Morse theory on Lie groupoids

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Resumo

Valencia, F. **Teoria de Morse em grupóides de Lie**. Tese (Doutorado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2024.

Estendemos a teoria de Morse clássica ao contexto dos grupóides de Lie e seus stacks diferenciáveis. Isto nos permite obter informações geométricas e topológicas dos objetos singulares representados pelos espaços de órbitas correspondentes, oferecendo uma abordagem unificada para estudar a teoria de Morse equivariante, bem como a teoria de Morse para orbifolds. Mostramos uma versão do lema de Morse em grupóides, descrevemos o comportamento topológico dos grupóides de Lie ao redor das órbitas críticas não degeneradas, estudamos a dinâmica de Morse-Smale e recuperamos a cohomologia de Bott–Shulman–Stasheff de um grupóide usando técnicas da teoria de Morse. Definimos funções de Morse em stacks, provando assim análogos dos resultados anteriores no contexto dos stacks diferenciáveis. Isto último nos permite obter desigualdades de tipo Morse para espaços de órbitas compactos associados a grupóides próprios. Para desenvolver uma teoria de Morse 2-equivariante em grupóides, introduzimos uma noção natural de ação isométrica de um 2-grupo de Lie em grupóides Riemannianos. As contrapartes globais e infinitesimais de tal noção são exploradas em detalhe. Também estudamos a existência de geodésicas fechadas em stacks Riemannianos e descrevemos construções que explicam como obter a cohomologia equivariante dos stacks simpléticos tóricos.

Palavras-chave: Teoria de Morse, grupóide Riemanniano, geodésica fechada, ação isométrica, cohomologia.

Abstract

Valencia, F. **Morse theory on Lie groupoids**. Thesis (PhD) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2024.

We extend classical Morse theory to the realm of Lie groupoids and their differentiable stacks. This allows us to obtain both topological and geometrical information of the singular objects represented by the corresponding orbit spaces, offering a unified approach to study equivariant Morse theory as well as Morse theory for orbifolds. We show a groupoid version of the Morse lemma, describe the topological behavior of Lie groupoids around nondegenerate critical orbits, study Morse–Smale dynamics, and recover the Bott–Shulman–Stasheff cohomology of a Lie groupoid by using Morse theory techniques. We define Morse stacky functions, thus proving analogues of the previous results in the context of differentiable stacks. The latter enables us to get Morse-like inequalities for compact orbit spaces of proper Lie groupoids. In order to develop a 2-equivariant Morse theory over Lie groupoids we introduce a natural notion of isometric Lie 2-group action on Riemannian groupoids. The global and infinitesimal counterparts of such a notion are explored in detail. We also study the existence of closed stacky geodesics on Riemannian stacks and describe constructions which explain how to obtain the equivariant cohomology of toric symplectic stacks.

Keywords: Morse theory, Riemannian groupoid, closed geodesic, isometric action, cohomology.

Contents

Introduction

The core of this thesis consists of extending classical Morse theory to the realm of Lie groupoids and their differentiable stacks, thus obtaining topological and geometrical information of the singular spaces that can be described with their underlying structure.

On the one hand, Morse theory can be thought of as a beautiful and natural extension of the minimum principle for continuous functions on compact spaces [22], so that it provides an important tool to study the topology of a manifold. Such a theory was named in honor to *Harold Calvin Marston Morse* (1892–1977) who ventured into the study of the critical points of a smooth function to understand geodesics which are actually critical points of the energy functional on paths [96]. These sort of techniques were later used in Bott's proof of his periodicity theorem [20, 21]. In fact, Bott's work [20] enables to get similar results as those from standard Morse theory but without assuming that the critical point sets are formed by isolated points. That is, the critical point sets of our functions may content submanifolds of positive dimension, each of which is assumed to be non-degenerate in the sense that its normal Hessian is a fiberwise non-degenerate bilinear symmetric form. The latter happens for instance when working with generic functions invariant by the action of a compact Lie group [116]. These ideas lead to the study of the so-called **Morse–Bott functions**. After Bott's seminal works, Morse theory has been applied to satisfactorily address several problems in both mathematics and physics as may be evidenced with the results of Smale [109] and Witten [119]. The first one studied Morse theory topological features from a more dynamical viewpoint by working with moduli spaces of gradient flow lines and the second one showed that the Morse inequalities can be obtained by considerations of a certain supersymmetric quantum mechanics Hamiltonian. Additionally, this theory has also led to several interesting geometrical results, including the existence of closed geodesics on compact Riemannian manifolds [66] as well as an infinite dimensional version of the Morse complex which led to Floer homology [8, 50].

On the other hand, groupoids were initially studied by Brandt [25] around the 1920s. However, in the context of differential geometry, Lie groupoids were introduced by Ehresmann [46] around the 1950s and the development of the Lie theory underlying their structure was initiated by Pradines [106]. The richness behind these geometric objects is given by the fact that they allow to encode symmetries of certain spaces which can not be captured by considering only a group. Lie groupoids encompass several classical geometries, as they generalize Lie group actions, surjective submersions, foliations, pseudogroups, vector and principal bundles, among others, giving lead to a new perspective on classical geometric questions and results [79, 80, 92]. Besides, these objects can be seen as an intermediate step in defining differentiable stacks, other geometric objects admitting singularities and generalizing both manifolds and orbifolds [17, 37]. The detailed treatment of the interaction between these two theories provides a clean way to perform differential geometry on certain singular spaces. In this regard, Lie groupoids equipped with geometric structures suitably compatible with Morita equivalence have been object of intense research since such structures descend to the quotient stack of a Lie groupoid. For instance, just to mention a few:

- the notion of Morita equivalence of VB-groupoids plays a role in defining vector bundles over differentiable stacks [42],
- Riemannian metrics on Lie groupoids were introduced in [40] showing that they behave well with respect to Morita equivalence, hence inducing a notion of Riemannian metric on the associated quotient stack [41],
- Lie algebroids over stacks are modeled by LA-groupoids [115] and the space of multiplicative sections of an LA-groupoid [100] has the structure of a Lie 2-algebra which is Morita invariant and, in particular, the space of vector fields on a differentiable stack has a natural structure of Lie algebra as conjectured in [58],
- 0-symplectic structures on Lie groupoids which are invariant by the action of foliation Lie 2-groups are also Morita invariant, so that they yield a notion of Hamiltonian differentiable stack [62], and
- stacky contact structures were introduced in [82] by looking at the notions of 0-shifted and +1-shifted contact structures over Lie groupoids. Both notions turn out to be Morita invariant as well.

In many situations a manifold comes with a Morse-Bott function whose critical submanifolds are not necessarily given by the orbits of a Lie group action but they still come from certain symmetries of the manifold. In this thesis we are interested in the case of Morse-Bott functions on a manifold whose critical submanifolds are given by the orbits of a Lie groupoid. Morse theory on singular spaces modeled by the orbit space of a Lie groupoid has been addressed by several authors. In fact, Lerman and Tolman studied torus actions on symplectic orbifolds for which they proved some results on Morse-Bott theory on orbifolds [73]. Similarly, Hepworth developed Morse theory on differentiable Deligne-Mumford stacks, showing for instance the Morse inequalities for orbifolds [57]. Also, Cho and Hong introduced the Morse-Smale-Witten complex for orbifolds [30]. Another approach known in the literature suitable for studying Morse theory on certain singular spaces is provided by the stratified Morse theory of Goresky and MacPherson [53].

Passing from a manifold to a differentiable stack means that one needs to replace a smooth function by a **stacky function**. It turns out that each of these stacky functions is completely determined by a real-valued Lie groupoid morphism and, in turn, by a **basic function**. The latter fact yields a simple way to establish a notion of Morse stacky function over differentiable stacks by imposing that the critical orbits of the corresponding basic function are non-degenerate in the sense of Morse–Bott theory. These natural notions will allow to extend the most important results of standard Morse theory to the context of Lie groupoids and their differentiable stacks. In the search for applications will also be desirable to develop some geometric and algebraic aspects revolving around the notion of Riemannian groupoid metric which comes to be one of the most important ingredients that constitutes this work.

This thesis is structured as explained right below.

Chapter 1 briefly introduces the general background about classical Morse theory, isometric Lie group actions, and Lie groupoids, providing many definitions, results and examples which we shall use throughout the main chapters. First, we exhibit the fundamental results and related constructions in Morse theory that we extend along this work, paying special attention to those involving Morse–Bott functions and closed geodesics. Some elementary facts about isometric Lie group actions are also addressed. Second, we introduce Lie groupoids, focusing on showing the main geometrical features derived from their structure. We define differentiable stacks as Lie groupoids up to Morita equivalence. After presenting some miscellany results and constructions in the Lie groupoid setting, we also mention some facts about Riemannian groupoids and their corresponding Riemannian stacks as well as Lie 2-groups and their associated Lie 2-algebras.

The major chapter of this thesis is Chapter 2. Here we study a natural notion of Morse Lie groupoid morphism which, as alluded to previously, enables us to extend the most important results in Morse theory to the realm of Lie groupoids and differentiable stacks. Of course, such a notion is Morita invariant. We show groupoid versions of: the Morse lemma, the fundamental theorems of Morse theory concerning the topological behavior of the level sets of a Morse function whether or not we cross through a nondegenerate critical point, Morse-Smale dynamics, the so-called stable/unstable manifold theorem, and the Morse cochain complex. Most of these results also have their counterpart when dealing with stacky Morse functions on differentiable stacks. We also prove Morse-like inequalities for certain compact orbits spaces associated to proper Lie groupoids, develop a 2-equivariant Morse theory, and provide several examples and constructions illustrating some of the extensions. For instance, we explain how to get the equivariant cohomology of toric symplectic stacks by using Morse theory techniques. It is worth saying that several interesting constructions concerning the notion of Riemannian groupoid metric due to del Hoyo–Fernandes as well as the stratification of the orbit space of a proper Lie groupoid by Morita types will be needed. Additionally, most of the results in this chapter have already been published in [101].

Chapter 3 aims at showing an existence result for closed stacky geodesic of positive length on certain separated Riemannian stacks. We start by introducing some general facts about stacky geodesics. Then, inspired by the Lusternik–Schnierelmann program to prove the existence of at least one closed geodesic of positive length on a compact Riemannian manifold, we carry out an study of the properties of the stacky energy functional over stacky curves, but restricted to the case of regular separated differentiable stacks, thus adapting techniques and results due to Guruprasad–Haefliger in the case of orbifolds. In particular, we verify that the stacky energy satisfies the so-called Palais–Smale condition (C), something that allows us to transfer some techniques from Morse theory for Hilbert manifolds to our context. After applying the desingularization of separated Riemannian stacks due to Posthuma–Tang–Wang we get our existence result. It is important to mention that the results of this chapter are based on joint work in progress with C. Ortiz and L. Vitagliano.

Finally, although it seems disconnected from the main purposes of this thesis, Chapter 4 arises due to the need to develop a 2-equivariant Morse theory suitably compatible with our main constructions. Here we further explore the properties of a natural notion of isometric Lie 2-group action on Riemannian groupoids which is introduced as well as used in Chapter 2. For instance, we prove 2-equivariant versions of the Slice Theorem and the Equivariant Tubular Neighborhood Theorem and construct bi-invariant groupoid metrics on compact Lie 2-groups. We also exhibit an infinitesimal description of an isometric Lie 2-group action which motivates us to define an algebra of transversal infinitesimal isometries associated to any Riemannian n-metric on a Lie groupoid. The latter turns out to be a Morita invariant algebraic object, so that it gives rise to a notion of geometric Killing vector field on a quotient Riemannian stack. If our Riemannian stack is separated then we show that the algebra formed by such geometric Killing vector fields is always finite dimensional. We provide natural examples and transfer other classical constructions to this setting. Most of the results in this chapter have already been published in [59] and they were obtained in joint work with J. S. Herrera-Carmona.

The short introduction at the beginning of each chapter as well as the small descriptions at the start of each such subsection will also give a brief but still comprehensive account of the content of each chapter.

Chapter 1

Preliminaries

This chapter is devoted to introduce the main ingredients and topics that this thesis is concerned about. Firstly, we discuss several concepts, results, and constructions appearing in classical Morse theory, focusing our attention on the notion of Morse–Bott function as well as the problem of showing the existence of at least one closed geodesic of positive length on compact Riemannian manifolds. Secondly, we briefly mention some facts about isometric Lie group actions which shall play an important role at several stages of our work. Thirdly, we present Lie groupoids and study the main geometric and topological properties underlying their structure, paying special attention to Morita equivalences. We use the latter notion to define differentiable stacks, some sort of objects which can be thought of as a generalization of the notion of manifold and allow to study higher symmetries and singular geometric features. Likely the most important ingredient that constitutes this work is the notion of Riemannian groupoid metric introduced by del Hoyo and Fernandes. Thus, we also define and study Riemannian groupoids and their associated Riemannian stacks. Finally, we say a few facts about (strict) Lie 2-groups and their corresponding Lie 2-algebras.

All the notions and objects mentioned above are illustrated with plenty of examples.

1.1 Classical Morse theory

In this section we briefly introduce the basic notions and results on Morse theory which will be used and extended throughout this work. Most of the classical results about this beautiful subject may be found for instance in [8, 20, 49, 88, 97].

Let $f : M \to \mathbb{R}$ be a smooth function on a manifold M. A point $x \in M$ is called a **critical point** of f if the differential $df(x) : T_xM \to \mathbb{R}$ is zero. The set of critical points of f is denoted by Crit(f). Given a critical point $x \in M$ one has a bilinear symmetric form $\mathcal{H}_x(f) : T_xM \times T_xM \to \mathbb{R}$ defined by the expression $\mathcal{H}_x(f)(v,w) = \tilde{v}(\tilde{w}f)(x)$ where $\tilde{v}$ and $\tilde{w}$ are vector fields on M with $\tilde{v}(x) = v$ and $\tilde{w}(x) = w$. The bilinear form $\mathcal{H}_x(f)$ is called the **Hessian** of f at x. A critical point $x \in M$ is said to be **nondegenerate** if the Hessian $\mathcal{H}_x(f)$ is nondegenerate. That is, $\mathcal{H}_x(f)(v,w) = 0$ for every $v \in T_xM$ if and only if $w = 0$.

Definition 1.1.1. A smooth function $f : M \to \mathbb{R}$ is called **Morse** if all its critical points are nondegenerate.

Remark 1.1.1. It is well known that if $b : V \times V \to \mathbb{R}$ is a symmetric and nondegenerate bilinear form on a real n-dimensional vector space V then there is at least one ordered basis $(e_1, \cdots, e_n)$ of V such that for any $v = \sum_j v^j e_j$ it holds that

$$b(v, v) = -(|v^1|^2 + \cdots + |v^\lambda|^2) + (|v^{\lambda+1}|^2 + \cdots + |v^n|^2).$$

The integer λ does not depend on the basis $(e_1, \cdots, e_n)$ and is called the **index** of b. It can be equivalently defined as the largest integer l such that there exists an l-dimensional subspace V_- of V with the property that the restriction of b to V_- is negative definite.

This fact motivates the following definition.

Definition 1.1.2. Let $x \in M$ be a nondegenerate critical point of $f : M \to \mathbb{R}$. The **index** of f at x is defined as the index of $\mathcal{H}_x(f)$. This will be denoted by $\lambda(f, x)$.

Some elementary examples of Morse functions are the following.

Example 1.1.1. Let $f : S^2 \to \mathbb{R}$ be the function that assigns to each point $(x, y, z) \in S^2$ its third component z. This is a Morse function which has only two critical points: the south pole $(0, 0, -1)$ with index 0 and the north pole $(0, 0, 1)$ with index 2.

Example 1.1.2. Let r and R be two positive real numbers such that $r < R$. We consider the 2-dimensional torus $\mathbb{T}^2$ given implicitly as

$$\mathbb{T}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + (\sqrt{y^2 + z^2} - R)^2 - r^2 = 0\}.$$

The height function $f : \mathbb{T}^2 \to \mathbb{R}$ defined by sending $(x, y, z) \mapsto z$ is a Morse function with just 4 critical points $(0, 0, -(R + r)), (0, 0, -(R - r)), (0, 0, R - r), (0, 0, R + r)$ having respective index $0, 1, 1, 2$.

Example 1.1.3. Let us consider the complex projective space $\mathbb{C}P^n$ with the function $f : \mathbb{C}P^n \to \mathbb{R}$ defined as $f(z_0 : \cdots : z_n) = \sum_j c_j |z_j|^2$ where the c_j's are distinct real constants. It follows that f is a Morse function with exactly $n + 1$ critical points of the form $p_k = (0 : 0 : \cdots : 1 : \cdots : 0)$ for $0 \leq k \leq n$. The index $\lambda(f, p_k)$ is twice the number of j with $c_j < c_k$. Note that every possible even index between 0 and $2n$ occurs exactly once since all the c_j's are different.

Next result gives a local form of a function around a nondegenerate critical point. Let $Q_{f,x_0} : T_{x_0}M \to \mathbb{R}$ denote the quadratic form associated to the Hessian $\mathcal{H}_{x_0}(f)$ at $x_0 \in M$. Recall that this is usually given by the formula $Q_{f,x_0}(x) = \frac{1}{2}\mathcal{H}_{x_0}(f)(x, x)$.

Theorem 1.1.1 (Morse Lemma). *Let $f : M \to \mathbb{R}$ be a smooth function and $x_0 \in M$ be a nondegenerate critical point of f. Then there are local coordinates $(x^1, \cdots, x^n)$ on a neighborhood $x_0 \in U \subseteq M$ with $x^j(x_0) = 0$ for every $j = 1, ..., n$ such that*

$$f(x) = f(x_0) + Q_{f,x_0}(x), \quad x \in U.$$

The previous result says that Morse functions are locally rigid in the sense that on a small enough open neighborhood of a critical point they are given by quadratic polynomials.

Corollary 1.1.1. *If $f : M \to \mathbb{R}$ is a Morse function then its critical points are isolated. Moreover, if M is compact then* $\mathrm{Crit}(f)$ *is a finite set.*

One of the key reasons for the topological versatility of Morse functions is their abundance, namely, they are generic. Indeed:

Theorem 1.1.2. *Let M be a compact manifold. Then the set of all Morse functions on M determines an open and dense subset in $C^\infty(M)$ with respect to the strong topology.*

Topological aspects

Roughly speaking, Morse functions allow us to describe the topological behavior of nicely behaved spaces whether or not we cross through a nondegenerate critical point. For instance, we can obtain substantial information about their homology as well as cohomology by studying the dynamics of the gradient flow of a Morse function once we have the presence of a Riemannian structure satisfying some transversality assumptions. Below we explicitly exhibit these facts.

Let $f : M \to \mathbb{R}$ be a Morse function. For each $a \in \mathbb{R}$ we can consider the **level set** of f below a which is given by the manifold with boundary

$$M^a := \{x \in M : f(x) \leq a\}.$$

The heart of Morse theory lies in the following fundamental result.

Theorem 1.1.3. *Let $f : M \to \mathbb{R}$ be a Morse function and $[a, b] \subset \mathbb{R}$ be a closed interval such that $f^{-1}[a, b]$ is compact. The following dichotomy holds true:*

- *if $f^{-1}[a, b]$ does not contain critical points of f then M^a is diffeomorphic to M^b. Moreover, the level set M^a is a deformation retract of M^b so that the inclusion $M^a \hookrightarrow M^b$ is a homotopy equivalence, or*
- *if the only critical point of f inside $f^{-1}(a, b)$ is x then the level set M^b is homotopy equivalent to $M^a \cup_{\partial D^{\lambda(f,x)}} D^{\lambda(f,x)}$ which is the result of gluing the level set M^a and a $\lambda(f, x)$-cell $D^{\lambda(f,x)}$ along its boundary $\partial D^{\lambda(f,x)}$.*

As a consequence of the previous result, one can see that a manifold equipped with a Morse function is homotopy equivalent to a CW-complex with exactly one λ-cell for every critical point of index λ. Another important fact which can be derived from Theorem 1.1.3 is given by the so-called **Morse inequalities**. On the one hand, any Morse function $f : M \to \mathbb{R}$ has an associated polynomial

$$M_f(\tau) := \sum_{x \in \mathrm{Crit}(f)} \tau^{\lambda(f,x)} = \sum \mu_f(\lambda)\tau^{\lambda},$$

where $\mu_f(\lambda)$ is the number of critical points of f of index λ. Such a polynomial is called the **Morse polynomial** of f. On the other hand, any compact manifold M has an associated polynomial containing topological information. This is the **Poincaré polynomial** of M which is defined by

$$P_M(\tau) = \sum_{j \geq 0} b_j \tau^j,$$

where $b_j = \dim_{\mathbb{F}}(H_j(M, \mathbb{F}))$ are the Betti numbers of M with respect to some coefficient field $\mathbb{F}$.

Suppose that M is a compact manifold. We may use Theorem 1.1.3 to compare the Morse and Poincaré polynomials as follows.

Theorem 1.1.4 (Morse Inequalities)**.** *There exists a polynomial $R(\tau)$ with non-negative integers coefficients such that*

$$M_f(\tau) - P_M(\tau) = (1 + \tau)R(\tau).$$

In particular, we have that $\mu_f(\lambda) \geq b_\lambda$ and that the Euler characteristic of M can be written as $\mathcal{X}(M) = \sum_{\lambda \geq 0}(-1)^{\lambda}\mu_f(\lambda)$.

Morse–Smale–Witten complex

Let $f : M \to \mathbb{R}$ be a Morse function on a compact and oriented Riemannian manifold M. The negative of the gradient vector field of f is denoted by $-\nabla f$. This generates a globally defined **descending flow** which we denote by $\Phi_\tau : M \to M$ for all $\tau \in \mathbb{R}$. Recall that from the Morse lemma it follows that the critical points of f are isolated and hence for any critical point $x \in \mathrm{Crit}(f)$ its **unstable** and **stable** manifolds are respectively defined by

$$W^u(x) = \{y \in M : \lim_{\tau \to -\infty} \Phi_\tau(y) = x\} \qquad \text{and} \qquad W^s(x) = \{y \in M : \lim_{\tau \to \infty} \Phi_\tau(y) = x\}.$$

It holds that $W^u(x)$ and $W^s(x)$ are indeed embedded submanifolds of M of dimension $\lambda(f, x)$ and $\dim(M) - \lambda(f, x)$, respectively. A classical result of Smale says that for a generic f the unstable manifold and the stable manifold of any pair of critical points intersect transversally. Namely, we may assume that $f : M \to \mathbb{R}$ is a **Morse-Smale function**, that is, f is a Morse function such that $W^u(x)$ intersects $W^s(y)$ transversally for every $x, y \in \mathrm{Crit}(f)$. Note that both $W^u(x)$ and $W^s(y)$ admit a canonical action of $\mathbb{R}$ by **flow translation**. The **moduli space of gradient flow lines** between $x, y \in \mathrm{Crit}(f)$ is defined as the quotient space

$$\mathcal{M}(x, y) := \big(W^s(y) \cap W^u(x)\big)/\mathbb{R}.$$

Let us sketch how to build a differential complex out of the moduli spaces of gradient lines. Firstly, from the transversality assumption it follows that $\mathcal{M}(x, y)$, for $x \neq y$, is actually a smooth manifold since the action by flow translation is free and proper. Secondly, we orient the unstable manifold of a critical point x and hence, due again to transversality, its stable manifold inherits an orientation as well. This induces an orientation on all moduli spaces of gradient lines. Let us now suppose that $\lambda(f, y) = \lambda(f, x) - 1$, so that $W^s(y) \cap W^u(x)$ is an oriented manifold of dimension 1. Therefore, the moduli space $\mathcal{M}(x, y)$ is just a collection of oriented critical points. For every $j \geq 0$ we let C_j to be the free $\mathbb{Z}$-module generated by critical points of index j. There is a boundary operator $\partial : C_j \to C_{j-1}$ defined on generators by

$$\partial(x) = \sum m_k x_k,$$

where x is a critical point of index j, the x_k's are critical points of index $j - 1$ and $m_k = \sharp(\mathcal{M}(x_k, x))$. Counting gradient lines between critical points shows that $\partial^2 = 0$. The associated homology is denoted $H_\bullet(C_\bullet, \partial)$ and it is referred to as the **Morse homology** of (M, f). In principle, the Morse homology depends on the function $f : M \to \mathbb{R}$, but the following result says that actually this is not the case and Morse homology recovers a classical homotopy invariant of the manifold M.

Theorem 1.1.5. *The homology of the complex $(C_\bullet, \partial)$ is naturally isomorphic to the singular homology of M. That is,*

$$H_\bullet(M, \mathbb{Z}) \cong H_\bullet(C_\bullet, \partial).$$

It is also possible to define a cochain complex of gradient lines. In that case, the corresponding cohomology is isomorphic to the singular cohomology of M. Observe that by using the previous result we have that the Morse inequalities follow as a simple corollary.

1.1.1 Morse–Bott functions

Suppose that K is a Lie group acting on a smooth manifold M and that $f : M \to \mathbb{R}$ is a K-invariant function. It is simple to check that if $x \in M$ is a critical point of f then the whole K-orbit $\mathcal{O}_x$ is contained in Crit(f). Therefore, from this we may deduce that classical Morse theory is no longer the right setting to extract topological information of a space out of a function which is invariant by a "group of symmetries" of such a space. This is due to the fact that critical points come in orbits, so that they are no longer non-degenerate (isolated) in general. The right framework to study an invariant Morse function on a manifold is given by the so-called **Morse-Bott theory**. Here, critical points are arranged in families of submanifolds which are non-degenerate in the sense that the normal Hessian is non-degenerate. More precisely, let $f : M \to \mathbb{R}$ be a smooth function such that Crit$(f) := \{x \in M : \, df(x) = 0\}$ contains a submanifold C of positive dimension. The choice of a Riemannian metric on M yields a decomposition

$$TM|_C = TC \oplus \nu(C),$$

where TC and $\nu(C)$ are the tangent bundle and the normal bundle of C, respectively. Let $\mathcal{H}_x(f)$ be the Hessian of f at $x \in C$, then $T_xC \subseteq \ker(\mathcal{H}_x(f))$. Indeed, if $v, w \in T_xC$ and $\tilde{w} \in \mathfrak{X}(M)$ is any extension of w, then

$$\mathcal{H}_x(f)(v, w) = v(\widetilde{w} \cdot f) = 0,$$

since $df(\widetilde{w})|_C = 0$ because $C \subseteq$ Crit(f). Therefore, the Hessian $\mathcal{H}_x(f)$ induces a well defined symmetric bilinear form on $\nu_x(C)$ referred to as the **normal Hessian** of f at $x \in C$ and which we shall denote again as $\mathcal{H}_x(f)$ only if there is no risk of confusion. A critical submanifold $C \subseteq M$ of f is called **non-degenerate** if the normal Hessian at every $x \in C$ is non-degenerate. This is equivalent to asking $\ker(\mathcal{H}_x(f)) = T_xC$ for every $x \in C$.

Definition 1.1.3. A smooth function $f : M \to \mathbb{R}$ is said to be **Morse–Bott** if Crit(f) is a disjoint union of connected submanifolds which are non-degenerate.

Remark 1.1.2. The main advantage of working with Morse–Bott functions is that they allow us to get similar results to those obtained with usual Morse functions but without assuming that their critical point set is formed by isolated points.

In what follows we shall elaborate on the previous fact by following [8, 13, 20, 97] closely. If C is a connected nondegenerate critical submanifold for f then we may define a quadratic form Q_f on $\nu(C)$ which regards as a function on the total space of the normal bundle $\nu(C)$ quadratic along the fibers. Namely,

$$Q_f(v) = \frac{1}{2}\mathcal{H}_{\pi(v)}(f)(v, v), \qquad v \in \nu(C), \tag{1.1}$$

where $\pi : \nu(C) \to C$ is the bundle projection.

The local behavior of a Morse–Bott function around a critical submanifold can be described as follows.

Proposition 1.1.1 (Morse–Bott lemma). *Let C be a connected nondegenerate critical submanifold for a smooth function $f : M \to \mathbb{R}$. Then there exists a tubular neighborhood $\phi : V \subset \nu(C) \to U \subset M$ of C such that*

$$\phi^* f = c + Q_f,$$

where c is the common value of f on C.

Choose a Riemannian metric $\eta^{(0)}$ on M and consider the induced metric $\overline{\eta}^{(0)}$ on $\nu(C)$. The fact that $\mathcal{H}(f)$ is a nondegenerate symmetric bilinear form on $\nu(C)$ allows us to define a self-adjoint automorphism $A_f : \nu(C) \to \nu(C)$ by means of the formula $\overline{\eta}^{(0)}(A_f(v), w) = \mathcal{H}_f(v, w)$ for all $v, w \in \nu(C)$. It follows that A_f has nonzero eigenvalues since $\mathcal{H}(f)$ is fiber-wise non-singular and produces a splitting

$$\nu(C) = \nu_+(C) \oplus \nu_-(C),$$

where $\nu_\pm(C)$ is spanned by the eigenvectors of $\mathcal{H}_f/A_f$ corresponding to positive/negative eigenvalues. Because of the connectedness of C the rank of $\nu_-(C)$ is an invariant of C called the **index** of the critical submanifold C. This will be denoted by $\lambda(f, C)$.

Let us give some elementary but instructive examples.

Example 1.1.4. Any Morse function $f : M \to \mathbb{R}$ is a Morse–Bott function where the critical submanifolds are just the critical points of f. Also, any constant function $f = c \in \mathbb{R}$ determines a Morse–Bott function. In particular, $f = 0$ is a Morse–Bott function.

Example 1.1.5. Let S^2 be the 2-sphere and $f : S^2 \to \mathbb{R}$ be defined by $f(x, y, z) = z^2$. A simple computation shows that

$$\text{Crit}(f) = \{(x, y, z) \in S^2 : \ z = 0 \text{ or } z = \pm 1\}.$$

It follows that f is a Morse–Bott function on S^2 since the components of Crit(f) are the poles $N = (0, 0, 1)$ and $S = (0, 0, -1)$ which are nondegenerate critical points of index 2 and the equator $E \cong S^1$ that is a nondegenerate critical submanifold of index 0. More generally, let us consider the complex projective space $\mathbb{C}P^n$ together with the map $f : \mathbb{C}P^n \to \mathbb{R}$ defined by $f(z_0 : \cdots : z_n) = |z_n|^2$. This also provides us with an example of Morse–Bott function having a critical point $(0 : 0 : \cdots : 1)$ of index $2n$ and a critical submanifold $\mathbb{C}P^{n-1}$ of index 0.

Example 1.1.6. Let M be a smooth manifold and consider a Lie group K acting on M. A smooth function $f : M \to \mathbb{R}$ is said to be **invariant** if $f(kx) = f(x)$ for all $(k, x) \in K \times M$. It is clear that if $x \in M$ is a critical point of f then every point in the orbit through x is also a critical point of f. Therefore, Crit(f) is formed by a disjoint union of K-orbits. For instance, consider $M = \mathbb{C}^2$ and $K = U(1) \times SU(2)$. It is simple to check that $U(1) \times SU(2)$ acts on $\mathbb{C}^2$ by matrix multiplication and that such an action is norm-preserving. The orbit of an element $w \in \mathbb{C}^2 \backslash \{0\}$ has the form $\{w \in \mathbb{C}^2 : \|w\| = r\} \cong S^3$ for some $r > 0$. Let $f : \mathbb{C}^2 \to \mathbb{R}$ be defined as $f(w) = -\|w\|^2 + \|w\|^4$. This is a K-invariant Morse–Bott function with nondegenerate critical submanifolds $w = 0$ and $\{w \in \mathbb{C}^2 : \|w\| = \frac{1}{2}\sqrt{2}\}$.

An interesting result of Wasserman in [116] asserts that K-invariant functions on M with nondegenerate critical orbits are generic provided K is a compact Lie group.

Example 1.1.7 (Key example)**.** If $f : N \to \mathbb{R}$ is a Morse–Bott function and $\pi : M \to N$ is a surjective submersion, then the pullback function $f \circ \pi : M \to \mathbb{R}$ is a Morse–Bott function as well. In this case, if C is a nondegenerate critical submanifold for f then $\pi^{-1}(C)$ is a nondegenerate critical submanifold for $\pi^* f$ and the equality of indexes $\lambda(f, C) = \lambda(f \circ \pi, \pi^{-1}(C))$ holds true. This is because the formula

$$\mathcal{H}_x(f \circ \pi) = d\pi(x)^T \cdot \mathcal{H}_{\pi(x)}(f) \cdot d\pi(x), \tag{1.2}$$

is satisfied at every critical point x of $f \circ \pi$.

Example 1.1.8. Let (M,ω) be a symplectic manifold and assume that there exists a Hamiltonian action of an n-torus $\mathbb{T}^n$ on M with moment map $\mu : M \to \mathfrak{t}$. It is not a trivial fact to prove that the component function $\hat{\mu}(\xi) : M \to \mathbb{R}$ is a Morse–Bott function for each $\xi \in \mathfrak{t}$. Furthermore, the critical point set of $\hat{\mu}(\xi)$ has components of even index, see for instance [97, s. 3.5] or [7, c. IV]. A similar result can be proven for Hamiltonian torus actions on presymplectic manifolds as studied in [76].

We can also use Morse–Bott functions to study the topology features of a manifold around a nondegenerate critical submanifold. Let C be a nondegenerate critical submanifold for f and respectively denote by $D_-(C)$ and $\partial D_-(C)$ the unit closed disk and unit sphere bundles associated to $\nu_-(C)$ after restricting the Riemannian metric $\overline{\eta}^{(0)}$. One has that the fibers of $D_-(C)$ are $\lambda(f,C)$-disks and the fibers of $\partial D_-(C)$ are $(\lambda(f,C)-1)$-spheres for which sometimes $\partial D_-(C)$ is referred to as the boundary of $D_-(C)$.

Theorem 1.1.6. *Let $f : M \to \mathbb{R}$ be a Morse–Bott function and $[a,b] \subset \mathbb{R}$ be a closed interval such that $f^{-1}[a,b]$ is compact. The following dichotomy holds true:*

- *if $f^{-1}[a,b]$ does not contain critical points of f then M^a is diffeomorphic to M^b. Moreover, the level set M^a is a deformation retract of M^b so that the inclusion $M^a \hookrightarrow M^b$ is a homotopy equivalence, or*
- *if the only nondegenerate critical submanifold of f inside $f^{-1}(a,b)$ is C then the level set M^b is homotopy equivalent to $M^a \cup_{\partial D_-(C)} D_-(C)$ which is the result of gluing the level set M^a and $D_-(C)$ along $\partial D_-(C)$.*

As expected, Morse type inequalities can also be extended to this new context. In fact, a way to obtain them is by using Morse–Bott–Smale dynamics, in order to recover the de Rham cohomology of the manifold. This is done by means of a cochain complex which is defined in terms of the de Rham complex of the critical point sets and gradient flow line spaces. We finish this subsection by sketching such a construction.

The Austin–Braam complex

The main ideas and results presented below can be found in [8]. Suppose that $(M,\eta^{(0)})$ is compact Riemannian manifold. For a given Morse–Bott function $f : M \to \mathbb{R}$ we denote by S_i the disjoint union of the connected components of Crit(f) of index i. Let us consider the negative gradient vector field $-\nabla f$ and its descending flow $\Phi_\tau : M \to M$ for all $\tau \in \mathbb{R}$. The **stable** and **unstable** manifolds of S_i are respectively defined as

$$W^s(S_i) := \{x \in M : \lim_{\tau\to\infty} \Phi_\tau(x) \in S_i\} \quad \text{and} \quad W^u(S_i) := \{x \in M : \lim_{\tau\to-\infty} \Phi_\tau(x) \in S_i\}.$$

These are smooth manifolds of respective dimension $\dim(S_i) + i$ and $\dim M - i$ with the property that the smooth **endpoint maps** $u^i : W^u(S_i) \to S_i$ and $l^i : W^s(S_i) \to S_i$ defined as

$$u^i(x) := \lim_{\tau\to-\infty} \Phi_\tau(x) \quad \text{and} \quad l^i(x) := \lim_{\tau\to\infty} \Phi_\tau(x),$$

are locally trivial fiber bundles with fibers diffeomorphic to the disks D^i and $D^{\dim M - \dim(S_i) - i}$. As in the Morse theory case, there exists a natural action of $\mathbb{R}$ on the intersection $W^u(S_i) \cap W^s(S_j)$ which is also referred to as **action by flow translation**. The **moduli space of gradient flow lines** in M associated to the nondegenerate critical submanifolds S_i and S_j is defined to be the quotient space obtained from the action by flow translation:

$$\mathcal{M}(S_i,S_j) := (W^u(S_i) \cap W^s(S_j))/\mathbb{R}.$$

In the sequel it will be necessary to make the following assumption.

Assumption 1.1.1.
i. f is weakly self-indexing, meaning that $\mathcal{M}(S_i, S_j) = \emptyset$ if $i \leq j$,

ii. for all i, j and for all $z \in S_i$, the fiber $W_i^u(z) := (u^i)^{-1}(z)$ intersects $W^s(S_j)$ transversally,

iii. both the critical submanifolds S_i and their negative normal bundle $\nu_-(S_i)$ are orientable for all i, and

iv. M is orientable.

The requirement from item ii. is known as **Morse–Smale transversality**. This immediately implies that $\mathcal{M}(S_i, S_j)$ is a smooth manifold of dimension $i-j+\dim S_i$ (if $\mathcal{M}(S_i, S_j) \neq \emptyset$), since the action by flow translation is free and proper. Furthermore, the induced endpoint maps $l_j^i : \mathcal{M}(S_i, S_j) \to S_j$ and $u_j^i : \mathcal{M}(S_i, S_j) \to S_i$ respectively given by

$$l_j^i([x]) = l^j(x) \qquad \text{and} \qquad u_j^i([x]) = u^i(x), \tag{1.3}$$

are well defined smooth maps such that u_j^i has the structure of a locally trivial fibration. The other items in Assumption 1.1.1 are needed to induce an orientation on the moduli space of gradient flow lines as well as on the fibers of u_j^i.

Let $C^{i,j} := \Omega^j(S_i)$ denote the space of j-forms on S_i. We define the operator $\partial_r : C^{i,j} \to C^{i+r,j-r+1}$ as

$$\partial_r(\omega) = \begin{cases} d\omega & \text{if} \quad r = 0 \\ (-1)^j (u_i^{i+r})_* (l_i^{i+r})^* (\omega) & \text{otherwise,} \end{cases}$$

where $(u_i^{i+r})_*$ is integration along the fiber of the fibration (1.3). It follows that $\sum_{m=0}^k \partial_{k-m} \circ \partial_m = 0$ for each k, hence the maps ∂_r fit together into a cochain complex

$$C^k := \bigoplus_{i=0}^k C^{i,k-i}; \quad \partial := \partial_1 \oplus \cdots \oplus \partial_m.$$

More importantly, as shown by Austin–Braam in [8] we obtain that:

Theorem 1.1.7. *The cohomology of the complex $(C^\bullet, \partial)$ is isomorphic to the de Rham cohomology of M. That is,*

$$H^\bullet(C^\bullet, \partial) \cong H^\bullet_{\text{dR}}(M).$$

It is worth mentioning that in the case when f is just a generic Morse function $C^i = \Omega^0(S_i)$ is the vector space generated by the critical points of index i. This is because critical points are isolated. Moreover, for dimensional reasons, we get that $\partial_r = 0$ for all $r \neq 1$. Therefore, in this specific case $\mathcal{M}(S_{i+1}, S_i)$ is an oriented 0-manifold, that is, a collection of signed points and integrating over the endpoint maps simply counts these points with orientations. Thus, we have recovered the ordinary Morse complex.

As a consequence of Theorem 1.1.7 it is possible to get the so-called **Morse–Bott inequalities**.

Corollary 1.1.2. *There is a polynomial $R(\tau)$ with non-negative coefficients such that*

$$\sum_{i,j} \dim H^j(S_i; \mathbb{R}) \tau^{i+j} = \sum_k \dim H^k(M; \mathbb{R}) \tau^k + (1+\tau) R(\tau).$$

Let us look at an illustrative example, see [8] for the specific details regarding the arguments involving spectral sequences.

Example 1.1.9. Consider the Morse–Bott functions on S^2 and $\mathbb{C}P^n$ introduced in Example 1.1.5. On the one hand, recall that $f : S^2 \to \mathbb{R}$ has 2 critical points N and S of index 2 and the equator S^1 is a critical submanifold of index 0. Therefore, the E_1 term of the spectral sequence appearing in the Austin–Braam's constructions is

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ \mathbb{R} & 0 & 0 & 0 \\ \mathbb{R} & 0 & \mathbb{R}^2 & 0. \end{array}$$

This implies that the differential $\partial_2 : \mathbb{R}[S^1] \to \mathbb{R}[N] \oplus \mathbb{R}[S]$ is given by integrating the volume form ω (line element) on S^1 along the moduli spaces $\mathcal{M}(N, S^1)$ and $\mathcal{M}(S, S^1)$ which are both circles so that $\partial_2(\omega) = (1, -1)$ for orientation reasons. In consequence, the E_∞ term of the sequence which corresponds to the cohomology of S^2 is

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathbb{R} & 0 & \mathbb{R} & 0. \end{array}$$

On the other hand, recall that $f : \mathbb{C}P^n \to \mathbb{R}$ has 1 critical points p of index $2n$ and $\mathbb{C}P^{n-1}$ is a critical submanifold of index 0, thus obtaining the E_1 term of the spectral sequence looks like

$$\begin{array}{ccccc} 2n-2 & \mathbb{R} & 0 & \cdots & \\ 2n-3 & 0 & 0 & \cdots & \\ 2n-4 & \mathbb{R} & 0 & \cdots & \\ \vdots & \vdots & \vdots & \cdots & \\ 0 & \mathbb{R} & 0 & \cdots & \mathbb{R} \\ & 0 & 1 & \cdots & 2n. \end{array}$$

Note that all higher differential vanish for dimensional reasons which implies that E_1 agrees with E_∞ which, once again, corresponds to the cohomology of $\mathbb{C}P^n$.

Another interesting application of the Austin–Braam's construction has to do with recovering the equivariant cohomology associated to an action by a compact Lie group. Let K be a compact Lie group acting on M. If $EK \to BK$ denotes the universal K-fibration determined by K then the quotient space $M_K := EK \times_K M$ by the diagonal action of K on $EK \times M$ is called **homotopy quotient**. This is a fiber bundle over BK with fiber M. The K-**equivariant cohomology** of M is defined to be

$$H^\bullet_K(M) := H^\bullet(M_K, \mathbb{R}),$$

where the right hand side stands for the standard singular cohomology with real coefficients of the homotopy quotient M_K. For instance, if $M = *$ is just a point then $H^\bullet_K(*) = H^\bullet(BK, \mathbb{R})$. Also, if K acts freely on M then $H^\bullet_K(M) = H^\bullet(M/K, \mathbb{R})$.

There is a convenient model for equivariant cohomology which we can work with. Let us denote by $\mathfrak{k}$ the Lie algebra of K. The space of K-**equivariant differential forms** of M is defined by $\Omega^\bullet_K(M) = (\Omega^\bullet(M) \otimes S(\mathfrak{k}^*))^K$, the space of K-invariant elements in $\Omega^\bullet(M) \otimes S(\mathfrak{k}^*)$, where the action of K on $\Omega^\bullet(M)$ is by pullback and on $S(\mathfrak{k}^*)$ is by the co-adjoint representation of K. The elements in the symmetric algebra $S(\mathfrak{k}^*)$, that is, the linear polynomials on $\mathfrak{k}$, have degree two, thus turning $\Omega_K^\bullet(M)$ into a bi-graded complex. If we pick a basis (X_α) of $\mathfrak{k}$ with dual basis (ϕ^α) then the differential is a graded derivation defined on generators by

$$d_K(\omega) = d\omega - \sum_\alpha \phi^\alpha \iota_{\tilde{X}_\alpha}\omega \quad \text{and} \quad d_K(\phi^\alpha) = 0,$$

for $\omega \in \Omega^\bullet(M)$ and $\tilde{X}_\alpha$ the fundamental vector field on M induced by X_α. Cartan showed that $(\Omega_K^\bullet(M), d_K)$ in fact defines a cochain complex and $H_K^\bullet(M) \cong H^\bullet(\Omega_K^\bullet(M), d_K)$. In particular, if $M = *$ then we remarkably get that $H^\bullet(BK, \mathbb{R}) = S(\mathfrak{k}^*)^K$.

Let $\eta^{(0)}$ be a Riemannian metric on M for which the action of K on M becomes isometric. It is well known that these kinds of metrics always exist since K is compact, visit Section 1.2. Suppose that $f : M \to \mathbb{R}$ is a K-invariant function with nondegenerate critical orbits, see Example 1.1.6. The critical point set S_i of index i is K-invariant so that it is given by a disjoint union of K-orbits. Furthermore, the moduli space of gradient flow lines $\mathcal{M}(S_i, S_j)$ is also K-invariant.

We assume now that the stable and unstable submanifolds intersect transversally, see item ii. from Assumption 1.1.1. Let us consider again the induced endpoint maps $l_j^i : \mathcal{M}(S_i, S_j) \to S_j$ and $u_j^i : \mathcal{M}(S_i, S_j) \to S_i$, which in this case have the additional property of being K-equivariant. Due to the K-invariance of f it follows that the weakly self-indexing requirement from Assumption 1.1.1 is implied by the transversality assumption. Similarly, the assumption which asks the endpoint maps to induce fibrations is an immediate consequence of the presence of a transitive K-action on the components of the critical point set.

We will form a complex which computes the K-equivariant cohomology of M by using equivariant forms on the critical submanifolds. Indeed, by using the Cartan model for equivariant cohomology we define the complex $(C_K^\bullet, \partial_K)$ with

$$C_K^q = \sum_{i+j+2p=q} (\Omega^j(S_i) \otimes S^p(\mathfrak{k}^*))^K; \qquad \partial_K = \sum_m (\partial_K)_m,$$

where, as before, for $m = 0$ we set $(\partial_K)_0(\omega \otimes \phi) = d_K(\omega \otimes \phi)$ and for $m > 0$ we set $(\partial_K)_m(\omega \otimes \phi) = \partial_m(\omega) \otimes \phi$, for all $\omega \otimes \phi \in (\Omega^j(S_i) \otimes S^p(\mathfrak{k}^*))^K$.

Theorem 1.1.8. *The cohomology of the complex $(C_K^\bullet, \partial_K)$ is isomorphic to the K-equivariant cohomology of M. That is,*

$$H^\bullet(C_K^\bullet, \partial_K) \cong H_K^\bullet(M).$$

As always, the existence of this complex leads to equivariant Morse inequalities, namely, a similar result as that of Corollary 1.1.2 but considering instead K-equivariant cohomology.

1.1.2 Closed geodesics

The problem about showing the existence of closed geodesics of positive length on a Riemannian manifold has been the object of intensive research since the beginning of global differential geometry during the last century. In particular, to study geodesics and to relate them to properties of the ambient space has historically been one of the directional forces in the development of the calculus of variations. It can be said that this kind of problem was the initial motivation of Marston Morse who ventured into the study of the critical points of a smooth function to understand geodesics which in fact are critical points of the energy functional on paths [96]. In fact, as we shall mention below, every compact manifold contains a closed geodesic [66] and a beautiful and elegant proof of this fact is presented in [88] where Morse theory is applied to the energy functional on the space of closed curves to prove existence of closed geodesics on compact manifolds. The aim of this subsection is to provide the ingredients used to better stating the previous result. We shall follow [66, 72, 88] closely.

Let (M, η^M) be a Riemannian manifold. It is well known that there exists a unique torsion free connection $D : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ which leaves parallel the Riemannian metric η^M, i.e. $D\eta^M = 0$. More precisely,

$$[X,Y] = D_X Y - D_Y X \quad \text{and} \quad Z \cdot \eta^M(X,Y) - \eta^M(D_Z X, Y) - \eta^M(X, D_Z Y) = 0,$$

for all $X, Y, Z \in \mathfrak{X}(M)$. As the value of $D_X Y$ at a point $x \in M$ depends only on the value $X(x) \in T_x M$ and on the value of Y along some smooth curve tangent to $X(x)$ then we can induce a covariant derivative $D_\tau : \Gamma(\alpha^* TM) \to \Gamma(\alpha^* TM)$ on vector fields along any smooth curve $\alpha : I \subset \mathbb{R} \to M$. In these terms, a **geodesic** on M is a smooth curve α verifying $D_\tau \dot{\alpha} = 0$. Locally, the previous equation determines a second order non-linear ordinary differential equation system with smooth coefficients. Therefore, by using general existence and uniqueness results from the theory of ordinary differential equations it follows that the existence of geodesics is guaranteed after fixing smooth initial conditions at any point with a predetermined direction. In particular, it is possible to define the exponential map $\exp_x : V \subseteq T_x M \to M$ of any point $x \in M$ as $\exp_x(v) := \alpha(1)$ where $\alpha : (1-\epsilon, 1+\epsilon) \to M$ is the unique geodesic passing through x with speed v. Here V stands for a small open neighborhood of $0 \in T_x M$. It is well known that the exponential map is a local diffeomorphism, so that every $x \in M$ is connected to any nearby point by a unique "short" geodesic, consult [72, c. 4-5].

For each pair of distinct points $x, y \in M$ we denote by $\Omega^c_{x,y}(M)$ (resp. $\Omega_{x,y}(M)$) the **space of continuous (resp. smooth) paths** starting at x and ending at y. That is to say, the space of continuous (resp. smooth) curves $\alpha : [0,1] \to M$ such that $\alpha(0) = x$ and $\alpha(1) = y$. It follows that $\Omega^c_{x,y}(M)$ is a Banach (resp. Hilbert) manifold and the tangent space at a curve α is

$$T_\alpha \Omega^c_{x,y}(M) = \{X \in \Gamma^c(\alpha^*(TM)) : X(0) = 0,\ X(1) = 0\}.$$

The tangent space $T_\alpha \Omega_{x,y}(M)$ is similarly obtained by considering instead smooth vector fields along α vanishing at the endpoints. More precisely, each element $X \in T_\alpha \Omega_{x,y}(M)$ determines a curve $\alpha_X : J \subseteq \mathbb{R} \to \Omega_{x,y}(M)$ (with $0 \in J$ small enough) defined by $\alpha_X(\lambda)(\tau) := \exp_{\alpha(\tau)}(\lambda X(\tau))$ which satisfies $\alpha_X(0) = \alpha$ and $\partial_\lambda(\alpha_X(\lambda)(\tau))|_{\lambda=0} = X(\tau)$ for all $\tau \in [0,1]$, see [66]. The **energy functional** along smooth paths $E : \Omega_{x,y}(M) \to \mathbb{R}$ is defined to be

$$E(\alpha) = \frac{1}{2}\int_0^1 \|\dot{\alpha}(\tau)\|^2 d\tau.$$

This may be regarded as a differentiable map whose derivative at α is the linear map $dE(\alpha) : T_\alpha \Omega_{x,y}(M) \to \mathbb{R}$ given by the expression

$$dE(\alpha)(X) = \int_0^1 \eta^M(D_\tau X, \dot{\alpha}) d\tau = -\int_0^1 \eta^M(X, D_\tau \dot{\alpha}) d\tau. \tag{1.4}$$

From this formula it easily follows that α is a critical point of E if and only if $D_\tau \dot{\alpha} = 0$, i.e. α is a geodesic starting at x and ending at y.

The **space of closed curves** at $x \in M$ is defined as $\Omega_x(M) := \{\alpha \in \Omega_{x,x}(M) : \dot{\alpha}(0) = \dot{\alpha}(1)\}$. This may be identified with the space of all smooth pointed maps $(S^1, e) \to (M, x)$. The space of all closed curves is defined by $\Omega(M) = \bigcup_{x \in M} \Omega_x(M)$. It can be turned into a Riemannian Hilbert manifold with respect to the following scalar product. As expected, the tangent space $T_\alpha \Omega_x(M)$ at α is given by all the elements in $X \in \Gamma(\alpha^*(TM))$ vanishing at e, i.e. $X(0) = X(1) = 0$. The Riemannian metric on $\Omega(M)$ is determined by the scalar product

$$(X, X)(\alpha) = \int_{S^1} \eta^M(X(\tau), X(\tau))d\tau + \int_{S^1} \eta^M((D_\tau X)(\tau), (D_\tau X)(\tau))d\tau.$$

Using the Arzelà-Ascoli theorem it is possible to prove that the induced Riemannian metric $(\cdot,\cdot)_\Lambda$ on $\Omega(M)$ is complete, compare [66, Thm. 1.4.5]. More importantly, the energy functional E satisfies the so-called **Condition (C)** of Palais and Smale with respect to the distance d_Λ on $\Omega(M)$ associated to $(\cdot,\cdot)_\Lambda$, see [102]. Namely, motivated by Equation (1.4) and with the help of the Riemannian metric $(\cdot,\cdot)_\Lambda$ we define the **gradient vector field** of E on $\Omega(M)$ by the expression

$$(\mathrm{grad} E(\alpha), X)_\Lambda = dE_\alpha(X) = \int_{S^1} \eta^{(0)}(\dot{\alpha}, D_\tau X)d\tau, \qquad X \in T_\alpha \Omega(M).$$

The Palais–Smale Condition (C) reads as follows. Let us consider a sequence of curves $\{\alpha_n\}$ in $\Omega(M)$ such that

i. the sequence $\{E(\alpha_n)\}$ is bounded, and

ii. the sequence $\{\|\mathrm{grad} E(\alpha_n)\|_\Lambda\}$ tends to zero.

Then the sequence $\{\alpha_n\}$ has limit points and any limit point is a critical point of E, i.e. a geodesic, consult [66, Thm. 1.4.7]. Some of the properties derived from the previous condition, involving the negative gradient flow $\partial_\lambda \Phi_\lambda = -\mathrm{grad} E(\Phi_\lambda)$, come in order below, see [66, s. 1.4]. Let Crit(E) denote the set of critical points of the energy functional E. That is, the set of closed geodesics on M. For each $a \geq 0$ we consider the sublevel set of E below a as $\Omega(M)^a = \{\alpha \in \Omega(M) : E(\alpha) \leq a\}$. Then:

- The flow Φ_λ is well defined for all $\lambda \geq 0$,
- Crit$(E) \cap \Omega(M)^a$ is compact, and
- $\partial\Omega(M)^0 \cong M \subset \Omega(M)^\epsilon$ is a strong deformation retract via Φ_λ for $\epsilon > 0$ small enough.

Another interesting result of topological nature that can be obtained in this context is the following.

Proposition 1.1.2. *The natural inclusions* $\Omega(M) \hookrightarrow \Omega^c(M)$ *and* $\Omega_{x,y}(M) \hookrightarrow \Omega^c_{x,y}(M)$ *are homotopy equivalences.*

Additional results describing the Morse theory underlying the energy functional E can be found in [88, c. 3] and [66, s. 2.4].

Let us now quickly introduce the notion of Φ-family, which is associated to the negative gradient flow Φ. Suppose that M is compact and denote by P either $\Omega(M)$ or $\Omega_{x,y}(M)$. A Φ-**family** is a collection $\mathcal{A}$ of nonempty subsets of P such that $E|_A$ is bounded on each $A \in \mathcal{A}$ and $\mathcal{A}$ is closed under Φ in the sense that if $A \in \mathcal{A}$ then $\Phi_\lambda(A) \in \mathcal{A}$ for all $\lambda \geq 0$.

Pick $a \in \mathbb{R}$ and choose $\epsilon > 0$ such that E has no critical values in $(a, a+\epsilon]$. We define a **Φ-family of** P mod P^a to be a collection of nonempty subsets $\mathcal{A}$ of P such that $\mathcal{A}$ is a Φ-family and each member of $\mathcal{A}$ is not contained in $P^{a+\epsilon}$. The **critical value** of a Φ-family $\mathcal{A}$ of P mod P^a is defined as

$$\alpha_{\mathcal{A}} = \inf_{A \in \mathcal{A}} \sup E|_A.$$

Hence, in these terms we get that:

Theorem 1.1.9. *It holds that $\alpha_{\mathcal{A}} > a$ and there exists a critical point α of E such that $E(\alpha) = \alpha_{\mathcal{A}}$.*

Furthermore, it follows that E restricted to a connected component of P assumes its infimum in some point and such a point is actually a critical point of E.

As explained for instance in [66, c. 2], the previous results are part of what is called **Lusternik–Schnirelmann theory** and they can be applied to study the existence of at least one closed geodesic of positive length on a compact Riemannian manifold.

Theorem 1.1.10. *Let (M, η^M) be a compact Riemannian manifold. If the fundamental group of M has an element of infinite order or is finite then there exists at least a closed geodesic of positive length on M.*

It is worth mentioning that Birkhoff, Gromoll–Meyer, Hingston, Lusternik–Fet, Vigué-Poirrier–Sullivan, Schnirelmann, among others, developed other powerful methods to prove the existence of one or possibly many (even infinitely many) closed geodesics on Riemannian manifolds [18, 55, 60, 78, 114]. For instance, the case of simply connected Riemannian manifolds (particularly on the simply connected compact rank 1 symmetric spaces) was studied by Hingston in [60] by applying equivariant Morse theory to the energy functional over the space of closed curves. Our purpose in this thesis is to try to provide a similar result as that described in Theorem 1.1.10, but in the context of separated differentiable stacks whose underlying orbit space is compact.

1.2 Isometric Lie group actions

Isometric actions of Lie groups on Riemannian manifolds have been widely studied in the literature, as they constitute a powerful tool applied to deal with several interesting problems in both mathematics and physics, yielding important geometric-topological consequences and describing certain physical phenomena in nature. They appear in all branches of science where symmetries preserving length and angle measures play a role.

This short section aims at providing some classical results around the notion of isometric action of a Lie group on a Riemannian manifold. We mainly follow [2]. Let $\theta^0 : K \times M \to M$ be a smooth action of a Lie group K on a Riemannian manifold (M, η^M). For each $k \in K$ we denote by $\theta^0_k : M \to M$ the diffeomorphism defined by sending $x \mapsto kx$. Accordingly, for each $x \in M$ we denote by $\theta^0_x : G \to M$ the smooth map given by sending $k \mapsto kx$.

Definition 1.2.1. The action θ^0 is said to be **isometric** if θ^0_k is an isometry of (M, η^M) for all $k \in K$.

A couple of comments come in order.

- If the action θ^0 is additionally free and proper then there exists a unique metric $\overline{\eta}^M$ on the quotient manifold M/K such that the orbit projection $\pi : M \to M/K$ becomes a

Riemannian submersion. Such a metric is defined as

$$\overline{\eta}^M_{[x]}(v,u) := \eta^M_x(\tilde{v},\tilde{u}),$$

where x is any point in the fiber $\pi^{-1}([x])$ and $\tilde{v},\tilde{u}$ are the unique vectors in $\ker(d\pi(x))^{\perp_{\eta^M}}$ satisfying $d\pi(x)(\tilde{v}) = v$ and $d\pi(x)(\tilde{u}) = u$.

- Let $\theta^0 : K \times M \to M$ be a smooth action of a Lie group K on a smooth manifold M. If K is compact then there always exists a Riemannian metric η^M on M for which θ^0 becomes an isometric action. Indeed, let us consider any Riemannian metric η on M which can be constructed by using partitions of unity and take the normalized Haar measure μ on K. Then, η^M is defined by averaging

  $$\eta^M := \int_K (\theta^0_k)^* \eta d\mu(k).$$

  A similar construction can be obtained when weaken the compactness of K just by requiring that the action θ^0 is proper, visit [65].

The equivariant tubular neighborhood theorem is one of the most interesting results that can be derived from the existence of Riemannian metrics which are invariant by the action of certain compact Lie group. In order to state this result we need to introduce the following terminology.

Definition 1.2.2. If $\theta^0 : K \times M \to M$ is a smooth action of a Lie group K on a smooth manifold M then a **standard slice** at $x_0 \in M$ is an embedded submanifold S_{x_0} containing x_0 and satisfying the following properties:

i. $T_{x_0}M = T_{x_0}(K \cdot x_0) \oplus T_{x_0}S_{x_0}$ and $T_xM = T_x(K \cdot x) + T_xS_{x_0}$ for all $x \in S_{x_0}$,

ii. S_{x_0} is $\mathrm{Iso}_K(x_0)$-invariant, and

iii. if $x \in S_{x_0}$ and $k \in K$ are such that $\theta_k(x) \in S_{x_0}$ then $k \in \mathrm{Iso}_K(x_0)$.

Here $K \cdot x_0$ and $\mathrm{Iso}_K(x_0)$ respectively denote the K-orbit and the K-isotropy group at x_0 with respect to θ^0.

As an important fact we have that:

Proposition 1.2.1. *Let $\theta^0 : K \times M \to M$ be a proper Lie group action and $x_0 \in M$. Then there exists a slice S_{x_0} at x_0.*

This slice can be constructed by considering a Riemannian metric on M for which the restricted action of the isotropy group $\mathrm{Iso}_K(x_0)$ on M becomes isometric, see [2, s. 3.2] for specific details. The subspace $\mathrm{Tub}(K \cdot x_0) := \theta^0(K, S_{x_0})$ is an open neighborhood in M containing the orbit $K \cdot x_0$. We shall refer to this open subspace as the **tubular neighborhood** of $K \cdot x_0$.

Let us suppose that $\theta^0 : K \times M \to M$ is a proper Lie group action and consider a slice S_{x_0} at $x_0 \in M$. On the one hand, by condition ii. from Definition 1.2.2 it follows that θ^0 restricts to a well defined action of $\mathrm{Iso}_K(x_0)$ on S_{x_0}. On the other hand, we have that the orbit projection $\pi : K \to K/\mathrm{Iso}_K(x_0)$ canonically determines a $\mathrm{Iso}_K(x_0)$-principal bundle. Hence, we can construct an associated fiber bundle $K \times_{\mathrm{Iso}_K(x_0)} S_{x_0}$ over $K/\mathrm{Iso}_K(x_0)$ with fiber S_{x_0}. The total space $K \times_{\mathrm{Iso}_K(x_0)} S_{x_0}$ is defined as the quotient manifold $(K \times$

$S_{x_0})/\mathrm{Iso}_K(x_0)$ with respect to the action $k \cdot (k', x) = (k'k^{-1}, kx)$ for all $k \in \mathrm{Iso}_K(x_0)$, $k' \in K$ and $x \in S_{x_0}$. The projection $\overline{\pi} : K \times_{\mathrm{Iso}_K(x_0)} S_{x_0} \to K/\mathrm{Iso}_K(x_0)$ is given by $\overline{\pi}([k, x]) = \pi(k)$.

The following result shows that manifolds equipped with proper K-actions are locally K-equivariant to associated bundles.

Theorem 1.2.1 (Equivariant tubular neighborhood theorem)**.** *Let $\theta^0 : K \times M \to M$ be a proper Lie group action. Then, for every $x_0 \in M$ there exists a K-equivariant diffeomorphism* $\mathrm{Tub}(K \cdot x_0) \xrightarrow{\cong} K \times_{\mathrm{Iso}_K(x_0)} S_{x_0}$.

Such a result can be used to study orbit types of proper actions, thus obtaining a way to induce stratifications for M as well as for the orbit space M/K, see [2, c. 3].

We introduce now the notion of orthogonal (or quadratic) Lie group which will be a key ingredient in several constructions of this thesis. Let K be a real connected Lie group and $\mathfrak{k}$ be its Lie algebra. For every $k \in K$, we denote by $L_k : K \to K$ (resp. $R_k : K \to K$) the left (resp. right) multiplication by k on K. A Riemannian metric $\langle \cdot, \cdot \rangle$ over K is called **bi-invariant** if L_k and R_k are isometries of $(K, \langle \cdot, \cdot \rangle)$ for all $k \in K$. If $\mathrm{Ad} : K \to \mathrm{GL}(\mathfrak{k})$ and $\mathrm{ad} : \mathfrak{k} \to \mathfrak{gl}(\mathfrak{k})$ denote the adjoint representations of K and $\mathfrak{k}$, respectively, then to have a bi-invariant metric $\langle \cdot, \cdot \rangle$ on K is equivalent to having an inner product $\langle \cdot, \cdot \rangle_0 : \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}$ such that any of the following statements are satisfied:

- $\mathrm{Ad}(k) : \mathfrak{k} \to \mathfrak{k}$ is a linear isometry of $(\mathfrak{k}, \langle \cdot, \cdot \rangle_0)$ for all $k \in K$. That is,

$$\langle \mathrm{Ad}(k)(x), \mathrm{Ad}(k)(y) \rangle_0 = \langle x, y \rangle_0, \qquad k \in K, \quad x, x \in \mathfrak{k}. \tag{1.5}$$

- $\mathrm{ad}(x) : \mathfrak{k} \to \mathfrak{k}$ is an infinitesimal isometry of $(\mathfrak{k}, \langle \cdot, \cdot \rangle_0)$ for all $x \in \mathfrak{k}$, meaning that

$$\langle \mathrm{ad}(x)(y), z \rangle_0 + \langle y, \mathrm{ad}(x)(z) \rangle_0 = 0, \qquad x, y, z \in \mathfrak{k}. \tag{1.6}$$

An inner product satisfying Identity (1.5) is said to be Ad-**invariant**.

Definition 1.2.3. The pair:

i. $(K, \langle \cdot, \cdot \rangle)$ is called an **orthogonal Lie group** if $\langle \cdot, \cdot \rangle$ is a bi-invariant Riemannian metric on K, and

ii. $(\mathfrak{k}, \langle \cdot, \cdot \rangle_0)$ where $\mathfrak{k}$ is a finite-dimensional Lie algebra and $\langle \cdot, \cdot \rangle_0$ is an Ad-invariant inner on $\mathfrak{k}$ is named to be an **orthogonal Lie algebra**. Clearly, such a property is equivalent to asking $\langle \cdot, \cdot \rangle_0$ to verify Identity (1.6).

It is simple to check that there exists a one-to-one correspondence between simply connected orthogonal Lie groups and orthogonal Lie algebras, see [85, 89]. In particular, if $\langle \cdot, \cdot \rangle_0 : \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}$ is an Ad-invariant inner product, then the formula

$$\langle v, u \rangle(k) := \langle d(L_{k^{-1}})_k(v), d(L_{k^{-1}})_k(u) \rangle_0 \qquad k \in K,$$

defines a bi-invariant Riemannian metric on K. Some of the most interesting examples of orthogonal Lie groups are provided by compact Lie groups, semisimple Lie groups, cotangent bundles of Lie groups, and the so-called λ-oscillator groups [85].

Let us introduce two general constructions which will be generalized later on.

Example 1.2.1 (Principal connection warping)**.** Suppose that $\pi : P \to M$ is a principal K-bundle over a Riemannian manifold (M, η^M) such that the structural group K is orthogonal with bi-invariant metric $\langle \cdot, \cdot \rangle$. There exists a Riemannian metric on P for which the action of K on P is isometric and such that the projection $\pi : P \to M$ becomes a Riemannian submersion, compare [98]. Indeed, let $\langle \cdot, \cdot \rangle_0$ be the induced Ad-invariant inner product on the Lie algebra $\mathfrak{k}$ of K and let $\omega \in \Omega^1(P, \mathfrak{k})$ denote any connection 1-form on P. Then we set

$$\eta^P(v, w) = \eta^M(d\pi(v), d\pi(w)) + \langle \omega(v), \omega(w) \rangle_0.$$

It is simple to check that this expression yields a well defined Riemannian metric on P with the desired properties. This is because π is constant along the action orbits, ω is of Ad-invariant type, and $\ker(d\pi)^{\perp_{\eta^P}} = \ker(\omega)$.

Example 1.2.2 (Cheeger deformation)**.** Consider a compact Lie group K acting isometrically on a Riemannian manifold (M, η). Let us fix a bi-invariant metric $\langle \cdot, \cdot \rangle$ on K and endow the manifold $M \times K$ with the product metric $\eta \oplus \frac{1}{\tau} \langle \cdot, \cdot \rangle$. There is a natural isometric Lie group action of K on $M \times K$ given by $k' \cdot (x, k) = (k'x, k'k)$ for all $k, k' \in K$ and $x \in M$. Thus, we know that the quotient manifold $(M \times K)/K$ can be equipped with a unique Riemannian metric η_τ such that the orbit projection $\pi : M \times K \to (M \times K)/K$ is a Riemannian submersion. After some simple identifications it follows that the manifold $(M \times K)/K$ is diffeomorphic to M, so that we have actually obtained another Riemannian metric η_τ on M for which the map $\pi : \left(M \times K, \eta \oplus \frac{1}{\tau} \langle \cdot, \cdot \rangle\right) \to (M, \eta_\tau)$, given by sending $(x, k) \mapsto k^{-1}x$, is a Riemannian submersion. A couple of interesting properties of the Riemannian manifold (M, η_τ) are the mentioned below.

- The original K-action on (M, η) is also isometric on (M, η_τ). This is because there is another isometric action of the Lie group K on $\left(M \times K, \eta \oplus \frac{1}{\tau} \langle \cdot, \cdot \rangle\right)$, given by $k' * (x, k) = (x, kk'^{-1})$, which commutes with the previous one, and hence descends to an isometric K-action on the corresponding orbit space (M, η_τ). Note that $k' \cdot \pi(x, k) = k'k^{-1}x = \pi(k' * (x, k))$, so that the K-action induced by $*$ is the original K-action on M.

- The 1-parameter family of metrics η_τ on M varies smoothly with τ and extends smoothly to $\tau = 0$ with $\eta_0 = \eta$. Therefore, the collection $\{\eta_\tau\}_{\tau \geq 0}$ determines a deformation of τ by other K-invariant metrics on M. This is the so-called **Cheeger deformation** of η.

Example 1.2.3 (Equivariant Morse theory)**.** As evidenced in Subsection 1.1.1, Riemannian metrics on a manifold which are invariant by the action of a Lie group are a key ingredient when developing an equivariant Morse theory for invariant functions. This well known fact was used by Wasserman [116] and by Austin–Braam [8] in order to show some of the classical Morse theory results in equivariant topology.

1.2.1 Killing vector fields

Let us finish this section by mentioning a few facts about the Lie algebra of infinitesimal isometries on a Riemannian manifold. We shall follow [67, c. VI, s. 3] closely. It is well known that an **isometry** of a Riemannian manifold (M, η) is a diffeomorphism $\phi : M \to M$ leaving η invariant, i.e. $\phi^*\eta = \eta$. The set of isometries of (M, η) will be denoted by Iso(M, η). This is clearly a group under the composition of maps. A vector field v on M is called an **infinitesimal isometry** or a **Killing vector field** if the local 1-parameter subgroup generated by its flow in a neighborhood of each point of M consists of local isometries. Note that the latter is equivalent to asking $L_v\eta = 0$, where L stands for the Lie derivative of tensors on M. The set of Killing vector fields on (M, η) will be usually denoted by $\mathfrak{o}(M, \eta)$. This is a Lie subalgebra of the Lie algebra of vector fields on M since

$$L_{[v,u]}\eta = L_v \circ L_u\eta - L_u \circ L_v\eta = 0, \quad v, u \in \mathfrak{o}(M, \eta).$$

Furthermore:

Theorem 1.2.2. *The Lie algebra $\mathfrak{o}(M, \eta)$ of Killing vector fields on a connected Riemannian manifold (M, η) is of dimension at most $n(n+1)/2$ where $n = \dim M$.*

In other words, the Lie algebra of Killing vector fields on a connected Riemannian manifold is always finite dimensional. More importantly, we have the following result.

Theorem 1.2.3. *Let (M, η) be a Riemannian manifold with a finite number of connected components. Then the group* Iso(M, η) *of isometries of (M, η) is a Lie group with respect to the compact-open topology in M. Moreover, the Lie algebra of* Iso(M, η) *is isomorphic to the Lie algebra of all complete Killing vector fields in $\mathfrak{o}(M, \eta)$.*

Observe that if M is compact then the Lie algebra of Iso(M, η) is naturally isomorphic to $\mathfrak{o}(M, \eta)$.

