mathrm{Sym}(n)} \prod_{1 \leqslant i \leqslant n} [\mathrm{x}(\lambda_{\tau(i)}), \mathrm{y}(\mu_i)] \quad \in \mathrm{S}(\check{\mathfrak{h}}),$$

where (λ_i) and (μ_i), $1 \leqslant i \leqslant n$ are arbitrary enumerations of λ and μ, respectively.

Lemma 3.1. Suppose that $\lambda, \mu \in \mathcal{P}$ and $|\lambda| = |\mu|$.

i. $\mathrm{S}(\lambda, \mu) = \mathrm{S}(\mu, \lambda)$;
ii. if, in addition, λ and μ are homogeneous of degree-d in t, then

$$\mathrm{S}(\lambda^\star, \mu) = \mathrm{S}(\lambda, \mu^\star).$$

Proof. Let $n = |\lambda| = |\mu|$, and choose some enumerations (λ_i), (μ_i), $1 \leqslant i \leqslant n$ of λ and μ. The anti-involution ω point-wise fixes $\mathrm{S}(\check{\mathfrak{h}})$, and so ω fixes $\mathrm{S}(\lambda,\mu)$. On the other hand,

$$\begin{aligned}
\omega(\mathrm{S}(\lambda,\mu)) &= \sum_{\tau\in\mathrm{Sym}(n)} \prod_{1\leqslant i\leqslant n} \omega([\mathrm{x}(\lambda_{\tau(i)}),\mathrm{y}(\mu_i)]) \\
&= \sum_{\tau\in\mathrm{Sym}(n)} \prod_{1\leqslant i\leqslant n} [\mathrm{x}(\mu_i),\mathrm{y}(\lambda_{\tau(i)})] \\
&= \mathrm{S}(\mu,\lambda)
\end{aligned}$$

proving part (i). Suppose that λ, μ are homogeneous of degree-d in t. For each $1 \leqslant i \leqslant n$, let $\epsilon_i, \gamma_i \in \mathcal{C}$ be such that

$$\lambda_i = (\epsilon_i, d), \qquad \mu_i = (\gamma_i, d).$$

Then

$$\mathrm{S}(\lambda^\star,\mu) = \sum_{\tau\in\mathrm{Sym}(n)} \prod_{1\leqslant i\leqslant n} [\mathrm{x}(\epsilon_{\tau(i)})\otimes \mathrm{t}^{\mathrm{N}-d}, \mathrm{y}(\gamma_i)\otimes \mathrm{t}^d]. \tag{3.2}$$

For any $1 \leqslant i \leqslant n$ and $\tau \in \mathrm{Sym}(n)$,

$$\begin{aligned}
[\mathrm{x}(\epsilon_{\tau(i)})\otimes \mathrm{t}^{\mathrm{N}-d}, \mathrm{y}(\gamma_i)\otimes \mathrm{t}^d] &= [\mathrm{x}(\epsilon_{\tau(i)}), \mathrm{y}(\gamma_i)]\otimes \mathrm{t}^{\mathrm{N}} \\
&= [\mathrm{x}(\epsilon_{\tau(i)})\otimes \mathrm{t}^d, \mathrm{y}(\gamma_i)\otimes \mathrm{t}^{\mathrm{N}-d}],
\end{aligned}$$

and hence $\mathrm{S}(\lambda^\star,\mu) = \mathrm{S}(\lambda,\mu^\star)$ by equation (3.2), proving part (ii). □

Proposition 3.3. Suppose that $L \in \mathcal{L}$, and that $\lambda,\mu \in \mathcal{P}_L$. Then

$$\mathbf{B}(\mathrm{y}(\lambda),\mathrm{y}(\mu)) = \prod_{0\leqslant d\leqslant \mathrm{N}} \prod_{\alpha\in\Delta_+} \mathrm{S}(\lambda^{\alpha,d},(\mu^{\alpha,d})^\star) \tag{3.4}$$

and $\mathbf{B}(\mathrm{y}(\lambda),\mathrm{y}(\mu)) = \mathbf{B}(\mathrm{y}(\mu),\mathrm{y}(\lambda))$.

Proof. Let $\lambda,\mu \in \mathcal{P}_L$, $L \in \mathcal{L}$, and let $0 \leqslant d \leqslant \mathrm{N}$. Then

$$|\lambda^{\alpha,d}| = |\mu^{\alpha,d}| = L_{\alpha,d}, \quad \alpha \in \Delta_+.$$

Write $l = |\lambda^d| = |\mu^d|$. Then by Lemma 2.7 (page 38), applied to the Lie algebra with triangular decomposition $(\check{\mathfrak{g}}, \mathfrak{h}_0, \check{\mathfrak{h}}, \check{\mathfrak{g}}_+, \omega)$,

$$\deg_{\check{\mathfrak{h}}} \mathbf{B}(\mathrm{y}(\lambda^d),\mathrm{y}(\mu^d)) \leqslant l,$$

and the degree-l component of $\mathbf{B}(\mathrm{y}(\lambda^d),\mathrm{y}(\mu^d))$ is given by

$$\prod_{\alpha\in\Delta_+} \mathrm{S}(\lambda^{\alpha,d},(\mu^{\alpha,d})^\star), \tag{3.5}$$

since $\mathbf{B}\big(\mathrm{y}(\lambda^d), \mathrm{y}(\mu^d)\big) = \mathbf{q}(\mathrm{x}(\lambda^d)\mathrm{y}(\mu^{d\star}))$. By Lemma 1.1, and since $ld + l(\mathrm{N} - d) = l\mathrm{N}$,

$$\mathbf{B}\big(\mathrm{y}(\lambda^d), \mathrm{y}(\mu^d)\big) \in \mathcal{U}(\check{\mathfrak{h}})_{l\mathrm{N}}.$$

Therefore $\deg_{\check{\mathfrak{h}}}\mathbf{B}\big(\mathrm{y}(\lambda^d), \mathrm{y}(\mu^d)\big) \geqslant l$, since $\deg_{\mathrm{t}}\phi \leqslant \mathrm{N}$ for any $\phi \in \check{\mathfrak{h}}$; and so $\mathbf{B}\big(\mathrm{y}(\lambda^d), \mathrm{y}(\mu^d)\big)$ is homogeneous of degree-l in $\check{\mathfrak{h}}$, and is equal to the expression (3.5). By Proposition 2.5,

$$\mathbf{B}\big(\mathrm{y}(\lambda), \mathrm{y}(\mu)\big) = \prod_{0 \leqslant d \leqslant \mathrm{N}} \mathbf{B}\big(\mathrm{y}(\lambda^d), \mathrm{y}(\mu^d)\big),$$

and so the equation (3.4) follows. The symmetry of $\mathbf{B}|_{\mathrm{span}(\mathcal{P}_L)}$ follows from equation (3.4),

$$\begin{aligned}
\mathbf{B}\big(\mathrm{y}(\lambda), \mathrm{y}(\mu)\big) &= \prod_{0 \leqslant d \leqslant \mathrm{N}} \prod_{\alpha \in \Delta_+} \mathrm{S}\big(\lambda^{\alpha,d}, (\mu^{\alpha,d})^\star\big) \\
&= \prod_{0 \leqslant d \leqslant \mathrm{N}} \prod_{\alpha \in \Delta_+} \mathrm{S}\big((\mu^{\alpha,d})^\star, \lambda^{\alpha,d}\big) \\
&= \prod_{0 \leqslant d \leqslant \mathrm{N}} \prod_{\alpha \in \Delta_+} \mathrm{S}\big(\mu^{\alpha,d}, (\lambda^{\alpha,d})^\star\big) \\
&= \mathbf{B}\big(\mathrm{y}(\mu), \mathrm{y}(\lambda)\big)
\end{aligned}$$

and parts (i) and (ii) of Lemma 3.1. □

3.2. Tensor powers of bilinear forms. If U, V are vector spaces, and $\phi : U \times V \to \Bbbk$ is a bilinear map, write

$$\tilde{\phi} : U \bigotimes V \to \Bbbk \tag{3.6}$$

for the unique linear map such that $\tilde{\phi}(u \otimes v) = \phi(u, v)$ for all $u \in U$, $v \in V$.

Proposition 3.7. Suppose that U, V are vector spaces with bilinear forms. Then the vector space $U \bigotimes V$ carries a bilinear form defined by

$$(\, u_1 \otimes v_1 \mid u_2 \otimes v_2 \,) = (\, u_1 \mid u_2 \,)(\, v_1 \mid v_2 \,), \tag{3.8}$$

for all $u_1, u_2 \in U$ and $v_1, v_2 \in V$. Moreover, if the forms on U and V are non-degenerate, then so is the form on $U \bigotimes V$.

Proof. Let $\phi : U \times U \to \Bbbk$, $\psi : V \times V \to \Bbbk$ denote the bilinear forms on U, V, respectively. Let

$$\nu : (U \bigotimes U) \times (V \bigotimes V) \to \Bbbk$$

be given by

$$\nu(u_1 \otimes u_2, v_1 \otimes v_2) = \tilde{\phi}(u_1 \otimes u_2)\tilde{\psi}(v_1 \otimes v_2),$$

for all $u_1, u_2 \in U$, $v_1, v_2 \in V$, where the maps $\tilde{\phi}, \tilde{\psi}$ are defined by (3.6). Then ν is bilinear, and so defines a linear map

$$\tilde{\nu} : (U \bigotimes U) \bigotimes (V \bigotimes V) \to \Bbbk$$

by (3.6). Since

$$(U \bigotimes U) \bigotimes (V \bigotimes V) \cong (U \bigotimes V) \bigotimes (U \bigotimes V),$$

the map $\tilde{\nu}$ may be considered as a bilinear form

$$(\cdot \mid \cdot) : (U \bigotimes V) \times (U \bigotimes V) \to \Bbbk.$$

Now if $u_1, u_2 \in U$, $v_1, v_2 \in V$, then

$$\begin{aligned} (\, u_1 \otimes v_1 \mid u_2 \otimes v_2 \,) &= \tilde{\nu}(u_1 \otimes u_2 \otimes v_1 \otimes v_2) \\ &= \nu(u_1 \otimes u_2, v_1 \otimes v_2) \\ &= \tilde{\phi}(u_1 \otimes u_2)\tilde{\psi}(v_1 \otimes v_2) \\ &= \phi(u_1, u_2)\psi(v_1, v_2), \end{aligned}$$

and so this is the required bilinear form. The non-degeneracy claim follows immediately from the definition (3.8) of the form. $\square$

For any vector space U and non-negative integer n, write

$$\mathrm{T}^n(U) = U \bigotimes \cdots \bigotimes U, \quad (n \text{ times})$$

for the space of homogeneous degree-n tensors in U. For any $u_i \in U$, $1 \leqslant i \leqslant n$, write

$$\textstyle\bigotimes_{i=1}^{n} u_i = u_1 \otimes \cdots \otimes u_n,$$

so that

$$\mathrm{T}^n(U) = \mathrm{span}\,\{ \textstyle\bigotimes_{i=1}^{n} u_i \mid u_i \in U,\ 1 \leqslant i \leqslant n \}.$$

Write

$$u_1 \cdots u_n = \prod_{i=1}^{n} u_i = \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}(n)} \textstyle\bigotimes_{i=1}^{n} u_{\sigma(i)}$$

for the symmetric tensor in $u_i \in U$, $1 \leqslant i \leqslant n$, and let

$$\mathrm{S}^n(U) = \mathrm{span}\,\{ \prod_{i=1}^{n} u_i \mid u_i \in U,\ 1 \leqslant i \leqslant n \}$$

denote the space of degree-n symmetric tensors in U. Let

$$\mathrm{T}(U) = \bigoplus_{n \geq 0} \mathrm{T}^n(U), \qquad \mathrm{S}(U) = \bigoplus_{n \geq 0} \mathrm{S}^n(U),$$

denote the tensor and symmetric algebras over U, respectively.

Proposition 3.9. Suppose that U is a vector space endowed with a bilinear form, and that $n \geqslant 0$. Then $\mathrm{S}^n(U)$ carries a bilinear form defined by

$$(\prod_{i=1}^{n} u_i \mid \prod_{i=1}^{n} v_i) = \frac{1}{n!} \sum_{\tau \in \mathrm{Sym}(n)} \prod_{i=1}^{n} (u_i \mid v_{\tau(i)}),$$

for any $u_i, v_i \in U$, $1 \leqslant i \leqslant n$. Moreover, if the form on U is non-degenerate, then so is the form on $\mathrm{S}^n(U)$.

Proof. Let $\mathrm{A}(U)$ denote the two-sided ideal of $\mathrm{T}(U)$ generated by the elements of the set

$$\{ u_1 \otimes u_2 - u_2 \otimes u_1 \mid u_1, u_2 \in U \}.$$

Then $\mathrm{T}(U) = \mathrm{S}(U) \bigoplus \mathrm{A}(U)$ is a direct sum of graded vector spaces. Hence, for any $n \geqslant 0$,

$$\mathrm{T}^n(U) = \mathrm{S}^n(U) \bigoplus \mathrm{A}^n(U) \tag{3.10}$$

is a direct sum of vector spaces, where $\mathrm{A}^n(U)$ denotes the homogeneous degree-n component of $\mathrm{A}(U)$. By Proposition 3.7, the tensor power $\mathrm{T}^n(U)$ carries a bilinear form defined by

$$(\bigotimes_{i=1}^{n} u_i \mid \bigotimes_{i=1}^{n} v_i) = \prod_{i=1}^{n} (u_i \mid v_i), \qquad u_i, v_i \in U, \quad 1 \leqslant i \leqslant n. \tag{3.11}$$

Observe that for any $u_i, v_i \in U$, $1 \leqslant i \leqslant n$,

$$\sum_{\sigma \in \mathrm{Sym}(n)} \prod_{i=1}^{n} (u_i \mid v_{\sigma(i)})$$

is independent of the enumeration of the elements $v_1, \ldots, v_n$. It follows that the direct sum (3.10) is orthogonal, with respect to the bilinear form (3.11). A form is defined on $\mathrm{S}^n(U)$ by restriction of the form on $\mathrm{T}^n(U)$. For any $u_i, v_i \in U$, $1 \leqslant i \leqslant n$,

$$\begin{aligned}
(\prod_{i=1}^{n} u_i \mid \prod_{i=1}^{n} v_i) &= (\frac{1}{n!})^2 \sum_{\sigma \in \mathrm{Sym}(n)} \sum_{\tau \in \mathrm{Sym}(n)} \prod_{i=1}^{n} (u_{\sigma(i)} \mid v_{\tau(i)}) \\
&= \frac{1}{n!} \sum_{\tau \in \mathrm{Sym}(n)} \prod_{i=1}^{n} (u_i \mid v_{\tau(i)}).
\end{aligned}$$

Hence $\mathrm{S}^n(U)$ carries the required bilinear form. If the form on U is non-degenerate, then by Proposition 3.7 the form on $\mathrm{T}^n(U)$ is non-degenerate, and since the sum (3.10) is orthogonal, the restriction of the form to $\mathrm{S}^n(U)$ is non-degenerate also. □

3.3. Lie algebras with non-degenerate pairing. A Lie algebra with triangular decomposition $(\mathfrak{g}, \mathfrak{h}_0, \mathfrak{h}, \mathfrak{g}_+, \omega)$ is said to have *non-degenerate pairing* if for all $\alpha \in \Delta$, there exists a non-zero $\mathbf{h}(\alpha) \in \mathfrak{h}$, and a non-degenerate bilinear form

$$(\cdot \mid \cdot)_\alpha : \mathfrak{g}^\alpha \times \mathfrak{g}^\alpha \to \Bbbk,$$

such that

$$[x_1, \omega(x_2)] = (x_1 \mid x_2)_\alpha \mathbf{h}(\alpha), \tag{3.12}$$

for all $x_1, x_2 \in \mathfrak{g}^\alpha$.

If $\mathfrak{g}$ has a non-degenerate pairing, then for any $\alpha \in \Delta$, the space

$$[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] = [\mathfrak{g}^{-\alpha}, \mathfrak{g}^\alpha]$$

is one-dimensional, and so the elements $\mathbf{h}(\alpha)$ and $\mathbf{h}(-\alpha)$ can differ only by a non-zero scalar.