Let us now consider an action $\theta^0 : K \times M \to M$ of a Lie group K on a manifold M. It is well known that the set Diff(M) of diffeomorphisms on M has the structure of an infinite dimensional Lie group and that its Lie algebra may be identified with the Lie algebra $\mathfrak{X}(M)$ of vector fields on M. Therefore, most of the features of θ^0 may be interpreted in terms of the Lie group homomorphism $K \to$ Diff(M) defined by sending $k \mapsto \theta^0_k$. Differentiating at the identity we get what we call an **infinitesimal action** of the Lie algebra $\mathfrak{k}$ of K on M. That is, a Lie algebra homomorphism $\mathfrak{k} \to \mathfrak{X}(M)$. For the action θ^0, such an infinitesimal action is defined by sending $\xi \mapsto \tilde{\xi}$, where $\tilde{\xi}$ denotes the fundamental vector field on M associated to $\xi \in \mathfrak{k}$. From Theorem 1.2.3 it follows that if (M, η) is a Riemannian manifold and K acts on it isometrically then we obtain now a Lie group homomorphism $K \to$ Iso(M, η) together with a Lie algebra homomorphism $\mathfrak{k} \to \mathfrak{o}(M, \eta)$. It is clear that the vector field $\tilde{\xi}$ is always complete and it automatically satisfies $L_{\tilde{\xi}}\eta = 0$ since the action θ^0 is isometric.

1.3 Lie groupoids

In this section we discuss the basics on Lie groupoid theory, paying special attention to the relation between Lie groupoids and differentiable stacks. These geometric objects provide a framework suitable to perform differential geometry on certain singular spaces. We shall be following [37, 80, 92] closely.

A concise way to define groupoids is as follows.

Definition 1.3.1. A **groupoid** is a small category in which every morphism is invertible.

A groupoid can be equivalently described by a set M of objects, a set G of morphisms/arrows between objects, together with structural maps $s, t : G \to M$ source and target, a partial composition $m : G^{(2)} \to G$ from the set of composable arrows $G^{(2)} = G \times_M G$, an inversion $i : G \to G$ and a unit map $u : M \to G$, satisfying the axioms of a category. The collection of maps mentioned above is called **structural maps** of the groupoid. We usually denote a groupoid by $G \rightrightarrows M$ or simply by G provided the set of objects M is understood from the context. An arrow g with source x and target y will be sometimes depicted either by $y \xleftarrow{g} x$ or $x \xrightarrow{g} y$.

Definition 1.3.2. A **Lie groupoid** is a groupoid $G \rightrightarrows M$ where both M and G are smooth manifolds, the structural maps are smooth and $s, t : G \to M$ are surjective submersions.

By convention, all our manifolds (including all our groupoids) are second countable and Hausdorff. Let us describe some features concerning the structure of a Lie groupoid. Let $G \rightrightarrows M$ be a Lie groupoid. For each $x \in M$, its **isotropy group** is defined as $G_x := s^{-1}(x) \cap t^{-1}(x)$. There is an equivalence relation on M defined by $x \sim y$ if there exists $g \in G$ with $s(g) = x$ and $t(g) = y$. The corresponding equivalence class of $x \in M$ is denoted by $\mathcal{O}_x \subseteq M$ and called the **orbit** of x. The previous equivalence relation defines a quotient space M/G called the **orbit space** of $G \rightrightarrows M$. This space equipped with the quotient topology is in general a *singular space*, that is, it does not carry a differentiable structure making the quotient projection $M \to M/G$ a surjective submersion. The first fundamental aspects concerning the geometric information underlying a Lie groupoid are:

- the isotropies G_x are Lie groups,
- the orbits $\mathcal{O}_x$ are immersed submanifolds of M, and
- the s-fiber at x is a principal G_x-bundle over $\mathcal{O}_x$ with projection determined by the restriction of the target map t.

The partition of M into connected components of the orbits of $G \rightrightarrows M$ forms a foliation $\mathcal{F}_M$ of M, which is possibly singular, in the sense that different leaves might have different dimension. The pullback foliation $\mathcal{F}_G := s^*\mathcal{F}_M = t^*\mathcal{F}_M$ of G may be also singular, as its leaves in G have the same codimension as the leaves of $\mathcal{F}_M$ in M. We shall refer to these foliations as the **characteristic foliations** of $G \rightrightarrows M$.

Example 1.3.1. Any Lie group K can be viewed as a Lie groupoid with just one object $K \rightrightarrows *$. Clearly, the source and target maps are trivial and the multiplication and inversion are given by the Lie group structure of K.

Example 1.3.2. Any manifold M defines a Lie groupoid $M \rightrightarrows M$ where all of the structural maps are given by the identity map on M. This will be called the **unit groupoid** associated to M.

Example 1.3.3. A smooth action of a Lie group K on a manifold M gives rise to a Lie groupoid $K \ltimes M \rightrightarrows M$ named as **action groupoid**. The arrows $K \ltimes M$ stands for the product manifold $K \times M$, the source and target maps are respectively defined by $s(k, x) = x$ and $t(k, x) = kx$, and the composition is set by $(k, k'x)(k', x) = (kk', x)$. Note that the orbits and the isotropy groups of the action groupoid coincide with the usual notions for smooth actions.

This is one of the best sources of examples which allow to expose the rich geometric information that can be encoded in a Lie groupoid. For instance, let the circle S^1 act on the plane $\mathbb{R}^2$ by rotations. It follows that the leaves of the singular foliation on the plane corresponding to the associated action groupoid are the origin and the concentric circles centered on it. In this case, the orbit space of the induced action groupoid is Hausdorff, something which might not occur in general. Indeed, suppose that $(\mathbb{R}^+, \cdot)$ acts on the plane $\mathbb{R}^2$ by scalar multiplication. The orbits of the corresponding action groupoid are the origin and the radial open half-lines. It holds that the point determined by the origin is dense inside the orbit space of such an action groupoid.

Example 1.3.4. A subjective submersion $\pi : M \to N$ induces the so-called **submersion groupoid** $M \times_N M \rightrightarrows M$. The source and target maps are given by the projections from the pullback manifold $M \times_N M$ onto M and the composition is defined as $(x, y)(y, z) = (x, z)$. In this case, the orbits are the fibers of π, the isotropies are trivial, and the orbit space can be naturally identified with N. Particular instances of submersion groupoids are the following. Firstly, associated to the constant map $M \to *$ is the unit groupoid $M \rightrightarrows M$. Secondly, the identity map $M \to M$ gives rise to the **pair groupoid** $M \times M \rightrightarrows M$. Thirdly, if (U_i) is an open cover for a manifold M then the groupoid $\coprod_{j,i} U_j \cap U_i \rightrightarrows \coprod_i U_i$ arising from the canonical submersion $\coprod_i U_i \to M$ is called the $\check{C}$**ech groupoid** associated to the open cover.

Example 1.3.5. Let $\pi : P \to M$ be a principal K-bundle. It is well known that we can recover $\pi : P \to M$ as the quotient projection $P \to P/K$ of the free and proper action of K on P. The **gauge groupoid** of P is obtained as the quotient groupoid of the pair groupoid $P \times P \rightrightarrows P$ by the canonical action of K. It is simple to see that gauge groupoids have only one orbit. Conversely, every Lie groupoid $G \rightrightarrows M$ having only one orbit is the gauge groupoid of some s-fiber. Indeed, just consider the gauge groupoid determined by the principal G_x-bundle $t : s^{-1}(x) \to \mathcal{O}_x$ for any $x \in M$.

A particular example of this kind of Lie groupoid can be built as follows. Let us consider a vector bundle $E \to M$. We can define a **general linear groupoid** $\mathrm{GL}(E) \rightrightarrows M$ for E which generalizes the Lie group of automorphisms of a finite-dimensional vector space. Namely, the arrows between two objects $x, y \in M$ consist of linear isomorphisms $E_y \leftarrow E_x$ between the fibers of E. The structural maps are the obvious ones. By using parallel translation with respect to a connection on E we may check that $\mathrm{GL}(E) \rightrightarrows M$ has only one orbit. Of course, such a groupoid can be recovered as the gauge groupoid associated to the frame bundle $F(E) \to M$ of E. Note that if the vector bundle E additionally carries a Riemannian metric then we may construct an **orthonormal linear groupoid** $\mathrm{O}(E) \rightrightarrows M$ in a similar fashion.

Example 1.3.6. Let M be a smooth manifold. The **fundamental groupoid** of M is defined to be $\Pi_1(M) \rightrightarrows M$ where $\Pi_1(M)$ stands for the manifold of homotopy classes of paths with fixed endpoints. The source and target maps are respectively defined by $s([\sigma]) = \sigma(0)$ and $t([\sigma]) = \sigma(1)$ and the composition of homotopy classes is given by the concatenation of representative paths. The isotropy at $x \in M$ corresponds to the fundamental group $\Pi_1(M, x)$. If M is connected then we may think of the fundamental groupoid of M as the gauge groupoid determined by the principal $\Pi_1(M, x)$-bundle $p : \tilde{M} \to M$, that is, its universal covering.

Example 1.3.7. For a regular foliation $\mathcal{F}$ on a manifold M we can define the **monodromy groupoid** $\mathrm{Mon}(\mathcal{F}) \rightrightarrows M$ whose manifold of arrows $\mathrm{Mon}(\mathcal{F})$ is formed by leafwise homotopy classes of paths with fixed endpoints. The structural maps are defined in a similar fashion as in the previous example. Therefore, the orbits and the isotropy groups correspond to the leaves of $\mathcal{F}$ and their fundamental groups, respectively. The **holonomy groupoid** $\mathrm{Hol}(\mathcal{F}) \rightrightarrows M$ may be defined analogously, except that one considers instead the holonomy classes of paths for arrows of the homotopy classes. Indeed, each arrow in $\mathrm{Mon}(\mathcal{F})$ induces the germ of a transverse diffeomorphism (the holonomy of a path) and the quotient of the monodromy groupoid by holonomy classes is still a Lie groupoid, namely, the one that we called holonomy groupoid, see [92, s. 2.1].

Example 1.3.8. A Lie groupoid $G \rightrightarrows M$ gives rise to a new Lie groupoid $TG \rightrightarrows TM$ after applying the tangent functor to each of its structural maps. This is known as the **the tangent groupoid** of $G \rightrightarrows M$.

Since Lie groupoids are small categories with some additional smooth structure, it is natural to define morphism between them simply as functors, but taking into account the smooth structures involved.

Definition 1.3.3. A **Lie groupoid morphism** between $G \rightrightarrows M$ and $G' \rightrightarrows M'$ is a pair $\phi := (\phi^1, \phi^0)$ where $\phi^1 : G \to G'$ and $\phi^0 : M \to M'$ are smooth maps commuting with both source and target maps and preserving the composition map. Accordingly, $G \rightrightarrows M$ is said to be a **Lie subgroupoid** of $G' \rightrightarrows M'$ if ϕ is an injective immersive Lie groupoid morphism.

There is still an important piece of information underlying a Lie groupoid that we want to introduce. It concerns the transversal geometry of the orbit space which is encoded in the normal representation of any orbit. In order to do so we need the following terminology.

Definition 1.3.4. A **left Lie groupoid action** of $G \rightrightarrows M$ with **moment map** $\mu : P \to M$ is a smooth map $\theta : G \times_M P \to P$ verifying the following conditions:

i. $\mu(\theta_g(p)) = t(g)$,

ii. $\theta_g(\theta_h(p)) = \theta_{gh}(p)$ when $(g,h) \in G^{(2)}$, and

iii. $\theta_{u(x)}(p) = p$.

Right Lie groupoid actions can be defined in a similar manner. If $\mu : E \to M$ is a vector bundle and $\theta : G \times_M E \to E$ is linear, meaning that for each arrow $y \xleftarrow{g} x$ the induced map $E_y \xleftarrow{\theta_g} E_x$ is a linear isomorphism, then we get what is called a **representation** of $G \rightrightarrows M$ on E. Equivalently, such a representation may be described by a Lie groupoid morphism $(G \rightrightarrows M) \to (\mathrm{GL}(E) \rightrightarrows M)$ covering the identity map on M.

Let $G \rightrightarrows M$ be a Lie groupoid. If $S \subset M$ is a saturated submanifold, i.e. it is given by the union of orbits, then we can restrict the groupoid structure to $G_S = s^{-1}(S) = t^{-1}(S)$, thus obtaining a Lie subgroupoid $G_S \rightrightarrows S$ of $G \rightrightarrows M$. The Lie groupoid structure of the tangent groupoid $TG \rightrightarrows TM$ induces a new Lie groupoid $\nu(G_S) \rightrightarrows \nu(S)$ on the normal bundles, having the property that all of its structural maps are fiberwise isomorphisms. In particular, if $S = \mathcal{O}$ is any orbit, the source and target maps of the normal Lie groupoid $\nu(G_{\mathcal{O}}) \rightrightarrows \nu(\mathcal{O})$ yield vector bundle isomorphisms $\overline{ds} : \nu(G_{\mathcal{O}}) \to s^*\nu(\mathcal{O})$ and $\overline{dt} : \nu(G_{\mathcal{O}}) \to t^*\nu(\mathcal{O})$. Furthermore,

$$\overline{dt} \circ \overline{ds}^{-1} : s^*\nu(\mathcal{O}) \to t^*\nu(\mathcal{O}), \tag{1.7}$$

defines a representation of $G_{\mathcal{O}} \rightrightarrows \mathcal{O}$ on the normal bundle $\nu(\mathcal{O}) \to \mathcal{O}$. As a consequence, for every $x \in M$ the isotropy group G_x has a canonical linear representation on the normal fiber $\nu_x(\mathcal{O}_x)$. More concretely, such a representation can be seen as $g \cdot [v] := [dt(g)(w)]$ where $w \in T_gG$ is any vector with $ds(g)(w) = v$. We will refer to the action $G_x \curvearrowright \nu_x(\mathcal{O}_x)$ as the **normal representation** at $x \in M$ and to the collection $(M/G, \{G_x \curvearrowright \nu_x(\mathcal{O}_x)\}_{x \in M})$ as the **transversal data** of $G \rightrightarrows M$.

It is simple to see that the transversal data associated to any Lie groupoid is functorial. That is, a Lie groupoid morphism $\phi : (G \rightrightarrows M) \to (G' \rightrightarrows M')$ always induces a continuous map $M/G \to M'/G'$ between orbit spaces, a Lie group morphism $G_x \to G'_{\phi^0(x)}$ between isotropies, and a morphism of representations $(G_x \curvearrowright \nu_x(\mathcal{O}_x)) \to (G'_{\phi^0(x)} \curvearrowright \nu_{\phi^0(x)}(\mathcal{O}'_{\phi^0(x)}))$ between normal representations. Accordingly, Lie groupoid isomorphisms yield isomorphisms $(M/G, \{G_x \curvearrowright \nu_x(\mathcal{O}_x)\}_{x \in M}) \cong (M'/G', \{G'_{\phi^0(x)} \curvearrowright \nu_{\phi^0(x)}(\mathcal{O}'_{\phi^0(x)})\}_{x \in M})$. However, there is a larger class of groupoid morphisms which gives rise to isomorphisms between transversal data. These are the so-called Morita maps.

Definition 1.3.5. A Lie groupoid morphism $\phi : (G \rightrightarrows M) \to (G' \rightrightarrows M')$ is called a **Morita map** if it is **fully faithful** and **essentially surjective**, in the sense that the source/target maps $(s, t) : G \to M \times M$ and $(s', t') : G' \to M' \times M'$ define a fiber product of manifolds $G \cong (M \times M) \times_{(M' \times M')} G'$ and the map $G' \times_{M'} M \to M$ sending $(y \xleftarrow{g} \phi^0(x), x) \mapsto y$ is a surjective submersion. Morita maps shall be usually denoted by $(G \rightrightarrows M) \underset{\sim}{\to} (G' \rightrightarrows M')$.

The following deep result proved in [37] provides a complete geometric characterization of Morita maps in terms of transversal data.

Theorem 1.3.1. *A Lie groupoid morphism $\phi : (G \rightrightarrows M) \to (G' \rightrightarrows M')$ is a Morita map if and only if it induces an isomorphism between the transversal data associated to $G \rightrightarrows M$ and $G' \rightrightarrows M'$.*

It is worth mentioning that the previous result as well as its consequences play an important role in our treatment and study of the differential geometry/topology that may be carried out over the orbit space of a Lie groupoid and its associated differentiable stack.

Differentiable stacks

Roughly, a differentiable stack can be thought of as a generalization of the notion of manifold which allows us to study higher symmetries and singular geometric features. Our definition of stack strongly depends on the notion of Morita equivalence. Let $G \rightrightarrows M$ and $G' \rightrightarrows M'$ be two Lie groupoids. A **groupoid fraction** from G to G' is by definition a pair of Lie groupoid morphisms $(G \rightrightarrows M) \underset{\sim}{\xleftarrow{\phi}} (H \rightrightarrows N) \xrightarrow{\psi} (G' \rightrightarrows M')$ where ϕ is a Morita map. We shall denote fractions by $\psi/\phi : G \to G'$. In these terms, we set up the following definition.

Definition 1.3.6. Two Lie groupoids G and G' are **Morita equivalent** if there exists a fraction $\psi/\phi : G \to G'$ where both ϕ and ψ are Morita maps.

First of all, it is important to comment that Morita equivalence is in fact an equivalence relation. Second, one can always assume that both ϕ and ψ are **Morita fibrations**, in the sense that the maps on object $\phi^0 : N \to M$ and $\psi^0 : N \to M'$ are surjective submersions [37, 92]. Third, it follows from Theorem 1.3.1 that two Lie groupoids are Morita equivalent if and only if they have isomorphic transversal data.

Definition 1.3.7. A **differentiable stack** is defined to be a Lie groupoid up to Morita equivalence. We write $[M/G]$ for the differentiable stack presented by the Lie groupoid $G \rightrightarrows M$.

Before going further let us take a look at some instructive examples. We start by exhibiting some Morita maps.

Example 1.3.9. Two Lie groups $K \rightrightarrows *$ and $K' \rightrightarrows *$ are Morita equivalent if and only if they are isomorphic. Also, two manifolds $M \rightrightarrows M$ and $M' \rightrightarrows M'$ are Morita equivalent if and only if they are diffeomorphic.

Example 1.3.10. Let $G \rightrightarrows M$ be a Lie groupoid having only one orbit. It follows that for any $x \in M$ the inclusion $(G_x \rightrightarrows *) \hookrightarrow (G \rightrightarrows M)$ is a Morita map since the respective orbits spaces contain only one point and the normal representations are trivially isomorphic.

Example 1.3.11. Let $\pi : M \to N$ be a surjective submersion and consider its associated submersion groupoid $M \times_N M \rightrightarrows M$. It is simple to check that the Lie groupoid morphism $\phi : (M \times_N M \rightrightarrows M) \to (N \rightrightarrows N)$ defined by $\phi^1(x, y) = \pi(y)$ and $\phi^0(x) = \pi(x)$ is a Morita fibration. In particular, any open cover (U_i) for a manifold M determines a Lie groupoid $\coprod_{j,i} U_j \cap U_i \rightrightarrows \coprod_i U_i$ which is Morita equivalent to $M \rightrightarrows M$. The latter example may be generalized in the following way. Given a Lie groupoid $G \rightrightarrows M$ and an open cover $\mathcal{U} = (U_i)$ for M we can define a new Lie groupoid $G_{\mathcal{U}} := \left(\coprod_{j,i} G(U_j, U_i) \rightrightarrows \coprod_i U_i\right)$ with arrows $(y, j) \xleftarrow{(g,j,i)} (x, i)$ for $x \in U_i$, $y \in U_j$ and $y \xleftarrow{g} x$ an arrow in G. The composition is set by $(g, k, j)(h, j, i) = (gh, k, i)$. It follows that the canonical projection $\pi_{\mathcal{U}} : G_{\mathcal{U}} \to G$ also defines a Morita fibration.

Example 1.3.12. Let $G \rightrightarrows M$ be a Lie groupoid and $\mathcal{O}$ be the orbit through $x \in M$. Consider the action groupoids $G_x \ltimes \nu_x(\mathcal{O}) \rightrightarrows \nu_x(\mathcal{O})$ and $G_{\mathcal{O}} \ltimes \nu(\mathcal{O}) \rightrightarrows \nu(\mathcal{O})$ respectively determined by the normal representation at x and the representation of $G_{\mathcal{O}} \rightrightarrows \mathcal{O}$ on the normal bundle $\nu(\mathcal{O}) \to \mathcal{O}$, see Equation (1.7). It is simple to check that the inclusion

$$(G_x \ltimes \nu_x(\mathcal{O}) \rightrightarrows \nu_x(\mathcal{O})) \hookrightarrow (G_{\mathcal{O}} \ltimes \nu(\mathcal{O}) \rightrightarrows \nu(\mathcal{O})),$$

is fully faithful and essentially surjective, so that it is a Morita map.

Example 1.3.13. If K is a Lie group acting freely and properly on a smooth manifold M then the orbit space M/K can be endowed with a unique smooth structure such that the canonical projection $\pi : M \to M/K$ is a surjective submersion. It follows that the map $K \ltimes M \to M \times_{M/K} M$ sending $(k, x) \to (x, kx)$ induces a Lie groupoid isomorphism between the action groupoid $K \ltimes M \rightrightarrows M$ and the submersion groupoid $M \times_{M/K} M \rightrightarrows M$. Therefore, by Example 1.3.11 we get that $K \ltimes M \rightrightarrows M$ is Morita equivalent to the unit groupoid $M/K \rightrightarrows M/K$.

Example 1.3.14. Let $\mathcal{F}$ be a regular foliation on a smooth manifold M. The monodromy groupoid $\mathrm{Mon}(\mathcal{F}) \rightrightarrows M$ and the holonomy groupoid $\mathrm{Hol}(\mathcal{F}) \rightrightarrows M$ are not Morita equivalent in general, as there may be nontrivial loops with trivial holonomy. Nevertheless, the canonical projection $\mathrm{Mon}(\mathcal{F}) \to \mathrm{Hol}(\mathcal{F})$ still determines a homeomorphism between the orbit(leaf) spaces and an isomorphism on the normal directions to the leaves.

Some elementary examples of differentiable stacks are the following.

Example 1.3.15. Smooth manifolds can be identified with "separated" differentiable stacks that have no isotropy.

Example 1.3.16. The differentiable stack $[*/K]$ associated to the a Lie group $K \rightrightarrows *$ is known as the **classifying stack** of K. It is worth mentioning that this is a finite-dimensional stacky model for the usual infinite-dimensional classifying space BK.

Example 1.3.17. Let $K \curvearrowright M$ be a smooth Lie group action. The differentiable stack $[M/K]$ associated to the action groupoid $K \ltimes M \rightrightarrows M$ encodes the equivariant geometry of the action. If the action is free and proper then $[M/K]$ is identified with the quotient manifold M/K.

Example 1.3.18. Orbifolds are spaces locally modeled by quotients of Euclidean spaces by finite group actions. An **orbifold chart** (U, K, φ) for a space O consists of a connected open subset U inside some Euclidean space $\mathbb{R}^d$, a finite group K of diffeomorphisms of U and an open embedding $\varphi : U/K \to O$. An **orbifold** is defined to be a space O equipped with an orbifold atlas, which is just a collection $\mathcal{U} = (U_i, K_i, \varphi_i)$ of compatible orbifold charts. Two atlases define the same orbifold if they are compatible, in the sense that their union is again an atlas. Out of an orbifold O with a numerable atlas $\mathcal{U}$ we can define a Lie groupoid $G \rightrightarrows M$ whose manifold of objects M is given by $\coprod_i U_i$ and manifold of arrows G consists of germs of compositions of maps in some K_i, endowed with the sheaf-like manifold structure. If we consider two different numerable atlases $\mathcal{U}$ and $\mathcal{U}'$ of O such that $\mathcal{U}$ refines $\mathcal{U}'$ then a choice of inclusions leads to a Morita map $(G \rightrightarrows M) \to (G' \rightrightarrows M')$. In particular, common refinements of different orbifold atlases determine Morita equivalences between the induced Lie groupoids. From the stack viewpoint, orbifolds are defined as "separated" stacks with finite isotropy groups. For specific details visit [92].

Let us now quickly introduce the notion of stacky map. Let $G \rightrightarrows M$ and $G' \rightrightarrows M'$ be Lie groupoids. Two Lie groupoid morphisms $\phi, \psi : (G \rightrightarrows M) \to (G' \rightrightarrows M')$ are said to be **isomorphic** if there exists a smooth natural transformation between them. That is, there is a smooth map $\alpha : M \to G'$ with $s' \circ \alpha = \phi$ and $t' \circ \alpha = \psi$, verifying $\psi(g)\alpha(x) = \alpha(y)\phi(g)$ for each arrow $y \xleftarrow{g} x$ in G. It is simple to see that isomorphic Lie groupoid morphisms induce the same map between orbit spaces and determine linear maps between the normal representations which are related by conjugations. Two fractions $(G \rightrightarrows M) \xleftarrow[\sim]{\phi_1} (H_1 \rightrightarrows N_1) \xrightarrow{\psi_1} (G' \rightrightarrows M')$ and $(G \rightrightarrows M) \xleftarrow[\sim]{\phi_2} (H_2 \rightrightarrows N_2) \xrightarrow{\psi_2} (G' \rightrightarrows M')$ are **equivalent** if there are Morita maps $\alpha_1 : (H_3 \rightrightarrows N_3) \to (H_1 \rightrightarrows N_1)$ and $\alpha_2 : (H_3 \rightrightarrows N_3) \to (H_2 \rightrightarrows N_2)$ such that $\psi_1 \circ \alpha_1$ is isomorphic to $\psi_2 \circ \alpha_2$ and $\phi_1 \circ \alpha_1$ is isomorphic to $\phi_2 \circ \alpha_2$. It holds that the previous relation is also an equivalence relation [37, 92]. A class of fractions $[\psi/\phi] : [M/G] \to [M'/G']$ is named to be a **stacky map**.

Remark 1.3.1. The geometric interpretation of differentiable stacks as well as stacky maps between them that we shall adopt throughout this thesis is as follows.

- The characterization of Morita equivalences in terms of transversal data gives us some geometric intuition about the notion of differentiable stack $[M/G]$. Namely, this is an enhanced version of the orbit space M/G endowed with certain smooth information which is encoded by the normal representations $G_x \curvearrowright \nu_x(\mathcal{O}_x)$. This perspective allows us to think of points in $[M/G]$ as elements lying inside M/G (i.e. orbits), so that their tangent spaces can be modeled as follows. The **coarse tangent space** of the differentiable stack $[M/G]$ at $[x] = \mathcal{O}$ is by definition the coarse orbit space $\nu_x(\mathcal{O})/G_x \cong \nu(\mathcal{O})/G_{\mathcal{O}}$ of the action groupoid determined by the normal representation on the orbit $\mathcal{O}$ through $x \in M$, compare [42]. This will be denoted by $T_{[x]}[M/G]$. Such a space is well defined in the sense that two points in the same orbit have isomorphic normal representations and Morita equivalences preserve transversal data.

- Let $F := [\psi/\phi] : [M/G] \to [M'/G']$ be a stacky map. On the one side, the **coarse map** $\overline{F}$ of F is defined to be the composition $\overline{\psi} \circ \overline{\phi}^{-1} : M/G \to M'/G'$. On the other side, the **coarse differential** $\overline{dF}_{[x]}$ of F at $[x] \in [M/G]$ is defined by the composition $\overline{d\psi}_{[z]} \circ \overline{d\phi}_{[z]}^{-1} : T_{[x]}[M/G] \to T_{\overline{F}([x])}[M'/G']$ where z is a representative of the uniquely determined class $[z]$ verifying $\overline{\phi}([z]) = [x]$. The well definition of $\overline{F}$ and $\overline{dF}_{[x]}$ follows from the fact that $\overline{\psi}, \overline{\phi}, \overline{d\psi}_{[z]}, \overline{d\phi}_{[z]}$ are invariant under isomorphisms of Lie groupoid morphisms, so that it does not depend on the representative fraction of F.

Remark 1.3.2. We warn the reader that there is another way to introduce and study differentiable stacks. Roughly speaking, a stack is a category fibered in groupoids over the category of smooth manifolds with conditions of gluing objects and gluing morphisms. Any smooth manifold M can be interpreted as the stack given by $\mathrm{Hom}(-, M)$. In these terms, a differentiable stack is a pair $(\mathfrak{M}, M)$ where $\mathfrak{M}$ is a stack and M is a smooth manifold for which there exists a morphism of stacks $M \to \mathfrak{M}$ that is representable and has local sections [17]. Although our focus on this thesis does not follow the previous terminology, it is important to mention that there is a dictionary between such a viewpoint for differentiable stacks and the one provided by Lie groupoids up to Morita equivalences. This says that, in a certain sense, Lie groupoids are like "local charts" on a differentiable stack [17]. Such a dictionary will be used without further comments in Section 2.5 in order to interpret within our language some of the results proven in [58].

Miscellany on Lie groupoids

The aim of this subsection is to collect some notions, results and constructions from the theory of Lie groupoids that we will be using throughout. All of the above will depend on the class of Lie groupoid which we are interested in. Let $G \rightrightarrows M$ be a Lie groupoid.

Proper. We say that $G \rightrightarrows M$ is **proper** if the source/target map $(s, t) : G \to M \times M$ is proper. In this case, the isotropy groups are compact, the orbits are embedded and the orbit space is Hausdorff, second-countable, and paracompact [37, 92]. Two important features of proper Lie groupoids are that they admit proper Haar measure systems and are linearizable around saturated submanifolds. Firstly, a **Haar measure system** on $G \rightrightarrows M$ is a family $\{\mu^x\}_{x\in M}$ of measures on the s-fibers $s^{-1}(x)$ such that they are:

i. **right-invariant**: for all $g \in G$ the right multiplication $R_g : s^{-1}(t(g)) \to s^{-1}(s(g))$ satisfies $R_g^*(\mu^{s(g)}) = \mu^{t(g)}$, and

ii. **smooth**: for all $f \in C_c^\infty(G)$ the map $x \to \mu^x(f|_{s^{-1}(x)})$ is smooth.

Given a Haar system μ, the **support** $\mathit{supp}_x(\mu)$ at x is defined to be the smallest closed set K in the fiber $s^{-1}(x)$ such that $\mu^x(f) = 0$ for all $f \in C_c^\infty(s^{-1}(x)\backslash K)$. The **support** of μ is thus defined as $\mathit{supp}(\mu) = \cup_{x\in M}\mathit{supp}_x(\mu)$. Accordingly, μ is said to be **proper** if the restriction of s to $\mathit{supp}(\mu)$ is a proper map. It is well known that a Lie groupoid admits a proper Haar measure system if and only if it is proper [110]. The main reason we introduce this notion is because we want to be able to integrate, so that we can use averaging techniques.

Secondly, let S be a saturated submanifold in M. A **groupoid neighborhood** of $G_S \rightrightarrows S$ is given by an open Lie subgroupoid $(\tilde{U} \rightrightarrows U) \subseteq (G \rightrightarrows M)$ with $G_S \subset \tilde{U}$ and $S \subset U$. If our groupoid is proper then we can assume that the groupoid neighborhoods are **full** in the sense that $\tilde{U} = G_U = s^{-1}(U) \cap t^{-1}(U)$, see [40]. Let $\nu(G_S) \rightrightarrows \nu(S)$ be the normal groupoid associated to S. We say that $G \rightrightarrows M$ is **linearizable** around S if there are full groupoid neighborhoods $(G_U \rightrightarrows U) \subseteq (G \rightrightarrows M)$ of $G_S \rightrightarrows S$ and $(\nu(G_S)_V \rightrightarrows V) \subseteq (\nu(G_S) \rightrightarrows \nu(S))$ of $G_S \rightrightarrows S$ seen as the zero section, and a Lie groupoid isomorphism $\phi : (\nu(G_{\mathcal{O}})_V \rightrightarrows V) \to (G_U \rightrightarrows U)$ which is the identity on $G_S \rightrightarrows S$. We shall sometimes refer to ϕ as a **full Lie groupoid tubular neighborhood** of G_S. The *linearization theorem* asserts that every proper Lie groupoid is linearizable around any of its orbits. It is worth mentioning that this result generalizes other classical and important results in differential geometry such as the Ehresmann's theorem for submersions, the local Reeb stability for foliations, and the tube theorem for proper Lie group actions. The linearization problem was first addressed by Weinstein in [118] for the case of regular proper groupoids by reducing it to the fixed point case. A first complete proof of this result was provided by Zung in [120] with the extra assumption of source locally triviality. The latter hypothesis and variants of it were treated later by Crainic and Struchiner in [36]. Other novel approaches that lead to much more geometric proofs of the linearization theorem are given in [40, 39, 87]. In particular, if $S = \mathcal{O}_x$ is the orbit through any $x \in M$ then there is an open neighborhood U of x in M diffeomorphic to $O \times V_x$ where O is an open ball in the orbit $\mathcal{O}_x$ centered at x and V_x is a G_x-invariant open ball in $\nu_x(\mathcal{O}_x)$ centered at the origin. Under this diffeomorphism $G_U \rightrightarrows U$ is isomorphic to the product of the pair groupoid $O \times O \rightrightarrows O$ and the action groupoid $G_x \ltimes V_x \rightrightarrows V_x$, see [103, Cor. 3.11].

Another important fact to take into account is that if one of any two Morita equivalent Lie groupoids is proper then the other one is proper as well. Differentiable stacks presented by proper Lie groupoids will be called **separated**. Some elementary examples of proper groupoids are provided by compact Lie groups, submersions groupoids, action groupoids induced by proper Lie group actions, holonomy groupoids of regular Riemannian foliations, and restrictions of already known proper groupoids.

Étale. The Lie groupoid $G \rightrightarrows M$ is said to be **étale** if either s or t is a local diffeomorphism. This is clearly equivalent to asking that $\dim M = \dim G$. It is obvious that all the structural maps of an étale groupoid are local diffeomorphisms. Furthermore, it holds that the s-fibers, the t-fibers, the isotropy groups, and the orbits are discrete. More importantly, a Lie groupoid is Morita equivalent to an étale one if and only if it has discrete isotropy groups [33],[92, p. 136]. If we further assume that $G \rightrightarrows M$ is proper then any $x \in M$ has an open neighborhood U in M with an action of the isotropy group G_x such that there is an isomorphism between the étale groupoids $G_U \rightrightarrows U$ and $G_x \ltimes U \rightrightarrows U$, see [92, p. 142]. This implies that any proper étale groupoid defines an orbifold structure on its space of orbits. Accordingly, from the Lie groupoid viewpoint, orbifolds are thought of as proper étale groupoids up to Morita equivalence.

Some examples of étale groupoids are provided by manifolds, discrete groups, and action groupoids induced by Lie group actions of discrete groups.

Transitive. We say that $G \rightrightarrows M$ is **transitive** if it has only one orbit. It is simple to check that this is the case if and only if the source/target map $(s,t) : G \to M \times M$ is a surjective submersion. We already commented in Example 1.3.5 that every transitive groupoid is isomorphic to the gauge groupoid of a suitable principal bundle. Transitivity is a Morita invariant property. These kinds of groupoids are extensively discussed in [79].

Examples of transitive Lie groupoids are provided by action groupoids induced by transitive Lie group actions, fundamental groupoids associated to connected manifolds, restrictions of groupoid structures to the orbits, and obviously, gauge groupoids induced by principal bundles.

Regular. The Lie groupoid $G \rightrightarrows M$ is named to be **regular** if for each $x \in M$ the restriction of target map t to the fiber $s^{-1}(x)$ has locally constant rank. In such a case, all the orbits have the same dimension so that their connected components determine a regular foliation of M. There exists a classification result which says that any regular Lie groupoid G fits into a short exact sequence $K \to G \to E$ with K a bundle of Lie groups and E a foliation groupoid [91]. Moreover, if G is proper then so are K and E. On the one hand, a **bundle of Lie groups** is a Lie groupoid for which the source and target maps agree. In this case, each arrow lies inside an isotropy group, all isotropy groups are isomorphic and the groupoid itself is a fiber bundle with a Lie group as fiber. These groupoids are proper if and only if the fiber Lie group is compact. On the other hand, a **foliation groupoid** is a Lie groupoid whose isotropy groups are discrete. This is equivalent to asking that its combined source/target map has discrete fibers. As we already mentioned, a Lie groupoid is a foliation groupoid if and only if it is Morita equivalent to an étale Lie groupoid [33].

Lie algebroids. It is well known that any Lie group can be differentiated at the unit in order to obtain its infinitesimal counterpart which is a Lie algebra. The corresponding notion for a Lie groupoid is that of a Lie algebroid. A **Lie algebroid** over a manifold M consists of a vector bundle A together with a bundle map $\rho_A : A \to TM$ and a Lie bracket on the space of sections $\Gamma(A)$ satisfying the Leibniz identity:

$$[a, fb]_A = f[a, b] + L_{\rho_A(a)}(f)b, \quad a, b \in \Gamma(A),\ f \in C^\infty(M).$$

After conveniently using the latter identity together with the Jacobi identity of the Lie bracket on sections one can show that the induced map $\rho_A : \Gamma(A) \to \mathfrak{X}(M)$ is a Lie algebra homomorphism. Examples of Lie algebroids are provided by tangent bundles, Lie algebras, involutive distributions (foliations), bundles of Lie algebras, infinitesimal actions, Atiyah sequences, Poisson structures, among others [32, 79, 80]. More importantly, the Lie algebroid associated to the Lie groupoid $G \rightrightarrows M$ is defined to be the vector bundle $A := \ker(ds)|_M \subset TG$ with anchor map $\rho_A : A \to TM$ obtained by restricting $dt : TG \to TM$ to A and Lie bracket on $\Gamma(A)$ induced by the bracket of vector fields on G, see [32, s. 1.4]. We omit for now bringing additional details about this infinitesimal counterpart as well as the interesting theory it plays, leaving the study of its underlying properties for our specific purposes in Chapter 4. Classical references regarding this subject are for instance [32, 79, 80, 92].

Differentiable groupoid cohomology. There are several cohomology theories associated to groupoids which allow to extract different topological or geometrical information of the corresponding stacks depending on the purposes one has. Let us briefly introduce the differentiable cohomology with coefficients in a representation of a Lie groupoid by following [31, 111] closely.

Throughout several stages of this thesis a Lie groupoid will be also denoted as $G^{(1)} \rightrightarrows G^{(0)}$. This permits us to work with the simplicial structure underlying a Lie groupoid in a more concise way. Namely, the nerve $G^{(\bullet)}$ of $G^{(1)} \rightrightarrows G^{(0)}$ can be depicted as

$$\begin{tikzcd}
G^{(\bullet)} : \cdots \arrow[r, shift left=4] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[r, shift right=4] & G^{(n)} \arrow[r, shift left=4] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[r, shift right=4] & \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & G^{(2)} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & G^{(1)} \arrow[r, shift left=1] \arrow[r, shift right=1] & G^{(0)}.
\end{tikzcd}$$

The manifolds $G^{(n)}$ of n-**composable arrows** are given by

$$G^{(n)} = G^{(1)} \times_{G^{(0)}} \cdots \times_{G^{(0)}} G^{(1)} = \{(g_1, \cdots, g_n) :\ s(g_j) = t(g_{j+1});\ j = 1, \cdots, n-1\},$$

and the left arrows represent the **face maps** $d_k^n : G^{(n)} \to G^{(n-1)}$ for $k = 0, \cdots, n$. These are defined as $d_0^1 = t$, $d_1^1 = s$ and

$$d_k^n(g_1, \cdots, g_n) = \begin{cases} (g_2, \cdots, g_n) & \text{if} \quad k = 0 \\ (g_1, \cdots, g_i g_{i+1}, \cdots, g_n) & \text{if} \quad 0 < k < n \\ (g_1, \cdots, g_{n-1}) & \text{if} \quad k = n. \end{cases}$$

The face maps are surjective submersions and they satisfy the so-called **semi-simplicial identities**

$$d_k^{n-1} \circ d_{k'}^n = d_{k'-1}^{n-1} \circ d_k^n, \qquad k < k'. \tag{1.8}$$

Also, there are **vertex maps** $\lambda_j^n : G^{(n)} \to G^{(0)}$ defined by $\lambda_j^n = t \circ \mathrm{pr}_j$ for $j = 1, \cdots, n$ and $\lambda_{n+1}^n = s \circ \mathrm{pr}_n$. Let us consider a representation of $G^{(1)} \rightrightarrows G^{(0)}$ on a vector bundle $\pi : E \to G^{(0)}$. For any integer $n \geq 0$ we define a **smooth** n**-chain** on G with values in E

to be a smooth section of the pullback vector bundle of $E \to G^{(0)}$ along $\lambda_1^n : G^{(n)} \to G^{(0)}$. The set of those smooth n-chains is denoted by $C^n(G, E) := \Gamma(G^{(n)}, (\lambda_1^n)^* E)$. It turns out that the collection $C^\bullet(G, E)$ forms a cochain complex whose co-differential operator $\mathrm{d}^\bullet : C^\bullet(G, E) \to C^{\bullet+1}(G, E)$ is defined by

$$\begin{aligned}(\mathrm{d}^n c)(g_1, \cdots, g_{n+1}) &= g_1 \cdot c(g_2, \cdots, g_{n+1}) + \sum_{j=1}^{n} (-1)^j c(g_1, \cdots, g_j g_{j+1}, \cdots, g_{n+1}) \\ &+ (-1)^{n+1} c(g_1, \cdots, g_n),\end{aligned}$$

and $(\mathrm{d}^0 c)(g) = g \cdot c(s(g)) - c(t(g))$. One can check easily that $\mathrm{d}^2 = 0$ so that we have an associated cohomology $H^\bullet_{\mathrm{diff}}(G, E)$ which is known as the **differentiable groupoid cohomology** of G with coefficients in E. Note that $H^0_{\mathrm{diff}}(G, E) = \Gamma(E)^G$, that is, G-invariant sections of E.

Firstly, it simple to see that for the specific case of Lie groups we recover the usual differential cohomology. Secondly, if $G^{(1)} \rightrightarrows G^{(0)}$ is proper then $H^n_{\mathrm{diff}}(G, E) = 0$ for all $n \geq 1$. Thirdly, the differential groupoid cohomology is Morita invariant in the sense that if $\phi : G \to G'$ is a Morita map then $H^\bullet_{\mathrm{diff}}(G, \phi^* E) \cong H^\bullet_{\mathrm{diff}}(G', E)$, see [31].

Principal groupoid bi-bundles. Another approach to equivalences of Lie groupoids is specified in terms of principal bi-bundles. A **left G-bundle** over a manifold N is given by a left Lie groupoid action $\theta : (G \rightrightarrows M) \curvearrowright P \to M$ with a surjective submersion $\alpha : P \to N$ whose fibers are invariant by θ, that is, $\alpha(\theta_g(p)) = \alpha(p)$. A left G-bundle is said to be **principal** if θ is a free action and its orbits are exactly the fibers of α. Principal right G-bundles are similarly defined. In these terms, a **principal groupoid bi-bundle** $(P, \alpha, \alpha') : G \dashrightarrow G'$ is given by a left groupoid action $\theta : (G \rightrightarrows M) \curvearrowright P \to M$ and a right groupoid action $\theta' : M' \leftarrow P \curvearrowleft (G' \rightrightarrows M')$ verifying that the moment maps $\alpha : P \to M$ and $\alpha' : P \to M'$ are surjective submersions, θ yields a principal left G-bundle over $\alpha' : P \to M'$ and θ' yields principal right G'-bundle over $\alpha : P \to M$, and, when defined, the actions θ and θ' commute.

An **isomorphism** of principal bi-bundles $P, P' : G \dashrightarrow G'$ is a diffeomorphism $P \to P'$ which commutes with both action maps and both submersions. This allows to provide an alternative definition of Morita equivalence by considering instead an isomorphism class of principal bi-bundles. Actually, there exists a one-to-one correspondence between Morita equivalences of Lie groupoids and principal groupoid bi-bundles [17, 37, 92]. The approach for Morita equivalences in terms of isomorphism classes of principal bi-bundles will be also useful at some stages of this thesis.

1.3.1 Riemannian groupoids

The notion of Riemannian metric on a Lie groupoid that we will be dealing with along this thesis was introduced in [40] (see also [51, 103]). Such a notion is compatible with the groupoid composition so that it plays an important role in several parts of our work. We start by recalling that a submersion $\pi : (M, \eta^M) \to N$ with (M, η^M) a Riemannian manifold is said to be **Riemannian** if the fibers of it are equidistant (transverse condition). In this case the base N gets an induced metric $\eta^N := \pi_* \eta^M$ for which the linear map $d\pi(x) : (\ker(d\pi(x)))^\perp \to T_{\pi(x)} N$ is an isometry for all $x \in M$. If $(\eta^M)^*$ denotes the dual metric associated to η^M then the condition for a Riemannian submersion can be rephrased as follows. For all $x \in M$ the map $d\pi(x)^* : T^*_{\pi(x)} N \to \ker(d\pi(x))^\circ$ is an isometry, where $\ker(d\pi(x))^\circ$ denotes the annihilator of the vectors tangent to the fiber. If $\pi : M \to N$ is a surjective submersion then a Riemannian metric η^M on M is said to be **transverse** to π if for all $a \in N$ and all $x_1, x_2 \in \pi^{-1}(a)$ we have that the map

$$d\pi(x_1)^* \circ (d\pi(x_2)^*)^{-1} : \ker(d\pi(x_2))^\circ \to T^*_a N \to \ker(d\pi(x_1))^\circ, \tag{1.9}$$

is a linear isometry. In this case, there exists a unique metric η^N on N such that π becomes a Riemannian submersion. Such a metric is defined by the expression $\eta^N(d\pi(v), d\pi(w)) := \eta^M(v, w)$ for $v, w \in \ker(d\pi)^\perp$ and is called the **push-forward metric**. The notation we shall be using for the previous Riemannian metric is $\eta^N := \pi_* \eta^M$.

It is well known that given a Lie groupoid $G \rightrightarrows M$ every pair of composable arrows in $G^{(2)}$ may be identified with an element in the space of commutative triangles so that it admits an action of S_3 determined by permuting the vertices of such triangles. In these terms, we set up the following definition which is due to del Hoyo–Fernandes in [40].

Definition 1.3.8. A **Riemannian groupoid** is a pair $(G \rightrightarrows M, \eta)$ where $G \rightrightarrows M$ is a Lie groupoid and $\eta = \eta^{(2)}$ is a Riemannian metric on $G^{(2)}$ that is invariant by the S_3-action and transverse to the composition map $m : G^{(2)} \to M$.

The permutations $\sigma_1 = (xyz)$, $\sigma_2 = (xy)(z)$, and $\sigma_3 = (yz)(x)$ determine diffeomorphisms $(g, h) \mapsto (h^{-1}g^{-1}, g)$, $(g, h) \mapsto (gh, h^{-1})$, and $(g, h) \mapsto (g^{-1}, gh)$ on $G^{(2)}$ respectively intertwining the pairs of maps (π_1, π_2), (π_1, m), and (π_2, m) which go from $G^{(2)} \to G$. Therefore, the metric $\eta^{(2)}$ induces metrics $\eta^{(1)} = (\pi_2)_* \eta^{(2)} = m_* \eta^{(2)} = (\pi_1)_* \eta^{(2)}$ on G and $\eta^{(0)} = s_* \eta^{(1)} = t_* \eta^{(1)}$ on M such that $\pi_2, m, \pi_1 : G^{(2)} \to G$ and $s, t : G \to M$ are Riemannian submersions and $i : G \to G$ is an isometry. The metric $\eta^{(j)}$, for $j = 2, 1, 0$, is called a j-**metric**. On the one side, 0-metrics are Riemannian metrics transversely invariant under the action of G on M. In other words, the normal representation (1.7) is by linear isometries. Such metrics and their properties have been studied in [103]. On the other side, 1-metrics were first introduced in [51]. In a few words, these are Riemannian metrics invariant under inversion and for which the source and target fibers are equidistant.

Let us see some elementary examples.

Example 1.3.19. Firstly, any transitive groupoid admits 0-metrics. This is because they have only one orbit, so that the condition becomes vacuous. Similarly, Lie group bundles admit 0-metrics since their orbits are just points and the normal representations are trivial. Secondly, 0-metrics on submersions groupoids correspond to transverse Riemannian metrics with respect to the defining surjective submersion. Thirdly, a 0-metric on an étale Lie groupoid is the same as a Riemannian metric for which the bisections act by isometries, see [40].

Example 1.3.20. Consider a regular foliation $\mathcal{F}$ on a smooth manifold M. To have a 0-metric on $\mathrm{Hol}(\mathcal{F}) \rightrightarrows M$ is equivalent to having a Riemannian metric on M for which $\mathcal{F}$ becomes a Riemannian foliation. As shown in [40], there always exists a way of extending 0-metrics to 1-metrics in this case.

Example 1.3.21. Let K be a compact Lie group endowed with a bi-invariant metric η^K and suppose that it acts on a Riemannian manifold (M, η^M) by isometries. It follows that $\eta^M \oplus \eta^K$ does not define a 1-metric on the action groupoid $K \ltimes M \rightrightarrows M$ in general, as the inversion might not be an isometry. The **gauge trick** introduced in [40] yields a way to overcome this difficulty. For instance, by construction, the Riemannian metric $\eta^{(1)}$ on $G = K \ltimes M$ is such that the following are Riemannian submersions:

$$\begin{tikzcd}
(K\times K\times M, \eta^K\oplus\eta^K\oplus\eta^M) \arrow[r, "\pi_2"] \arrow[d, "(k_1,k_2,x)\mapsto(k_2k_1^{-1},k_1x)"] & (K\times M, \eta^K\oplus\eta^M) \arrow[d, "\pi"] \\
(G,\eta^{(1)}) \arrow[r, "s"'] & (M,\eta^M).
\end{tikzcd}$$

Example 1.3.22. If (M,η^0) is a Riemannian manifold then $\eta^0\oplus\eta^0\oplus\eta^0$ defines a 2-metric on the pair groupoid $M\times M\rightrightarrows M$.

Example 1.3.23. Let $\pi:(M,\eta^{(0)})\to(N,\eta^N)$ be a Riemannian submersion and $M\times_N M\rightrightarrows M$ be its associated submersion groupoid. Then, the expression

$$\eta^{(1)}:=\mathrm{pr}_1^*\eta^{(0)}+\mathrm{pr}_2^*\eta^{(0)}-(\pi\circ\mathrm{pr}_1)^*\eta^N,$$

defines a Riemannian metric on $M\times_N M$ in such a way both pr_1 and pr_2 become Riemannian submersions. It is simple to check that the inversion of $M\times_N M\rightrightarrows M$ is an isometry on $(M\times_N M,\eta^{(1)})$, so that we have obtained a 1-metric. A 2-metric on $M\times_N M\rightrightarrows M$ can be built in a similar fashion by using instead $\eta^{(1)}$.

The most important features about Riemannian groupoids we shall be using throughout come in order below. Specific details can be found in [40]. Let $(G\rightrightarrows M,\eta)$ be a Riemannian groupoid. Then:

- The units $u(M)\hookrightarrow G$ form a totally geodesic submanifold since it agrees with the set of fixed points of the inversion i which is an isometry.
- The characteristic foliations $\mathcal{F}_M$ on $(M,\eta^{(0)})$ and $\mathcal{F}_G$ on $(G,\eta^{(1)})$ are singular Riemannian foliations. This mainly follows from the fact that s and t are Riemannian submersions, so that their fibers naturally determine regular Riemannian foliations on $(G,\eta^{(1)})$.
- As commented before, the normal representation along orbits (1.7) is by linear isometries with respect to the induced bundle metric $\overline{\eta}^{(0)}$.
- The gauge trick mentioned above can be adapted to construct 2-metrics on more general Lie groupoids. In particular, every proper Lie groupoid admits 2-metrics.
- We can use the exponential maps determined by groupoid Riemannian 2-metrics to prove that proper Lie groupoids are linearizable. Namely, let us further assume that $G\rightrightarrows M$ is proper and let S denote a saturated submanifold in M. Then there are open neighborhoods $S\subset U\subset M$ and $S\subset V\subset\nu(S)$ such that

  $$\exp=(\exp^{\eta^{(1)}},\exp^{\eta^{(0)}}):(\nu(S)_V\rightrightarrows V)\to(G_U\rightrightarrows U),$$

  is a Lie groupoid isomorphism. Of course, $\exp$ restricts to the identity on $G_S\rightrightarrows S$. Additionally, if $\eta^{(0)}$ is complete then the obtained linearization is **invariant**, that is, U and V can be chosen to be saturated, visit [39].

Remark 1.3.3. The notion of n**-metric** on Lie groupoids for $n\geq 3$ was also introduced in [40]. This is just a Riemannian metric on the set of n-composable arrows $G^{(n)}$ that is invariant by the canonical S_{n+1}-action on $G^{(n)}$ and transverse to one (hence to all) face map $G^{(n)}\to G^{(n-1)}$. We can push this n-metric forward with the different face maps

$G^{(n)} \to G^{(n-1)}$ to define an $(n-1)$-metric on $G^{(n-1)}$ in such a way these face maps become Riemannian submersions. One can use this process to obtain r-metrics $\eta^{(r)}$ on $G^{(r)}$ for all $0 \leq r \leq n-1$ so that we get Riemannian submersions $(G^{(r)}, \eta^{(r)}) \to (G^{(r-1)}, \eta^{(r-1)})$. As expected, proper Lie groupoids can always be equipped with Riemannian n-metrics. To work with Riemannian groupoid metrics in this generality will be necessary in Subsections 2.3.1 and 2.4.2 where the whole nerve of our Lie groupoid is considered.

Riemannian stacks

We finish this subsection by commenting that the notion of Riemannian groupoid metric we have been speaking about is Morita invariant. There is a notion of equivalence of 2-metrics on a given Lie groupoid and Morita equivalence yields a one-to-one correspondence between equivalence classes of 2-metrics [41]. This allows to induce a notion of metric on differentiable stacks which generalizes the usual concepts of Riemannian metric on manifolds and orbifolds.

Let $G \rightrightarrows M$ be a Lie groupoid and $[M/G]$ denote its associated differentiable stack.

Definition 1.3.9. Two Riemannian 2-metrics η_1 and η_2 on $G \rightrightarrows M$ are said to be **equivalent** if they induce the same inner products on the normal vector spaces over the groupoid orbits of G.

More generally, we can define a **Riemannian Morita map** (resp. **fibration**) $\phi : (H \rightrightarrows N) \to (G \rightrightarrows M)$ as a Morita map between Riemannian groupoids that induces isometries on the normal vector spaces to the groupoid orbits $\nu_z(\mathcal{O}^H) \to \nu_{\phi(z)}(\mathcal{O}^G)$ (resp. Riemannian submersion at the level of objects). By using this terminology we have that η_1 and η_2 are equivalent if and only if the identity $\mathrm{id} : (G \rightrightarrows M, \eta_1) \to (G \rightrightarrows M, \eta_2)$ is a Riemannian Morita map.

Let us consider a Morita fraction $G \underset{\sim}{\overset{\phi}{\leftarrow}} H \underset{\sim}{\overset{\psi}{\rightarrow}} G'$ where both ϕ and ψ are assumed to be Morita fibrations. Suppose that G can be endowed with a 2-metric η^G. As proven in [41], there exists a 2-metric η^H on H that makes the Morita fibration $\phi : H \to G$ Riemannian. We can slightly modify η^H by a cotangent averaging procedure so that we get another 2-metric $\tilde{\eta}^H$ on H which descends to G' defining a 2-metric $\eta^{G'}$ making of the Morita fibration $\psi : H \to G'$ Riemannian. It turns out that these pullback and pushforward constructions are well-defined and mutually inverse modulo equivalence of 2-metrics. This is because η^H and $\tilde{\eta}^H$ turn out to be equivalent. In this case, we refer to (G, η^G) and $(G', \eta^{G'})$ as being **Morita equivalent Riemannian groupoids**. As mentioned above, it suggests a definition for Riemannian metrics over differentiable stacks. Namely:

Definition 1.3.10. A **stacky metric** on the orbit stack $[M/G]$ is defined to be an equivalence class $[\eta]$ of a 2-metric η on $G \rightrightarrows M$.

Let us look at some basic examples.

Example 1.3.24. Let $\pi : M \to N$ be a surjective submersion and $M \times_N M \rightrightarrows M$ be its corresponding submersion groupoid. Recall that a 2-metric η on $M \times_N M$ induces Riemannian metrics η^M on M and η^N on N so that π becomes a Riemannian submersion. It is simple to check that two Riemannian 2-metrics η and η' are equivalent if and only if $\eta^N = \eta'^N$. Therefore, stacky metrics on $[M/M \times_N M]$ are just standard Riemannian metrics on N.

Example 1.3.25. Suppose that $G \rightrightarrows M$ is an étale Lie groupoid. In this case, the normal spaces are the same as the tangent spaces, so that two 2-metrics are equivalent if and only if they are equal. If G is additionally proper then the notion of stacky metric on the orbifold $[M/G]$ recovers the classical notion of orbifold metric, as studied for instance in [56].

Example 1.3.26. Any two 2-metrics on a transitive Lie groupoid are equivalent. Therefore, stacky metrics on Lie groups are trivial, since such a notion does not detect any relevant information on the isotropies and only sees the transverse directions.

The notion of stacky metric introduced in Definition 1.3.10 opens up the possibility to extend to singular spaces modeled by stacks classical concepts in Riemannian geometry. For instance, a theory of stacky geodesics on Riemannian stacks was developed in [38] allowing to establish a stacky version of the Hopf–Rinow Theorem. We will use stacky metrics to study some Morse theoretical features over differentiable stacks in Subsection 2.5, to study the problem regarding the existence of closed stacky geodesics on separated Riemannian stacks in Chapter 3, and to define a notion of geometric Killing vector field on Riemannian stacks in Section 4.2.

1.3.2 Lie 2-groups

In this subsection we introduce the notion of (strict) Lie 2-group as well as its infinitesimal counterpart which is known as (strict) Lie 2-algebra. These geometric/algebraic objects will play an important role when describing the corresponding 2-equivariant analog of several of the results this thesis is mainly concerned.

As it was explicitly mentioned by Baez and Lauda in [10], the notion of Lie 2-group goes back to Brown and Spencer in [28] where it became clear that classical group theory is just the beginning of a larger subject that sometimes is called higher-dimensional group theory. In many contexts where we are tempted in using groups to tackle certain symmetries-involved problems, it turns out actually to be more natural to use a richer kind of structure where, in addition to group elements describing symmetries, we also have isomorphisms between these, thus describing symmetries between symmetries. In this spirit, we may concisely define:

Definition 1.3.11. A **Lie 2-group** is a group internal to the category of Lie groupoids.

In other words, a Lie 2-group is a Lie groupoid $K^{(1)} \rightrightarrows K^{(0)}$ where both $K^{(1)}$ and $K^{(0)}$ are Lie groups and the structural maps of $K^{(1)} \rightrightarrows K^{(0)}$ are Lie group homomorphisms [10, 28]. We will denote by $*$ the composition of arrows in $K^{(1)} \rightrightarrows K^{(0)}$ and by $\cdot$ the product of arrows in $K^{(1)}$. For instance, the fact that the multiplication is a morphism of groups amounts to the exchange law:

$$(k_1 * k_2) \cdot (k_1' * k_2') = (k_1 \cdot k_1') * (k_2 \cdot k_2'), \quad \text{for all } (k_1, k_2), (k_1', k_2') \in K^{(2)}, \tag{1.10}$$

where $K^{(2)}$ is the space of pairs of composable arrows of $K^{(1)} \rightrightarrows K^{(0)}$. As expected, **morphisms** between Lie 2-groups are by definition Lie groupoid morphisms that preserve the Lie group structures involved.

There are several alternative ways to think of Lie 2-groups and one of them is described in terms of crossed modules.

Definition 1.3.12. A **crossed module of Lie groups** is a quadruple (K, H, ρ, α) consisting of a Lie group homomorphism $\rho : H \to K$ together with an action α of K on H, $(k, h) \mapsto$

kh, by Lie group automorphisms such that:

$$\rho(kh) = k\rho(h)k^{-1} \quad \text{and} \quad \rho(h)h' = hh'h^{-1},$$

for all $k \in K$ and $h, h' \in H$.

On the one hand, given a Lie 2-group $K^{(1)} \rightrightarrows K^{(0)}$ one has an associated crossed module of Lie groups (K, H, ρ, α), where $K = K^{(0)}$, $H = \ker(s_K)$, $\rho := t_K|_H : H \to K$ and K acts on H by conjugation via the identity bisection $u_K : K^{(0)} \to K^{(1)}$, that is, $kh := 1_k \cdot h \cdot 1_{k^{-1}}$. On the other hand, if (K, H, ρ, α) is a crossed module of Lie groups then one can define a Lie 2-group by setting the objects as the Lie group $K^{(0)} := K$, the arrows as the semi-direct product $K^{(1)} := H \rtimes K$ with respect to the induced representation $\alpha : K \to \mathrm{Aut}(H)$ so that:

$$(h_1, k_1) \cdot (h_2, k_2) := (h_1(k_1h_2), k_1k_2),$$

and the groupoid structure as the one obtained by the action groupoid induced by the action of H on K through ρ. That is, the structural maps are

$$s_K(h, k) = k, \quad t_K(h, k) = \rho(h)k, \quad (h_1, \rho(h_2)k_2) * (h_2, k_2) = (h_1h_2, k_2).$$

The correspondence above defines an equivalence of categories between the category of Lie 2-groups and the category of crossed modules of Lie groups [10].

Some elementary examples of Lie 2-groups can be provided by the following crossed modules of Lie groups.

Example 1.3.27. Let $\alpha : K \to \mathrm{GL}(V)$ be a linear representation of a Lie group K on a vector space V. If V is viewed as an abelian Lie group with the sum of vectors and $\rho : V \to K$ is the trivial Lie group homomorphism then (K, V, ρ, α) determines a crossed module of Lie groups. The Lie 2-group $V \ltimes K \rightrightarrows K$ is isomorphic to the Lie group bundle induced by the trivial vector bundle $\pi : V \times K \to K$. That is, both source and target maps agree with π and the composition is determined by the addition in the fiber V. In particular, the tangent bundle and cotangent bundles of K can be seen as the Lie 2-groups $TK \rightrightarrows K$ and $T^*K \rightrightarrows K$ respectively induced by the adjoint and coadjoint representations of K, as they are trivializable.

Example 1.3.28. Let us consider a split central extension of Lie groups $1 \to \tilde{K} \xrightarrow{\iota} H \xrightarrow{\rho} K \to 1$. That is, there is a Lie group homomorphism $\sigma : K \to H$ such that $\rho \circ \sigma = \mathrm{id}_K$. Then, the representation $\alpha : K \to \mathrm{Aut}(H)$ given by $\alpha_k(h) = \sigma(k)h\sigma(k)^{-1}$ is such that (K, H, ρ, α) defines a crossed module of Lie groups. This follows by noting that $\rho : H \to K$ is a principal $\tilde{K}$-bundle, so that for each $h \in H$ there exists a unique $\tilde{k} \in \tilde{K}$ such that $\sigma(\rho(h)) = h\iota(\tilde{k})$.

Example 1.3.29. Suppose that H is a normal Lie subgroup of a Lie group K. In this case, the inclusion $\iota : H \hookrightarrow K$ and the representation by conjugation $c : K \to \mathrm{Aut}(H)$ define a crossed module of Lie groups (K, H, ι, c). On the one side, if H is the trivial subgroup then we get the unit groupoid $K \rightrightarrows K$ which is a Lie 2-group in a canonical way. On the other side, if H is the whole K then we obtain the action groupoid $K \ltimes K \rightrightarrows K$ defined through the action of K on itself by left translations. This becomes a Lie 2-group after considering the Lie group structure on $K \times K$ induced by conjugations.

Remark 1.3.4. It is worth mentioning that if (K, H, ρ, α) is a crossed module of Lie groups then $\ker(\rho)$ is a K-invariant central Lie subgroup of H and $\mathrm{im}(\rho)$ is a normal Lie subgroup of K.

Let us now introduce the infinitesimal counterpart of a Lie 2-group which is known as Lie 2-algebra.

Definition 1.3.13. A **Lie 2-algebra** is a Lie algebra internal to the category of Lie groupoids.

More concretely, a Lie 2-algebra is a Lie groupoid $\mathfrak{k}^{(1)} \rightrightarrows \mathfrak{k}^{(0)}$ where both $\mathfrak{k}^{(1)}$ and $\mathfrak{k}^{(0)}$ are Lie algebras and the structural maps of $\mathfrak{k}^{(1)} \rightrightarrows \mathfrak{k}^{(0)}$ are Lie algebra homomorphisms. **Morphisms** between Lie 2-algebras are defined to be Lie groupoid morphisms that preserve the Lie algebra structures involved. As expected, the Lie functor yields a correspondence between Lie 2-groups and Lie 2-algebras.

Crossed module of Lie groups have associated infinitesimal objects as well.

Definition 1.3.14. A **crossed module of Lie algebras** is a quadruple $(\mathfrak{k}, \mathfrak{h}, \partial, \mathcal{L})$ where $\mathfrak{k}$ and $\mathfrak{h}$ are Lie algebras and $\partial : \mathfrak{h} \to \mathfrak{k}$ and $\mathcal{L} : \mathfrak{k} \to \mathrm{Der}(\mathfrak{h})$ are Lie algebra homomorphisms verifying

$$\partial(\mathcal{L}_x y) = [x, \partial(y)]_{\mathfrak{k}} \quad \text{and} \quad \mathcal{L}_{\partial(x)} y = [x, y]_{\mathfrak{h}}. \tag{1.11}$$

A Lie 2-algebra $\mathfrak{k}^{(1)} \rightrightarrows \mathfrak{k}^{(0)}$ has an associated crossed module given by the data $\mathfrak{h} = \ker(s)$, $\mathfrak{k} = \mathfrak{k}^{(0)}$, $\partial = t|_{\mathfrak{h}}$ and $\mathcal{L}_x = \mathrm{ad}^1_{u(x)}$ for all $x \in \mathfrak{k}$. Conversely, a crossed module of Lie algebras $(\mathfrak{k}, \mathfrak{h}, \partial, \mathcal{L})$ has an associated Lie 2-algebra which in turn is given by the data $\mathfrak{k}^{(1)} = \mathfrak{h} \rtimes \mathfrak{k}$ with Lie algebra structure provided by the semi-direct product with respect to $\mathcal{L}$, $\mathfrak{k}^{(0)} = \mathfrak{k}$, and structural maps

$$s(x, y) = y, \quad t(x, y) = \partial(x) + y, \quad u(y) = (0, y), \quad i(x, y) = (-x, y + \partial(x)),$$

$$m((x', y + \partial(x)), (x, y)) = (x + x', y).$$

The infinitesimal counterparts of the examples of crossed modules of Lie groups presented above serve to give examples of crossed modules of Lie algebras and, in turn, examples of Lie 2-algebras. One of the most important example of Lie 2-algebra this thesis will be concerned about is the so-called Lie 2-algebra of multiplicative vector fields [100]. A **multiplicative vector field** on a Lie group $G \rightrightarrows M$ is defined to be a pair of vector fields $(\xi, v) \in \mathfrak{X}(G) \times \mathfrak{X}(M)$ such that $\xi : G \to TG$ determines a Lie groupoid morphism covering $v : M \to TM$, see [81]. The set of multiplicative vector field on G is denoted by $\mathfrak{X}_m(G)$. This can be endowed with a Lie algebra structure in a natural way.

Example 1.3.30. Let $A \to M$ be the Lie algebroid of $G \rightrightarrows M$. For each section $a \in \Gamma(A)$ we may define **right invariant** a^r and **left invariant** a^l vector fields on G by using the composition map of the tangent groupoid $TG \rightrightarrows TM$. Indeed,

$$a^r(g) = a(t(g)) * 0_g \quad \text{and} \quad a^l(g) = -0_g * i(a(s(g))), \qquad g \in G.$$

There exists a crossed module of Lie algebras $(\mathfrak{X}_m(G), \Gamma(A), \delta, D)$ where $\delta : \Gamma(A) \to \mathfrak{X}_m(G)$ and $D : \mathfrak{X}_m(G) \to \mathrm{Der}(\Gamma(A))$ are respectively defined by the expressions

$$\delta(a) = (a^r - a^l, \rho(a)) \quad \text{and} \quad D_{(\xi,v)}(a) = [\xi, a^r]|_M,$$

for all $a \in \Gamma(A)$ and $(\xi, v) \in \mathfrak{X}_m(G)$. Here $\rho : A \to TM$ stands for the anchor map of A.

The notion of morphism between crossed modules is as follows.

Definition 1.3.15. A **morphism** of:

i. crossed module of Lie groups $F : (K, H, \rho, \alpha) \to (K', H', \rho', \alpha')$ consists of two Lie group homomorphisms $F_0 : K \to K'$ and $F_1 : H \to H'$ such that $F_0 \circ \rho = \rho' \circ F_1$ and $F_1(\alpha_k(h)) = \alpha'_{F_0(k)}(F_1(h))$; and

ii. crossed module of Lie algebras $f : (\mathfrak{g}, \mathfrak{h}, \partial, \mathcal{L}) \to (\mathfrak{g}', \mathfrak{h}', \partial', \mathcal{L}')$ consists of two Lie algebra homomorphisms $f_0 : \mathfrak{g} \to \mathfrak{g}'$ and $f_1 : \mathfrak{h} \to \mathfrak{h}'$ such that $f_0 \circ \partial = \partial' \circ f_1$ and $f_1(\mathcal{L}_x y) = \mathcal{L}'_{f_0(x)}(f_1(y))$.

Of course, there is a one to one correspondence between morphisms of Lie 2-groups (resp. 2-algebras) and morphisms of crossed modules of Lie groups (resp. algebras). Note that a morphism of crossed modules of Lie algebras induces a pair of Lie algebra homomorphisms ker$(\partial) \to$ ker(∂') and coker$(\partial) \to$ coker(∂'). We say that a morphism of crossed modules is a **quasi-isomorphism** if both of these Lie algebra morphisms are isomorphisms. Accordingly, the **derived category** of crossed modules of Lie algebras is defined to be the localization of the category of crossed modules of Lie algebras obtained by inverting all quasi-isomorphisms. In these terms, we can state the following nice result proved in [100].

Theorem 1.3.2. *If $G \rightrightarrows M$ and $G' \rightrightarrows M'$ are Morita equivalent Lie groupoids then the crossed modules of Lie algebras $(\mathfrak{X}_m(G), \Gamma(A), \delta, D)$ and $(\mathfrak{X}_m(G'), \Gamma(A'), \delta', D')$ are isomorphic in the derived category of crossed modules. In consequence, the following quotient spaces are isomorphic as Lie algebras:*

$$\mathfrak{X}_m(G)/\mathrm{im}(\delta) \cong \mathfrak{X}_m(G')/\mathrm{im}(\delta').$$

This motivates the following definition.

Definition 1.3.16. A **geometric vector field** is an element of the Lie algebra $\mathfrak{X}_m(G)/\mathrm{im}(\delta)$.

It is important to mention that such a notion of vector field recovers the classical notions of vector field on both manifolds and orbifolds as well as the notion of transverse vector field for regular foliations [100]. Besides, due to Theorem 1.3.2 it makes sense to define the Lie algebra of vector fields on a differentiable stack $[M/G]$ presented by $G \rightrightarrows M$ as $\mathfrak{X}([M/G]) := \mathfrak{X}_m(G)/\mathrm{im}(\delta)$.