Example 3.13. Let $\mathfrak{g}$ be a symmetrisable Kac-Moody Lie algebra over $\Bbbk$ (cf. Example 1.3, page 33), and let $(\cdot \mid \cdot)$ denote a standard bilinear form on $\mathfrak{g}$ (as per [**20**, page 20]). The restriction of this form to $\mathfrak{h}$ is non-degenerate. Therefore, for any $\chi \in \mathfrak{h}^*$, there exists a unique $\mathbf{h}(\chi) \in \mathfrak{h}$ such that

$$\langle \chi, h \rangle = (\mathbf{h}(\chi) \mid h) \qquad h \in \mathfrak{h}.$$

The map $\mathbf{h} : \mathfrak{h}^* \to \mathfrak{h}$ is a linear isomorphism. For any $\alpha \in \Delta$, let

$$(\cdot \mid \cdot)_\alpha : \mathfrak{g}^\alpha \times \mathfrak{g}^\alpha \to \Bbbk,$$

be given by

$$(x_1 \mid x_2)_\alpha = (x_1 \mid \omega(x_2)), \qquad x_1, x_2 \in \mathfrak{g}^\alpha.$$

Then for any $\alpha \in \Delta$, the form $(\cdot \mid \cdot)_\alpha$ is non-degenerate, and is such that equation (3.12) holds (see, for example, Theorem 2.2 of [**20**]). Hence $\mathfrak{g}$ carries a non-degenerate pairing.

Example 3.14. Suppose that $\mathfrak{g}$ is a Lie algebra with triangular decomposition, such that for any root $\alpha \in \Delta$,

$$\dim \mathfrak{g}^\alpha = \dim \mathfrak{g}^{-\alpha} = 1, \quad \text{and} \quad [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] \neq 0.$$

Then for each $\alpha \in \Delta$, we may choose an arbitrary non-zero

$$\mathbf{h}(\alpha) \in [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$$

and let the form $(\cdot \mid \cdot)_\alpha : \mathfrak{g}^\alpha \times \mathfrak{g}^\alpha \to \Bbbk$ be defined by equation (3.12).

Example 3.15. Let $\mathfrak{g}$ denote the Virasoro algebra (cf. Example 1.4, page 1.4). Let $\alpha \in \Delta$, and let m be the non-zero integer such that $\alpha = m\delta$. Then

$$\mathfrak{g}^{\alpha} = \Bbbk \mathrm{L}_m, \qquad \mathfrak{g}^{-\alpha} = \Bbbk \mathrm{L}_{-m},$$

and $[\, \mathrm{L}_m, \mathrm{L}_{-m} \,] = 2m\mathrm{L}_0 + \psi(m)\mathrm{c}$ is non-zero. Therefore, by Example 3.14, $\mathfrak{g}$ carries a non-degenerate pairing, with

$$\mathbf{h}(\alpha) = 2m\mathrm{L}_0 + \psi(m)\mathrm{c}, \qquad \alpha = m\delta.$$

Example 3.16. The Heisenberg Lie algebra $\mathfrak{a}$ carries a non-degenerate pairing (cf. Example 1.5, page 34). Let $\alpha \in \Delta$, and let m be the non-zero integer such that $\alpha = m\delta$. Then

$$\mathfrak{a}^{\alpha} = \Bbbk \mathrm{a}_m, \qquad \mathfrak{a}^{-\alpha} = \Bbbk \mathrm{a}_{-m},$$

and $[\, \mathrm{a}_m, \mathrm{a}_{-m} \,] = m\hbar$ is non-zero. Therefore, by Example 3.14, $\mathfrak{a}$ carries a non-degenerate pairing, with

$$\mathbf{h}(\alpha) = m\hbar, \qquad \alpha = m\delta.$$

Suppose that $(\mathfrak{g}, \mathfrak{h}_0, \mathfrak{h}, \mathfrak{g}_+, \omega)$ is a Lie algebra with triangular decomposition and non-degenerate pairing, and let $\check{\mathfrak{g}}$ denote the truncated current Lie algebra with nilpotency index N associated to $\mathfrak{g}$. Non-degenerate bilinear forms are defined on the homogeneous degree components of the roots spaces of $\check{\mathfrak{g}}$ in the following manner. For all $\alpha \in \Delta_+$ and $0 \leqslant d \leqslant \mathrm{N}$, define a non-degenerate bilinear form $(\,\cdot \mid \cdot\,)_{\alpha,d}$ on $\mathfrak{g}^{\alpha} \otimes \mathrm{t}^d$ by

$$(\, x_1 \otimes \mathrm{t}^d \mid x_2 \otimes \mathrm{t}^d \,)_{\alpha,d} = (\, x_1 \mid x_2 \,)_{\alpha}, \qquad x_1, x_2 \in \mathfrak{g}^{\alpha}. \tag{3.17}$$

For all $\alpha \in \Delta_+$ and $0 \leqslant d \leqslant \mathrm{N}$, let

$$\hat{\mathcal{C}}_{\alpha,d} = \{\, \gamma \in \hat{\mathcal{C}} \mid \Delta(\gamma) = \alpha, \quad \deg_{\mathrm{t}}\gamma = d \,\}.$$

Lemma 3.18. Let $\alpha \in \Delta_+$ and let $0 \leqslant d \leqslant \mathrm{N}$. Then,

$$[\, \mathrm{x}(\phi), \mathrm{y}(\psi^{\star}) \,] = (\, \mathrm{x}(\phi) \mid \mathrm{x}(\psi) \,)_{\alpha,d}\mathbf{h}(\alpha) \otimes \mathrm{t}^{\mathrm{N}},$$

for all $\phi, \psi \in \hat{\mathcal{C}}_{\alpha,d}$.

Proof. Let $\phi', \psi' \in \mathcal{C}$ be such that $\phi = (\phi', d)$ and $\psi = (\psi', d)$. Then

$$\mathrm{y}(\psi^{\star}) = \mathrm{y}(\psi') \otimes \mathrm{t}^{\mathrm{N}-d} = \omega(\mathrm{x}(\psi')) \otimes \mathrm{t}^{\mathrm{N}-d},$$

and $\mathrm{x}(\phi) = \mathrm{x}(\phi') \otimes \mathrm{t}^d$. Therefore

$$\begin{aligned} [\,\mathrm{x}(\phi), \mathrm{y}(\psi^\star)\,] &= [\,\mathrm{x}(\phi'), \omega(\mathrm{x}(\psi'))\,] \otimes \mathrm{t}^{\mathrm{N}} \\ &= (\,\mathrm{x}(\phi') \,|\, \mathrm{x}(\psi')\,)_\alpha \mathbf{h}(\alpha) \otimes \mathrm{t}^{\mathrm{N}} \\ &= (\,\mathrm{x}(\phi) \,|\, \mathrm{x}(\psi)\,)_{\alpha,d} \mathbf{h}(\alpha) \otimes \mathrm{t}^{\mathrm{N}}, \end{aligned}$$

by equation (3.17). □

3.4. Recognition of the restrictions $\mathbf{B}|_{\mathbf{span}(\mathcal{P}_L)}$. For any $L \in \mathcal{L}$, let

$$\mathrm{A}_L = \bigotimes_{0 \leqslant d \leqslant \mathrm{N}} \bigotimes_{\alpha \in \Delta_+} \mathrm{S}^{L_{\alpha,d}}(\mathfrak{g}^\alpha \otimes \mathrm{t}^d).$$

The vector space A_L has a basis parameterised by the partitions in $\mathcal{P}_L$:

$$\mathcal{P}_L \ni \quad \lambda \quad \leftrightarrow \quad \bar{\mathrm{x}}(\lambda) \quad \in \mathrm{A}_L,$$

where, for all $\lambda \in \mathcal{P}$,

$$\bar{\mathrm{x}}(\lambda) = \bigotimes_{0 \leqslant d \leqslant \mathrm{N}} \bigotimes_{\alpha \in \Delta_+} \prod_{\gamma \in \lambda^{\alpha,d}} \mathrm{x}(\gamma).$$

Proposition 3.19. Let $\alpha \in \Delta_+$ and $0 \leqslant d \leqslant \mathrm{N}$. If $\lambda, \mu \in \mathcal{P}$ are partitions with components in $\hat{\mathcal{C}}_{\alpha,d}$ such that $|\lambda| = |\mu| = k$, then

$$\mathrm{S}(\lambda, \mu^\star) = k!\,(\mathbf{h}(\alpha) \otimes \mathrm{t}^{\mathrm{N}})^k\,(\,\bar{\mathrm{x}}(\lambda) \,|\, \bar{\mathrm{x}}(\mu)\,)$$

where $(\,\cdot \,|\, \cdot\,)$ is the form on $\mathrm{S}^k(\mathfrak{g}^\alpha \otimes \mathrm{t}^d)$ defined by the form on $\mathfrak{g}^\alpha \otimes \mathrm{t}^d$ and Proposition 3.9.

Proof. The claim follows from Lemma 3.18 and the definition of the form on $\mathrm{S}^k(\mathfrak{g}^\alpha \otimes \mathrm{t}^d)$. Let (λ_i) and (μ_i), $1 \leqslant i \leqslant k$ be any enumerations of λ and μ, respectively. Then:

$$\begin{aligned} \mathrm{S}(\lambda, \mu^\star) &= \sum_{\tau \in \mathrm{Sym}(k)} \prod_{1 \leqslant i \leqslant k} [\,\mathrm{x}(\lambda_{\tau(i)}), \mathrm{y}(\mu_i{}^\star)\,] \\ &= \sum_{\tau \in \mathrm{Sym}(k)} \prod_{1 \leqslant i \leqslant k} (\,\mathrm{x}(\lambda_{\tau(i)}) \,|\, \mathrm{x}(\mu_i)\,)_{\alpha,d} \mathbf{h}(\alpha) \otimes \mathrm{t}^{\mathrm{N}} \\ &= k!\,(\mathbf{h}(\alpha) \otimes \mathrm{t}^{\mathrm{N}})^k \frac{1}{k!} \sum_{\tau \in \mathrm{Sym}(k)} \prod_{1 \leqslant i \leqslant k} (\,\mathrm{x}(\lambda_{\tau(i)}) \,|\, \mathrm{x}(\mu_i)\,)_{\alpha,d} \\ &= k!\,(\mathbf{h}(\alpha) \otimes \mathrm{t}^{\mathrm{N}})^k\,(\,\bar{\mathrm{x}}(\lambda) \,|\, \bar{\mathrm{x}}(\mu)\,). \end{aligned}$$

□

For any $L \in \mathcal{L}$, the vector spaces $\mathrm{span}(\mathcal{P}_L)$ and A_L are isomorphic by linear extension of the correspondence

$$\mathrm{y}(\lambda) \quad \leftrightarrow \quad \bar{\mathrm{x}}(\lambda), \qquad \lambda \in \mathcal{P}_L.$$

Let $(\,\cdot\,|\,\cdot\,)$ denote the non-degenerate form on A_L defined by the forms (3.17) on $\mathfrak{g}^\alpha \otimes \mathrm{t}^d$ and by Propositions 3.7 and 3.9. So

$$(\,\bar{\mathrm{x}}(\lambda) \,|\, \bar{\mathrm{x}}(\mu)\,) = \prod_{0 \leqslant d \leqslant \mathrm{N}} \prod_{\alpha \in \Delta_+} (\,\bar{\mathrm{x}}(\lambda^{\alpha,d}) \,|\, \bar{\mathrm{x}}(\mu^{\alpha,d})\,),$$

for all $\lambda, \mu \in \mathcal{P}_L$, where $(\,\bar{\mathrm{x}}(\lambda^{\alpha,d}) \,|\, \bar{\mathrm{x}}(\mu^{\alpha,d})\,)$ is defined by Proposition 3.9. Let J_L be the bilinear form on $\mathrm{span}(\mathcal{P}_L)$ given by bilinear extension of

$$\mathrm{J}_L(\,\mathrm{y}(\lambda) \,|\, \mathrm{y}(\mu)\,) = (\,\bar{\mathrm{x}}(\lambda) \,|\, \bar{\mathrm{x}}(\mu)\,), \qquad \lambda, \mu \in \mathcal{P}_L.$$

Then the form J_L is non-degenerate.

Theorem 3.20. For any $L \in \mathcal{L}$,

$$\mathbf{B}|_{\mathrm{span}(\mathcal{P}_L)} = \mathbf{h}(L) \cdot \mathrm{J}_L$$

where $\mathbf{h}(L) \in \mathrm{S}(\check{\mathfrak{h}})$ is given by

$$\mathbf{h}(L) = \prod_{0 \leqslant d \leqslant \mathrm{N}} \prod_{\alpha \in \Delta_+} (L_{\alpha,d}!)\,(\mathbf{h}(\alpha) \otimes \mathrm{t}^{\mathrm{N}})^{L_{\alpha,d}}.$$

Proof. Let $\lambda, \mu \in \mathcal{P}_L$. Then:

$$\begin{aligned}
\mathbf{B}(\mathrm{y}(\lambda), \mathrm{y}(\mu)) &= \prod_{0 \leqslant d \leqslant \mathrm{N}} \prod_{\alpha \in \Delta_+} \mathrm{S}(\lambda^{\alpha,d}, (\mu^{\alpha,d})^\star) \\
&\quad \text{(by Proposition 3.3)} \\
&= \prod_{0 \leqslant d \leqslant \mathrm{N}} \prod_{\alpha \in \Delta_+} (L_{\alpha,d}!)\,(\mathbf{h}(\alpha) \otimes \mathrm{t}^{\mathrm{N}})^{L_{\alpha,d}} (\,\bar{\mathrm{x}}(\lambda^{\alpha,d}) \,|\, \bar{\mathrm{x}}(\mu^{\alpha,d})\,) \\
&\quad \text{(by Proposition 3.19)} \\
&= \Big[\prod_{0 \leqslant d \leqslant \mathrm{N}} \prod_{\alpha \in \Delta_+} (L_{\alpha,d}!)\,(\mathbf{h}(\alpha) \otimes \mathrm{t}^{\mathrm{N}})^{L_{\alpha,d}} \Big] \cdot (\,\bar{\mathrm{x}}(\lambda) \,|\, \bar{\mathrm{x}}(\mu)\,) \\
&= \mathbf{h}(L) \cdot \mathrm{J}_L(\,\mathrm{y}(\lambda) \,|\, \mathrm{y}(\mu)\,).
\end{aligned}$$

The set $\{\,\mathrm{y}(\lambda) \mid \lambda \in \mathcal{P}_L\,\}$ is a basis for $\mathrm{span}(\mathcal{P}_L)$, and so the equality follows. $\square$

4. Reducibility of Verma Modules

Let $(\mathfrak{g}, \mathfrak{h}_0, \mathfrak{h}, \mathfrak{g}_+, \omega)$ denote a Lie algebra with triangular decomposition and non-degenerate pairing, and let $(\check{\mathfrak{g}}, \mathfrak{h}_0, \check{\mathfrak{h}}, \check{\mathfrak{g}}_+, \omega)$ denote the truncated current Lie algebra of nilpotency index N associated to $\mathfrak{g}$. In this section we establish reducibility criterion for a Verma module $\mathfrak{V}(\Lambda)$ for $\check{\mathfrak{g}}$ in terms of evaluations of the functional $\Lambda \in \check{\mathfrak{h}}^*$. We then interpret this result separately for the semisimple finite-dimensional Lie algebras, for the affine Kac-Moody Lie algebras, for the symmetrisable Kac-Moody Lie algebras, for the Virasoro algebra and for the Heisenberg algebra.

Theorem 4.1. Let $\Lambda \in \check{\mathfrak{h}}^*$ and let $\chi \in \mathcal{Q}_+$.

i. The Verma module $\mathfrak{V}(\Lambda)$ for $\check{\mathfrak{g}}$ contains a non-zero primitive vector of weight $\Lambda|_{\mathfrak{h}_0} - \chi$ if and only if

$$\langle \Lambda_{\mathrm{N}}, \mathbf{h}(\alpha) \rangle = 0 \tag{4.2}$$

for some $\alpha \in \Delta_+$ such that $\chi - \alpha \in \mathcal{Q}_+$;

ii. $\mathfrak{V}(\Lambda)$ is reducible if and only if equation (4.2) holds for some $\alpha \in \Delta$.