Lie 2-group actions

After having defined Lie 2-groups we are lead now to briefly introduce strict 2-actions over Lie groupoids. Such a notion shall be crucial at several stages of this thesis. Let $K^{(1)} \rightrightarrows K^{(0)}$ be a Lie 2-group and $G \rightrightarrows M$ be a Lie groupoid.

Definition 1.3.17. A **left Lie 2-group action** of $K^{(1)} \rightrightarrows K^{(0)}$ on $G \rightrightarrows M$ is defined to be a Lie groupoid morphism $\theta = (\theta^1, \theta^0) : (K^{(1)} \times G \rightrightarrows K^{(0)} \times M) \to (G \rightrightarrows M)$ such that both maps θ^1 and θ^0 are usual left Lie group actions. Right Lie 2-group actions can be similarly defined.

In the previous definition $K^{(1)} \times G \rightrightarrows K^{(0)} \times M$ denotes the product Lie groupoid. We shall say that θ is a **free** (resp. **proper**) Lie 2-group action if both θ^1 and θ^0 are free (resp. proper) actions. Note that given a left Lie 2-group action of $K^{(1)} \rightrightarrows K^{(0)}$ on $G \rightrightarrows M$ we immediately get that the structural maps of $G \rightrightarrows M$ are equivariant with respect the structural maps of $K^{(1)} \rightrightarrows K^{(0)}$. More precisely, if $g \in G, x \in M$ and $k \in K^{(1)}, k_0 \in K^{(0)}$ then we obtain for instance that

$$s_G(kg) = s_K(k)s_G(g), \quad t_G(kg) = t_K(k)t_G(g), \quad \text{and} \quad u_G(k_0 x) = u_K(k_0)u_G(x). \tag{1.12}$$

Moreover, we have that the action on arrows is **multiplicative**, meaning that for pairs of composable arrows $(g, \tilde{g}) \in G^{(2)}$ and $(k, \tilde{k}) \in K^{(2)}$ the following formula holds true

$$(k * \tilde{k})(g * \tilde{g}) = (kg) * (\tilde{k}\tilde{g}). \tag{1.13}$$

Here we are denoting by $m_G(g, \tilde{g}) = g * \tilde{g}$ and $m_K(k, \tilde{k}) = k * \tilde{k}$.

Remark 1.3.5. For the sake of completeness we only mention that there is a reformulation of the notion of Lie 2-group actions on Lie groupoids in terms of actions of crossed modules of Lie groups on Lie groupoids as introduced in [62]. However, the viewpoint used in Definition 1.3.17 is more suitable for our purposes in the forthcoming chapters.

Chapter 2

Morse theory on Lie groupoids

The aim of this chapter is to develop a Morse theory in the realm of Lie groupoids and their differentiable stacks. Roughly speaking, we are mainly interested in studying Morse theory for stacky maps $[M/G] \to \mathbb{R}$ where we think of $\mathbb{R}$ as a differentiable stack presented by the unit groupoid $\mathbb{R} \rightrightarrows \mathbb{R}$. It turns out that each of these stacky functions is completely determined by a Lie groupoid morphism $G \to \mathbb{R}$ and, in turn, by a basic function $M \to \mathbb{R}$. The latter fact yields a simple way to establish a notion of Morse stacky function over $[M/G]$ by imposing that the critical orbits of the corresponding basic function are nondegenerate in the sense of Morse–Bott theory.

We start by introducing Morse Lie groupoid morphisms, thus studying their main properties. As an important feature we show that this notion is Morita invariant, so that it gives rise to a well defined notion of Morse function over differentiable stacks. Among the main results that we shall obtain in this chapter we have a groupoid version of the Morse lemma which is used to describe the topological behavior of the critical subgroupoid levels of a Morse Lie groupoid morphism around its nondegenerate critical orbits, a groupoid version of the so-called stable/unstable manifold theorem which suggests that the moduli space of gradient flow lines has a natural structure of Lie groupoid, Morse type inequalities for certain separated differentiable stacks, and the construction of a Morse double complex whose total cohomology is isomorphic to the Bott–Shulman–Stasheff cohomology of the underlying Lie groupoid. We also establish a 2-equivariant Morse theory over Lie groupoids and compare our approach to study Morse theory over the orbit space of a proper Lie groupoid with the approach provided by stratified Morse theory, thus exposing why our focus becomes more suitable, natural and cleaner for our purposes. Throughout the sections we provide several examples and applications, focusing our attention on some aspects coming from stacky symplectic geometry.

It is worth mentioning that most of the results mentioned in this chapter have already been published in [101], so that they are reproduced with permission from Springer Nature.

2.1 Morse Lie groupoid morphisms

In this section we will focus on introducing and studying the notion of Morse type morphism in the Lie groupoid setting which we are interested to work with throughout this thesis.

Let $G \rightrightarrows M$ be a Lie groupoid. We start by paying attention to Lie groupoid morphisms of the form $F : (G \rightrightarrows M) \to (\mathbb{R} \rightrightarrows \mathbb{R})$. It is simple to check that any F is completely determined by a smooth function $f : M \to \mathbb{R}$ such that F is given either by $F = s^*f$

or $F = t^*f$. That is, real valued Lie groupoids morphisms are completely determined by basic functions. The space of **basic functions** on M is defined to be

$$C^\infty(M)^G := \{f \in C^\infty(M) : s^*f = t^*f\}.$$

In other words, a basic function $f : M \to \mathbb{R}$ is just a function which is constant along the groupoid orbits of G, so that it induces a well defined continuous function on the orbit space $\overline{f} : M/G \to \mathbb{R}$.

From now on we identify the space of real valued Lie groupoids morphisms with that of basic functions. It follows that the space of basic functions is Morita invariant since it satisfies that $H^0_{\text{diff}}(G) = C^\infty(M)^G$, where $H^0_{\text{diff}}(G)$ stands for the 0^{th}-degree differentiable groupoid cohomology with coefficients in the trivial bundle [31]. Hence, we may think of $H^0_{\text{diff}}(G) = C^\infty(M)^G$ as the space of smooth functions on the quotient stack $[M/G]$.

Let us look at some elementary examples of real valued Lie groupoid morphisms.

Example 2.1.1. If M is a smooth manifold and $M \rightrightarrows M$ is its underlying unit Lie groupoid then $C^\infty(M)^M = C^\infty(M)$. That is, we recover usual smooth functions on M.

Example 2.1.2. If K is a Lie group acting on a smooth manifold M and $K \ltimes M \rightrightarrows M$ is the corresponding action groupoid then $C^\infty(M)^{K\times M}$ is given by the set of K-invariant smooth functions on M. That is, smooth functions $f : M \to \mathbb{R}$ verifying $f(k \cdot x) = f(x)$ for all $k \in K$ and $x \in M$.

Example 2.1.3. Suppose that $\pi : M \to N$ is a surjective submersion with corresponding submersion groupoid $M \times_N M \rightrightarrows M$. In this case, $C^\infty(M)^{M\times_N M}$ equals the set of smooth functions on M that are constant on the fibers of π, meaning that $f(x) = f(y)$ for all $x, y \in \pi^{-1}(b)$ and $b \in N$.

Example 2.1.4. Let $G \rightrightarrows M$ be a proper Lie groupoid with proper Haar measure system $\{\mu^x\}_{x\in M}$. For any smooth function $f : M \to \mathbb{R}$ it follows that the averaging

$$f^\mu(x) := \int_{g\in s^{-1}(x)} (f \circ t)(g)\mu^x(g), \qquad x \in M,$$

defines a basic function on M. Indeed, the properness of the Haar system $\{\mu^x\}_{x\in M}$ ensures that the integral defining f^μ is finite, the smoothness tells us that f^μ is also smooth and, moreover, the right-invariance and the identity $t \circ m = t \circ \pi_1$ imply that for all $h \in G$

$$\begin{aligned} f^\mu(t(h)) &= \int_{g\in s^{-1}(t(h))} (f \circ t)(g)\mu^{t(h)}(g) = \int_{g\in s^{-1}(s(h))} (f \circ t)(gk)\mu^{s(h)}(g) \\ &= \int_{g\in s^{-1}(s(h))} (f \circ t)(g)\mu^{s(h)}(g) = f^\mu(s(h)). \end{aligned}$$

The reader should be wondering why not to consider multiplicative functions on G instead of basic functions on M. A quick answer for this is as follows.

Remark 2.1.1. A smooth function $F : G \to \mathbb{R}$ is said to be **multiplicative** if and only if it satisfies $F(gh) = F(g) + F(h)$ for all $(g, h) \in G^{(2)}$. These kinds of functions are in one-to-one correspondence with Lie groupoid morphisms $F : (G \rightrightarrows M) \to (\mathbb{R} \rightrightarrows *)$. Observe that multiplicative functions do not allow us to establish a well behaved notion of smooth function on the stacky quotient $[M/G]$ since, for instance, in the most elementary case of the unit groupoid $M \rightrightarrows M$ we get that those functions turn out to be trivial. That is, we can not even recover the usual smooth functions on M in a natural way. Same inconvenient appears when analyzing other basic examples.

Let $F : G \to \mathbb{R}$ be a Lie groupoid morphism covering a basic function $f : M \to \mathbb{R}$. Note that if x is a critical point of f then its orbit $\mathcal{O}_x$ is a critical submanifold of f. Hence the critical point set $\mathrm{Crit}(f) \subset M$ is saturated. More importantly:

Lemma 2.1.1. *There exists a natural topological groupoid structure* $\mathrm{Crit}(F) \rightrightarrows \mathrm{Crit}(f)$.

Proof. Let us suppose that $g \in G$ is a critical arrow of F. It is simple to see that both $s(g)$ and $t(g)$ are critical points of f since both s and t are surjective submersions. This in turn implies that g^{-1} is a critical arrow of F as well. Also, if $x \in \mathrm{Crit}(f)$ one easily sees that $1_x \in \mathrm{Crit}(F)$. Finally, using the identities $s \circ m = s \circ \pi_2$ and $t \circ m = t \circ \pi_1$, we conclude that if $(g, h) \in G^{(2)}$ with either g or h a critical arrow of F then the composition gh is also a critical arrow of F. □

It follows from the previous lemma that $s^{-1}\mathrm{Crit}(f) = t^{-1}\mathrm{Crit}(f) = \mathrm{Crit}(F)$. In particular, if $\mathcal{O} \subseteq M$ is a critical orbit of f then $G_{\mathcal{O}}$ is a critical submanifold for F so that the restricted Lie groupoid $G_{\mathcal{O}} \rightrightarrows \mathcal{O}$ can be thought of as a critical Lie subgroupoid of $G \rightrightarrows M$. Furthermore, we have that $\mathcal{O} \subseteq M$ is a non-degenerate critical orbit for f if and only if $G_{\mathcal{O}} \subseteq G$ is a non-degenerate critical submanifold for F. This follows from the fact that $\overline{ds} : \nu(G_{\mathcal{O}}) \to \nu(\mathcal{O})$ is a fiberwise isomorphism and Formula (1.2) holds true. Motivated by the previous facts we set up the following definition.

Definition 2.1.1. Let $F : G \to \mathbb{R}$ be a Lie groupoid morphism covering a basic function $f : M \to \mathbb{R}$. We say that F is a **Morse Lie groupoid morphism** if every critical orbit $\mathcal{O} \subseteq M$ of f is non-degenerate.

In other words, one can say that $F : G \to \mathbb{R}$ is a Morse Lie groupoid morphism if and only if every critical subgroupoid $G_{\mathcal{O}} \rightrightarrows \mathcal{O}$ is non-degenerate in the sense that both $\mathcal{O} \subset M$ and $G_{\mathcal{O}} \subseteq G$ are nondegenerate critical submanifolds.

Given the simplicity of our definition we can show now that the notion of Morse Lie groupoid morphism is Morita invariant. Recall that a Morita equivalence between $G \rightrightarrows M$ and $G' \rightrightarrows M'$ yields an isomorphism between 0^{th}-degree differentiable groupoid cohomology, that is, an isomorphism between the corresponding spaces of basic functions. More precisely, if $K \rightrightarrows N$ is a Lie groupoid together with Morita fibrations $\phi : K \to G$ and $\psi : K \to G'$, then $\psi^* f' \in C^\infty(N)^K$ for every $f' \in C^\infty(M')^{G'}$. Also, there exists a unique $f \in C^\infty(M)^G$ with $\phi^* f = \psi^* f'$. This clearly defines an isomorphism $C^\infty(M')^{G'} \to C^\infty(M)^G$ by sending $f' \mapsto f$. In particular, the previous assignment naturally yields another isomorphism

$$\mathrm{Hom}_{Gpds}(G', \mathbb{R}) \to \mathrm{Hom}_{Gpds}(G, \mathbb{R}); F' = s^* f' \mapsto F = s^* f. \tag{2.1}$$

Our main goal now is to show that these isomorphisms give rise to an isomorphism between basic Morse–Bott functions, hence between Morse Lie groupoid morphisms.

Proposition 2.1.1. *Let $G \leftarrow K \to G'$ be a Morita equivalence covering surjective submersions at the level of objects. The isomorphism (2.1) preserves Morse Lie groupoid morphisms.*

Proof. Suppose that $f \in C^\infty(M)^G$ allows us to define a Morse Lie groupoid morphism. To prove that the corresponding $f' \in C^\infty(M')^{G'}$ induces another Morse Lie groupoid morphism it suffices to show that $\phi^* F$ is a Morse Lie groupoid morphism since $\phi^* f = \psi^* f'$ and both ϕ^0 and ψ^0 are surjective submersions. Indeed, the Lie groupoid morphism $\phi^* F : (K \rightrightarrows N) \to (\mathbb{R} \rightrightarrows \mathbb{R})$ is given by the pair $(F \circ \phi^1, f \circ \phi^0)$. Note that if $x \in M$ is a critical point of $f \circ \phi^0$, then $\phi^0(x)$ is a critical point of f since ϕ^0 is a surjective submersion. Thus, from Identity (1.2) we get at x that

$$\mathcal{H}_x(f \circ \phi^0) = d\phi^0(x)^T \cdot \mathcal{H}_{\phi^0(x)}(f) \cdot d\phi^0(x).$$

From Theorem 1.3.1 we know that if ϕ is a Morita map then $\overline{d\phi^0} : \nu(\mathcal{O}_x) \to \nu(\mathcal{O}'_{\phi^0(x)})$ is a fiberwise isomorphism. Thus, as $\mathcal{H}_{\phi^0(x)}(f)$ is non-degenerate when restricted to $\nu_{\phi^0(x)}(\mathcal{O}'_{\phi^0(x)})$ we conclude that $\mathcal{H}_x(f \circ \phi^0)$ is non-degenerate when restricted to $\nu_x(\mathcal{O}_x)$, as desired. □

We provide below some elementary and interesting examples of Morse Lie groupoid morphisms, focusing our attention on their existence as well as their occurrence through the notion of moment maps for 0-symplectic groupoids.

Example 2.1.5. If M is a smooth manifold then every Morse function $f : M \to \mathbb{R}$ induces a Morse Lie groupoid morphism on the unit groupoid $M \rightrightarrows M$.

Example 2.1.6. Suppose that $G \rightrightarrows M$ is a Lie group bundle, i.e. $s = t$. Therefore, any Morse function $f : M \to \mathbb{R}$ induces a Morse Lie groupoid morphism on $G \rightrightarrows M$ since its orbits are points. In particular, on a Lie group $K \rightrightarrows *$ every Lie groupoid morphism $K \to \mathbb{R}$ is necessarily constant. Hence, there are no interesting examples of Morse Lie groupoid morphisms on Lie groups. Combining this with Proposition 2.1.1 we conclude the same for any transitive Lie groupoid $G \rightrightarrows M$ since transitive Lie groupoids are always Morita equivalent to Lie groups.

Example 2.1.7. Let K be a compact Lie group acting on a smooth manifold M and consider the action groupoid $K \ltimes M \rightrightarrows M$. Wasserman showed in [116] that the set of K-invariant Morse functions on M is dense in the set of K-invariant functions. Hence, there always exist Morse Lie groupoid morphisms on $K \ltimes M \rightrightarrows M$.

Example 2.1.8. Let $(M, \mathcal{F})$ be a complete transverse parallel foliated connected manifold (see for instance [95, s. 4.5] or [92, s. 4.1.2]). Let $p : \widetilde{M} \to M$ denote the universal covering of M and consider the induced foliation $\widetilde{\mathcal{F}} = p^*\mathcal{F}$ on $\widetilde{M}$. This foliation is simple so that we have that $X = \widetilde{M}/\widetilde{\mathcal{F}}$ is a Hausdorff manifold and the canonical projection $\pi_{\text{bas}} : \widetilde{M} \to X$ is a surjective submersion. It turns out that with this data it is possible to obtain a natural structure of principal groupoid bi-bundle $(\widetilde{M}, p, \pi_{\text{bas}}) : \text{Hol}(M, \mathcal{F}) \dashrightarrow \pi_1(M) \ltimes X$ between the holonomy groupoid $\text{Hol}(M, \mathcal{F}) \rightrightarrows M$ and the action groupoid $\pi_1(M) \ltimes X \rightrightarrows X$, that is, a Morita equivalence. Therefore, if M has finite fundamental group then as a consequence of Wasserman's result and Proposition 2.1.1 we get that there exist Morse Lie groupoid morphisms on $\text{Hol}(M, \mathcal{F}) \rightrightarrows M$.

Next particular case shows that there may be no interesting examples of Morse Lie groupoid morphisms over certain non-proper Lie groupoids.

Example 2.1.9. Consider the action of $\mathbb{Z}$ on S^1 given by $n \cdot e^{i\theta} = e^{i(\theta + 2\pi n\alpha)}$ where $\alpha \in [0, 1]$ is some irrational number. This action is free but not proper. Moreover, every orbit of such an action is dense in S^1. Therefore, if there is a $\mathbb{Z}$-invariant function on S^1 then it must be constant. As a consequence of Morita invariance, the foliation groupoid on the 2-torus $\mathbb{T}^2$ associated to the Kronecker foliation admits no Morse Lie groupoid morphisms different from the constant functions.

2.1.1 The proper groupoid case

The aim of this subsection is to study the existence of Morse Lie groupoid morphisms on proper groupoids from two different perspectives. The first one follows classical ideas corresponding to standard Morse theory as well as its equivariant counterpart, whereas the second one uses a canonical bridge that we may construct between our approach to study Morse theory over the orbit space of a proper Lie groupoid and the approach provided by stratified Morse theory. In next chapters we plan to expose why our focus becomes more suitable, natural and cleaner for our purposes.

The strong topology on basic functions

It is well known that the set of Morse functions over a compact manifold M form a open and dense subset of $C^\infty(M)$ with respect to the strong topology. As we mentioned before, Wasserman showed in [116] a similar result for the set of K-invariant Morse functions placed inside the set of all K-invariant functions when the Lie group K acting on M is compact. For the case of orbifolds, Hepworth introduced in [57] a modified strong topology on the set of functions over an orbifold and proved that the subset of those that are Morse is also open and dense with respect to such a topology [57, s. 6].

When we have proper actions of non-compact Lie groups some pathologies may occur.

Remark 2.1.2. If K is a non-compact Lie group acting properly on a smooth manifold M then the strong topology induced on the set of all K-invariant smooth functions on M becomes discrete, see [64, Prop. 4.7]. As consequence, the set of K-invariant Morse functions on M can not be dense with respect to the strong topology induced on $C^\infty(M)^K$.

A simple example in which the previous phenomena comes about is the following.

Example 2.1.10. Consider the usual free and proper action of $\mathbb{Z}$ on $\mathbb{R}$ given by translations. In this case the quotient map (exponential) $\pi : \mathbb{R} \to S^1$ is clearly not proper. It follows that the set of $\mathbb{Z}$-invariant Morse functions on $\mathbb{R}$ can not be dense in $C^\infty(\mathbb{R})^{\mathbb{Z}}$ with respect to the strong topology.

It is well known that if K is a compact Lie group acting on a smooth manifold M then the canonical orbit projection $M \to M/K$ is a proper map. Let us consider a proper Lie groupoid $G \rightrightarrows M$. Motivated by Remark 2.1.2 and Example 2.1.10 it seems reasonable to further assume that the canonical projection $\pi : M \to M/G$ is a proper map in order to show that Morse Lie groupoid morphisms are dense in $C^\infty(M)^G$ with respect to the induced strong topology. Our aim now is to prove that this is in fact the case. Moreover, if M/G is compact then we show that they actually form an open subset. Such a result clearly recovers both the classical and the equivariant cases and partially recovers the case of orbifolds. This is because the modified strong topology defined by Hepworth only agrees with the classical strong topology on $C^\infty(M)^G$ when $\pi : M \to M/G$ is proper [57, Prop. 6.5]. Note that if G is a proper Lie groupoid over a compact manifold M then the previous requirements are clearly fulfilled. In particular, an interesting example to have in mind is given by the holonomy groupoid $\mathrm{Hol}(M, \mathcal{F}) \rightrightarrows M$ induced by a regular Riemannian foliation $\mathcal{F}$ over a compact manifold M.

Lemma 2.1.2. *The set $C^\infty(M)^G$ is a Baire space in $C^\infty(M)$ with respect to the strong topology.*

Proof. Let us take a proper Haar measure system $\{\mu^x\}_{x\in M}$ for $G \rightrightarrows M$. By averaging with respect to $\{\mu^x\}_{x\in M}$ we can define a surjective linear continuous operator $P^\mu : C^\infty(M) \to$

$C^\infty(M)^G$ by sending f to f^μ as defined in Example 2.1.4. This is consequence of having that $C^\infty(M)$ is a Fréchet space and the average operator P^μ is a projection, i.e. $(P^\mu)^2 = P^\mu$. In particular, by the Banach-Schauder Theorem it follows that P^μ is open. Mather proved in [84, Prop. 3.1] that $C^\infty(M)$ is a Baire space so that the previous facts imply that the space of basic functions $C^\infty(M)^G$ is also a Baire space in $C^\infty(M)$, as claimed. □

Although the proof of next result uses some terminology to be introduced later on we state it here because of our purposes in this subsection.

Lemma 2.1.3. *Let $G_U \rightrightarrows U$ be an open subgroupoid of $G \rightrightarrows M$ and $K \subset U/G_U \subset M/G$ be a compact subset. Then the set of basic functions $f : M \to \mathbb{R}$ such that $f|U$ has no degenerate critical orbits inside the compact $\pi^{-1}(K)$ is an open subset in $C^\infty(M)^G$.*

Proof. The proof of this result is a straightforward adaptation of the proof of [14, Lem. 5.32] by considering instead the coarse differential $dF_{[x]} : T_{[x]}[M/G] \to \mathbb{R}$ and the coarse Hessian $H_{[x]}(F) : T_{[x]}[M/G] \times T_{[x]}[M/G] \to \mathbb{R}$ as defined in Section 2.5. □

Theorem 2.1.1. *Suppose that $G \rightrightarrows M$ is a proper Lie groupoid such that the canonical projection $\pi : M \to M/G$ is a proper map. Then Morse Lie groupoid morphisms on G are dense in the space of all Lie groupoid morphisms $G \to \mathbb{R}$. Moreover, if M/G is compact then they form an open subset.*

Proof. We know that there is an open neighborhood U_α of any x in M which is diffeomorphic to $O \times V_x$ where O is an open ball in the orbit $\mathcal{O}_x$ centered at x and V_x is a G_x-invariant open ball in $\nu_x(\mathcal{O}_x)$ centered at the origin. Under this diffeomorphism $G_{U_\alpha} \rightrightarrows U_\alpha$ is isomorphic to the product of the pair groupoid $O \times O \rightrightarrows O$ and the action groupoid $G_x \ltimes V_x \rightrightarrows V_x$, see [103, Cor. 3.11]. The pair groupoid admits Morse Lie groupoid morphisms since this is Morita equivalent to a manifold and the action groupoid admits Morse Lie groupoid morphisms as consequence of Wasserman's density result. Therefore, by Proposition 2.1.1 we have that there exist Morse Lie groupoid morphisms on $G_{U_\alpha} \rightrightarrows U_\alpha$ and they are actually dense. Recall that the canonical projection $\pi : M \to M/G$ is an open map. Thus, we may assume that the open cover $\{U_\alpha/G_\alpha\}$ of M/G we obtain from above is countable and it satisfies that for every α there exists a compact subset $K_\alpha \subset U_\alpha/G_\alpha$ such that $\{K_\alpha\}$ also covers M/G since this is Hausdorff and paracompact.

Let $\mathcal{M}_\alpha(G)$ denote the subset formed by basic functions $f \in C^\infty(M)^G$ for which $f|_{U_\alpha}$ has no degenerate critical orbits inside the compact set $\pi^{-1}(K_\alpha)$. Note that the set $\mathcal{M}(G)$ of all basic functions defining Morse Lie groupoid morphisms on $G \rightrightarrows M$ agrees with $\bigcap_\alpha \mathcal{M}_\alpha(G)$ since $\{\pi^{-1}(K_\alpha)\}$ covers M. If we show that $\mathcal{M}_\alpha(G)$ is open and dense for all α then $\mathcal{M}(G)$ will be dense since $C^\infty(M)^G$ is a Baire space. On the one hand, the fact that $\mathcal{M}_\alpha(G)$ is open follows from Lemma 2.1.3. On the other hand, pick $f \in C^\infty(M)^G$ and let $\mathcal{N}$ be an open neighborhood of f. If we prove that $\mathcal{N} \cap \mathcal{M}_\alpha(G)$ is nonempty then we will have that $\mathcal{M}_\alpha(G)$ is dense. Consider the restriction $f|_{U_\alpha}$ of f on U_α. By taking average with respect to $\{\mu^x\}_{x\in M}$ it follows that given the closed subset $\pi^{-1}(K_\alpha)$ and some open neighborhood $W_\alpha \subseteq U_\alpha$ of $\pi^{-1}(K_\alpha)$ there exists a basic function $\phi_\alpha : U_\alpha \to \mathbb{R}$ such that $\phi_\alpha \equiv 1$ in a small neighborhood of $\pi^{-1}(K_\alpha)$ and such that its support is compact and contained in W_α, use [34, Prop. 9] with [57, Lem. 3.12]. Also, since ϕ_α has compact support it follows that by using again an average process we can consider the map $C^\infty(U_\alpha)^{G_{U_\alpha}} \to C^\infty(M)^G$ given by $g \mapsto \widetilde{\phi_\alpha g}$, where the symbol $\widetilde{\ }$ denotes smooth G-invariant extension by zero, compare with [57, Lem 3.11 and Lem. 6.12]. This map is continuous so that there exists a small enough open neighborhood $\mathcal{N}'$ in $C^\infty(U_\alpha)^{G_{U_\alpha}}$ such that $f(1 - \widetilde{\phi_\alpha}) + \widetilde{\phi_\alpha g} \in \mathcal{N}$ for all $g \in \mathcal{N}'$. We already know that Morse Lie groupoid morphisms on $G_{U_\alpha} \rightrightarrows U_\alpha$ are dense so that we may assume that there exists $g \in \mathcal{N}'$ defining a Morse Lie groupoid morphism such that $f(1-\widetilde{\phi_\alpha}) + \widetilde{\phi_\alpha g} \in \mathcal{N}$. Recall that by construction $\phi_\alpha \equiv 1$ in a neighborhood of $\pi^{-1}(K_\alpha) \subset U_\alpha$ for which $f(1-\widetilde{\phi_\alpha}) + \widetilde{\phi_\alpha g}$ restricts to g over such a neighborhood, meaning that $f(1-\widetilde{\phi_\alpha}) + \widetilde{\phi_\alpha g} \in \mathcal{N} \cap \mathcal{M}_\alpha(G)$. That is, $\mathcal{N} \cap \mathcal{M}_\alpha(G)$ is nonempty as claimed.

Finally, if M/G is compact then the intersection $\mathcal{M}(G) = \bigcap_\alpha \mathcal{M}_\alpha(G)$ becomes finite. This completes the proof. □

Stratified Morse theory

Let us now establish a bridge between the approach we propose to study Morse theory over the orbit space of a proper Lie groupoid and the approach provided by the well known stratified Morse theory in the sense of Goresky and MacPherson in [53]. Let N be a smooth manifold and let $X \subseteq N$ be a subset with a Whitney stratification $\mathcal{S}$, visit [34, s. 4.1]. For $\mathcal{S}_j, \mathcal{S}_k \in \mathcal{S}$ we write $\mathcal{S}_j \leq \mathcal{S}_k$ to denote the usual ordering given by $\mathcal{S}_j \subseteq \overline{\mathcal{S}_k}$. For each $x \in X$ we denote by $\mathcal{S}_x$ the stratum of $\mathcal{S}$ containing x. Fix a smooth function $\tilde{f} : N \to \mathbb{R}$ and set $f := \tilde{f}|_X$. Such an f is simply referred to as a smooth function on X. The **stratified critical point set** of f is by definition $\mathrm{Crit}_{\mathcal{S}}(f) := \bigcup_j \mathrm{Crit}_{\mathcal{S}_j}(f)$ where $\mathrm{Crit}_{\mathcal{S}_j}(f)$ stands for the usual critical point set of f on $\mathcal{S}_j$. For $\mathcal{S}_j$ we define the **conormal space to $\mathcal{S}_j$ in N** as the vector bundle $T^*_{\mathcal{S}_j}N$ over $\mathcal{S}_j$ whose fiber at $x \in \mathcal{S}_j$ consist of the covector in T^*_xN which vanish on the tangent space to $\mathcal{S}_j$ at x. Note that $x \in \mathrm{Crit}_{\mathcal{S}}(f)$ if and only if $d\tilde{f}(x) \in T^*_{\mathcal{S}_x}N$. The set of **degenerate conormal covectors** to a stratum $\mathcal{S}_j$ is defined to be

$$D^*_{\mathcal{S}_j}N := T^*_{\mathcal{S}_j}N \cap \bigcup_{\mathcal{S}_j < \mathcal{S}_k} \overline{T^*_{\mathcal{S}_k}N} = \left(\bigcup_{\mathcal{S}_j < \mathcal{S}_k} \overline{T^*_{\mathcal{S}_k}N} \right) |_{\mathcal{S}_j}.$$

That is, the fiber $(D^*_{\mathcal{S}_x}N)_x$ consists of limits at x of conormal covectors to larger strata or, equivalently, conormal covectors to $\mathcal{S}_x$ at x which vanish on limiting tangent spaces from larger strata. In these terms we define:

Definition 2.1.2. A point $x \in X$ is a said to be a **nondegenerate critical point** of f if and only if x is a nondegenerate critical point of $f|_{\mathcal{S}_x}$ and $d\tilde{f}(x) \notin D^*_{\mathcal{S}_x}N$. Accordingly, the function f is called a **stratified Morse function** if and only if all of its critical points are nondegenerate.

It is worth mentioning the following subtle fact which turns out to be one of the differences derived from our approach.

Remark 2.1.3. The classical definition of stratified Morse function also assumes f to be proper and to have distinct critical values, see [53]. However, because of our purposes in this subsection such additional conditions will not be required.

Let $G \rightrightarrows M$ be a proper Lie groupoid. It is well known that the orbit space M/G of G has several natural Whitney stratifications, compare [34, 105]. In particular, canonical stratifications for both M and M/G by Morita types, which generalize the canonical stratifications induced by a proper Lie group action, were given in [34]. Namely, the **Morita type equivalence** is the equivalence relation on M given by $x \sim_{\mathcal{M}} y$ if and only if the normal representations $G_x \curvearrowright \nu_x(\mathcal{O}_x)$ and $G_y \curvearrowright \nu_y(\mathcal{O}_y)$ are isomorphic. The **partition by Morita types**, denoted by $\mathcal{P}_{\mathcal{M}}(M)$, is defined to be the resulting partition. Each member of $\mathcal{P}_{\mathcal{M}}(M)$ is called a **Morita type**. If $[G_x, \nu_x(\mathcal{O}_x)] = \alpha$ denotes the isomorphism class of the normal representation $G_x \curvearrowright \nu_x(\mathcal{O}_x)$ then the element in $\mathcal{P}_{\mathcal{M}}(M)$ corresponding to

α will be denoted by $M_{(\alpha)} = \{x \in M : [G_x, \nu_x(\mathcal{O}_x)] = \alpha\}$. We also denote by $M_{(x)}$ the Morita type of a point $x \in M$. It is simple to see that points in the same orbit belong to the same Morita type and hence we also obtain a partition by Morita types on the orbit space, $\mathcal{P}_{\mathcal{M}}(M/G)$. The orbit projection map $\pi : M \to M/G$ takes Morita types $M_{(\alpha)}$ in M into Morita types $X_{(\alpha)} = \pi(M_{(\alpha)})$ in M/G. Observe that by its very definition, the Morita type of $\mathcal{O} \in M/G$ only depends on Morita invariant information. The **canonical Whitney stratification** on M, denoted by $\mathcal{S}_G(M)$, is the partition on M obtained by passing to connected components of $\mathcal{P}_{\mathcal{M}}(M)$. The canonical Whitney stratification on the orbit space M/G, denoted by $\mathcal{S}(M/G)$, is the partition on M/G obtained by passing to connected components of $\mathcal{P}_{\mathcal{M}}(M/G)$, see [34, s. 4.6]. As important features of these canonical Whitney stratifications we have that if our Lie groupoid has only one Morita type then the orbit space M/G is a smooth manifold and the canonical projection $\pi : M \to M/G$ becomes a submersion whose fibers are the orbits. More importantly, let $G_{(\alpha)} \rightrightarrows M_{(\alpha)}$ denote the groupoid $G_{(\alpha)} = s^{-1}(M_{(\alpha)})$ over a fixed Morita type $M_{(\alpha)}$. It follows that $G_{(\alpha)} \rightrightarrows M_{(\alpha)}$ is a Lie groupoid, $X_{(\alpha)}$ is a smooth manifold and the canonical projection $\pi : M_{(\alpha)} \to X_{(\alpha)}$ is a submersion. Furthermore, any Morita equivalence between two proper Lie groupoids induces an isomorphism of differentiable stratified spaces between their orbit spaces, visit [34].

Suppose either that the orbit space M/G is compact or has a finite number of Morita types. In these cases it follows that M/G can be embedded into an affine space $\mathbb{R}^d$, compare [48] and [34], respectively. We are now in conditions to establish a simple correspondence between our approach to study Morse theory over the orbit space M/G and the approach provided by stratified Morse theory. Indeed:

Proposition 2.1.2. *Let $G \rightrightarrows M$ be a proper Lie groupoid having either compact orbit space M/G or else a finite number of Morita types. There is a natural one-to-one correspondence between Morse Lie groupoid morphisms on G and stratified Morse functions on M/G as those defined in Definition 2.1.2.*

Proof. Let $F : G \to \mathbb{R}$ be a Morse Lie groupoid morphism induced by a basic function $f : M \to \mathbb{R}$ and let $\overline{f} : M/G \to \mathbb{R}$ denote the corresponding function over the orbit space M/G. By using the embedding $M/G \hookrightarrow \mathbb{R}^d$ we can think of $\overline{f}$ as a smooth function on M/G (see [34, p. 827]), so that we may use standard partitions of unity on $\mathbb{R}^d$ to construct a smooth function $\tilde{f} : \mathbb{R}^d \to \mathbb{R}$ such that $\tilde{f}|_{M/G} = \overline{f}$. Every critical point $[x]$ of $\overline{f}$ in M/G belongs to some stratum $X_{(\alpha)}$ and we have the relation $\overline{f}|_{X_{(\alpha)}} = f|_{M_{(\alpha)}} \circ \pi$. This automatically guarantees that $[x]$ is a stratified nondegenerate critical point of $\overline{f}$ since f is Morse–Bott with nondegenerate critical orbits, $\pi|_{M_{(\alpha)}}$ is a surjective submersion, and the normal Hessian of f is invariant under the normal representation, see Lemma 2.2.2. The condition $d\tilde{f}(x) \notin D^*_{X_{(\alpha)}}\mathbb{R}^d$ follows from the fact that $f : M \to \mathbb{R}$ is basic which implies it preserves the stratification $\mathcal{S}(M/G)$. Conversely, if $\overline{f} : M/G \to \mathbb{R}$ is a stratified Morse function induced by $\tilde{f} : \mathbb{R}^d \to \mathbb{R}$ then by taking average with respect to some proper Haar measure system over G as applied in [34, s. 3] we can construct a basic smooth function $f : M \to \mathbb{R}$ such that $\overline{f} = f \circ \pi$. So, the results follow by arguing similarly as above. $\square$

In [83] it was shown that proper Lie groupoids are real analytic. Hence, after adapting the results obtained in [34] to the real analytic world we may get as an application of Proposition 2.1.2 together with the Pignoni's density result proved in [104] that:

Corollary 2.1.1. *Under the same hypothesis of Proposition 2.1.2 it follows that the set of smooth functions $\mathbb{R}^d \to \mathbb{R}$ which restrict to stratified Morse functions on M/G and induce Morse Lie* *groupoid morphisms on G form an open and dense subset with respect to the strong topology on $C^\infty(\mathbb{R}^d)$.*

2.1.2 Moment maps on 0-symplectic groupoids.

It is known that the component functions of moment maps associated to classical Hamiltonian torus actions always determine examples of Morse–Bott functions, see Example 1.1.8. Our goal now is to show that moment maps for Hamiltonian Lie 2-group actions on 0-symplectic groupoids in the sense of [62] induce Morse–Bott Lie groupoid morphisms. This will be consequence of the results proved in [76] for Hamiltonian actions on presymplectic manifolds. We start by briefly introducing some necessary terminology which can be found in [62]. Recall that a foliation groupoid is a Lie groupoid $G \rightrightarrows M$ whose space of objects M is Hausdorff and whose isotropy groups G_x are discrete for all $x \in M$. For instance, every étale Lie groupoid with Hausdorff objects manifold is a foliation groupoid. The converse is not true, however every foliation groupoid is Morita equivalent to an étale groupoid. As shown in [31, 33], being a foliation groupoid is equivalent to the associated Lie algebroid anchor map $\rho : A \to TM$ being injective. As a consequence, the manifold M comes with a regular foliation $\mathcal{F}$ tangent to the leaves of $\mathrm{im}(\rho) \subseteq TM$. Note that if $G \rightrightarrows M$ is source-connected then the leaves of $\mathrm{im}(\rho) \subseteq TM$ coincide with the groupoid orbits.

A **basic** 2-form on a foliation groupoid $G \rightrightarrows M$ is given by a pair of 2-forms $\omega = (\omega_1, \omega_0)$ with $\omega_1 \in \Omega^2(G), \omega_0 \in \Omega^2(M)$ satisfying $s^*\omega_0 = \omega_1 = t^*\omega_0$. We say that ω is **non-degenerate** if $\ker(\omega_0) = \mathrm{im}(\rho) \subseteq TM$. A basic 2-form $\omega = (\omega_1, \omega_0)$ is **closed** if ω_0 is closed.

Next definition was recently introduced and is due to Hoffman–Sjamaar in [62].

Definition 2.1.3. A 0-**symplectic groupoid** is a foliation groupoid equipped with a closed and non-degenerate basic 2-form ω.

These are also known in the literature as 0-shifted symplectic structures. It is important to point out that this notion of symplectic groupoid differs from that of Weinstein introduced in [117], since $\omega_1 \in \Omega^2(G)$ is not necessarily non-degenerate nor multiplicative. It follows immediately from Definition 2.1.3 that (M, ω_0) is a pre-symplectic manifold with $\ker(\omega_0) = T\mathcal{F}$ in the sense of [76]. Additionally, there is a left action of the product groupoid $G \times G \rightrightarrows M \times M$ on G along (s, s) given by $(g, h)f = gfh^{-1}$. The components of the orbits of this action define a regular foliation $\mathcal{F}_1$ of G satisfying $T\mathcal{F}_1 = \ker(ds) + \ker(dt)$. In particular, ω in Definition 2.1.3 is non-degenerate if and only if $\ker(\omega_1) = \ker(ds) + \ker(dt)$. As a consequence, (G, ω_1) is also a pre-symplectic manifold with $\ker(\omega_1) = T\mathcal{F}_1$.

Suppose that $K^{(1)} \rightrightarrows K^{(0)}$ is a foliation Lie 2-group with associated crossed module of Lie groups (K, H, ∂, α) and Lie 2-algebra $\mathfrak{k}^{(1)} \rightrightarrows \mathfrak{k}^{(0)}$. In this case we have that $\mathrm{Lie}(\partial) : \mathfrak{h} \to \mathfrak{k} = \mathfrak{k}_0$ is injective. As the Lie algebra $\mathrm{Lie}(\partial)(\mathfrak{h}) \cong \mathfrak{h}$ is an ideal in $\mathfrak{k}$ then we may consider the quotient Lie algebra $\mathfrak{k}/\mathfrak{h}$. Let us denote by $\pi : \mathfrak{k} \to \mathfrak{k}/\mathfrak{h}$ the quotient map. The following key result was proven in [62].

Lemma 2.1.4. *If $K^{(1)} \rightrightarrows K^{(0)}$ is a foliation Lie 2-group then the pair $(\pi \circ \mathrm{Lie}(t), \pi) : (\mathfrak{k}^{(1)} \rightrightarrows \mathfrak{k}^{(0)}) \to (\mathfrak{k}/\mathfrak{h} \rightrightarrows \mathfrak{k}/\mathfrak{h})$ is a Morita map of Lie 2-algebras.*

Here by **Morita map** of Lie 2-algebras we mean a Morita map that is at the same time a Lie 2-algebra morphism. As a consequence of the previous result, the Lie groupoid morphism $\mathrm{Ad} : (K^{(1)} \times \mathfrak{k}^{(1)} \rightrightarrows K^{(0)} \times \mathfrak{k}^{(0)}) \to (\mathfrak{k}^{(1)} \rightrightarrows \mathfrak{k}^{(0)})$, which is formed by the adjoint actions Ad_j of $K^{(j)}$ on $\mathfrak{k}^{(j)}$ (for $j = 0, 1$), descends to a well defined Lie 2-group action of $K^{(1)} \rightrightarrows K^{(0)}$ on $\mathfrak{k}/\mathfrak{h} \rightrightarrows \mathfrak{k}/\mathfrak{h}$. By abuse of language, we will call this induced 2-action as the **adjoint action** and denote it by Ad as well. Accordingly, this notion of adjoint action allows us to speak about the **coadjoint action** $\mathrm{Ad}^* : (K^{(1)} \times (\mathfrak{k}/\mathfrak{h})^* \rightrightarrows K^{(0)} \times (\mathfrak{k}/\mathfrak{h})^*) \to ((\mathfrak{k}/\mathfrak{h})^* \rightrightarrows (\mathfrak{k}/\mathfrak{h})^*)$ which is nothing but the Lie 2-group action whose component maps are the coadjoint actions Ad^*_j of $K^{(j)}$ on $(\mathfrak{k}/\mathfrak{h})^*$ induced by the identification we mentioned above.

One of the needed ingredients to define Hamiltonian Lie 2-group actions is given by the notion of fundamental vector field associated to a 2-action. Namely, consider a Lie 2-group action of $K^{(1)} \rightrightarrows K^{(0)}$ on $G \rightrightarrows M$. It is simple to check that for every $\xi \in \mathfrak{k} = \mathfrak{k}^{(0)}$ the pair $(\mathrm{Lie}(u)(\xi)_G, \xi_M)$, formed by the fundamental vector fields of the respective Lie group actions, determines a multiplicative vector field on $G \rightrightarrows M$. Therefore, if $K^{(1)} \rightrightarrows K^{(0)}$ is a foliation Lie 2-group and $G \rightrightarrows M$ is a foliation groupoid then the **fundamental vector field** associated to the 2-action above is by definition the basic vector field on $G \rightrightarrows M$ determined by the pair $(\mathrm{Lie}(u)(\xi)_G, \xi_M)$, see [62, s. 5.4]. By applying this procedure it is possible to show that there exists a Lie algebra anti-morphism from $\mathfrak{k}/\mathfrak{h}$ to the Lie algebra of basic vector fields on $G \rightrightarrows M$; see [62, Pro. 6.9.2] for further details.

The notion of moment map introduced in [62] is as follows.

Definition 2.1.4. Let $(G \rightrightarrows M, \omega)$ be a 0-symplectic groupoid and let $K^{(1)} \rightrightarrows K^{(0)}$ be a foliation Lie 2-group with associated crossed module (K, H, ∂, α). A 2-action of $K^{(1)} \rightrightarrows K^{(0)}$ on $(G \rightrightarrows M, \omega)$ is said to be **Hamiltonian** if the following conditions hold:

i. the action of $K^{(0)}$ on M is presymplectic, and

ii. there is a morphism of Lie groupoids called **moment map**

$$\mu = (\mu_1, \mu_0) : (G \rightrightarrows M) \to ((\mathfrak{k}/\mathfrak{h})^* \rightrightarrows (\mathfrak{k}/\mathfrak{h})^*),$$

verifying

   i) for all $\xi \in \mathfrak{k}/\mathfrak{h}$ it satisfies $d\mu_0^\xi = \iota_{\xi_M}\omega_0$, and

   ii) μ is equivariant with respect to the Lie 2-group action of $K^{(1)} \rightrightarrows K^{(0)}$ on $(G \rightrightarrows M, \omega)$ and the coadjoint action of $K^{(1)} \rightrightarrows K^{(0)}$ on $(\mathfrak{k}/\mathfrak{h})^* \rightrightarrows (\mathfrak{k}/\mathfrak{h})^*$.

If all these conditions are satisfied then we say that $(G \rightrightarrows M, \omega)$ is a **Hamiltonian** $(K^{(1)} \rightrightarrows K^{(0)})$**-groupoid** with moment map μ.

Some observations about the previous definition come in order. Firstly, as we are working with a 2-action and ω is basic then we immediately get that the action of $K^{(1)}$ on G is also presymplectic. Additionally, $d\mu_1^\xi = \iota_{\mathrm{Lie}(u)(\xi)_G}\omega_1$. This follows from the fact that $(\mathrm{Lie}(u)(\xi)_G, \xi_M)$ is a multiplicative vector field, ω is basic, and either $\mu_0 \circ s = \mu_1$ or $\mu_0 \circ t = \mu_1$. Secondly, one also observes that

$$s^*(\mu_0^\xi)(x) = \mu_0(s(x))(\xi) = \mu_1(x)(\xi) = \mu_0(t(x))(\xi) = t^*(\mu_0^\xi)(x).$$

Therefore, for each $\xi \in \mathfrak{k}/\mathfrak{h}$ we have a well defined Lie groupoid morphism $\mu^\xi : (G \rightrightarrows M) \to (\mathbb{R} \rightrightarrows \mathbb{R})$ given either by $s^*(\mu_0^\xi)$ or $t^*(\mu_0^\xi)$.

The required condition that will allow us to ensure that μ^ξ is a Morse–Bott Lie groupoid morphism is determined in terms of the notion of "cleanness" introduced in [76]. Consider a left action of a connected Lie group K on a presymplectic manifold (M, ω) with foliation $\mathcal{F}$ and set $\mathfrak{n}(\mathcal{F}) = \{\xi \in \mathfrak{k} : (\xi_M)(x) \in T_x\mathcal{F} \text{ for all } x \in M\}$. This space is an ideal in $\mathfrak{k}$. Let $N(\mathcal{F})$ be the connected immersed Lie subgroup in K with Lie algebra $\mathfrak{n}(\mathcal{F})$.

Definition 2.1.5. The action of K on M is **clean** if

$$T_x(\mathcal{O}_{N(\mathcal{F})}(x)) = T_x(\mathcal{O}_K(x)) \cap T_x\mathcal{F},$$

for all $x \in M$.

Remark 2.1.4. Suppose that $F : G \to \mathbb{R}$ is a Lie groupoid morphism covering a basic $K^{(0)}$-invariant function $f : M \to \mathbb{R}$. Note that the action of $K^{(0)}$ on M imposes additional symmetries in M different from the ones we already had associated to the Lie groupoid structure. This in particular can make the dimension of the connected components of Crit(f) to increase since they may content more than one groupoid orbit. If this is the case then we say that $F : G \to \mathbb{R}$ is a **Morse–Bott Lie groupoid morphism** if f is a Morse–Bott function in the usual sense.

Summing up, we are in conditions to state:

Proposition 2.1.3. *Let $(G \rightrightarrows M, \omega)$ be a Hamiltonian $(K^{(1)} \rightrightarrows K^{(0)})$-groupoid with moment map $\mu : G \to (\mathfrak{k}/\mathfrak{h})^*$. Suppose that $K^{(0)}$ is a torus and the action of $K^{(0)}$ on M is clean. Then, for every $\xi \in \mathfrak{k}/\mathfrak{h}$ the map $\mu^\xi : (G \rightrightarrows M) \to (\mathbb{R} \rightrightarrows \mathbb{R})$ is a Morse–Bott Lie groupoid morphism with even index at every non-degenerate critical submanifold.*

Proof. Let $\xi \in \mathfrak{k}/\mathfrak{h}$ be fixed. The fact that μ^ξ is a Lie groupoid morphism implies that the critical point set of μ_0^ξ is saturated in M. Thus, when applying [76, Thm. 3.4.5] to our situation we get that every critical component in Crit(μ_0^ξ) is a non-degenerate saturated submanifold of even index since the action of $K^{(0)}$ on M is clean. Therefore, μ^ξ is a Morse–Bott Lie groupoid morphism with the required property. □

In particular, this result recovers the case of toric actions on symplectic orbifolds studied for instance in [63]. Additionally, a specially interesting application of Proposition 2.1.3 will be provided in Example 2.4.1.

Example 2.1.11. Let us exhibit an interesting example that can be found throughout [62]. Let $K = \mathbb{T}^n$ be the n-torus and $N \subseteq K$ be an immersed Lie subgroup. It is well known that $X = \mathbb{C}^n$ is naturally a Hamiltonian K-manifold with the symplectic form $\omega = \frac{1}{2\pi i}\sum_{j=1}^n dz_j \wedge d\bar{z}_j$ and the moment map $\mu(z) = \sum_{j=1}^n |z_j|^2 e_j^*$. Here $\{e_j^*\}$ stands for the dual of the standard basis of $\mathbb{R}^n$. If $\iota : \mathfrak{n} \hookrightarrow \mathfrak{k}$ denotes the canonical Lie algebra inclusion and $\iota^* : \mathfrak{k}^* \to \mathfrak{n}^*$ its corresponding dual projection then (M, ω_0, K, μ_0) with

$$M = (\iota^* \circ \mu)^{-1}(0), \quad \omega_0 = \omega|_M, \quad K = \mathbb{T}^n, \quad \mu_0 = \mu|_M,$$

is a pre-symplectic Hamiltonian K-manifold. Note that μ_0 takes values in $\mathfrak{n}^\circ$.

Let $\pi : \tilde{N} \to N$ be an étale Lie group homomorphism. We may consider $\tilde{N} = N$, $\mathfrak{n}$ (universal covering of the identity component of N), or $p^{-1}(N)$ where $p : \mathbb{R}^n \to \mathbb{T}^n$ (universal covering). On the one hand, the action of $\tilde{N}$ on M via π determines an action groupoid $\tilde{N} \ltimes M \rightrightarrows M$ which is in fact a foliation Lie groupoid. On the other hand, the action of $\tilde{N}$ on K by left multiplications via π determines another action groupoid $\tilde{N} \ltimes K \rightrightarrows K$ that turns out to be a Lie 2-torus. Here $\tilde{N} \ltimes K$ is equipped with the product Lie group structure. Finally, it is simple to check that $(\tilde{N} \ltimes M \rightrightarrows M, \omega_0)$ is a Hamiltonian $(\tilde{N} \ltimes K \rightrightarrows K)$-groupoid with moment map $\mu = s^*\mu_0$. The Lie 2-group action we are considering here is set by $(n, k) \cdot (n', x) = (nn', k \cdot x)$.

Plenty of examples can be also obtained by considering the classification of toric symplectic stacks that was carried out by Hoffman in [61].

2.2 Extending classical results

The aim of this section is to extend the fundamental results of Morse theory to the realm of Lie groupoids. We start by showing our groupoid version of the Morse lemma as well as exploring its consequences. For instance, we describe critical sub-levels of Morse Lie groupoid morphisms in terms of attaching groupoids. For the latter we also need to study gradient vector fields of real valued Lie groupoid morphisms with respect to a Riemannian 2-metric. As an important feature, these gradient vector fields will enjoy the nice property of being multiplicative.

2.2.1 The Morse lemma

Recall that given a Morse-Bott function $f : M \to \mathbb{R}$ and a non-degenerate critical submanifold $C \subseteq M$, the Morse-Bott lemma gives a local normal form for f around C. Namely, on a suitable neighborhood of C the function f looks like the quadratic fiberwise form $Q_f : \nu(C) \to \mathbb{R}$ defined in Equation (1.1) up to a constant, see for instance [12] or [49, App.B]. The main goal of this subsection is to state a version of the Morse lemma in the Lie groupoid setting as well as to describe other important features of the normal form of a Morse Lie groupoid morphism around a nondegenerate critical orbit.

Let $G \rightrightarrows M$ be a Lie groupoid and let $F : G \to \mathbb{R}$ denote a Lie groupoid morphism covering a basic function $f : M \to \mathbb{R}$. Suppose that $\mathcal{O} \subseteq M$ is a critical orbit for f and consider the restricted Lie groupoid $G_{\mathcal{O}} \rightrightarrows \mathcal{O}$. As $G_{\mathcal{O}} \subseteq G$ is also a critical submanifold for F the construction of Equation (1.1) applies to both f and F yielding quadratic fiberwise forms $Q_F : \nu(G_{\mathcal{O}}) \to \mathbb{R}$ and $Q_f : \nu(\mathcal{O}) \to \mathbb{R}$. Furthermore:

Lemma 2.2.1. *The pair* $(Q_F, Q_f) : (\nu(G_{\mathcal{O}}) \rightrightarrows \nu(\mathcal{O})) \to (\mathbb{R} \rightrightarrows \mathbb{R})$ *is a Lie groupoid morphism.*

Proof. Let $g \in G_{\mathcal{O}}$ be a critical arrow. Then, from Identity (1.2) we get that

$$\mathcal{H}_g(f \circ s) = \overline{ds}(g)^T \cdot \mathcal{H}_{s(g)}(f) \cdot \overline{ds}(g).$$

Given that $f : M \to \mathbb{R}$ is basic we actually have $\overline{ds}(g)^T \cdot \mathcal{H}_{s(g)}(f) \cdot \overline{ds}(g) = \overline{dt}(g)^T \cdot \mathcal{H}_x(f) \cdot \overline{dt}(g)$. Thus, it follows from the definition of Q_F and Q_f that the previous two identities immediately imply that $Q_{f \circ s} = Q_f \circ \overline{ds}$ and $Q_{f \circ t} = Q_f \circ \overline{dt}$ as required. □

Let us further assume that $G \rightrightarrows M$ is a proper Lie groupoid and suppose now that $G_{\mathcal{O}} \rightrightarrows \mathcal{O}$ is a non-degenerate critical subgroupoid for $F : G \to \mathbb{R}$. If $\phi : (\nu(G_{\mathcal{O}})_V \rightrightarrows V) \xrightarrow{\cong} (G_U \rightrightarrows U)$ is a full Lie groupoid tubular neighborhood of $G_{\mathcal{O}} \rightrightarrows \mathcal{O}$ then the **local model** of F around $G_{\mathcal{O}}$ is defined as the Lie groupoid morphism

$$\tilde{F} := \phi^* F : (\nu(G_{\mathcal{O}})_V \rightrightarrows V) \to (\mathbb{R} \rightrightarrows \mathbb{R}). \tag{2.2}$$

Note that the zero section is a non-degenerate critical subgroupoid of $\tilde{F}$. In these terms we can state:

Theorem 2.2.1 (Morse lemma)**.** *Let* $F : G \to \mathbb{R}$ *be a Morse Lie groupoid morphism covering* $f : M \to \mathbb{R}$. *If* G *is proper, then around a non-degenerate critical subgroupoid* $G_{\mathcal{O}} \rightrightarrows \mathcal{O}$ *there is a full Lie groupoid tubular neighborhood* $\phi : (\nu(G_{\mathcal{O}})_V \rightrightarrows V) \xrightarrow{\cong} (G_U \rightrightarrows U)$ *such that*

$$\tilde{F} = c + Q_F.$$

Proof. Let $x \in \mathcal{O}$. By the Slice Theorem for proper Lie groupoids we know that there is a transversal T to the orbit $\mathcal{O}$ such that the restricted subgroupoid $G_T \rightrightarrows T$ is isomorphic to the action groupoid $G_x \ltimes B \rightrightarrows B$ for some open set $0 \in B \subset \nu_x(\mathcal{O})$, see [36, 37, 103, 120]. Also, it is known that $G_T \rightrightarrows T$ is Morita equivalent to $G_U \rightrightarrows U$ where U is the open saturation of T. Let us consider the restriction $F_U : G_U \to \mathbb{R}$ of the Morse Lie groupoid morphism F to $G_U \rightrightarrows U$ and transfer it to a Morse Lie groupoid morphism F'_U on $G_x \ltimes W \rightrightarrows W$ by using Proposition 2.1.1. Since G is proper, G_x is compact so $G_x \ltimes W \rightrightarrows W$ is also a proper groupoid. Therefore, by the equivariant version of the Morse lemma [73, 87, 116], it follows that there is a full groupoid tubular neighborhood around the corresponding nondegenerate critical orbit $\mathcal{O}'$ on which $\tilde{F}'_U$ agrees up to constant with $Q_{F'_U}$. Hence, as consequence of Proposition 2.1.1 and the fact that linearization is Morita invariant [36, Prop. 3.7 and Col. 3.9], we get that there is a full groupoid tubular neighborhood around $\mathcal{O}$ on which $\tilde{F_U}$ agrees up to constant with Q_{F_U}. This completes the proof. ☐

We claim that there is another approach which we may follow in order to prove a groupoid version of the Morse lemma. This relies on the nice geometric proof of existence of groupoid tubular neighborhoods around an orbit due to Meinrenken in [87] by constructing Euler-like multiplicative vector fields. We will exhibit such an approach below for later use and because we find the ideas around it really interesting.

A vector field on a manifold M is called **Euler-like** with respect to a submanifold C if it vanishes along the submanifold and its linear approximation is the Euler vector field $\mathcal{E}$ on the normal bundle $\nu(C)$, i.e. $\mathcal{E}$ is the vector field having scalar multiplication by e^{-t} as its flow at the normal direction. It is important to mention that an Euler-like vector field X for (M, C) determines a unique maximal tubular neighborhood embedding $\phi : \nu(C) \to M$ such that $\phi^* X = \mathcal{E}$; compare with [19]. A nice application mentioned in [87] says that if $f : M \to \mathbb{R}$ is a Morse–Bott function with non-degenerate critical submanifold $C \subseteq M$ then there exists an Euler-like vector field X for (M, C) such that $\mathcal{L}_X f = 2f$ near C. Therefore, in the resulting tubular neighborhood embedding ϕ associated to X we have $\phi^* X = \mathcal{E}$, thus obtaining that $\mathcal{L}_{\mathcal{E}} \phi^* f = 2\phi^* f$. This means that $\phi^* f$ is homogeneous of degree 2 and hence coincides with its quadratic approximation $c + Q_f$ (here c is the common value of f on C). As a consequence, we have obtained that $\phi^* f = c + Q_f$ which is just the Morse–Bott lemma.

Given a Lie groupoid $G \rightrightarrows M$ together with an orbit $\mathcal{O} \subseteq M$ it follows from [87] that if $X : G \to TG$ is an Euler-like multiplicative vector field for $(G, G_{\mathcal{O}})$ then the tubular neighborhood embedding $\phi : \nu(G_{\mathcal{O}}) \to G$ defined by X determines a Lie groupoid tubular neighborhood of $G_{\mathcal{O}} \rightrightarrows \mathcal{O}$. In particular, if $G \rightrightarrows M$ is proper then there always exists such an Euler-like multiplicative vector field for $(G, G_{\mathcal{O}})$ and the corresponding tubular neighborhood can be taken to be full.

Another proof of Theorem 2.2.1. We shall follow closely Meinrenken's ideas to construct an Euler-like multiplicative vector field for $(G, G_{\mathcal{O}})$ combined with his application regarding the proof of the Morse-Bott lemma mentioned above. As the groupoid orbit $\mathcal{O}$ is nondegenerate for f there exists an Euler-like vector field Y for $(M, \mathcal{O})$ such that $\mathcal{L}_Y f = 2f$ near $\mathcal{O}$. Since s is a surjective submersion there is a vector field X on G such that X is s-related with Y. The fact that $\overline{ds} : \nu(G_{\mathcal{O}}) \to \nu(\mathcal{O})$ is a fiberwise isomorphism implies that X is Euler-like for $(G, G_{\mathcal{O}})$. Furthermore, $\mathcal{L}_X F = 2F$ near $G_{\mathcal{O}}$ because $F = s^* f$ and X is s-related with Y.

Let us now consider a proper Haar measure system $\{\mu^x\}_{x \in M}$ for $G \rightrightarrows M$. By following results due to Crainic and Struchiner in [36] we can construct a multiplicative vector field

$\overline{X} : G \to TG$ by taking the average with respect to $\{\mu^x\}_{x \in M}$:

$$\overline{X}(g) = \int_{a \in t^{-1}(s(g))} dm_{(ga,a^{-1})}(X(ag), di(a)(X(a))\mu(a).$$

This vector field is s-related with the vector field $\overline{Y}$ on M defined as

$$\overline{Y}(x) = \int_{a \in t^{-1}(x)} dt_a(X(a))\mu(a).$$

Meinrenken showed that both $\overline{X}$ and $\overline{Y}$ are Euler-like vector fields, thus proving that $G \rightrightarrows M$ is linearizable around $G_{\mathcal{O}} \rightrightarrows \mathcal{O}$ (see [87, Lem. 4.2 and Thm. 4.5]). Therefore, our result will follow once we prove that $\mathcal{L}_{\overline{Y}} f = 2f$ and $\mathcal{L}_{\overline{X}} F = 2F$ near $\mathcal{O}$ and $G_{\mathcal{O}}$, respectively. Indeed, on the one hand it follows that

$$\begin{aligned} \mathcal{L}_{\overline{Y}} f(x) &= \int_{a \in t^{-1}(x)} d(f \circ t)_a(X(a))\mu(a) = \int_{a \in t^{-1}(x)} df_{s(a)}(Y(s(a))\mu(a) \\ &= 2\int_{a \in t^{-1}(x)} (f \circ s)(a)\mu(a) = \int_{a \in t^{-1}(x)} (f \circ t)(a)\mu(a) = 2f(x), \end{aligned}$$

since X is s-related with Y and f is basic. On the other hand, as $\overline{X}$ and $\overline{Y}$ are s-related then from the previous computation we trivially obtain that $\mathcal{L}_{\overline{X}} F = 2F$ near $G_{\mathcal{O}}$, as desired. ☐

Remark 2.2.1. For the sake of completeness we mention that there is still another possible approach to prove a groupoid version of the Morse Lemma. This involves the weakly linearization around an orbit provided by the exponential map of a 2-metric in the sense of del Hoyo and Fernandes in [40], combined with the classical ideas for the proof of the Morse-Bott lemma that can be found for instance in [49, App.B]. Although the latter approach is less simple than the two we exhibited above it has the advantage of not requiring our Lie groupoid to be proper.

Negative normal groupoid

Let us now describe other important features of the normal form of a Lie groupoid morphism $F : G \to \mathbb{R}$ around a nondegenerate critical orbit which are derived from putting a groupoid metric into play. Suppose that $G \rightrightarrows M$ can be equipped with a Riemannian 2-metric $\eta^{(2)}$ on $G^{(2)}$ and consider the induced 1-metric $\eta^{(1)}$ on G and 0-metric $\eta^{(0)}$ on M, see Subsection 1.3.1. If $\mathcal{O}$ is a groupoid orbit then we can restrict the 1-metric $\eta^{(1)}$ to $\nu(G_{\mathcal{O}})$ and the 0-metric $\eta^{(0)}$ to $\nu(\mathcal{O})$ and use them to identify $\nu(G_{\mathcal{O}}) \cong TG_{\mathcal{O}}^{\perp}$ and $\nu(\mathcal{O}) \cong T\mathcal{O}^{\perp}$. In particular, if $\mathcal{O}$ is a nondegenerate critical orbit of a Morse Lie groupoid morphism $F : G \to \mathbb{R}$ covering a basic function $f : M \to \mathbb{R}$ then $\mathcal{H}_g(f \circ s) = \overline{ds}(g)^{-1} \cdot \mathcal{H}_x(f) \cdot \overline{ds}(g)$ for all $g \in G_{\mathcal{O}}$, since $s : G \to M$ is a Riemannian submersion. Thus, the indexes $\lambda(G_{\mathcal{O}}, F)$ and $\lambda(\mathcal{O}, f)$ agree.

The following important fact will be crucial at several stages of this thesis.

Lemma 2.2.2. *The Hessian $\mathcal{H}(f) : \nu(\mathcal{O}) \oplus \nu(\mathcal{O}) \to \mathbb{R}$ and the fiberwise quadratic form $Q_f : \nu(\mathcal{O}) \to \mathbb{R}$ are invariant under the normal representation* (1.7).

Proof. Similar arguments as those used in Lemma 2.2.1 show that the Hessian $\mathcal{H}(F) : \nu(G_{\mathcal{O}}) \oplus \nu(G_{\mathcal{O}}) \to \mathbb{R}$ is a Lie groupoid morphism covering $\mathcal{H}(f) : \nu(\mathcal{O}) \oplus \nu(\mathcal{O}) \to \mathbb{R}$. In other words,

$$\mathcal{H}(F) = (\overline{ds} \oplus \overline{ds})^* H(f) = (\overline{dt} \oplus \overline{dt})^* H(f). \tag{2.3}$$

Pick $x \in \mathcal{O}$ and let $g, h \in G_x$, $w_1 \in T_gG$ and $w_2 \in T_hG$ be such that $ds(g)(w_1) = v_1$ and $ds(h)(w_2) = v_2$. Therefore, by using the identities $t \circ \pi_1 = t \circ m$ and $s \circ \pi_2 = s \circ m$ we obtain

$$\begin{array}{rcl}
\mathcal{H}_x(f)(g \cdot [v_1], h \cdot [v_2]) & = & \mathcal{H}_x(f)([dt(g)(w_1)], [dt(h)(w_2)]) \\
 & = & \mathcal{H}_x(f)([dt(g)(w_1)], [d(t \circ \pi_1)_{(h,h^{-1}g)}(w_2, (ds(h^{-1}g))^{-1}(ds(h)(w_2))]) \\
 & = & \mathcal{H}_x(f)([dt(g)(w_1)], [dt(g)(dm_{(h,h^{-1}g)}(w_2, (ds(h^{-1}g))^{-1}(ds(h)(w_2))]) \\
\text{By Identity (2.3)} & = & \mathcal{H}_x(f)([ds(g)(w_1)], [ds(g)(dm_{(h,h^{-1}g)}(w_2, (ds(h^{-1}g))^{-1}(ds(h)(w_2))]) \\
 & = & \mathcal{H}_x(f)([ds(g)(w_1)], [ds(h^{-1}g)((ds(h^{-1}g))^{-1}(ds(h)(w_2))]) \\
 & = & \mathcal{H}_x(f)([ds(g)(w_1)], [ds(h)(w_2)]) = \mathcal{H}_x(f)([v_1], [v_2]).
\end{array}$$

□

It follows immediately from the this lemma that the index of a non-degenerate critical orbit $\mathcal{O} \subseteq M$ is well-defined even if $\mathcal{O}$ is not connected.

Definition 2.2.1. Let $F : G \to \mathbb{R}$ be a Morse Lie groupoid morphism covering a basic function $f : M \to \mathbb{R}$. If $\mathcal{O} \subseteq M$ is a nondegenerate critical orbit then any arrow $g \in G_{\mathcal{O}}$ will be called a **non-degenerate critical arrow** of F and its **index** is defined as $\lambda(g, F) := \lambda(\mathcal{O}, f)$.

Let us fix a nondegenerate critical orbit $\mathcal{O}$ of a Morse Lie groupoid morphism $F : G \to \mathbb{R}$ covering $f : M \to \mathbb{R}$. We can use the Riemannian metrics $\eta^{(1)}$ and $\eta^{(0)}$ respectively to split $\nu(G_{\mathcal{O}}) = \nu_+(G_{\mathcal{O}}) \oplus \nu_-(G_{\mathcal{O}})$ and $\nu(\mathcal{O}) = \nu_+(\mathcal{O}) \oplus \nu_-(\mathcal{O})$ into subbundles which are fiberwise defined by the eigenvectors corresponding to the positive/negative eigenvalues of $\mathcal{H}(F)$ and $\mathcal{H}(f)$. Besides, we get that the groupoid structure on $\nu(G_{\mathcal{O}})$ is preserved by such splittings, as the following result shows.