Proof. Let $\Lambda \in \check{\mathfrak{h}}^*$ and let $\alpha \in \Delta_+$. By Proposition 2.4 (page 37), the Verma module $\mathfrak{V}(\Lambda)$ has a non-zero primitive vector of weight $\Lambda|_{\mathfrak{h}_0} - \chi$ if and only if the form $\mathbf{F}_\chi(\Lambda)$ is degenerate. The determinants $\det \mathbf{F}_\chi$ and $\det \mathbf{B}_\chi$ can differ only in sign, and

$$\det \mathbf{F}_\chi(\Lambda) = \langle \Lambda, \det \mathbf{F}_\chi \rangle.$$

Hence such a primitive vector exists if and only if $\langle \Lambda, \det \mathbf{B}_\chi \rangle$ vanishes. Now

$$\begin{aligned}
\langle \Lambda, \det \mathbf{B}_\chi \rangle &= \langle \Lambda, \prod_{L \in \mathcal{L}_\chi} \det \mathbf{B}|_{\mathrm{span}(\mathcal{P}_L)} \rangle \\
&\quad \text{(by Corollary 2.2)} \\
&= \langle \Lambda, \prod_{L \in \mathcal{L}_\chi} \det \mathrm{J}_L \cdot \mathbf{h}(L)^{|\mathcal{P}_L|} \rangle \\
&\quad \text{(by Theorem 3.20)} \\
&= \prod_{L \in \mathcal{L}_\chi} \det \mathrm{J}_L \cdot \langle \Lambda, \mathbf{h}(L) \rangle^{|\mathcal{P}_L|}.
\end{aligned}$$

For any $L \in \mathcal{L}_\chi$, the form J_L is non-degenerate, and so $\det \mathrm{J}_L$ is a non-zero scalar. Hence $\langle \Lambda, \det \mathbf{B}_\chi \rangle$ vanishes if and only if $\langle \Lambda, \mathbf{h}(L) \rangle$ vanishes for some $L \in \mathcal{L}_\chi$. As

$$\mathbf{h}(L) = \prod_{0 \leqslant d \leqslant \mathrm{N}} \prod_{\alpha \in \Delta_+} (L_{\alpha,d}!)\, (\mathbf{h}(\alpha) \otimes \mathrm{t}^{\mathrm{N}})^{L_{\alpha,d}},$$

$\langle \Lambda, \det \mathbf{B}_\chi \rangle$ vanishes if and only if $\langle \Lambda, \mathbf{h}(\alpha) \otimes \mathrm{t}^{\mathrm{N}} \rangle$ is zero for some $\alpha \in \Delta_+$ for which there exists $L \in \mathcal{L}_\chi$ and $0 \leqslant d \leqslant \mathrm{N}$ with $L_{\alpha,d} > 0$. This condition on α is equivalent to requiring that there exist some partition $\mu \in \mathcal{P}_\chi$ for which $|\mu^\alpha| > 0$, which occurs precisely when $\chi - \alpha \in \mathcal{Q}_+$. Hence the first part is proven; as $\mathbf{h}(\alpha)$ and $\mathbf{h}(-\alpha)$ are proportional, for any $\alpha \in \Delta$, the second part follows. □

It is apparent from Theorem 4.1 that the reducibility of a Verma module $\mathfrak{V}(\Lambda)$ for $\breve{\mathfrak{g}}$ depends only upon Λ_{N}.

4.1. Symmetrisable Kac-Moody Lie algebras. Let $\mathfrak{g}$ be a symmetrisable Kac-Moody Lie algebra as per Examples 1.3 (page 33) and 3.13. The map

$$\mathbf{h} : \mathfrak{h}^* \to \mathfrak{h},$$

from Example 3.13 transports the non-degenerate form $(\cdot \mid \cdot)$ on $\mathfrak{h}$ to the space $\mathfrak{h}^*$ via

$$(\chi \mid \gamma) = (\mathbf{h}(\chi) \mid \mathbf{h}(\gamma)), \qquad \chi, \gamma \in \mathfrak{h}^*.$$

Hence, for any $\Lambda \in \breve{\mathfrak{h}}^*$ and $\alpha \in \Delta$,

$$\langle \Lambda_{\mathrm{N}}, \mathbf{h}(\alpha) \rangle = (\mathbf{h}(\Lambda_{\mathrm{N}}) \mid \mathbf{h}(\alpha)) = (\Lambda_{\mathrm{N}} \mid \alpha),$$

by definition of the map $\mathbf{h}$. The following Corollary of Theorem 4.1 may be viewed as a generalisation of Corollary 4.4.

Corollary 4.3. Let $\mathfrak{g}$ be a symmetrisable Kac-Moody Lie algebra, and let $\breve{\mathfrak{g}}$ denote the truncated current Lie algebra of nilpotency index N associated to $\mathfrak{g}$. Then, for any $\Lambda \in \breve{\mathfrak{h}}^*$, the Verma module $\mathfrak{V}(\Lambda)$ for $\breve{\mathfrak{g}}$ is reducible if and only if Λ_{N} is orthogonal to some root of $\mathfrak{g}$ with respect to the symmetric bilinear form.

4.2. Finite-dimensional semisimple Lie algebras. The following Corollary is a special case of Corollary 4.3.

Corollary 4.4. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra, and let $\breve{\mathfrak{g}}$ denote the truncated current Lie algebra of nilpotency index N associated to $\mathfrak{g}$. Then, for any $\Lambda \in \breve{\mathfrak{h}}^*$, the Verma module $\mathfrak{V}(\Lambda)$ for $\breve{\mathfrak{g}}$ is reducible if and only if Λ_{N} is orthogonal to some root of $\mathfrak{g}$ in the geometry defined by the Killing form.

Hence the reducibility criterion for Verma modules for $\breve{\mathfrak{g}}$ can be described by a finite union of hyperplanes in $\mathfrak{h}^*$.

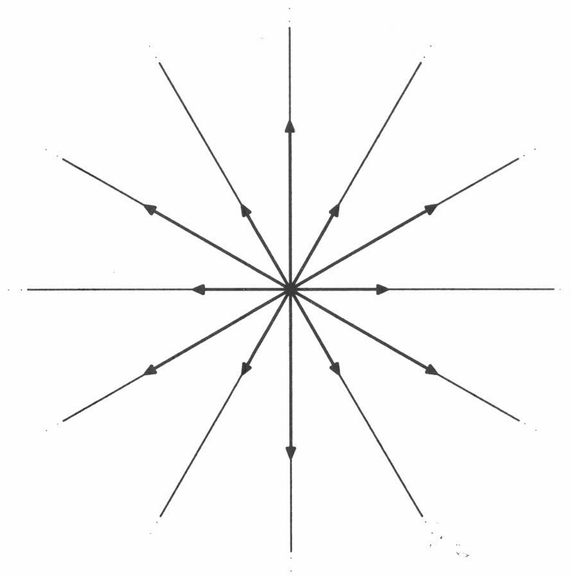


Figure 1: Reducibility criterion for Verma modules of $\breve{\mathfrak{g}}$, where $\mathfrak{g}$ is of type G_2

Example 4.5. Figure 1(a) of page 12 and Figure 1 of page 65 illustrate the reducibility criterion for the Lie algebras $\mathfrak{g}$ over $\mathbb{R}$ with root systems A_2 and G_2, respectively. Roots are drawn as arrows. A Verma module $\mathfrak{V}(\Lambda)$ for $\breve{\mathfrak{g}}$ is reducible if and only if Λ_{N} belongs to the union of hyperplanes indicated.

4.3. Affine Kac-Moody Lie algebras. We refine the criterion of Corollary 4.3 for the affine Kac-Moody Lie algebras. Let $\dot{\mathfrak{g}}$ denote a finite-dimensional semisimple Lie algebra over the field $\mathbb{k}$ with Cartan subalgebra $\dot{\mathfrak{h}}$, root system $\dot{\Delta}$ and Killing form $(\,\cdot\,|\,\cdot\,)$. Let $\mathfrak{g}$ denote the affinisation of $\dot{\mathfrak{g}}$,

$$\mathfrak{g} = \dot{\mathfrak{g}} \otimes \mathbb{k}[\mathrm{s}, \mathrm{s}^{-1}] \oplus \mathbb{k}\mathrm{c} \oplus \mathbb{k}\mathrm{d},$$

with Lie bracket relations

$$\begin{aligned} [\,x \otimes \mathrm{s}^m, y \otimes \mathrm{s}^n\,] &= [\,x, y\,] \otimes \mathrm{s}^{m+n} + m\delta_{m,-n}(\,x \,|\, y\,)\mathrm{c}, \qquad [\,\mathrm{c}, \mathfrak{g}\,] = 0, \\ [\,\mathrm{d}, x \otimes \mathrm{s}^m\,] &= mx \otimes \mathrm{s}^m, \end{aligned}$$

for all $x, y \in \dot{\mathfrak{g}}$ and $m, n \in \mathbb{Z}$. Let Δ denote the root system of $\mathfrak{g}$, and let

$$\mathfrak{h} = \dot{\mathfrak{h}} \oplus \mathbb{k}\mathrm{c} \oplus \mathbb{k}\mathrm{d},$$

denote the Cartan subalgebra. Consider any $\Lambda \in \dot{\mathfrak{h}}^*$ as a functional on $\mathfrak{h}$ by declaring

$$\Lambda(\mathrm{c}) = \Lambda(\mathrm{d}) = 0.$$

This identifies $\dot{\mathfrak{h}}^*$ with a subspace of $\mathfrak{h}^*$. Let $\delta, \tau \in \mathfrak{h}^*$ be given by

$$\begin{aligned} \langle \delta, \dot{\mathfrak{h}} \rangle &= 0, & \langle \delta, \mathrm{c} \rangle &= 0, & \langle \delta, \mathrm{d} \rangle &= 1, \\ \langle \tau, \dot{\mathfrak{h}} \rangle &= 0, & \langle \tau, \mathrm{c} \rangle &= 1, & \langle \tau, \mathrm{d} \rangle &= 0, \end{aligned}$$

so that

$$\mathfrak{h}^* = \dot{\mathfrak{h}}^* \oplus \Bbbk\delta \oplus \Bbbk\tau. \tag{4.6}$$

The symmetric bilinear form $(\,\cdot\,|\,\cdot\,)$ on $\mathfrak{h}^*$ may be obtained as an extension of the Killing form on $\dot{\mathfrak{h}}^*$, via

$$(\,\delta\,|\,\dot{\mathfrak{h}}^*\,) = (\,\tau\,|\,\dot{\mathfrak{h}}^*\,) = 0, \qquad (\,\delta\,|\,\delta\,) = (\,\tau\,|\,\tau\,) = 0, \qquad (\,\delta\,|\,\tau\,) = 1. \tag{4.7}$$

The sum (4.6) is orthogonal with respect to this form. For any $\Lambda \in \mathfrak{h}^*$, let $\widetilde{\Lambda} \in \dot{\mathfrak{h}}^*$ denote the projection of Λ on to $\dot{\mathfrak{h}}^*$ defined by the decomposition (4.6). The root system $\Delta = \Delta^{\mathrm{re}} \cup \Delta^{\mathrm{im}}$ of $\mathfrak{g}$ is given by,

$$\Delta^{\mathrm{re}} = \{\,\alpha + m\delta \mid \alpha \in \dot{\Delta},\ m \in \mathbb{Z}\,\}, \qquad \Delta^{\mathrm{im}} = \{\,m\delta \mid m \in \mathbb{Z},\ m \neq 0\,\}. \tag{4.8}$$

Corollary 4.9. Let $\mathfrak{g}$ denote an affine Kac-Moody Lie algebra, and let $\breve{\mathfrak{g}}$ denote the truncated current Lie algebra of nilpotency index N associated to $\mathfrak{g}$. Then, for any $\Lambda \in \breve{\mathfrak{h}}^*$, the Verma module $\mathfrak{V}(\Lambda)$ for $\breve{\mathfrak{g}}$ is reducible if and only if $\langle \Lambda_{\mathrm{N}}, \mathrm{c} \rangle = 0$ or $(\,\widetilde{\Lambda_{\mathrm{N}}}\,|\,\alpha\,) = m\,\langle \Lambda_{\mathrm{N}}, \mathrm{c} \rangle$ for some $\alpha \in \dot{\Delta}$ and $m \in \mathbb{Z}$.

Proof. It is immediate from (4.6) and (4.7) that

$$\Lambda = \widetilde{\Lambda} + (\,\Lambda\,|\,\tau\,)\delta + (\,\Lambda\,|\,\delta\,)\tau.$$

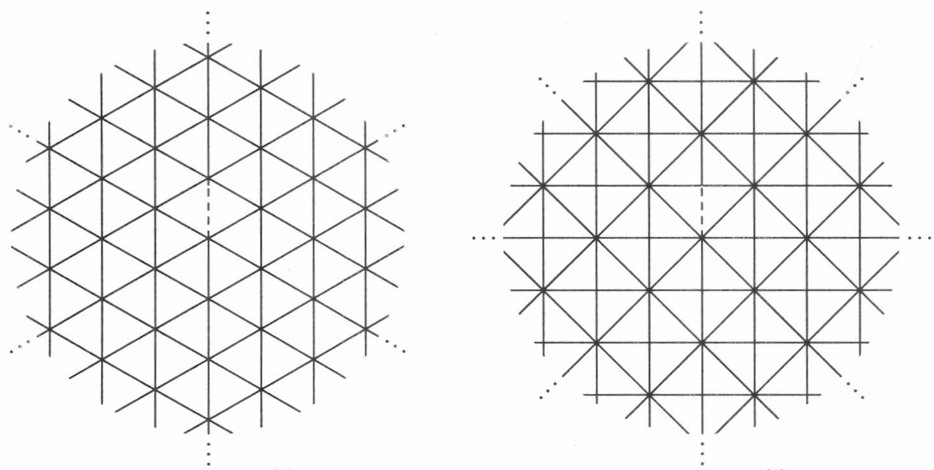
Hence

$$\langle \Lambda, \mathrm{c} \rangle = (\,\Lambda\,|\,\delta\,)\langle \tau, \mathrm{c} \rangle = (\,\Lambda\,|\,\delta\,).$$

Therefore $\langle \Lambda, \mathrm{c} \rangle = 0$ if and only if $(\,\Lambda\,|\,\beta\,) = 0$ for some $\beta \in \Delta^{\mathrm{im}}$. For $\alpha \in \dot{\Delta}$ and $m \in \mathbb{Z}$,

$$(\,\Lambda\,|\,\alpha + m\delta\,) = (\,\Lambda\,|\,\alpha\,) + m(\,\Lambda\,|\,\delta\,) = (\,\widetilde{\Lambda}\,|\,\alpha\,) + m\langle \Lambda, \mathrm{c} \rangle$$

and so $(\,\Lambda\,|\,\alpha + m\delta\,) = 0$ if and only if $(\,\widetilde{\Lambda}\,|\,\alpha\,) = -m\langle \Lambda, \mathrm{c} \rangle$. The claim now follows from (4.8) and Corollary 4.3. □



(a) $\mathfrak{g}$ of type $\mathrm{A}_2^{(1)}$ (b) $\mathfrak{g}$ of type $\mathrm{B}_2^{(1)}$

Figure 2: Reducibility criterion for the Verma modules of $\breve{\mathfrak{g}}$, where $\mathfrak{g}$ is of type $\mathrm{A}_2^{(1)}$ or $\mathrm{B}_2^{(1)}$

Example 4.10. Figures 2(a) and 2(b) of page 67 and Figure 1(b) of page 12 illustrate the reducibility criterion of Corollary 4.9 in the case where $\Bbbk = \mathbb{R}$ and $\dot{\mathfrak{g}}$ is the Lie algebra with root systems A_2, B_2 and G_2, respectively. Thus, respectively, $\mathfrak{g}$ is the affine Kac-Moody Lie algebra of type $\mathrm{A}_2^{(1)}$, $\mathrm{B}_2^{(1)}$ and $\mathrm{G}_2^{(1)}$. A Verma module $\mathfrak{V}(\Lambda)$ for $\breve{\mathfrak{g}}$ is reducible if and only if $\widetilde{\Lambda_{\mathrm{N}}}$ belongs to the described infinite union of hyperplanes, where the length of the dashed line segment is $|\langle \Lambda_{\mathrm{N}}, \mathrm{c}\rangle|$ times the length of a short root for $\dot{\mathfrak{g}}$.

4.4. The Virasoro Algebra. The following Corollary is immediate from Theorem 4.1 and Examples 1.4 (page 1.4) and 3.15.

Corollary 4.11. Let $\mathfrak{g}$ denote the Virasoro algebra, and let $\breve{\mathfrak{g}}$ denote the truncated current Lie algebra of nilpotency index N associated to $\mathfrak{g}$. Then, for any $\Lambda \in \breve{\mathfrak{h}}^*$, the Verma module $\mathfrak{V}(\Lambda)$ for $\breve{\mathfrak{g}}$ is reducible if and only if

$$2m\langle \Lambda_{\mathrm{N}}, \mathrm{L}_0\rangle + \psi(m)\langle \Lambda_{\mathrm{N}}, \mathrm{c}\rangle = 0,$$

for some non-zero integer m.

Hence, if ψ is defined by $\psi(m) = \frac{m^3-m}{12}$ and $\Bbbk = \mathbb{R}$, a Verma module $\mathfrak{V}(\Lambda)$ for $\breve{\mathfrak{g}}$ is reducible if and only if Λ_{N} belongs to the infinite union of hyperplanes indicated in Figure 3. The extension of a functional in the horizontal and vertical directions is determined by evaluations at c and L_0, respectively.

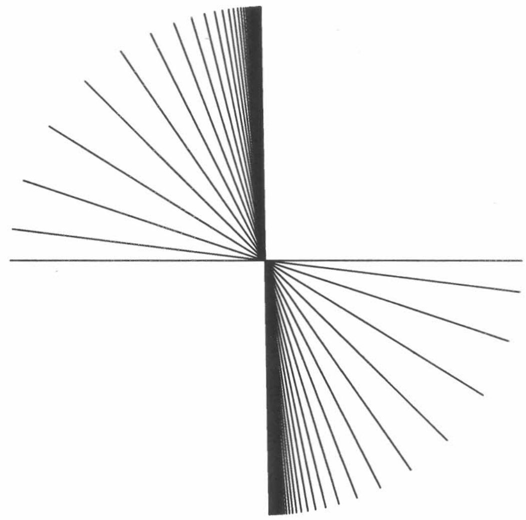


Figure 3: Reducibility criterion for Verma modules of $\breve{\mathfrak{g}}$, where $\mathfrak{g}$ is the Virasoro algebra

4.5. The Heisenberg Algebra. The following Corollary is immediate from Theorem 4.1 and Examples 1.5 (page 34) and 3.16.

Corollary 4.12. Let $\hat{\mathfrak{a}}$ denote the truncated current Lie algebra of nilpotency index N associated to the Heisenberg algebra $\mathfrak{a}$. Then, for any $\Lambda \in \breve{\mathfrak{h}}^*$, a Verma module $\mathfrak{V}(\Lambda)$ for $\hat{\mathfrak{a}}$ is reducible if and only if $\langle \Lambda_{\mathrm{N}}, \hbar \rangle = 0$.

4.A. Characters of Irreducible Highest-Weight Modules

Let $\mathfrak{g}$ denote a Lie algebra with triangular decomposition and non-degenerate pairing, and let $\breve{\mathfrak{g}}$ denote the truncated current Lie algebra of nilpotency index N associated to $\mathfrak{g}$. Theorem 4.1 describes a reducibility criterion for Verma modules for $\breve{\mathfrak{g}}$, but provides little information on the size of the maximal submodule. This appendix describes the characters of the irreducible highest-weight $\breve{\mathfrak{g}}$-modules under the assumption that $\mathfrak{h}$ is one-dimensional (and hence $\mathfrak{h}_0 = \mathfrak{h}$). For example, $\mathfrak{g}$ may be the Lie algebra sl(2), the Witt algebra, or a modified Heisenberg algebra.

For any $\gamma \in \mathfrak{h}^*$, let $\mathfrak{L}(\gamma)$ denote the irreducible highest-weight $\mathfrak{g}$-module of highest-weight γ, and for any $\Lambda \in \breve{\mathfrak{h}}^*$, let $\mathfrak{L}(\Lambda)$ denote the irreducible highest-weight $\breve{\mathfrak{g}}$-module of highest weight Λ. Let

$$\{ \mathrm{e}^{\chi} \mid \chi \in \mathfrak{h}^* \}$$

denote a multiplicative copy of the additive group $\mathfrak{h}^*$, so that

$$\mathrm{e}^{\chi} \cdot \mathrm{e}^{\gamma} = \mathrm{e}^{\chi+\gamma}, \qquad \chi, \gamma \in \mathfrak{h}^*.$$

If M is a vector space graded by $\mathfrak{h}^*$, $M = \oplus_{\chi\in\mathfrak{h}^*} M^{\chi}$, such that all components M^{χ} are finite-dimensional, write

$$\operatorname{char} M = \sum_{\chi\in\mathfrak{h}^*} (\dim M^{\chi})\, \mathrm{e}^{\chi}.$$

Proposition 4.A.1. Let $\mathfrak{g}, \breve{\mathfrak{g}}$ be as above, and let $\Lambda \in \breve{\mathfrak{h}}^*$. Let $0 \leqslant m \leqslant \mathrm{N}$ be minimal such that $\Lambda_n = 0$ for all $m < n \leqslant \mathrm{N}$. Then, if $m > 0$,

$$\operatorname{char} \mathfrak{L}(\Lambda) = \mathrm{e}^{\Lambda_0} \cdot (\operatorname{char} \mathcal{U}(\mathfrak{g}_-))^m,$$

and if $m = 0$, then $\mathfrak{L}(\Lambda)$ is a $\mathfrak{g}$-module isomorphic to $\mathfrak{L}(\Lambda_0)$.

Proof. Suppose that $m = 0$. Since $\mathfrak{g}$ is the quotient of $\breve{\mathfrak{g}}$ by the ideal $\oplus_{0<i\leqslant \mathrm{N}} \mathfrak{g} \otimes \mathrm{t}^i$, the $\mathfrak{g}$-module $\mathfrak{L}(\Lambda_0)$ is a natural $\breve{\mathfrak{g}}$-module. Moreover, $\mathfrak{L}(\Lambda_0)$ is an irreducible highest-weight $\breve{\mathfrak{g}}$-module of highest-weight Λ, and so $\mathfrak{L}(\Lambda_0) \cong \mathfrak{L}(\Lambda)$.

Suppose instead that $m > 0$. Then it must be that $\Lambda_m \neq 0$. Let $\breve{\mathfrak{g}}'$ denote the truncated current Lie algebra of nilpotency index m associated to $\mathfrak{g}$. Let

$$\Lambda' = (\Lambda_0, \ldots, \Lambda_m) \quad \in (\breve{\mathfrak{h}}')^*,$$

and let $\mathfrak{V}(\Lambda')$ denote the Verma module for $\breve{\mathfrak{g}}'$ of highest-weight Λ'. Since $\breve{\mathfrak{g}}'$ is the quotient of $\breve{\mathfrak{g}}$ by the ideal $\oplus_{m<i\leqslant \mathrm{N}} \mathfrak{g} \otimes \mathrm{t}^i$, the $\breve{\mathfrak{g}}'$-module $\mathfrak{V}(\Lambda')$ is a natural $\breve{\mathfrak{g}}$-module. Moreover, $\mathfrak{V}(\Lambda')$ is of highest-weight Λ as a $\breve{\mathfrak{g}}$-module. Since $\mathfrak{h}$ is one-dimensional, $\mathfrak{V}(\Lambda')$ is an irreducible $\breve{\mathfrak{g}}'$-module, by Theorem 4.1. Hence $\mathfrak{V}(\Lambda')$ is the irreducible $\breve{\mathfrak{g}}$-module of highest-weight Λ, i.e. $\mathfrak{L}(\Lambda) \cong \mathfrak{V}(\Lambda')$ as $\breve{\mathfrak{g}}$-modules. In particular, $\mathfrak{L}(\Lambda)$ and $\mathfrak{V}(\Lambda')$ are isomorphic as $\breve{\mathfrak{h}}^*$-graded vector spaces. Now

$$\operatorname{char} \mathfrak{V}(\Lambda') = \mathrm{e}^{\Lambda_0} \cdot (\operatorname{char} \mathcal{U}(\mathfrak{g}_-))^m$$

by Proposition 2.3 part (ii) (page 36), and so the claim follows. □

4.B. Imaginary Highest-Weight Theory for Truncated Current Lie Algebras

Let $\dot{\mathfrak{g}}$ denote the finite-dimensional Lie algebra sl(2) over the field $\Bbbk$, with root system $\dot{\Delta} = \{\pm\alpha\}$, and let $\mathfrak{g}$ denote the affinisation of $\dot{\mathfrak{g}}$ (cf. Subsection 4.3). Let $\mathfrak{h}$ denote the

Cartan subalgebra of $\mathfrak{g}$, let Δ denote the root system, and let δ denote the fundamental imaginary root. Let

$$\Delta_+ = \{\, \alpha + m\delta \mid m \in \mathbb{Z} \,\} \cup \{\, m\delta \mid m \in \mathbb{Z},\ m > 0 \,\},$$

so that $\Delta = \Delta_+ \cup -\Delta_+$. Let

$$\mathfrak{g}_+ = \oplus_{\beta \in \Delta_+} \mathfrak{g}^{\beta}, \qquad \mathfrak{g}_- = \oplus_{\beta \in \Delta_+} \mathfrak{g}^{-\beta},$$

so that

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+. \tag{4.B.1}$$

The subset $\Delta_+ \subset \Delta$ is the imaginary partition of the root system (cf. Section 2 of Chapter 1). The decomposition (4.B.1) defined by Δ_+ does not satisfy the axioms of a triangular decomposition in the sense of Chapter 3, nor in the sense of [**24**]: the additive semigroup $\mathcal{Q}_+$ generated by Δ_+ is not generated by any linearly independent subset of $\mathcal{Q}_+$. Let $\breve{\mathfrak{g}}$ denote the truncated current Lie algebra of nilpotency index N associated to $\mathfrak{g}$. We investigate the difficulty inherent in employing our techniques to derive reducibility criterion for the Verma modules $\mathfrak{V}(\Lambda)$ for $\breve{\mathfrak{g}}$, $\Lambda \in \breve{\mathfrak{h}}^*$. The Theorem 3.20 holds in this setting. However, as we shall see, the degeneracy of an evaluation $\langle \Lambda, \mathbf{B}_\chi \rangle$ of the modified Shapovalov form may not be deduced from the degeneracy of the evaluations $\langle \Lambda, \mathbf{B}|_{\mathrm{span}(\mathcal{P}_L)} \rangle$, where $L \in \mathcal{P}_\chi$.

Let $\mathrm{N} = 1$, and as per Chapter 3 and Section 1, let

$$\mathcal{C} = \Delta_+, \qquad \hat{\mathcal{C}} = \mathcal{C} \times \{\, 0, 1 \,\}.$$

All root spaces of $\mathfrak{g}$ are one-dimensional, so the choice of basis for $\mathfrak{g}_+$

$$\mathcal{C} \ni \quad \beta \quad \leftrightarrow \quad \mathrm{x}(\beta) \quad \in \mathfrak{g}^{\beta}$$

is unique up to scalar multiples. Fix the order of the basis elements $\{\, \mathrm{x}(\gamma) \mid \gamma \in \hat{\mathcal{C}} \,\}$ by firstly comparing degree in the indeterminate t, and secondly by the following order of $\{\, \mathrm{x}(\beta) \mid \beta \in \mathcal{C} \,\}$:

$$\cdots \mathrm{x}(\alpha - 2\delta),\ \mathrm{x}(\alpha - \delta),\ \mathrm{x}(\alpha),\ \mathrm{x}(\alpha + \delta),\ \mathrm{x}(\alpha + 2\delta), \cdots \qquad \cdots \mathrm{x}(\delta),\ \mathrm{x}(2\delta), \cdots$$

For any integer $m > 0$, define partitions

$$\begin{aligned} \mu_m &= \{\, (\alpha - m\delta, 0) \,\} \cup \{\, (\delta, 0) \ \ (m \text{ times}) \,\}, \\ \gamma_m &= \{\, (\alpha - m\delta, 1) \,\} \cup \{\, (\delta, 1) \ \ (m \text{ times}) \,\}, \end{aligned}$$

and $\lambda = \{\, (\alpha, 0) \,\}$. Let $\chi = \alpha \in \mathcal{Q}_+$. Then

$$\{\, \lambda \,\} \cup \{\, \mu_m, \gamma_m \mid m > 0 \,\} \subset \mathcal{P}_\chi.$$

Elementary computation using the Lie bracket relations shows that, for any $m > 0$,

$$\mathbf{B}\big(\mathrm{y}(\lambda), \mathrm{y}(\gamma_m)\big) = (-2)^m \mathbf{h}(\alpha - m\delta) \otimes \mathrm{t}^0, \qquad \mathbf{B}\big(\mathrm{y}(\mu_m), \mathrm{y}(\lambda)\big) = (-2)^m \mathbf{h}(\alpha - m\delta) \otimes \mathrm{t}^1.$$

Hence, if the basis $\{\, \mathrm{y}(\mu) \mid \mu \in \mathcal{P}_\chi \,\}$ is to be linearly ordered so that the matrix representation of $\mathbf{B}_\chi$ is upper triangular, then it must be that both

$$\mathrm{y}(\mu_m) < \mathrm{y}(\lambda), \qquad \mathrm{y}(\lambda) < \mathrm{y}(\gamma_m),$$

for all $m > 0$. Thus the matrix of $\mathbf{B}_\chi$ would be bilaterally infinite.

The degeneracy of a bilaterally-infinite upper-triangular matrix can not be determined from its diagonal entries, as the following simple example demonstrates. Let V denote the vector space with basis the symbols

$$\{\, \mathrm{v_m} \mid m \in \mathbb{Z} \,\} \tag{4.B.2}$$

and let $\Phi : V \to V$ be defined by linear extension of the rule

$$\Phi : \mathrm{v}_m \mapsto \mathrm{v}_{m+1}, \qquad m \in \mathbb{Z}.$$

Then Φ is an automorphism of V. Order the basis elements (4.B.2) by their indices. Then the matrix representation M of Φ with respect to this ordered basis will be upper triangular, in the sense that $M_{i,j} = 0$ whenever $i > j$. However, the diagonal entries $M_{i,i}$ are all identically zero.

CHAPTER 5

Characters of Exponential-Polynomial Modules

1. Preliminaries

For any positive integer r, denote by $\mathbb{Z}_r$ the additive group of integers considered modulo r, by $\mathfrak{R}(r)$ the set of primitive roots of unity of order r, and by ζ_r some fixed element of $\mathfrak{R}(r)$. Denote by $\operatorname{ord}\eta$ the order of a finite-order automorphism η. If η is an endomorphism of a vector space V, write

$$V|_\lambda^\eta = \{\, v \in V \mid \eta(v) = \lambda v \,\}$$

for the eigenspace of eigenvalue λ, for any $\lambda \in \Bbbk$. Let $\mathcal{A} = \Bbbk[\mathrm{t}, \mathrm{t}^{-1}]$.