Lemma 2.2.3. *The Lie groupoid structure of $\nu(G_{\mathcal{O}}) \rightrightarrows \nu(\mathcal{O})$ can be restricted to define two new Lie subgroupoids $\nu_-(G_{\mathcal{O}}) \rightrightarrows \nu_-(\mathcal{O})$ and $\nu_+(G_{\mathcal{O}}) \rightrightarrows \nu_+(\mathcal{O})$.*

Proof. We will prove why it is possible to restrict the groupoid structure in the first case since the second one is completely analogous. We already know that $\mathcal{H}_g(f \circ s) = \overline{ds}(g)^{-1} \cdot \mathcal{H}_x(f) \cdot \overline{ds}(g)$. If v is an eigenvector of $\mathcal{H}_g(f \circ s)$ with negative eigenvalue c, then

$$\mathcal{H}_x(f)(\overline{ds}(g)(v)) = \overline{ds}(g)(\mathcal{H}_g(f \circ s)(v)) = \overline{ds}(g)(c \cdot v) = c \cdot \overline{ds}(g)(v).$$

That is, $\overline{ds}(g)(v)$ is an eigenvector of $\mathcal{H}_x(f)$ with eigenvalue c. Same conclusion may be obtained by arguing with the Riemannian submersion t. Let v and u be eigenvectors of $\mathcal{H}_g(f \circ s)$ and $\mathcal{H}_h(f \circ s)$ with respective negative eigenvalues c_1 and c_2 such that $\overline{dm}_{(g,h)}(v, u)$ is well defined. Let us say $z \overset{g}{\leftarrow} x \overset{h}{\leftarrow} y$. Thus, by using the formula $s \circ m = s \circ \pi_2$ we get that

$$\begin{array}{rcl}
\mathcal{H}_{gh}(f \circ s)(\overline{dm}_{(g,h)}(v, u)) & = & \overline{ds}(gh)^{-1}(\mathcal{H}_y(f)(\overline{ds}(gh)(\overline{dm}_{(g,h)}(v, u)))) \\
 & = & \overline{ds}(gh)^{-1}(\mathcal{H}_y(f)(\overline{d(s \circ m)}_{(g,h)}(v, u))) = \overline{ds}(gh)^{-1}(\mathcal{H}_y(f)(\overline{ds}(h)(u))) \\
 & = & \overline{ds}(gh)^{-1}(\overline{ds}(h)(\mathcal{H}_h(f \circ s)(u))) = c_2 \cdot \overline{ds}(gh)^{-1}(\overline{ds}(h)(u)) \\
 & = & c_2 \cdot \overline{ds}(gh)^{-1}(\overline{ds}_{\pi_2(g,h)}((\overline{d\pi_2})_{(g,h)}(v, u))) \\
 & = & c_2 \cdot \overline{ds}(gh)^{-1}(\overline{d(s \circ m)}_{(g,h)}(v, u)) = c_2 \cdot \overline{dm}_{(g,h)}(v, u).
\end{array}$$

If we assume that $F = t^*f$ then by using the identity $t \circ m = s \circ \pi_1$ we conclude that $\mathcal{H}_{gh}(f \circ t)(\overline{dm}_{(g,h)}(v, u)) = c_1 \cdot \overline{dm}_{(g,h)}(v, u)$. This computation implies that the composition $\overline{dm} : (\nu_-(G_{\mathcal{O}}))^{(2)} \to \nu_-(G_{\mathcal{O}})$ is well defined when considering $\overline{ds}$ and $\overline{dt}$ restricted to

$\nu_-(G_\mathcal{O})$. The restriction of the unit map as $\overline{du} : \nu_-(\mathcal{O}) \to \nu_-(G_\mathcal{O})$ is also well defined since $s \circ u = id$ holds true. Indeed, if v is an eigenvector of $\mathcal{H}_x(f)$ with negative eigenvalue c we have that

$$\begin{aligned}\mathcal{H}_{1_x}(f \circ s)(\overline{du}(x)(v)) &= \overline{ds}(1_x)^{-1}(\mathcal{H}_x(f)(\overline{ds}(1_x)(\overline{du}(x)(v)))) = \overline{ds}(1_x)^{-1}(\mathcal{H}_x(f)(\overline{d(s \circ u)}(x)(v)))\\ &= c \cdot \overline{ds}(1_x)^{-1}(v) = c \cdot \overline{ds}(1_x)^{-1}(\overline{d(s \circ u)}(x)(v) = c \cdot \overline{du}(x)(v).\end{aligned}$$

So, $\overline{du}(x)(v)$ is an eigenvector of $\mathcal{H}_{1_x}(f \circ s)$ with eigenvalue c. Finally, with similar computations, using the identities $t = s \circ i$, $s = t \circ i$, and $\mathcal{H}(f \circ s) = \mathcal{H}(f \circ t)$, we obtain that if v is an eigenvector of $\mathcal{H}_g(f \circ s)$ with negative eigenvalue c then $\mathcal{H}_{g^{-1}}(f \circ s)(\overline{di}(g)(v)) = c \cdot \overline{di}(g)(v)$. Thus, the restriction of the inverse map as $\overline{di} : \nu_-(G_\mathcal{O}) \to \nu_-(G_\mathcal{O})$ is also well defined. The properties required to be satisfied by the composition, the inverse, and the unit map follow from those of $\nu(G_\mathcal{O}) \rightrightarrows \nu(\mathcal{O})$. □

Consider the **negative unit disk bundle** $D_-(G_\mathcal{O})$ defined by

$$D_-(G_\mathcal{O}) = \{v \in \nu_-(G_\mathcal{O}) : \|v\|_1 \leq 1\},$$

where $\|\cdot\|_1$ is the norm on $\nu_-(G_\mathcal{O})$ induced by $\eta^{(1)}$. The positive unit disk bundle $D_+(G_\mathcal{O})$ is defined accordingly. Also, one has unit disk bundles at the level of objects $D_-(\mathcal{O})$ and $D_+(\mathcal{O})$ defined by the induced metric $\eta^{(0)}$ on M. As both the ranks of $\nu_-(G_\mathcal{O})$ and $\nu_-(\mathcal{O})$ agree and they are actually the index λ of the non-degenerate critical submanifolds, the fibers of the negative unit disk bundles are λ-dimensional disks. Moreover, the unit disk bundles define groupoids of the normal groupoid.

Lemma 2.2.4. *The Lie groupoid structure of $\nu_-(G_\mathcal{O}) \rightrightarrows \nu_-(\mathcal{O})$ restricts to the unit disk bundle yielding a topological subgroupoid $D_-(G_\mathcal{O}) \rightrightarrows D_-(\mathcal{O})$.*

Proof. Recall that $s, t : G \to M$ as well as $\pi_1, m, \pi_2 : G^{(2)} \to G$ are Riemannian submersions and that the inversion map $i : G \to G$ is an isometry. Therefore, if we consider the norms $\|\cdot\|_1$ and $\|\cdot\|_0$ with respect to the metrics $\eta^{(1)}$ and $\eta^{(0)}$ restricted to the normal bundles $\nu(G_\mathcal{O})$ and $\nu(\mathcal{O})$, respectively, then we get the identities

$$\|v\|_1 = \|\overline{ds}(v)\|_0, \qquad \|v\|_1 = \|\overline{dt}(v)\|_0, \qquad \|(v, w)\|_2 = \|\overline{dm}(v, w)\|_1 = \|\overline{ds}(w)\|_0 = \|w\|_1$$

$$\|(v, w)\|_2 = \|\overline{dm}(v, w)\|_1 = \|\overline{dt}(v)\|_0 = \|v\|_1, \qquad \|\overline{di}(v)\|_1 = \|v\|_1, \qquad \|\overline{du}(v)\|_1 = \|v\|_0.$$

To deduce these formulas it is important to have in mind the identities $s \circ m = s \circ \pi_2$, $t \circ m = t \circ \pi_1$, $t = s \circ i$, $s = t \circ i$, and $s \circ u = id$. Hence, by mimicking the steps followed in Lemma 2.2.3 it is simple to see that $D_-(G_\mathcal{O}) \rightrightarrows D_-(\mathcal{O})$ is a topological subgroupoid of $\nu_-(G_\mathcal{O}) \rightrightarrows \nu_-(\mathcal{O})$. □

The groupoid introduced in Lemma 2.2.4 will be called the **unit disk groupoid** of $\nu_-(G_\mathcal{O}) \rightrightarrows \nu_-(\mathcal{O})$ with respect the 2-metric $\eta^{(2)}$. Note that this can be thought of as a topological groupoid with boundary. Indeed, the boundary of $D_-(G_\mathcal{O})$ is the unit sphere bundle

$$\partial D_-(G_\mathcal{O}) = \{v \in \nu(G_\mathcal{O}) : \|v\|_1 = 1\},$$

with the natural projection onto $G_\mathcal{O}$. The unit sphere bundle $\partial D_-(\mathcal{O})$ is similarly defined by using instead the norm $\|\cdot\|_0$ induced by $\eta^{(0)}$. Observe that the fibers of these sphere bundles are indeed $(\lambda-1)$-dimensional spheres. It is simple to check that there is a well defined Lie groupoid $\partial D_-(G_\mathcal{O}) \rightrightarrows \partial D_-(\mathcal{O})$ whose structural maps are the induced ones. This Lie groupoid will be called the **unit sphere groupoid** of $\nu_-(G_{\mathcal{O}}) \rightrightarrows \nu_-(\mathcal{O})$ with respect the 2-metric $\eta^{(2)}$ and we shall usually refer to it as the boundary of $D_-(G_{\mathcal{O}}) \rightrightarrows D_-(\mathcal{O})$.

Recall that the action groupoid of the normal representation $G_x \curvearrowright \nu_x(\mathcal{O})$ canonically sits inside the local model $\nu(G_{\mathcal{O}}) \rightrightarrows \nu(\mathcal{O})$ and the inclusion $(G_x \ltimes \nu_x(\mathcal{O}) \rightrightarrows \nu_x(\mathcal{O})) \hookrightarrow (\nu(G_{\mathcal{O}}) \rightrightarrows \nu(\mathcal{O}))$ is a Morita map, see Example 1.3.12. Hence, as consequence of Lemma 2.2.2 and the fact that the normal representation acts by isometries when we are equipped with a 2-metric [40], the following result becomes clear.

Proposition 2.2.1. *The normal representation* (1.7) *induces a Lie groupoid representation of* $G_{\mathcal{O}} \rightrightarrows \mathcal{O}$ *along* $\nu_-(\mathcal{O}) \to \mathcal{O}$ *and a groupoid* (*resp. Lie groupoid*) *action of* $G_{\mathcal{O}} \rightrightarrows \mathcal{O}$ *along* $D_-(\mathcal{O}) \to \mathcal{O}$ (*resp.* $\partial D_-(\mathcal{O}) \to \mathcal{O}$). *Moreover, the action groupoids of these normal actions*

$$G_x \ltimes \nu_-(\mathcal{O})_x \rightrightarrows \nu_-(\mathcal{O})_x, \qquad G_x \ltimes D_-(\mathcal{O})_x \rightrightarrows D_-(\mathcal{O})_x, \qquad G_x \ltimes \partial D_-(\mathcal{O})_x \rightrightarrows \partial D_-(\mathcal{O})_x,$$

canonically sit inside their respective local models and the inclusions are Morita. In particular, there are homeomorphisms between the orbit spaces $D_-(\mathcal{O})_x/G_x \cong D_-(\mathcal{O})/D_-(G_{\mathcal{O}})$ *and* $\partial D_-(\mathcal{O})_x/G_x \cong \partial D_-(\mathcal{O})/\partial D_-(G_{\mathcal{O}})$.

It is simple to see that the same conclusions can be obtained about the groupoids defined in terms of positive eigenvalues. It is worth saying from this point on that Lemma 2.2.2 and Proposition 2.2.1 will be crucial when dealing with Morse theory for differentiable stacks.

2.2.2 Level subgroupoids

After having described some of the local features of a Morse Lie groupoid morphism around a non-degenerate critical orbit, in this subsection we deal with one of the most important results of Morse theory that, in turn, addresses the topological behavior of a Lie groupoid around a non-degenerate critical Lie subgroupoid. We start by introducing a notion of attaching groupoid and then we study multiplicative gradient vector fields with respect to Riemannian groupoid metrics.

Attaching groupoid

Let us quickly explain how to construct topological groupoids by an attaching procedure between Lie groupoids.

Definition 2.2.2. Let $G \rightrightarrows M$ be a Lie groupoid. A **groupoid attaching data** on G consists of:

i) a Lie groupoid $G' \rightrightarrows M'$,

ii) closed submanifolds $\partial G' \subset G'$ and $\partial M' \subset M'$ such that $\partial G' \rightrightarrows \partial M'$ is a Lie subgroupoid of $G' \rightrightarrows M'$, and

iii) a Lie groupoid morphism $(B, b) : (\partial G' \rightrightarrows \partial M') \to (G \rightrightarrows M)$.

A groupoid attaching data defines two topological spaces $G \sqcup_B G'$ and $M \sqcup_b M'$ given by the usual attaching construction. Namely, the quotient space $G \sqcup_B G'$ is defined by taking the disjoint union $G \sqcup G'$ and then identifying $g' \sim B(g')$ for all $g' \in \partial G'$. The attaching space $M \sqcup_b M'$ is defined in the same way.

One immediately observes that there is a natural topological groupoid $G \sqcup G' \rightrightarrows M \sqcup M'$ whose structural maps are defined as the disjoint union of the corresponding structural maps of G and G'. Our main goal in what follows is to show that this groupoid structure descends to the attaching spaces giving rise to a topological groupoid $G \sqcup_B G' \rightrightarrows M \sqcup_b M'$. Since $(B, b) : (\partial G' \rightrightarrows \partial M') \to (G \rightrightarrows M)$ is a Lie groupoid morphism, it follows that the source and target maps $s \sqcup s', t \sqcup t' : G \sqcup G' \to M \sqcup M'$ pass to the quotient, yielding surjective open maps $\overline{s}, \overline{t} : G \sqcup_B G' \to M \sqcup_b M'$. As usual, we define the set of composable arrows $(G \sqcup_B G')^{(2)}$ as the fibered product induced by $\overline{s}$ and $\overline{t}$. Therefore, to define the composition map $\overline{m} : (G \sqcup_B G')^{(2)} \to G \sqcup_B G'$ we have to consider the following four cases. Recall that $(1, x') \sim (b(x'), 2)$ are the only kinds of elements related in $M \sqcup M'$ so that we set

$$\overline{m}([(g, 2)]_B, [(h, 2)]_B) := [(gh, 2)]_B, \quad \overline{m}([(1, g')]_B, [(1, h')]_B) := [(1, g'h')]_B,$$

$$\overline{m}([(1, g')]_B, [(g, 2)]_B) := [(B(g')g, 2)]_B, \quad \overline{m}([(g, 2)]_B, [(1, g')]_B) := [(gB(g'), 2)]_B.$$

It is simple to check that the well definition of this composition follows from the fact that $\overline{s}$ and $\overline{t}$ are also well defined and (B, b) is a Lie groupoid morphism. The associative property of $\overline{m}$ is satisfied because we have that m and m' are associative in G and G', respectively and $B : \partial G' \to G$ satisfies $B \circ m' = m \circ (B \times B)$. Indeed, we only have to be careful when verifying the following case:

$$\begin{aligned} [(1, g')]_B \cdot ([(1, h')]_B \cdot [(g, 2)]_B) &= [(1, g')]_B \cdot [(B(h')g, 2)]_B = [(B(g')(B(h')g), 2)]_B \\ &= [((B(g')B(h'))g, 2)]_B = [(B(g'h')g, 2)] \\ &= [(1, g'h')]_B \cdot [(g, 2)]_B = ([(1, g')]_B \cdot [(1, h')]_B) \cdot [(g, 2)]_B. \end{aligned}$$

The case $[(g, 2)]_B \cdot ([(1, g')]_B \cdot [(1, h')]_B) = ([(g, 2)]_B \cdot [(1, g')]_B) \cdot [(1, h')]_B$ may be verified in a similar fashion and the other ones follow more directly.

As expected, the unit map $\overline{u} : M \sqcup_b M' \to G \sqcup_B G'$ and the inverse $\overline{i} : G \sqcup_B G' \to G \sqcup_B G'$ are defined by passing to the quotient $u \sqcup u' : M \sqcup M' \to G \sqcup G'$ and $i \sqcup i' : G \sqcup G' \to G \sqcup G'$, respectively. It follows easily that both $\overline{u}$ and $\overline{i}$ satisfy the required conditions of the groupoid axioms.

Summing up, we have obtained:

Proposition 2.2.2. *There exists a natural topological groupoid $G \sqcup_B G' \rightrightarrows M \sqcup_b M'$ whose structural maps are given by the ones $(\overline{s}, \overline{t}, \overline{m}, \overline{u}, \overline{i})$ defined as above.*

This groupoid will be called the **attaching groupoid** of $G \rightrightarrows M$ with respect to the Lie groupoid morphism (B, b). A very special case is obtained when taking both $G' = M' = D^\lambda$ a closed λ-disk and $\partial G' = \partial M' = \partial D^\lambda$ its corresponding $(\lambda - 1)$-sphere with their underlying structure of unit groupoids. In this case we attach cells of the same dimension at both arrows and objects extending the groupoid structure.

The gradient vector field

We define now the gradient vector field of a Lie groupoid morphism $F : G \to \mathbb{R}$ with respect to a 2-metric. Let $G \rightrightarrows M$ be a Lie groupoid equipped with a 2-metric $\eta^{(2)}$ on $G^{(2)}$ and consider a Lie groupoid morphism $F : G \to \mathbb{R}$ covering a basic function $f : M \to \mathbb{R}$. The **gradient** of F is defined as the pair $\nabla F = (\nabla(s^* f), \nabla f)$, where $\nabla(s^* f)$ and ∇f are the gradient vector fields of s^*f and f with respect to the induced 1-metric $\eta^{(1)}$ on G and the induced 0-metric $\eta^{(0)}$ on M, respectively.

We start by proving the following key result.

Proposition 2.2.3. *The gradient ∇F is a multiplicative vector field on G.*

Proof. We know that $s, t : G \to M$ as well as $\pi_1, m, \pi_2 : G^{(2)} \to G$ are Riemannian submersions. Thus, the vector fields $\nabla(f\circ s)$ and ∇f are s-related, $\nabla(f\circ t)$ and ∇f are t-related, and $\nabla((f\circ s)\circ m)$ and $\nabla(f\circ s)$ are m-related. Here $\nabla((f\circ s)\circ m)$ denotes the gradient vector field of $(f \circ s)\circ m$ with respect to $\eta^{(2)}$ on $G^{(2)}$ and we have that $dm\circ\nabla((f\circ s)\circ m) = \nabla(f\circ s)\circ m$. The crucial point is to show that for $(g, h) \in G^{(2)}$ the identity

$$\nabla((f \circ s) \circ m)(g, h) = (\nabla(f \circ s)(g), \nabla(f \circ t)(h)),$$

holds true as long as $ds(g)(\nabla(f \circ s)(g)) = \nabla f(s(g)) = \nabla f(t(h)) = dt(h)(\nabla(f \circ t)(h))$. However, the vector fields $\nabla((f\circ s)\circ\pi_1)$ and $\nabla(f\circ s)$ are π_1-related and $\nabla((f\circ t)\circ\pi_2)$ and the vector fields $\nabla(f \circ t)$ are π_2-related. Therefore, as $(f \circ s) \circ m = (f \circ s) \circ \pi_1 = (f \circ t) \circ \pi_2$ since f is basic, we obtain that

$$d\pi_1 \circ \nabla((f \circ s) \circ m) = \nabla(f \circ s) \circ \pi_1 \quad \text{and} \quad d\pi_2 \circ \nabla((f \circ s) \circ m) = \nabla(f \circ t) \circ \pi_2.$$

That is, we get what we required and because of this $dm \circ (\nabla(f \circ s) \times \nabla(f \circ s)) = \nabla(f \circ s) \circ m$. □

It is clear that we also may define the gradient vector field as the pair $\nabla F = (\nabla(t^*f), \nabla f)$.

Remark 2.2.2. The fact of ∇F being multiplicative gives rise to other important properties involving the Lie groupoid structure of G. For instance, ∇F is a multiplicative vector field if and only if the pair of flows $(\Phi_\tau^{\nabla(s^*f)}, \Phi_\tau^{\nabla f})$ induces (local) automorphisms on the Lie groupoid [81]. Namely, the following identities always hold true in our case:

$$s \circ \Phi_\tau^{\nabla(s^*f)} = \Phi_\tau^{\nabla f} \circ s, \qquad t \circ \Phi_\tau^{\nabla(t^*f)} = \Phi_\tau^{\nabla f} \circ t, \qquad \Phi_\tau^{\nabla(f\circ s)} \circ m = m \circ (\Phi_\tau^{\nabla(f\circ s)} \times \Phi_\tau^{\nabla(f\circ s)}).$$

Let us now define the subgroupoid levels of the kind of Lie groupoid morphisms we are working with. Let $F : G \to \mathbb{R}$ be a Lie groupoid morphism covering a basic function $f : M \to \mathbb{R}$. Take $a \in \mathbb{R}$ and consider the level set $M^a = \{x \in M : f(x) \leq a\}$. It is simple to check that $s^{-1}(M^a) = t^{-1}(M^a)$ which is equivalent to saying that M^a is a saturated submanifold. This implies that $G^a = G_{M^a}$, with G^a denoting the level set of F below a, so that we get a well defined topological subgroupoid $G^a \rightrightarrows M^a$ of $G \rightrightarrows M$ that we call the **level subgroupoid** of F below a. It is important to notice that $G^a \rightrightarrows M^a$ can be thought of as a Lie groupoid with boundary in the sense that if $\partial M^a = \{x \in M : f(x) = a\}$ and $\text{int}(M^a) = \{x \in M : f(x) < a\}$ then both $\partial G^a \rightrightarrows \partial M^a$ and $\text{int}(G^a) \rightrightarrows \text{int}(M^a)$ are clearly Lie subgroupoids of $G \rightrightarrows M$.

It is well known that the gradient vector field $X = \nabla f$ of a smooth function f on a Riemannian manifold $(M, \eta^{(0)})$ is actually a **gradient-like vector field**. That is, it satisfies both $\text{Zeros}(X) = \text{Crit}(f)$ and $df(X) > 0$ on $M \backslash \text{Crit}(f)$. Thus, as consequence of Proposition 2.2.3 we obtain:

Proposition 2.2.4. *Suppose that $G \rightrightarrows M$ is a proper groupoid, $[a, b]$ is a closed interval which does not contain critical values of f and $f^{-1}[a, b]$ is compact. Then $G^a \rightrightarrows M^a$ and $G^b \rightrightarrows M^b$ are isomorphic groupoids. Furthermore, $G^a \rightrightarrows M^a$ is a deformation retraction of $G^b \rightrightarrows M^b$.*

Proof. Let us follow some ideas stated in [88, 97] about the proof of this result in the classical case. It is clear that if $f^{-1}[a,b] \cap \mathrm{Crit}(f) = \emptyset$, then $(s^*f)^{-1}[a,b] \cap \mathrm{Crit}(F) = \emptyset$. We know that $\widetilde{X} = \nabla(f \circ s)$ and $X = \nabla f$ are gradient-like vector fields on G and M, respectively. Thus, consider the smooth function $\mu : M \to [0,\infty)$ which is defined by $\|Xf\|^{-1}$ in $f^{-1}[a,b]$ and that vanishes outside a compact neighborhood of this set. We can similarly construct a smooth function $\widetilde{\mu} : G \to [0,\infty)$ by using instead $\widetilde{X}$ and $(s^*f)^{-1}[a,b]$. On the one hand, observe that since $\nabla F = (\widetilde{X}, X)$ is a multiplicative vector field we get that

$$\begin{aligned} \mu(s(g)) &= \|(Xf)(s(g))\|^{-1} = \|df(s(g))(X(s(g))\|^{-1} \\ &= \|df(s(g))(ds(g)(\widetilde{X}(g)))\|^{-1} = \|d(f\circ s)(g)(\widetilde{X}(g))\|^{-1} \\ &= \|(\widetilde{X}F)(g)\|^{-1} = \widetilde{\mu}(g), \end{aligned}$$

what means that we have $\widetilde{\mu} = \mu \circ s$. On the other hand, let us consider the vectors fields $-\widetilde{\mu}\widetilde{X}$ and $-\mu X$. From [88, Lem. 2.4] and [35, Lem. 4.4] we obtain that they are complete vector fields since the pair $(-\widetilde{\mu}\widetilde{X}, -\mu X)$ defines again a multiplicative vector field. Indeed, as consequence of Proposition 2.2.3 we get

$$(ds \circ \widetilde{\mu}\widetilde{X})(g) = \widetilde{\mu}(g)ds(g)(\widetilde{X}(g)) = \widetilde{\mu}(g)X(s(g)) = \mu(s(g))X(s(g)) = (\mu X \circ s)(g).$$

Given that $\nabla(f \circ s) = \nabla(f \circ t)$ we actually have that $\widetilde{\mu} = \mu \circ s = \mu \circ t$ and thus we can analogously obtain that $dt \circ \widetilde{\mu}\widetilde{X} = \mu X \circ t$. To prove the identity regarding the composition map observe that the formulas $s \circ m = s \circ \pi_2$ and $t \circ m = t \circ \pi_1$ imply that $\widetilde{\mu}(m(g,h)) = \widetilde{\mu}(g) = \widetilde{\mu}(h)$ for all $(g,h) \in G^{(2)}$. Therefore,

$$\begin{aligned} (dm \circ (\widetilde{\mu}\widetilde{X} \times \widetilde{\mu}\widetilde{X}))(g,h) &= dm_{(g,h)}(\widetilde{\mu}(g)\widetilde{X}(g), \widetilde{\mu}(h)\widetilde{X}(h)) \\ &= \widetilde{\mu}(m(g,h))dm_{(g,h)}(\widetilde{X}(g), \widetilde{X}(h)) \\ &= \widetilde{\mu}(m(g,h))\widetilde{X}(m(g,h)) \\ &= (\widetilde{\mu}\widetilde{X} \circ m)(g,h). \end{aligned}$$

For all $\tau \in \mathbb{R}$, let $\widetilde{\Phi}_\tau : G \to G$ and $\Phi_\tau : M \to M$ denote the respective flows generated by the pair $(-\widetilde{\mu}\widetilde{X}, -\mu X)$. From [81] we know that $(\widetilde{\Phi}_\tau, \Phi_\tau)$ is a Lie groupoid isomorphism for all $\tau \in \mathbb{R}$. As in the classical case (see [88, 97]), we get diffeomorphisms

$$\Phi_{b-a}(M^b) = M^a, \qquad \Phi_{a-b}(M^a) = M^b \quad \text{and} \quad \widetilde{\Phi}_{b-a}(G^b) = G^a, \qquad \widetilde{\Phi}_{a-b}(G^a) = G^b$$

so that $(\widetilde{\Phi}_{b-a}, \Phi_{b-a}) : (G^b \rightrightarrows M^b) \to (G^a \rightrightarrows M^a)$ defines a Lie groupoid isomorphism with obvious inverse $(\widetilde{\Phi}_{a-b}, \Phi_{a-b})$. Finally, we can define two deformation retractions $H : [0,1] \times M^b \to M^b$ and $\widetilde{H} : [0,1] \times G^b \to G^b$ respectively as

$$H(\tau, x) = \Phi_{\tau\cdot(f(x)-a)^+}(x) \qquad \text{and} \qquad \widetilde{H}(\tau, g) = \widetilde{\Phi}_{\tau\cdot((s^*f)(g)-a)^+}(g).$$

Here $r^+ := \max\{r, 0\}$ for every real number r. Hence, we obtain again a Lie groupoid morphism which allows us to conclude that $G^a \rightrightarrows M^a$ is a deformation retraction of $G^b \rightrightarrows M^b$. ☐

We analyze now the case in which $f^{-1}[a,b] \cap \mathrm{Crit}(f) \neq \emptyset$. The main reference for the basic ideas we will be following in this case may be found for instance in [49, App. B].

Let $\mathcal{O}$ be a non-degenerate critical orbit of f. We will denote by $\xi_0 : \nu(\mathcal{O}) \to \nu_-(\mathcal{O})$ and $\eta_0 : \nu(\mathcal{O}) \to \nu_+(\mathcal{O})$ the two mutually complementary projections. They induce bundle morphisms between the bundle projections $\pi : \nu(\mathcal{O}) \to \mathcal{O}$ and $\pi_\pm : \nu_\pm(\mathcal{O}) \to \mathcal{O}$, respectively. Moreover, for every $v \in \nu(\mathcal{O})$ we have that $v = \xi_0(v) + \eta_0(v)$ and the expression

$$\|v\|_0^2 = -Q_f(\xi_0(v)) + Q_f(\eta_0(v)),$$

defines a positive definite quadratic form (i.e. a norm) on $\nu(\mathcal{O})$. As expected, we can define respective mutually complementary projections $\xi_1 : \nu(G_\mathcal{O}) \to \nu_-(G_\mathcal{O})$ and $\eta_1 : \nu(G_\mathcal{O}) \to \nu_+(G_\mathcal{O})$ which enjoy similar properties as above. Namely, every $\widetilde{v} \in \nu(G_\mathcal{O})$ can be rewritten as $\widetilde{v} = \xi_1(\widetilde{v}) + \eta_1(\widetilde{v})$ and we have the norm $\|\widetilde{v}\|_1^2 = -Q_F(\xi_1(\widetilde{v})) + Q_F(\eta_1(\widetilde{v}))$ on $\nu(G_\mathcal{O})$. Furthermore:

Lemma 2.2.5. *Both $(\xi_1, \xi_0) : (\nu(G_\mathcal{O}) \rightrightarrows \nu(\mathcal{O})) \to (\nu_-(G_\mathcal{O}) \rightrightarrows \nu_-(\mathcal{O}))$ and $(\eta_1, \eta_0) : (\nu(G_\mathcal{O}) \rightrightarrows \nu(\mathcal{O})) \to (\nu_+(G_\mathcal{O}) \rightrightarrows \nu_+(\mathcal{O}))$ are Lie groupoid morphisms. Moreover, for all $x \in \mathcal{O}$ it holds that ξ_0 and η_0 are G_x-equivariant with respect to the action induced by the normal representation (1.7).*

Proof. If $\widetilde{v} \in \nu(G_{\mathcal{O}_x})$ then we get

$$(\xi_0 \circ \overline{ds})(\widetilde{v}) = (\xi_0 \circ \overline{ds})(\xi_1(\widetilde{v}) + \eta_1(\widetilde{v})) = \xi_0(\overline{ds}(\xi_1(\widetilde{v})) + \overline{ds}(\eta_1(\widetilde{v}))) = \overline{ds}(\xi_1(\widetilde{v})).$$

We can analogously prove that $\xi_0 \circ \overline{dt} = \overline{dt} \circ \xi_1$. Now, if $\widetilde{v}, \widetilde{u} \in \nu(G_{\mathcal{O}_x})$ are such that $\overline{dm}(\widetilde{v}, \widetilde{u})$ is defined then

$$\begin{aligned}(\xi_1 \circ \overline{dm})(\widetilde{v}, \widetilde{u}) &= (\xi_1 \circ \overline{dm})(\xi_1(\widetilde{v}) + \eta_1(\widetilde{v}), \xi_1(\widetilde{u}) + \eta_1(\widetilde{u})) \\ &= \xi_1(\overline{dm}(\xi_1(\widetilde{v}), \xi_1(\widetilde{u})) + \overline{dm}(\eta_1(\widetilde{v}), \eta_1(\widetilde{u}))) = \overline{dm}(\xi_1(\widetilde{v}), \xi_1(\widetilde{u})),\end{aligned}$$

so that $\xi_1 \circ \overline{dm} = \overline{dm} \circ (\xi_1 \times \xi_1)$.

The fact that (η_1, η_0) is a Lie groupoid morphism may be shown in a similar way and the G_x-equivariance of ξ_0 and η_0 is consequence of Lemma 2.2.2. □

With all of this in mind we have:

Theorem 2.2.2. *Suppose that $G \rightrightarrows M$ is a proper groupoid and that $[a, b]$ is a closed interval such that $f^{-1}[a, b]$ is compact and the only non-degenerate critical orbit inside $f^{-1}(a, b)$ is $\mathcal{O}$. Then $G^a \cup_B D_-(G_\mathcal{O}) \rightrightarrows M^a \cup_b D_-(\mathcal{O})$ is a deformation retraction of $G^b \rightrightarrows M^b$.*

Proof. Given that $\mathcal{O} \subset f^{-1}(a, b) \subset M$ is closed and embedded we get that $\mathcal{O}$ is also compact. Furthermore, $(s^*f)^{-1}[a, b]$ is also compact since G is proper so that $G_\mathcal{O}$ is the only compact non-degenerate critical submanifold inside $(s^*f)^{-1}(a, b)$.

Consider the mutually complementary projections $\xi_1 : \nu(G_\mathcal{O}) \to \nu_-(G_\mathcal{O})$, $\eta_1 : \nu(G_\mathcal{O}) \to \nu_+(G_\mathcal{O})$ and $\xi_0 : \nu(\mathcal{O}) \to \nu_-(\mathcal{O})$, $\eta_0 : \nu(\mathcal{O}) \to \nu_+(\mathcal{O})$ as well as the norms

$$\|\widetilde{v}\|_1^2 = -Q_F(\xi_1(\widetilde{v})) + Q_F(\eta_1(\widetilde{v})) \qquad \text{and} \qquad \|v\|_0^2 = -Q_f(\xi_0(v)) + Q_f(\eta_0(v)),$$

for all $\widetilde{v} \in \nu(G_\mathcal{O})$ and $v \in \nu(\mathcal{O})$. Because of Theorem 2.2.1 there is a full Lie groupoid tubular neighborhood $\phi : (\nu(G_\mathcal{O})_V \rightrightarrows V) \xrightarrow{\cong} (G_U \rightrightarrows U)$ such that $(\phi_1^* F, \phi_0^* f) = (c + Q_F, c + Q_f)$, where $c = f(\mathcal{O}) = F(G_\mathcal{O})$ is the common value of f and F on $G_\mathcal{O}$ and $\mathcal{O}$, respectively. By identifying $\nu(G_\mathcal{O})_V \cong G_U$ and $V \cong U$ we may think of F on G_U and f on U as respectively given by

$$F(\widetilde{v}) = c + Q_F(\widetilde{v}) = c - \|\xi_1(\widetilde{v})\|_1^2 + \|\eta_1(\widetilde{v})\|_1^2, \qquad f(v) = c + Q_f(v) = c - \|\xi_0(v)\|_0^2 + \|\eta_0(v)\|_0^2.$$

Let us now follow some ideas stated in [49, 88, 97] about the proof of this result in the classical case. We choose $\epsilon > 0$ small enough so that the interval $(c-\epsilon, c+\epsilon)$ is contained in $[a, b]$ and all points $\widetilde{v} \in \nu(G_{\mathcal{O}})$ with $\|\widetilde{v}\|_1^2 \leq 2\epsilon$ belong to the neighborhood G_U and $v \in \nu(\mathcal{O})$ with $\|v\|_0^2 \leq 2\epsilon$ belong to the neighborhood U. Similarly to how it was done in [88, p. 16-19] we construct two smooth functions which are a modification of the pair (F, f). Namely, let $\mu : \mathbb{R} \to \mathbb{R}$ be the smooth function verifying the conditions

$$\begin{aligned} \mu(0) &> \epsilon, \\ \mu(r) &= 0 && \text{for all } r \geq 2\epsilon, \\ -1 < \mu'(r) &\leq 0 && \text{for all } r. \end{aligned}$$

Consider the functions $F_1 : G \to \mathbb{R}$ and $f_1 : M \to \mathbb{R}$ which respectively coincide with F and f outside of G_U and U but within those neighborhoods they are given as

$$F_1(\widetilde{v}) = F(\widetilde{v}) - \mu(\|\xi_1(\widetilde{v})\|_1^2 + 2\|\eta_1(\widetilde{v})\|_1^2) = c - \|\xi_1(\widetilde{v})\|_1^2 + \|\eta_1(\widetilde{v})\|_1^2 - \mu(\|\xi_1(\widetilde{v})\|_1^2 + 2\|\eta_1(\widetilde{v})\|_1^2),$$

and

$$f_1(v) = f(v) - \mu(\|\xi_0(v)\|_0^2 + 2\|\eta_0(v)\|_1^2) = c - \|\xi_0(v)\|_0^2 + \|\eta_0(v)\|_0^2 - \mu(\|\xi_0(v)\|_0^2 + 2\|\eta_0(v)\|_1^2).$$

These are well defined smooth functions. Let us prove that $(F_1, f_1) : (G \rightrightarrows M) \to (\mathbb{R} \rightrightarrows \mathbb{R})$ is also a Lie groupoid morphism. Since ϕ is a Lie groupoid isomorphism and (F_1, f_1) agrees with (F, f) outside (G_U, U) we only have to see what happens inside (G_U, U). As consequence of the classical Morse–Bott lemma we may identify the norms $\|\cdot\|_1$ and $\|\cdot\|_0$ with those norms defined by the restrictions of $\eta^{(1)}$ and $\eta^{(0)}$ on $\nu(G_{\mathcal{O}})$ and $\nu(\mathcal{O})$, respectively. As $s, t : G \to M$ are Riemannian submersions we obtain that $\|\widetilde{v}\|_1 = \|ds(\widetilde{v})\|_0$ and $\|\widetilde{v}\|_1 = \|dt(\widetilde{v})\|_0$. It is worth noticing that if we do not identify the norms as we did above we can also get the previous statement by using the fact that (Q_F, Q_f), (ξ_1, ξ_0), and (η_1, η_0) are Lie groupoid morphisms. Therefore, by using again the latter fact we obtain that

$$\begin{aligned} (\overline{ds}^* f_1)(\widetilde{v}) &= f_1(\overline{ds}(\widetilde{v})) = c + Q_f(\overline{ds}(\widetilde{v})) - \mu(\|\xi_0(\overline{ds}(\widetilde{v}))\|_0^2 + 2\|\eta_0(\overline{ds}(\widetilde{v})\|_0^2) \\ &= c + (Q_f \circ \overline{ds})(\widetilde{v}) - \mu(\|\overline{ds}(\xi_1(\widetilde{v}))\|_0^2 + 2\|\overline{ds}(\xi_1(\widetilde{v})\|_0^2) \\ &= c + Q_F(\widetilde{v}) - \mu(\|\xi_1(\widetilde{v})\|_1^2 + 2\|\eta_1(\widetilde{v})\|_1^2) = F_1. \end{aligned}$$

We can analogously show that $\overline{dt}^* f_1 = F_1$ which implies what we desired.

Just as it was proved in [88, p. 16-19] we have the following assertions.

- The regions $F_1^{-1}(-\infty, c+\epsilon]$ and $f_1^{-1}(-\infty, c+\epsilon]$ coincide with the regions $G^{c+\epsilon}$ and $M^{c+\epsilon}$, respectively.
- The functions F and F_1 have the same critical arrows. Similarly, the functions f and f_1 have the same critical points.
- The sets $F_1^{-1}[c-\epsilon, c+\epsilon]$ and $f_1^{-1}[c-\epsilon, c+\epsilon]$ contain no critical points of F_1 and f_1, respectively.

Thus, from Proposition 2.2.4 we conclude that the Lie subgroupoid level $F_1^{-1}(-\infty, c-\epsilon] \rightrightarrows f_1^{-1}(-\infty, c-\epsilon]$ is a deformation retraction of $G^{c+\epsilon} \rightrightarrows M^{c+\epsilon}$.

Let us now consider the negative disk bundle $D_-(\mathcal{O}) = \{v \in U : \|\xi_0(v)\|_0^2 \leq \epsilon,\ \eta_0(v) = 0\}$. As the value of f_1 on any point of $D_-(\mathcal{O})$ is less than $c-\epsilon$ we have that $M^{c-\epsilon} \cup D_-(\mathcal{O}) \subset$

$f_1^{-1}(-\infty, c-\epsilon]$. Moreover, the intersection $D_-(\mathcal{O}) \cap M^{c-\epsilon}$ coincides with the negative sphere bundle $\partial D_-(\mathcal{O}) = \{v \in U : \|\xi_0(v)\|_0^2 = \epsilon,\ \eta_0(v) = 0\}$. Therefore, we can consider the attaching space $M^{c-\epsilon} \cup_b D_-(\mathcal{O})$ with respect to the bundle projection $b : \partial D_-(\mathcal{O}) \to M^{c-\epsilon}$ which actually we may be seen as the union $M^{c-\epsilon} \cup D_-(\mathcal{O})$ inside $f_1^{-1}(-\infty, c-\epsilon]$. As it was argued in [88, p. 18-19], we have that the union $M^{c-\epsilon} \cup D_-(\mathcal{O})$ is a deformation retract of $f_1^{-1}(-\infty, c-\epsilon]$. The deformation retraction $r_\tau : f_1^{-1}(-\infty, c-\epsilon] \to f_1^{-1}(-\infty, c-\epsilon]$ is identical outside U but within U it acts as follows.

- In the domain $\|\xi_0(v)\|_0^2 \leq \epsilon$ (i.e. in $D_-(\mathcal{O})$) the deformation r_t is given by the formula
$$r_\tau(v) = \xi_0(v) + \tau\eta_0(v).$$
- In the domain $\epsilon \leq \|\xi_0(v)\|_0^2 \leq \|\eta_0(v)\|_0^2 + \epsilon$ we define r_τ by
$$r_\tau(v) = \xi_0(v) + s_\tau(v)\eta_0(v),$$
where the number $s_\tau(v) \in [0,1]$ is defined by $s_\tau(v) = \tau + (1-\tau)\sqrt{\dfrac{\|\xi_0(v)\|_0^2 - \epsilon}{\|\eta_0(v)\|_0^2}}$. The map r_0 takes values in $f^{-1}(c-\epsilon)$.
- Within the domain $\|\eta_0(v)\|_0^2 + \epsilon \leq \|\xi_0(v)\|_0^2$ (i.e. in $M^{c-\epsilon}$) we set r_τ to be the identity map, $\tau \in [0,1]$.

The expressions above agree on the intersection of the three domains and thus they define a continuous map r_τ so that r_1 is the identity map and r_0 is a retraction of $f_1^{-1}(-\infty, c-\epsilon]$ onto $M^{c-\epsilon} \cap D_-(\mathcal{O})$. As expected, we can similarly define the attaching space $G^{c-\epsilon} \cup_B D_-(G_{\mathcal{O}})$ by using the bundle projection $B : \partial D_-(G_{\mathcal{O}}) \to G^{c-\epsilon}$ as well as construct a deformation retraction $\widetilde{r_\tau} : F_1^{-1}(-\infty, c-\epsilon] \to F_1^{-1}(-\infty, c-\epsilon]$ by using the mutually complementary projections ξ_1, η_1 and the norm $\|\cdot\|_1$ to produce same formulas as we did above.

The key point now is to prove that $(\widetilde{r_\tau}, r_\tau)$ defines a groupoid morphism. As $\widetilde{r_\tau}$ and r_τ are the identity outside G_U and U, respectively, we only have to check this on the three special cases mentioned above. We will use again the fact that (ξ_1, ξ_0) and (η_1, η_0) are Lie groupoid morphisms. In the first domain we have that
$$(r_\tau \circ \overline{ds})(\widetilde{v}) = \xi_0(\overline{ds}(\widetilde{v})) + \tau\eta_0(\overline{ds}(\widetilde{v})) = \overline{ds}(\xi_1(\widetilde{v})) + \tau\overline{ds}(\eta_1(\widetilde{v})) = (\overline{ds} \circ \widetilde{r_\tau})(\widetilde{v}).$$

We can analogously prove that $r_\tau \circ \overline{dt} = \overline{dt} \circ \widetilde{r_\tau}$. Now, if $\widetilde{v}$ and $\widetilde{u}$ are such that $dm(\widetilde{v}, \widetilde{u})$ is defined then
$$\begin{aligned}
\widetilde{r_\tau}(\overline{dm}(\widetilde{v}, \widetilde{u})) &= \xi_1(\overline{dm}(\widetilde{v}, \widetilde{u})) + \tau\eta_1(\overline{dm}(\widetilde{v}, \widetilde{u})) = \overline{dm}(\xi_1(\widetilde{v}), \xi_1(\widetilde{u})) + \tau\overline{dm}(\eta_1(\widetilde{v}), \eta_1(\widetilde{u}))\\
&= \overline{dm}((\xi_1(\widetilde{v}), \xi_1(\widetilde{u})) + \tau(\eta_1(\widetilde{v}), \eta_1(\widetilde{u}))) = \overline{dm}(\xi_1(\widetilde{v}) + \tau\eta_1(\widetilde{v}), \xi_1(\widetilde{u}) + \tau\eta_1(\widetilde{u}))\\
&= \overline{dm}(\widetilde{r_\tau}(\widetilde{v}), \widetilde{r_\tau}(\widetilde{u})),
\end{aligned}$$

what means that $\widetilde{r_\tau} \circ \overline{dm} = \overline{dm} \circ (\widetilde{r_\tau} \times \widetilde{r_\tau})$. To verify the assertion in the second domain we only have to check that $s_\tau(\overline{ds}(\widetilde{v})) = \widetilde{s_\tau}(\widetilde{v})$, $s_\tau(\overline{dt}(\widetilde{v})) = \widetilde{s_\tau}(\widetilde{v})$ and $\widetilde{s_\tau}(\widetilde{v}) = \widetilde{s_\tau}(\overline{dm}(\widetilde{v}, \widetilde{u})) = \widetilde{s_\tau}(\widetilde{u})$. These formulas follows from the fact that (ξ_1, ξ_0) and (η_1, η_0) are Lie groupoid morphisms plus the identities
$$\|\overline{dm}(\widetilde{v}, \widetilde{u})\|_1 = \|\overline{ds}(\widetilde{u})\|_0 = \|\widetilde{u}\|_1 \qquad \text{and} \qquad \|\overline{dm}(\widetilde{v}, \widetilde{u})\|_1 = \|\overline{dt}(\widetilde{v})\|_0 = \|\widetilde{v}\|_1$$

which can be obtained by either the fact that s, t, m, π_1 and π_2 are Riemannian submersions or (Q_F, Q_f), (ξ_1, ξ_0), (η_1, η_0) are Lie groupoid morphisms. The remaining computations are similar to those done when looking at the first domain. Finally, in the third domain the assertion directly follows since $\widetilde{r_\tau}$ and r_τ are the identity maps over there.

In conclusion, the topological groupoid $G^{c-\epsilon} \cup_B D_-(G_{\mathcal{O}}) \rightrightarrows M^{c-\epsilon} \cup_b D_-(\mathcal{O})$ is a deformation retraction of the level subgroupoid $F_1^{-1}(-\infty, c-\epsilon] \rightrightarrows f_1^{-1}(-\infty, c-\epsilon]$ which in turn, as consequence of Proposition 2.2.4, is a deformation retraction of the level subgroupoid $G^{c+\epsilon} \rightrightarrows M^{c+\epsilon}$. Hence, $G^{c-\epsilon} \cup_B D_-(G_{\mathcal{O}}) \rightrightarrows M^{c-\epsilon} \cup_b D_-(\mathcal{O})$ is a deformation retraction of $G^{c+\epsilon} \rightrightarrows M^{c+\epsilon}$. □

Remark 2.2.3. It is important to point out that the assumption from Proposition 2.2.4 and Theorem 2.2.2 asking for $f^{-1}[a, b]$ to be compact in M may be relaxed just by requiring $\overline{f}^{-1}[a, b]$ to be compact in the orbit space M/G. Here $\overline{f} : M/G \to \mathbb{R}$ denotes the underlying continuous function defined through the basic function $f : M \to \mathbb{R}$. This fact will be clarified in Section 2.5 where we will study some Morse theoretical features over differentiable stacks.

It is simple to deduce from the proof of Theorem 2.2.2 that Theorem 2.2.1 and Lemmas 2.2.2 and 2.2.5 imply that:

Corollary 2.2.1. *Around the local model to the orbit $\mathcal{O}$ the deformation retraction r_τ is G_x-equivariant with respect to the action induced by the normal representation* (1.7).

We finish this subsection by providing some observations about the advantages that our approach to study Morse theory over M/G have in contrast with the approach given by stratified Morse theory, see Subsection 2.1.1.

Remark 2.2.4. Firstly, our focus aims at extending classical Morse theory to the context of differentiable stacks, so that its study passing through Lie groupoids is more suitable for our purposes. Also, the latter provides a cleaner way to extract several fundamental results of Morse theory without assuming properness, compactness or finiteness on the Morita types equivalences, which are requirements that seem to be necessary in order to guarantee that the orbit space admits some kind of embedding into an affine space. Evidences of this fact are mentioned in Remark 2.2.1, Subsection 2.1.2, Proposition 2.2.1. Even though sometimes we further assume properness for the Lie groupoids we are working with, results as those provided in Proposition 2.2.4 and Theorem 2.2.2 in the groupoid category are clearly natural and easier to obtain by following the procedure we propose. Secondly, as we will see in next sections/chapters, our approach also provides a way to recover some algebraic/topological constructions underlying the Lie groupoid structure in a natural way. For instance, the total cohomology of the Bott-Shulman-Stasheff double complex as well as its 2-equivariant version can be obtained by using our techniques. The latter construction can be applied to compute the equivariant cohomology of certain toric symplectic stacks, see Example 2.4.1. Thirdly, our approach seems to be applicable to satisfactorily extend other topological and geometrical constructions derived from classical Morse theory to more general contexts, e.g. Novikov type inequalities for certain separated differentiable stacks as done by the thesis' author in [113].

2.3 Morse–Smale dynamics

Let us now adapt some notions of the Morse–Smale dynamics to the Lie groupoid setting. Our goal here is to define the stable and unstable Lie groupoids of a Morse Lie groupoid morphism as well as to study some of their elementary properties. For specific details regarding the classical notions the reader is recommended to visit [8, 13, 97] and references therein. Let $F : G \to \mathbb{R}$ be a Morse Lie groupoid morphism covering a smooth basic function $f : M \to \mathbb{R}$ and let $\eta^{(2)}$ denote a 2-metric on $G \rightrightarrows M$. Throughout this section, apart from assuming that $G \rightrightarrows M$ is proper, we will also assume that one of either G or M is compact. This automatically implies that the other one is compact as well. Under this additional assumption it is clear that the multiplicative gradient vector field ∇F is given by a pair of complete vector fields.

Remark 2.3.1. As consequence of Theorems 4.15 and 5.12 in [58] together with the Dictionary Lemmas from [17, s. 2.6], it follows that the compactness assumption for either M or G may be relaxed by requiring only that M/G is compact. This is enough to have globally defined gradient flows since ∇F is a multiplicative vector field. It is worth mentioning that this fact will be clarified in Section 2.5. However, along this section we assume the compactness of our Lie groupoid for simplicity.

Let $\widetilde{\Phi_\tau} : G \to G$ and $\Phi_\tau : M \to M$ denote the flows of the vector fields $-\nabla(s^*f)$ and $-\nabla f$, respectively. These are the so-called **descending flows**. As usual, if $x \in \mathrm{Crit}(f)$ then the **stable manifold** $W^s(x)$ and the **unstable manifold** $W^u(x)$ of f at x are respectively defined as

$$W^s(x) = \{y \in M : \lim_{\tau\to\infty} \Phi_\tau(y) = x\} \quad \text{and} \quad W^u(x) = \{y \in M : \lim_{\tau\to-\infty} \Phi_\tau(y) = x\}. \tag{2.4}$$

The stable and unstable manifolds of a critical arrow $g \in \mathrm{Crit}(F)$ are similarly defined by using the descending flow $\widetilde{\Phi_\tau}$. Let $S_\lambda \subseteq \mathrm{Crit}(f)$ denote the set formed by the orbits in M with same index λ. We may assume that S_λ consists of orbits with the same dimension, otherwise we split S_λ into components consisting of orbits with the same dimension. Hence, we may assume that S_λ is a manifold which, being saturated, yields a well-defined Lie groupoid $G_{S_\lambda} \rightrightarrows S_\lambda$ defined by the restriction of $G \rightrightarrows M$ to S_λ. It is important to observe that S_λ is a non-degenerate critical submanifold for f of index λ and as consequence of what we did before we may conclude that G_{S_λ} is also a non-degenerate critical submanifold for F with same index λ. Thus, as we have that every point in S_λ is a critical point for f we define the stable and unstable submanifolds of S_λ as the respective disjoint unions

$$W^s(S_\lambda) = \bigcup_{x\in S_\lambda} W^s(x) \qquad \text{and} \qquad W^u(S_\lambda) = \bigcup_{x\in S_\lambda} W^u(x).$$

The stable and unstable submanifolds $W^s(G_{S_\lambda})$ are $W^u(G_{S_\lambda})$ are defined in the same way since every arrow in G_{S_λ} is a critical arrow for F. As a consequence of the multiplicativity of $-\nabla F = (-\nabla(s^*f), -\nabla f)$ we obtain the following expected result.

Lemma 2.3.1. *The Lie groupoid structure of $G \rightrightarrows M$ can be naturally restricted to define two Lie groupoids $W^s(G_{S_\lambda}) \rightrightarrows W^s(S_\lambda)$ and $W^u(G_{S_\lambda}) \rightrightarrows W^u(S_\lambda)$.*

Proof. We will only show why the Lie groupoid structure may be well restricted for defining $W^s(G_{S_\lambda}) \rightrightarrows W^s(S_\lambda)$ since the other case follows analogously. This is carried out in the following steps. First, if $h \in W^s(G_{S_\lambda})$, then $h \in W^s(g)$ for some critical arrow $g \in G_{S_\lambda}$. Thus, the identity $s \circ \widetilde{\Phi_\tau} = \Phi_\tau \circ s$ implies that

$$\lim_{\tau\to\infty} \Phi_\tau(s(h)) = \lim_{\tau\to\infty} s(\widetilde{\Phi_\tau}(h)) = s\left(\lim_{\tau\to\infty} \widetilde{\Phi_\tau}(h)\right) = s(g).$$

This means that $s(h) \in W^s(s(g)) \subset W^s(S_\lambda)$ and therefore $s : W^s(G_{S_\lambda}) \to W^s(S_\lambda)$ is well restricted. As for each critical arrow $g \in G_{S_\lambda}$ we have that $t(g) \in S_\lambda$ is a critical point, then by arguing in the exactly same way with the identity $t \circ \widetilde{\Phi_\tau} = \Phi_\tau \circ t$ we get that $t : W^s(G_{S_\lambda}) \to W^s(S_\lambda)$ is well restricted. Now, let us consider the fibered product space $(W^s(G_{S_\lambda}))^{(2)}$ defined through s and t. If $(h_1, h_2) \in (W^s(G_{S_\lambda}))^{(2)}$ then there exists a pair of critical arrows $(g_1, g_2) \in G_{S_\lambda} \times G_{S_\lambda}$ such that $h_1 \in W^s(g_1)$ and $h_2 \in W^s(g_2)$. On the one hand, observe that $(g_1, g_2) \in G_{S_\lambda}^{(2)}$. Indeed, since $s(h_1) = t(h_2)$ and $(\widetilde{\Phi_\tau}, \Phi_\tau)$ is a Lie groupoid morphism we obtain

$$s(g_1) = s\left(\lim_{\tau\to\infty} \widetilde{\Phi_\tau}(h_1)\right) = \lim_{\tau\to\infty} \Phi_\tau(s(h_1)) = \lim_{\tau\to\infty} \Phi_\tau(t(h_2)) = t\left(\lim_{\tau\to\infty} \widetilde{\Phi_\tau}(h_2)\right) = t(g_2).$$

On the other hand, from the identity $\widetilde{\Phi_\tau} \circ m = m \circ (\widetilde{\Phi_\tau} \times \widetilde{\Phi_\tau})$ we get that

$$\lim_{\tau\to\infty} \widetilde{\Phi_\tau}(h_1 h_1) = \lim_{\tau\to\infty} m(\widetilde{\Phi_\tau}(h_1), \widetilde{\Phi_\tau}(h_2)) = m\left(\lim_{\tau\to\infty} \widetilde{\Phi_\tau}(h_1), \lim_{\tau\to\infty} \widetilde{\Phi_\tau}(h_2)\right) = g_1 g_2.$$

Given that we know that the composition of two composable critical arrows produces again a critical arrow we have that $h_1 h_2 \in W^s(g_1 g_2) \subset W^s(G_{S_\lambda})$. Hence, the composition map $m : (W^s(G_{S_\lambda}))^{(2)} \to W^s(G_{S_\lambda})$ is also well restricted. Finally, let us now consider the inversion map $i : G \to G$ and the unit map $u : M \to G$ of the Lie groupoid $G \rightrightarrows M$. As $(\widetilde{\Phi_\tau}, \Phi_\tau)$ is a Lie groupoid morphism we have that $\widetilde{\Phi_\tau} \circ i = i \circ \widetilde{\Phi_\tau}$ and $\widetilde{\Phi_\tau} \circ u = u \circ \Phi_\tau$. Recall that if g is a critical arrow, then g^{-1} is also a critical arrow and if x is a critical point, then 1_x is a critical arrow. So, by arguing as in the previous items we obtain that $i : W^s(G_{S_\lambda}) \to W^s(G_{S_\lambda})$ and $u : W^s(S_\lambda) \to W^s(G_{S_\lambda})$ are well restricted as well.

□

The Lie groupoids introduced in Lemma 2.3.1 will be respectively called **stable** and **unstable Lie groupoids** associated to the critical submanifold S_λ. Consider now the **endpoint maps** $l_0 : W^s(S_\lambda) \to S_\lambda$ and $u_0 : W^u(S_\lambda) \to S_\lambda$ which are respectively defined by

$$l_0(x) = \lim_{\tau\to\infty} \Phi_\tau(x) \qquad \text{and} \qquad u_0(x) = \lim_{\tau\to-\infty} \Phi_\tau(x). \tag{2.5}$$

It was shown for instance in [8, Prop. 3,2] that the endpoint maps (2.5) are smooth locally trivial fibrations. In order to state the groupoid analogue of this property, we need the following definition which can be found for instance in [80].

Definition 2.3.1. A **Lie groupoid fibration** is a Lie groupoid morphism $\phi^1 : G \to G'$ covering a surjective submersion $\phi^0 : M \to M'$ with the property that the map $\hat{\phi} : G \to G' \times_{M'} M$, defined by sending $g \mapsto (\phi^1(g), s(g))$, is a surjective submersion.

At the level of arrows, we can similarly define smooth endpoint maps $l_1 : W^s(G_{S_\lambda}) \to G_{S_\lambda}$ and $u_1 : W^u(G_{S_\lambda}) \to G_{S_\lambda}$ by using the descending flow $\widetilde{\Phi_\tau}$. Recall that we have the equality of indexes $\lambda(G_{S_\lambda}, F) = \lambda(S_\lambda, f) = \lambda$. Thus, as consequence of [8, Prop. 3,2] we obtain that u_1 and l_1 are smooth fiber bundles with fibers diffeomorphic to the disks D^λ and $D^{k-\lambda}$, respectively. Here $k = \operatorname{codim}(G_{S_\lambda}) = \operatorname{codim}(S_\lambda)$ seen as submanifolds of G and M, respectively. So, motivated by these facts and Lemma 2.3.1 one has the following result.

Proposition 2.3.1. *The endpoint maps* $l = (l_1, l_0) : (W^s(G_{S_\lambda}) \rightrightarrows W^s(S_\lambda)) \to (G_{S_\lambda} \rightrightarrows S_\lambda)$ *and* $u = (u_1, u_0) : (W^u(G_{S_\lambda}) \rightrightarrows W^u(S_\lambda)) \to (G_{S_\lambda} \rightrightarrows S_\lambda)$ *are Lie groupoid fibrations.*

Proof. On the one hand, the fact that both l and u define Lie groupoid morphisms follows by arguing exactly as we did in Lemma 2.3.1. Indeed, let us consider for instance the case of u. As the pair $(\widetilde{\Phi_\tau}, \Phi_\tau)$ determines a Lie groupoid morphism we have that

$$(s \circ u_1)(h) = s\left(\lim_{\tau \to -\infty} \widetilde{\Phi_\tau}(h)\right) = \lim_{\tau \to -\infty} s(\widetilde{\Phi_\tau}(h)) = \lim_{\tau \to -\infty} \Phi_\tau(s(h)) = (u_0 \circ s)(h).$$

We can analogously get that $t \circ u_1 = u_0 \circ t$. If $(h_1, h_2) \in (W^u(G_{S_\lambda}))^{(2)}$ then

$$\begin{aligned}(m \circ (u_1 \times u_1))(h_1, h_2) &= m\left(\lim_{\tau \to -\infty} \widetilde{\Phi_\tau}(h_1), \lim_{\tau \to -\infty} \widetilde{\Phi_\tau}(h_2)\right) = \lim_{\tau \to -\infty} (m \circ (\widetilde{\Phi_\tau} \times \widetilde{\Phi_\tau}))(h_1, h_2) \\ &= \lim_{\tau \to -\infty} (\widetilde{\Phi_\tau}(m(h_1, h_2)) = (u_1 \circ m)(h_1, h_2).\end{aligned}$$

On the other hand, let us consider the fiber product given by the diagram below

$$\begin{tikzcd} G_{S_\lambda} \times_{S_\lambda} W^u(S_\lambda) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & W^u(S_\lambda) \arrow[d, "u_0"] \\ G_{S_\lambda} \arrow[r, "s"'] & S_\lambda. \end{tikzcd}$$

We already know that $u_0 : W^u(S_\lambda) \to S_\lambda$ is a surjective submersion. It remains to check that the map $\hat{u} : W^u(G_{S_\lambda}) \to G_{S_\lambda} \times_{S_\lambda} W^u(S_\lambda)$, defined by mapping $h \mapsto (u_1(h), s(h))$, is a surjective submersion. Nevertheless, this fact follows directly since π_1 and π_2 are surjective submersions, $\pi_1 \circ \hat{u} = u_1$ and $\pi_2 \circ \hat{u} = s$, so that $\hat{u}$ is a surjective submersion because u_1 and s are so.

As expected, we can prove that l also defines a Lie groupoid fibration by arguing with similar computations. □

It is clear that the canonical projections $\pi_1^- : \nu_-(G_{S_\lambda}) \to G_{S_\lambda}$ and $\pi_0^- : \nu_-(S_\lambda) \to S_\lambda$ allow us to define a groupoid fibration from $\nu_-(G_{S_\lambda}) \rightrightarrows \nu_-(S_\lambda)$ onto $G_{S_\lambda} \rightrightarrows S_\lambda$. Obviously, for the positive normal groupoid a similar fibration π^+ onto $G_{S_\lambda} \rightrightarrows S_\lambda$ can be defined. Therefore, the naturality of the previous result is behind the following interesting fact, which can be thought of as the groupoid version of the so-called stable/unstable manifold theorem, compare for instance [8, Thm A.9] or else [14, Thm 4,15].

Theorem 2.3.1 (Stable/unstable groupoid theorem)**.** *There exists a Lie groupoid isomorphism from a full Lie groupoid open neighborhood of the zero section $G_{S_\lambda} \rightrightarrows S_\lambda$ in $\nu_-(G_{S_\lambda}) \rightrightarrows \nu_-(S_\lambda)$ and a full Lie groupoid open neighborhood of $G_{S_\lambda} \rightrightarrows S_\lambda$ in $W^u(G_{S_\lambda}) \rightrightarrows W^u(S_\lambda)$ which intertwines the groupoid fibrations π^- and u. Such an isomorphism behaves as the identity over $G_{S_\lambda} \rightrightarrows S_\lambda$. Analogously, same assertion holds true between the groupoid fibrations π^+ and l onto $G_{S_\lambda} \rightrightarrows S_\lambda$.*

Proof. First of all, it follows that around a base point $y \in S_\lambda$ there are coordinates in an open neighborhood $U \subset M$ such that $U \cong S_\lambda \times \nu_+(S_\lambda)_s \times \nu_-(S_\lambda)_s$ so that each element $z \in U$ may be rewritten as $z = (z_0, z_+, z_-)$, see [8, Appx. A.3]. We focus on proving the statement for π^- and u since the other asserted case follows in a similar fashion. Pick $y \in S_\lambda$ and $v_- \in \nu_+(S_\lambda)_{y'}$. As consequence of [8, Thm A.9] (see also [14, Thm 4,15]), we know that there exists a unique integral curve $c(\tau) = \Phi_{y'}(\tau) := \Phi_\tau(y')$ of $-\nabla f$ starting at $y' \in M$ such that $c(0)_- = v_-$ and $\lim_{\tau \to -\infty} c(\tau) = \lim_{\tau \to -\infty} \Phi_\tau(y') = y$. Furthermore, after shrinking U if necessary, for varying (y, v_-) the map ψ^0 sending $y \mapsto c(0)_0$ defines a diffeomorphism between an open neighborhood U of the zero section S_λ in $\nu_-(S_\lambda)$ and an open neighborhood V of S_λ in $W^u(S_\lambda)$ which intertwines the projection π_0^- and endpoint map u_0. Again, after shrinking U if necessary and by using instead the descending flow of $-\nabla F$, we may similarly define around $g \in G_{S_\lambda}$ an open neighborhood $\tilde{U}$ of the zero section G_{S_λ} in $\nu_-(G_{S_\lambda})$ of the form $\tilde{U} = s^{-1}(U) \cap t^{-1}(U) \cong G_{S_\lambda} \times \nu_+(G_{S_\lambda})_g \times \nu_-(G_{S_\lambda})_g$ enjoying of the following properties. Each element $\tilde{z} \in \tilde{U}$ may be rewritten as $\tilde{z} = (\tilde{z}_0, \tilde{z}_+, \tilde{z}_-)$ and there is a uniquely defined map ψ^1 sending $g \mapsto \tilde{c}(0)_0$ which defines a diffeomorphism between $\tilde{U}$ and an open neighborhood $\tilde{V}$ of G_{S_λ} in $W^u(G_{S_\lambda})$ that intertwines the projection π_1^- and endpoint map u_1.

Let us check that $\psi = (\psi^1, \psi^0)$ defines a Lie groupoid morphism over the open Lie groupoid neighborhood $\tilde{U} \rightrightarrows U$ of $G_{S_\lambda} \rightrightarrows S_\lambda$. This will be consequence of having that ∇F is a multiplicative vector field. Take $g \in \tilde{U}$, $\tilde{v}_- \in \nu_-(G_{S_\lambda})_{g'}$ and $\tilde{c}(\tau) = \widetilde{\Phi}_{g'}(\tau) = \widetilde{\Phi}_\tau(g')$ as described above. We know that $s \circ \widetilde{\Phi}_\tau = \Phi_\tau \circ s$ which in turn yields the identity $s \circ \widetilde{\Phi}_h = \Phi_{s(h)}$ for all $h \in G$. The latter formula implies that $s(g) = \lim\limits_{\tau \to -\infty} \Phi_{s(g')}(\tau) = \lim\limits_{\tau \to -\infty} \Phi_\tau(s(g'))$ and $\overline{ds}(g')(\tilde{v}_-) = c(0)_-$ where in this case we are denoting $c(\tau) = \Phi_{s(g')}(\tau) = \Phi_\tau(s(g'))$. Thus, by the uniqueness in [8, Thm A.9] it follows that

$$(s \circ \psi^1)(g) = s(\tilde{c}(0)_0) = (s \circ \widetilde{\Phi}_{g'})(0)_0 = \Phi_{s(g')}(0)_0 = c(0)_0 = (\psi^0 \circ s)(g).$$

The identity $t \circ \psi^1 = \psi^0 \circ t$ can be verified in a similar manner. Let us now consider pairs $g \in \tilde{U}$, $\tilde{v}_- \in \nu_-(G_{S_\lambda})_{g'}$ and $h \in \tilde{U}$, $\tilde{w}_- \in \nu_-(G_{S_\lambda})_{h'}$ such that $s(g) = t(h)$, $s(g') = t(h')$, $\overline{ds}(g')(\tilde{v}_-) = \overline{dt}(h')(\tilde{w}_-)$ and $\tilde{a}(\tau) = \widetilde{\Phi}_{g'}(\tau) = \widetilde{\Phi}_\tau(g')$ with $\tilde{b}(\tau) = \widetilde{\Phi}_{h'}(\tau) = \widetilde{\Phi}_\tau(h')$ as described above. It is simple to verify that the equality $\widetilde{\Phi}_\tau \circ m = m \circ (\widetilde{\Phi}_\tau \times \widetilde{\Phi}_\tau)$ implies the formula $m \circ (\widetilde{\Phi}_{h_1} \times \widetilde{\Phi}_{h_2}) = \widetilde{\Phi}_{h_1 h_2}$ for all $(h_1, h_2) \in G^{(2)}$. It follows that $gh = \lim\limits_{\tau \to -\infty} \tilde{\Phi}_{g'h'}(\tau) = \lim\limits_{\tau \to -\infty} \tilde{\Phi}_\tau(g'h')$ and $\overline{dm}(g', h')(\tilde{v}_-, \tilde{w}_-) = \tilde{a}(0)_-$ with $\tilde{c}(\tau) = \tilde{\Phi}_{g'h'}(\tau) = \tilde{\Phi}_\tau(g'h')$. Hence, again by the uniqueness in [8, Thm A.9] we obtain that

$$m(\psi^1(g), \psi^1(h)) = m(\tilde{a}(0)_0, \tilde{b}(0)_0) = m(\widetilde{\Phi}_{g'}(0), \widetilde{\Phi}_{h'}(0))_0 = \widetilde{\Phi}_{h_1 h_2}(0)_0 = \tilde{c}(0)_0 = \psi^1(gh).$$

Observe that ψ is the identity over $G_{S_\lambda} \rightrightarrows S_\lambda$ since the descending flows fix critical points. The proof is completed by observing that in a proper groupoid every groupoid neighborhood contains a full groupoid neighborhood, see for instance [40, Lem. 5.3].

□

We consider now the notion of moduli space of gradient flow lines in the Lie groupoid context. Some of the ideas stated below will be used to construct a double complex which in turn will allow us to recover the total cohomology of the Bott–Shulman–Stasheff double complex of the Lie groupoid we are working with. Note that there exists a natural action of $\mathbb{R}$ on $W^u(S_\lambda)$ (resp. on $W^s(S_\lambda)$) defined by $r \cdot y = \Phi_r(y)$ for all $r \in \mathbb{R}$. This is well defined since if $y \in W^u(S_\lambda)$ then we have that

$$\lim_{\tau \to -\infty} \Phi_\tau(r \cdot y) = \lim_{\tau \to -\infty} \Phi_\tau(\Phi_r(y)) = \lim_{\tau \to -\infty} \Phi_{\tau + r}(y) \in W^u(S_\lambda). \tag{2.6}$$

In particular, it is simple to see that these actions induce a well defined free action of $\mathbb{R}$ on any intersection $W^u(S_{\lambda_i}) \cap W^s(S_{\lambda_j})$, for $\lambda_i \neq \lambda_j$. We will refer to this action as **action by flow translation**. The **moduli space of gradient flow lines** in M associated to the non-degenerate critical saturated submanifolds S_{λ_i} and S_{λ_j} is defined as the quotient space obtained from the action by flow translation:

$$\mathcal{M}(S_{\lambda_i}, S_{\lambda_j}) := (W^u(S_{\lambda_i}) \cap W^s(S_{\lambda_j}))/\mathbb{R}.$$

The moduli space of gradient flow lines $\mathcal{M}(G_{S_{\lambda_i}}, G_{S_{\lambda_j}})$ is equally defined by using instead the action induced by the descending flow $\widetilde{\Phi_\tau}$. Thus, motivated by the Morse–Bott transversality condition (see [8, 13]), we set up the following definition.

Definition 2.3.2. A Morse Lie groupoid morphism $F : G \to \mathbb{R}$ which covers a basic function $f : M \to \mathbb{R}$ is said to satisfy the **Morse–Smale transversality** condition with respect to a 2-metric on $G \rightrightarrows M$ if for any two critical non-degenerate saturated submanifolds S_{λ_i} and S_{λ_j} with respect to f we have that $W^u(y) \pitchfork W^s(S_{\lambda_j})$ for all $y \in S_{\lambda_i}$.

It is important to notice that, unlike the classical Morse case, in the Morse–Bott case it is not always possible to perturb a Riemannian metric to make a given Morse-Bott function satisfy the Morse-Bott-Smale transversality condition. See [71, Rmk. 2.4] for an interesting counterexample. More importantly, such a condition is not always satisfied for G-invariant Morse functions; compare [71, s. 5]. As a consequence, not every Morse Lie groupoid morphism satisfies the Morse–Smale transversality condition in general. As it is argued in [8], when assuming the Morse–Smale transversality condition from Definition 2.3.2, we get that the spaces $W^u(S_{\lambda_i}) \cap W^s(S_{\lambda_j})$ and $\mathcal{M}(S_{\lambda_i}, S_{\lambda_j})$ are smooth manifolds since the action by flow translation is free and proper. The endpoint maps (2.5) descend to the moduli spaces yielding new endpoint maps $(l_0)^i_j : \mathcal{M}(S_{\lambda_i}, S_{\lambda_j}) \to S_{\lambda_j}$ and $(u_0)^i_j : \mathcal{M}(S_{\lambda_i}, S_{\lambda_j}) \to S_{\lambda_i}$ given respectively by

$$(l_0)^i_j([x]) = l_0(x) \qquad \text{and} \qquad (u_0)^i_j([x]) = u_0(x), \tag{2.7}$$

One can see that these maps are well defined in a similar manner as in (2.6). Also, the endpoint maps $(u_0)^i_j$ on the moduli spaces are locally trivial fibrations, see [8].