1.1. Ramanujan sums. Let μ denote the Möbius function, i.e. the function

$$\mu : \mathbb{N} \to \{\, -1, 0, 1 \,\}$$

such that $\mu(d) = (-1)^l$ if d is the product of l distinct primes, $l \geqslant 0$, and $\mu(d) = 0$ otherwise. For any $r > 0$, the function μ satisfies the fundamental property

$$\sum_{d|r} \mu(d) = \delta_{r,1}, \tag{1.1}$$

where δ denotes the Kronecker function. A summation $\sum_{d|r} a_d$ is to be understood as the sum of all the a_d where d is a positive divisor of r. Let $\phi : \mathbb{N} \to \mathbb{N}$ denote Euler's totient function, so that

$$\phi(d) = \#\{\, 0 < k \leqslant d \mid \gcd(k, d) = 1 \,\}, \qquad d > 0.$$

For any positive integer d and $n \in \mathbb{Z}$, the quantity $\mathrm{c}_d(n)$ defined by (4.3) (page 13) is called a *Ramanujan sum*, a *von Sterneck function*, or a *modified Euler number*. These quantities have extensive applications in number theory (see, for example, [**29**], [**25**]), although we require only the most basic properties, such as those described in [**11**]. In particular, we note the identities

$$\mathrm{c}_r(n) = \sum_{\zeta \in \mathfrak{R}(r)} \zeta^n = \sum_{d|\gcd(r,n)} d\,\mu(\frac{r}{d}) \tag{1.2}$$

where $n \in \mathbb{Z}$ and $r > 0$. The function $\mathrm{c}_d(\cdot) : \mathbb{Z} \to \Bbbk$ is a *d-even* arithmetic function, i.e.

$$\mathrm{c}_d(n) = \mathrm{c}_d(\gcd(d, n)), \quad n \in \mathbb{Z}.$$

Any r-even arithmetic function may be expressed as a linear combination of the functions $\mathrm{c}_d(\cdot)$, where d is a divisor of r [**10**]; such an expression is called a *Ramanujan-Fourier transform.*

1.2. Exponential-polynomial functions. Define an endomorphism τ of the vector space $\mathcal{F}$ via

$$(\tau \cdot \varphi)(m) = \varphi(m+1), \qquad m \in \mathbb{Z}, \quad \varphi \in \mathcal{F}.$$

The rule $\mathrm{t} \mapsto \tau$ endows $\mathcal{F}$ with the structure of an $\mathcal{A}$-module. For any $\varphi \in \mathcal{F}$, the action of $\hat{\mathfrak{h}}$ on $\mathbf{H}(\varphi)$, via $\tilde{\varphi}$, may be equivalently defined by

$$(\mathrm{h} \otimes a) \cdot b = (a \cdot \varphi)(0)(ab), \qquad a \in \mathcal{A}, \quad b \in \operatorname{im} \tilde{\varphi} \subset \mathcal{A}.$$

Define $\mathcal{E} \subset \mathcal{F}$ by

$$\mathcal{E} = \{\, \varphi \in \mathcal{F} \mid c \cdot \varphi = 0 \ \text{ for some } \ c \in \Bbbk[\mathrm{t}] \,\} \tag{1.3}$$

For any $\varphi \in \mathcal{E}$, the annihilator $\operatorname{ann}(\varphi) \subset \Bbbk[\mathrm{t}]$ is a non-zero ideal of $\Bbbk[\mathrm{t}]$; the unique monic generator $\mathrm{c}_\varphi \in \operatorname{ann}(\varphi)$ is called the *characteristic polynomial* of φ. The equivalence of these definitions and those given in Section 1 of Chapter 1 is demonstrated by Proposition 1.8. The definition (1.3) implies that $\mathcal{E}$ is a submodule of the $\mathcal{A}$-module $\mathcal{F}$.

The exponential-polynomial functions are those whose values solve a homogeneous linear recurrence relation with constant coefficients. Indeed, suppose that $c(\mathrm{t}) \in \Bbbk[\mathrm{t}]$ is a non-zero polynomial of degree q, and write $c(\mathrm{t}) = \sum_{k=0}^{q} c_k \mathrm{t}^k$. Then $c \cdot \varphi = 0$ if and only if

$$0 = (c \cdot \varphi)(m) = c_0\varphi(m) + c_1\varphi(m+1) + \cdots + c_q\varphi(m+q), \tag{1.4}$$

for all $m \in \mathbb{Z}$, i.e. precisely when the values of φ satisfy the recurrence relation (1.4) defined by c. In particular, a solution φ to $c \cdot \varphi = 0$ is determined by any q of its consecutive values. Therefore, if $\varphi \in \mathcal{E}$ is non-zero, then the support of φ is not wholly contained in any of the infinite subsets of consecutive integers $\mathbb{N}, -\mathbb{N} \subset \mathbb{Z}$. It follows from Lemma 1.5 that the submonoid of $\mathbb{Z}$ generated by the support of φ is of the form $r\mathbb{Z}$, for some $r > 0$. Equivalently, $\operatorname{im} \tilde{\varphi} = \Bbbk[\mathrm{t}^r, \mathrm{t}^{-r}]$, and so $\varphi \in \mathcal{F}'$. Thus $\mathcal{E} \setminus \{\, 0 \,\} \subset \mathcal{F}'$.
Lemma 1.5. Suppose that A is a submonoid of $\mathbb{Z}$ such that $\mathbb{N}, -\mathbb{N} \not\subset A$. Then $A = r\mathbb{Z}$, where $r \in A$ is any non-zero element of minimal absolute value.

Proof. Let $r \in A \cap \mathbb{N}$ be of minimal absolute value. For any $m \in A \cap -\mathbb{N}$, we have that $m + kr \in A$ where k is the unique positive integer such that

$$0 \leqslant m + k_0 r < r.$$

Thus $m + kr = 0$ by the minimality of r; it follows that r divides m, for any $m \in A \cap -\mathbb{N}$. Moreover,

$$-r = m + (k-1)r \qquad \in A$$

since $k - 1$ is non-negative. It follows therefore that $-r$ is the element of minimal absolute value in $A \cap -\mathbb{N}$. The argument above with inequalities reversed shows that $-r$ divides all positive elements of A, and so $A \subset r\mathbb{Z}$. The opposite inclusion is obvious since $r, -r \in A$ and A is closed under addition. $\square$

For any $k \geqslant 0$ and $\lambda \in \mathbb{k}^\times$, define the function $\theta_{\lambda,k} \in \mathcal{F}$ by

$$\theta_{\lambda,k}(m) = m^k \lambda^m, \qquad m \in \mathbb{Z}.$$

Lemma 1.6. For any $\lambda, \mu \in \mathbb{k}^\times$ and $k \geqslant 0$,

i. $(\mathrm{t} - \mu) \cdot \theta_{\lambda,k} = (\lambda - \mu)\theta_{\lambda,k} + \lambda \sum_{j=0}^{k-1} \binom{k}{j} \theta_{\lambda,j}$;
ii. $(\mathrm{t} - \lambda)^k \cdot \theta_{\lambda,k} = k! \lambda^k \theta_{\lambda,0}$.

Proof. For any $m \in \mathbb{Z}$,

$$\begin{aligned}(\mathrm{t} \cdot \theta_{\lambda,k})(m) = \theta_{\lambda,k}(m+1) &= (m+1)^k \lambda^{m+1} \\ &= \lambda \textstyle\sum_{j=0}^{k} \binom{k}{j} m^j \lambda^m \\ &= \lambda \textstyle\sum_{j=0}^{k} \binom{k}{j} \theta_{\lambda,j}(m).\end{aligned}$$

Therefore,

$$(\mathrm{t} - \mu) \cdot \theta_{\lambda,k} = \lambda \textstyle\sum_{j=0}^{k} \binom{k}{j} \theta_{\lambda,j} - \mu\theta_{\lambda,k},$$

and so part (i) is proven. Part (ii) is proven by induction. The claim is trivial if $k = 0$, so suppose that the claim holds for some $k \geqslant 0$. Then

$$\begin{aligned}(\mathrm{t} - \lambda)^{k+1} \cdot \theta_{\lambda,k+1} &= (\mathrm{t} - \lambda)^k \cdot \lambda \textstyle\sum_{j=0}^{k} \binom{k+1}{j} \theta_{\lambda,j} \\ &= \lambda \binom{k+1}{k} k! \lambda^k \theta_{\lambda,0} \quad \text{(by inductive hypothesis)} \\ &= (k+1)! \lambda^{k+1} \theta_{\lambda,0},\end{aligned}$$

where part (i) is used in obtaining the first and second equalities. Therefore the claim holds for all $k \geqslant 0$ by induction. $\square$

Proposition 1.7. The set $\{\, \theta_{\lambda,k} \mid \lambda \in \mathbb{k}^\times,\ k \geqslant 0 \,\} \subset \mathcal{F}$ is linearly independent.

Proof. Suppose that $\gamma_{\lambda,k} \in \Bbbk$, $\lambda \in \Bbbk^\times$, $k \geqslant 0$, are scalars such that the sum

$$\varphi = \sum_{\lambda \in \Bbbk^\times} \sum_{k \geqslant 0} \gamma_{\lambda,k} \theta_{\lambda,k},$$

is finite and equal to zero. Write $Z = \{\, \lambda \in \Bbbk^\times \mid \gamma_{\lambda,k} \neq 0 \ \text{ for some } \ k \geqslant 0 \,\}$, and let

$$n_\lambda = \max\{\, k \mid \gamma_{\lambda,k} \neq 0 \,\}, \quad \lambda \in Z.$$

Then, for any $\lambda \in Z$,

$$\begin{aligned} 0 &= \prod_{\mu \in Z} (\mathrm{t} - \mu)^{n_\mu + 1 - \delta_{\lambda,\mu}} \cdot \varphi \\ &= \prod_{\mu \in Z} (\mathrm{t} - \mu)^{n_\mu + 1 - \delta_{\lambda,\mu}} \cdot (\gamma_{\lambda,n_\lambda} \theta_{\lambda,n_\lambda}) \\ &= \gamma_{\lambda,n_\lambda} \lambda^{n_\lambda} n_\lambda! \prod_{\mu \in Z, \mu \neq \lambda} (\lambda - \mu)^{n_\mu + 1} \cdot \theta_{\lambda,0}, \end{aligned}$$

by Lemma 1.6. Therefore $\gamma_{\lambda,n_\lambda} = 0$ for all $\lambda \in Z$, which is absurd, unless Z is the empty set. □

Proposition 1.8. Suppose that $\varphi \in \mathcal{E}$. Then φ has a unique expression

$$\varphi = \sum_{\lambda \in \Bbbk^\times} \varphi_\lambda \mathrm{EXP}(\lambda), \tag{1.9}$$

as a finite sum of products of polynomials functions φ_λ and exponential functions $\mathrm{EXP}(\lambda)$, $\lambda \in \Bbbk^\times$. Moreover,

$$\mathrm{c}_\varphi(\mathrm{t}) = \prod_{\lambda \in Z} (\mathrm{t} - \lambda)^{\deg \varphi_\lambda + 1}, \tag{1.10}$$

where $Z = \{\, \lambda \in \Bbbk^\times \mid \varphi_\lambda \neq 0 \,\}$.

Proof. Let $c \in \Bbbk[\mathrm{t}]$ be of degree q. The equation $c \cdot \varphi = 0$ is equivalent to the relation (1.4), and so the space consisting of all solutions φ is at most q-dimensional. Now write $Z \subset \Bbbk^\times$ for the set of all roots of c. The field $\Bbbk$ is algebraically closed, and so

$$c(\mathrm{t}) \sim_{\Bbbk^\times} \prod_{\lambda \in Z} (\mathrm{t} - \lambda)^{m_\lambda}, \tag{1.11}$$

where m_λ is the multiplicity of the root $\lambda \in Z$. Lemma 1.6 shows that the set

$$\{\, \theta_{\lambda,k} \mid \lambda \in Z, \ \ 0 \leqslant k < m_\lambda \,\},$$

which is of size $\sum_{\lambda \in Z} m_\lambda = q$, consists of solutions to $c \cdot \varphi = 0$. By Proposition 1.7, this set is linearly independent, and hence is a basis for the solution space. Therefore any $\varphi \in \mathcal{E}$ has a unique expression (1.9).

Now suppose that φ has the form (1.9), let $c \in \Bbbk[\mathrm{t}]$ be non-zero, and write c in the form (1.11). By Lemma 1.6 part (i), $c \cdot \varphi = 0$ if and only if $m_\lambda > \deg \varphi_\lambda$ whenever $\varphi_\lambda \neq 0$. The polynomial (1.10) is the minimal degree monic polynomial that satisfies this condition, and hence is the characteristic polynomial. □

2. Loop-Module Realisation of $\mathbf{N}(\varphi)$

For $\varphi \in \mathcal{F}$, let $\Bbbk \mathrm{v}_\varphi$ be the one-dimensional $\hat{\mathfrak{h}}$-module defined by

$$\mathrm{h} \otimes a \cdot \mathrm{v}_\varphi = (a \cdot \varphi)(0)\mathrm{v}_\varphi, \qquad a \in \mathcal{A}.$$

Let $\hat{\mathfrak{g}}_+ \cdot \mathrm{v}_\varphi = 0$, and denote by

$$\mathsf{V}(\varphi) = \mathrm{Ind}_{\hat{\mathfrak{h}} \oplus \hat{\mathfrak{g}}_+}^{\hat{\mathfrak{g}}} \Bbbk \mathrm{v}_\varphi$$

the induced $\hat{\mathfrak{g}}$-module. This definition is equivalent to the definition (1.5) of Chapter 1. The module $\mathsf{V}(\varphi)$ and its unique irreducible quotient $\mathsf{L}(\varphi)$ are not $\mathbb{Z}$-graded. In this section, it is shown that if $\varphi \in \mathcal{F}'$, then $\mathbf{N}(\varphi)$ is isomorphic to an irreducible constituent of the loop module $\widehat{\mathsf{L}(\varphi)}$, and moreover that this constituent may be described in terms of the semi-invariants of an action of the cyclic group $\mathbb{Z}_r$ on $\mathsf{L}(\varphi)$, $r = \deg \varphi$. The results of this section are due to Chari and Pressley [**9**] (see also [**7**]).

2.1. Cyclic group action on $\mathsf{L}(\varphi)$.

Lemma 2.1. Suppose that $\varphi \in \mathcal{F}'$, that $r = \deg \varphi$, and that $\zeta \in \Bbbk^\times$ is such that $\zeta^r = 1$. Then for all $a \in \mathcal{A}$,

$$(a(\zeta \mathrm{t}) \cdot \varphi)(0) = (a \cdot \varphi)(0).$$

Proof. The support of φ is contained in $r\mathbb{Z}$. Therefore, if $a(\mathrm{t}) = \sum_i a_i \mathrm{t}^i$, then

$$(a(\zeta \mathrm{t}) \cdot \varphi)(0) = \sum_{i \equiv 0 \pmod r} a_i \zeta^i \varphi(i) = \sum_{i \equiv 0 \pmod r} a_i \varphi(i) = (a \cdot \varphi)(0). \qquad \square$$

Proposition 2.2. Suppose that $\varphi \in \mathcal{F}'$, and that $r = \deg \varphi$. Then there exists an order-r automorphism $\eta = \eta_\varphi$ of the vector space $\mathsf{L}(\varphi)$ defined by $\eta(\mathrm{v}_\varphi) = \mathrm{v}_\varphi$ and

$$\eta(x \otimes a \cdot w) = x \otimes a(\zeta^{-1}\mathrm{t}) \cdot \eta(w), \qquad x \in \mathfrak{g}, \quad a \in \mathcal{A}, \quad w \in \mathsf{L}(\varphi),$$

where $\zeta = \zeta_r$. Moreover, η decomposes $\mathsf{L}(\varphi)$ as a direct sum of eigenspaces

$$\mathsf{L}(\varphi) = \bigoplus_{i \in \mathbb{Z}_r} \mathsf{L}(\varphi)|^{\eta}_{\zeta^i}$$

in a manner compatible with the weight-space decomposition induced by $\mathrm{h} \otimes \mathrm{t}^0$.

Proof. The rule $\mathrm{t} \mapsto \zeta^{-1}\mathrm{t}$ extends to an automorphism of $\mathcal{A}$, which defines an automorphism of the loop algebra $\hat{\mathfrak{g}}$. This automorphism in turn defines an automorphism η of $\mathcal{U}(\hat{\mathfrak{g}})$. The universal module $\mathsf{V}(\varphi)$ may be realised as the quotient of $\mathcal{U}(\hat{\mathfrak{g}})$ by the left ideal I generated by $\hat{\mathfrak{g}}_+$ and by the elements of the set

$$\{ \mathrm{h} \otimes a - (a \cdot \varphi)(0) \mid a \in \mathcal{A} \}.$$

The map η preserves this set by Lemma 2.1:

$$\begin{aligned} \eta(\mathrm{h} \otimes a - (a \cdot \varphi)(0)) &= \mathrm{h} \otimes a(\zeta^{-1}\mathrm{t}) - (a \cdot \varphi)(0) \\ &= \mathrm{h} \otimes a(\zeta^{-1}\mathrm{t}) - (a(\zeta^{-1}\mathrm{t}) \cdot \varphi)(0). \end{aligned}$$

Clearly η preserves $\hat{\mathfrak{g}}_+$, and so $\eta(I) = I$. Therefore η is well-defined on the quotient $\mathsf{V}(\varphi)$ of $\mathcal{U}(\hat{\mathfrak{g}})$. The monomial

$$\mathrm{f} \otimes \mathrm{t}^{n_1} \cdots \mathrm{f} \otimes \mathrm{t}^{n_k} \cdot \mathrm{v}_\varphi \quad \in \mathsf{V}(\varphi)$$

is an eigenvector of eigenvalue ζ^{-m} where $m = \sum_{i=1}^{k} n_i$, and so the Poincaré-Birkhoff-Witt Theorem guarantees a decomposition

$$\mathsf{V}(\varphi) = \bigoplus_{i \in \mathbb{Z}_r} \mathsf{V}(\varphi)|^{\eta}_{\zeta^i} \tag{2.3}$$

of $\mathsf{V}(\varphi)$ into eigenspaces for η. It is easy to check that η commutes with the action of $\mathrm{h} \otimes \mathrm{t}^0$, and that if U is a submodule of $\mathsf{V}(\varphi)$, then so is $\eta(U)$. Thus, if U is a proper submodule, then so is $\eta(U)$. Hence η preserves the maximal submodule of $\mathsf{V}(\varphi)$, and so is defined on the quotient $\mathsf{L}(\varphi)$. This induced map is of order r, by construction, and decomposes $\mathsf{L}(\varphi)$ in the manner claimed by (2.3). □

2.2. Irreducible constituents of the loop module. For any $\varphi \in \mathcal{F}'$, define an automorphism $\hat{\eta}_\varphi$ of the vector space $\widehat{\mathsf{L}(\varphi)}$ via

$$\hat{\eta}_\varphi(u \otimes a) = \eta_\varphi(u) \otimes a(\zeta_r \mathrm{t}), \qquad u \in \mathsf{L}(\varphi), \quad a \in \mathcal{A},$$

where $r = \deg \varphi$.

Theorem 2.4. Suppose that $\varphi \in \mathcal{F}'$, and that $r = \deg \varphi$. Let $\zeta = \zeta_r$ and $\hat{\eta} = \hat{\eta}_\varphi$. Then:

i. $\hat{\eta}$ is automorphism of the $\mathbb{Z}$-graded $\hat{\mathfrak{g}}$-module $\widehat{\mathsf{L}(\varphi)}$ of order r;

ii. $\hat{\eta}$ decomposes $\widehat{\mathsf{L}(\varphi)}$ as a direct sum of eigenspaces

$$\widehat{\mathsf{L}(\varphi)} = \bigoplus_{i \in \mathbb{Z}_r} \widehat{\mathsf{L}(\varphi)}|^{\hat{\eta}}_{\zeta^i},$$

where

$$\widehat{\mathsf{L}(\varphi)}|^{\hat{\eta}}_{\zeta^i} = \bigoplus_{m \in \mathbb{Z}} \mathsf{L}(\varphi)|^{\eta_\varphi}_{\zeta^{i-m}} \otimes \mathrm{t}^m, \qquad i \in \mathbb{Z}_r;$$

iii. For any $i \in \mathbb{Z}_r$, the $\mathbb{Z}$-graded $\hat{\mathfrak{g}}$-modules $\widehat{\mathsf{L}(\varphi)}|^{\hat{\eta}}_{\zeta^i}$ and $\mathbf{N}(\varphi)$ are isomorphic.

Proof. For any $x \in \mathfrak{g}$, $u \in \mathsf{L}(\varphi)$, $a \in \mathcal{A}$ and $m \in \mathbb{Z}$,

$$\begin{aligned}
\hat{\eta}(x \otimes \mathrm{t}^m \cdot u \otimes a) &= \hat{\eta}((x \otimes \mathrm{t}^m \cdot u) \otimes \mathrm{t}^m a) \\
&= \zeta^m \eta(x \otimes \mathrm{t}^m \cdot u) \otimes \mathrm{t}^m a(\zeta \mathrm{t}) \\
&= \zeta^m \zeta^{-m} (x \otimes \mathrm{t}^m \cdot \eta(u)) \otimes \mathrm{t}^m a(\zeta \mathrm{t}) \\
&= x \otimes \mathrm{t}^m \cdot (\eta(u) \otimes a(\zeta \mathrm{t})) \\
&= x \otimes \mathrm{t}^m \cdot \hat{\eta}(u \otimes a),
\end{aligned}$$

where $\eta = \eta_\varphi$. The map $\hat{\eta}$ is of order r by definition, and so part (i) is proven. Part (ii) follows immediately from Proposition 2.2. Let $i \in \mathbb{Z}_r$, and write $U = \widehat{\mathsf{L}(\varphi)}|^{\hat{\eta}}_{\zeta^i}$. The generating weight spaces $U^{\varphi(0)\alpha} \subset U$ and $\mathbf{H}(\varphi) \subset \mathbf{N}(\varphi)$ are isomorphic as $\mathbb{Z}$-graded $\hat{\mathfrak{h}}$-modules, via

$$\mathrm{v}_\varphi \otimes \mathrm{t}^{mr+i} \mapsto \mathrm{t}^{mr}, \qquad m \in \mathbb{Z}.$$

This map extends uniquely to an epimorphism of $\mathbb{Z}$-graded $\hat{\mathfrak{g}}$-modules $U \to \mathbf{N}(\varphi)$. Therefore it is sufficient to prove that U is an irreducible $\mathbb{Z}$-graded $\hat{\mathfrak{g}}$-module. Suppose that W is a graded submodule of U. Then W contains a non-zero homogeneous maximal vector $v \otimes \mathrm{t}^n$. The $\hat{\mathfrak{g}}$-module epimorphism $U \to \mathsf{L}(\varphi)$ that is induced by $\mathrm{t} \mapsto 1$ maps this element to a non-zero maximal vector of $\mathsf{L}(\varphi)$. Therefore $v = \lambda \mathrm{v}_\varphi$ is a non-zero scalar multiple of the highest-weight vector. Hence W has non-trivial intersection with the generating weight space $U^{\varphi(0)\alpha}$ of U. The $\mathbb{Z}$-graded $\hat{\mathfrak{h}}$-module $U^{\varphi(0)\alpha}$ is irreducible, so $U^{\varphi(0)\alpha} \subset W$, and thus $W = U$. Therefore U is irreducible. □

2.3. Characters and semi-invariants. Theorem 2.4 describes the modules $\mathbf{N}(\varphi)$ in terms of the semi-invariants of $\mathsf{L}(\varphi)$ with respect to the action of the cyclic group $\mathbb{Z}_r$ defined by η, where $r = \deg \varphi$. In particular, we have the following description of the character of an exponential-polynomial module.

Corollary 2.5. Suppose that $\varphi \in \mathcal{E}$ is non-zero and that $\deg \varphi = r$. Then

$$\operatorname{char} \mathbf{N}(\varphi) = \sum_{k \geqslant 0} \sum_{n \in \mathbb{Z}} \dim \mathsf{L}(\varphi)_k|^{\eta}_{\zeta^n} \mathrm{X}^k \mathrm{Z}^n,$$

where $\zeta = \zeta_r$.

3. Semi-invariants of Actions of Finite Cyclic Groups

A $\mathbb{Z}_+$*-graded vector-space* is a vector space V over $\Bbbk$ with a decomposition $V = \bigoplus_{k \geqslant 0} V(k)$ of V into finite-dimensional subspaces indexed by $\mathbb{Z}_+$. If V is a $\mathbb{Z}_+$-graded vector space and r is a positive integer, then the tensor power

$$V^r := V \otimes \cdots \otimes V \qquad (r \text{ times})$$

is also a $\mathbb{Z}_+$-graded vector space, with the decomposition

$$V^r = \bigoplus\nolimits_{k \geqslant 0} V^r(k), \qquad V^r(k) = \bigoplus\nolimits_{k_1 + \cdots + k_r = k} V(k_1) \otimes \cdots \otimes V(k_r).$$

The finite cyclic group $\mathbb{Z}_r$ acts on V^r by cycling homogeneous tensors; the generator $1 \in \mathbb{Z}_r$ acts via the vector space automorphism

$$\sigma_r : v_1 \otimes \cdots \otimes v_r \mapsto v_r \otimes v_1 \cdots \otimes v_{r-1}, \qquad v_i \in V,$$

and this action preserves the grading, so that $\sigma_r(V^r(k)) = V^r(k)$, for any $k \geqslant 0$. For any $U \subset V^r$, let

$$U_n = U|^{\sigma_r}_{\zeta_r^n}, \qquad n \in \mathbb{Z}.$$

The automorphism σ_r decomposes V^r as a direct sum of $\mathbb{Z}_+$-graded vector spaces

$$V^r = \bigoplus\nolimits_{n \in \mathbb{Z}_r} V^r_n, \qquad V^r_n = \bigoplus\nolimits_{k \geqslant 0} V^r_n(k), \qquad V^r_n(k) = (V^r(k))_n.$$

Associated to any $\mathbb{Z}_+$-graded vector space U is the generating function

$$\mathscr{P}_U(\mathrm{X}) = \sum_{k \geqslant 0} \dim U_k \mathrm{X}^k \quad \in \mathbb{Z}_+[[\mathrm{X}]].$$

Theorem. For any $\mathbb{Z}_+$-graded vector space V, $r > 0$ and $n \in \mathbb{Z}$,

$$\mathscr{P}_{V^r_n}(\mathrm{X}) = \frac{1}{r} \sum_{d \mid r} \mathrm{c}_d(n) \left(\mathscr{P}_V(\mathrm{X}^d) \right)^{\frac{r}{d}}.$$

In this section, we describe an elementary proof of this statement. In the particular case where U is the regular representation of $\mathbb{Z}_r$ and $V = \mathrm{S}(U)$ is the symmetric algebra, the statement follows from Molien's Theorem and the identity (1.2).

Fix a $\mathbb{Z}_+$-graded vector space V, let

$$\mathrm{B} = \{ (k,s) \in \mathbb{Z}_+^2 \mid 1 \leqslant s \leqslant \dim V(k) \}$$

and for each $k \geqslant 0$, choose a basis $\{ v_s^k \}_{1 \leqslant s \leqslant \dim V(k)}$ for $V(k)$. For any $r > 0$ and $k \geqslant 0$, let

$$\mathrm{D}_{r,k} = \{ ((k_1,s_1),\ldots,(k_r,s_r)) \in \mathrm{B}^r \mid \textstyle\sum_{i=1}^r k_i = k \}.$$

The elements of $\mathrm{D}_{r,k}$ parameterise a graded basis of $V^r(k)$:

$$\mathrm{D}_{r,k} \ni \quad ((k_1,s_1),\ldots,(k_r,s_r)) = I \quad \leftrightarrow \quad v_I = v_{s_1}^{k_1} \otimes \cdots \otimes v_{s_r}^{k_r} \quad \in V(k).$$

Define an automorphism τ_r of the sets $\mathrm{D}_{r,k}$ via the rule

$$\tau_r : ((k_1,s_1),\ldots,(k_r,s_r)) \mapsto ((k_r,s_r),(k_1,s_1),\ldots,(k_{r-1},s_{r-1})).$$

The automorphisms σ_r and τ_r are compatible in the sense that

$$\sigma_r(v_I) = v_{\tau_r(I)}, \qquad I \in \mathrm{D}_{r,k}, \quad k \geqslant 0.$$

For $I \in \mathrm{D}_{r,k}$, write $\operatorname{ord} I = d$ for the minimal positive integer such that $(\tau_r)^d(I) = I$. For any positive divisor d of r, let

$$\mathrm{O}_{r,d}(k) = \#\{ I \in \mathrm{D}_{r,k} \mid \operatorname{ord} I = d \}, \qquad k \geqslant 0,$$

and write $\mathscr{O}_{r,d}(\mathrm{X}) = \sum_{k \geqslant 0} \mathrm{O}_{r,d}(k)\mathrm{X}^k$ for the generating function. It is apparent that

$$\mathscr{P}_{V^r}(\mathrm{X}) = (\mathscr{P}_V(\mathrm{X}))^r = \sum_{d \mid r} \mathscr{O}_{r,d}(\mathrm{X}) \tag{3.1}$$

Lemma 3.2. Suppose that l, r are positive integers and that $l \mid r$. Then

$$\{ d \mid d > 0, \ \ r/l \mid d \ \text{ and } \ d \mid r \} = \{ r/d' \mid d' > 0, \ \ d' \mid l \}.$$

Proof. If $d > 0$ and $\frac{r}{l} \mid d$, then there exists some positive integer s such that

$$d = \frac{r}{l} s = \frac{r}{(l/s)};$$

if in addition $d \mid r$, then $d' := l/s$ is a positive integer, and so $d = r/d'$ with $d' \mid l$. Conversely, if $d' \mid l$, then $r/l \mid r/d'$, and it is obvious that $r/d' \mid r$. $\square$

Proposition 3.3. For any $\mathbb{Z}_+$-graded vector space V and any $r > 0$,

$$\mathscr{P}_{V_n^r}(\mathrm{X}) = \frac{1}{r} \sum_{d \mid \gcd(r,n)} d\,\mathscr{O}_{r,\frac{r}{d}}(\mathrm{X}).$$

for all $n \in \mathbb{Z}$.