Lemma 2.3.2. *Let $F : G \to \mathbb{R}$ be a Morse Lie groupoid morphism covering a basic function $f : M \to \mathbb{R}$ and having the Morse-Smale transversality property. Then the non-degenerate critical submanifolds $G_{S_{\lambda_i}}$ and $G_{S_{\lambda_j}}$ satisfy the Morse-Smale transversality condition with respect to F.*

Proof. Let $y \xleftarrow{g} x$ be an arrow in $W^u(G_{S_{\lambda_i}})$ such that $g \in W^u(h) \cap W^s(G_{S_{\lambda_j}})$. Given that $s : G \to M$ is a submersion we have that it is transverse to any submanifold in M so that, in particular, we obtain that $s \pitchfork W^u(s(h))$ and $s \pitchfork W^s(S_{\lambda_j})$. Since $x \in W^u(s(h)) \cap W^s(S_{\lambda_j})$ the Morse-Smale transversality condition with respect to f gives us

$$\begin{aligned} T_g W^u(h) + T_g W^s(G_{S_{\lambda_j}}) &= T_g(s^{-1}(W^u(s(h)))) + T_g(s^{-1}(W^s(S_{\lambda_j}))) \\ &= ds_g^{-1}(T_x W^u(s(h))) + ds_g^{-1}(T_x W^s(S_{\lambda_j})) \\ &= ds_g^{-1}(T_x W^u(s(h)) + T_x W^s(S_{\lambda_j})) \\ &= ds_g^{-1}(T_x M) = T_g G. \end{aligned}$$

So, the result follows. □

The previous fact implies that we can analogously define smooth manifolds $W^u(G_{S_{\lambda_i}}) \cap W^s(G_{S_{\lambda_j}})$ and $\mathcal{M}(G_{S_{\lambda_i}}, G_{S_{\lambda_j}})$ and smooth endpoint maps $(l_1)^i_j : \mathcal{M}(G_{S_{\lambda_i}}, G_{S_{\lambda_j}}) \to G_{S_{\lambda_j}}$ and $(u_1)^i_j : \mathcal{M}(G_{S_{\lambda_i}}, G_{S_{\lambda_j}}) \to G_{S_{\lambda_i}}$, so that each $(u_1)^i_j$ has the structure of a locally trivial fibration.

It is simple to check that by the nature of the structure of our stable and unstable Lie groupoids we may naturally define a Lie groupoid $W^u(G_{S_{\lambda_i}}) \cap W^s(G_{S_{\lambda_j}}) \rightrightarrows W^u(S_{\lambda_i}) \cap W^s(S_{\lambda_j})$. More importantly:

Proposition 2.3.2 (Groupoid of gradient flow lines)**.** *There exists a unique structure of Lie groupoid $\mathcal{M}(G_{S_{\lambda_i}}, G_{S_{\lambda_j}}) \rightrightarrows \mathcal{M}(S_{\lambda_i}, S_{\lambda_j})$ on the moduli spaces of gradient lines, making the canonical projection $W^u(G_{S_{\lambda_i}}) \cap W^s(G_{S_{\lambda_j}}) \to \mathcal{M}(G_{S_{\lambda_i}}, G_{S_{\lambda_j}})$ a Lie groupoid fibration.*

Proof. The action of $\mathbb{R}$ on $W^u(G_{S_{\lambda_i}}) \cap W^s(G_{S_{\lambda_j}}) \rightrightarrows W^u(S_{\lambda_i}) \cap W^s(S_{\lambda_j})$ is given by Lie groupoid automorphisms. Since this action is free and proper, then the quotient inherits a Lie groupoid structure with the desired properties. In Proposition 4.1.1 we will sketch the proof of a more general version of the latter assertion.

□

As an immediate application of Proposition 2.3.1 we easily get:

Corollary 2.3.1. *The pair of endpoint maps $u^i_j = ((u_1)^i_j, (u_0)^i_j) : (\mathcal{M}(G_{S_{\lambda_i}}, G_{S_{\lambda_j}}) \rightrightarrows \mathcal{M}(S_{\lambda_i}, S_{\lambda_j})) \to (G_{S_{\lambda_i}} \rightrightarrows S_{\lambda_i})$ defines a Lie groupoid fibration.*

Note that because of the well definition of index we have that $W^u(S_{\lambda_i}) \cap W^s(S_{\lambda_j})$ does not contain critical points of f since $S_{\lambda_i} \cap S_{\lambda_j} = \emptyset$ for $\lambda_i \neq \lambda_j$. This in particular implies that the action by flow translation on this space is free. Namely, the latter statement follows from the fact that a point is a singularity of a vector field if and only if its flow fixes such a point. In particular, the gradient vector field of f is nonzero at the regular points of f which implies that its flow does not fix such points. Same conclusion can be obtained at the level of arrows. This key observation allows us to define the composition map of the Lie groupoid from Proposition 2.3.2 explicitly. Indeed, the source and target maps are respectively defined by setting $\overline{s} : \mathcal{M}(G_{S_{\lambda_i}}, G_{S_{\lambda_j}}) \to \mathcal{M}(S_{\lambda_i}, S_{\lambda_j})$ as $\overline{s}[g] = [s(g)]$ and $\overline{t} : \mathcal{M}(G_{S_{\lambda_i}}, G_{S_{\lambda_j}}) \to \mathcal{M}(S_{\lambda_i}, S_{\lambda_j})$ as $\overline{t}[g] = [t(g)]$. They are well-defined since our gradient vector field is multiplicative. Furthermore, if we take $[g]$ and $[h]$ in $\mathcal{M}(G_{S_{\lambda_i}}, G_{S_{\lambda_j}})$ such that $\overline{s}[g] = \overline{t}[h]$ then there exists $r \in \mathbb{R}$ such that $\Phi_r(s(g)) = t(h)$ which is equivalent to have $s(\widetilde{\Phi_r}(g)) = t(h)$, so that we can multiply $m(\widetilde{\Phi_r}(g), h) = \widetilde{\Phi_r}(g)h$. Thus, we define $\overline{m} : \mathcal{M}(G_{S_{\lambda_i}}, G_{S_{\lambda_j}})^{(2)} \to \mathcal{M}(G_{S_{\lambda_i}}, G_{S_{\lambda_j}})$ as $\overline{m}([g],[h]) := [\widetilde{\Phi_r}(g)h]$. This description may be used in order to "compactify" the Lie groupoid structure over each $\mathcal{M}(G_{S_{\lambda_i}}, G_{S_{\lambda_j}})$. Let us briefly sketch the proof of this assertion without going into too much details. As it can be viewed for instance in [8], the Morse–Smale transversality assumption allows to show that there exists a way for constructing a compactification of these moduli spaces of gradient flow lines. It can be done by applying a series of fiber products with respect to the endpoint maps u and l. Therefore, as consequence of Proposition 2.3.2, [8, Lem. 3.3] and [37, Prop. 4.4.1] we may extend the Lie groupoid structure previously described to the respective compactifications. Here it is important to have in mind that u and l define Lie groupoid morphisms where u is composed by locally trivial fibrations.

2.3.1 The double complex of a Morse Lie groupoid morphism

Recall that given a Morse function $f : M \to \mathbb{R}$, the Morse-Smale-Witten complex of the pair (M, f) is a complex whose homology computes the singular homology of the manifold M, see Subsection 1.1. In the setting of Morse-Bott theory, there are several versions of the Morse-Smale-Witten complex which compute either the homology or the de Rham cohomology of the manifold [8, 13, 50]. The main aim of this subsection is to introduce the groupoid version of the Morse complex. Namely, given a Morse Lie groupoid morphism $F : G \to \mathbb{R}$ we construct a double cochain complex whose total cohomology recovers the total cohomology of the Bott–Shulman–Stasheff double complex associated to G. Our construction relies on the Austin–Braam's version of the Morse-Bott complex [8], the one that was sketched in Subsection 1.1.1.

Throughout this subsection a Lie groupoid G is denoted as $G^{(1)} \rightrightarrows G^{(0)}$. Let $F_1 : (G^{(1)} \rightrightarrows G^{(0)}) \to (\mathbb{R} \rightrightarrows \mathbb{R})$ be a Morse Lie groupoid morphism which is induced by a smooth basic function $F_0 : G^{(0)} \to \mathbb{R}$. We assume that $G^{(1)} \rightrightarrows G^{(0)}$ is proper and either $G^{(1)}$ or $G^{(0)}$ is compact, compare Remark 2.3.1. Let us consider the simplicial terminology introduced in Subsection 1.3. The nerve $G^{(\bullet)}$ of $G^{(1)} \rightrightarrows G^{(0)}$ is depicted as

$$\begin{tikzcd} G^{(\bullet)} : \cdots \arrow[r, Rrightarrow] & G^{(n)} \arrow[r, Rrightarrow] & \cdots \arrow[r, Rrightarrow] & G^{(2)} \arrow[r, Rrightarrow] & G^{(1)} \arrow[r, shift left] \arrow[r, shift right] & G^{(0)}. \end{tikzcd}$$

By our initial assumptions it follows that each manifold of composable arrows $G^{(n)}$ is compact since it is closed and is contained inside $G^{(1)} \times \cdots \times G^{(1)}$. From [40] we know that the Lie groupoid $G^{(1)} \rightrightarrows G^{(0)}$ admits an n-metric $\eta^{(n)}$ on $G^{(n)}$. It is important to remember that we can push $\eta^{(n)}$ forward with the different face maps $d^n_k : G^{(n)} \to G^{(n-1)}$ to define an $(n-1)$-metric $\eta^{(n-1)}$ on $G^{(n-1)}$ where $\eta^{(n-1)} = (d^n_k)_*\eta^{(n)} = (d^n_{k'})_*\eta^{(n)}$ for all k, k' and every $d^n_k : (G^{(n)}, \eta^{(n)}) \to (G^{(n-1)}, \eta^{(n-1)})$ becomes a Riemannian submersion. One can use this process to obtain r-metrics $\eta^{(r)}$ on $G^{(r)}$ in such a way that $d^r_k : (G^{(r)}, \eta^{(r)}) \to (G^{(r-1)}, \eta^{(r-1)})$ is a Riemannian submersion for every $0 \leq r \leq n-1$.

Applying the nerve functor to $F_1 : G^{(1)} \to \mathbb{R}$ yields a simplicial function $F_\bullet = (F_n)_{n\in\mathbb{N}}$ between the nerves $G^{(\bullet)} \to \mathbb{R}$:

$$\begin{tikzcd} F_\bullet : \cdots \arrow[r, Rrightarrow] & G^{(n)} \arrow[r, Rrightarrow] \arrow[d, "F_n"'] & \cdots \arrow[r, Rrightarrow] \arrow[d] & G^{(2)} \arrow[r, Rrightarrow] \arrow[d, "F_2"'] & G^{(1)} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "F_1"'] & G^{(0)} \arrow[d, "F_0"] \\ \cdots \arrow[r] & \mathbb{R} \arrow[r] & \cdots \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R}, \end{tikzcd}$$

which, for $n \geq 2$, is inductively defined as $F_n = (d^n_k)^* F_{n-1} = (d^n_{k'})^* F_{n-1}$ for all k, k'. These smooth functions are well defined since F_1 is a Lie groupoid morphism and the simplicial identities (1.8) hold true. Recall that, in our case, the set of critical points of F_0 is given by a disjoint union of finite compact and connected Lie groupoid orbits $\mathrm{Crit}(F_0) = \dot{\bigcup} \mathcal{O}$ since $G^{(0)}$ is compact and $G^{(1)} \rightrightarrows G^{(0)}$ is proper. Therefore:

Lemma 2.3.3. *There exists a sub-nerve $G_i^{(\bullet)}$ of $G^{(\bullet)}$ formed by non-degenerate critical submanifolds of index i for $F_\bullet$ of the form:*

$$\begin{tikzcd} G_i^{(\bullet)} : \cdots \arrow[r, Rrightarrow] & G_i^{(n)} \arrow[r, Rrightarrow] & \cdots \arrow[r, Rrightarrow] & G_i^{(2)} \arrow[r, Rrightarrow] & G_i^{(1)} \arrow[r, shift left] \arrow[r, shift right] & G_i^{(0)}, \end{tikzcd}$$

where $G_i^{(n)} = \dot{\bigcup}\{G_{\mathcal{O}}^{(n)} \subset \mathrm{Crit}(F_n) : \mathrm{index}(F_n, G_{\mathcal{O}}^{(n)}) = i\}$ and $G_{\mathcal{O}}^{(n)}$ denotes the manifold of n-composable arrows of $G_{\mathcal{O}} \rightrightarrows \mathcal{O}$.

Proof. This follows by applying simple arguments from the previous sections since F_1 is a Morse Lie groupoid morphism. Namely, $G_i^{(0)}$ is a saturated submanifold with $G_i^{(1)} = s^{-1}(G_i^{(0)}) = t^{-1}(G_i^{(0)})$ and all the face maps of $G^{(\bullet)}$ are Riemannian submersions verifying the simplicial identities (1.8). $\square$

Let us now consider the collection of vector fields $-\nabla F_\bullet = (-\nabla F_n)_{n\in\mathbb{N}}$ where $-\nabla F_n$ denotes the negative gradient vector field of F_n on $G^{(n)}$ with respect to $\eta^{(n)}$. As the face maps $d_k^n : G^{(n)} \to G^{(n-1)}$ are Riemannian submersions we actually have that $-\nabla F_\bullet$ defines a simplicial vector field on the nerve $G^{(\bullet)}$ since $\eta^{(n-1)} = (d_k^n)_* \eta^{(n)}$ so that $d(d_k^n) \circ \nabla F_n = \nabla F_{n-1} \circ d_k^n$. It turns out that the collection of descending flows $\Phi_\tau^n : G^{(n)} \to G^{(n)}$, which are defined for all $\tau \in \mathbb{R}$ since $G^{(n)}$ is compact, verifies the relations

$$d_k^n \circ \Phi_\tau^n = \Phi_\tau^{n-1} \circ d_k^n, \qquad k = 0, \cdots, n \quad \text{and} \quad \forall \tau \in \mathbb{R}, \tag{2.8}$$

thus obtaining a simplicial automorphism $\Phi_\tau^\bullet : G^{(\bullet)} \to G^{(\bullet)}$ for every $\tau \in \mathbb{R}$. The collection of smooth endpoint maps associated to the collection of stable and unstable submanifolds $W^u(G_i^{(n)})$ and $W^s(G_i^{(n)})$ will be denoted by $u_i(n) : W^u(G_i^{(n)}) \to G_i^{(n)}$ and $l_i(n) : W^s(G_i^{(n)}) \to G_i^{(n)}$. Recall that these maps are locally trivial fiber bundles respectively defined by sending $A \mapsto \lim_{\tau \to -\infty} \Phi_\tau^n(A)$ and $A \mapsto \lim_{\tau \to \infty} \Phi_\tau^n(A)$. It follows directly from Identities (2.8) that

$$d_k^n \circ u_i(n) = u_i(n-1) \circ d_k^n \quad \text{and} \quad d_k^n \circ l_i(n) = l_i(n-1) \circ d_k^n, \tag{2.9}$$

for all $k = 0, \cdots, n$. For indexes i and j we can consider again the moduli spaces of gradient flow lines

$$\mathcal{M}^n(G_i^{(n)}, G_j^{(n)}) = W^u(G_i^{(n)}) \cap W^s(G_j^{(n)})/\mathbb{R},$$

which are the quotient spaces defined through the action by flow translation.

Lemma 2.3.4. *The simplicial structure of the nerve $G^{(\bullet)}$ may be restricted to define the stable and unstable sub-nerves $W^s(G_i^{(\bullet)})$ and $W^u(G_i^{(\bullet)})$ of the sub-nerve $G_i^{(\bullet)}$ and a topological sub-nerve structure $\mathcal{M}^\bullet(G_i^{(\bullet)}, G_j^{(\bullet)})$ between the moduli spaces of gradient flow lines.*

Proof. Note that the first statement follows by flowing to ∞ and $-\infty$ at both sides of Equations (2.8) so that $d_k^n : W^s(G_i^{(n)}) \to W^s(G_i^{(n-1)})$ and $d_k^n : W^u(G_i^{(n)}) \to W^u(G_i^{(n-1)})$ are well restricted. Furthermore, the maps $\overline{d_k^n} : \mathcal{M}^n(G_i^{(n)}, G_j^{(n)}) \to \mathcal{M}^{n-1}(G_i^{(n-1)}, G_j^{(n-1)})$ defined by sending $[A]_n \mapsto [d_k^n(A)]_{n-1}$ are well defined, open and surjective since d_k^n are so. ☐

We are interested in inducing well behaved structures of smooth manifold and orientability over our moduli spaces of gradient flow lines. To obtain this we require the following analogue of the Austin-Braam assumption, see Subsection 1.1.1.

Assumption 2.3.1. i. $\mathcal{M}^0(G_i^{(0)}, G_j^{(0)}) = \emptyset$ if $i \leq j$ (F_0 is weakly self-indexing).

ii. For all i, j and $x \in G_i^{(0)}$ we have that $W_i^u(x) := u_i(0)^{-1}(x)$ intersects $W^s(G_j^{(0)})$ transversally:

$$W_i^u(x) \pitchfork W^s(G_j^{(0)}).$$

iii. Both the critical submanifolds $G_i^{(0)}$ and the negative normal bundles $\nu_-(G_i^{(0)})$ are orientable for all i.

iv. $G^{(n)}$ is orientable for all $n \in \mathbb{N}$.

Although condition iv. from the previous assumption may seem to be somewhat restrictive there are many cases where such a requirement can be achieved. For instance, if $G^{(0)}$ is orientable then every manifold conforming the nerve of the unit groupoid, the pair groupoid, the action groupoid defined through a smooth action of a Lie group on $G^{(0)}$, and étale Lie groupoids over $G^{(0)}$, is orientable.

Lemma 2.3.5. *Conditions* i.-ii.-iii. *from Assumption 2.3.1 are satisfied at every level of the nerve configuration.*

Proof. First, by Lemma 2.3.4 and proceeding by induction over n it is simple to check that $\mathcal{M}^{n-1}(G_i^{(n-1)}, G_j^{(n-1)}) = \emptyset$ if $i \leq j$ implies that $\mathcal{M}^n(G_i^{(n)}, G_j^{(n)}) = \emptyset$ if $i \leq j$. The step $n = 1$ is consequence of i. and the well definition of $\overline{d_k^1}$.

Proceeding again by induction over n, we may prove that if the Morse–Smale transversality condition ii. holds true at the level $n-1$ then it also holds true at the level n. The case $n = 1$ was argued in Lemma 2.3.2. The inductive cases follow by arguing in the exactly same way but with the submersions d_k^n.

Finally, let us now look at the orientability requirements iii.. On the one hand, by the Transverse Submanifold Theorem, it is well known that because of conditions iii. and iv. together the fact that d_k^n are submersions with $G_i^{(n)} = (d_k^n)^{-1}(G_i^{(n-1)})$, it follows that if $G_i^{(n-1)}$ is orientable then so is $G_i^{(n)}$. The induced orientation on $G_i^{(n)}$ is the one obtained from the orientability of $G^{(n)}$, $G^{(n-1)}$ and $G_i^{(n-1)}$. On the other hand, recall that in Lemma 2.2.3 we proved that the Lie groupoid structure of $\nu(G_i^{(1)}) \rightrightarrows \nu(G_i^{(0)})$ can be well restricted to define a Lie groupoid between $\nu_-(G_i^{(1)}) \rightrightarrows \nu_-(G_i^{(0)})$. To do so it was mainly used the fact that the structural maps of our Lie groupoid are Riemannian submersions, the simplicial identities (1.8), and the relation between the Hessian forms of F_1 and F_0. In consequence, by using Lemma 2.3.3 we may easily extend some of these arguments to construct a subnerve $\nu_-(G_i^{(\bullet)})$ of $\nu_-(G^{(\bullet)})$ where the face maps at every level are the collection of fiberwise isomorphisms $\overline{d(d_k^n)}$. Therefore, proceeding again by induction over n we will have that if $\nu_-(G_i^{(n-1)})$ is orientable, then so is $\nu_-(G_i^{(n)})$. Indeed, by following [54, c. VII] and [75, c. 8], if $\iota_x : \nu_-(G_i^{(0)})_x \hookrightarrow \nu_-(G_i^{(0)})$ denotes the canonical inclusion of each fiber and ω is an i-form on $\nu_-(G_i^{(0)})$ inducing orientations $\iota_x^*\omega_x$ at every fiber $\nu_-(G_i^{(0)})_x$, then when pulling $\iota_x^*\omega_x$ back by the isomorphism $\overline{ds}_g : \nu_-(G_i^{(1)})_g \to \nu_-(G_i^{(0)})_x$ we get orientations $(\iota_x \circ \overline{ds}_g)^*\omega_x = (\overline{ds} \circ \iota_g)^*\omega_x$ for each fiber $\iota_g : \nu_-(G_i^{(1)})_g \hookrightarrow \nu_-(G_i^{(1)})$ so that $\overline{ds}^*\omega$ is an i-form inducing an orientation on $\nu_-(G_i^{(1)})$. Hence, the other orientations are inductively obtained by using any of the fiberwise isomorphisms $\overline{d(d_k^n)}$. $\square$

The first important consequence of the previous result is that, because of ii., for every $n \in \mathbb{N}$ the moduli space of gradient flow lines $\mathcal{M}^n(G_i^{(n)}, G_j^{(n)})$ is a smooth manifold where the new endpoint maps

$$u_j^i(n) : \mathcal{M}^n(G_i^{(n)}, G_j^{(n)}) \to G_i^{(n)} \quad \text{and} \quad l_j^i(n) : \mathcal{M}^n(G_i^{(n)}, G_j^{(n)}) \to G_j^{(n)} \tag{2.10}$$

given by $u_j^i(n)([A]_n) := u_i(n)(A)$ and $l_j^i(n)([A]_n) := l_j(n)(A)$, are well defined smooth maps such that $u_j^i(n)$ has the structure of locally trivial bundle; see [8]. Therefore, as the action by flow translations is free and proper we get that the maps $\overline{d_k^n}$ are smooth and from Identities (2.9) we obtain that the following diagrams are commutative

$$\begin{tikzcd}
\mathcal{M}^n(G_i^{(n)}, G_j^{(n)}) \arrow[r, "\overline{d_k^n}"] \arrow[d, "u_j^i(n)"'] & \mathcal{M}^{n-1}(G_i^{(n-1)}, G_j^{(n-1)}) \arrow[d, "u_j^i(n-1)"] \\
G_i^{(n)} \arrow[r, "d_k^n"'] & G_i^{(n-1)}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{M}^n(G_i^{(n)}, G_j^{(n)}) \arrow[r, "\overline{d_k^n}"] \arrow[d, "l_j^i(n)"'] & \mathcal{M}^{n-1}(G_i^{(n-1)}, G_j^{(n-1)}) \arrow[d, "l_j^i(n-1)"] \\
G_j^{(n)} \arrow[r, "d_k^n"'] & G_j^{(n-1)}
\end{tikzcd}$$

so that we have the commutative identities

$$d_k^n \circ u_j^i(n) = u_j^i(n-1) \circ \overline{d_k^n} \quad \text{and} \quad d_k^n \circ l_j^i(n) = l_j^i(n-1) \circ \overline{d_k^n}. \tag{2.11}$$

In other words, we have obtained simplicial smooth maps

$$u^i_j(\bullet) : \mathcal{M}^\bullet(G_i^{(\bullet)}, G_j^{(\bullet)}) \to G_i^{(\bullet)} \text{ and } l^i_j(\bullet) : \mathcal{M}^\bullet(G_i^{(\bullet)}, G_j^{(\bullet)}) \to G_j^{(\bullet)} \tag{2.12}$$

with $u^i_j(\bullet)$ a locally trivial simplicial fibration.

Let us define our double cochain complex. For that, we will define maps which are the composition of the pullback operation followed by the integration along the fibers via (2.12). For each $n \in \mathbb{N}$ we set $C^{i,j}(G^{(n)}) := \Omega^j(G_i^{(n)})$ and define the operator $\partial^n_r : C^{i,j}(G^{(n)}) \to C^{i+r,j-r+1}(G^{(n)})$ as

$$\partial^n_r(\omega) = \begin{cases} d\omega & \text{if} \quad r = 0 \\ (-1)^j (u_i^{i+r}(n))_* (l_i^{i+r}(n))^*(\omega) & \text{otherwise,} \end{cases}$$

where $(u_i^{i+r}(n))_*$ is integration along the fiber of the bundle (2.10). We also set

$$C^p(G^{(n)}) := \bigoplus_{i+j=p} C^{i,j}(G^{(n)}) = \bigoplus_{i+j=p} \Omega^j(G_i^{(n)}) \quad \text{with} \quad \partial^n = \sum \partial^n_i. \tag{2.13}$$

As it was shown for the classical case in [8], the operator ∂^n is a boundary operator, i.e., $(\partial^n)^2 = 0$, visit Theorem 1.1.7. Let us now consider the simplicial differentials of the sub-nerve $G_i^{(\bullet)}$ from Lemma 2.3.3:

$$\delta^n_i = \sum (-1)^k (d^n_k)^* : C^{i,j}(G^{(n-1)}) \to C^{i,j}(G^{(n)}).$$

Lemma 2.3.6. *The following diagram commutes for all i and r*

$$\begin{tikzcd} \Omega^j(G_i^{(n-1)}) \arrow[r, "\partial^{n-1}_r"] \arrow[d, "\delta^n_i"'] & \Omega^{j-r+1}(G_{i+r}^{(n-1)}) \arrow[d, "\delta^n_{i+r}"] \\ \Omega^j(G_i^{(n)}) \arrow[r, "\partial^n_r"'] & \Omega^{j-r+1}(G_{i+r}^{(n)}). \end{tikzcd}$$

Proof. Observe that if $r = 0$ then

$$\delta^n_{i+0} \circ \partial^{n-1}_0 = \partial^n_0 \circ \delta^n_i \iff \delta^n_i \circ d = d \circ \delta^n_i,$$

since the simplicial differential already commute with the de Rham differentials. Otherwise,

$$(\partial^n_r \circ \delta^n_i)(\omega) = \partial^n_r \left(\sum (-1)^k (d^n_k)^*(\omega) \right) = \sum (-1)^k \partial^n_r((d^n_k)^*(\omega)),$$

where, as consequence of the base-change formula of the integration along the fiber operation together with Identities (2.11), we obtain

$$\begin{aligned} \partial^n_r((d^n_k)^*(\omega)) &= (-1)^j (u_i^{i+r}(n))_* (l_i^{i+r}(n))^*((d^n_k)^*(\omega)) \\ &= (-1)^j (u_i^{i+r}(n))_* (d^n_k \circ l_i^{i+r}(n))^*(\omega) \\ &= (-1)^j (u_i^{i+r}(n))_* (l_i^{i+r}(n-1) \circ \overline{d^n_k})^*(\omega) \\ &= (-1)^j (u_i^{i+r}(n))_* (\overline{d^n_k})^*((l_i^{i+r}(n-1))^*(\omega)) \\ &= (-1)^j (d^n_k)^* (u_i^{i+r}(n-1))_*((l_i^{i+r}(n-1))^*(\omega)) \\ &= (d^n_k)^*((-1)^j (u_i^{i+r}(n-1))_*((l_i^{i+r}(n-1))^*(\omega))) \\ &= (d^n_k)^*(\partial^{n-1}_r(\omega)). \end{aligned}$$

Thus, we have that

$$(\partial_r^n \circ \delta_i^n)(\omega) = \sum (-1)^k \partial_r^n((d_k^n)^*(\omega)) = \sum (-1)^k (d_k^n)^*(\partial_r^{n-1}(\omega)) = (\delta_{i+r}^n \circ \partial_r^{n-1})(\omega).$$

□

Having the previous fact in mind we define $\overline{\delta}^n : C^p(G^{(n-1)}) \to C^p(G^{(n)})$ as

$$\overline{\delta}^n(\omega) = \begin{cases} \delta_i^n(\omega) & \text{if} \quad w \in \Omega^j(G_i^{(n-1)}) \\ 0 & \text{otherwise.} \end{cases}$$

This operator may be thought of as $\overline{\delta}^n = \sum \delta_i^n$ verifying $\delta_i^n \circ \delta_{i'}^n = \delta_{i'}^n \circ \delta_i^n = 0$ since if $i \neq i'$ then $\delta_i^n \circ \delta_{i'}^n = 0$ by definition and if $i = i'$ then $(\delta_i^n)^2 = 0$ also holds because δ_i^n is the simplicial differential of the sub-nerve $G_i^{(\bullet)}$. In particular, we have that $(\overline{\delta}^n)^2 = 0$.

Note that we have defined two boundary operators ∂ and $\overline{\delta}$ verifying $\partial^2 = 0$, $\overline{\delta}^2 = 0$ and, moreover, from the commutativity property stated in Lemma 2.3.6 we get that they also satisfy $\partial \circ \overline{\delta} = \overline{\delta} \circ \partial$.

Summing up, we have obtained that:

Proposition 2.3.3. *The triple $(C^\bullet(G^{(\bullet)}), \partial, \overline{\delta})$ determines a double cochain complex which may be depicted as*

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
C^2(G^{(0)}) \arrow[r, "\overline{\delta}"] \arrow[u, "\partial"] & C^2(G^{(1)}) \arrow[r, "\overline{\delta}"] \arrow[u, "\partial"] & C^2(G^{(2)}) \arrow[r, "\overline{\delta}"] \arrow[u, "\partial"] & \cdots \\
C^1(G^{(0)}) \arrow[r, "\overline{\delta}"] \arrow[u, "\partial"] & C^1(G^{(1)}) \arrow[r, "\overline{\delta}"] \arrow[u, "\partial"] & C^1(G^{(2)}) \arrow[r, "\overline{\delta}"] \arrow[u, "\partial"] & \cdots \\
C^0(G^{(0)}) \arrow[r, "\overline{\delta}"] \arrow[u, "\partial"] & C^0(G^{(1)}) \arrow[r, "\overline{\delta}"] \arrow[u, "\partial"] & C^0(G^{(2)}) \arrow[r, "\overline{\delta}"] \arrow[u, "\partial"] & \cdots
\end{tikzcd}$$

We can make a total complex out of this double cochain complex by setting $C_T^n(G) = \bigoplus_{p+q=n} C^p(G^{(q)})$ and defining the total differential $\partial_T : C_T^n(G) \to C_T^{n+1}(G)$ as

$$\partial_T(\omega) = (\overline{\delta} + (-1)^q \partial)(\omega), \quad \omega \in C^p(G^{(q)}). \tag{2.14}$$

The sing change is introduced in order to have that $\partial_T^2 = 0$. So, we establish the following definition.

Definition 2.3.3. The **groupoid Morse cohomology** associated to the Morse Lie groupoid $F_1 : (G^{(1)} \rightrightarrows G^{(0)}) \to (\mathbb{R} \rightrightarrows \mathbb{R})$ and the n-metric $\eta^{(n)}$ on $G^{(n)}$ is defined to be the cohomology of $(C_T^\bullet(G), \partial_T)$.

Let us now exhibit a morphism of double complexes between the Bott–Shulman-Stasheff double cochain complex of $G^{(1)} \rightrightarrows G^{(0)}$ and the double cochain complex constructed above. The **Bott-Shulman-Stasheff double complex** $(\Omega^\bullet(G^{(\bullet)}), d, \delta)$ may be depicted as

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\Omega^2(G^{(0)}) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^2(G^{(1)}) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^2(G^{(2)}) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
\Omega^1(G^{(0)}) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^1(G^{(1)}) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^1(G^{(2)}) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
\Omega^0(G^{(0)}) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^0(G^{(1)}) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^0(G^{(2)}) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots
\end{tikzcd}$$

where the vertical differential is the de Rham differential and the horizontal differential is the simplicial differential associated to the nerve $G^{(\bullet)}$ of $G^{(1)} \rightrightarrows G^{(0)}$. By using Formula 2.14 we can similarly define a total cohomology for this double cochain complex which is usually denoted by $H^\bullet_{dR}(G)$. It is worth mentioning that such a cohomology is Morita invariant in the sense that Morita equivalent Lie groupoids have corresponding isomorphic total cohomologies, compare for instance [16].

Let us consider the collection of maps $\Psi^\bullet := \{\Psi^n\}_{n\in\mathbb{N}}$ constructed as follows. Recall that, for each $n \in \mathbb{N}$, the map $u_i(n) : W^u(G_i^{(n)}) \to G_i^{(n)}$ has the structure of locally trivial bundle. Thus, by using again integration along the fiber, we define $\Psi_i^n : \Omega^p(G^{(n)}) \to C^{i,p-i}(G^{(n)})$ as

$$\Psi_i^n(\omega) := (u_i(n))_* \left(\omega|_{W^u(G_i^{(n)})}\right) \qquad \omega \in \Omega^p(G^{(n)}),$$

and $\Psi^n = \bigoplus \Psi_i^n : \Omega^p(G^{(n)}) \to C^p(G^{(n)})$. As consequence of the results due to Austin and Braam in [8] we have that $\Psi^\bullet \circ d = \partial \circ \Psi^\bullet$ and, more importantly, this collection of maps induces isomorphisms between the cohomology groups

$$H^\bullet(C^\bullet(G^{(n)}), \partial) \cong H^\bullet_{dR}(\Omega^\bullet(G^{(n)}), d).$$

To see that the collection $\Psi^\bullet$ defines a morphism of double complexes it remains to check that $\Psi^\bullet \circ \delta = \overline{\delta} \circ \Psi^\bullet$. This will be consequence of showing the following identity.

Lemma 2.3.7. *For all i and n, the following commutativity property holds true*

$$\Psi_i^n \circ \delta = \overline{\delta} \circ \Psi_i^{n-1}.$$

Proof. Because of Identities (2.9), which in turn allow us to conclude that every face map d_k^n is well restricted to the stable/unstable manifolds, and by using the base-change property of the integration along the fiber operation, we obtain that

$$\begin{aligned}
(\Psi_i^n \circ \delta)(\omega) &= \Psi_i^n \left(\sum (-1)^k (d_k^n)^*(\omega)\right) = \sum (-1)^k \Psi_i^n((d_k^n)^*(\omega)) \\
&= \sum (-1)^k (u_i(n))_* \left((d_k^n)^*(\omega)|_{W^u(G_i^{(n)})}\right) \\
&= \sum (-1)^k (d_k^n)^*(u_i(n-1))_* \left(\omega|_{W^u(G_i^{(n-1)})}\right) = (\overline{\delta} \circ \Psi_i^{n-1})(\omega),
\end{aligned}$$

for all $\omega \in \Omega^p(G^{(n-1)})$ as desired. □

So, $\Psi^\bullet$ defines a morphism of double cochain complexes between $(\Omega^\bullet(G^{(\bullet)}), d, \delta)$ and $(C^\bullet(G^{(\bullet)}), \partial, \overline{\delta})$ inducing isomorphisms between the vertical cohomologies of the complexes. Therefore, as consequence of all the facts stated above, by using a usual argument of spectral sequences (see for instance [24, p. 108]), we conclude that:

Theorem 2.3.2. *The total cohomology of the double cochain complex $(C^\bullet(G^{(\bullet)}), \partial, \overline{\delta})$ is isomorphic to the total cohomology of the Bott–Shulman-Stasheff double cochain complex of $G^{(1)} \rightrightarrows G^{(0)}$:*

$$H_T^\bullet(G, \partial_T) \cong H_{dR}^\bullet(G).$$

Some important observations derived from the previous result come in order. Firstly, the total cohomology on the left hand side turns out to be somehow independent of the choice of both the Morse Lie groupoid morphism F and the Riemannian groupoid n-metric η. Secondly, such a total cohomology is Morita invariant, something that should be expected since the notion of Morse Lie groupoid morphism we are working with is also Morita invariant. Thirdly, by arguing as before, it is simple to check that if $G \leftarrow H \rightarrow G'$ is a Morita equivalence between compact proper Lie groupoids such that $G^{(n)}$, $H^{(n)}$, and $G'^{(n)}$ are orientable for all $n \in \mathbb{N}$ then Assumption 2.3.1 is preserved by Proposition 2.1.1. This key fact can be used to compute the Bott–Shulman-Stasheff cohomology of certain Lie groupoids by using elementary examples of Morse Lie groupoid morphisms.

Example 2.3.1. Let M be a smooth manifold and $f : M \to \mathbb{R}$ be a smooth function satisfying Assumption 1.1.1. From [8] it follows that the groupoid Morse complex associated to the induced Morse Lie groupoid morphism $F : (M \rightrightarrows M) \to (\mathbb{R} \rightrightarrows \mathbb{R})$ is the standard Morse complex of (M, f). Also, the Bott–Shulman–Stasheff complex of $M \rightrightarrows M$ is the ordinary de Rham complex of M. Therefore, as explained above, the computation of the Morse complex can be used to obtain the Bott–Shulman-Stasheff cohomology of those compact and orientable Lie groupoids which are Morita equivalent to M. See [8, s. 3.6] for explicit computations in the standard case.

Example 2.3.2. Suppose that K is a compact Lie group acting on a compact oriented manifold M. Let $f : M \to \mathbb{R}$ be a K-invariant Morse function satisfying the Morse–Smale transversality condition. The Bott–Shulman-Stasheff cohomology of the action groupoid $K \ltimes M \rightrightarrows M$ is isomorphic to K-equivariant cohomology associated to the Cartan complex, i.e. the equivariant cohomology of the action [16]. Once again, from [8] we get that the groupoid Morse complex of the induced Morse Lie groupoid morphism $F : (K \ltimes M \rightrightarrows M) \to (\mathbb{R} \rightrightarrows \mathbb{R})$ recovers the standard K-equivariant cohomology of the action. This fact can be viewed as a particular instance of the construction in Subsection 2.4.2. Consult [8, s. 5.4] and [71, s. 5] for explicit computations.

Example 2.3.3. If $G \rightrightarrows M$ is a proper and étale Lie groupoid then its Bott–Shulman-Stasheff cohomology agrees with the basic cohomology of G, which turns out to be isomorphic to the singular cohomology $H^\bullet(M/G, \mathbb{R})$, compare [103, 112]. Thus, Morse Lie groupoid morphisms on $G \rightrightarrows M$ verifying Assumption 2.3.1 allow us to recover the singular cohomology of the orbit space M/G. A particularly interesting instance of this fact was recently checked in [30, 74] for the case of effective orientable orbifolds by providing an explicit isomorphism. Let K be a compact connected Lie group acting on a compact oriented manifold M by orientation preserving diffeomorphisms. If the action is effective and locally free then the quotient space M/K has the structure of an effective orbifold, so that the differentiable stack $[M/K \ltimes M]$ represented by the action groupoid $K \ltimes M \rightrightarrows M$ can be thought of as an orientable orbifold. Hence, the groupoid Morse cohomology associated to a Morse Lie groupoid morphism $(K \ltimes M \rightrightarrows M) \to (\mathbb{R} \rightrightarrows \mathbb{R})$ verifying the Morse–Smale transversality condition equals the singular cohomology $H^\bullet(M/K, \mathbb{R})$ since the action is locally free, see [74].

We finish this subsection by commenting a possible future application.

Remark 2.3.2. Fukaya in [50] gave a construction which is similar to Austin–Braam's model, but using singular chains instead of differential forms. One of the main features of his construction is that it can be directly generalized to the infinite-dimensional case so that it allowed him to address problems in Floer homology. We plan to apply Fukaya's approach to our setting with the hope of starting the study of Floer homology in the context of Lie groupoids and their differentiable stacks.

2.4 2-Equivariant Morse theory

In this section we develop a 2-equivariant Morse theory over Lie groupoids. Namely, we study real-valued Lie groupoid morphisms which are invariant by the action of a compact Lie 2-group and satisfy a Morse–Bott nondegeneracy condition along its critical invariant saturated submanifolds.

2.4.1 Isometric Lie 2-group actions

In order to describe a 2-equivariant version of the results proved in the previous sections we need to introduce a notion of isometric action of a Lie 2-group on a Riemannian groupoid. The results we will introduce in this subsection were obtained in joint work with J. S. Herrera-Carmona in [59]. Due to our purposes we only plan to mention a few facts about these results which will be further developed as well as explored in Chapter 4.

In the sequel we shall denote by $K^{(1)} \rightrightarrows K^{(0)}$ a Lie 2-group and by $(G^{(1)} \rightrightarrows G^{(0)}, \eta)$ a Riemannian groupoid.

Definition 2.4.1. A Lie 2-group action of $K^{(1)} \rightrightarrows K^{(0)}$ on $(G^{(1)} \rightrightarrows G^{(0)}, \eta)$ is said to be **isometric** if $K^{(2)}$ acts by isometries on $(G^{(2)}, \eta^{(2)})$.

As an immediate consequence of the previous definition we can prove the following.

Lemma 2.4.1. *The action of $K^{(1)}$ on $(G^{(1)}, \eta^{(1)})$ is by isometries. Consequently, the action of $K^{(0)}$ on $(G^{(0)}, \eta^{(0)})$ is also by isometries.*

Proof. For simplicity we shall verify that if $K^{(1)}$ acts on $(G^{(1)}, \eta^{(1)})$ by isometries then $K^{(0)}$ acts on $(G^{(0)}, \eta^{(0)})$ by isometries as well. Let $v, w \in T_xG^{(0)}$ and $k_0 \in K^{(0)}$. It is clear that there are $\tilde{v}, \tilde{w} \in \ker(d(s_G)_g)^{\perp_{\eta^{(1)}}}$ with $s_G(g) = x$ such that $d(s_G)_g(\tilde{v}) = v$ and $d(s_G)_g(\tilde{w}) = w$ and there is $k \in K^{(1)}$ such that $s_K(k) = k_0$. Therefore,

$$\begin{aligned}
(\theta^0_{k_0})^*_x\eta^{(0)}(v,w) &= \eta^{(0)}_{k_0x}(d(\theta^0_{k_0})_x(v), d(\theta^0_{k_0})_x(w)) = \eta^{(0)}_{k_0x}(d(\theta^0_{k_0} \circ s_G)_g(\tilde{v}), d(\theta^0_{k_0} \circ s_G)_g(\tilde{w})) \\
\star &= \eta^{(0)}_{k_0x}(d(s_G \circ \theta^1_k)_g(\tilde{v}), d(s_G \circ \theta^1_k)_g(\tilde{w})) = \eta^{(1)}_{kg}(d(\theta^1_k)_g(\tilde{v}), d(\theta^1_k)_g(\tilde{w})) \\
&= \eta^{(1)}_g(\tilde{v}, \tilde{w}) = \eta^{(0)}_x(v,w).
\end{aligned}$$

In the equality $\star$ above we used the fact that s_G is a Riemannian submersion and that the action θ^1 preserves the horizontal distribution $\ker(ds_G)^{\perp_{\eta^{(1)}}}$. Note that this computation does not depend on the choice of $\tilde{v}, \tilde{w}, g$ and k. Furthermore, we may obtain the same conclusion by choosing $\eta^{(0)} = (t_G)_*\eta^{(1)}$ instead of $\eta^{(0)} = (s_G)_*\eta^{(1)}$. The fact that $K^{(1)}$ acts on $(G^{(1)}, \eta^{(1)})$ by isometries provided that $K^{(2)}$ acts on $(G^{(1)}, \eta^{(2)})$ by isometries can be similarly shown by using that $\eta^{(1)} = (\pi_{2,G})_*\eta^{(2)} = (m_G)_*\eta^{(2)} = (\pi_{1,G})_*\eta^{(2)}$ and that $\pi_{2,G}, m_G, \pi_{1,G} : G^{(2)} \to G^{(1)}$ are Riemannian submersions verifying analogous equivariant relations as those in Equation (1.12) with respect to the maps $\pi_{2,K}, m_K, \pi_{1,K} : K^{(2)} \to K^{(1)}$. $\square$

More generally:

Remark 2.4.1. By using exactly the same arguments as in Lemma 2.4.1 it is simple to check that if $\eta^{(n)}$ is an n-metric on $G^{(n)}$ and the induced left action θ^n of $K^{(n)}$ on $(G^{(n)}, \eta^{(n)})$ is by isometries then the action θ^r of $K^{(r)}$ on $(G^{(r)}, \eta^{(r)})$ will be by isometries for all $0 \leq r \leq n-1$.

As an interesting consequence derived from this notion of isometric Lie 2-group action we can obtain 2-equivariant weak linearizations around $K^{(0)}$-invariant saturated submanifolds in $G^{(0)}$. Namely, let θ be a 2-action of $K^{(1)} \rightrightarrows K^{(0)}$ on $G^{(1)} \rightrightarrows G^{(0)}$ and let $S \subset G^{(0)}$ be a $K^{(0)}$-invariant saturated submanifold. We say that $G^{(1)} \rightrightarrows G^{(0)}$ is 2**-equivariant weakly linearizable** at S if there are K-invariant Lie groupoid neighborhoods $\widetilde{V} \rightrightarrows V$ of $G_S \rightrightarrows S$ in $\nu(G_S) \rightrightarrows \nu(S)$ (seen as the zero section) and $\widetilde{U} \rightrightarrows U$ of $G_S \rightrightarrows S$ in $G^{(1)} \rightrightarrows G^{(0)}$, and a 2-equivariant Lie groupoid isomorphism $\phi : (\widetilde{V} \rightrightarrows V) \xrightarrow{\cong} (\widetilde{U} \rightrightarrows U)$ which is the identity on $G_S \rightrightarrows S$.

The K-invariant property of the Lie groupoid neighborhood in $\nu(G_S) \rightrightarrows \nu(S)$ used above makes sense because of the following facts. Let us assume that we are given with a 2-metric $\eta^{(2)}$ on $G^{(2)}$ and that $K^{(2)}$ acts on $(G^{(2)}, \eta^{(2)})$ by isometries. It is simple to check that every 2-action $\theta = (\theta^1, \theta^0)$ of $K^{(1)} \rightrightarrows K^{(0)}$ on $G^{(1)} \rightrightarrows G^{(0)}$ induces a 2-action $T\theta = (T\theta^1, T\theta^0)$ of $K^{(1)} \rightrightarrows K^{(0)}$ on $TG^{(1)} \rightrightarrows TG^{(0)}$ by differentiating the actions θ^1 and θ^0. Let us pick a $K^{(0)}$-invariant saturated submanifold S in $G^{(0)}$. This clearly implies that G_S is $K^{(1)}$-invariant and that $G_S^{(2)}$ is $K^{(2)}$-invariant. If we respectively use $\eta^{(2)}$, $\eta^{(1)}$, and $\eta^{(0)}$ to identify $\nu(G_S)^{(2)} \cong \nu(G_S^{(2)})$ with $(T(G_S)^{(2)})^{\perp}$, $\nu(G_S)$ with $TG_S^{\perp}$, and $\nu(S)$ with $TS^{\perp}$ then it follows that the 2-action $T\theta$ restrict to a well defined 2-action $\overline{T\theta}$ of $K^{(1)} \rightrightarrows K^{(0)}$ on $\nu(G_S) \rightrightarrows \nu(S)$ since θ is isometric. Furthermore, the latter fact also implies that the exponential maps $\exp^{(2)}$, $\exp^{(1)}$, and $\exp^{(0)}$ are equivariant local diffeomorphisms. Therefore, as isometries preserve (horizontal) geodesics then by following the classical proofs of the equivariant tubular neighborhood theorem in [26, VI. Thm. 2.2] and [64, 65, Thm. 4.4] with the proof of the weak linearization theorem given in [40, Thm. 5.11] we easily obtain:

Proposition 2.4.1 (2-equivariant weak groupoid linearization)**.** *Let $K^{(1)} \rightrightarrows K^{(0)}$ be a Lie 2-group acting by isometries on a Riemannian groupoid $(G^{(1)} \rightrightarrows G^{(0)}, \eta)$. Then there exists a 2-equivariant weak linearization of $G^{(1)} \rightrightarrows G^{(0)}$ around any $K^{(0)}$-invariant saturated submanifold S in $G^{(0)}$.*

We follow now the classical approach used to guarantee the existence of invariant Riemannian metrics on K-manifolds to provide a similar construction in our context. In order to do so we need to have in mind the following key observation about the averaging of a collection of transverse Riemannian metrics with respect to a fixed surjective submersion, see Equation (1.9).

Remark 2.4.2. Denote by η^* the dual metric associated to a Riemannian metric η. From [40, Prop. 2.2] we know that if $\pi : E \to B$ is a surjective submersion and $\{\eta_1, \cdots, \eta_j\}$ is a collection of π-transverse metrics then its tangent average $\frac{1}{j}\sum_{l=1}^{j}\eta_l$ fails to be π-transverse again in general. Nevertheless, its cotangent average $\frac{1}{j}\left(\sum_{l=1}^{j}\eta_l^*\right)^*$ is always π-transverse which sometimes makes more advantageous to take a cotangent space point of view in the study of Riemannian submersions.

Let us assume for a moment that $K^{(1)}$ is compact so that $K^{(0)}$ is also compact. If μ_1 is the normalized Haar measure on $K^{(1)}$ then, by uniqueness, the pushforward measure $s_{K*}\mu_1$ agrees with the normalized Haar measure μ_0 on $K^{(0)}$ since s_K a surjective Lie group homomorphism and the identity

$$\int_{K^{(0)}} f d(s_{K*}\mu_1) = \int_{K^{(1)}} f \circ s_K d\mu_1, \tag{2.15}$$

holds true for each continuous function $f : K^{(0)} \to \mathbb{R}$. Analogously, $t_{K*}\mu_1 = \mu_0$ and $i_{K*}\mu_1 = \mu_1$. Note also that the fact that $i_{K*}\mu_1 = \mu_1$ and $s_K \circ i_K = t_K$ immediately implies that $t_{K*}\mu_1 = s_{K*}\mu_1$. Thus, we are in conditions to state:

Theorem 2.4.1. *Suppose that $K^{(1)} \rightrightarrows K^{(1)}$ is a Lie 2-group acting on a Riemannian groupoid $(G^{(1)} \rightrightarrows G^{(0)}, \eta)$. If $K^{(1)}$ compact then there exists another groupoid metric $\overline{\eta}$ on $G^{(1)} \rightrightarrows G^{(0)}$ for which the 2-action θ becomes isometric.*

Proof. To simplify the computations we shall suppose that $\eta = \eta^{(1)}$ is a 1-metric on $G^{(1)}$. By dual averaging we define the metrics:

$$(\overline{\eta}^{(1)})^* := \int_{K^{(1)}} (\theta^1)_k^*(\eta^{(1)})^* d\mu_1(k) \quad \text{and} \quad (\overline{\eta}^{(0)})^* := \int_{K^{(0)}} (\theta^0)_{k_0}^*(\eta^{(0)})^* d\mu_0(k_0).$$

The result will follow from the fact that θ is a 2-action, η is already a 1-metric, and Identity (2.15) holds true. Indeed, on the one hand, a straightforward computation using the fact that i_G is an isometry of $(G^{(1)}, \eta^{(1)})$ verifying both Identity (1.12) and $i_{K*}\mu_1 = \mu_1$ implies that i_G is an isometry of $(G^{(1)}, \overline{\eta}^{(1)})$ as well. The fact that $s_G : G^{(1)} \to G^{(0)}$ is a Riemannian submersion tells us that $ds_G(g)^* : T^*_{s_G(g)}G^{(0)} \to \ker(ds_G(g))^\circ$ is an isometry for all $g \in G^{(1)}$ where $\ker(ds_G(g))^\circ$ denotes the annihilator of the vectors tangent to the fiber. Given $g \in G^{(1)}$, covectors $\alpha, \beta \in T^*_{s_G(g)}G^{(0)}$, and $k \in K^{(1)}$ such that $s_K(k) = k_0$ we get the following chain of equalities:

$$\begin{aligned}
(\overline{\eta}^{(0)}_{s_G(g)})^*(\alpha,\beta) &= \int_{K^{(0)}} (\eta^{(0)}_{k_0 s_G(g)})^*((\theta^0_{k_0})^*(\alpha), (\theta^0_{k_0})^*(\beta)) d\mu_0(k_0)\\
\text{(2.15)} \quad &= \int_{K^{(1)}} (\eta^{(0)}_{s_K(k)s_G(g)})^*((\theta^0_{s_K(k)})^*(\alpha), (\theta^0_{s_K(k)})^*(\beta)) d\mu_1(k)\\
&= \int_{K^{(1)}} (\eta^{(1)}_{kg})^*(ds_G(kg)^*((\theta^0_{s_K(k)})^*(\alpha)), ds_G(kg)^*((\theta^0_{s_K(k)})^*(\beta))) d\mu_1(k)\\
&= \int_{K^{(1)}} (\eta^{(1)}_{kg})^*((\theta^1_k)^*(ds_G(g)^*(\alpha)), (\theta^1_k)^*(ds_G(g)^*(\beta))) d\mu_1(k)\\
&= (\overline{\eta}^{(1)}_g)^*(ds_G(g)^*(\alpha), ds_G(g)^*(\beta)),
\end{aligned}$$

from which we conclude that s_G is also Riemannian for the averaged metrics. With analogous computations we get a similar conclusion for $t_G : G^{(1)} \to G^{(0)}$ so that the result follows as claimed. □

Remark 2.4.3. It is worth noticing that if we consider an n-metric on $G^{(n)}$ for $n \geq 2$ and take into account the simplicial approach described in Remark 2.4.1 then by applying similar averaging arguments as in the proof of Theorem 2.4.1 it is possible to show the existence of another n-metric on $G^{(n)}$ that is invariant by the action of $K^{(n)}$ on $G^{(n)}$. For instance, let $\eta^{(2)}$ be a 2-metric on $G^{(2)}$ and let μ_2 denote the normalized Haar measure on $K^{(2)}$. On the one hand, by dual averaging we define

$$(\overline{\eta}^{(2)})^* := \int_{K^{(2)}} (\theta^2)^*_{(k_1,k_2)} (\eta^{(2)})^* d\mu_2(k_1, k_2).$$

On the other hand, it is simple to see that we can obtain analogous formulas as that in Identity (2.15) by using instead $\pi_{2,K}, m_K, \pi_{1,K} : K^{(2)} \to K^{(1)}$ as well as the Lie group isomorphisms $K^{(2)} \to K^{(2)}$ determined by the canonical action of S_3 on $K^{(2)}$. Hence, by similar computations as those in the main part of Theorem 2.4.1 it follows that $\overline{\eta}^{(2)}$ is a 2-metric on $G^{(2)}$ for which the action of $K^{(2)}$ on $(G^{(2)}, \overline{\eta}^{(2)})$ becomes isometric.

Recall that in Subsection 1.3.1 we commented that every proper Lie groupoid can be equipped with a groupoid metric, see [40, Thm. 4.13]. Hence, we get that:

Corollary 2.4.1. *Let θ be a 2-action of a Lie 2-group $K^{(1)} \rightrightarrows K^{(0)}$ on a proper groupoid $G^{(1)} \rightrightarrows G^{(0)}$. If $K^{(1)}$ is compact then there always exists a Riemannian groupoid metric on $G^{(1)} \rightrightarrows G^{(0)}$ for which θ becomes an isometric 2-action.*

As we mentioned at the beginning of this subsection, the notion of isometric Lie 2-group action on a Riemannian groupoid introduced in Definition 2.4.1 will be further explored in Chapter 4.

2.4.2 2-Invariant Morse Lie groupoid morphisms

In [59, Ex. 3.30] we briefly commented how to use the notion of isometric Lie 2-group action defined in the previous subsection in order to develop a 2-equivariant analogue of the Morse theoretical results on Lie groupoids we have obtained so far. Our aim in this subsection is to elaborate on those ideas with enough details.

Let $G^{(1)} \rightrightarrows G^{(0)}$ be a Lie groupoid and $F_1 : G^{(1)} \to \mathbb{R}$ be a Lie groupoid morphism covering a basic function $F_0 : G^{(0)} \to \mathbb{R}$. Suppose that $K^{(1)} \rightrightarrows K^{(0)}$ is a Lie 2-group, with $K^{(1)}$ compact, determining a 2-action on $G^{(1)} \rightrightarrows G^{(0)}$ such that F_0 is $K^{(0)}$-invariant. Thus, it is simple to check that F_1 is $K^{(1)}$-invariant and $F_2 = m_G^* F_1 = \pi_{1,G}^* F_1 = \pi_{2,G}^* F_1$ is $K^{(2)}$-invariant. Note that Crit(F_0) is $K^{(0)}$-invariant, so that we may assume that its components S are given by $K^{(0)}$-invariant saturated submanifold since F_0 is also basic. Besides, there is a canonical Lie 2-group action of $K^{(1)} \rightrightarrows K^{(0)}$ on the topological groupoid Crit$(F_1) \rightrightarrows$ Crit(F_0).

Suppose that $F_1 : G^{(1)} \to \mathbb{R}$ is a Morse–Bott Lie groupoid morphism (see Remark 2.1.4), meaning that every $K^{(0)}$-invariant saturated component S of Crit(F_0) is a nondegenerate critical submanifold for $F_0 : G^{(0)} \to \mathbb{R}$. Observe that the Lie groupoid morphism $(Q_{F_1}, Q_{F_0}) : (\nu(G_S) \rightrightarrows \nu(S)) \to (\mathbb{R} \rightrightarrows \mathbb{R})$ is 2-invariant with respect to the canonical Lie 2-group action of $K^{(1)} \rightrightarrows K^{(0)}$ on $\nu(G_S) \rightrightarrows \nu(S)$ since F_0 is both basic and $K^{(0)}$-invariant.

Proposition 2.4.2 (2-equivariant Morse lemma)**.** *If $G^{(1)}$ is proper then around a 2-invariant non-degenerate critical subgroupoid $G_S \rightrightarrows S$ there is a full 2-equivariant groupoid tubular neigh-*

borhood $\phi^K : (\nu(G_S)_V \rightrightarrows V) \xrightarrow{\cong} (G_U \rightrightarrows U)$ *such that*

$$(\phi^K)^* F_1 = c + Q_{F_1}.$$

Proof. Let us consider the Euler-like multiplicative vector field $(\overline{X}, \overline{Y})$ for (G, G_S) which was constructed in the second proof of Theorem 2.2.1. Recall that it was obtained by considering only one groupoid orbit but the arguments go through for the pair (G, G_S) after obvious modifications. By averaging $(\overline{X}, \overline{Y})$ with respect to the normalized Haar measures on $K^{(1)}$ and $K^{(0)}$, with similar formulas as those used in Theorem 2.4.1, we can define another pair $(\tilde{X}, \tilde{Y})$ which is formed by vector fields that are still Euler-like for (G, G_S) and invariant by the 2-action of $K^{(1)} \rightrightarrows K^{(0)}$, compare [87, p. 230] for the classical equivariant argument. It is simple to check that $(\tilde{X}, \tilde{Y})$ remains being multiplicative since $(\overline{X}, \overline{Y})$ is so. Therefore, the result follows after considering the full groupoid tubular neighborhood of $G_S \rightrightarrows S$ induced by $(\tilde{X}, \tilde{Y})$, see [87, Thm. 4.5]. □

Remark 2.4.4. Just as it was commented in Remark 2.2.1 it follows that a different proof of the 2-equivariant Morse lemma may be provided by considering the 2-equivariant weak groupoid linearization of Proposition 2.4.1. Recall that such a linearization is induced by the existence of a 2-metric on $G^{(1)} \rightrightarrows G^{(0)}$ which is invariant by the 2-action of $K^{(1)} \rightrightarrows K^{(0)}$.

Let $\eta^{(2)}$ be a 2-metric on $G^{(2)}$ such that the Lie 2-group action of $K^{(1)} \rightrightarrows K^{(0)}$ on $G^{(1)} \rightrightarrows G^{(0)}$ is isometric. On the one hand, let S be a nondegenerate $K^{(0)}$-invariant saturated component of Crit(F_0). Recall that we can use the Riemannian metrics $\eta^{(1)}$ and $\eta^{(0)}$ respectively to split $\nu(G_S) = \nu_+(G_S) \oplus \nu_-(G_S)$ and $\nu(S) = \nu_+(S) \oplus \nu_-(S)$ into subbundles which are fiberwise defined by the eigenvectors corresponding to the positive/negative eigenvalues of $\mathcal{H}(F_1)$ and $\mathcal{H}(F_0)$. As the 2-action of $K^{(1)} \rightrightarrows K^{(0)}$ on $G^{(1)} \rightrightarrows G^{(0)}$ is isometric it follows from the 2-invariance of F_1 that the Hessian Lie groupoid morphism $H(F) : \nu(G_S) \oplus \nu(G_S) \to \mathbb{R}$ covering $H(f) : \nu(S) \oplus \nu(S) \to \mathbb{R}$ is 2-invariant with respect to the standard diagonal action of $K^{(1)} \rightrightarrows K^{(0)}$ on $\nu(G_S) \oplus \nu(G_S) \rightrightarrows \nu(S) \oplus \nu(S)$. In particular, the Lie 2-group action preserves the splittings above. More importantly, we can summarize the isometric features of these splittings in the next straightforward result.

Proposition 2.4.3. *There are natural* 2*-actions of the Lie* 2*-group* $K^{(1)} \rightrightarrows K^{(0)}$ *on the negative normal Lie groupoid* $\nu_-(G_S) \rightrightarrows \nu_-(S)$*, the negative unit disk topological groupoid* $D_-(G_S) \rightrightarrows D_-(S)$*, and the negative unit sphere Lie groupoid* $\partial D_-(G_S) \rightrightarrows \partial D_-(S)$*. Similar statements hold true for the positive counterpart. Besides, the two mutually complementary Lie groupoid projections* $(\xi_1, \xi_0) : (\nu(G_S) \rightrightarrows \nu(S)) \to (\nu_-(G_S) \rightrightarrows \nu_-(S))$ *and* $(\eta_1, \eta_0) : (\nu(G_S) \rightrightarrows \nu(S)) \to (\nu_+(G_S) \rightrightarrows \nu_+(S))$ *become* 2*-equivariant.*

On the other hand, by Proposition 2.2.3 we know that the gradient vector field ∇F_1 is multiplicative. But F_1 (resp. F_0) is $K^{(1)}$-invariant (resp. $K^{(0)}$-invariant) so that we actually get that ∇F_1 is a 2-equivariant multiplicative vector field. Note that this is equivalent to saying that the pair of flows $(\Phi_\tau^{\nabla F_1}, \Phi_\tau^{\nabla F_0})$ induces (local) 2-equivariant automorphisms on $G^{(1)} \rightrightarrows G^{(0)}$; see [81].

Let us now consider the level subgroupoids of F_1. Pick $a \in \mathbb{R}$. Again, from the 2-invariance of F_1 it follows that the Lie 2-group $K^{(1)} \rightrightarrows K^{(0)}$ naturally acts on the level subgroupoid $(G^{(1)})^a \rightrightarrows (G^{(0)})^a$. So, at this point we are in conditions to describe, in a 2-equivariant way, how are the topology changes of $G^{(1)} \rightrightarrows G^{(0)}$ whether or not we cross by a $K^{(0)}$-invariant saturated nondegenerate critical component of F_1. Indeed:

Proposition 2.4.4. *Suppose that* $G^{(1)} \rightrightarrows G^{(0)}$ *is a proper groupoid and* $[a, b]$ *is a closed interval such that the* $K^{(0)}$*-invariant space* $F_0^{-1}[a, b]$ *is compact. The following dichotomy holds true:*

- *if* $F_0^{-1}[a,b]$ *does not contain critical points of* F_0 *then* $(G^{(1)})^a \rightrightarrows (G^{(0)})^a$ *and* $(G^{(1)})^b \rightrightarrows (G^{(0)})^b$ *are 2-equivariant isomorphic groupoids. Furthermore,* $(G^{(1)})^a \rightrightarrows (G^{(0)})^a$ *is a 2-equivariant deformation retraction of* $(G^{(1)})^b \rightrightarrows (G^{(0)})^b$*, or*
- *if the only non-degenerate* $K^{(0)}$*-invariant component of* $\mathrm{Crit}(F_0)$ *inside* $F_0^{-1}(a,b)$ *is* S *then* $(G^{(1)})^a \cup_{\partial D_-(G_S)} D_-(G_S) \rightrightarrows (G^{(0)})^a \cup_{\partial D_-(S)} D_-(S)$ *is a 2-equivariant deformation retraction of* $(G^{(1)})^b \rightrightarrows (G^{(0)})^b$*.*

Proof. Firstly, if $F_0^{-1}[a,b]$ does not contain critical points of F_0 then we may assume that the function $\mu : G^{(0)} \to \mathbb{R}$ (resp. $\tilde{\mu} : G^{(1)} \to \mathbb{R}$) used in Proposition 2.2.4 is $K^{(0)}$-invariant (resp. $K^{(1)}$-invariant). Otherwise, we can use an average argument with the normalized Haar measure on $K^{(0)}$ (resp. on $K^{(1)}$) in order to construct another smooth function with the same properties which is additionally $K^{(0)}$-invariant. Therefore, the multiplicative vector field $(-\tilde{\mu}\tilde{X}, \mu X)$ that we constructed in Proposition 2.2.4 becomes 2-equivariant, so that the first assertion follows. Secondly, if the only non-degenerate $K^{(0)}$-invariant component of $\mathrm{Crit}(F_0)$ inside $F_0^{-1}(a,b)$ is S then the second assertion directly follows from the arguments in Theorem 2.2.2 since by Proposition 2.4.3 the auxiliary Morse Lie groupoid morphism $\tilde{F}_1$ as well as the groupoid deformation retraction $(\tilde{r}_\tau, r_\tau)$ constructed therein are 2-equivariant. □

2-Equivariant Morse–Bott double complex

Motivated by the Austin–Braam's equivariant Morse complex (see the end of Subsection 1.1.1 and [8, s. 5.1]), in this subsection we apply the constructions from Subsection 2.3.1 to recover the equivariant cohomology associated to a Lie 2-group action on a Lie groupoid as defined in [77]. Let us consider a Lie 2-group $K^{(1)} \rightrightarrows K^{(0)}$, with $K^{(1)}$ compact, acting on a proper Lie groupoid $G^{(1)} \rightrightarrows G^{(0)}$. Suppose that $F_1 : G^{(1)} \to \mathbb{R}$ is a Morse–Bott Lie groupoid morphism covering a $K^{(0)}$-invariant basic function $F_0 : G^{(0)} \to \mathbb{R}$ which satisfies the assumptions we considered in Subsection 2.3.1. The set of n-composable arrows $K^{(n)}$ inherits a canonical Lie group structure from the direct product $(K^{(1)})^n$ and the Lie 2-group action above allows us to define a smooth left action of $K^{(n)}$ on $G^{(n)}$ in a canonical way. In other words, we have a well defined simplicial left action of the nerve $K^{(\bullet)}$ on the nerve $G^{(\bullet)}$. It is simple to check that the face maps $(d_k^n)_K : K^{(n)} \to K^{(n-1)}$ and $(d_k^n)_G : G^{(n)} \to G^{(n-1)}$ satisfy the equivariant relations

$$(d_k^n)_G(k \cdot g) = (d_k^n)_K(k) \cdot (d_k^n)_G(g), \tag{2.16}$$

for all $k \in K^{(n)}$ and $g \in G^{(n)}$. This is consequence of the simplicial identities on $G^{(\bullet)}$ and the fact that we are working with a 2-action. Therefore, Formula (2.16) implies that the simplicial function $F_\bullet$ on $G^{(\bullet)}$ is $K^{(\bullet)}$-invariant in the sense that F_n is $K^{(n)}$-invariant for all n since F_0 is $K^{(0)}$-invariant. Let us now consider an n-metric $\eta^{(n)}$ on $G^{(n)}$ such that the 2-action of $K^{(1)} \rightrightarrows K^{(0)}$ on $G^{(1)} \rightrightarrows G^{(0)}$ is isometric, see Definition 2.4.1. Recall that by Lemma 2.4.1 and Remark 2.4.1 we know that the fact that $K^{(n)}$ acts on $(G^{(n)}, \eta^{(n)})$ isometrically is enough to guarantee that the action of $K^{(r)}$ on $(G^{(r)}, \eta^{(r)})$ is isometric as well, for all $0 \leq r \leq n-1$. Moreover, by Corollary 2.4.1 it follows that when $K^{(1)}$ is compact and $G^{(1)} \rightrightarrows G^{(0)}$ is proper then invariant n-metrics in the sense of Definition 2.4.1 always exist.

Given that the functions F_n are $K^{(n)}$-invariant it follows that their negative vector fields $-\nabla F_n$ are $K^{(n)}$-invariant so that their descending flows Φ_τ^n are $K^{(n)}$-equivariant. Note that the saturated submanifold $G_i^{(0)}$ which is formed by the non-degenerate groupoid orbits of $G^{(1)} \rightrightarrows G^{(0)}$ having index i is $K^{(0)}$-invariant since the fact that F_0 is $K^{(0)}$-invariant implies that $\mathrm{Crit}(F_0)$ is formed by Lie group $K^{(0)}$-orbits. In consequence, it is simple to check that for each $n \in \mathbb{N}$ it holds that $G_i^{(n)}$ is $K^{(n)}$-invariant, thus obtaining that the endpoint maps $u_i(n)$ and $l_i(n)$ are $K^{(n)}$-equivariant since our descending flows are $K^{(n)}$-equivariant. More importantly, if we consider the left action of $K^{(n)}$ on $\mathcal{M}^n(G_i^{(n)}, G_j^{(n)})$ defined by $k \cdot [A]_n = [k \cdot A]_n$ then we get two new $K^{(n)}$-equivariant endpoint maps $u_j^i(n)$ and $l_j^i(n)$.

Remark 2.4.5. On the one hand, as it was commented in [8], it follows that due to the $K^{(0)}$-invariance of F_0 our weakly self-indexing requirement from Assumption 2.3.1 is implied by the transversality assumption. Similarly, the assumption which asks that the endpoint maps induce fibrations is an immediate consequence of the presence of a transitive $K^{(0)}$-action on the components of the critical point set Crit(F_0). On the other hand, as perhaps it was already expected by the reader, it is possible to prove 2-equivariant versions of the results exhibited in Section 2.3, without so many changes along their proofs.

Let us briefly introduce the notion of 2-equivariant cohomology associated to a Lie 2-group action on a Lie groupoid as defined in [77].

Definition 2.4.2. A **double Lie groupoid** consists of a square of Lie groupoids

$$\begin{tikzcd}
D \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & H \arrow[d, shift left] \arrow[d, shift right] \\
G \arrow[r, shift left] \arrow[r, shift right] & M,
\end{tikzcd}$$

in which the structural maps of $D \rightrightarrows H$ are groupoid morphisms over the structural maps of $G \rightrightarrows M$ and also the structural maps of $D \rightrightarrows G$ are groupoid morphisms over the structural maps of $H \rightrightarrows M$.

Firstly, it is simple to see that a Lie 2-group is exactly a double groupoid where the base groupoids are singletons:

$$\begin{tikzcd}
K^{(1)} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & K^{(0)} \arrow[d, shift left] \arrow[d, shift right] \\
\{*\} \arrow[r, shift left] \arrow[r, shift right] & \{*\}.
\end{tikzcd}$$

Secondly, the Lie 2-group action naturally allows us to define another double Lie groupoid

$$\begin{tikzcd}
K^{(1)} \times G^{(1)} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & G^{(1)} \arrow[d, shift left] \arrow[d, shift right] \\
K^{(0)} \times G^{(0)} \arrow[r, shift left] \arrow[r, shift right] & G^{(0)},
\end{tikzcd}$$

where the horizontals are given by action groupoids and the verticals are Lie groupoid products. If we consider the nerve configuration associated to the latter double Lie groupoid then we obtain a **bisimplicial smooth manifold** so that we may work with the triple cochain complex $C^{\bullet,\bullet,\bullet}$ where

$$C^{n,p,q} = \Omega^p((K^{(n)})^q \times G^{(n)}),$$

with differentials given by the de Rham differential, the simplicial differential associated to the actions groupoids, and the simplicial differential associated to the product groupoids. The following notion has been introduced in [77].

Definition 2.4.3. The **2-equivariant cohomology** of the Lie 2-group action of $K^{(1)} \rightrightarrows K^{(0)}$ on $G^{(1)} \rightrightarrows G^{(0)}$ is defined to be the total cohomology determined by the triple cochain complex mentioned above. This will be denoted by $H_K^\bullet(G)$.