Proof. Suppose that $k \geqslant 0$, and write $\mathrm{D}_{r,k} = \bigsqcup_{O\in\mathcal{P}} O$ for the decomposition of $\mathrm{D}_{r,k}$ into a disjoint union of orbits for the action of $\mathbb{Z}_r$ defined by τ_r. Then

$$V^r(k) = \bigoplus_{O\in\mathcal{P}} U_O, \qquad U_O = \operatorname{span}\{\, v_I \mid I \in O \,\},$$

and moreover $\sigma_r(U_O) = U_O$. For any orbit $O \in \mathcal{P}$, the action of σ_r on U_O defines the regular representation of $\mathbb{Z}_d$, where $d = \#O$ is the size of the orbit; in particular, the eigenvalues of σ_r on U_O are precisely the roots of unity ζ such that $\zeta^d = 1$, each with multiplicity 1. Now ζ_r^n is of order $\frac{r}{\gcd(r,n)}$. Therefore,

$$\begin{aligned}\dim V_n^r(k) &= \#\{\, O\in\mathcal{P} \mid \tfrac{r}{\gcd(r,n)} \mid \#O \,\}\\ &= \#\{\, O\in\mathcal{P} \mid \#O = \tfrac{r}{d} \text{ for some } d \mid \gcd(r,n) \,\}\end{aligned}$$

where the last equality follows from Lemma 3.2 with $l = \gcd(r,n)$. The number of orbits $O \in \mathcal{P}$ of size r/d is precisely $d/r \cdot \mathrm{O}_{r,r/d}(k)$. It follows therefore that

$$\dim V_n^r(k) = \sum_{d\mid\gcd(r,n)} \tfrac{d}{r}\mathrm{O}_{r,\frac{r}{d}}(k),$$

which yields the required equality of generating functions. $\square$

Proposition 3.4. For any $\mathbb{Z}_+$-graded vector space V and positive integers r, d with $d \mid r$,

$$\mathscr{O}_{r,d}(\mathrm{X}) = \mathscr{O}_{d,d}(\mathrm{X}^{\frac{r}{d}})$$

Proof. Suppose that $k \geqslant 0$, that

$$I = ((k_1, s_1), \ldots, (k_r, s_r)) \in \mathrm{D}_{r,k}$$

and that $\operatorname{ord} I = d$. Then

$$I' = ((k_1, s_1), \ldots, (k_d, s_d)) \in \mathrm{D}_{d,\frac{kd}{r}}.$$

and $\operatorname{ord} I' = d$. This establishes a bijection between order-d elements of the sets $\mathrm{D}_{r,k}$ and $\mathrm{D}_{d,\frac{kd}{r}}$, and so $\mathrm{O}_{r,d}(k) = \mathrm{O}_{d,d}(\frac{kd}{r})$. Therefore

$$\begin{aligned}\mathscr{O}_{r,d}(\mathrm{X}) &= \sum_{k\geqslant 0} \mathrm{O}_{d,d}(\tfrac{kd}{r})\mathrm{X}^k\\ &= \sum_{k\geqslant 0} \mathrm{O}_{d,d}(k)\,(\mathrm{X}^{\frac{r}{d}})^k\\ &= \mathscr{O}_{d,d}(\mathrm{X}^{\frac{r}{d}}).\end{aligned}$$

$\square$

It follows immediately from Proposition 3.4 and equation (3.1) that

$$(\mathscr{P}_V(\mathrm{X}))^r = \sum_{d|r} \mathscr{O}_{d,d}(\mathrm{X}^{\frac{r}{d}}). \tag{3.5}$$

Proposition 3.6. For any $\mathbb{Z}_+$-graded vector space V and $r > 0$,

$$\mathscr{O}_{r,r}(\mathrm{X}) = \sum_{d|r} \mu(d) \left(\mathscr{P}_V(\mathrm{X}^d)\right)^{\frac{r}{d}}.$$

Proof. The claim is trivial if $r = 1$, so suppose that $s > 1$ and that the claim holds for all $0 < r < s$. Then:

$$\begin{aligned}
\mathscr{O}_{s,s}(\mathrm{X}) &= (\mathscr{P}_V(\mathrm{X}))^s - \sum_{d|s,d\neq s} \mathscr{O}_{d,d}(\mathrm{X}^{\frac{r}{d}}) \quad \text{(by equation (3.5))} \\
&= (\mathscr{P}_V(\mathrm{X}))^s - \sum_{d|s,d\neq s} \sum_{d'|d} \mu(d') \left(\mathscr{P}_V(\mathrm{X}^{\frac{sd'}{d}})\right)^{\frac{d}{d'}} \quad \text{(by inductive hypothesis)} \\
&= (\mathscr{P}_V(\mathrm{X}))^s - \sum_{e|s,e\neq 1} \Big(\sum_{d|e,d\neq e} \mu(d)\Big) (\mathscr{P}_V(\mathrm{X}^e))^{\frac{s}{e}} \\
&\quad \text{(write } e = \tfrac{sd'}{d} \text{ and use Lemma 3.2)} \\
&= (\mathscr{P}_V(\mathrm{X}))^s - \sum_{e|s,e\neq 1} (-\mu(e)) (\mathscr{P}_V(\mathrm{X}^e))^{\frac{s}{e}} \quad \text{(by equation (1.1))} \\
&= \sum_{e|s} \mu(e) (\mathscr{P}_V(\mathrm{X}^e))^{\frac{s}{e}},
\end{aligned}$$

and so the claim holds for s also. □

Theorem 3.7. For any $\mathbb{Z}_+$-graded vector space V, $r > 0$ and $n \in \mathbb{Z}$,

$$\mathscr{P}_{V_n^r}(\mathrm{X}) = \frac{1}{r} \sum_{d|r} \mathrm{c}_d(n) \left(\mathscr{P}_V(\mathrm{X}^d)\right)^{\frac{r}{d}}.$$

Proof. For any $n \in \mathbb{Z}$,

$$\begin{aligned}
\mathscr{P}_{V_n^r}(\mathrm{X}) &= \frac{1}{r} \sum_{d|\gcd(r,n)} d\,\mathscr{O}_{r,\frac{r}{d}}(\mathrm{X}) \quad \text{(by Proposition 3.3)} \\
&= \frac{1}{r} \sum_{d|\gcd(r,n)} d\,\mathscr{O}_{\frac{r}{d},\frac{r}{d}}(\mathrm{X}^d) \quad \text{(by Proposition 3.4)} \\
&= \frac{1}{r} \sum_{d|\gcd(r,n)} d \sum_{d'|\frac{r}{d}} \mu(d') \left(\mathscr{P}_V(\mathrm{X}^{dd'})\right)^{\frac{r}{dd'}} \quad \text{(by Proposition 3.6)} \\
&= \frac{1}{r} \sum_{e|r} \Big(\sum_{d|\gcd(e,n)} d\mu(\frac{e}{d})\Big) \left(\mathscr{P}_V(\mathrm{X}^e)\right)^{\frac{r}{e}} \\
&= \frac{1}{r} \sum_{e|r} \mathrm{c}_e(n) \left(\mathscr{P}_V(\mathrm{X}^e)\right)^{\frac{r}{e}},
\end{aligned}$$

where the last equality follows from equation (1.2). □

4. Exponential-Polynomial Modules

In this section, we show that if $\varphi \in \mathcal{E}$, then the module $\mathsf{L}(\varphi)$ is an irreducible highest-weight module for the truncated current Lie algebra $\mathfrak{g}(\varphi)$. An explicit formula for the character of such a module was obtained in Chapter 4. Therefore, we are able to derive an explicit formula for $\operatorname{char} \mathbf{N}(\varphi)$ by employing the results of Sections 2 and 3.

4.1. Modules for truncated current Lie algebras.

Proposition 4.1. *Suppose that $\varphi \in \mathcal{E}$. Then the defining ideal $\mathfrak{g} \otimes \mathrm{c}_\varphi \mathcal{A} \subset \hat{\mathfrak{g}}$ acts trivially on the $\hat{\mathfrak{g}}$-module $\mathsf{L}(\varphi)$, and so $\mathsf{L}(\varphi)$ is a $\mathfrak{g}(\varphi)$-module.*

Proof. Let $\Bbbk \mathrm{v}_+$ denote the one-dimensional $\hat{\mathfrak{h}}$-module defined by

$$\mathrm{h} \otimes a \cdot \mathrm{v}_+ = (a \cdot \varphi)(0)\mathrm{v}_+, \qquad a \in \mathcal{A}.$$

Then by definition of the characteristic polynomial c_φ, the subalgebra $\mathfrak{h} \otimes \mathrm{c}_\varphi \mathcal{A} \subset \hat{\mathfrak{h}}$ acts trivially upon v_+, and so $\Bbbk \mathrm{v}_+$ may be considered as an $\mathfrak{h}(\varphi)$-module. Let $\mathfrak{g}_+(\varphi) \cdot \mathrm{v}_+ = 0$, and let

$$M = \operatorname{Ind}_{\mathfrak{h}(\varphi) \oplus \mathfrak{g}_+(\varphi)}^{\mathfrak{g}(\varphi)} \Bbbk \mathrm{v}_+$$

denote the induced $\mathfrak{g}(\varphi)$-module. Denote by L the unique irreducible quotient of M. Then L is a $\hat{\mathfrak{g}}$-module, via the canonical epimorphism $\hat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}(\varphi)$, and is irreducible with highest-weight defined by the function φ. Hence $\mathsf{L}(\varphi) \cong L$ as $\hat{\mathfrak{g}}$-modules, and the claim follows from the construction of L. $\square$

4.2. Tensor products.

Proposition 4.2. Let $\varphi_1, \varphi_2 \in \mathcal{E}$. Then

$$\mathsf{L}(\varphi_1 + \varphi_2) \cong \mathsf{L}(\varphi_1) \bigotimes \mathsf{L}(\varphi_2),$$

as $\hat{\mathfrak{g}}$-modules if c_{φ_1} and c_{φ_2} are co-prime.

Proof. Let $\varphi = \varphi_1 + \varphi_2$. Then $\mathrm{c}_\varphi = \mathrm{c}_{\varphi_1}\mathrm{c}_{\varphi_2}$ since c_{φ_1} and c_{φ_2} are co-prime. By Proposition 4.1, $\mathsf{L}(\varphi)$ is an irreducible module for $\mathfrak{g}(\varphi)$, and by the Chinese Remainder Theorem,

$$\mathfrak{g}(\varphi) \cong \mathfrak{g}(\varphi_1) \oplus \mathfrak{g}(\varphi_2). \tag{4.3}$$

By Proposition, 4.1 $\mathsf{L}(\varphi_i)$ is a module for $\mathfrak{g}(\varphi_i)$, $i = 1, 2$. The Lie algebra $\mathfrak{g}(\varphi_i)$ is finite-dimensional, and $\Bbbk$ is algebraically closed, and so $\mathcal{U}(\mathfrak{g}(\varphi_i))$ is Schurian [**27**], $i = 1, 2$. Thus $\mathcal{U}(\mathfrak{g}(\varphi_i))$ is tensor-simple [**2**], and so $\mathsf{L}(\varphi_1) \bigotimes \mathsf{L}(\varphi_2)$ is an irreducible module for $\mathcal{U}(\mathfrak{g}(\varphi_1)) \bigotimes \mathcal{U}(\mathfrak{g}(\varphi_2))$. The decomposition (4.3) and the Poincaré-Birkhoff-Witt Theorem imply that

$$\mathcal{U}(\mathfrak{g}(\varphi_1)) \bigotimes \mathcal{U}(\mathfrak{g}(\varphi_2)) \cong \mathcal{U}(\mathfrak{g}(\varphi)),$$

and so $\mathsf{L}(\varphi_1) \bigotimes \mathsf{L}(\varphi_2)$ is an irreducible module for $\mathfrak{g}(\varphi)$. The irreducible highest-weight modules $\mathsf{L}(\varphi)$ and $\mathsf{L}(\varphi_1) \bigotimes \mathsf{L}(\varphi_2)$ are of equal highest weight, by the Leibniz rule, and hence are isomorphic. $\square$

4.3. Semi-invariants of the modules $\mathsf{L}(\varphi)$.

Lemma 4.4. Suppose that $\varphi \in \mathcal{E}$ is non-zero. Then $\varphi \in \mathcal{F}'$, and $\varphi_\lambda = \varphi_{\zeta\lambda}$ whenever $\lambda, \zeta \in \Bbbk^\times$ and $\zeta^r = 1$, $r = \deg \varphi$. Moreover, there exists $\psi \in \mathcal{E}$ such that

i. $\varphi = \wp_r \psi$, and
ii. $\mathrm{c}_\varphi = \prod_{i \in \mathbb{Z}_r} \mathrm{c}_\psi(\zeta_r^i \mathrm{t})$ is a decomposition of c_φ into co-prime factors.

Proof. According to the discussion of subsection 1.2, $\varphi \in \mathcal{F}'$ and $r = \deg \varphi > 0$. Thus the support of φ is contained in the support $r\mathbb{Z}$ of $\wp_r$ and so $\varphi = \frac{1}{r}\wp_r\varphi$. Hence

$$\varphi_\lambda = (\frac{1}{r}\wp_r\varphi)_\lambda = \frac{1}{r}\sum_{i \in \mathbb{Z}_r} \varphi_{(\zeta_r^i\lambda)},$$

for any $\lambda \in \Bbbk^\times$. If $\zeta^r = 1$, then the expression on the right-hand side is invariant under the substitution $\lambda \mapsto \zeta\lambda$, and so the first claim is proven.

Multiplication by ζ_r decomposes $\Bbbk^\times$ into a disjoint union of orbits for the cyclic group $\mathbb{Z}_r$, and all orbits are of size r. Choose any set B of representatives, so that $\Bbbk^\times = \bigsqcup_{i\in\mathbb{Z}_r} \zeta_r^i B$. Then $\psi = \sum_{\lambda\in B} \varphi_\lambda \mathrm{EXP}(\lambda)$ has the required property, by Proposition 1.8. $\square$

Remark 4.5. The function $\psi \in \mathcal{E}$ of Lemma 4.4 is not unique. Indeed, if

$$\psi = \sum_i a_i \mathrm{EXP}(\mu_i)$$

has the required property, then so does $\psi' = \sum_i a_i \mathrm{EXP}(\zeta_r^{n_i}\mu_i)$ for any $n_i \in \mathbb{Z}_r$.

For any $\varphi \in \mathcal{F}$, consider $\mathsf{L}(\varphi)$ as a $\mathbb{Z}_+$-graded vector space via

$$\mathsf{L}(\varphi) = \bigoplus_{k\geqslant 0} \mathsf{L}(\varphi)(k), \qquad \mathsf{L}(\varphi)(k) = \mathsf{L}(\varphi)^{(\varphi(0)-k)\alpha}, \quad k \geqslant 0.$$

Proposition 4.6. Suppose that $\varphi \in \mathcal{E}$ is non-zero and that $\varphi = \wp_r\psi$, where $r = \deg\varphi$ and $\psi \in \mathcal{E}$, as per Lemma 4.4. Then there exists an isomorphism

$$\Omega : \mathsf{L}(\varphi) \to \mathsf{L}(\psi)^r$$

of $\mathbb{Z}_+$-graded vector spaces such that $\sigma_r = \Omega \circ \eta_\varphi \circ \Omega^{-1}$.

Proof. For $j \in \mathbb{Z}_r$, write $\psi^j = \mathrm{EXP}(\zeta_r^{-j})\psi$. Then $c_\varphi = \prod_{j\in\mathbb{Z}_r} c_{\psi^j}$ is a decomposition of c_φ into co-prime factors, and $\varphi = \sum_{j\in\mathbb{Z}_r} \psi^j$. By the Chinese Remainder Theorem, there exists a finite linearly independent set $\{\, a_i \mid i \in I \,\} \subset \mathcal{A}$ such that $\{\, a_i + c_\psi\mathcal{A} \mid i \in I \,\}$ is a basis for $\mathcal{A}/c_\psi\mathcal{A}$ and

$$a_i \equiv 0 \pmod{c_{\psi^j}}, \quad j \not\equiv 0 \pmod{r}, \quad i \in I, \quad j \in \mathbb{Z}_r.$$

Write $a_{i,j}(\mathrm{t}) = a_i(\zeta_r^{-j}\mathrm{t})$, $i \in I$, $j \in \mathbb{Z}_r$. Then by symmetry, $\{\, a_{i,j} + c_{\psi^j}\mathcal{A} \mid i \in I \,\}$ is a basis for $\mathcal{A}/c_{\psi^j}\mathcal{A}$ and

$$a_{i,j} \equiv 0 \pmod{c_{\psi^k}}, \quad j \not\equiv k \pmod{r}, \quad i \in I, \quad j,k \in \mathbb{Z}_r.$$

For any $i \in I$ and $j \in \mathbb{Z}_r$,

$$\eta_\varphi(\mathrm{f} \otimes a_{i,j} \cdot w) = \mathrm{f} \otimes a_{i,j}(\zeta_r^{-1}\mathrm{t}) \cdot \eta_\varphi(w) = \mathrm{f} \otimes a_{i,j+1} \cdot \eta_\varphi(w), \qquad w \in \mathsf{L}(\varphi). \tag{4.7}$$

By Proposition 4.2, there exists an isomorphism

$$\Upsilon : \mathsf{L}(\varphi) \to \bigotimes_{j\in\mathbb{Z}_r} \mathsf{L}(\psi^j)$$

of $\hat{\mathfrak{g}}$-modules, and we may assume that $\Upsilon(\mathrm{v}_\varphi) = \otimes_{j\in\mathbb{Z}_r}\mathrm{v}_{\psi^j}$. For any $k \in \mathbb{Z}_r$, identify

$$\mathsf{L}(\psi^k) = 1 \otimes \cdots \otimes \mathsf{L}(\psi^k) \otimes \cdots \otimes 1 \subset \bigotimes_{j\in\mathbb{Z}_r} \mathsf{L}(\psi^j). \tag{4.8}$$