Two important features of this cohomology is that it is Morita invariant and can be recovered by the **Cartan model** as follows. Let us consider the nerve configuration $\mathfrak{k}^{(\bullet)}$ associated to the Lie 2-algebra $\mathfrak{k}^{(1)} \rightrightarrows \mathfrak{k}^{(0)}$ of $K^{(1)} \rightrightarrows K^{(0)}$. Motivated by the notion of simplicial equivariant forms introduced by Meinrenken in [86, Appx. C] we consider the double cochain complex $(C_{CM}^{\bullet,\bullet}, d_K, \delta_K)$ given by

$$C_{CM}^{n,k} = \Omega^k_{K^{(n)}}(G^{(n)}) = \bigoplus_{k=2p+q} (S^p((\mathfrak{k}^{(n)})^*) \otimes \Omega^q(G^{(n)}))^{K^{(n)}}, \tag{2.17}$$

where $(\mathfrak{k}^{(n)})^*$ denotes the dual vector space of the Lie algebra $\mathfrak{k}^{(n)}$, d_K is the Cartan differential, and $\delta_K : \Omega^k_{K^{(n)}}(G^{(n)}) \to \Omega^k_{K^{(n+1)}}(G^{(n+1)})$ is defined by $\delta_K = \delta_{\mathfrak{k}} \otimes \delta_G$ with $\delta_{\mathfrak{k}}$ the simplicial differential of $\mathfrak{k}^{(\bullet)}$ and δ_G the simplicial differential of $G^{(\bullet)}$. As it was proven in [77] the total cohomology of this double complex is isomorphic to the equivariant cohomology $H_K^\bullet(G)$ since $K^{(1)}$ is assumed to be compact.

We claim that it is possible to recover the equivariant cohomology defined above by following the ideas from [8] together with what we did in Subsection 2.3.1. For that, we consider a Morse–Bott Lie groupoid morphism $F_1 : G^{(1)} \to \mathbb{R}$ covering a basic function $F_0 : G^{(0)} \to \mathbb{R}$ which is $K^{(0)}$-invariant. Using the Cartan model we define the 2-equivariant version of the double cochain complex (2.13):

$$C^p(G^{(n)}) = \bigoplus_{i+r=p} \Omega^r_{K^{(n)}}(G_i^{(n)}) = \bigoplus_{i+j+2v=p} (\Omega^j(G_i^{(n)}) \otimes S^v((\mathfrak{k}^{(n)})^*))^{K^{(n)}}, \tag{2.18}$$

with differential operators ∂_K^n and $\overline{\delta}_K^n$ defined as follows. On the one hand, as before we split $\partial_K^n : C^p(G^{(n)}) \to C^{p+1}(G^{(n)})$ as the sum $\partial_K^n = \sum_v (\partial_K^n)_v$ where, for $\omega \otimes \phi \in (\Omega^j(G_i^{(n)}) \otimes S^j((\mathfrak{k}^{(n)})^*))^{K^{(n)}}$, we have that $(\partial_K^n)_0(\omega \otimes \phi) = d_K(\omega \otimes \phi)$ is the Cartan differential and for $v > 0$ we set $(\partial_K^n)_v(\omega \otimes \phi) = \partial_v^n \omega \otimes \phi$. On the other hand, we define $\overline{\delta}_K^n : C^p(G^{(n)}) \to C^p(G^{(n+1)})$ as $\overline{\delta}_K^n(\omega \otimes \phi) = \overline{\delta}^n \omega \otimes \delta_{\mathfrak{k}} \phi$. It is simple to check that these two operators ∂_K and $\overline{\delta}_K$ commute and that $\overline{\delta}_K^2 = 0$. Moreover, from [8] we also get that $\partial_K^2 = 0$. Therefore, we have actually obtained a double cochain complex $(C^\bullet(G^{(\bullet)}), \partial_K, \overline{\delta}_K)$, as claimed above.

Finally, let us now exhibit a morphism of double complexes between the double cochain complex $(C_{CM}^{\bullet,\bullet}, d_K, \delta_K)$ which is obtained by using the Cartan model (2.17) and $(C^\bullet(G^{(\bullet)}), \partial_K, \overline{\delta}_K)$ defined in (2.18). Let $\Theta^\bullet : C^\bullet(G^{(\bullet)}) \to C_{CM}^{\bullet,\bullet}$ be defined as a collection of maps $\{\Theta^n\}_{n\in\mathbb{N}}$ defined by $\Theta^n(\omega \otimes \phi) = \Psi^n(\omega) \otimes \phi$. From a straightforward computation it follows that $\Theta^\bullet \circ \delta_K = \overline{\delta}_K \circ \Theta^\bullet$. Also, as a consequence of what it was proven in [8] we have that $\Theta^\bullet \circ d_K = \partial_K \circ \Theta^\bullet$, and, more importantly, this collection of maps induces isomorphisms between the cohomology groups

$$H^\bullet(C^\bullet(G^{(n)}), \partial_K) \cong H^\bullet(\Omega^\bullet_{K^{(n)}}(G^{(n)}), d_K) = H^\bullet_{K^{(n)}}(G^{(n)}), \qquad n \in \mathbb{N}.$$

Here $H^\bullet_{K^{(n)}}(G^{(n)})$ denotes the equivariant cohomology obtained through the Cartan model associated to the action of $K^{(n)}$ on $G^{(n)}$. So, $\Theta^\bullet$ defines a morphism of double cochain complexes between $(C_{CM}^{\bullet,\bullet}, d_K, \delta_K)$ and $(C^\bullet(G^{(\bullet)}), \partial_K, \overline{\delta}_K)$ inducing isomorphisms between the vertical cohomologies of the complexes. Just as in the non equivariant case, by means of a spectral sequence argument, we conclude that:

Proposition 2.4.5. *The total cohomology $H^\bullet_T(G,\partial_{TK})$ of the double cochain complex $(C^\bullet(G^{(\bullet)}),\partial_K,\overline{\delta}_K)$ is isomorphic to the total cohomology of the double cochain complex determined by the Cartan model associated to the Lie 2-group action of $K^{(1)} \rightrightarrows K^{(0)}$ on $G^{(1)} \rightrightarrows G^{(0)}$. That is,*

$$H^\bullet_T(G,\partial_{TK}) \cong H^\bullet_K(G).$$

In particular, we can recover the equivariant cohomology of a Lie group acting on a differentiable stack which was introduced in [15]. Additionally, toric symplectic stacks [61] are presented by 0-symplectic groupoids with a Hamiltonian action of a 2-torus [62]. Hence, as an application of Proposition 2.4.5 one can compute the equivariant cohomology of a toric symplectic stack by means of groupoid Morse theory.

Example 2.4.1 (**Equivariant cohomology of toric symplectic stacks**)**.** Let $G^{(1)} \rightrightarrows G^{(0)}$ be a 0-symplectic groupoid equipped with a Hamiltonian $(K^{(1)} \rightrightarrows K^{(0)})$-groupoid 2-action with moment map μ verifying Proposition 2.1.3. For every $\xi \in \mathfrak{k}/\mathfrak{h}$ we have a Morse Lie groupoid morphism $\mu^\xi : (G^{(1)} \rightrightarrows G^{(0)}) \to (\mathbb{R} \rightrightarrows \mathbb{R})$ for which μ^ξ_0 is $K^{(0)}$-invariant since $K^{(0)}$ is abelian. Therefore, if there exists $\xi \in \mathfrak{k}/\mathfrak{h}$ such that μ^ξ verifies Assumption 2.3.1, then Proposition 2.4.5 allows to compute the equivariant cohomology associated to the 2-action of the foliation Lie 2-group $(K^{(1)} \rightrightarrows K^{(0)})$ on the 0-symplectic groupoid $G^{(1)} \rightrightarrows G^{(0)}$ by using the equivariant version of the groupoid Morse cohomology. However, this is in general a difficult task since powerful tools as a 2-equivariant version of the Atiyah–Bott localization theorem are necessary to compute 2-equivariant cohomology groups [77]. Let us visualize this situation in the simplest case. Let $(G^{(0)},\omega)$ be a compact S^1-Hamiltonian pre-symplectic manifold with moment map $\mu : G^{(0)} \to \mathbb{R}$ satisfying the Morse–Smale transversality condition, see [76, 107]. Assume that the S^1 action is clean and denote by $\mathcal{F}$ the foliation of $G^{(0)}$ associated to $\ker(\omega)$. From [76] we know that the critical point set of μ equals the set of fixed leaves $\mathcal{L}$ of $\mathcal{F}$ by the S^1 action

$$\mathrm{Crit}(\mu) := G^{(0)}(\mathcal{F})^{S^1} = \{\mathcal{L} : k\cdot\mathcal{L} = \mathcal{L} \text{ for all } k \in S^1\} = \{x \in G^{(0)} : \partial_\theta(x) \in T_x\mathcal{F}\}.$$

Here ∂_θ stands for the fundamental vector field of the S^1 action. We denote the set of leaves of index i by $G^{(0)}(\mathcal{F})^{S^1}_i$. Recall that i is always even. Let us check that $(\partial^0_{S^1})_v = 0$ for all $v > 0$, provided that the S^1 action fixes the points of the critical leaves. Firstly, the moduli space $\mathcal{M}^0(G^{(0)}(\mathcal{F})^{S^1}_{j+v}, G^{(0)}(\mathcal{F})^{S^1}_j)$ inherits an S^1 action which commutes with the endpoint maps. But such an action fixes the endpoints, so that

$$\iota_{\partial_\theta}(l(0)^{j+v}_j)^*(\beta) = (l(0)^{j+v}_j)^*\iota_{\partial_\theta}(\beta) = 0, \qquad \beta \in \Omega^\bullet(G^{(0)}(\mathcal{F})^{S^1}_j).$$

Secondly, the vector field ∂_θ is tangent to the fiber of $u(0)^{j+v}_j : \mathcal{M}^0(G^{(0)}(\mathcal{F})^{S^1}_{j+v}, G^{(0)}(\mathcal{F})^{S^1}_j) \to G^{(0)}(\mathcal{F})^{S^1}_{j+v}$ and therefore

$$\partial^0_v(\beta) = (u(0)^{j+v}_j)_*(l(0)^{j+v}_j)^*(\beta) = 0.$$

Hence, $\partial^{(0)}_{S^1} = (\partial^{(0)}_{S^1})_0$ and $H^\bullet_{S^1}(G^0,\partial^{(0)}_{S^1}) = \bigoplus_i H^{\bullet-i}_{S^1}(G^{(0)}(\mathcal{F})^{S^1}_i)$. This is exactly the equivariant Morse complex associated the unit Lie 2-group $S^1 \rightrightarrows S^1$ acting over the unit 0-symplectic groupoid $G^{(0)} \rightrightarrows G^{(0)}$.

We can use the previous data to obtain a more interesting Hamiltonian Lie groupoid. Indeed, Theorem 7.2.1 in [62] shows how to build a Hamiltonian Lie groupoid from a Hamiltonian pre-symplectic manifold. In the specific case described above we recover the unit Lie 2-group $S^1 \rightrightarrows S^1$ and any source-connected foliation Lie groupoid $G^{(1)} \rightrightarrows G^{(0)}$

integrating $\mathcal{F}$. That is to say, a Lie groupoid whose orbits in $G^{(0)}$ agree with the leaves of $\mathcal{F}$ and the Lie 2-group action of $S^1 \rightrightarrows S^1$ on $G^{(1)} \rightrightarrows G^{(0)}$ is obtained by extending that of S^1 on $G^{(0)}$. In particular, one may choose $G^{(1)}$ to be the Holonomy groupoid associated to the foliation $\mathcal{F}$. Additionally, we get an S^1-invariant Morse–Bott Lie groupoid morphism $\mu_1 : G^{(1)} \to \mathbb{R}$ by setting $\mu_1 = s^*\mu$. We endow $G^{(1)} \rightrightarrows G^{(0)}$ with a groupoid Riemannian metric, so that $\mathcal{F}$ becomes a Riemannian foliation. This implies that $G^{(1)}$ is proper since each leaf has a finite holonomy group (see Theorem 2.6 and page 141 in [92]). Hence, under these assumptions our constructions apply and we can similarly check that $(\partial^n_{S^1})_v = 0$ for all $n \geq 1$ and $v > 0$.

2.5 The stacky perspective

Recall that a stack can be thought of as a generalization of the notion of manifold which allows us to study higher symmetries and singular geometric features. The aim of this section is to adapt some of the Morse theory results obtained in the previous sections for Lie groupoids to the setting of differentiable stacks. It is worth mentioning that the fact that our notion of Morse Lie groupoid morphism is Morita invariant makes the passage clearer. As an interesting consequence of the notions studied below we will get Morse-like inequalities for certain separated differentiable stacks. The reader is recommended to visit Subsection 1.3 in order to refresh the notions that we need to use; in particular see Remark 1.3.1.

2.5.1 Stacky Morse functions

Let $[M/G]$ be a differentiable stack presented by a Lie groupoid $G \rightrightarrows M$. We are interested in studying Morse theory for **stacky functions**. More precisely, stacky maps from $[M/G]$ to $\mathbb{R}$ where we think of $\mathbb{R}$ as a differentiable stack presented by the unit groupoid $\mathbb{R} \rightrightarrows \mathbb{R}$. Since $\mathbb{R}$ has trivial isotropies every fraction $(G \rightrightarrows M) \underset{\sim}{\overset{\phi}{\leftarrow}} (H \rightrightarrows N) \xrightarrow{\psi} (\mathbb{R} \rightrightarrows \mathbb{R})$ sends arrows over identities so that it descends to a usual Lie groupoid morphism. In other words, every stacky function $[M/G] \to \mathbb{R}$ is completely determined by a Lie groupoid morphism $F : G \to \mathbb{R}$ and, in turn, by a basic function $f : M \to \mathbb{R}$. This provides us with a simple way to establish a notion of Morse stacky map over $[M/G]$ as we do below.

Recall also that the coarse tangent space $T_{[x]}[M/G]$ of $[M/G]$ at $[x] = \mathcal{O}$ is by definition the coarse orbit space $\nu_x(\mathcal{O})/G_x \cong \nu(\mathcal{O})/G_{\mathcal{O}}$ of the action groupoid determined by the normal representation on the orbit $\mathcal{O}$ through $x \in M$. Therefore, if $F : [M/G] \to \mathbb{R}$ is a stacky function presented by a basic function $f : M \to \mathbb{R}$ then the coarse differential at $[x] \in [M/G]$ is the map $dF_{[x]} : T_{[x]}[M/G] \to \mathbb{R}$ defined by $dF_{[x]}([v]) := df(x)(v)$. This is well defined in the sense that for $g \in G_x$ and $w \in T_xM$ such that $ds(g)(w) = v$ we have

$$dF_{[x]}(g \cdot [v]) = df(x)(dt(g)(w)) = df(x)(ds(g)(w)) = dF_{[x]}([v]),$$

since f is basic. In consequence, we say that $[x] \in [M/G]$ is a **critical point** of $F : [M/G] \to \mathbb{R}$ if $d_{[x]}F([v]) = 0$ for all $[v] \in T_{[x]}[M/G]$. This is clearly equivalent to requiring that $\mathcal{O}$ is a critical submanifold of $f : M \to \mathbb{R}$. We also define the **stacky Hessian** of $F : [M/G] \to \mathbb{R}$ at a critical point $[x] \in [M/G]$ as the pairing $\mathcal{H}_{[x]}(F) : T_{[x]}[M/G] \times T_{[x]}[M/G] \to \mathbb{R}$ given by

$$\mathcal{H}_{[x]}(F)([v_1], [v_2]) = \overline{\mathcal{H}_x(f)}([v_1], [v_2]),$$

where $\overline{\mathcal{H}_x(f)}$ denotes the restriction of the Hessian $\mathcal{H}_x(f)$ to the normal direction of $\nu_x(\mathcal{O})$.

From Lemma 2.2.2 it follows that the stacky Hessian $\mathcal{H}_{[x]}(F)$ is a well defined "form" over $T_{[x]}[M/G]$. Furthermore, it is simple to check that if $y \in \mathcal{O}$ then for $g \in G$ such that $t(g) = y$ it holds the identity

$$\overline{\mathcal{H}_y(f)} = (\overline{ds(g)} \circ \overline{dt(g)}^{-1})^T \cdot \overline{\mathcal{H}_x(f)} \cdot (\overline{ds(g)} \circ \overline{dt(g)}^{-1}), \tag{2.19}$$

so that $\mathcal{H}_x(f)$ is nondegenerate if and only if $\mathcal{H}_y(f)$ is nondegenerate. This justifies the following definition.

Definition 2.5.1. A critical point $[x] \in [M/G]$ of a stacky function $F : [M/G] \to \mathbb{R}$ is said to be **nondegenerate** if $\mathcal{H}_{[x]}(F)$ nondegenerate. Accordingly, a **Morse stacky function** is a stacky function for which all of its critical points are nondegenerate.

In other words, a stacky function $F : [M/G] \to \mathbb{R}$ is Morse if and only if it is presented by a Morse Lie groupoid morphism $G \to \mathbb{R}$.

Let us assume from now on that $[M/G]$ is separated, i.e. it is presented by a proper groupoid. Consider the quadratic form $Q_{[x]} : T_{[x]}[M/G] \to \mathbb{R}$ which is defined by the expression

$$Q_{[x]}(F)([v]) = \frac{1}{2}\mathcal{H}_{[x]}(F)([v],[v]).$$

Using the stacky terminology introduced in [41, s. 6] we can state a stacky version of the Morse lemma as follows.

Proposition 2.5.1 (Stacky Morse lemma). *Let $[x] \in [M/G]$ be a nondegenerate critical point of a stacky function $F : [M/G] \to \mathbb{R}$. Then there are stacky neighborhoods $[V/\nu(G_{\mathcal{O}})_V]$ and $[U/G_U]$ of $[x]$ in $[\nu(\mathcal{O})/\nu(G_{\mathcal{O}})]$ and $[M/G]$, respectively, and a stacky isomorphism $\varphi : [V/\nu(G_{\mathcal{O}})_V] \to [U/G_U]$ fixing $[x]$ such that*

$$\varphi^* F = c + Q_{[x]}(F).$$

Proof. The Lie groupoid tubular neighborhood from Theorem 2.2.1 which is constructed around the nondegenerate critical orbit $\mathcal{O} = [x]$ in such a way f equals $c + Q_f$ near $\mathcal{O}$ induces the desired data, see Proposition 6.4.2 in [41]. □

Observe that after shrinking if necessary the stacky neighborhoods mentioned in the previous proposition we have that $[0] \in T_{[x]}[M/G]$ is the only critical point of $\varphi^* F$ inside $[V/\nu(G_{\mathcal{O}})_V]$. So, we get that:

Corollary 2.5.1. *The nondegenerate critical points of a Morse stacky function are isolated in M/G.*

Furthermore, just as in both the classical and the equivariant cases (compare [116, Prop. 4,2]), from the previous fact it follows that:

Corollary 2.5.2. *If M/G is compact then any stacky Morse function $F : [M/G] \to \mathbb{R}$ has a finite amount of nondegenerate critical points.*

We define the **index data** of a nondegenerate critical point $[x]$ of a Morse stacky function $F : [M/G] \to \mathbb{R}$ as the pair $\lambda(F,[x]) := (\lambda(f,\mathcal{O}), G_x)$ where $\lambda(f,\mathcal{O})$ is the integer number $\mathrm{rk}(\nu_-(\mathcal{O}))$ for any basic function $f : M \to \mathbb{R}$ presenting F and G_x is the isotropy group at x. This is well defined in the sense that if $y \in \mathcal{O}$ then the normal representations $G_x \curvearrowright \nu_x(\mathcal{O})$ and $G_y \curvearrowright \nu_y(\mathcal{O})$ are isomorphic and the Identity (2.19) holds. Accordingly, based on Proposition 2.2.1, the **index** of $[x]$ will be defined as

$$\dim \nu_-(\mathcal{O}_x)_x/G_x = 2\dim \nu_-(\mathcal{O}_x)_x - \dim \nu_-(\mathcal{O}_x)_x \times G_x = \lambda(f,\mathcal{O}_x) - \dim G_x.$$

Let $F : [M/G] \to \mathbb{R}$ be a stacky function and fix $a \in \mathbb{R}$. We define the **stacky level** of F below a as the set $[M/G]^a = \{[x] \in [M/G] : F([x]) \leq a\}$. One can describe $[M/G]^a$ as a substack with boundary in the following sense. Suppose that $H \rightrightarrows N$ is a Lie groupoid together with Morita fibrations $\phi : H \to G$ and $\psi : H \to G'$. Recall that for every $f' \in C^\infty(M')^{G'}$ we have that $\psi^* f' \in C^\infty(N)^H$. Also, there exists a unique $f \in C^\infty(M)^G$ with $\phi^* f = \psi^* f'$. This defines an isomorphism $C^\infty(M')^{G'} \to C^\infty(M)^G$ by sending $f' \mapsto f$, which actually preserves our Morse–Bott type condition along critical orbits, see Proposition 2.1.1. Consider the level set $M^a = \{x \in M : f(x) \leq a\}$ with its boundary $\partial M^a = \{x \in M : f(x) = a\}$. It follows that M^a is saturated so that we get a level subgroupoid $G^a = s^{-1}(M^a) = t^{-1}(M^a)$ of G. We can analogously define level subgroupoids H^a and G'^a of H and G', respectively, where H^a is defined by either $\phi^* f$ or $\psi^* f'$. Note that we have well defined Morita fractions

$$\mathrm{int}(G^a) \underset{\sim}{\overset{\phi}{\leftarrow}} \mathrm{int}(H^a) \underset{\sim}{\overset{\psi}{\rightarrow}} \mathrm{int}(G'^a) \quad \text{and} \quad \partial G^a \underset{\sim}{\overset{\phi}{\leftarrow}} \partial H^a \underset{\sim}{\overset{\psi}{\rightarrow}} \partial G'^a,$$

so that we may think of $G^a \underset{\sim}{\overset{\phi}{\leftarrow}} H^a \underset{\sim}{\overset{\psi}{\rightarrow}} G'^a$ as a Morita fraction preserving boundaries. Hence, if $F : [M/G] \to \mathbb{R}$ is a stacky function presented by a basic function $f : M \to \mathbb{R}$ then the stacky level of F below $a \in \mathbb{R}$ equals

$$[M^a/G^a] = [\mathrm{int}(M^a)/\mathrm{int}(G^a)] \cup [\partial M^a/\partial G^a].$$

Let $[\eta]$ be a stacky metric on $[M/G]$, e.i. an equivalence class of a Riemannian 2-metric η on $G \rightrightarrows M$, see Subsection 1.3.1. Consider the **tangent stack** of $[M/G]$ which is by definition the differentiable stack $T[M/G] := [TM/TG]$ that is presented by the tangent groupoid $TG \rightrightarrows TM$, compare [58]. The multiplicative vector field defined by the gradient of $F : G \to \mathbb{R}$ with respect to the groupoid metric η defines, by means of the Dictionary Lemmas from [17, s. 2.6], a stacky vector field on $[M/G]$ in the sense of [58, Def. 4.14]. By using the identification of the coarse tangent space $T_{[x]}[M/G]$ with the orbit space $\nu_x(\mathcal{O})/G_x \cong \nu(\mathcal{O})/G_{\mathcal{O}}$ we may think of this stacky vector field as a stacky map $\nabla F : [M/G] \to T[M/G]$ given by $[x] \mapsto [\nabla f(x)]$, which is clearly well defined. We shall refer to it as the **stacky gradient vector field** of $F : [M/G] \to \mathbb{R}$ with respect to $[\eta]$. Observe that ∇F satisfies

$$[\eta](\nabla F[x], [v]) = [\eta]([\nabla f(x)], [v]) = \eta(\nabla f(x), v) = df(x)(v) = dF_{[x]}([v]).$$

Remark 2.5.1. From the point of view of Lie groupoids, Theorem 4.15 in [58] says that the category of multiplicative vector fields on G depends, up to equivalence, only on the Morita equivalence class of G. If we think of the differentiable stack $[M/G]$ as the equivalence class of G in the enlarged 2-category of Lie groupoids, principal bi-bundle and isomorphisms then the category of stacky vector field on $[M/G]$ is equivalent to category of multiplicative vector fields on G.

Let us consider the underlying continuous map $\overline{f} : M/G \to \mathbb{R}$.

Proposition 2.5.2. *Let $[M/G]$ be a separated stack and $[a, b] \subset \mathbb{R}$ be a real interval such that $\overline{f}^{-1}[a, b]$ is compact in M/G. If $\overline{f}^{-1}[a, b]$ has no critical points of F then $[M/G]^a$ and $[M/G]^b$ are stacky isomorphic. Furthermore, $[M/G]^a$ is a stacky deformation retraction of $[M/G]^b$.*

Proof. We shall follow the proof of [57, Thm. 7.5] closely and apply some of the results proved in [58] for stacky vector fields and flows. On the one hand, a simple computation shows that the smooth function $\|\nabla f\|^2 : M \to \mathbb{R}$ is basic since f is so and the gradient vector field ∇F on G is horizontal with respect to both Riemannian submersions s and t. On the other hand, since $[M/G]$ is a separated stack we may construct "stacky" partitions of unity for $[M/G]$, see [58, Def. 2.13 & Prop. 2.14]. In consequence, from [57, Lem. 3.11 & Lem. 3.12] we may find a stacky function $\rho : [M/G] \to \mathbb{R}$ with compact support in M/G and with $\overline{\rho} = 1/\overline{\|\nabla f\|^2}$ in $\overline{f}^{-1}[a,b]$. Therefore, we can form the stacky vector field $\tilde{X} = \rho\nabla F$ on $[M/G]$, which has compact support in M/G, and then take its stacky flow $\Phi : [M/G] \times \mathbb{R} \to [M/G]$ by using [58, Thm. 5.12].

The vector field $\tilde{X}$ can, by [58, Thm. 4.15] and the proof of [58, Prop. 4.17], be presented by a multiplicative vector field $X = (X_1, X_0)$ on $G \rightrightarrows M$ that is not equal to the zero section only on a subgroupoid of G whose image in M/G has compact closure. That is to say, the vector fields X_1 and X_0 are compactly-supported in $F^{-1}[a,b] \subset G$ and $f^{-1}[a,b] \subset M$, respectively. Thus, motivated by the proof of Proposition 2.2.4, we may clearly identify $X_1 = \tilde{\rho}\nabla F$ and $X_0 = \rho\nabla f$ where $\rho : M \to \mathbb{R}$ is the compactly-supported smooth function on M with $\rho = 1/\|\nabla f\|^2$ inside $f^{-1}[a,b]$. This is clearly basic and $\tilde{\rho} : G \to \mathbb{R}$ is given by either $s^*\rho$ or $t^*\rho$. Observe that $\tilde{\rho}$ is also compactly-supported in $F^{-1}[a,b]$.

Hence, the arguments above allow us to think of the stacky vector field $\tilde{X} : [M/G] \to T[M/G]$ as being determined by the assignment $[x] \mapsto [X_0(x)]$. Also, by using the Dictionary Lemmas from [17, s. 2.6] as in the proof of [58, Prop. 6.2], the stacky flow $\Phi : [M/G] \times \mathbb{R} \to [M/G]$ of $\tilde{X}$ is determined by $\Phi([x], \tau) = [\varphi^0_\tau(x)]$ where $(\varphi^1_\tau, \varphi^0_\tau)$, for all $\tau \in \mathbb{R}$, is the 1-parameter family of Lie groupoid automorphisms on G determined by the flow of the multiplicative vector field X. In particular, by using these identifications it holds that if $\Phi([x], \tau) \in \overline{f}^{-1}[a,b]$ then $\tilde{X} \cdot F(\Phi([x], \tau)) = 1$ so that the result follows either by proceeding without any chance as in the proof of [88, Thm. 3.1] or by using Proposition 2.2.4 directly. $\square$

Let us pick an orbit $\mathcal{O}$ of G and denote by $\mathcal{O}^H = \phi^{-1}(\mathcal{O})$ and $\mathcal{O}' = \psi(\mathcal{O}^H)$ the corresponding orbits of H and G'. From [41, Prop. 6.4.1] it follows that there is an induced Morita fraction

$$\nu(G_{\mathcal{O}}) \xleftarrow[\sim]{\overline{d\phi}} \nu(H_{\mathcal{O}^H}) \xrightarrow[\sim]{\overline{d\psi}} \nu(G'_{\mathcal{O}'}).$$

Observe that if in addition $\mathcal{O}$ is a nondegenerate critical orbit then the previous fraction induces a Morita fraction between the negative normal groupoids $\nu_-(G_{\mathcal{O}}) \xleftarrow[\sim]{\overline{d\phi}} \nu_-(H_{\mathcal{O}^H}) \xrightarrow[\sim]{\overline{d\psi}} \nu_-(G'_{\mathcal{O}'})$, compare Lemma 2.2.3. More importantly, if G and G' are the Morita equivalent Riemannian groupoids as we described above then there are Morita fractions between the unit and sphere groupoids:

$$D_-(G_{\mathcal{O}}) \xleftarrow[\sim]{\overline{d\phi}} D_-(H_{\mathcal{O}^H}) \xrightarrow[\sim]{\overline{d\psi}} D_-(G'_{\mathcal{O}'}) \quad \text{and} \quad \partial D_-(G_{\mathcal{O}}) \xleftarrow[\sim]{\overline{d\phi}} \partial D_-(H_{\mathcal{O}^H}) \xrightarrow[\sim]{\overline{d\psi}} \partial D_-(G'_{\mathcal{O}'}).$$

The stack $e^\lambda_{[x]} := [D_-(\mathcal{O})/D_-(G_{\mathcal{O}})]$ will be called **stacky λ-cell** at $[x]$ and its boundary $\partial e^\lambda_{[x]} := [\partial D_-(\mathcal{O})/\partial D_-(G_{\mathcal{O}})]$ will be called **stacky $(\lambda - 1)$-sphere** at $[x]$.

Remark 2.5.2. We may think of the attaching space $[M/G]^a \cup_{\partial e^\lambda_{[x]}} e^\lambda_{[x]}$ as being a topological stack presented by the attaching groupoid $G^a \cup_B D_-(G_{\mathcal{O}})$. This is because we can consider the topological Morita fraction between the attaching groupoids

$$G^a \cup_B D_-(G_{\mathcal{O}}) \xleftarrow[\sim]{\phi \cup \overline{d\phi}} H^a \cup_B D_-(H_{\mathcal{O}^H}) \xrightarrow[\sim]{\psi \cup \overline{d\psi}} G'^a \cup_B D_-(G'_{\mathcal{O}'}).$$

Whit this notation we have:

Proposition 2.5.3. *Let $[M/G]$ be a separated stack and $[a,b] \subset \mathbb{R}$ be a real interval such that $\overline{f}^{-1}[a,b]$ is compact in M/G. If $\overline{f}^{-1}[a,b]$ contains no critical points besides $[x]$ of index data $\lambda(F,[x])$ then $[M/G]^b$ is stacky homotopy equivalent to $[M/G]^a \cup_{\partial e^{\lambda}_{[x]}} e^{\lambda}_{[x]}$.*

Proof. We start by noting that by Lemma 2.2.2 and Proposition 2.2.1 we may split the coarse tangent space as $T_{[x]}[M/G] = T^-_{[x]}[M/G] \oplus T^+_{[x]}[M/G]$ where $T^-_{[x]}[M/G] = \nu_-(\mathcal{O})_x/G_x$ and $T^+_{[x]}[M/G] = \nu_+(\mathcal{O})_x/G_x$. In particular, by Lemma 2.2.5 it follows that the two mutually complementary projections $\xi_0 : \nu(\mathcal{O}) \to \nu_-(\mathcal{O})$ and $\eta_0 : \nu(\mathcal{O}) \to \nu_+(\mathcal{O})$ descend to define coordinates over $T^-_{[x]}[M/G]$ and $T^+_{[x]}[M/G]$, respectively, and a norm on $T_{[x]}[M/G]$ as:

$$\|[v]\|^2 = -Q_{[x]}(F)(\xi_0([v])) + Q_{[x]}(F)(\eta_0([v])),$$

which agrees with the norm induced by $[\eta]$. Now, by Proposition 2.5.1 we have that there are stacky neighborhoods $[V/\nu(G_{\mathcal{O}})_V]$ and $[U/G_U]$ of $[x]$ in $[\nu(\mathcal{O})/\nu(G_{\mathcal{O}})]$ and $[M/G]$, respectively, and a stacky isomorphism $\varphi : [V/\nu(G_{\mathcal{O}})_V] \to [U/G_U]$ fixing $[x]$ such that $\varphi^* F = c + Q_{[x]}(F)$. Under the identification $[V/\nu(G_{\mathcal{O}})_V] \cong [U/G_U]$ we may think of F on $[U/G_U]$ as given respectively by

$$F([v]) = c + Q_{[x]}(F)([v]) = c - \|\xi_0([v])\|^2 + \|\eta_0([v])\|^2.$$

Let us consider the stacky function $F_1 : [M/G] \to \mathbb{R}$ determined by the basic function $f_1 : M \to \mathbb{R}$ constructed in Theorem 2.2.2. Such a stacky function can be described in the way we just did above with F. Therefore, the result follows by following the same steps in [57, Thm. 7.6] after using Proposition 2.5.2, Theorem 2.2.2 together the Dictionary Lemmas from [17, s. 2.6], and Corollary 2.2.1. □

Recall that as consequence of Proposition 2.2.1 we know that the orbit spaces associated to $e^{\lambda}_{[x]}$ and $\partial e^{\lambda}_{[x]}$ are respectively given by $D_-(\mathcal{O})_x/G_x$ and $\partial D_-(\mathcal{O})_x/G_x$. Thus:

Corollary 2.5.3. *The orbit space $(M/G)^b$ has the homotopy type of $(M/G)^a$ with a copy of $D_-(\mathcal{O})_x/G_x$ attached along $\partial D_-(\mathcal{O})_x/G_x$.*

2.5.2 Morse inequalities for the orbit space

By following similar arguments as those used by Hepworth in [57] it is possible to prove Morse-like inequalities for certain separated stacks. We shall follow [57, Sec. 7.3] closely. Suppose that $[M/G]$ is a separated differentiable stack with M/G compact and let $F : [M/G] \to \mathbb{R}$ be a stacky Morse function presented by a basic function $f : M \to \mathbb{R}$. From Corollary 2.5.1 it follows that the critical points of F are isolated so that we may take a finite sequence $q_0 < q_1 < q_2 < \cdots < q_r \in \mathbb{R}$ such that each interval (q_j, q_{j+1}) contains only one critical value of F and such that all critical values lie inside such intervals. Let us denote by $[x^j_1], \cdots, [x^j_{k_j}]$ the critical points inside $\overline{f}^{-1}(q_j, q_{j+1})$. Therefore, by an inductive process it follows from Corollary 2.5.3 that:

Corollary 2.5.4. *There is a decomposition $M/G = \bigcup_{j=1}^r (M/G)^{q_j}$ where each $(M/G)^{q_{j+1}}$ has the homotopy type of $(M/G)^{q_j}$ with copies of $D_-(\mathcal{O}_{x^j_l})_{x^j_l}/G_{x^j_l}$ attached along $\partial D_-(\mathcal{O}_{x^j_l})_{x^j_l}/G_{x^j_l}$ for $l = 1, \cdots, k_j$.*

Recall that the **Betti numbers** of M/G are by definition $b_j = \dim H_j(M/G, \mathbb{R})$ and the **Poincaré polynomial** is given by

$$\mathcal{P}_\tau(M/G) = \sum b_j \tau^j.$$

Definition 2.5.2. A critical point $[x]$ of a stacky Morse function $F : [M/G] \to \mathbb{R}$ is said to be **orientable** if the action of G_x on $\nu_-(\mathcal{O})_x$ from Proposition 2.2.1 is orientation-preserving. The **Morse polynomial** of F is defined as

$$\mathcal{M}_\tau(M/G) = \sum_{[x]\in \mathrm{Crit}(f)_{o-p}} \tau^{\dim \nu_-(\mathcal{O}_x)_x/G_x},$$

where $\mathrm{Crit}(f)_{o-p}$ stands for the set of orientable critical points of F.

We are now in conditions to state that:

Theorem 2.5.1. *There is a polynomial $\mathcal{R}_\tau(M/G)$ with non-negative integer coefficients such that*

$$\mathcal{M}_\tau(M/G) = \mathcal{P}_\tau(M/G) + (1+\tau)\mathcal{R}_\tau(M/G).$$

In particular, if $\mathcal{M}_\tau(M/G)$ has no consecutive powers of τ then $\mathcal{M}_\tau(M/G) = \mathcal{P}_\tau(M/G)$.

Proof. The proof of this result is similar to [57, Thm. 7.11]. We shall sketch its main ideas here for the sake of completeness. It is well known that the function

$$S_j(X,Y) = \dim H_j(X,Y;\mathbb{R}) - \dim H_{j-1}(X,Y;\mathbb{R}) + \cdots \pm \dim H_0(X,Y;\mathbb{R}), \tag{2.20}$$

is subadditive in the sense that if $Z \subset Y \subset X$ then we get $S_j(X,Z) \leq S_j(X,Y) + S_j(Y,Z)$. Thus, from Corollary 2.5.4 it follows that

$$S_j(M/G,\emptyset) = S_j((M/G)^{q_r},(M/G)^{q_0}) \leq \sum_v S_j((M/G)^{q_v},(M/G)^{q_{v-1}}).$$

Using again Corollary 2.5.4 and the excision theorem for relative homology we obtain

$$\begin{aligned} H_j((M/G)^{q_v},(M/G)^{q_{v-1}};\mathbb{R}) &= H_j\left((M/G)^{q_{v-1}} \cup \bigcup_l D_-(\mathcal{O}_{x_l^v})_{x_l^v}/G_{x_l^v}, (M/G)^{q_{v-1}};\mathbb{R}\right) \\ &= \bigoplus_l H_j\left(D_-(\mathcal{O}_{x_l^v})_{x_l^v}/G_{x_l^v}, \partial D_-(\mathcal{O}_{x_l^v})_{x_l^v}/G_{x_l^v};\mathbb{R}\right) \end{aligned}$$

where the homology $H_j\left(D_-(\mathcal{O}_{x_l^v})_{x_l^v}/G_{x_l^v}, \partial D_-(\mathcal{O}_{x_l^v})_{x_l^v}/G_{x_l^v};\mathbb{R}\right)$ equals $\mathbb{R}$ if $[x_l^v]$ is orientable and $j = \dim \nu_-(\mathcal{O}_{x_l^v})_{x_l^v}/G_{x_l^v}$. Otherwise, it equals 0. Hence, the sum (2.20) becomes

$$S_j(M/G,\emptyset) \leq A_j - A_{j-1} + \cdots \pm A_0, \tag{2.21}$$

where A_l denotes the number of orientable critical points $[x]$ of F with $l = \dim \nu_-(\mathcal{O}_x)_x/G_x$. However, as consequence of [14, Lem. 3.43] it holds that Inequality (2.21) is actually equivalent to the claim of the proposition so that the result follows. □

It is clear that the polynomial equality in the previous proposition only depends on the Morita equivalence class of G. Therefore, we have actually obtained Morse-like inequalities for the differentiable stack $[M/G]$.

Chapter 3

Closed geodesics on Riemannian stacks

The main purpose of this chapter is to show an existence result for stacky closed geodesic of positive length over certain separated Riemannian stacks. In order to do so, we need to apply as well as adapt some techniques from the classical Morse theory for Hilbert manifolds to the realm of differentiable stacks presented by proper Lie groupoids. Roughly speaking, our strategy to address such an existence result relies on first studying the case of regular proper Riemannian groupoids and then using the Riemannian desingularization of proper Riemannian groupoids due to Posthuma–Tang–Wang.

We study the set of closed stacky curves over a regular separated Riemannian stack, proving that it admits a natural structure of Hilbert manifold. In consequence, we verify that the corresponding stacky energy functional has as critical points the set of closed geodesics on the regular stack. After developing some topological constructions in which the stacky energy functional is involved, we show that it satisfies the so-called Palais–Smale condition (C). The consequences derived from this property allow us to obtain our existence result by imposing some reasonable assumptions over the Posthuma–Tang–Wang desingularization which come motivated by the case of orbifolds, first studied by Guruprasad–Haefliger.

It is worth mentioning that the results of this chapter are based on joint work in progress with C. Ortiz and L. Vitagliano.

3.1 Stacky geodesics

In this short section we introduce the terminology and some of the results regarding the notion of stacky geodesic which was recently introduced in [38]. We shall be following [38, 40, 41, 56] closely. Let $G \rightrightarrows M$ be a proper Lie groupoid and denote by $[M/G]$ its associated separated differentiable stack. As usual, the structural maps of $G \rightrightarrows M$ will be denoted by (s, t, m, u, i). It follows that $G \rightrightarrows M$ can be equipped with a Riemannian 2-metric $\eta = \eta^{(2)}$ on $G^{(2)}$ which induces a Riemannian 1-metric $\eta^{(1)}$ on G as well as a Riemannian 0-metric $\eta^{(0)}$ on M such that $\pi_2, m, \pi_1 : G^{(2)} \to G$ and $s, t : G \to M$ are Riemannian submersions and $i : G \to G$ in an isometry. From Subsection 1.3.1 we know that the characteristic foliations $\mathcal{F}_M$ of M and $\mathcal{F}_G = s^*\mathcal{F}_M = t^*\mathcal{F}_M$ of G are singular Riemannian foliations, the units $u(M) \subset G$ form a totally geodesic submanifold and the normal representation on orbits is by linear isometries, consult [40].

We shall be working with stacky maps using a **cocycle description**, see Example 1.3.11. In these terms, a **stacky curve** $\alpha : I \to [M/G]$ is a stacky map from a real interval I viewed as a stack via the unit groupoid $I \rightrightarrows I$. If $\mathcal{U} = (U_i)$ is an open cover of I then it is simple to check that α is given by the class of a cocycle $(a, \mathcal{U})$ determined by paths $a_{ji} : U_j \cap U_i \to G$

verifying

$$a_{kj}(\tau)a_{ji}(\tau)=a_{ki}(\tau),\qquad \tau\in U_k\cap U_j\cap U_i.$$

This implies in particular that $a_{ii}(\tau)$ is a unit of G so that a_{ii} can be considered as a path $a_i:U_i\to M$, thus obtaining arrows $a_j(\tau)\xleftarrow{a_{ji}(\tau)} a_i(\tau)$. The situation above may be depicted with the fraction diagram

$$(I\rightrightarrows I)\underset{\sim}{\xleftarrow{\pi_{\mathcal{U}}}}\left(\coprod_{j,i}U_j\cap U_i\rightrightarrows\coprod_i U_i\right)\xrightarrow{a}(G\rightrightarrows M),$$

where the Lie groupoid in the middle is the Čech groupoid $I_{\mathcal{U}}$ associated to the open cover $\mathcal{U}$ of I and $\pi_{\mathcal{U}}:I_{\mathcal{U}}\to I$ is the canonical projection which turns out to be a Morita fibration. It is important to mention that the class of the cocycle $(a,\mathcal{U})$ is defined with respect to the following equivalence relation. Let $\mathcal{U}'$ be another open cover of I and let $(a',\mathcal{U}')$ denote another cocycle. We say that $(a,\mathcal{U})$ and $(a',\mathcal{U}')$ are **equivalent** if there exists a common refinement $\mathcal{U}''$ of $\mathcal{U}$ and $\mathcal{U}'$ together with an isomorphism between the groupoid maps $a|_{I_{\mathcal{U}''}}\cong a'|_{I_{\mathcal{U}''}}:I_{\mathcal{U}''}\to G$. We call $a=(a_{ji})$ a **good cocycle** if it is supported on a dimension 1 cover, namely $i\in\mathbb{Z}$ with $U_j\cap U_i=\emptyset$ except for consecutive i,j and there are no triple intersections $U_k\cap U_j\cap U_i$ for different i,j,k. Of course, any stacky curve can be presented by a good cocycle, and two good cocycles define the same stacky curve if they restrict to isomorphic good cocycles on a common refinement, see [38].

Let us exhibit some examples which were already described in [38].

Example 3.1.1. Suppose that $\pi:M\to N$ is a surjective submersion and denote by $G\rightrightarrows M$ its corresponding submersion Lie groupoid. A good cocycle $(a_{ji}):I_{\mathcal{U}}\to G$ is a collection of local lifts a_i to M of a given curve on N. The transitions $a_{i+1,i}$ are completely determined by the curves a_i, as there is no isotropy. Furthermore, two cocycles are equivalent if they are determined by the same curve on N.

Example 3.1.2. Let $G\rightrightarrows M$ be a proper étale groupoid, so that $[M/G]$ is an orbifold. A curve $I\to[M/G]$ is classically defined as a continuous curve $\alpha:I\to M/G$ that can be locally lifted to smooth curves $a_i:I_i\to U_i$ on orbifold charts over U_i. It follows that a stacky curve $\alpha:I\to[M/G]$ induces a curve in this classic sense. Indeed, if (a_{ji}) is a cocycle representing α then the curves a_i serve as local lifts into orbifold charts.

Example 3.1.3. Let K be a Lie group acting on a smooth manifold M and consider the corresponding action groupoid $K\ltimes M\rightrightarrows M$. A good cocycle (a_{ji}) presenting a stacky curve $\alpha:I\to[M/G]$, with $G=K\ltimes M$, can be expressed as a family of curves $a_i:U_i\to M$ and $k_{i+1,i}:U_{i+1}\cap U_i\to K$ verifying the condition $k_{i+1,i}(\tau)\cdot a_i(\tau)=a_{i+1}(\tau)$ for all $\tau\in U_{i+1}\cap U_i$. It is simple to check that the collection of curves (k_{ji}) defines a K-cocycle over the covering (U_i) of I. But every principal K-bundle over I is trivial since it is contractible, so that we can integrate the cocycle in such a way we gain a global representative $a:I\to M$ for any stacky curve α.

Example 3.1.4. Consider a regular foliation $\mathcal{F}$ of a smooth manifold M. By following [23] we get that the foliation $\mathcal{F}$ can be described in terms of a family of submersion $\pi_i:V_i\to W_i$ where $V_i\subset M$ and $W_i\subset\mathbb{R}^q$ are such that if $V_j\cap V_i\neq\emptyset$ then there is a diffeomorphism $\gamma_{ji}:W_i\to W_j$ verifying $\gamma_{ji}\pi_i=\pi_j$. Let us consider the differentiable stack $[M/\mathrm{Hol}(M,\mathcal{F})]$ presented by the holonomy groupoid $\mathrm{Hol}(M,\mathcal{F})\rightrightarrows M$ and fix a defining submersion $\pi_i:V_i\to W_i$. In these terms, a stacky curve $\alpha:I\to[M/\mathrm{Hol}(M,\mathcal{F})]$ can be presented as the class of a good cocycle (a_{ji}) where (U_i) is a good open cover of I such that $a_i:U_i\to V_i$

and $a_{ji}(\tau)$ equals the diffeomorphism γ_{ji} at $a_i(\tau)$. It follows that the relevant information of each curve a_i is in the composition $b_i = \pi_i \circ a_i : U_i \to W_i$ and a stacky curve on the leaf space is the same as a family of curves $b_i : U_i \to W_i$ that are connected by the defining cocycle (γ_{ji}).

Let us now define both the velocity and the speed of a stacky curve $\alpha : I \to [M/G]$ at every time $\tau_0 \in I$. We denote by $[\eta]$ the stacky metric on $[M/G]$ presented by η. Recall that the coarse tangent space of the differentiable stack $[M/G]$ at $[x]$ is by definition the coarse orbit space $\nu_x(\mathcal{O})/G_x \cong \nu(\mathcal{O})/G_{\mathcal{O}}$ of the action groupoid determined by the normal representation on the orbit $\mathcal{O}$ through $x \in M$, compare [42]. Suppose that α is presented by the class of a good cocycle (a_{ji}) which is supported over the good open cover (U_i) of I. The **velocity** of a stacky curve $\alpha : I \to [M/G]$ at $\tau_0 \in I$ is

$$\dot{\alpha}(\tau_0) = d_{[\tau_0]}\alpha[\partial_\tau|_{\tau_0}] = [\dot{a}_k(\tau_0)] \in T_{[a_k(\tau_0)]}[M/G], \quad \tau_0 \in U_k.$$

Given that any groupoid metric η on $G \rightrightarrows M$ yields a G_x-invariant inner product on $\nu_x(\mathcal{O})$, we can define the normal norm $\|v\|_N$ of any vector $v \in T_xM$, so that we set the **speed** of α at τ_0 as $\|\dot{\alpha}(\tau_0)\| = \|\dot{a}_k(\tau_0)\|_N$. The first key result in [38] says that this speed varies continuously so that it makes sense to define the **length** of a stacky curve $\alpha : I \to [M/G]$ as $L(\alpha) = \int_I \|\dot{\alpha}(\tau)\| d\tau$. By using the class of good cocycle $(a, \mathcal{U})$ presenting α we obtain

$$L(\alpha) = \int_I \|\dot{\alpha}(\tau)\| d\tau = \sum_i \int_{U_i} \|\dot{a}_i(\tau)\|_N d\tau - \sum_i \int_{U_{i+1}\cap U_i} \|\dot{a}_{i+1,i}(\tau)\|_N d\tau.$$

This expression makes sense since there are no triple intersections and the first sum is counting twice each overlap $U_{i+1} \cap U_i$. Also, it is important to remember that for each $\tau \in U_{i+1} \cap U_i$, the source and target maps yield linear isometries $\nu_{a_{i+1,i}(\tau)}(G_{\mathcal{O}}) \to \nu_{a_i(\tau)}(\mathcal{O})$ and $\nu_{a_{i+1,i}(\tau)}(G_{\mathcal{O}}) \to \nu_{a_{i+1}(\tau)}(\mathcal{O})$ which send $[\dot{a}_{i+1,i}(\tau)]$ to $[\dot{a}_i(\tau)]$ and $[\dot{a}_{i+1}(\tau)]$, respectively. It follows that the stacky length allows us to recover a normal pseudo-distance on the coarse orbit space M/G of a proper groupoid $G \rightrightarrows M$ equipped with a metric on the units that is transversally invariant, i.e. a 0-metric, as studied in [103]. Namely, given a Riemannian groupoid $(G \rightrightarrows M, \eta)$ this normal pseudo-distance d_N can be defined by considering chains $(x_0, \cdots, x_{2n+1})$ such that x_{2i}, x_{2i+1} are in the same orbit and x_{2i-1}, x_{2i} are in the same component:

$$d_N([x],[y]) = \inf \left\{ \sum_{i=1}^{n} d(x_{2i-1}, x_{2i}) : \ (x_i) \text{ chain from } x \text{ to } y \right\}, \tag{3.1}$$

for $[x],[y] \in M/G$. Here d denotes the distance on M induced by the Riemannian metric $\eta^{(0)}$. If $G \rightrightarrows M$ is proper and M/G is connected then d_N defines a honest distance on M/G [103]. More importantly, if M/G is connected then $d_N([x],[y])$ is the infimum of the lengths $L(\alpha)$ of stacky curves $\alpha : I \to [M/G]$ joining $[x]$ and $[y]$. As consequence, equivalent metrics on the same Lie groupoid $G \rightrightarrows M$ yield the same normal pseudo-distance on the coarse orbit space M/G, consult [38].

We are ready to establish the main definition this chapter is concerned with. By normal geodesic to a singular Riemannian foliation $\mathcal{F}$ we will mean a geodesic verifying the property that if it is normal to $\mathcal{F}$ at a given time then it remains normal to $\mathcal{F}$ at every time.

Definition 3.1.1. A stacky curve $\alpha : I \to [M/G]$ is said to be **stacky geodesic** if it can be presented by the class of a cocycle $\alpha = (a, \mathcal{U})$ on which each $a_{ji} : U_j \cap U_i \to G$ is a normal geodesic of G with respect to $\mathcal{F}_G$.

This requirement automatically implies that the curves $a_i : U_i \to M$ are also normal geodesics of M with respect to $\mathcal{F}_M$. It is simple to check that every stacky geodesic can be represented by a good cocycle of normal geodesics, and two stacky geodesics are equivalent if they yield the same good cocycle on a common refinement.

Let us illustrate this notion with some basic examples.

Example 3.1.5. Let $\pi : M \to N$ be a Riemannian submersion and consider the corresponding submersion Riemannian groupoid $M \times_N M \rightrightarrows M$. If $\alpha : I \to [M/G]$, with $G = M \times_N M$, is a stacky geodesic presented by the class of a good cocycle (a_{ji}) then each geodesic $a_i : U_i \to M$ is a horizontal geodesic and it projects to a geodesic on N. Conversely, for any geodesic on N it holds that the horizontal local lifts are geodesics and these can be used to construct a good cocycle (a_{ji}) for a stacky geodesic $\alpha : I \to [M/G]$. This can be done by following the procedure explained in Example 3.1.1. In consequence, the notion of stacky geodesic recovers the usual notion for manifolds.

Example 3.1.6. Suppose that $G \rightrightarrows M$ is a proper étale Riemannian groupoid, so that its associated stack $[M/G]$ becomes a Riemannian orbifold. By the classical definition, an orbifold geodesic $\alpha : I \to [M/G]$ is a continuous curve $\alpha : I \to M/G$ that locally lifts to a smooth geodesic $a_i : J \to U_i$ into some Riemannian orbifold chart (U_i, G_i, φ). This definition corresponds to the notion of stacky geodesic, as the local lifts (a_i) determines a cocycle for a stacky geodesic $I \to [M/G]$ at the level of objects and and it can always be extended at the level of arrows, see Lemma 7 and Corollary 8 in [38].

Example 3.1.7. Let K be a compact Lie group endowed with a bi-invariant metric η^K and suppose that it acts on a Riemannian manifold (M, η^M) by isometries. By using the gauge trick introduced in [40] and the Riemannian metrics η^K and η^M, it is possible to build a structure of Riemannian groupoid over the action groupoid $K \ltimes M \rightrightarrows M$. It follows that for a stacky geodesic $\alpha : I \to [M/G]$, with $G = K \ltimes M$, each curve a_i yields a normal geodesic in M and each a_{ji} gives rise to an element k_{ji} in K such that $k_{ji} \cdot a_i|_{U_j \cap U_i} = a_j|_{U_j \cap U_i}$. But the action of K on M is by isometries for which any stacky geodesic α can be presented by a single normal geodesic $\alpha : I \to M$. In other words, we are just translating the local pieces a_i by using the action of K in order to get a single a.

Example 3.1.8. Let us now consider a regular Riemannian foliation $\mathcal{F}$ on a Riemannian manifold M. In the terminology of Example 3.1.4, the submersions π_i have to be Riemannian submersions and the diffeomorphisms γ_{ji} have to be isometries. After fixing the defining Riemannian submersions $\pi_i : (V_i, \eta^{V_i}) \to (W_i, \eta^{W_i})$, a stacky geodesic $\alpha : I \to [M/\mathrm{Hol}(M, \mathcal{F})]$ can be given by normal geodesics $a_i : U_i \to V_i$ which satisfy $\pi_j a_j = \gamma_{ji} \pi_i a_i$. The transversal information of each geodesic a_i is captured by the geodesics $b_i : U_i \to W_i$. Furthermore, under this description, we can think of a geodesic on the leaf space $M/\mathcal{F}$ as a family of geodesics $b_i : U_i \to (W_i, \eta^{W_i})$ that are glued by the defining cocycle $\gamma_{i+1,i}$, consult [3].

Existence and uniqueness results regarding stacky geodesics were proven in [38]. More importantly, the authors characterized stacky geodesics as locally minimizing curves and established a stacky version of the so-called Hopf–Rinow Theorem. Such a minimizing property is as follows. We say that a stacky curve $\alpha : I \to [M/G]$ is **minimizing** at $\tau_0 \in I$ if the length of α from τ_0 to τ equals the normal pseudo-distance between $\alpha(\tau)$ and $\alpha(\tau_0)$ for every τ near enough τ_0. It follows that if $G \rightrightarrows M$ is a proper Riemannian groupoid then α is a stacky geodesic if and only if it is minimizing at every $\tau_0 \in I$, i.e. it is locally minimizing; compare [38].

This chapter aims at showing an existence result for stacky closed geodesic of positive length over certain separated Riemannian stacks. In order to do so we need to apply as well as adapt some techniques from the classical Morse theory for Hilbert manifolds to our setting.

Definition 3.1.2. A stacky curve $\alpha : [0,1] \to [M/G]$ is named to be **closed** if $\alpha(0) = \alpha(1)$ and $\dot{\alpha}(0) = \dot{\alpha}(1)$.

Note that the conditions $\alpha(0) = \alpha(1)$ and $\dot{\alpha}(0) = \dot{\alpha}(1)$ allow us to think of closed stacky curves as "pointed" stacky maps $\alpha : (S^1, e) \to ([M/G], [x])$ which are presented by classes of good cocycles supported by open covers of S^1, where the circle S^1 is viewed as a stack via the unit groupoid $S^1 \rightrightarrows S^1$. Here by pointed we mean that $\alpha(e) = [x]$.

3.2 The case of regular proper Lie groupoids

The main idea we have to attack the problem regarding the existence of stacky closed geodesics of positive length on separated Riemannian stacks is based on first studying the case of regular proper Riemannian groupoids and then using the Riemannian desingularization of proper Riemannian groupoids recently introduced in [105]. In order to be precise we need to introduce some necessary terminology. Let $(G \rightrightarrows M, \eta^{(2)})$ be a proper Riemannian groupoid presenting a separated Riemannian stack $([M/G], [\eta])$. Roughly speaking, the Riemannian desingularization result from [105] states that there exists a regular proper Riemannian groupoid $(\tilde{G} \rightrightarrows \tilde{M}, \tilde{\eta}^{(2)})$ and a proper surjective groupoid morphism $\pi : \tilde{G} \to G$ which is an isometry almost everywhere, i.e. an isometry over a dense and open saturated submanifold in M. The triple $(\tilde{G}, \pi, \tilde{\eta}^{(2)})$ is called a **Riemannian desingularization** of $(G, \eta^{(2)})$. Such a desingularization can be obtained by a successive blow-up construction with respect to the stratification by dimensions induced by the leaves of the singular Riemannian foliation $\mathcal{F}_M$. As an important feature, we have that Morita equivalent Riemannian groupoids admit Morita equivalent Riemannian desingularizations, so that we may think of the Riemannian stack desingularization of $([G/M], [\eta])$ as

$$[\pi] : ([\tilde{G}/\tilde{M}], [\tilde{\eta}]) \to ([G/M], [\eta]),$$

which, in the terminology introduced in [41], is a stacky isometry almost everywhere.

Because of our purposes, it will be necessary at least to sketch the fist step, i.e. what happens after the first blow-up in the construction of the Riemannian 2-metric $\tilde{\eta}$. Firstly, we need to say how to build the blow-up along a submanifold.

Remark 3.2.1 (Blow-up along a submanifold)**.** Let S be a closed submanifold of a smooth manifold M and $\xi : V \subset \nu(S) \to U \subset M$ be a tubular neighborhood of S. If $\mathbb{P}(\nu(S))$ denotes the real projectivization of the normal bundle $\nu(S)$ then we consider the set $\tilde{V} = \{(v, l) \in V \times \mathbb{P}(\nu(S)) : v \in l\}$ and the map $\pi : \tilde{V} \to V$ sending $(v, l) \mapsto v$. They satisfy the following properties: $\tilde{V}$ is a smooth manifold such that π is a smooth proper map, $E = \pi^{-1}(S)$ is a smooth manifold of dimension $\dim M - 1$ which is diffeomorphic to $\mathbb{P}(\nu(S))$, $\pi|_{\tilde{V}-E} : \tilde{V} - E \to V - S$ is a diffeomorphism, and $\pi|_E : E \to S$ is a submersion. The manifold E is called **exceptional divisor**. The **blow-up** of M along S is defined to be $(\tilde{M}, \pi)$ where $\tilde{M} := \tilde{V} \cup_{\xi\circ\pi} (M - S)$ and $\pi := (\xi \circ \pi) \cup_{\xi\circ\pi} \mathrm{id} : \tilde{M} \to M$. It follows that $\tilde{M}$ is a smooth manifold and π is a smooth proper map verifying that $\pi|_{\tilde{M}-E} : \tilde{M} - E \to M - S$ is a diffeomorphism and that $\pi|_E : E \to S$ is a submersion. The isomorphism class of the blow-up does not depend on the choice of the tubular neighborhood for $S \subset M$. For more details the reader is recommended to consult [45, s. 2.9].

We consider now the proper Lie groupoid $G \rightrightarrows M$ and pick a closed saturated submanifold $S \subset M$. Let $S \subset U \subset M$ and $S \subset V \subset \nu(S)$ be a tubular neighborhood of S such that $G_U \cong \nu(G_S)_V$, which exists by the linearization theorem for proper Lie groupoids, see Subsection 1.3. Let $\tilde{M}$ denote the blow up of M along S that can be constructed by using the tubular neighborhood above. From [105] we know that G acts smoothly on $\tilde{M}$ along π and that the corresponding action groupoid $G \times_M \tilde{M}$ is isomorphic to the blow-up of G along G_S.

As done in [105] we set:

Definition 3.2.1. The action groupoid $\tilde{G} := G \times_M \tilde{M} \rightrightarrows \tilde{M}$ is defined to be the **blow-up** of G along S.

Secondly, we need to explain how is defined the stratification by dimensions used in [105].

Remark 3.2.2 (Stratification by dimensions). Let $G \rightrightarrows M$ be a proper Lie groupoid. For $0 \leq k \leq \dim M$ we set $S^k = \{x \in M : \text{codim}(\mathcal{O}_x) = k\}$. The connected components of S^k form a stratification of M. If $j = \max\{0 \leq k \leq \dim M : S^k \neq \emptyset\}$ and $m = \min\{0 \leq k \leq \dim M : S^k \neq \emptyset\}$ then S^j and S^m are referred to as the **most singular stratum** and the **most regular stratum** of M, respectively. In this case S^m is open, dense and connected in M, see Proposition 3.6 in [105]. Let $S \subset S^j$ be a most singular stratum. Then the blow-up $\tilde{G}$ of G along S is a proper Lie groupoid. Moreover, the blow-down map $\pi : \tilde{G} \to G$ is a smooth surjective proper groupoid morphism, which is an isomorphism almost everywhere and the leaves $L \subset \pi^{-1}(S)$ of the characteristic foliation $\mathcal{F}_{\tilde{M}}$ satisfy that $\pi(L)$ is a leaf of the characteristic foliation $\mathcal{F}_M$ in S with $\dim L > \dim \pi(L)$. After blowing up a finite amount of times, through the same procedure, we get a regular proper groupoid, also denoted by $\tilde{G} \rightrightarrows \tilde{M}$, and a proper surjective groupoid morphism $\pi : \tilde{G} \to G$ which turns out to be an isomorphism almost everywhere. See [105] for specific details.

Recall that a groupoid n-metric, for $n \geq 0$, can be thought of as a sort of **simplicial metric**. That is, as a collection of Riemannian metrics $\eta = \{\eta^{(n)}\}_{n \geq 0}$ where $\eta^{(n)}$ is a Riemannian metric on the component $G^{(n)}$ of the nerve $G^{(\bullet)}$ of G such that each face map $G^{(n)} \to G^{(n-1)}$ is a Riemannian submersion and the group S_{n+1} acts on $G^{(n)}$ by isometries. We already know that any proper Lie groupoid admits a simplicial metric [40]. For instance, in Example 1.3.23 we mentioned that the following construction allows us to obtain simplicial metrics on submersion groupoids.

Remark 3.2.3. Consider the pullback diagram of manifolds

$$\begin{tikzcd}
M \times_N M' \arrow[r, "p'"] \arrow[d, "p"'] & (M', \eta') , \arrow[d, "f'"] \\
(M, \eta) \arrow[r, "f"'] & (N, \eta_N)
\end{tikzcd}$$

where f is a Riemannian submersion. It follows that the expression

$$\eta * \eta' := p^*\eta + p'^*\eta' - (f \circ p)^*\eta_N,$$

defines a Riemannian metric on $M \times_N M'$ in such a way p' becomes a Riemannian submersion.

Let us fix a simplicial metric η on $G \rightrightarrows M$. We are now in conditions to sketch the first step in the construction of $\tilde{\eta}$ on the desingularization $\tilde{G} \rightrightarrows \tilde{M}$.

- Let $S \subset S^j$ denote a most singular stratum of $G \rightrightarrows M$ with respect to dimension stratification of G and E be its corresponding exceptional divisor. There exists a Riemannian metric $\overline{\eta}^{(0)}$ on $\tilde{M}$ such that $\pi|_E : E \to S$ is a Riemannian submersion and $\pi : \tilde{M} \to M$ is an isometry outside an open neighborhood of E. This metric was obtained in Proposition 6.5 from [105] by mimicking Alexandrino's construction in [1] with respect to the singular Riemannian foliation $\mathcal{F}_M$.
- Let $G \times_M G \rightrightarrows G$ be the submersion groupoid determined by the source map $s : G \to M$. The $(n-1)$-th manifold of its nerve is denoted by $G^{[n]}$. There are diffeomorphisms $\psi_k : G^{[n]} \to G^{(n)}$ for all n defined by $\psi_n(g_n, \cdots, g_1) = (g_n g_{n-1}^{-1}, \cdots, g_2 g_1^{-1}, g_1)$. By Lemma 6.6 from [105] it follows that the Riemannian metrics $\psi_n^*(\eta^{(n)})$ form a simplicial metric on $G \times_M G \rightrightarrows G$.
- Let $\tilde{G} \times_{\tilde{M}} \tilde{G} \rightrightarrows \tilde{G}$ denote the submersion groupoid determined by source map $\tilde{s} : \tilde{G} \to \tilde{M}$. It is simple to check that $\tilde{G}^{[n]} \cong G^{[n]} \times_M \tilde{M}$. The pullback metrics $\overline{\eta}^{(n)} = \psi_n^*(\eta^{(n)}) * \overline{\eta}^{(0)}$ define a simplicial metric on $\tilde{G} \times_{\tilde{M}} \tilde{G} \rightrightarrows \tilde{G}$ and the projections $\pi^n : \tilde{G}^{[n]} \to G^{[n]}$ are Riemannian submersions when restricted to the exceptional divisor E. This is the content of Lemma 6.7 from [105].
- It follows that $\tilde{G}$ acts along the source map $\tilde{G}^{[n]} \to \tilde{M}$ via the formula $(\tilde{g}_n, \cdots, \tilde{g}_1) \cdot \tilde{g}_0 = (\tilde{g}_n \tilde{g}_0, \cdots, \tilde{g}_1 \tilde{g}_0)$ with quotient map $\tilde{\phi}_n : \tilde{G}^{[n]} \to \tilde{G}^{(n-1)}$ given by $\tilde{\phi}_n(\tilde{g}_n, \cdots, \tilde{g}_1) = (\tilde{g}_n \tilde{g}_{n-1}^{-1}, \cdots, \tilde{g}_2 \tilde{g}_1^{-1})$. By the averaging process introduced in Definition 4.10 from [40] we can get Riemannian metrics $\mathrm{Av}(\overline{\eta}^{(n+1)})$ on $\tilde{G}^{[n+1]}$ and then push them forward with $\tilde{\phi}_{n+1}$ to define the Riemannian metrics $\tilde{\eta}^{(n)} = (\tilde{\phi}_{n+1})_*(\mathrm{Av}(\overline{\eta}^{(n+1)}))$ on $\tilde{G}^{(n)}$. This produces a simplicial metric $\tilde{\eta}$ on $\tilde{G} \rightrightarrows \tilde{M}$, the blow-up of G along S, such that $\pi : \tilde{G}|_E \to G_S$ is a Riemannian submersion when restricted to E and $\pi : \tilde{G} \to G$ is an isometry outside an open neighborhood of E, see Proposition 6.8 from [105].

This motivates the observation below.

Remark 3.2.4. Firstly, note that in the construction described in the previous items it has been used the n-metric on $G^{(n)}$ to get the $(n-1)$-metric on $\tilde{G}^{(n-1)}$. This explains why the authors in [105] used simplicial metrics instead of 2-metrics. Secondly, the actual simplicial metric on the regularization $\tilde{G} \rightrightarrows \tilde{M}$ is obtained by the procedure described above after a finite amount of times, where in each step the blow-ups of all strata with the minimal dimension are applied. Thirdly, it is worth mentioning that the transversal geometry described by $\tilde{\eta}^{(0)}$ is similar to (linear isometric to) the transversal geometry described by $\overline{\eta}^{(0)}$ and, in turn, it is similar to the transversal geometry described by $\eta^{(0)}$. This is consequence of Theorem 1.2 in [1], Proposition 4.11 in [40] and the proof of Proposition 6.8 in [105]. The reader is recommended to visit [105] for further enlighten details.

The following result justifies our interest in first desingularizing proper Riemannian groupoids.

Proposition 3.2.1. *Let $\alpha : I \to [\tilde{M}/\tilde{G}]$ be a stacky geodesic of positive length presented by the class of a good cocycle $(a, \mathcal{U})$. Then, $[\pi]_*\alpha : I \to [M/G]$ is a stacky geodesic of positive length presented by the class of the push-forward cocycle $(\pi_* a, \mathcal{U})$.*

Proof. It is clear that the class of the cocycle $(\pi_* a, \mathcal{U})$ determined by the paths $(\pi \circ a_{ji})$ supported over the open cover $\mathcal{U}$ is well defined since π is a Lie groupoid morphism. Let us consider the first step of the blow-up construction described above. As π is an isometry outside a neighborhood of E we only have to see what happens around E. By Theorem 1.2 in [1] it follows that if a_i is a (unit speed) geodesic which is orthogonal to E then $\pi \circ a_i$ is a (unit speed) geodesic which is orthogonal to S. Additionally, the lifts of horizontal geodesics of the singular Riemannian foliation $(S, (\mathcal{F}_M)|_S, \eta^{(0)})$ are horizontal geodesics of the singular Riemannian foliation $(E, (\mathcal{F}_{\tilde{M}})|_E, \overline{\eta}^{(0)})$. Therefore, if the geodesics a_{ji} are orthogonal to $(\tilde{G}|_E, (\mathcal{F}_{\tilde{G}})|_{\tilde{G}|_E}, \tilde{\eta}^{(1)})$ then we get that $\pi \circ a_{ji}$ are orthogonal to $(G|_S, (\mathcal{F}_G)|_{G|_S}, \eta^{(1)})$ and the normal length is preserved. This is because, up to linear isometries, all the metrics $\tilde{\eta}^{(1)}$, $\overline{\eta}^{(1)}$, $\overline{\eta}^{(0)}$, $\eta^{(0)}$, and $\eta^{(1)}$ determine the same inner products along the normal directions to the leaves of the corresponding singular Riemannian foliations. This finishes the proof. □

Remark 3.2.5. As mentioned before, the previous result suggests us to study first the problem of showing the existence of closed stacky geodesics of positive length for Riemannian stacks presented by regular proper Riemannian groupoids, which behave somehow similar to Riemannian orbifolds, see [105, p. 1283].

3.2.1 The energy functional on stacky curves

We are now interested in studying some properties of what we shall call the stacky energy functional. This will be a function defined over the "spaces" of stacky curves on $[M/G]$. Therefore, motivated by the classical theory, we need first to describe the differentiable structure we can put over the set of stacky curves. Let us pick $[x], [y] \in M/G$ and denote by $\Omega^c_{[x],[y]}([M/G])$ the set of stacky curves $\alpha : [0,1] \to [M/G]$ presented by classes of continuous cocycles such that $\alpha(0) = [x]$ and $\alpha(1) = [y]$. Recall that if we require that $\alpha(0) = \alpha(1)$ then the set of closed stacky curves $\Omega^c_{[x]}([M/G])$ can be thought of as the set of pointed stacky maps $(S^1, e) \to ([M/G], [x])$ presented by classes of continuous cocycles where the circle S^1 is viewed as a stack via the unit groupoid $S^1 \rightrightarrows S^1$. Let us set $\Omega^c([M/G]) = \bigcup_{[x]\in[M/G]} \Omega^c_{[x]}([M/G])$.

Just as happens in the cases of Riemannian manifolds and orbifolds [66, 56] we can show that:

Proposition 3.2.2. *Suppose that $(G \rightrightarrows M, \eta^{(2)})$ is a regular proper Riemannian groupoid. Then, the sets $\Omega^c([M/G])$ and $\Omega^c_{[x],[y]}([M/G])$ have structure of Banach manifolds.*

Proof. We prove these facts by transferring some of the ideas in the proof of [56, Prop. 3.1.1] to our setting. Take a closed stacky curve $\alpha : S^1 \to [M/G]$ at $[x]$ presented by the class of a continuous cocycle $(a, \mathcal{U})$ and consider the bundle a^*TG onto S^1 given by the space $\coprod_{j,i} a^*_{ji}TG$ quoted by the equivalence relation $(\tau, \xi_{ji}(\tau)) \sim (\tau', \xi_{lk}(\tau'))$ if and only if $\tau = \tau'$, $i = l$, and the composition $\xi_{ji}(\tau)\xi_{ik}(\tau) \in T_{a_{jk}(\tau)}G$ is defined in the tangent groupoid $TG \rightrightarrows TM$. The projection onto S^1 is canonically defined as $[(\tau, \xi_{ji}(\tau))] \mapsto \tau \bmod 1$. On the one hand, observe that two equivalent cocycles $(a, \mathcal{U})$ and $(a', \mathcal{U}')$ respectively determine isomorphic vector bundles a^*TG and a'^*TG. Indeed, if $\mathcal{U}''$ is a common refinement of $\mathcal{U}$ and $\mathcal{U}'$ and $\phi : \coprod_i U''_i \to G$ induces an isomorphism between $a|_{I_{\mathcal{U}''}}, a'|_{I_{\mathcal{U}''}} : I_{\mathcal{U}''} \to G$ then the assignment $[(\tau, \xi_{ji}(\tau))] \mapsto [(\tau, \dot{\phi}(\tau, j)\xi_{ji}(\tau)\dot{\phi}(\tau, i)^{-1})]$ is the desired isomorphism. On the other hand, the metric $\eta^{(1)}$ induces inner products on the fiber of a^*TG:

$$\langle [(\tau, \xi_{ji}(\tau))], [(\tau, \xi'_{ji}(\tau))] \rangle_\tau := \overline{\eta}^{(1)}_{a_{ji}(\tau)}([\xi_{ji}(\tau)], [\xi'_{ji}(\tau)]), \tag{3.2}$$

where $\overline{\eta}^{(1)}_{a_{ji}(\tau)}$ is the induced inner product on the normal direction to the leave in $\mathcal{F}_G$ through $a_{ji}(\tau)$. This is well defined because $\pi_2, m, \pi_1 : G^{(2)} \to G$ are Riemannian submersions, so that they yield linear isometries on the normal directions. In consequence, the Banach space of continuous sections $C^0(S^1, a^*TG)$ with the sup norm will play the role of the tangent space of $\Omega^c([M/G])$ at α. Note that the elements of $C^0(S^1, a^*TG)$ can be thought of as stacky vector fields along α in the sense that they are presented by classes of continuous cocycles $(X, \mathcal{U})$ with values in $TG \rightrightarrows TM$ which are determined by continuous vector fields $X_{ji} : U_j \cap U_i \to TG$ along $a_{ji} : U_j \cap U_i \to G$ verifying

$$X_{kj}(\tau)X_{ji}(\tau) = X_{ki}(\tau), \qquad \tau \in U_k \cap U_j \cap U_i. \tag{3.3}$$

Let us now choose $\epsilon > 0$ small enough such that, for all $\tau \in S^1$, the exponential map $\exp^{(1)}$ is a Lie groupoid isomorphism from an open ball $B_\epsilon(a_{ji}(\tau))$ centered at 0 into $\nu(G_{\mathcal{O}_{a_i(\tau)}})$, see [37]. Let $\tilde{U}^c_\epsilon$ be the open ball of radius ϵ centered at the origin in $C^0(S^1, a^*TG)$. By mimicking the classical case (see Theorem 1.2.9 in [66]), we define the chart $\exp^c_\epsilon : \tilde{U}^c_\epsilon \to U^c_\epsilon$ by sending a section $(X, \mathcal{U})$ as above to the class of the continuous cocycle $(a^X, \mathcal{U})$ defined by $a^X_{ji}(\tau) = \exp^{(1)}_{a_{ji}(\tau)}[X_{ji}(\tau)]$ for all $\tau \in S^1$. This establishes a bijection, so that the images U^c_ϵ for distinct α inside $\Omega^c([M/G])$ form a basis for its topology. Furthermore, the change of charts are differentiable. Hence, $\Omega^c([M/G])$ becomes a Banach manifold and the tangent space $T_\alpha\Omega^c([M/G])$ turns out to be the space of continuous sections of the vector bundle a^*TG over S^1.

The Banach structure on $\Omega^c_{[x],[y]}([M/G])$ may be defined in a similar fashion. In this case, if $\alpha : [0,1] \to [M/G]$ is a stacky curve from $[x]$ to $[y]$ then the tangent space of $\Omega^c_{[x],[y]}([M/G])$ at α can be identified with the space of continuous "stacky vector fields" along α which vanish at 0 and 1.

□

Let us formalize some of the terminology introduced in the previous result. We denote by $T[M/G] := [TM/TG]$ the differentiable stack presented by the tangent groupoid $TG \rightrightarrows TM$.

Definition 3.2.2. Let $\alpha : I \to [M/G]$ be a stacky curve presented by the class of a cocycle $(a, \mathcal{U})$. A **stacky vector field along** α is a stacky map $X : I \to T[M/G]$ presented by the class of cocycle $(X, \mathcal{U})$ with values in $TG \rightrightarrows TM$. This amounts to asking for vector fields $X_{ji} : U_j \cap U_i \to TG$ along $a_{ji} : U_j \cap U_i \to G$ verifying Equation (3.3).

Of course, if α is presented by a good cocycle then so is X. Note that we may think of the velocity of α as a stacky vector field along itself when describing it with the class of the tangent cocycle $(\dot{a}, \mathcal{U})$, meaning that $\dot{\alpha} = (\dot{a}_{ji})$.

We are now ready to introduce the energy functional for stacky curves. Let $\alpha : I \to [M/G]$ denote a stacky curve presented by the class of a good cocycle $(a, \mathcal{U})$. As the speed of a stacky curve varies continuously, we can define the **energy** of α as $E(\alpha) = \frac{1}{2}\int_I \|\dot{\alpha}(\tau)\|^2 d\tau$. By using the good cocycle representing α we obtain

$$E(\alpha) = \frac{1}{2}\int_I \|\dot{\alpha}(\tau)\|^2 d\tau = \frac{1}{2}\left(\sum_i \int_{U_i} \|\dot{a}_i(\tau)\|^2_N d\tau - \sum_i \int_{U_{i+1}\cap U_i} \|\dot{a}_{i+1,i}(\tau)\|^2_N d\tau\right).$$

Once again, this expression makes sense since there are no triple intersections, so that the first sum is counting twice each overlap $U_{i+1} \cap U_i$ and for each $\tau \in U_{i+1} \cap U_i$, the source and target maps yield linear isometries $\nu_{a_{i+1,i}(\tau)}(G_{\mathcal{O}}) \to \nu_{a_i(\tau)}(\mathcal{O})$ and $\nu_{a_{i+1,i}(\tau)}(G_{\mathcal{O}}) \to \nu_{a_{i+1}(\tau)}(\mathcal{O})$ which send $[\dot{a}_{i+1,i}(\tau)]$ to $[\dot{a}_i(\tau)]$ and $[\dot{a}_{i+1}(\tau)]$, respectively.

Lemma 3.2.1. *The length and the energy of a stacky curve $\alpha : I \to [M/G]$ are related each other through the inequality*

$$L(\alpha)^2 \leq 2E(\alpha)\ell(I),$$

with the equality holding if and only if α has constant speed. Here $\ell(I)$ denotes the length of I.

Proof. As in the classical case, the result follows from the continuity of the speed together with the so-called Cauchy–Schwarz inequality. □

Suppose that for any other stacky curve $\gamma : I \to [M/G]$ of constant speed we have that $E(\alpha) \leq E(\gamma)$. That is, the stacky curve α somehow "minimizes" the energy E. Therefore, Lemma 3.2.1 implies that $L(\alpha)^2 \leq L(\gamma)^2$. Thus, as consequence of Lemma 7 and Theorem 7 from [38] we may interpret this as α being locally minimizing so that α should be a stacky geodesic. Observe that α must have constant speed for otherwise it would not minimize the energy functional by the same Lemma 3.2.1 and the condition of equality thereto pertaining.

The following key result will play an important role along this chapter.

Lemma 3.2.2. *Let $(G \rightrightarrows M, \eta^{(2)})$ be a regular proper Riemannian groupoid. Then, the following assertions hold true:*

1. *every stacky curve $\alpha : I \to [M/G]$ can be presented by the class of a good cocycle $(a^{\perp},\tilde{\mathcal{U}})$ such that each curve $a^{\perp}_{ji} : \tilde{U}_j \cap \tilde{U}_i \to G$ is orthogonal to the leaves of the regular Riemannian foliation $\mathcal{F}_G$, and*

2. *The normal speed $\|\dot{\alpha}(\tau)\|$ varies smoothly.*

Proof. By Lemma 4.5 in [39] we may assume that the 2-metric $\eta^{(2)}$ is such that we may split TG as

$$TG = R \oplus (\ker(ds) \cap R^{\perp}) \oplus (\ker(dt) \cap R^{\perp}) \oplus (\ker(ds) + \ker(dt))^{\perp},$$

where $R = \ker(ds) \cap \ker(dt)$. More precisely, we can construct an equivalent 2-metric verifying the previous condition. Let $a : I \to M$ be a smooth curve and let $\tilde{a} : J \subseteq I \to G$ denote its horizontal lift with respect to the source map $s : G \to M$ starting at the unit $u(a(\tau_0))$ for some $\tau_0 \in I$. Note that the expression $a^{\perp}(\tau) = t(\tilde{a}(\tau))$ for $\tau \in J$ determines a smooth curve in M which is canonically isomorphic to a and orthogonal to the regular foliation $\mathcal{F}_M$ of M. Therefore, by Lemma 7 and Corollary 8 in [38], if $\alpha : I \to [M/G]$ is a stacky curve presented by the class a good cocycle $(a,\mathcal{U})$ then by arguing in a similar fashion as above also by using the Riemannian submersions $\pi_1, \pi_2 : G^{(2)} \to G$ we can construct another good cocycle $(a^{\perp},\tilde{\mathcal{U}})$, equivalent to $(a,\mathcal{U})$, such that each curve $a^{\perp}_{ji} : \tilde{U}_j \cap \tilde{U}_i \to G$ is orthogonal to the leaves of the regular Riemannian foliation $\mathcal{F}_G$.

Finally, as it was commented in [38, p. 414], it follows from the previous facts that the normal speed $\|\dot{\alpha}(\tau)\|$ varies smoothly since we have a well defined orthogonal projection. □

As a first consequence we have that:

Remark 3.2.6. Lemma 3.2.2 allows us to rewrite the inner products (3.2) from Proposition 3.2.2 in terms of the Riemannian metric $\eta^{(1)}$, and in turn in terms of $\eta^{(0)}$, just as in the case of orbifolds.

We denote by $\Omega([M/G])$ (resp. $\Omega_{[x],[y]}([M/G])$) the set of closed stacky curves (resp. stacky curves from $[x]$ to $[y]$) presented by classes of smooth cocycles. It is clear that these spaces are also Banach manifolds (actually Hilbert manifolds as we will comment later on). The second interesting consequence of Lemma 3.2.2 is that the energy E can be naturally regarded as a differentiable function $E : \Omega([M/G]) \to \mathbb{R}$ (or $E : \Omega_{[x],[y]}([M/G]) \to \mathbb{R}$). In order to characterize stacky geodesics in the concrete case of regular separated Riemannian stacks as critical points of the energy functional E we need to introduce the following terminology. A **stacky variation** of a stacky curve $\alpha : [0,1] \to [M/G]$ joining $\alpha(0) = [x]$ and $\alpha(1) = [y]$ is a collection of stacky curves $\alpha_\lambda : [0,1] \to [M/G]$ with $\lambda \in (-\epsilon, \epsilon)$ such that $\alpha_0 = \alpha$ and $\alpha_\lambda(0) = [x]$, $\alpha_\lambda(1) = [y]$ for all $\lambda \in (-\epsilon, \epsilon)$. Note that we may think of a stacky variation as a stacky map $H : [0,1] \times (-\epsilon, \epsilon) \to [M/G]$ determining a family $\alpha_\lambda(\tau) = H(\tau, \lambda)$ of stacky curves with their endpoints fixed. By using the open cover $(U_i \times (-\epsilon, \epsilon))$ of $[0,1] \times (-\epsilon, \epsilon)$ we can look at the stacky variation H as the class of a cocycle (H_{ji}) where $H_{ji} : U_j \cap U_i \times (-\epsilon, \epsilon) \to G$ is such that $(H_{ji}|_\lambda)$ determines both a class of a cocycle for the stacky curve α_λ and variations for the curves forming the class of the cocycle (a_{ji}) representing α. If $H : [0,1] \times (-\epsilon, \epsilon) \to [M/G]$ is a stacky variation of $\alpha : [0,1] \to [M/G]$ then we define its **stacky variational field** $\partial_\lambda H|_{\lambda=0} : [0,1] \to T[G/M]$ as the stacky vector field along α determined by the class of the tangent cocycle $(\partial_\lambda H_{ij}|_{\lambda=0})$. Here $\partial_\lambda H_{ij}|_{\lambda=0}$ denotes the variational field associated to H_{ji} which is actually a vector field along the curve a_{ji}.

Suppose that $(G \rightrightarrows M, \eta)$ is a regular proper Riemannian groupoid. Let $X : [0,1] \to T[M/G]$ be a stacky vector field along $\alpha : [0,1] \to [M/G]$ which is presented by the class of a good cocycle $(X, \mathcal{U})$. By using a similar horizontal lifting argument as in the proof of Lemma 3.2.2 we may assume that the vector fields $X_{ji} : U_j \cap U_i \to TG$ along $a_{ji} : U_j \cap U_i \to G$ are such that $X_{ji}(\tau)$ is orthogonal to $\mathcal{F}_G$ for all $\tau \in [0,1]$. Therefore, if $X \in T_\alpha \Omega_{[x],[y]}([M/G])$ then we have a stacky variation $H^X : [0,1] \times (-\epsilon, \epsilon) \to [M/G]$ presented by the class of the cocycle $H^X_{ji}(\tau, \lambda) = \exp^{(1)}_{a_{ji}(\tau)}(\lambda X_{ji}(\tau))$ which satisfies that $H^X_{ji}(\tau, 0) = a_{ji}(\tau)$ and $\partial_\lambda H^X_{ji}(\tau, \lambda)|_{\lambda=0} = X_{ji}(\tau)$. By Lemma 3.2.2 we may further assume that $H^X_{ji}(\tau, \lambda)$ is orthogonal to $\mathcal{F}_G$. Thus, by computing the differential of E we obtain that:

Theorem 3.2.1. *Let $(G \rightrightarrows M, \eta^{(2)})$ be a regular proper Riemannian groupoid. A stacky curve $\alpha : [0,1] \to [M/G]$ is a critical point of the energy functional E if and only if it is a stacky geodesic.*

Proof. Working locally, we can assume that α and X are in fact respectively presented by a curve $a : [0,1] \to M$ and a vector field $X : [0,1] \to TM$ along a such that $\|\dot{\alpha}(\tau)\| = \|\dot{a}(\tau)\|_N = \|\dot{a}(\tau)\|$ and $X(\tau)$ is orthogonal to $\mathcal{F}_M$ for all $\tau \in [0,1]$. In consequence,

$$E(H^X(\cdot, \lambda)) = \frac{1}{2}\int_0^1 \|\dot{H^X}(\tau, \lambda)\|_N^2 \, d\tau = \frac{1}{2}\int_0^1 \|\dot{H^X}(\tau, \lambda)\|^2 \, d\tau,$$

so that

$$dE(\alpha)(X) = \partial_\lambda E(H^X(\tau, \lambda))|_{\lambda=0} = -\int_0^1 \eta^{(0)}(X, D_\tau \dot{a}) \, d\tau.$$

Here D_τ stands for the covariant derivative along curves induced by the Levi–Civita connection of $\eta^{(0)}$. Therefore, α is a critical point of E if and only if $D_\tau \dot{a} = 0$. That is, a is a geodesic in M which is also orthogonal to $\mathcal{F}_M$ as desired. □

3.2.2 The Palais–Smale condition (C)

The next step in our study is to look at the so-called Palais–Smale condition (C) for the stacky energy functional, also for the proper regular case. Thus, in what follows, we will suppose that $(G \rightrightarrows M, \eta^{(2)})$ is a regular proper Riemannian groupoid without further comments. Let $X : [0,1] \to T[M/G]$ be a stacky vector field along a stacky curve $\alpha : [0,1] \to [M/G]$. Suppose that α (resp. X) is presented by the class of a good cocycle $(a, \mathcal{U})$ (resp. $(X, \mathcal{U})$). By Lemma 3.2.2 we may assume that both $a_{ji}(\tau)$ and $X_{ji}(\tau)$ are orthogonal to $\mathcal{F}_G$ for all j, i and $\tau \in [0,1]$. This automatically implies that $a_i(t)$ and $X_i(t)$ are orthogonal to $\mathcal{F}_M$ for all i and $\tau \in [0,1]$ since s and t are Riemannian submersions.

Definition 3.2.3. The **normal covariant derivative** $D_\tau X$ of X is defined to be the stacky vector field along α presented by the good cocycle $(D_\tau^{(1)\perp} X_{ji})$ supported over $\mathcal{U}$. Here $D_\tau^{(1)}$ denotes the covariant derivative of vector fields along paths determined by the Levi–Civita connection of $\eta^{(1)}$.

The well definition of $(D_\tau X, \mathcal{U})$ follows from Theorem 1 in [99] since $\dot{\alpha}$ and X do not have vertical components. That is, the following identities hold

$$ds_{a_{ji}(\tau)}((D_\tau^{(1)\perp} X_{ji})(\tau)) = (D_\tau^{(0)} X_i)(\tau) \quad \text{and} \quad dt_{a_{ji}(\tau)}((D_\tau^{(1)\perp} X_{ji})(\tau)) = (D_\tau^{(0)} X_j)(\tau). \tag{3.4}$$

This is because the orthogonal component of $D_\tau^{(1)} X_{ji}$ turns our to be the horizontal lifting of both $D_\tau^{(0)} X_i$ and $D_\tau^{(0)} X_j$. The latter fact implies that $D_\tau^{(0)} X_i$ and $D_\tau^{(0)} X_j$ are orthogonal to $\mathcal{F}_M$ since both s and t are Riemannian submersions.

We want now to define a Riemannian Hilbert structure over the space $\Omega([M/G])$ (or $\Omega_{[x],[y]}([M/G])$) and we will use the previous facts in order to do so. As the classical case suggests [65, p. 8] (see also [56]), for each element $X \in T_\alpha(\Omega_{[x]}([M/G]))$ we define the scalar product

$$\begin{aligned}
(X,X)(\alpha) &= \int_{S^1} \overline{\eta}^{(1)}([X(\tau)],[X(\tau)])d\tau + \int_{S^1} \overline{\eta}^{(1)}([(D_\tau X)(\tau)],[(D_\tau X)(\tau)])d\tau \\
&= \sum_i \int_{U_i} \eta^{(0)}_{a_i(\tau)}(X_i(\tau),X_i(\tau))d\tau - \sum_i \int_{U_{i+1}\cap U_i} \eta^{(1)}_{a_{i+1,i}(\tau)}(X_{i+1,i}(\tau),X_{i+1,i}(\tau))d\tau \\
&+ \sum_i \int_{U_i} \eta^{(0)}_{a_i(\tau)}(D_\tau^{(0)} X_i(\tau), D_\tau^{(0)} X_i(\tau))d\tau \\
&- \sum_i \int_{U_{i+1}\cap U_i} \eta^{(1)}_{a_{i+1,i}(\tau)}(D_\tau^{(1)} X_{i+1,i}(\tau), D_\tau^{(1)} X_{i+1,i}(\tau))d\tau \\
&= \int_{S^1} \eta^{(1)}(X(\tau),X(\tau))d\tau + \int_{S^1} \eta^{(1)}((D_\tau X)(\tau),(D_\tau X)(\tau))d\tau.
\end{aligned}$$

Once again, the chain of equalities above follows because s and t are Riemannian submersions. This scalar product is well defined and does not depend on the cocycles (a_{ji}) and (X_{ji}) representing α and X, for the groupoid version of the normal representations are by isometries. Just as happens in the classical case of manifold [65, Thms. 1.2.9] and orbifolds [56, s. 3.3.2], by arguing as in Proposition 3.2.2 it follows that $\Omega([M/G])$ admits a structure of Hilbert manifold for which the scalar product $(\cdot,\cdot)$ determines a Riemannian metric $(\cdot,\cdot)_\Lambda$. Similarly, $\Omega_{[x],[y]}([M/G])$ is a Riemannian Hilbert manifold.

Motivated by the proof of Theorem 3.2.1, with the help of the Riemannian metric $(\cdot,\cdot)_\Lambda$

on $\Omega([M/G])$ we define the gradient vector field grad$E$ of the stacky energy by the expression

$$(\text{grad}E(\alpha), X)_\Lambda = dE_\alpha(X) = \int_{S^1} \eta^{(0)}(\dot{\alpha}, D_\tau X) d\tau, \qquad X \in T_\alpha(\Omega_{[x]}([M/G])).$$

From now on we suppose that the orbit space M/G is compact and connected. The following properties are straightforward adaptations of the classical case, see for instance [65, p. 22-24]. Let $\alpha, \alpha' : S^1 \to [M/G]$ be two stacky curves. The d_∞**-metric** between α and α' in $\Omega^c([M/G])$ is defined to be

$$d_\infty(\alpha, \alpha') = \max_\tau d_N(\alpha(\tau), \alpha'(\tau)).$$

Here d_N denotes the normal pseudo-distance on M/G which can be recovered by using the stacky length, see Equation (3.1). Observe that this metric is complete since d_N is so. In fact, this is consequence of M/G being compact, see [38, Cor. 20].

Lemma 3.2.3. *Let $\alpha, \alpha \in \Omega([M/G])$. Then*

1. $d_N^2(\alpha(\tau_0), \alpha(\tau_1)) \leq |\tau_1 - \tau_0| 2E(\alpha)$.
2. $d_\infty^2(\alpha, \alpha') \leq 2d_\Lambda^2(\alpha, \alpha')$ *where d_Λ is the distance on $\Omega_{[x]}([M/G])$ derived from the Riemannian metric $(\cdot, \cdot)_\Lambda$.*
3. $|\sqrt{2E(\alpha)} - \sqrt{2E(\alpha')}| \leq d_\Lambda(\alpha, \alpha')$.
4. *The inclusion $\Omega([M/G]) \hookrightarrow \Omega^c([M/G])$ is continuous and compact, i.e. the image of every bounded subset of $\Omega([M/G])$ has compact closure in $\Omega^c([M/G])$.*
5. *$\Omega([M/G])$ is a complete metric space with respect to the metric d_Λ.*

Proof. Assertion 1. directly follows from Theorem 3 in [38] together with Lemma 3.2.1. Assertions 2. and 3. follows by mimicking Propositions 1.4.2 and 1.4.3 in [65], respectively. Note that the continuity in assertion 4. is consequence of item 2. The compactness can be derived from the properties 1. and 3. by similar arguments as those in Lemma 1.4.4 in [65]. Finally, as every Cauchy sequence is bounded, by item 4. together with the completeness of d_∞ we deduce that d_Λ is also complete, compare Theorem 1.4.5 in [65]. □

More importantly, we have:

Proposition 3.2.3. *The natural inclusions $\Omega([M/G]) \hookrightarrow \Omega^c([M/G])$ and $\Omega_{[x],[y]}([M/G]) \hookrightarrow \Omega^c_{[x],[y]}([M/G])$ are homotopy equivalences.*

Proof. We shall closely follow the classical arguments from [88, s. 17] and Proposition 3.3.5 in [56], together with the necessary stacky facts from [38]. Let us denote either $\Omega([M/G])$ or $\Omega_{[x],[y]}([M/G])$ by P. For a positive integer k we let P_k (resp. P_k^c) to be the subspace of P (resp. P^c) formed by stacky curves α presented by classes of cocycles $(a, \mathcal{U}^k)$, where each sub-intervals U_j^k in $\mathcal{U}^k$ satisfies the following requirements:

- for each j there is $\tau_j \in U_{j+1,j}$ such that the interval $[\tau_j, \tau_{j+1}]$ has length $1/2^k$, compare with Remark 2 in [38] and Proposition 3.3.5 in [56], and
- $\alpha(\tau_j) \in M/G$ is the center of a "geodesically convex ball" containing the image of $\pi \circ a_j$. Here $\pi : M \to M/G$ stands for the orbit projection. This ball can be constructed as consequence of Proposition 18 and Theorem 19 in [38].

The subspaces P_k and P_k^c are open and give rise to sequences of open subsets

$$P_1 \subset P_2 \subset P_3 \subset \cdots \quad \text{and} \quad P_1^c \subset P_2^c \subset P_3^c \subset \cdots$$

whose union equal to P and P^c, respectively. Denote by $\iota : P \hookrightarrow P^c$ the canonical inclusion, so that $P_k = \iota^{-1}(P_k^c)$. By using the properties described in Lemma 3.2.3 one can mimic the construction from [88, p. 91] to continuously deform such α to another stacky curve $\overline{\alpha}$ presented by the class of cocycle $(\overline{a}, \mathcal{U}^k)$ where each $\overline{a}_{j,j-1}$ is an orthogonal geodesic from $a_{j,j-1}(\tau_{j-1})$ to $a_{j,j-1}(\tau_j)$. Hence, passing to equivalence classes, we obtain a continuous deformation of $\iota|_{P_k} : P_k \hookrightarrow P_k^c$ which actually determines a homotopy equivalence. As P and P^c are given by the increasing union of the open subset P^k and P_k^c for $k = 1, 2, \cdots$, it follows that ι is itself a homotopy equivalence, see Appendix in [88]. □

As we already mentioned in Subsection 1.1.2, the Palais–Smale Condition (C) is the crucial property which allows an extension of the classical Morse theory to the context of Hilbert manifolds [102]. Therefore, our goal now is to show that the stacky energy functional E satisfies such a condition. That is, we need to check the following. Let $\{\alpha_n\}$ be a sequence of stacky curves in $\Omega([M/G])$ (or $\Omega_{[x],[y]}([M/G])$) such that

- the sequence $\{E(\alpha_n)\}$ is bounded, and
- the sequence $\{\|\text{grad} E(\alpha_n)\|_\Lambda\}$ tends to zero.

Then the sequence $\{\alpha_n\}$ has limit points and any limit point is a critical point of E, i.e. a stacky geodesic.

Proposition 3.2.4. *The triple* $(\Omega([M/G]), (\cdot,\cdot)_\Lambda, E)$ *satisfies the Palais–Smale condition* (C).

Proof. The result follows mainly by using the properties exhibited in Lemma 3.2.3. On the one hand, as $\{\alpha_n\}$ is a sequence on which E is bounded then from Lemma 3.2.1 it follows that $\sup_n L(\alpha_n) < \infty$. Also, $d_\infty(\alpha_n(\tau), \alpha_n(0)) \leq L(\alpha_n)$ for all $n \geq 0$. This implies that the sequence $\{\alpha_n(\tau)\}$ is d_∞-bounded for all $\tau \in S^1$. But M/G is d_∞-complete, actually, compact, so that $\{\alpha_n(\tau)\}$ is relatively compact. On the other hand, by item 1. from Lemma 3.2.3 we get that the sequence $\{\alpha_n\}$ is equi-continuous. Therefore, from the Arzelà–Ascoli's theorem it holds that (up to sub-sequence) $\{\alpha_n\}$ has a d_∞-limit element (converges uniformly to) α in $\Omega([M/G])$. Working locally, we may assume that all the elements of $\{\alpha_n\}$ belong to a domain $(\exp, U_\epsilon)$, as those described in Proposition 3.2.2, and that they form a Cauchy sequence with respect to the metric d_∞. By arguing similarly as in the proof of Theorem 1.4.7 in [65], it follows that the sequence $\{X_n = \exp^{-1}\alpha_n\}$ also form a Cauchy sequence with respect to the metric d_Λ. But d_Λ is complete by item 5. from Lemma 3.2.3. Thus, we obtain that $\{X_n\}$ converges to $\exp^{-1}(\alpha)$, meaning that $\{\alpha_n\}$ d_Λ-converges to α. Finally, by Theorem 3.2.1 we get that α is a stacky geodesic since the sequence $\{\|\text{grad} E(\alpha_n)\|_\Lambda\}$ tends to zero. □

The first interesting consequence we can get from the previous result is as follows.

Corollary 3.2.1. *The set of critical points of* E *in the subspaces* $E^{-1}[0, a]$ *of either* $\Omega([M/G])$ *or* $\Omega_{[x],[y]}([M/G])$ *are compact for every* $a \geq 0$.

Proof. Similar to Proposition 1.4.9 in [65]. □

Recall that M/G is compact. Hence, the negative gradient vector field $-\text{grad}E$ generates a local flow Φ_λ on $\Omega([M/G])$ (or $\Omega_{[x],[y]}([M/G])$) which is defined for all $\lambda \geq 0$, compare Theorem 1.4.11 in [65]. More generally, it becomes simple now to transfer several results and properties of $-\text{grad}E$ and its flow Φ to our context without so many changes along the proofs of the corresponding ones in the classical case. One of those results, which is really important to our next purposes, has to do with the notion of Φ-family, see Subsection 1.1.2 or [65, p. 32]. For convenience, we define again what this notion means. Let P denote either $\Omega([M/G])$ or $\Omega_{[x],[y]}([M/G])$ and let $P^a = \{\alpha \in P : E(\alpha) \leq a\}$ be the set level of E below $a \in \mathbb{R}$. A Φ-**family** is a collection $\mathcal{A}$ of nonempty subsets of P such that $E|_A$ is bounded on each $A \in \mathcal{A}$ and $\mathcal{A}$ is closed under Φ in the sense that if $A \in \mathcal{A}$ then $\Phi_\lambda(A) \in \mathcal{A}$ for all $\lambda \geq 0$. Pick $a \in \mathbb{R}$ and choose $\epsilon > 0$ such that E has no critical values in $(a, a+\epsilon]$. We define a Φ-**family of** P mod P^a to be a collection of nonempty subsets $\mathcal{A}$ of P such that $\mathcal{A}$ is a Φ-family and each member of $\mathcal{A}$ is not contained in $P^{a+\epsilon}$. Observe that a Φ-family is always a Φ-family of P mod P^a for $a < 0$.

The **critical value** of Φ-family $\mathcal{A}$ of P mod P^a is defined as

$$\alpha_{\mathcal{A}} = \inf_{A \in \mathcal{A}} \sup E|_A.$$

The following result can be proven by mimicking the proof of Theorem 2.1.1 in [65].

Proposition 3.2.5. *It holds that $\alpha_{\mathcal{A}} > a$ and there exists a critical point of α of E such that $E(\alpha) = \alpha_{\mathcal{A}}$.*

As an important consequence we get that:

Corollary 3.2.2. *The stacky energy E restricted to a connected component of P assumes its infimum in some point and such a point is actually a critical point of E.*

Proof. This follows by applying the previous result to the Φ-family formed by the points of a connected component of P and for $a < 0$. □

3.3 Existence of closed geodesics

Our aim now is to apply the results described in the previous sections to show the existence of at least one stacky closed geodesic of positive length on certain separated Riemannian stacks. The approach we shall follow comes motivated by some ideas due to Guruprasad–Haefliger in [56] to attack this problem in the orbifold case.

Let us start by transferring the idea of fundamental group of G-paths to the context of stacky curves. A **(piecewise) smooth** G**-path** is defined as a sequence of alternating (piecewise) smooth paths $\gamma_k : x_k \rightsquigarrow y_k$ in M and arrows $g_k : y_k \to x_{k+1}$ in G. We recommend the reader to consult [93, s. 3.3] and [27, c. G; s. 3] to get familiar with this notion as well as some of the results known around it. From Remark 2 in [38] we know that there is a one-to-one correspondence between good cocycles for stacky curves in $[M/G]$ and smooth G-paths in M. On the one hand, given a good cocycle (a_{ji}) for a stacky curve we can build a smooth G-path by splitting the interval I after choosing $\tau_k \in U_{k+1} \cap U_k$ and then setting $\gamma_k = a_k|_{[\tau_k, \tau_{k+1}]}$ and $g_k = a_{k+1,k}(\tau_k)$. On the other hand, a smooth G-path gives rise to a good cocycle by first extending g_k to a smooth curve $\tilde{g}_k : (\tau_k - \epsilon, \tau_k + \epsilon) \to G$ such that $\tilde{g}_k(\tau_k) = g_k$ and then modifying γ_k and γ_{k+1} near τ_k so as to agree with $s \circ \tilde{g}_k$ and $t \circ \tilde{g}_k$. Even though these operations depend on choices they are well defined up to equivalence classes of cocycles and small deformations of G-paths.

Let $\alpha_0, \alpha_1 : [0,1] \to [M/G]$ be two stacky curves joining the points $[x]$ and $[y]$ in M/G.

Definition 3.3.1. We say that α_0 and α_1 are **stacky homotopic** if the exists a stacky map $H : [0,1] \times [0,1] \to [M/G]$ determining a family $H(\cdot,\lambda) : [0,1] \to [M/G]$ of stacky curves with their endpoints fixed and equaling $[x]$ and $[y]$ for each $\lambda \in [0,1]$ and such that $\alpha_0 = H(\cdot,0)$ and $\alpha_1 = H(\cdot,1)$.

Let $\mathcal{U} = (U_i)$ be a good open cover of $[0,1]$. As expected, by using the open cover $(U_i \times [0,1])$ of $[0,1] \times [0,1]$ we may think of a stacky homotopy H as the class of a good cocycle (H_{ji}) where $H_{ji} : U_j \cap U_i \times [0,1] \to G$ is such that $(H_{ji}(\cdot,\lambda))$ determines both a class of a good cocycle for the stacky curve $\alpha_\lambda = H(\cdot,\lambda)$ and a homotopy for the curves forming the classes of good cocycles $(a_{ji,0})$ and $(a_{ji,1})$ presenting α_0 and α_1, respectively. Therefore, after using standard formulas from homotopy theory it becomes clear that such a notion of stacky homotopy allows to define an equivalence relation in $\Omega_{[x],[y]}([M/G])$. More importantly, it is simple to check that the correspondence between classes of good cocycles and small deformations of G-paths yields in turn a correspondence between this notion of homotopy for stacky curves and the notion of G-homotopy for G-paths, see [93, s. 3.3].

If $\alpha, \beta : [0,1] \to [M/G]$ are two continuous stacky curves such that $\alpha(1) = \beta(0)$ then their **concatenation** is the continuous stacky curve $\alpha * \beta : [0,1] \to [M/G]$ defined as follows. Suppose that α and β are respectively presented by the classes of good cocycles (a_{ji}) and (b_{ji}), supported over the open cover $\mathcal{U}$, such that $a_{i_1}(1) = b_{i_0}(0)$. This implies that the usual concatenation $a_{i_1} * b_{i_0}$ is well defined on $U_{i_1} \cup U_{i_0}$. Thus, $\alpha * \beta$ is presented by the class of the cocycle $(c_{ij}, \tilde{\mathcal{U}})$ where $\tilde{\mathcal{U}}$ is obtained from $\mathcal{U}$ by deleting U_{i_1} and U_{i_0} and then by adding $\tilde{U}_{i_1 \cup i_0} = U_{i_1} \cup U_{i_0}$. Also, $c_{ji} = a_{ji} \sqcup b_{ji}$ on $\tilde{U}_j \cap \tilde{U}_i = U_j \cap U_i$ for all i, j with at least one of them different from the index that we denoted by $i_1 \cup i_0$ and $c_{i_1 \cup i_0} = a_{i_1} * b_{j_0}$ on $\tilde{U}_{i_1 \cup i_0}$. Firstly, this operation does not depend on the choice of representants in the classes of the good cocycles. Indeed, let $\{(a,\mathcal{U}), (a',\mathcal{U}')\}$ and $\{(b,\mathcal{U}), (b',\mathcal{U}')\}$ be two pairs of equivalent cocycles presenting α and β, respectively. For simplicity, we have assumed that a' and b' are also supported over the same open cover. Similarly, we suppose that $\mathcal{U}''$ is a common refinement of $\mathcal{U}$ and $\mathcal{U}'$ and that $\phi, \psi : \coprod_i U''_i \to G$ respectively induce isomorphisms between $a|_{I_{\mathcal{U}''}}, a'|_{I_{\mathcal{U}''}} : I_{\mathcal{U}''} \to G$ and $b|_{I_{\mathcal{U}''}}, b'|_{I_{\mathcal{U}''}} : I_{\mathcal{U}''} \to G$. Therefore, by using the previous construction with each of the pairs of cocycles (a,b) and (a',b'), it follows that $(c_{ij}, \tilde{\mathcal{U}})$ and $(c'_{ij}, \tilde{\mathcal{U}}')$ are equivalent cocycles since $\tilde{\mathcal{U}}''$ is a common refinement for $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{U}}'$ and the map $\phi * \psi : \coprod_i \tilde{U}''_i \to G$, defined by $\phi \sqcup \psi$ on $\tilde{U}''_j \cap \tilde{U}''_i = U''_j \cap U''_i$ for all i, j with at least one of them different from the index that we denoted by $i_1 \cup i_0$ and by the concatenation $\phi|_{\tilde{U}''_{i_1 \cup i_0}} * \psi|_{\tilde{U}''_{i_1 \cup i_0}}$ over $\tilde{U}''_{i_1 \cup i_0} = U''_{i_1} \cup U''_{i_0}$, induces an isomorphism between $c|_{I_{\tilde{\mathcal{U}}''}}, c'|_{I_{\tilde{\mathcal{U}}''}} : I_{\tilde{\mathcal{U}}''} \to G$. For this last statement to make sense, it is important to keep in mind that $a_{i_1}(1) = b_{i_0}(0)$ and $a'_{i_1}(1) = b'_{i_0}(0)$. Secondly, once again, by using standard results from homotopy theory it is simple to check that if α, α' and β, β' are pairs of homotopic stacky curves such that $\alpha(1) = \beta(0)$ and $\alpha'(1) = \beta'(0)$ then $\alpha * \beta$ and $\alpha' * \beta'$ are stacky homotopic as well.

From the above we deduce that it is possible to induce a group structure $*$ on the set $\Pi_1([M/G],[x])$ of all stacky homotopy classes of elements belonging to $\Omega^c_{[x]}([M/G])$. We shall refer to this group as the **stacky fundamental group** of $[M/G]$ at $[x] \in M/G$. As expected, the correspondence between classes of good cocycles and small deformations of G-paths gives rise in to an isomorphism between $\Pi_1([M/G],[x])$ and the fundamental group of G-homotopy classes of G-loops at $x \in M$, which is usually denoted by $\Pi_1(G,x)$, see [93, p. 188-189]. We shall assume the $[M/G]$ is connected in the sense that any two distinct points $[x], [y] \in M/G$ can be joined by a stacky curve. In this case it is simple to check that $\Pi_1([M/G],[x])$ and $\Pi_1([M/G],[y])$ are isomorphic, so that we can refer to any of these groups as the stacky fundamental group of $[M/G]$.

Remark 3.3.1. It is important to point out that, in general, the stacky fundamental group $\Pi_1([M/G],[x])$ is not the same as the standard fundamental group of the underlying orbit space $\Pi_1(M/G,[x])$. For instance, let Γ be a discrete group acting properly on a simply connected manifold M and let Γ^0 be the normal subgroup of Γ generated by the elements which have fixed points in M. If $G \rightrightarrows M$ stands for the action groupoid $\Gamma \ltimes M \rightrightarrows M$ then $\Pi_1([M/G],[x]) \cong \Gamma$ and $\Pi_1(M/G,[x]) \cong \Gamma/\Gamma^0$, see [6].

Before stating our main result in this chapter we need to introduce some additional terminology. A Riemannian orbifold O is said to be **developable** it there exists a connected Riemannian manifold M having a discrete subgroup of isometries Γ such that O is presented by the action groupoid $\Gamma \ltimes M \rightrightarrows M$. Recall that Riemannian structures over such action groupoids can be obtained by using the gauge trick introduced in [40] and orbifold geodesics are described as in Example 3.1.7. Guruprasad–Haefliger provided in [56] existence results for closed geodesics of positive length, focusing on the case of non-developable orbifolds. The case of developable orbifolds is known to be quite difficult and existence results have been obtained only in particular cases, for instance:

- Dragomir showed in [43] the existence of closed geodesics of positive length on compact developable orbifolds of nonpositive or nonnegative curvature, consult also [44].
- Riemannian manifolds (orbifolds) all of whose geodesics are closed are known in the literature as **Besse manifolds** (**orbifolds**) [18]. Amman–Lange–Radeschi proved in [4] that any odd-dimensional Besse orbifold is a developable orbifold of the form $(M,\eta)/\Gamma$ for some Riemannian manifold M homeomorphic to a sphere, a Besse metric η, and some finite group of isometries of Γ of (M,η).
- The existence of closed geodesics of positive length on 2-orbifolds was shown by Lange in [68, 69]. In particular, developable 2-orbifolds admits closed geodesics.
- Recently, Lange–Zwickler have proved in [70] that every odd-dimensional compact Riemannian orbifold has a nontrivial closed geodesic. Remarkably, this result covers the case of odd-dimensional developable orbifolds as well.

Remark 3.3.2. By the criterion for developability stated in [27, p. 613] we have that $G \rightrightarrows M$ is developable, i.e. it is equivalent to the regular Lie groupoid associated to an action of $\Pi_1(G,x_0)$ on a simply connected space E if and only if each point of E has a simply connected open neighborhood U such that: if $g \in G$ is not a unit and $s(g), t(g) \in U$ and if $\alpha : [0,1) \to U$ is a continuous path with $\alpha(1) = t(g)$ and $\alpha(0) = s(g)$, then the homotopy class of the G-loop (α, g) is non-trivial. Thus, if $G \rightrightarrows M$ is additionally proper then this turns our to be a criterion for developability of orbifolds since $\Pi_1(G,x_0)$ is discrete, meaning that the corresponding isotropies are finite.

We are now ready to state our existence result.

Theorem 3.3.1. *Let $[M/G]$ be a separated Riemannian stack with M/G compact and $[\tilde{M}/\tilde{G}]$ be its stacky desingularization. Then, there exists at least one closed stacky geodesic on $[M/G]$ of positive length if $[\tilde{M}/\tilde{G}]$ satisfies one of the following conditions:*

i. $[\tilde{M}/\tilde{G}]$ is not a developable orbifold,

ii. $[\tilde{M}/\tilde{G}]$ is a developable orbifold for which the existence of closed geodesics is already known in the literature. In particular, if its stacky fundamental group has an element of infinite order or is finite.

Proof. We shall only show the first assertion since the second one follows from the well known results in the literature for the case of orbifolds after applying Proposition 3.2.1. For this case we need to use all the machinery developed in the previous sections for regular proper Riemannian groupoids while mimicking step by step the proof of Theorem 5.1.1 in [56], thus applying once again Proposition 3.2.1. We shall sketch the main arguments here for the sake of completeness. By Remark 3.3.2 it holds that there exists $x \in \tilde{M}$ and a non-trivial element g inside the isotropy $\tilde{G}_x$ such that the stacky loop α associated to the closed $\tilde{G}$-loop at x presented by $(1_x, a, g)$, where $a : [0,1] \to M$ is the constant path at x, is homotopically trivial. Therefore, there is a continuous path in $\Omega([\tilde{M}/\tilde{G}])$ joining α to a point α' presenting a constant stacky loop. Note that these two points are in different components of $\Omega([\tilde{M}/\tilde{G}])$ and, moreover, the components are all compact, compare Corollary 3.2.1. The results proved in Subsection 3.2.2 can be used in a similar fashion to prove an analog of Corollary 1.4.16 from [66] in our context, thus obtaining that the set of points in $\Omega([\tilde{M}/\tilde{G}])$ for which the stacky energy is smaller than ϵ, for various $\epsilon > 0$, form a fundamental system of neighborhoods of $\Omega([\tilde{M}/\tilde{G}])$. This implies that the family of paths in $\Omega([\tilde{M}/\tilde{G}])$ joining α to α' determines a Φ-family mod $\Omega([\tilde{M}/\tilde{G}])$. Hence, the existence of at least one closed stacky geodesic of positive length on $[\tilde{M}/\tilde{G}]$ follows by applying Proposition 3.2.5 and Corollary 3.2.2.

□

With the hope of describing other interesting results regarding the existence of closed stacky geodesics of positive length it is expected to use Theorem 3.3.1 together with Moerdijk's classification result for regular Lie groupoids [90, 91], which says that any regular Lie groupoid $\tilde{G}$ fits into a short exact sequence $K \to \tilde{G} \to E$ with K a bundle of Lie groups and E a foliation groupoid.

Chapter 4

Isometric Lie 2-group actions on Riemannian groupoids

In this chapter we exhibit a deeper study of the notion of isometric action of a Lie 2-group on a Riemannian groupoid which was introduced in Subsection 2.4.1 in order to develop a 2-equivariant Morse theory over Lie groupoids.

We already know that Riemannian groupoid metrics invariant by the action of compact Lie 2-groups on proper groupoids always exist. In particular, this implies that proper Lie groupoids are 2-equivariantly linearizable around any invariant saturated submanifold. In this chapter we show 2-equivariant versions of the Slice Theorem and the Equivariant Tubular Neighborhood Theorem and construct bi-invariant groupoid metrics on compact Lie 2-groups. Additionally, with the idea in mind of describing isometric Lie 2-group actions infinitesimally, we introduce an algebra of transversal infinitesimal isometries associated to any Riemannian groupoid metric which turns out to be Morita invariant. Therefore, such an algebra gives rise to a notion of geometric Killing vector field on a quotient Riemannian stack. As an important feature we prove that if the Riemannian stack we are working with is separated then the algebra formed by such geometric Killing vector fields is always finite dimensional. Throughout the sections we provide several examples and transfer to this new setting some classical constructions such as principal connection warpings and Cheeger deformations.

It is worth mentioning that most of the results mentioned in this chapter have already been published in [59], so that they are reproduced with permission from Springer Nature. Besides, such results were obtained in joint work with J. S. Herrera-Carmona.

4.1 Isometric Lie 2-group actions revisited

In what follows we will denote by $K^{(1)} \rightrightarrows K^{(0)}$ a Lie 2-group, by $(G^{(1)} \rightrightarrows G^{(0)}, \eta)$ a Riemannian groupoid, and by $\theta = (\theta^1, \theta^0) : (K^{(1)} \times G^{(1)} \rightrightarrows K^{(0)} \times G^{(0)}) \to (G^{(1)} \rightrightarrows G^{(0)})$ a Lie 2-group action of $K^{(1)} \rightrightarrows K^{(0)}$ on $G^{(1)} \rightrightarrows G^{(0)}$.

Let us recall the main definition concerning this chapter.

Definition 4.1.1. A 2-action of $K^{(1)} \rightrightarrows K^{(0)}$ on $(G^{(1)} \rightrightarrows G^{(0)}, \eta)$ is said to be **isometric** if the induced action of $K^{(2)}$ on $(G^{(2)}, \eta^{(2)})$ is by isometries.

Some of the immediate consequences and results derived from the previous definition come in order, see Subsection 2.4.1.

- The action of $K^{(1)}$ on $(G^{(1)},\eta^{(1)})$ is by isometries. Consequently, the action of $K^{(0)}$ on $(G^{(0)},\eta^{(0)})$ is also by isometries.
- There exists a 2-equivariant weak linearization of $G^{(1)} \rightrightarrows G^{(0)}$ around any $K^{(0)}$-invariant saturated submanifold in $G^{(0)}$.
- Let $K^{(1)} \rightrightarrows K^{(0)}$ be a Lie 2-group acting on a Riemannian groupoid $(G^{(1)} \rightrightarrows G^{(0)},\eta)$. If $K^{(1)}$ is compact then there exists another groupoid metric $\overline{\eta}$ on $G^{(1)} \rightrightarrows G^{(0)}$ for which the Lie 2-group action θ becomes isometric.

We warn the reader that for the sake of simplicity at several stages of this chapter will be enough to consider only 1-metrics $\eta^{(1)}$ on $G^{(1)}$ so that in those cases we proceed by supposing that $K^{(1)} \rightrightarrows K^{(0)}$ acts isometrically on $(G^{(1)} \rightrightarrows G^{(0)},\eta)$ if $K^{(1)}$ acts by isometries on $(G^{(1)},\eta^{(1)})$. This is because the computations as well as the arguments will be both canonical and analogous if we consider 2-metrics or even n-metrics in general. Indeed, most of the notions and results we will introduce below shall have an analogous statement if the simplicial approach from Remark 2.4.1 is considered.

Another interesting consequence that comes up from our definition of isometric Lie 2-group action is that we can pass to the quotient groupoid metrics provided that the 2-action is additionally free and proper. This allows us to obtain examples of Riemannian groupoid submersions as defined in [41, Def. 3.2.1]. Namely:

Proposition 4.1.1. *If θ is a free and proper isometric 2-action of $K^{(1)} \rightrightarrows K^{(0)}$ on $(G^{(1)} \rightrightarrows G^{(0)},\eta)$ then there is a unique structure of Riemannian groupoid $(G^{(1)}/K^{(1)} \rightrightarrows G^{(0)}/K^{(0)},\overline{\eta})$ so that the canonical projection $\pi=(\pi_1,\pi_0):(G^{(1)} \rightrightarrows G^{(0)}) \to (G^{(1)}/K^{(1)} \rightrightarrows G^{(0)}/K^{(0)})$ becomes a Riemannian groupoid submersion.*

Proof. We carry out the proof for 1-metrics on $G^{(1)}$ since the computations are analogous if we consider instead 2-metrics on $G^{(2)}$. We start by exhibiting the Lie groupoid structure on $G^{(1)}/K^{(1)} \rightrightarrows G^{(0)}/K^{(0)}$. This fact was stated for instance in [52, Prop. 3.6] but without proof so that we exhibit a proof of it by sake of completeness. It is well known that $G^{(j)}/K^{(j)}$ admits a unique manifold structure so that π_j (for $j=0,1$) is a surjective submersion. We define source and target maps respectively as $\overline{s}([g])=[s_G(g)]$ and $\overline{t}([g])=[t_G(g)]$ for all $g\in G^{(1)}$. As consequence of Identities (1.12) these maps are well defined and, moreover, both of them are surjective submersions since π_j, s_G, and t_G are so. We have that $([g],[h])\in (G/K)^{(2)}$ if and only if $\overline{s}([g])=\overline{t}([h])$ which in turn holds true if and only if $s_G(g)=k_0t_G(h)$ for some $k_0\in K^{(0)}$. Thus, we define $\overline{m}([g],[h])=[m_G(g,kh)]$ for some $k\in K^{(1)}$ such that $t_K(k)=k_0$. It is simple to check that $\overline{m}$ does not depend on the choice of k and it is well defined because of Property (1.13). This is also clearly smooth and associative since m_G is associative and Identity (1.13) is satisfied. The unit map and the inversion are respectively defined by $\overline{u}([x])=[u_G(x)]$ and $\overline{i}([g])=[i_G(g)]$ for all $x\in G^{(0)}$ and $g\in G^{(1)}$. It is also easy to verify that these maps are well defined, smooth, and they satisfy the required groupoid conditions.

Consider, for $j=0,1$, the induced Riemannian metric $\overline{\eta}^{(j)}=(\pi_j)_*\eta^{(j)}$ on $G^{(j)}/K^{(j)}$ making of π_j a Riemannian submersion. Moreover, note that

$$\overline{i}_*\overline{\eta}^{(1)}=(\overline{i}\circ\pi_1)_*\eta^{(1)}=(\pi_1\circ i_G)_*\eta^{(1)}=(\pi_1)_*\eta^{(1)}=\overline{\eta}^{(1)},$$

since i is an isometry, and

$$\overline{s}_*\overline{\eta}^{(1)}=(\overline{s}\circ\pi_1)_*\eta^{(1)}=(\pi_0\circ s_G)_*\eta^{(1)}=(\pi_0)_*\eta^{(0)}=(\pi_0\circ t_G)_*\eta^{(1)}=(\overline{t}\circ\pi_1)_*\eta^{(1)}=\overline{t}_*\overline{\eta}^{(1)},$$

so that $\overline{s}_*\overline{\eta}^{(1)} = \overline{t}_*\overline{\eta}^{(1)} = \overline{\eta}^{(0)}$. Therefore, $(G^{(1)}/K^{(1)} \rightrightarrows G^{(0)}/K^{(0)}, \overline{\eta})$ with $\overline{\eta} = (\overline{\eta}^{(1)}, \overline{\eta}^{(0)})$ is a Riemannian groupoid and $\pi = (\pi_1, \pi_0)$ is a Riemannian submersion of groupoids, as desired. □

As an application of Theorem 2.4.1 and Proposition 4.1.1 we can prove analogous statements to those of the Slice Theorem and the Equivariant Tubular Neighborhood Theorem in our setting. For that it is necessary to remember the classical notion of "slice" associated to any Lie group action, see Definition 1.2.2.

Definition 4.1.2. Let θ be a Lie 2-group action of $K^{(1)} \rightrightarrows K^{(0)}$ on $G^{(1)} \rightrightarrows G^{(0)}$. A **groupoid slice** at $x_0 \in G^{(0)}$ is defined to be a Lie subgroupoid $S_{1_{x_0}} \rightrightarrows S_{x_0}$ of $G^{(1)} \rightrightarrows G^{(0)}$ such that S_{x_0} and $S_{1_{x_0}}$ are standard slices at x_0 and 1_{x_0}, respectively.

The following result is clear.

Lemma 4.1.1. *Take any $x_0 \in G^{(0)}$. There are natural structures of:*

- *Lie 2-subgroup between the K-isotropy groups $\mathrm{Iso}_{K^{(1)}}(1_{x_0}) \rightrightarrows \mathrm{Iso}_{K^{(0)}}(x_0)$, and*
- *Lie subgroupoid between the K-orbits $K^{(1)} \cdot 1_{x_0} \rightrightarrows K^{(0)} \cdot x_0$.*

Therefore, our version of the Slice Theorem is as follows.

Proposition 4.1.2 (Groupoid slice)**.** *Let θ be a proper 2-action of a Lie 2-group $K^{(1)} \rightrightarrows K^{(0)}$ on a Riemannian groupoid $(G^{(1)} \rightrightarrows G^{(0)}, \eta)$. Then there exists a groupoid slice $S_{1_{x_0}} \rightrightarrows S_{x_0}$ at each $x_0 \in G^{(0)}$.*

Proof. Let us consider the induced Lie 2-group action of $\mathrm{Iso}_{K^{(1)}}(1_{x_0}) \rightrightarrows \mathrm{Iso}_{K^{(0)}}(x_0)$ on $G^{(1)} \rightrightarrows G^{(0)}$. By applying Theorem 2.4.1 together with Remark 2.4.3 we may take the 2-metric η on $G^{(2)}$ and use it to construct another 2-metric $\tilde{\eta}$ on $G^{(2)}$ in such a way the Lie 2-group $\mathrm{Iso}_{K^{(1)}}(1_{x_0}) \rightrightarrows \mathrm{Iso}_{K^{(0)}}(x_0)$ acts isometrically on $(G^{(1)} \rightrightarrows G^{(0)}, \tilde{\eta})$. As in the classical case [8, Thm. 3.49], we define S_{x_0} by setting $S_{x_0} = \widetilde{\exp}^{(0)}_{x_0}(B_\epsilon(0))$ where $B_\epsilon(0)$ is an open ball of radius $\epsilon > 0$ around the origin in the normal space $\nu_{x_0}(K^{(0)} \cdot x_0)$ to the $K^{(0)}$-orbit through x_0 (normal domain).

As $K^{(1)} \cdot 1_{x_0} \rightrightarrows K^{(0)} \cdot x_0$ is a Lie subgroupoid of $G^{(1)} \rightrightarrows G^{(0)}$ we have a well defined Lie subgroupoid $\nu(K^{(1)} \cdot 1_{x_0}) \rightrightarrows \nu(K^{(0)} \cdot x_0)$ of $TG^{(1)} \rightrightarrows TG^{(0)}$ so that we may also consider the Lie groupoid $V_{B_\epsilon(0)} \rightrightarrows B_\epsilon(0)$ where $V_{B_\epsilon(0)} = \overline{ds}_{1_{x_0}}^{-1}(B_\epsilon(0)) \cap \overline{dt}_{1_{x_0}}^{-1}(B_\epsilon(0))$. Let us use the groupoid metric $\tilde{\eta}$ to identify $\nu(K^{(1)} \cdot 1_{x_0}) \cong T(K^{(1)} \cdot 1_{x_0})^{\perp}$ and $\nu(K^{(0)} \cdot x_0) \cong T(K^{(0)} \cdot x_0)^{\perp}$. By shrinking $B_\epsilon(0)$ if necessary we may assume that $V_{B_\epsilon(0)}$ is an open ball around the origin in the normal space $T_{1_{x_0}}(K^{(1)} \cdot 1_{x_0})^{\perp}$ to the $K^{(1)}$-orbit through 1_{x_0} on which $\widetilde{\exp}^{(1)}_{1_{x_0}}$ is well defined. Therefore, we now set $S_{1_{x_0}} = \widetilde{\exp}^{(1)}_{1_{x_0}}(V_{B_\epsilon(0)})$. Hence, by arguing with similar arguments as those used to prove the multiplicative property of the exponential maps associated to a Riemannian 2-metric in [40, Thm. 5.11] together with the equivariant property these exponential maps have, we conclude that $S_{1_{x_0}} \rightrightarrows S_{x_0}$ is the Lie subgroupoid of $G^{(1)} \rightrightarrows G^{(0)}$ we are looking for. □

The previous proposition will be used in order to obtain a 2-equivariant Tubular Neighborhood Theorem for the K-orbit groupoid $K^{(1)} \cdot 1_{x_0} \rightrightarrows K^{(0)} \cdot x_0$. For that we need to introduce some additional terminology regarding a notion of associated bundle in the groupoid framework.

The following definition can be found for instance in [29].

Definition 4.1.3. A **principal** 2**-bundle** $P^{(1)} \rightrightarrows P^{(0)}$ over a Lie groupoid $G^{(1)} \rightrightarrows G^{(0)}$ with **structural Lie** 2**-group** $K^{(1)} \rightrightarrows K^{(0)}$ is given by the data:

- a Lie groupoid fibration $\pi : (P^{(1)} \rightrightarrows P^{(0)}) \to (G^{(1)} \rightrightarrows G^{(0)})$ and
- a right Lie 2-group action of $K^{(1)} \rightrightarrows K^{(0)}$ on $P^{(1)} \rightrightarrows P^{(0)}$,

such that $\pi_1 : P^{(1)} \to G^{(1)}$ is a principal $K^{(1)}$-bundle and $\pi_0 : P^{(0)} \to G^{(0)}$ is a principal $K^{(0)}$-bundle.

By using this categorified notion of principal bundle we can introduce a natural construction which gives rise to what we call **associated groupoid bundle**.

Remark 4.1.1. Let $\pi : (P^{(1)} \rightrightarrows P^{(0)}) \to (G^{(1)} \rightrightarrows G^{(0)})$ be a groupoid principal 2-bundle with structural Lie 2-group $K^{(1)} \rightrightarrows K^{(0)}$ and assume that there exists a left Lie 2-group action of $K^{(1)} \rightrightarrows K^{(0)}$ over another Lie groupoid $F^{(1)} \rightrightarrows F^{(0)}$. Given this data we can construct two associated fiber bundles $E^{(j)} := P^{(j)} \times_{K^{(j)}} F^{(j)}$ over $G^{(j)}$ for $j = 0, 1$. These are defined as the quotient spaces $(P^{(j)} \times F^{(j)})/K^{(j)}$ with respect to the actions $k_j \cdot (p_j, f_j) = (p_j k_j^{-1}, k_j f_j)$, for all $k_j \in K^{(j)}$, $p_j \in P^{(j)}$ and $f_j \in F^{(j)}$, together with projections $\overline{\pi}_j([p_j, f_j]) = \pi_j(p_j)$ onto $G^{(j)}$. It is simple to check that there exists a natural Lie groupoid structure $E^{(1)} \rightrightarrows E^{(0)}$ for which the projection $\overline{\pi} : (E^{(1)} \rightrightarrows E^{(0)}) \to (G^{(1)} \rightrightarrows G^{(0)})$ becomes a Lie groupoid fibration. The source and target maps are the obvious ones, namely:

$$s_E([p, f]) = [s_P(p), s_F(f)] \qquad \text{and} \qquad t_E([p, f]) = [t_P(p), t_F(f)],$$

and the groupoid composition is defined as follows. If $s_E([p, f]) = t_E([q, l])$ then there is $k_0 \in K^{(0)}$ such that $(s_P(p)k_0^{-1}, k_0 s_F(f)) = (t_P(q), t_F(l))$. So, we set

$$m_E([p, f], [q, l]) = [(pk^{-1}) * q, (kf) * l],$$

for some k inside the s_K-fiber at k_0. From Formula (1.13) it follows that the latter equality is well defined and that it does not depend on the choice of $k \in s_K^{-1}(k_0)$. Furthermore, the same formula and the associativity of m_P, m_F together imply the associativity of the composition m_E. As expected, the inversion i_E and the unit map u_E are defined in the obvious way.

It is worth mentioning that a particular case of this construction can be found in [62, Lem. 9.1.2].

Let θ be a proper Lie 2-group action of $K^{(1)} \rightrightarrows K^{(0)}$ on $G^{(1)} \rightrightarrows G^{(0)}$. From Lemma 4.1.1, Proposition 4.1.2, the quotient construction described in Proposition 4.1.1, and Remark 4.1.1 we easily deduce that:

Lemma 4.1.2. *There is a principal groupoid* 2*-bundle* $\pi : (K^{(1)} \rightrightarrows K^{(0)}) \to (K^{(1)}/\mathrm{Iso}_{K^{(1)}}(1_{x_0}) \rightrightarrows K^{(0)}/\mathrm{Iso}_{K^{(0)}}(x_0))$ *with structural Lie* 2*-group* $\mathrm{Iso}_{K^{(1)}}(1_{x_0}) \rightrightarrows \mathrm{Iso}_{K^{(0)}}(x_0)$*, yielding an associated groupoid fibration*

$$(K^{(1)} \times_{\mathrm{Iso}_{K^{(1)}}(1_{x_0})} S_{1_{x_0}} \rightrightarrows K^{(0)} \times_{\mathrm{Iso}_{K^{(0)}}(x_0)} S_{x_0}) \to (K^{(1)}/\mathrm{Iso}_{K^{(1)}}(1_{x_0}) \rightrightarrows K^{(0)}/\mathrm{Iso}_{K^{(0)}}(x_0)),$$

with groupoid fiber $S_{1_{x_0}} \rightrightarrows S_{x_0}$*.*

Let us now consider the classical K-invariant tubular neighborhoods $\mathrm{Tub}_\epsilon^{\tilde{\eta}}(K^{(1)} \cdot 1_{x_0}) = \theta^1(K^{(1)}, S_{1_{x_0}})$ and $\mathrm{Tub}_\epsilon^{\tilde{\eta}}(K^{(0)} \cdot x_0) = \theta^0(K^{(0)}, S_{x_0})$ of $K^{(1)} \cdot 1_{x_0}$ and $K^{(0)} \cdot x_0$, respectively. Note that we have emphasized the inclusion of $\tilde{\eta}$ and ϵ, from the proof of Proposition 4.1.2, at the definition of the above tubular neighborhoods since the existence of our groupoid slice relies on them. It is clear that $\mathrm{Tub}^{\tilde{\eta}}_{\epsilon}(K^{(1)}\cdot 1_{x_0}) \rightrightarrows \mathrm{Tub}^{\tilde{\eta}}_{\epsilon}(K^{(0)}\cdot x_0)$ is a Lie subgroupoid of $G^{(1)} \rightrightarrows G^{(0)}$ and that the 2-action θ restricts well over it. Furthermore, $\mathrm{Tub}^{\tilde{\eta}}_{\epsilon}(K^{(1)}\cdot 1_{x_0}) \rightrightarrows \mathrm{Tub}^{\tilde{\eta}}_{\epsilon}(K^{(0)}\cdot x_0)$ determines an open Lie groupoid neighborhood of the K-orbit groupoid $K^{(1)}\cdot 1_{x_0} \rightrightarrows K^{(0)}\cdot x_0$.

We can state our version of the Equivariant Tubular Neighborhood Theorem in the following way:

Theorem 4.1.1 (Equivariant groupoid tubular neighborhood)**.** *Suppose that θ is a proper 2-action of a Lie 2-group $K^{(1)} \rightrightarrows K^{(0)}$ on a Riemannian groupoid $(G^{(1)} \rightrightarrows G^{(0)}, \eta)$. Then, for every $x_0 \in G^{(0)}$ there exists a 2-equivariant Lie groupoid isomorphism*

$$\Psi : (K^{(1)} \times_{\mathrm{Iso}_{K^{(1)}}(1_{x_0})} S_{1_{x_0}} \rightrightarrows K^{(0)} \times_{\mathrm{Iso}_{K^{(0)}}(x_0)} S_{x_0}) \xrightarrow{\cong} (\mathrm{Tub}^{\tilde{\eta}}_{\epsilon}(K^{(1)}\cdot 1_{x_0}) \rightrightarrows \mathrm{Tub}^{\tilde{\eta}}_{\epsilon}(K^{(0)}\cdot x_0)).$$

Proof. First of all, the left Lie 2-group action of $K^{(1)} \rightrightarrows K^{(0)}$ on $K^{(1)} \times_{\mathrm{Iso}_{K^{(1)}}(1_{x_0})} S_{1_{x_0}} \rightrightarrows K^{(0)} \times_{\mathrm{Iso}_{K^{(0)}}(x_0)} S_{x_0}$ we will consider in this case is the one given by $\overline{k_j}\cdot[k_j, f_j] = [\overline{k_j}k_j, f_j]$ for all $\overline{k_j}, k_j \in K^{(j)}$ and $f_j \in S_j$ for $j = 0, 1$. Here we are denoting $S_1 = S_{1_{x_0}}$ and $S_0 = S_{x_0}$. The Lie groupoid isomorphism Ψ is defined as

$$\Psi^j([k_j, f_j]) = \theta^j(k_j, f_j).$$

From the proof of Theorem 3.57 in [8] we already know that Ψ^j is a $K^{(j)}$-equivariant diffeomorphism. Thus, we only have to check that Ψ defines indeed a Lie groupoid morphism. By using the structural maps defined in Remark 4.1.1 we have that

$$(\Psi^0 \circ s_E)([k, f]) = \Psi^0([s_K(k), s_G(f)]) = s_K(k)s_G(f) = s_G(kf) = (s_G \circ \Psi^1)([k, f]).$$

We can similarly obtain that $\Psi^0 \circ t_E = t_G \circ \Psi^1$. Moreover, by applying Formula (1.13) we get

$$\begin{aligned} \Psi^1([k, f] * [k', l]) &= \Psi^1([(k\overline{k}^{-1}) * k', (\overline{k}f) * l]) = ((k\overline{k}^{-1}) * k') \cdot ((\overline{k}f) * l) \\ &= (kf) * (k'l) = \Psi^1([k, f]) * \Psi^1([k', l]). \end{aligned}$$

Hence, the result follows as desired. □

It is well known that the classical equivariant tubular neighborhood theorem can be used to study orbit types of proper actions, thus determining a way to induce stratifications for the manifold as well as for the corresponding orbit space [2, c. 3]. In our case we can stratify the Lie groupoid we are working with by topological subgroupoids as follows.

Remark 4.1.2. We say that x_0 and y_0 in $G^{(0)}$ have the same K-**orbit type** if there exists a Lie groupoid isomorphism Φ between $K^{(1)}\cdot 1_{x_0} \rightrightarrows K^{(0)}\cdot x_0$ and $K^{(1)}\cdot 1_{y_0} \rightrightarrows K^{(0)}\cdot y_0$ such that Φ^1 is $K^{(1)}$-equivariant. This automatically implies that Φ^0 is $K^{(0)}$-equivariant, so that we are actually speaking about a 2-equivariant isomorphism. It is clear that this K-orbit type requirement defines an equivalent relation $\sim$ on $G^{(0)}$ for which we denote by $G^{\sim}_{(x_0)}$ the equivalent class at $x_0 \in G^{(0)}$ associated to such a relation. We claim that $G^{\sim}_{(x_0)}$ is saturated in $G^{(0)}$, that is, $s_G^{-1}(G^{\sim}_{(x_0)}) = t_G^{-1}(G^{\sim}_{(x_0)})$. If $g \in s_G^{-1}(G^{\sim}_{(x_0)})$ then $s_G(g) \sim x_0$ so that there is a 2-equivariant isomorphism between $K^{(1)}\cdot 1_{s_G(g)} \rightrightarrows K^{(0)}\cdot s_G(g)$ and $K^{(1)}\cdot 1_{x_0} \rightrightarrows K^{(0)}\cdot x_0$. It is simple to check that Φ_g defined by $\Phi^1_g(k1_{t_G(g)}) = k1_{s_G(g)}$ and $\Phi^0_g(k_0 t_G(g)) = k_0 s_G(g)$ defines another 2-equivariant isomorphism between $K^{(1)}\cdot 1_{t_G(g)} \rightrightarrows K^{(0)}\cdot t_G(g)$ and $K^{(1)}\cdot$

$1_{s_g(g)} \rightrightarrows K^{(0)} \cdot s_G(g)$ so that by taking $\Phi \circ \Phi_g$ we conclude that $t_G(g) \sim x_0$. The other inclusion may be similarly checked. Thus, by setting $G^{\sim}_{(1_{x_0})} = s_G^{-1}(G^{\sim}_{(x_0)}) = t_G^{-1}(G^{\sim}_{(x_0)})$ we obtain a collection of topological groupoids $\{G^{\sim}_{(1_{x_0})} \rightrightarrows G^{\sim}_{(x_0)}\}_{x_0 \in G^{(0)}}$ which stratifies the Lie groupoid $G^{(1)} \rightrightarrows G^{(0)}$. This is because at the level of objects and arrows we are recovering the usual "local K-orbit type stratification" of the corresponding proper actions, see [2, s. 3.5].

The previous observation suggests that by combining classical ideas from [8, s. 3.5] with some of the results obtained in this section it could be possible to show that each of the $G^{\sim}_{(1_{x_0})} \rightrightarrows G^{\sim}_{(x_0)}$ is a honest Lie subgroupoid. Nevertheless, the local K-orbit type notion from the classical case does not extended directly in our case. *We conjecture that our guess is true*. That is, $G^{\sim}_{(1_{x_0})} \rightrightarrows G^{\sim}_{(x_0)}$ is Lie subgroupoid of $G^{(1)} \rightrightarrows G^{(0)}$ for all $x_0 \in G^{(0)}$.

4.1.1 Orthogonal Lie 2-groups

In this short subsection we derive another application of Theorem 2.4.1 which has to do with the construction of groupoid bi-invariant Riemannian metrics on compact Lie 2-groups. To simplify computations, here it will be enough to consider only 1-metrics. Let $K^{(1)} \rightrightarrows K^{(0)}$ be a Lie 2-group and consider the pairs $L = (L^1, L^0)$ and $R = (R^1, R^0)$ where L^j and R^j for $j = 0, 1$ are respectively the actions of $K^{(j)}$ on itself determined by left and right multiplications. It is simple to check that as consequence of Identity (1.10) and the fact that the structural maps of $K^{(1)} \rightrightarrows K^{(0)}$ are Lie group homomorphisms it follows that L and R determine left Lie 2-group actions of $K^{(1)} \rightrightarrows K^{(0)}$ on itself.