Then $\mathsf{L}(\psi^k)$ is generated by the action of the basis $\{\, \mathrm{f} \otimes a_{i,k} \mid i \in I\,\}$ of $\mathfrak{g}_+(\psi^k)$ on the highest-weight vector $\Upsilon(\mathrm{v}_\varphi)$. Therefore, modulo the identification (4.8),

$$(\Upsilon \circ \eta_\varphi \circ \Upsilon^{-1})(\mathsf{L}(\psi^k)) \subset \mathsf{L}(\psi^{k+1}), \qquad k \in \mathbb{Z}_r,$$

by equation (4.7). Since η_φ is an automorphism of the $\mathbb{Z}_+$-graded vector space $\mathsf{L}(\varphi)$, the restriction

$$(\Upsilon \circ \eta_\varphi \circ \Upsilon^{-1}) : \mathsf{L}(\psi^k) \to \mathsf{L}(\psi^{k+1}),$$

is an isomorphism of the $\mathbb{Z}_+$-graded vector spaces. These isomorphisms obviously induce isomorphisms $\epsilon_j : \mathsf{L}(\psi^j) \to \mathsf{L}(\psi^0) = \mathsf{L}(\psi)$, and

$$\epsilon_j : \prod_{i\in I}(\mathrm{f} \otimes a_{i,j})^{k_i} \cdot \mathrm{v}_{\psi^j} \mapsto \prod_{i\in I}(\mathrm{f} \otimes a_i)^{k_i} \cdot \mathrm{v}_\psi,$$

by equation (4.7). Let $\epsilon = \bigotimes_{j\in\mathbb{Z}_r} \epsilon_j$, and write Ω for the composition

$$\epsilon \circ \Upsilon : \mathsf{L}(\varphi) \to \mathsf{L}(\psi)^r.$$

The vector space $\mathsf{L}(\psi)^r$ is spanned by the homogeneous tensors

$$\bigotimes_{j\in\mathbb{Z}_r} \prod_{i\in I}(\mathrm{f} \otimes a_i)^{k_{i,j}}\mathrm{v}_\psi, \qquad k_{i,j} \geqslant 0.$$

For any homogeneous tensor of this form

$$\begin{aligned}
(\Omega \circ \eta_\varphi \circ \Omega^{-1}) \quad \cdot \quad & (\textstyle\bigotimes_{j\in\mathbb{Z}_r} \prod_{i\in I}(\mathrm{f} \otimes a_i)^{k_{i,j}}\mathrm{v}_\psi) \\
&= \epsilon \circ (\Upsilon \circ \eta_\varphi \circ \Upsilon^{-1})(\textstyle\bigotimes_{j\in\mathbb{Z}_r} \prod_{i\in I}(\mathrm{f} \otimes a_{i,j})^{k_{i,j}}\mathrm{v}_{\psi^j}) \\
&= \epsilon \circ (\Upsilon \circ \eta_\varphi \circ \Upsilon^{-1})(\textstyle\prod_{j\in\mathbb{Z}_r} \prod_{i\in I}(\mathrm{f} \otimes a_{i,j})^{k_{i,j}} \cdot \otimes_{j\in\mathbb{Z}_r}\mathrm{v}_{\psi^j}) \\
&= \epsilon(\textstyle\prod_{j\in\mathbb{Z}_r} \prod_{i\in I}(\mathrm{f} \otimes a_{i,j})^{k_{i,j-1}} \cdot \otimes_{j\in\mathbb{Z}_r}\mathrm{v}_{\psi^j}) \\
&= \epsilon(\textstyle\bigotimes_{j\in\mathbb{Z}_r} \prod_{i\in I}(\mathrm{f} \otimes a_{i,j})^{k_{i,j-1}}\mathrm{v}_{\psi^j}) \\
&= \textstyle\bigotimes_{j\in\mathbb{Z}_r} \prod_{i\in I}(\mathrm{f} \otimes a_i)^{k_{i,j-1}}\mathrm{v}_\psi \\
&= \sigma_r(\textstyle\bigotimes_{j\in\mathbb{Z}_r} \prod_{i\in I}(\mathrm{f} \otimes a_i)^{k_{i,j}}\mathrm{v}_\psi),
\end{aligned}$$

where the second and fourth equalities are by construction of the polynomials $a_{i,j}$ and the Leibniz rule. Therefore $\Omega \circ \eta_\varphi \circ \Omega^{-1} = \sigma_r$ as required. □

4.4. Character Formulae.

Theorem 4.9. Suppose that $a \in \mathcal{F}$ is a polynomial function and that $\varphi = a\mathrm{EXP}(\lambda)$ for some $\lambda \in \Bbbk^\times$. Then

$$\mathscr{P}_{\mathsf{L}(\varphi)}(\mathrm{X}) = \begin{cases} \frac{1-\mathrm{X}^{a+1}}{1-\mathrm{X}} & \text{if } a \in \mathbb{Z}_+, \\ (1-\mathrm{X})^{-(\deg a)-1} & \text{otherwise.} \end{cases}$$

Proof. Let $\mathrm{N} = \deg a$, and write $\varphi = \sum_{k=0}^{\mathrm{N}} a_k \theta_{\lambda,k}$. By Proposition 4.1, $\mathsf{L}(\varphi)$ is a module for the truncated current Lie algebra $\mathfrak{g}(\varphi)$. The Cartan subalgebra of $\mathfrak{g}(\varphi)$ has a basis

$$\{\, \mathrm{h} \otimes (\mathrm{t}-\lambda)^k \mid 0 \leqslant k \leqslant \mathrm{N} \,\}.$$

By Lemma 1.6, $\mathrm{h} \otimes (\mathrm{t}-\lambda)^{\mathrm{N}}$ acts on the highest-weight vector v_φ by the scalar

$$((\mathrm{t}-\lambda)^{\mathrm{N}} \cdot \varphi)(0) = \mathrm{N}!\lambda^{\mathrm{N}} a_{\mathrm{N}}. \tag{4.10}$$

If $\mathrm{N} = 0$, then (4.10) takes the value $a \in \Bbbk$, and so $\mathsf{L}(\varphi)$ is the irreducible $\mathfrak{g}$-module of highest weight a. Therefore

$$\mathscr{P}_{\mathsf{L}(\varphi)}(\mathrm{X}) = \begin{cases} \frac{1-\mathrm{X}^{a+1}}{1-\mathrm{X}} & \text{if } a \in \mathbb{Z}_+, \\ \frac{1}{1-\mathrm{X}} & \text{otherwise.} \end{cases}$$

If $\mathrm{N} > 0$, then a_{N} is non-zero; thus (4.10) is non-zero and the claim follows from Proposition 4.A.1 (page 69). □

Suppose that $\varphi \in \mathcal{E}$ is non-zero, $\deg \varphi = r$, and that $\psi \in \mathcal{E}$ is given by Lemma 4.4. Then

$$\psi = \sum_i a_i \mathrm{EXP}(\lambda_i) \tag{4.11}$$

for some finite collection of polynomial functions $a_i \in \mathcal{F}$ and distinct $\lambda_i \in \Bbbk^\times$, such that if $(\lambda_i/\lambda_j)^r = 1$, then $i = j$.

Theorem 4.12. Suppose that $\varphi \in \mathcal{E}$ is non-zero, $\deg \varphi = r$, and that

$$\varphi = \wp_r \sum_i a_i \mathrm{EXP}(\lambda_i)$$

where the $a_i \in \mathcal{F}$ and $\lambda_i \in \Bbbk^\times$ are given by (4.11). Let

$$\mathrm{P}_\varphi(\mathrm{X}) = \frac{\prod_{a_i \in \mathbb{Z}_+} (1 - \mathrm{X}^{a_i+1})}{(1-\mathrm{X})^M},$$

where $M = \sum_i (\deg a_i + 1)$ and the product is over those indices i such that $a_i \in \mathbb{Z}_+$. Then

$$\operatorname{char} \mathbf{N}(\varphi) = \frac{1}{r} \sum_{n \in \mathbb{Z}} \sum_{d \mid r} \mathrm{c}_d(n) \left(\mathrm{P}_\varphi(\mathrm{X}^d)\right)^{\frac{r}{d}} \mathrm{Z}^n.$$

Proof. By Corollary 2.5 and Proposition 4.6,

$$\operatorname{char} \mathbf{N}(\varphi) = \sum_{n\in\mathbb{Z}} \mathscr{P}_{\mathsf{L}(\psi)^r_n}(\mathrm{X})\mathrm{Z}^n,$$

and by Theorem 3.7

$$\mathscr{P}_{\mathsf{L}(\psi)^r_n}(\mathrm{X}) = \frac{1}{r}\sum_{d|r} \mathrm{c}_d(n)\left(\mathscr{P}_{\mathsf{L}(\psi)}(\mathrm{X}^d)\right)^{\frac{r}{d}}. \tag{4.13}$$

By Proposition 4.2, there is an isomorphism of $\hat{\mathfrak{g}}$-modules

$$\mathsf{L}(\psi) \cong \bigotimes_i \mathsf{L}(\psi^i), \qquad \psi^i = a_i \mathrm{EXP}(\lambda_i),$$

since the λ_i are distinct. In particular, $\mathscr{P}_{\mathsf{L}(\psi)} = \prod_i \mathscr{P}_{\mathsf{L}(\psi^i)}$, and so

$$\mathscr{P}_{\mathsf{L}(\psi)} = \mathrm{P}_\varphi$$

by Theorem 4.9. Therefore the claim follows from equation (4.13). $\square$

Index of Symbols

$\mathbb{N}$	$\{1, 2, \ldots\}$	
$\mathbb{Z}_+$	$\{0, 1, 2, \ldots\}$	
$\mathbb{Z}$	ring of integers	
$\mathbb{Z}_r$	ring of integers modulo r	
$\mathbb{R}$, $\mathbb{C}$	fields of real and complex numbers	
$\Bbbk^\times$	non-zero elements of a field $\Bbbk$	
$\mathfrak{R}(r)$	primitive roots of unity of order r	
ζ_r	fixed element of $\mathfrak{R}(r)$	
$\gcd(m, n)$	greatest common divisor	
$\mathrm{Sym}(n)$	symmetric group on n symbols	
$\mathrm{sgn}(\sigma)$	sign of $\sigma \in \mathrm{Sym}(n)$	
$\#S$	size of a finite set S	
δ	Kronecker function	
$\langle \Lambda, v \rangle$	evaluation of a functional Λ at v	
V^*	dual of the vector space V	
$\mathrm{End}\, V$	endomorphism algebra of V	
$\mathcal{A}$	ring of Laurent polynomials	
$\mathcal{U}(\mathfrak{g})$	universal enveloping algebra	
PBW	Poincaré-Birkhoff-Witt	
$\mathrm{S}(V)$	symmetric algebra of V	
$\mathrm{T}(V)$	tensor algebra of V	
$\mathrm{ad}\, x$	adjoint operator of $x \in \mathfrak{g}$	
$\mathfrak{g}^\alpha$	root space of $\mathfrak{g}$	
M^χ	weight space of a module M	
$V\|_\lambda^\eta$	space of eigenvectors of eigenvalue λ for $\eta \in \mathrm{End}\, V$	
$\mathrm{e}, \mathrm{h}, \mathrm{f}$	basis elements of sl(2)	1
α	positive root of sl(2)	2

Bibliography

[1] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[2] V. Bavula. Each Schurian algebra is tensor-simple. *Comm. Algebra*, 23(4):1363–1367, 1995.

[3] Y. Billig and K. Zhao. Weight modules over exp-polynomial Lie algebras. *J. Pure Appl. Algebra*, 191(1-2):23–42, 2004.

[4] P. Casati and G. Ortenzi. New integrable hierarchies from vertex operator representations of polynomial Lie algebras. arXiv:nlin.SI/0405040v1.

[5] P. Casati and G. Ortenzi. New integrable hierarchies from vertex operator representations of polynomial Lie algebras. *J. Geom. Phys.*, 56(3):418–449, 2006.

[6] V. Chari. Integrable representations of affine Lie-algebras. *Invent. Math.*, 85(2):317–335, 1986.

[7] V. Chari and J. Greenstein. Graded level zero integrable representations of affine lie algebras. arXiv:math.RT/ 0602514v2.

[8] V. Chari and J. Greenstein. Quantum loop modules. *Represent. Theory*, 7:56–80 (electronic), 2003.

[9] V. Chari and A. Pressley. New unitary representations of loop groups. *Math. Ann.*, 275(1):87–104, 1986.

[10] E. Cohen. A class of arithmetical functions. *Proc. Nat. Acad. Sci. U.S.A.*, 41:939–944, 1955.

[11] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. *Higher transcendental functions. Vol. III.* McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955. Based, in part, on notes left by Harry Bateman.

[12] S. Eswara Rao. Classification of loop modules with finite-dimensional weight spaces. *Math. Ann.*, 305(4):651–663, 1996.

[13] V. M. Futorny. The parabolic subsets of root system and corresponding representations of affine Lie algebras. In *Proceedings of the International Conference on Algebra, Part 2 (Novosibirsk, 1989)*, volume 131 of *Contemp. Math.*, pages 45–52, Providence, RI, 1992. Amer. Math. Soc.

[14] V. M. Futorny. Imaginary Verma modules for affine Lie algebras. *Canad. Math. Bull.*, 37(2):213–218, 1994.

[15] V. M. Futorny. *Representations of affine Lie algebras*, volume 106 of *Queen's Papers in Pure and Applied Mathematics.* Queen's University, Kingston, ON, 1997.

[16] F. Geoffriau. Sur le centre de l'algèbre enveloppante d'une algèbre de Takiff. *Ann. Math. Blaise Pascal*, 1(2):15–31 (1995), 1994.

[17] F. Geoffriau. Homomorphisme de Harish-Chandra pour les algèbres de Takiff généralisées. *J. Algebra*, 171(2):444–456, 1995.

[18] J. Greenstein. Characters of simple bounded modules over an untwisted affine Lie algebra. *Algebr. Represent. Theory*, 6(2):119–137, 2003.

[19] H. P. Jakobsen and V. G. Kac. A new class of unitarizable highest weight representations of infinite-dimensional Lie algebras. In *Nonlinear equations in classical and quantum field theory (Meudon/Paris, 1983/1984)*, volume 226 of *Lecture Notes in Phys.*, pages 1–20. Springer, Berlin, 1985.

[20] V. G. Kac. *Infinite-dimensional Lie algebras.* Cambridge University Press, Cambridge, third edition, 1990.

[21] V. G. Kac and A. K. Raina. *Bombay lectures on highest weight representations of infinite-dimensional Lie algebras*, volume 2 of *Advanced Series in Mathematical Physics.* World Scientific Publishing Co. Inc., Teaneck, NJ, 1987.

[22] H. Li. On certain categories of modules for affine Lie algebras. *Math. Z.*, 248(3):635–664, 2004.

[23] G. Magnano and F. Magri. Poisson-Nijenhuis structures and Sato hierarchy. *Rev. Math. Phys.*, 3(4):403–466, 1991.

[24] R. V. Moody and A. Pianzola. *Lie algebras with triangular decompositions.* Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons Inc., New York, 1995. , A Wiley-Interscience Publication.

[25] C. A. Nicol and H. S. Vandiver. A von Sterneck arithmetical function and restricted partitions with respect to a modulus. *Proc. Nat. Acad. Sci. U. S. A.*, 40:825–835, 1954.

[26] M. Pedroni and P. Vanhaecke. A Lie algebraic generalization of the Mumford system, its symmetries and its multi-Hamiltonian structure. *Regul. Chaotic Dyn.*, 3(3):132–160, 1998. J. Moser at 70 (Russian).

[27] D. Quillen. On the endormorphism ring of a simple module over an enveloping algebra. *Proc. Amer. Math. Soc.*, 21:171–172, 1969.

[28] M. Raïs and P. Tauvel. Indice et polynômes invariants pour certaines algèbres de Lie. *J. Reine Angew. Math.*, 425:123–140, 1992.

[29] S. Ramanujan. On certain trigonometrical sums and their applications in the theory of numbers [Trans. Cambridge Philos. Soc. **22** (1918), no. 13, 259–276]. In *Collected papers of Srinivasa Ramanujan*, pages 179–199. AMS Chelsea Publ., Providence, RI, 2000.

[30] N. Shapovalov. On a bilinear form on the universal enveloping algebra of a complex semisimple Lie algebra. *Funct. Anal. Appl.*, 6:65–70, 1972.

[31] S. J. Takiff. Rings of invariant polynomials for a class of Lie algebras. *Trans. Amer. Math. Soc.*, 160:249–262, 1971.