Definition 4.1.4. A Lie 2-group $K^{(1)} \rightrightarrows K^{(0)}$ is said to be **orthogonal** if it may be equipped with a 1-metric for which both L and R are isometric Lie 2-group actions. Such a 1-metric will be called **bi-invariant**.

Throughout this subsection we think of 1-metrics on a Lie 2-algebra $\mathfrak{k}^{(1)} \rightrightarrows \mathfrak{k}^{(0)}$ as pairs of inner products $\langle\cdot,\cdot\rangle = (\langle\cdot,\cdot\rangle^{(1)}, \langle\cdot,\cdot\rangle^{(0)})$ verifying the required conditions of 1-metric. Let $K^{(1)} \rightrightarrows K^{(0)}$ be a Lie 2-group with respective Lie 2-algebra $\mathfrak{k}^{(1)} \rightrightarrows \mathfrak{k}^{(0)}$. We denote by $\mathrm{Ad} = (\mathrm{Ad}^1, \mathrm{Ad}^0)$ the Lie 2-group action of $K^{(1)} \rightrightarrows K^{(0)}$ on $\mathfrak{k}^{(1)} \rightrightarrows \mathfrak{k}^{(0)}$ determined by the adjoint actions Ad^j of $K^{(j)}$ on $\mathfrak{k}^{(j)}$ for $j = 0, 1$. This 2-action will be called **adjoint Lie 2-group action** of $K^{(1)} \rightrightarrows K^{(0)}$.

The following result is expected.

Proposition 4.1.3. *A Lie 2-group $K^{(1)} \rightrightarrows K^{(0)}$ is orthogonal if and only if there exists a 1-metric $\langle\cdot,\cdot\rangle$ on its Lie 2-algebra $\mathfrak{k}^{(1)} \rightrightarrows \mathfrak{k}^{(0)}$ for which the adjoint Lie 2-group action is by linear isometries.*

Proof. It is clear that if η is a bi-invariant 1-metric on $K^{(1)} \rightrightarrows K^{(0)}$ then $\eta_e = (\eta^1_{e_1}, \eta^0_{e_0})$, where e_j is the identity element in $K^{(j)}$, defines a 1-metric on $\mathfrak{k}^{(1)} \rightrightarrows \mathfrak{k}^{(0)}$ for which the adjoint Lie 2-group action is by linear isometries. Conversely, given such a $\langle\cdot,\cdot\rangle$, we define η on $K^{(1)} \rightrightarrows K^{(0)}$ by setting

$$\eta^{(1)}_k(v, w) = \langle d(L^1_{k^{-1}})_k(v), d(L^1_{k^{-1}})_k(w)\rangle^{(1)}.$$

The metric $\eta^{(0)}$ has a similar defining formula but using L^0 and $\langle\cdot,\cdot\rangle^{(0)}$ instead of L^1 and $\langle\cdot,\cdot\rangle^{(1)}$. It is clear that $\eta^{(1)}$ is bi-invariant. Therefore, it remains to prove that η defines indeed a 1-metric. Firstly, for $k \in K^{(1)}$ and $v, w \in T_k K^{(1)}$ we have

$$\begin{aligned} i^*\eta^{(1)}(v,w)(k) &= \langle d(L^1_{i(k)^{-1}})_{i(k)}(di_k(v)), d(L^1_{i(k)^{-1}})_{i(k)}(di_k(w))\rangle^{(1)} \\ &= \langle d(L^1_{i(k^{-1})} \circ i)_k(v), d(L^1_{i(k^{-1})} \circ i)_k(w)\rangle^{(1)} \\ &= \langle d(i \circ L^1_{k^{-1}})_k(v), d(i \circ L^1_{k^{-1}})_k(w)\rangle^{(1)} \\ &= \langle di_{e_1}(d(L^1_{k^{-1}})_k(v)), di_{e_1}(d(L^1_{k^{-1}})_k(v))\rangle^{(1)} = \eta^{(1)}(v,w)(k). \end{aligned}$$

Secondly, let $k_0 \in K^{(0)}$ and $v, w \in T_{k_0}K^{(0)}$. It is clear that there are $\tilde{v}, \tilde{w} \in \ker(ds(k))^{\perp_{\eta^{(1)}}}$ with $s(k) = k_0$ such that $ds_k(\tilde{v}) = v$ and $ds_k(\tilde{w}) = w$. Thus

$$\begin{aligned} (s_*\eta^{(1)})(v,w)(k_0) &= \eta^{(1)}(\tilde{v},\tilde{w})(k) = \langle d(L^1_{k^{-1}})_k(\tilde{v}), d(L^1_{k^{-1}})_k(\tilde{w})\rangle^{(1)} \\ &= \langle ds_{e_1}(d(L^1_{k^{-1}})_k(\tilde{v})), ds_{e_1}(d(L^1_{k^{-1}})_k(\tilde{w}))\rangle^{(0)} \\ &= \langle d(s \circ L^1_{k^{-1}})_k(\tilde{v}), d(s \circ L^1_{k^{-1}})_k(\tilde{w})\rangle^{(0)} \\ &= \langle d(L^0_{k_0^{-1}} \circ s)_k(\tilde{v}), d(L^0_{k_0^{-1}} \circ s)_k(\tilde{w})\rangle^{(0)} \\ &= \langle d(L^0_{k_0^{-1}})_{k_0}(v), d(L^0_{k_0^{-1}})_{k_0}(v)\rangle^{(0)} = \eta^{(0)}(v,w)(k_0). \end{aligned}$$

Analogously, $t_*\eta^{(1)} = \eta^{(0)}$. So, the result follows. □

Let us now describe orthogonal Lie 2-groups infinitesimally. To simplify things from now on we assume that the Lie groups we are working with are connected. Recall that bi-invariant metrics on a Lie group are in one-to-one correspondence with inner products on its Lie algebra for which the adjoint representation determines infinitesimal isometries, consult Section 1.2 and [86, 88].

Definition 4.1.5. A Lie 2-algebra $\mathfrak{k}^{(1)} \rightrightarrows \mathfrak{k}^{(0)}$ is said to be **orthogonal** if it admits a 1-metric $\langle\cdot,\cdot\rangle = (\langle\cdot,\cdot\rangle^{(1)}, \langle\cdot,\cdot\rangle^{(0)})$ for which the adjoint representation $\mathrm{ad}^1 : \mathfrak{k}^{(1)} \to \mathrm{Der}(\mathfrak{k}^{(1)})$ acts by infinitesimal isometries on $(\mathfrak{k}^{(1)}, \langle\cdot,\cdot\rangle^{(1)})$.

As a consequence of Proposition 4.1.3 and [88, Lem. 7.2] we get:

Corollary 4.1.1. *A Lie 2-group is orthogonal if and only if its Lie algebra $\mathfrak{k}^{(1)} \rightrightarrows \mathfrak{k}^{(0)}$ is orthogonal.*

More importantly, from Theorem 2.4.1 and Corollary 2.4.1 we obtain that:

Corollary 4.1.2. *Every Lie 2-group $K^{(1)} \rightrightarrows K^{(0)}$ with $K^{(1)}$ compact can be endowed with a bi-invariant 1-metric.*

We may also compare the infinitesimal object introduced in Definition 4.1.5 with an existing notion of $\mathcal{L}$-orthogonal crossed module of Lie algebras already known in the literature. Namely, given an orthogonal Lie 2-algebra $\mathfrak{k}^{(1)} \rightrightarrows \mathfrak{k}^{(0)}$ we may split $\mathfrak{k}^{(1)}$ as a direct sum of ideals $\mathfrak{k}^{(1)} = \mathfrak{h} \oplus \mathfrak{h}^{\perp_1}$ where $\mathfrak{h} = \ker(s)$. As the unit map u is a canonical bisection we would expect that $u(x) \in \mathfrak{h}^{\perp_1}$ for all $x \in \mathfrak{k}^{(0)}$ but, however, this is not true in general unless we assume "non-canonical" identifications. We say that an orthogonal Lie 2-algebra is **trivial** if $\mathfrak{h}^{\perp_1} = u(\mathfrak{k}^{(0)})$. This notion comes up by the following simple result.

Proposition 4.1.4. *If $\mathfrak{k}^{(1)} \rightrightarrows \mathfrak{k}^{(0)}$ is a trivial orthogonal Lie 2-algebra then $\mathrm{ad}^1_{u(x)}|_{\mathfrak{h}} = 0$ for all $x \in \mathfrak{k}^{(0)}$. In consequence, $\mathfrak{h}$ is abelian and $\mathrm{im}(t|_{\mathfrak{h}})$ is a subspace of the center of $\mathfrak{k}^{(0)}$.*

Proof. On the one hand, note that for any $y \in \mathfrak{h}$ and $z \in \mathfrak{k}^{(1)}$ one gets

$$\langle \mathrm{ad}^1_{u(x)}(y), z\rangle^{(1)} = \langle u(x), \mathrm{ad}^1_y(z)\rangle^{(1)} = 0,$$

since $\mathfrak{h}$ is an ideal. Thus, as $\langle\cdot,\cdot\rangle^{(1)}$ is nondegenerate we get that $\mathrm{ad}^1_{u(x)}|_{\mathfrak{h}} = 0$ for all $x \in \mathfrak{k}^{(0)}$. On the other hand, for all $y, y' \in \mathfrak{h}$ and $x \in \mathfrak{k}^{(1)}$ it follows that $[y, y'] = \mathrm{ad}^1_{u(t(y))}(y') = 0$ and $0 = t([u(x), y]) = [x, t(y)]$ since t is a Lie algebra homomorphism. □

This fact in particular implies that the crossed module of Lie algebras associated to a trivial orthogonal Lie 2-algebra is of the form $(\mathfrak{k}, \mathfrak{h}, \partial, 0)$ with $\mathfrak{h}$ an abelian Lie algebra. In addition to that, it motivates the observation below.

Remark 4.1.3. Suppose that we have a crossed module of Lie algebras $(\mathfrak{k}, \mathfrak{h}, \partial, \mathcal{L})$ which is $\mathcal{L}$-orthogonal in the sense of [9, 47]. That is to say, there are inner products $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ and $\langle\cdot,\cdot\rangle_{\mathfrak{h}}$ such that the Lie algebra representation $\mathcal{L} : \mathfrak{k} \to \mathrm{Der}(\mathfrak{h})$ acts by infinitesimal isometries on $(\mathfrak{h}, \langle\cdot,\cdot\rangle_{\mathfrak{h}})$ and the adjoint representation of $\mathfrak{k}$ acts by infinitesimal isometries on $(\mathfrak{k}, \langle\cdot,\cdot\rangle_{\mathfrak{k}})$. Note that because of Formulas (1.11) this directly implies that the adjoint representation of $\mathfrak{h}$ acts by infinitesimal isometries on $(\mathfrak{h}, \langle\cdot,\cdot\rangle_{\mathfrak{h}})$. On the one hand, recall that the associated Lie 2-algebra constructed with the crossed module data has $\mathfrak{k}^{(1)} = \mathfrak{h} \rtimes \mathfrak{k}$ with Lie algebra structure provided by the semi-direct product with respect to $\mathcal{L}$. Therefore, a straightforward computation allows us to conclude that the adjoint representation of $\mathfrak{k}^{(1)}$ acts by infinitesimal isometries with respect to $\langle\cdot,\cdot\rangle_{\mathfrak{h}} + \langle\cdot,\cdot\rangle_{\mathfrak{k}}$ if and only if $\mathcal{L} = 0$. On the other hand, since the inversion $i : \mathfrak{k}^{(1)} \to \mathfrak{k}^{(1)}$ is given by $i(x, y) = (-x, y + \partial(x))$ for all $x, x' \in \mathfrak{h}$ and $y, y' \in \mathfrak{k}$ then it follows that i is an isometry with respect to $\langle\cdot,\cdot\rangle_{\mathfrak{h}} + \langle\cdot,\cdot\rangle_{\mathfrak{k}}$ if and only if $\partial = 0$. As consequence, we have noticed that there is no canonical correspondence between our notion of orthogonal Lie 2-algebras and the notion of $\mathcal{L}$-orthogonal crossed module of Lie algebras which is known in the literature.

4.1.2 Examples and related constructions

In this short subsection we exhibit some examples and interesting constructions in which isometric Lie 2-group actions naturally appear.

Example 4.1.1. As expected, classical isometric actions of Lie groups K on Riemannian manifolds (M, η) are recovered from isometric Lie 2-group actions of unit Lie 2-groups $K \rightrightarrows K$ acting upon unit Riemannian groupoids $(M \rightrightarrows M, \eta)$.

Example 4.1.2. Let $(G^{(1)} \rightrightarrows G^{(0)}, \eta)$ be a Riemannian groupoid for which there exists a complete multiplicative Killing vector field (ξ, v) on $G^{(1)} \rightrightarrows G^{(0)}$. Here we consider ξ a Killing vector field on $(G^{(1)}, \eta^{(1)})$ and v a Killing vector field on $(G^{(0)}, \eta^{(0)})$. From [81, Prop. 3.5] we know that the pair of flows defined by (ξ, v) determine global automorphisms on $G^{(1)} \rightrightarrows G^{(0)}$ so that we get a well defined isometric Lie 2-group action of $\mathbb{R} \rightrightarrows \mathbb{R}$ on $(G^{(1)} \rightrightarrows G^{(0)}, \eta)$. In particular, the flow of the multiplicative vector field $((1_\xi)_{G^{(1)}}, \xi_{G^{(0)}})$ formed by the fundamental vector fields of an isometric Lie 2-group action determines another isometric Lie 2-group action.

Example 4.1.3. Let K be an orthogonal Lie group and $H \leq K$ be a normal Lie subgroup. It is clear that H acts on K by left multiplication, leading to the action Lie groupoid $H \times K \rightrightarrows K$. Note that its space of arrows has a group structure, namely the semi-direct product by the conjugation action $c_k(h) = khk^{-1}$ of K on H so that we get a well defined Lie 2-group, see Example 1.3.29. More importantly, by applying the gauge trick construction behind Proposition 4.7 and Example 4.9 in [40] we conclude that it is possible to cook up explicitly a 1-metric on $H \rtimes K \rightrightarrows K$ made out from the initial bi-invariant metric on K, in such a way it becomes an orthogonal Lie 2-group.

Example 4.1.4. Let $(M,\mathcal{F})$ be a regular Riemannian foliation and consider a free and proper isometric foliated action $K \times (M,\mathcal{F}) \to (M,\mathcal{F})$ of a Lie group K. From [52, Thm. 3.7] it is known that the Lie 2-group $K \ltimes K \rightrightarrows K$, as defined in Example 4.1.3, determines a canonical Lie 2-group action on the holonomy groupoid $\mathrm{Hol}(M,\mathcal{F}) \rightrightarrows M$ of $(M,\mathcal{F})$ which extends the given action of K on M. The Riemannian metric on M completely determines a 0-metric on $\mathrm{Hol}(M,\mathcal{F}) \rightrightarrows M$ which can be extended to a 1-metric [40, Ex. 3.12]. Hence, the fact that K acts on M isometrically implies that the extended Lie 2-group action of $K \ltimes K \rightrightarrows K$ on $\mathrm{Hol}(M,\mathcal{F}) \rightrightarrows M$ is by isometries.

Example 4.1.5. If $(M,\eta^{(0)})$ is a Riemannian manifold then $\eta^{(1)} = \eta^{(0)} \oplus \eta^{(0)}$ defines a 1-metric on the pair groupoid $M \times M \rightrightarrows M$. Thus, if K is a Lie group acting freely, properly and isometrically on $(M,\eta^{(0)})$ then the unit Lie 2-group $K \rightrightarrows K$ acts isometrically on $M \times M \rightrightarrows M$ and, in light of Proposition 4.1.1, it follows that the quotient groupoid $(M\times M)/K \rightrightarrows M/K$ canonically inherits a 1-metric. If K admits a bi-invariant metric then the previous procedure yields an easy way to construct 1-metrics on the gauge groupoid associated to a principal K-bundle over a Riemannian manifold, see Example 4.1.6 below.

For the next construction we need to introduce the notion of multiplicative 2-connection on a groupoid principal 2-bundle as defined for instance in [29, s. 5].

Definition 4.1.6. Let $\pi : (P^{(1)} \rightrightarrows P^{(0)}) \to (G^{(1)} \rightrightarrows G^{(0)})$ be a groupoid principal 2-bundle with structural Lie 2-group $K^{(1)} \rightrightarrows K^{(0)}$. A **multiplicative 2-connection** on $P^{(1)} \rightrightarrows P^{(0)}$ is defined to be a pair $\omega = (\omega^1, \omega^0)$ where:

- $\omega^j \in \Omega^1(P^{(j)}, \mathfrak{k}^{(j)})$ is a connection 1-form on $P^{(j)}$ for $j = 0, 1$, and
- $\omega : (TP^{(1)} \rightrightarrows TP^{(0)}) \to (\mathfrak{k}^{(1)} \times P^{(1)} \rightrightarrows \mathfrak{k}^{(0)} \times P^{(0)})$ defines a Lie groupoid morphism.

Here $TP^{(1)} \rightrightarrows TP^{(0)}$ has the tangent groupoid structure and $\mathfrak{k}^{(1)} \times P^{(1)} \rightrightarrows \mathfrak{k}^{(0)} \times P^{(0)}$ has the product groupoid structure, where $\mathfrak{k}^{(1)} \rightrightarrows \mathfrak{k}^{(0)}$ stands for the Lie 2-algebra associated to $K^{(1)} \rightrightarrows K^{(0)}$.

Example 4.1.6 (Principal groupoid warping)**.** Suppose that we have a principal groupoid 2-bundle endowed with a multiplicative 2-connection as in the previous definition. Let us further assume that $G^{(1)} \rightrightarrows G^{(0)}$ is equipped with a 1-metric and that $K^{(1)} \rightrightarrows K^{(0)}$ is an orthogonal Lie 2-group. Motivated by Example 1.2.1 we claim that there exists a 1-metric $\overline{\eta}$ on $P^{(1)} \rightrightarrows P^{(0)}$ for which the Lie 2-group action of $K^{(1)} \rightrightarrows K^{(0)}$ is isometric and such that $\pi =: (P^{(1)} \rightrightarrows P^{(0)}) \to (G^{(1)} \rightrightarrows G^{(0)})$ is a Riemannian groupoid submersion. Indeed, consider the associated Ad-invariant 1-metric $\langle\cdot,\cdot\rangle$ on the Lie 2-algebra $\mathfrak{k}^{(1)} \rightrightarrows \mathfrak{k}^{(0)}$ and define

$$\overline{\eta}^{(1)}(v,w) = \eta^{(1)}(d\pi_1(v), d\pi_1(w)) + \langle \omega^1(v), \omega^1(w)\rangle^{(1)}.$$

The metric $\overline{\eta}^{(0)}$ is similarly defined by using instead $\eta^{(0)}$, π_0, $\langle\cdot,\cdot\rangle^{(0)}$, and ω^0. It is simple to check that this expression yields a well defined right $K^{(j)}$-invariant metric on $P^{(j)}$ for which $\pi_j : P^{(j)} \to G^{(j)}$ becomes a Riemannian submersion, for $j = 0, 1$. Recall that this is because π_j is constant along the action orbits, ω^j is of Ad^j-invariant type, and $\ker(d\pi_j)^{\perp_{\overline{\eta}^{(j)}}} = \ker(\omega^j)$, compare [98, p. 467]. Let us verify that $\overline{\eta} = (\overline{\eta}^{(1)}, \overline{\eta}^{(0)})$ determines a 1-metric on $P^{(1)} \rightrightarrows P^{(0)}$. Note that, by abusing a little on the notation, we may rewrite $\overline{\eta}^{(j)}$ in a simpler way as $\overline{\eta}^{(j)} = (\pi_j)^*\eta^{(j)} + (\omega^j)^*\langle\cdot,\cdot\rangle^{(j)}$. So, on the one hand we get

$$\begin{aligned} (i_P)^*\overline{\eta}^{(1)} &= (\pi_1 \circ i_P)^*\eta^{(1)} + (\omega^1 \circ i_P)^*\langle\cdot,\cdot\rangle^{(1)} = (i_G \circ \pi_1)^*\eta^{(1)} + (d(i_K)_{e_1} \circ \omega^1)^*\langle\cdot,\cdot\rangle^{(1)} \\ &= (\pi_1)^*\eta^{(1)} + (\omega^1)^*\langle\cdot,\cdot\rangle^{(1)} = \overline{\eta}^{(1)}, \end{aligned}$$

since both π and ω are Lie groupoid morphisms. On the other hand, if $v \in \ker(d(s_P)_p)^{\perp_{\overline{\eta}^{(1)}}}$ then it is simple to verify that the identities $\pi_0 \circ s_P = s_G \circ \pi_1$ and $\omega^0 \circ s_P = d(s_K)_{e_1} \circ \omega^1$ imply that $d(\pi_1)_p(v) \in \ker(d(s_G)_{\pi_1(p)})^{\perp_{\eta^{(1)}}}$ and $\omega^1(v) \in \ker(d(s_K)_{e_1})^{\perp_{\langle\cdot,\cdot\rangle^{(1)}}}$. Let us pick $v_1, v_2 \in \ker(d(s_P)_p)^{\perp_{\overline{\eta}^{(1)}}}$. Thus,

$$\begin{aligned}
\overline{\eta}^{(0)}_{s_P(p)}(ds_P(p)(v_1), ds_P(p)(v_2)) &= \eta^{(0)}_{\pi_0(s_P(p))}(d(\pi_0)_{s_P(p)}(ds_P(p)(v_1)), d(\pi_0)_{s_P(p)}(ds_P(p)(v_2))) \\
&+ \langle \omega^0(ds_P(p)(v_1)), \omega^0(ds_P(p)(v_2))\rangle^{(0)} \\
&= \eta^{(0)}_{s_G(\pi_1(p))}(d(s_G)_{\pi_1(p)}(d\pi_1(p)(v_1)), d(s_G)_{\pi_1(p)}(d\pi_1(p)(v_1))) \\
&+ \langle d(s_K)_{e_1}(\omega^1(v_1)), d(s_K)_{e_1}(\omega^1(v_2))\rangle^{(0)} \\
&= \eta^{(1)}_{\pi_1(p)}(d\pi_1(p)(v_1), d\pi_1(p)(v_1)) + \langle \omega^1(v_1), \omega^1(v_2)\rangle^{(0)} \\
&= \overline{\eta}^{(1)}(v_1, v_2).
\end{aligned}$$

Analogously, it follows that $(t_P)_*\overline{\eta}^{(1)} = \overline{\eta}^{(0)}$. Hence, we have shown that $(P^{(1)} \rightrightarrows P^{(0)}, \overline{\eta})$ is a Lie groupoid equipped with a 1-metric $\overline{\eta}$ for which $K^{(1)}$ acts isometrically on $(P^{(1)}, \overline{\eta})$ and such that π is a Riemannian groupoid submersion, as claimed.

After performing similar computations it is simple to verify an analogous result for 2-metrics instead. Namely, let us now suppose that $G^{(2)}$ can be endowed with a 2-metric $\eta^{(2)}$ and that $K^{(2)}$ admits a bi-invariant 2-metric with associated Ad-invariant 2-metric $\langle\cdot,\cdot\rangle^{(2)}$ on $\mathfrak{k}^{(2)}$. If $\omega^2 : TP^{(2)} \to \mathfrak{k}^{(2)} \times P^{(2)}$ denotes the induced connection 1-form by ω on the principal bundle $\pi_2 : P^{(2)} \to G^{(2)}$ with structural group $K^{(2)}$ then the formula

$$\overline{\eta}^{(2)}((v,w),(v',w')) = \eta^{(2)}(d\pi_2(v,w), d\pi_2(v',w')) + \langle \omega^2(v,w), \omega^2(v',w')\rangle^{(2)},$$

defines a 2-metric on $P^{(2)}$ for which the action of $K^{(2)}$ on $(P^{(2)}, \overline{\eta}^{(2)})$ is by isometries and π_2 becomes a Riemannian submersion.

A particular example in which the previous construction applies is the following.

Example 4.1.7. Let $(G^{(1)} \rightrightarrows G^{(0)}, \eta)$ be a Riemannian étale groupoid with $n = \dim G^{(0)} = \dim G^{(1)}$. We denote by $\pi_1 : O(G^{(1)}) \to G^{(1)}$ and $\pi_0 : O(G^{(0)}) \to G^{(0)}$ the principal $O(n,\mathbb{R})$-bundles of orthonormal frames over $(G^{(1)}, \eta^{(1)})$ and $(G^{(0)}, \eta^{(0)})$, respectively. There exists a canonical Lie groupoid structure $O(G^{(1)}) \rightrightarrows O(G^{(0)})$ with the property that $\pi : (O(G^{(1)}) \rightrightarrows O(G^{(0)})) \to (G^{(1)} \rightrightarrows G^{(0)})$ becomes a principal 2-bundle with structural Lie 2-group $O(n,\mathbb{R}) \rightrightarrows O(n,\mathbb{R})$. Moreover, it admits a canonical multiplicative 2-connection $\omega = (\omega^1, \omega^0)$, where ω^1 and ω^0 are respectively determined by the Levi-Civita connections on $(G^{(1)}, \eta^{(1)})$ and $(G^{(0)}, \eta^{(0)})$. The source and target maps of $O(G^{(1)}) \rightrightarrows O(G^{(0)})$ are given by

$$\tilde{s}(v_1, \cdots, v_n) = (ds(v_1), \cdots, ds(v_n)) \quad \text{and} \quad \tilde{t}(v_1, \cdots, v_n) = (dt(v_1), \cdots, dt(v_n)),$$

and the composition is

$$\tilde{m}((v_1, \cdots, v_n), (v'_1, \cdots, v'_n)) = (dm(v_1, v'_1), \cdots, dm(v_n, v'_n)).$$

The other structural maps can be easily defined in a similar fashion. Note that the principal groupoid warping construction from Example 4.1.6 can be applied in this case.

Next construction comes motivated by a beautiful notion known in the literature as Cheeger deformation, visit Example 1.2.2. The details for the classical construction can be found for instance in [8, s. 6.1].

Example 4.1.8 (Cheeger groupoid deformation)**.** Suppose that $(K^{(1)} \rightrightarrows K^{(0)}, \langle\cdot,\cdot\rangle)$ is an orthogonal Lie 2-group, with $K^{(1)}$ compact, acting isometrically on a Riemannian groupoid $(G^{(1)} \rightrightarrows G^{(0)}, \eta)$. On the Lie groupoid product $G^{(1)} \times K^{(1)} \rightrightarrows G^{(0)} \times K^{(0)}$ we can consider the 1-metric $\eta \oplus \frac{1}{\tau}\langle\cdot,\cdot\rangle$. There is a natural free Lie 2-group action of $K^{(1)} \rightrightarrows K^{(0)}$ on the product groupoid $G^{(1)} \times K^{(1)} \rightrightarrows G^{(0)} \times K^{(0)}$ where the $K^{(j)}$-action on $G^{(j)} \times K^{(j)}$ for $j = 1, 0$ is given by

$$k'_j \cdot (g_j, k_j) = (k'_j g_j, k'_j k_j), \qquad g_j \in G^{(j)} \quad \text{and} \quad k'_j, k_j \in K^{(j)}. \tag{4.1}$$

We claim that the quotient groupoid $\frac{G^{(1)} \times K^{(1)}}{K^{(1)}} \rightrightarrows \frac{G^{(0)} \times K^{(0)}}{K^{(0)}}$ determined by the previous actions is isomorphic to $G^{(1)} \rightrightarrows G^{(0)}$. Indeed, let us consider the groupoid principal 2-bundle $K^{(1)} \rightrightarrows K^{(0)}$ over the point groupoid $\{e_1\} \rightrightarrows \{e_0\}$ with structural Lie 2-group $K^{(1)} \rightrightarrows K^{(0)}$. As an application of Remark 4.1.1 we know that by using the Lie 2-group action of $K^{(1)} \rightrightarrows K^{(0)}$ on $G^{(1)} \rightrightarrows G^{(0)}$ we can construct the associated Lie groupoid bundle $(G^{(1)} \times_{K^{(1)}} K^{(1)} \rightrightarrows G^{(0)} \times_{K^{(0)}} K^{(0)}) \to (\{e_1\} \rightrightarrows \{e_0\})$. It is important to notice that $G^{(1)} \times_{K^{(1)}} K^{(1)} \rightrightarrows G^{(0)} \times_{K^{(0)}} K^{(0)}$ is precisely the quotient groupoid $\frac{G^{(1)} \times K^{(1)}}{K^{(1)}} \rightrightarrows \frac{G^{(0)} \times K^{(0)}}{K^{(0)}}$. Therefore, as the new Lie groupoid bundle has groupoid fiber $G^{(1)} \rightrightarrows G^{(0)}$ and base groupoid $\{e_1\} \rightrightarrows \{e_0\}$ then we have the desired isomorphism. Under this identification the canonical groupoid projection $\pi = (G^{(1)} \times K^{(1)} \rightrightarrows G^{(0)} \times K^{(0)}) \to (G^{(1)} \rightrightarrows G^{(0)})$ is formed by the maps $\pi_j(g_j, k_j) = k_j^{-1} g_j$.

Some interesting properties derived from the facts above come in order below.

- The Lie 2-group action (4.1) is also isometric. Thus, as consequence of Proposition 4.1.1, there is a unique 1-metric η_τ on $G^{(1)} \rightrightarrows G^{(0)}$ making of the projection $\pi = (G^{(1)} \times K^{(1)} \rightrightarrows G^{(0)} \times K^{(0)}, \eta \oplus \frac{1}{\tau}\langle\cdot,\cdot\rangle) \to (G^{(1)} \rightrightarrows G^{(0)}, \eta_\tau)$ a Riemannian groupoid submersion.

- The original Lie 2-group action of $K^{(1)} \rightrightarrows K^{(0)}$ on $(G^{(1)} \rightrightarrows G^{(0)}, \eta)$ is also isometric on $(G^{(1)} \rightrightarrows G^{(0)}, \eta_\tau)$. Indeed, there is another isometric Lie 2-group action of $K^{(1)} \rightrightarrows K^{(0)}$ on $(G^{(1)} \rightrightarrows G^{(0)}, \eta_\tau)$ given by the formula

  $$k'_j * (g_j, k_j) = (g_j, k_j k'^{-1}_j), \quad j = 0, 1.$$

  This 2-action commutes with the 2-action (4.1). Hence, it descends to an isometric Lie 2-group action on the corresponding quotient groupoid $(G^{(1)} \rightrightarrows G^{(0)}, \eta_\tau)$ since π is a Riemannian groupoid submersion. Note that $k'_j \cdot \pi_j(g_j, k_j) = k'_j k_j^{-1} g_j = \pi_j(k'_j * (g_j, k_j))$ for $j = 0, 1$, so that the Lie 2-group action induced by $*$ is the original Lie 2-group action of $K^{(1)} \rightrightarrows K^{(0)}$ on $G^{(1)} \rightrightarrows G^{(0)}$.

- By construction η_τ goes to η as τ goes to ∞. More importantly, the 1-parameter family of 1-metrics η_τ on $G^{(1)} \rightrightarrows G^{(0)}$ varies smoothly with τ and extends smoothly to $\tau = 0$ with $\eta_0 = \eta$. Hence, η_τ with $\tau \geq 0$ is a deformation of η by other $(K^{(1)} \rightrightarrows K^{(0)})$-invariant 1-metric on $G^{(1)} \rightrightarrows G^{(0)}$ which we certainly call **Cheeger groupoid deformation** of η.

Example 4.1.9 (2-Equivariant Morse theory on groupoids)**.** As we showed in Section 2.4, this notion of isometric Lie 2-group action can be used to develop a 2-equivariant Morse theory over Lie groupoids in a natural fashion. Actually, this was the initial motivation we had for establishing such a notion.

4.2 Geometric Killing vector fields

Recall that for any Lie group K acting isometrically on a Riemannian manifold $(M, \eta^{(0)})$ it holds that the fundamental vector field of each element in the Lie algebra $\mathfrak{k}$ of K determines a Killing vector field on M. In consequence, we get a Lie algebra homomorphism from $\mathfrak{k}$ to the finite dimensional Lie subalgebra $\mathfrak{o}(M, \eta^{(0)}) \leq \mathfrak{X}(M)$ of Killing vector fields on M, i.e. an infinitesimal action by isometries, see Subsection 1.2.1. The aim of this section is to bring an infinitesimal description of an isometric Lie 2-group action. Our approach will lead us to study an algebra of transversal infinitesimal isometries associated to any Riemannian n-metric on a Lie groupoid. It turns out that these transversal isometries will give rise to a notion of geometric Killing vector field on a quotient Riemannian stack. Some of the references we shall be following throughout are [80, 100] and [5, App. D].

Let us start by describing any Lie 2-group action in terms of a morphism of Lie 2-groups. To do so, we have to explain first what would be our attempt to set the diffeomorphisms group of a Lie groupoid. One of the key ingredients we need is the notion of bisection of a Lie groupoid, consult for instance [108] and [80, s. 1.4].

Definition 4.2.1. A **bisection** of a Lie groupoid $G^{(1)} \rightrightarrows G^{(0)}$ is a smooth map $\sigma : G^{(0)} \to G^{(1)}$ such that $s_G \circ \sigma = \mathrm{id}_{G^{(0)}}$ and the map $\iota_\sigma : G^{(0)} \to G^{(0)}$ defined by $\iota_\sigma(x) := t_G(\sigma(x))$ is a diffeomorphism. The set of all bisections of $G^{(1)} \rightrightarrows G^{(0)}$ will be denoted by $\mathrm{Bis}(G)$.

It is well known that $\mathrm{Bis}(G)$ has the structure of an infinite-dimensional Lie group where the multiplication of two bisections σ and σ' is given by

$$\sigma \bullet \sigma'(x) := \sigma((t_G \circ \sigma')(x)) * \sigma'(x), \quad x \in G^{(0)}.$$

Let us denote by $\mathrm{Aut}(G)$ the group of Lie groupoid automorphisms of $G^{(1)} \rightrightarrows G^{(0)}$. That is, the set of all Lie groupoid isomorphisms from $G^{(1)} \rightrightarrows G^{(0)}$ to itself with the group structure induced by the composition of maps. We may also think of $\mathrm{Aut}(G)$ as an infinite-dimensional Lie group whose structure is induced by that of $\mathrm{Diff}(G^{(0)})$ and $\mathrm{Diff}(G^{(1)})$. On the one hand, given any bisection $\sigma : G^{(0)} \to G^{(1)}$ one has an inner automorphism $I_\sigma : G^{(1)} \to G^{(1)}$ defined by

$$I_\sigma(g) := \sigma(t_G(g)) * g * i_G(\sigma(s_G(g))), \quad g \in G^{(1)}. \tag{4.2}$$

It is simple to check that I_σ covers the diffeomorphism $\iota_\sigma : G^{(0)} \to G^{(0)}$ from Definition 4.2.1 and that we get a well defined Lie group homomorphism $I : \mathrm{Bis}(G) \to \mathrm{Aut}(G)$ by sending $\sigma \mapsto (I_\sigma, \iota_\sigma)$. On the other hand, there is a Lie group homomorphism $\alpha : \mathrm{Aut}(G) \to \mathrm{Aut}(\mathrm{Bis}(G))$ given by $\alpha_{(\Phi,\phi)}(\sigma) := \Phi \circ \sigma \circ \phi^{-1}$ for all $(\Phi, \phi) \in \mathrm{Aut}(G)$ and $\sigma \in \mathrm{Bis}(G)$. More importantly, as proved in [5, App. D], we obtain that:

Lemma 4.2.1. *The quadruple* $(\mathrm{Aut}(G), \mathrm{Bis}(G), I, \alpha)$ *defines a crossed module of Lie groups.*

Such an object shall be called the **crossed module of automorphisms** of $G^{(1)} \rightrightarrows G^{(0)}$. Accordingly:

Definition 4.2.2. The **Lie 2-group of groupoid automorphisms** of $G^{(1)} \rightrightarrows G^{(0)}$ is defined to be the Lie 2-group $\mathrm{Bis}(G) \ltimes \mathrm{Aut}(G) \rightrightarrows \mathrm{Aut}(G)$ associated to the crossed module of Lie groups from Lemma 4.2.1.

One interesting property we can obtain of such a Lie 2-group is provided right below.

Proposition 4.2.1. *The orbit space of the Lie 2-group of groupoid automorphisms equals the set of Lie groupoid automorphisms up to smooth natural equivalences:*

$$\text{Aut}(G)/\text{Bis}(G) = \left\{[\Phi] \mid \Phi \in \text{Aut}(G), \quad \Psi \sim \Phi \Leftrightarrow \exists_\alpha \left(\Psi \overset{\alpha}{\Rightarrow} \Phi\right)\right\}.$$

Proof. Let us first describe the orbit of an element in $\Phi \in \text{Aut}(G)$ covering ϕ. If we pick $\sigma \in \text{Bis}(G)$ and another $\Psi \in \text{Aut}(G)$ covering ψ such that $I_\sigma(\Phi) = \Psi$ then it follows that

$$\Psi(g) = I_\sigma(\Phi(g)) = \sigma(\phi(t_G(g))) * \Phi(g) * i_G(\sigma(\phi(s_G(g)))),$$

for some $g \in G^{(1)}$ so that $\Psi(g) * \sigma(\phi(s_G(g))) = \sigma(\phi(t_G(g))) * \Phi(g)$. Therefore, by setting $\alpha := \sigma \circ \phi$ we get a smooth natural transformation $\Phi \overset{\alpha}{\Rightarrow} \Psi$. Conversely, note that if $\Phi \overset{\alpha}{\Rightarrow} \Psi$ is a smooth natural transformation then for some $g \in G^{(1)}$ it holds $\alpha(t_G(g)) * \Phi(g) = \Psi(g) * \alpha(s_G(g))$, thus obtaining that $t_G(\Phi(g)) = s_G(\alpha(t_G(g)))$ and $s_G(\Psi(g)) = t_G(\alpha(s_G(g)))$. On the one hand, by setting $\sigma := \alpha \circ \phi^{-1}$ it follows that $s_G \circ \sigma = \text{id}_{G^{(0)}}$. On the other hand, observe that

$$\psi(s_G(g)) = t_G(\alpha(s_G(g))) = t_G(\sigma(\phi(s_G)))(g) = \iota_\sigma(\phi(s_G(g))).$$

This identity implies that $\iota_\sigma = \psi \circ \phi^{-1}$ so that ι_σ is a diffeomorphism. □

A couple of examples which reflect the naturality of the previous result are the following.

Example 4.2.1. If $K \rightrightarrows *$ is a Lie group then $\text{Bis}(K) \simeq K$. We may think of $\text{Bis}(K)$ as the subset in $\text{Aut}(K)$ determined by conjugations with respect to the elements in K. This implies that the crossed module of automorphisms of K is $(\text{Aut}(K), K, j, c)$ where j is the inclusion and c is the identity representation. Thus, the orbit space $\text{Aut}(K)/K$ corresponds to the set of automorphisms of K up to conjugations. That is, the outer automorphisms of K.

Example 4.2.2. Let $\pi : M \to N$ be a surjective submersion and let $M \times_N M \rightrightarrows M$ denote its corresponding submersion groupoid. A straightforward computation shows that $\text{Aut}(M \times_N M)$ is in one-to-one correspondence with $\text{Aut}(\pi)$ which stands for the set of pairs $(\widetilde{f}, f) \in \text{Diff}(M) \times \text{Diff}(N)$ commuting with π and that $\text{Bis}(M \times_N M)$ corresponds to $\text{Gau}(\pi)$ that is the set of pairs $(\widetilde{f}, \text{id}) \in \text{Aut}(\pi)$. In this case the crossed module of automorphisms of $M \times_N M$ is $(\text{Aut}(\pi), \text{Gau}(\pi), j, c)$ where j is the inclusion and c is the representation by conjugations. Therefore, the orbit space

$$\text{Aut}(M \times_N M)/\text{Bis}(M \times_N M) \simeq \text{Aut}(\pi)/\text{Gau}(\pi) \simeq \text{Diff}(N)_M.$$

Here $\text{Diff}(N)_M$ stands for the set of diffeomorphisms $\phi : N \to N$ such that the pullback fibration $\phi^* M \to N$ is isomorphic to the fibration $\pi : M \to N$.

Let us now consider a Lie 2-group $K^{(1)} \rightrightarrows K^{(0)}$ acting on $G^{(1)} \rightrightarrows G^{(0)}$ from the left. In order to describe this Lie 2-group action infinitesimally we need to reinterpret it in terms of a morphism of Lie 2-groups, or equivalently, a morphism of crossed modules of Lie groups. With this idea in mind we show the following results.

Lemma 4.2.2. *The normal subgroup $H = \ker(s_K)$ acts on $G^{(1)} \rightrightarrows G^{(0)}$ by bisections and $K^{(0)}$ acts by Lie groupoid automorphisms. Moreover, the right multiplication map defined on s_G-fibers for each arrow is H-equivariant.*

Proof. Pick any $h \in H$ and define the map $\sigma_h : G^{(0)} \to G^{(1)}$ as $\sigma_h(x) := h1_x$. It is clear that $s_G(\sigma_h(x)) = x$ and $t_G(\sigma_h(x)) = t_K(h)x = \theta^0_{t_K(h)}(x)$ so that σ_h is a well defined bisection. Let us now take $k_0 \in K^{(0)}$ and define the map $\Sigma_{k_0} : G^{(1)} \to G^{(1)}$ as $\Sigma_{k_0}(g) := 1_{k_0}g$. Equation (1.13) implies that

$$1_{k_0}(g * g') = (1_{k_0} * 1_{k_0})(g * g') = (1_{k_0}g) * (1_{k_0}g'),$$

for all $k_0 \in K^{(0)}$ and $(g, g') \in G^{(2)}$, thus obtaining that $\Sigma_{k_0} : G^{(1)} \to G^{(1)}$ is a Lie groupoid morphism, covering $\theta^0_{k_0} : G^{(0)} \to G^{(0)}$, which clearly satisfies $(\Sigma_{k_0})^{-1} = \Sigma_{k_0^{-1}}$. Note that $s_G(hg) = s_G(g)$ for all $g \in G^{(1)}$ and $h \in H$ so that the left action of H on $G^{(1)}$ preserves the s_G-fibers. Therefore, for each $y \xleftarrow{g} x$ the right action $R_g : s_G^{-1}(y) \to s_G^{-1}(x)$ satisfies that

$$R_g(hg') = (hg') * (1_{e_0}g) = (h * 1_{e_0})(g * g') = hR_g(g').$$

In consequence, R_g is H-equivariant as claimed. □

It is clear that the same result can be obtained if we consider right 2-actions instead of left ones. Let (K, H, ρ, α) denote the crossed module of Lie groups associated to $K^{(1)} \rightrightarrows K^{(0)}$. Then:

Lemma 4.2.3. *There is a natural morphism of crossed modules of Lie groups* $(\sigma, \Sigma) : (K, H, \rho, \alpha) \to (\mathrm{Aut}(G), \mathrm{Bis}(G), I, \alpha)$ *where* σ_h *and* Σ_{k_0} *are defined as in Lemma 4.2.2.*

Proof. Let us check that $\Sigma \circ \rho = I \circ \sigma$ and $\sigma_{\alpha_{k_0}(h)} = \alpha_{(\Sigma_{k_0}, \theta^0_{k_0})}(\sigma_h)$ for all $k_0 \in K$ and $h \in H$. Firstly, for $g \in G^{(1)}$ and $h \in H$ we obtain

$$\begin{aligned} I_{\sigma_h}(g) &= \sigma_h(t_G(g)) * g * i_G(\sigma_h(s_G(g))) = ((h1_{t_G(g)}) * g) * i_G(h1_{s_G(g)}) \\ &= ((h * e_1)(1_{t_G(g)} * g)) * i_G(h1_{s_G(g)}) = hg * i_K(h)1_{s_G(g)} = 1_{\rho(h)}g = \Sigma_{\rho(h)}(g), \end{aligned}$$

since $\rho = t_K|_H$. Secondly, for $x \in G^{(0)}$, $k_0 \in K$ and $h \in H$ we get

$$\alpha_{(\Sigma_{k_0}, \theta^0_{k_0})}(\sigma_h)(x) = \Sigma_{k_0}(h1_{(k_0^{-1}x)}) = 1_{k_0}h1_{k_0^{-1}}1_x = \alpha_{k_0}(h)1_x = \sigma_{\alpha_{k_0}(h)}(x).$$

□

Recall the set of all diffeomorphisms $\mathrm{Diff}(M)$ of a smooth manifold M has the structure of Lie group whose Lie algebra can be identified with the Lie algebra of vector fields $\mathfrak{X}(M)$ on M. We already exhibited a natural structure of Lie 2-group over the set of all Lie groupoid automorphisms $\mathrm{Aut}(G)$ of a Lie groupoid $G^{(1)} \rightrightarrows G^{(0)}$. On the one hand, by results due to Mackenzie–Xu in [81], we get that the Lie algebra of $\mathrm{Aut}(G)$ can be identified with the Lie algebra of multiplicative vector fields $\mathfrak{X}_m(G)$ on $G^{(1)}$. On the other hand, the Lie algebra of the Lie group of bisections $\mathrm{Bis}(G)$ of $G^{(1)}$ can be identified with the Lie algebra $\Gamma(A_G)$ underlying the Lie algebroid $A_G \to G^{(0)}$ of $G^{(1)} \rightrightarrows G^{(0)}$, see [108]. Therefore, we conclude that the Lie 2-algebra of the Lie 2-group of groupoid automorphisms of $G^{(1)} \rightrightarrows G^{(0)}$ can be identified with the Lie 2-algebra of multiplicative vector fields on $G^{(1)} \rightrightarrows G^{(0)}$ described in Example 1.3.30, compare [100]. Recall that the associated crossed module of Lie algebras of $\mathfrak{X}_m(G)$ is $(\mathfrak{X}_m(G), \Gamma(A_G), \delta, D)$, where $\delta(\alpha) = (\alpha^r - \alpha^l, \rho(\alpha))$ and $D_{(\xi,v)}\alpha = [\xi, \alpha^r]|_{X_0}$ for all $\alpha \in \Gamma(A_G)$ and $(\xi, v) \in \mathfrak{X}_m(G)$.

We are now in conditions to define an **infinitesimal Lie 2-algebra action** associated to any right Lie 2-group action. From now on the symbol $\tilde{\xi}$ stands for the fundamental vector field associated to an element ξ in a Lie algebra with respect to a given Lie group action.

Theorem 4.2.1. *Let $K^{(1)} \rightrightarrows K^{(0)}$ be a Lie 2-group acting on a Lie groupoid $G^{(1)} \rightrightarrows G^{(0)}$ by the right. Let $(\mathfrak{k}, \mathfrak{h}, \partial, \mathcal{L})$ denote the crossed module of Lie algebras associated to the Lie 2-algebra of $K^{(1)} \rightrightarrows K^{(0)}$. Then, there is a canonical homomorphism of Lie 2-algebras $j = (j_{-1}, j_0) : (\mathfrak{k}, \mathfrak{h}, \partial, \mathcal{L}) \to (\mathfrak{X}_m(G), \Gamma(A_G), \delta, D)$ defined by $j_{-1}(\xi) = \tilde{\xi}\big|_{G^{(0)}}$ and $j_0(\zeta) = (\tilde{1_\zeta}, \tilde{\zeta})$ for all $\xi \in \mathfrak{h}$ and $\zeta \in \mathfrak{k}$.*

Proof. Let us first verify that j is a well defined morphism. To do so we need to check that for any $\xi \in \mathfrak{h}$ the fundamental vector field $\tilde{\xi}$ belongs to the set of right invariant vector fields tangent to the s_G-fibers $\mathfrak{X}^s_{inv}(G)$ and that for any $\zeta \in \mathfrak{k}$ it holds that $(\tilde{1_\zeta}, \tilde{\zeta})$ belongs to $\mathfrak{X}_m(G)$. The latter assertion is clear since if $\exp \tau\zeta \in K^{(0)}$ then Lemma 4.2.2 implies that the pair of flows $(\varphi_\tau^{\tilde{1_\zeta}}, \varphi_\tau^{\tilde{\zeta}})$ determines a Lie groupoid morphism. Now, if $\xi \in \mathfrak{h}$ then again from Lemma 4.2.2 it follows that the flow of its fundamental vector field lies inside the s-fibers since $s_G(\varphi_\tau^{\tilde{\xi}}(g)) = s_G(g\exp(\tau\xi)) = s_G(g)$ with $\exp(\tau\xi) \in H = \ker(s_K)$. That is, $\tilde{\xi}$ is tangent to the s_G-fibers. Furthermore, for $y \xleftarrow{g} x$ and $g' \in s_G^{-1}(y)$ one has that

$$R_g(\varphi_\tau^{\tilde{\xi}}(g')) = (g'\exp(\tau\xi)) * g = (g' * g)(\exp\tau\xi * 1_e) = \varphi_\tau^{\tilde{\xi}}(g' * g) = \varphi_\tau^{\tilde{\xi}}(R_g(g')),$$

thus obtaining that $d(R_g)_{g'}(\tilde{\xi}_{g'}) = \tilde{\xi}_{g' * g}$ so that $\tilde{\xi} \in \mathfrak{X}^s_{inv}(G)$ and $\tilde{\xi}\big|_{G^{(0)}} \in \Gamma(A_G)$.

We have to check now that for $\xi \in \mathfrak{h}$ and $\zeta \in \mathfrak{k}$ it satisfies that $\delta(j_{-1}(\xi)) = j_0(\partial\xi)$ and $j_{-1}(\mathcal{L}_\zeta\xi) = D_{j_0(\zeta)}(j_{-1}(\xi))$. On the one hand, by using the flow of the vector field $\delta(j_0(\xi)) = \tilde{\xi}\big|^r_{G^{(0)}} - \tilde{\xi}\big|^l_{G^{(0)}}$ and Equation (1.13) we get

$$\begin{aligned} \varphi_\tau^{\delta(j_{-1}(\xi))}(g) &= \varphi_\tau^{\tilde{\xi}}(1_{t_G(g)}) * g * i_G(\varphi_\tau^{\tilde{\xi}}(1_{s_G(g)})) = 1_{t_G(g)}\exp(\tau\xi) * g * i_G(1_{s_G(g)}\exp(\tau\xi)) \\ &= (1_{t_G(g)} * g)\exp(\tau\xi) * 1_{s_G(g)} i_K(\exp(\tau\xi)) = g(\exp(\tau\xi) * i_K(\exp(\tau\xi))) \\ &= g(1_{t_K(\exp(\tau\xi))}) = g1_{\exp(\tau\partial(\xi))} = g\exp(\tau 1_{\partial\xi}) = \varphi_\tau^{j_0(\partial\xi)}(g). \end{aligned}$$

Hence, $\delta(j_{-1}(\xi)) = j_0(\partial\xi)$. On the other hand, observe that

$$D_{j_0(\zeta)}(j_{-1}(\xi)) = \left[\tilde{1_\zeta}, \tilde{\xi}\right]\Big|_{G^{(0)}} = [\widetilde{1_\zeta, \xi}]\Big|_{G^{(0)}} = (\widetilde{\mathcal{L}_\zeta(\xi)})\Big|_{G^{(0)}} = j_{-1}(\mathcal{L}_\zeta\xi).$$

This finishes the proof. □

As we mentioned before, the previous result gives rise to an infinitesimal description of a Lie 2-group action. It turns out that such a description can be adapted to the Riemannian case in a natural fashion. Let us suppose that $G^{(1)} \rightrightarrows G^{(0)}$ can be equipped with a Riemannian 2-metric η. Consider the groups

$$\mathrm{Bis}_\eta(G) = \{\sigma \in \mathrm{Bis}(G) \mid \iota_\sigma^*\eta = \eta\} \quad \text{and} \quad \mathrm{Iso}(G, \eta) = \{(\Phi, \phi) \in \mathrm{Aut}(G) \mid \phi^*\eta = \eta\}.$$

Firstly, note that for defining these sets we are only using the induced 0-metric on $G^{(0)}$. This fact is necessary to establish the key results below. Secondly, $\mathrm{Bis}_\eta(G)$ and $\mathrm{Iso}(G, \eta)$ have natural Lie group structures induced from those of $\mathrm{Bis}(G)$ and $\mathrm{Aut}(G)$, respectively.

Proposition 4.2.2. *The quadruple $(\mathrm{Iso}(G, \eta), \mathrm{Bis}_\eta(G), I, \alpha)$ determines a sub-crossed module structure of $(\mathrm{Aut}(G), \mathrm{Bis}(G), I, \alpha)$.*

Proof. It is simple to see from the very definition that $I(\mathrm{Bis}_\eta(G)) \subseteq \mathrm{Iso}(G, \eta)$. As $I_{\alpha_\Phi(\sigma)} = \Phi I_\sigma \Phi^{-1}$ for all $\sigma \in \mathrm{Bis}_\eta(G)$ and $\Phi \in \mathrm{Iso}(G, \eta)$ then when restricting to unities we have that $\iota_{\alpha_\Phi(\sigma)} = \phi \circ \iota_\sigma \circ \phi^{-1}$, thus obtaining an isometry. □

This motivates the following definition.

Definition 4.2.3. The Lie 2-group associated to the crossed module $(\mathrm{Iso}(G,\eta), \mathrm{Bis}_\eta(G), I, \alpha)$ will be called **Lie 2-group of strong isometries** of $(G^{(1)} \rightrightarrows G^{(0)}, \eta)$.

Let us illustrate such a notion with an example.

Example 4.2.3. Let (M,η) be an n-dimensional Riemannian manifold and let $\pi : O(M) \to M$ denote the corresponding $O(n,\mathbb{R})$-principal bundle of orthonormal frames. After fixing an Ad-invariant inner product $\langle\cdot,\cdot\rangle$ on the Lie algebra $\mathfrak{o}(n,\mathbb{R})$ of $O(n,\mathbb{R})$ and taking the connection 1-form $\omega \in \Omega^1(O(M),\mathfrak{o}(n,\mathbb{R}))$ associated to the Levi–Civita connection on (M,η) we can define a Riemannian metric on $O(M)$ as

$$\tilde{\eta}(X,Y) = \eta(d\pi(X), d\pi(Y)) + \langle\omega(X),\omega(Y)\rangle.$$

We already commented in Example 1.2.1 that $\pi : (O(M),\tilde{\eta}) \to (M,\eta)$ becomes a Riemannian submersion. Thus, its corresponding submersion groupoid $O(M) \times_M O(M) \rightrightarrows O(M)$ inherits an induced 0-metric which we also denote by η [40, 98]. Therefore,

$$\mathrm{Bis}_\eta(O(M) \times_M O(M)) \simeq \mathrm{Gau}(O(M),\theta) \quad \text{and} \quad \mathrm{Iso}(O(M) \times_M O(M),\eta) \simeq \mathrm{Aut}(O(M),\theta),$$

where $\theta \in \Omega^1(O(M),\mathbb{R}^n)$ is the canonical 1-form, $\mathrm{Aut}(O(M),\theta)$ is the group of bundle isomorphisms preserving θ and $\mathrm{Gau}(O(M),\theta)$ is its normal subgroup of bundle isomorphisms covering the identity, compare Example 4.2.2. Hence, from [67, p. 236] we get that the orbit space

$$\mathrm{Iso}(O(M) \times_M O(M),\eta)/\mathrm{Bis}_\eta(O(M) \times_M O(M)) \simeq \mathrm{Iso}(M,\eta)_{O(M)}.$$

There is a natural way to associate to the Lie 2-group of strong isometries of $(G^{(1)} \rightrightarrows G^{(0)},\eta)$ an infinitesimal object in terms of Killing vector fields, i.e. infinitesimal isometries. Indeed, let us now consider the sets

$$\Gamma_\eta(A_G) = \left\{\alpha \in \Gamma(A_G) \,|\, \rho(\alpha) \in \mathfrak{o}(G^{(0)},\eta)\right\} \quad \text{and} \quad \mathfrak{o}_m(G) = \left\{(\xi,v) \in \mathfrak{X}_m(G) \,|\, v \in \mathfrak{o}(G^{(0)},\eta)\right\},$$

where $\mathfrak{o}(G^{(0)},\eta)$ denotes the Lie algebra of Killing vector fields of $(G^{(0)},\eta)$. In these terms we obtain that:

Proposition 4.2.3. *The quadruple $(\mathfrak{o}_m(G),\Gamma_\eta(A_G),\delta,D)$ defines a sub-crossed module structure of $(\mathfrak{X}_m(G),\Gamma(A_G),\delta,D)$.*

Proof. Observe that $\mathfrak{o}_m(G)$ is a Lie subalgebra of $\mathfrak{X}_m(G)$ since $\mathfrak{o}(G^{(0)},\eta)$ is a Lie algebra. Additionally, if $\alpha,\beta \in \Gamma_\eta(A_G)$ then $\rho([\alpha,\beta]) = [\rho(\alpha),\rho(\beta)] \in \mathfrak{o}(X_0,\eta)$ so that $[\alpha,\beta] \in \Gamma_\eta(A_G)$. If $(\xi,v) \in \mathfrak{o}_m(G)$ and $\alpha \in \Gamma_\eta(A_G)$ then we have by definition that $D_\xi(\alpha) = [\xi,\alpha^r]|_{G^{(0)}} \in \Gamma(A_G)$. However, the equivariance identity implies that $\delta(D_\xi(\alpha)) = [\xi,\delta(\alpha)]$. Therefore, it holds that $\delta(D_\xi\alpha)|_{G^{(0)}} = [\xi,\delta(\alpha)]|_{G^{(0)}}$ which is the same thing that saying $\rho(D_\xi\alpha) = [v,\rho(\alpha)] \in \mathfrak{o}(G^{(0)},\eta)$ since v is also a Killing vector field. □

In light of the previous result we define:

Definition 4.2.4. The Lie 2-algebra associated to the crossed module $(\mathfrak{o}_m(G),\Gamma_\eta(A_G),\delta,D)$ will be called **Lie 2-algebra of strong multiplicative Killing vector fields** of $(G^{(1)} \rightrightarrows G^{(0)},\eta)$.

Summing up, as consequence of Lemma 2.4.1, we can provide an infinitesimal description of an isometric Lie 2-group action as follows.

Corollary 4.2.1. *Let θ be an isometric right 2-action of a Lie 2-group $K^{(1)} \rightrightarrows K^{(0)}$ on a Riemannian groupoid $(G^{(1)} \rightrightarrows G^{(0)}, \eta)$. Denote by (K, H, ρ, α) the crossed module of Lie groups associated to $K^{(1)} \rightrightarrows K^{(0)}$ and by $(\mathfrak{k}, \mathfrak{h}, \partial, \mathcal{L})$ its corresponding crossed module of Lie algebras. Then:*

- *there is a morphism of crossed modules of Lie groups*

  $$(\sigma, \Sigma) : (K, H, \rho, \alpha) \to (\mathrm{Iso}(G, \eta), \mathrm{Bis}_\eta(G), I, \alpha),$$

  that is defined as in Lemma 4.2.3, and

- *there is a morphism of crossed modules of Lie algebras*

  $$(j_{-1}, j_0) : (\mathfrak{k}, \mathfrak{h}, \partial, \mathcal{L}) \to (\mathfrak{o}_m(G), \Gamma_\eta(A_G), \delta, D),$$

  which is defined in Theorem 4.2.1.

Note that to establish this result we only used the Riemannian metric on objects induced by η. This is because a condition asking for global isometries of a 2-metric on $G^{(2)}$, or even a 1-metric on $G^{(1)}$, seems to be too restrictive in our context. For instance, the sub-crossed module structures we defined above do not seem to have natural analogous for 2-metrics or 1-metric, unless we weaken the global condition of a diffeomorphism being a Riemannian isometry. In the search to solve this issue we came across a weak notion of "groupoid isometry" which allows us to say much more, namely, it allows us to speak about the infinitesimal isometries of a Riemannian stack.

4.2.1 Weak multiplicative Killing vector fields

Recall that we can think of a 0-metric on a Lie groupoid $G^{(1)} \rightrightarrows G^{(0)}$ as a Riemannian metric η on $G^{(0)}$ which is transversely invariant by the canonical left action of $G^{(1)} \rightrightarrows G^{(0)}$ on $G^{(0)}$, compare [40, 103]. Note that this is the same that requiring that η is transversely invariant by the action of the group of bisections $\mathrm{Bis}(G) \times G^{(0)} \to G^{(0)}$ which is defined by $\sigma \cdot x := \iota_\sigma(x)$. It is clear that this action preserves the orbits so that it induces a well defined action on the normal space of an orbit. In consequence, η is a 0-metric if and only if for all $\sigma \in \mathrm{Bis}(G)$ the map $\overline{d\iota_\sigma} : (\nu_x(\mathcal{O}), \overline{\eta}) \to (\nu_{\iota_\sigma(x)}(\mathcal{O}), \overline{\eta})$ is a linear isometry since we may identify $\nu(\mathcal{O}) \cong T\mathcal{O}^\perp$. Therefore, motivated by this fact and what we did in the previous section we now plan to weaken the condition for a diffeomorphism to be an isometry by imposing instead a transversal isometric condition along groupoid orbits. This will lead us to define a Lie 2-algebra of transverse infinitesimal isometries with respect to any Riemannian groupoid n-metric, which at the end turns out to be Morita invariant.

Let $(G^{(1)} \rightrightarrows G^{(0)}, \eta)$ be a Riemannian groupoid with $\eta = \eta^{(2)}$ a 2-metric on $G^{(0)}$. From now on we assume the identification $\nu(\mathcal{O}) \cong T\mathcal{O}^\perp$ for each groupoid orbit $\mathcal{O}$ in $(G^{(0)}, \eta^{(0)})$ without stating it explicitly unless it is necessary.

Definition 4.2.5. Let $\Phi : G^{(1)} \to G^{(1)}$ be a Lie groupoid automorphism covering $\phi : G^{(0)} \to G^{(0)}$. The diffeomorphism $\phi : G^{(0)} \to G^{(0)}$ is said to be a **transversal isometry** of $(G^{(0)}, \eta^{(0)})$ if $\overline{d\phi} : \nu(\mathcal{O}_x) \to \nu(\mathcal{O}_{\phi(x)})$ is a fiberwise isometry for every groupoid orbit $\mathcal{O}_x$ in $G^{(0)}$ with respect to $\eta^{(0)}$ restricted to the normal directions.

Consider the subgroup of $\mathrm{Aut}(G)$:

$$\mathrm{Iso}_{\mathrm{w}}(G,\eta) = \left\{(\Phi,\phi)\in \mathrm{Aut}(G)\,|\,\phi \text{ transversal isometry of } (G^{(0)},\eta^{(0)})\right\}.$$

Note that for every $\sigma\in \mathrm{Bis}(G)$ it follows that ι_σ is a transversal isometry of $(G^{(0)},\eta^{(0)})$, so that (I_σ,ι_σ) always belongs to $\mathrm{Iso}_{\mathrm{w}}(G,\eta)$. Thus, by arguing as in Proposition 4.2.2 we easily get that:

Lemma 4.2.4. *The quadruple* $(\mathrm{Iso}_{\mathrm{w}}(G,\eta),\mathrm{Bis}(G),I,\alpha)$ *determines a sub-crossed module structure of* $(\mathrm{Aut}(G),\mathrm{Bis}(G),I,\alpha)$.

This motivates the following definition.

Definition 4.2.6. The Lie 2-group determined by the crossed module $(\mathrm{Iso}_{\mathrm{w}}(G,\eta),\mathrm{Bis}(G),I,\alpha)$ is called **Lie 2-group of weak isometries** of $(G^{(1)}\rightrightarrows G^{(0)},\eta)$.

Observe that we are abusing on the convention by assuming that $\mathrm{Iso}_{\mathrm{w}}(G,\eta)$ has the structure of a Lie group. This fact is not obvious at first glance for which we skip it since our interest lies for the moment in defining the infinitesimal isometries of a Riemannian stack. The most expected infinitesimal object associated to the Lie 2-group of weak isometries of $(G^{(1)}\rightrightarrows G^{(0)},\eta)$ may be defined as follows. Let us denote by $\mathfrak{o}^{\mathrm{w}}(G^{(0)},\eta^{(0)})$ the set of vector fields on $G^{(0)}$ which are covered by multiplicative vector fields on $G^{(1)}$ whose flow determines a (local) transversal isometry of $(G^{(0)},\eta^{(0)})$. Define the set

$$\mathfrak{o}^{\mathrm{w}}_m(G) = \left\{(\xi,v)\in \mathfrak{X}_m(G)\,|\,v\in \mathfrak{o}^{\mathrm{w}}(G^{(0)},\eta^{(0)})\right\}.$$

On the one hand, if $v_1,v_2\in \mathfrak{o}^{\mathrm{w}}(G^{(0)},\eta^{(0)})$ then for τ small enough we have that the commutator flow $\varphi^{v_2}_{-\sqrt{\tau}}\varphi^{v_1}_{-\sqrt{\tau}}\varphi^{v_2}_{\sqrt{\tau}}\varphi^{v_1}_{\sqrt{\tau}}$ is a transversal isometry so that $\mathfrak{o}^{\mathrm{w}}(G^{(0)},\eta^{(0)})$ is a Lie subalgebra of $\mathfrak{X}(G^{(0)})$. On the other hand, from [108, Thm. D] we know that the Lie algebra of the Lie group of bisections $\mathrm{Bis}(G)$ is identified with $\Gamma(A_G)$. Therefore, by using similar arguments as those in Proposition 4.2.3 we obtain an infinitesimal description of the Lie 2-group of weak isometries of $(G^{(1)}\rightrightarrows G^{(0)},\eta)$. Namely:

Proposition 4.2.4. *The quadruple* $(\mathfrak{o}^{\mathrm{w}}_m(G),\Gamma(A_G),\delta,D)$ *defines a sub-crossed module structure of* $(\mathfrak{X}_m(G),\Gamma(A_G),\delta,D)$.

Thus, we define:

Definition 4.2.7. The Lie 2-algebra associated to the crossed module $(\mathfrak{o}^{\mathrm{w}}_m(G),\Gamma(A_G),\delta,D)$ will be called **Lie 2-algebra of weak multiplicative Killing vector fields** of $(G^{(1)}\rightrightarrows G^{(0)},\eta)$.

It is worth mentioning that the previous definition makes sense for any Riemannian n-metric on $G^{(n)}$. Indeed:

Remark 4.2.1. Let us suppose that we are equipped with an n-metric $\eta^{(n)}$ on $G^{(n)}$ and that (ξ,v) is a multiplicative vector field on $G^{(1)}\rightrightarrows G^{(0)}$ with $v\in \mathfrak{o}^{\mathrm{w}}(G^{(0)},\eta^{(0)})$. It is simple to see that the fact that $s_G:G^{(1)}\to G^{(0)}$ (or $t_G:G^{(1)}\to G^{(0)}$) is a Riemannian submersion clearly implies that $\xi\in \mathfrak{o}^{\mathrm{w}}(G^{(1)},\eta^{(1)})$. More importantly, if ξ_n denotes the vector field on $G^{(n)}$ induced by (ξ,v) for all $n\geq 2$ then if follows that $\xi_n\in \mathfrak{o}^{\mathrm{w}}(G^{(n)},\eta^{(n)})$ since the face maps $G^{(n)}\to G^{n-1}$ are Riemannian submersions. As a consequence of this we have that the notion of weak multiplicative Killing vector field can be extended to a notion associated to any Riemannian n-metric. This immediately implies that the Lie 2-algebra of weak multiplicative Killing vector fields $(\mathfrak{o}^{\mathrm{w}}_m(G),\Gamma(A_G),\delta,D)$ can be thought of as an algebraic/-geometric object associated to any Riemannian n-metric on $G^{(n)}$, as claimed.

Morita invariance

We want to apply now some of the results from [100] to our context in order to define a notion of geometric Killing vector field on a quotient Riemannian stack. Then, we also use some of the results from [35] to describe several interesting features that these kinds of vector fields have. In order to refresh the notions that we need to use below we recommend the reader to consult Section 1.3, focusing her/his attention on the notion of Riemannian stack.

Let $\phi : (Z^{(1)} \rightrightarrows Z^{(0)}) \to (G^{(1)} \rightrightarrows G^{(0)})$ be a Morita fibration. Following [100, s. 6], we denote the set of projectable sections by

$$\Gamma(A_Z)^{\phi} = \{\alpha \in \Gamma(A_Z) : \text{there exists } \alpha' \in \Gamma(A_G) \text{ such that } \phi_*\alpha = \alpha'\phi\}.$$

If $\alpha \in \Gamma(A_Z)$ then the surjectivity of ϕ at the level of objects implies that there exists at most one section $\alpha' \in \Gamma(A_G)$ such that $\phi_*\alpha = \alpha'\phi$, so that it follows that there is a natural linear map $\phi_* : \Gamma(A_Z)^{\phi} \to \Gamma(A_G)$. We denote by $\Gamma(A_Z)^{\phi} \hookrightarrow \Gamma(A_Z)$ the inclusion map. It is clear that we can similarly define the set of projectable multiplicative vector fields $\mathfrak{X}_m(Z)^{\phi}$, a natural map $\phi_* : \mathfrak{X}_m(Z)^{\phi} \to \mathfrak{X}_m(G)$ and an inclusion $\mathfrak{X}_m(Z)^{\phi} \hookrightarrow \mathfrak{X}_m(Z)$. Besides, as shown in Proposition 7.4 from [100] it follows that $(\mathfrak{X}_m(Z)^{\phi}, \Gamma(A_Z)^{\phi}, \delta, D)$ is a sub-crossed module of $(\mathfrak{X}_m(Z), \Gamma(A_Z), \delta, D)$ and both maps $\phi_* : (\mathfrak{X}_m(Z)^{\phi}, \Gamma(A_Z)^{\phi}, \delta, D) \to (\mathfrak{X}_m(G), \Gamma(A_G), \delta, D)$ and $(\mathfrak{X}_m(Z)^{\phi}, \Gamma(A_Z)^{\phi}, \delta, D) \hookrightarrow (\mathfrak{X}_m(Z), \Gamma(A_Z), \delta, D)$ are morphisms of crossed-modules. More importantly,

$$(\mathfrak{X}_m(Z), \Gamma(A_Z), \delta, D) \hookleftarrow (\mathfrak{X}_m(Z)^{\phi}, \Gamma(A_Z)^{\phi}, \delta, D) \xrightarrow{\phi_*} (\mathfrak{X}_m(G), \Gamma(A_G), \delta, D),$$

are quasi-isomorphisms of crossed modules. For specific details the reader is recommended to visit [100].

In these terms, we can state the following key result which is just the Riemannian analogous of the previous constructions.

Lemma 4.2.5. *Let $\phi : (Z^{(1)} \rightrightarrows Z^{(0)}, \eta^Z) \to (G^{(1)} \rightrightarrows G^{(0)}, \eta^G)$ be a Morita Riemannian fibration. Then:*

- $(\mathfrak{o}_m(Z)^{\phi}, \Gamma_\eta(A_Z)^{\phi}, \delta, D)$ *is a sub-crossed module of* $(\mathfrak{o}_m(Z), \Gamma_\eta(A_Z), \delta, D)$,
- *the inclusion* $(\mathfrak{o}_m(Z)^{\phi}, \Gamma_\eta(A_Z)^{\phi}, \delta, D) \hookrightarrow (\mathfrak{o}_m(Z), \Gamma_\eta(A_Z), \delta, D)$ *is a morphism of crossed modules, and*
- *the projection* $\phi_* : (\mathfrak{o}_m(Z)^{\phi}, \Gamma_\eta(A_Z)^{\phi}, \delta, D) \to (\mathfrak{o}_m(G), \Gamma_\eta(A_G), \delta, D)$ *is a morphism of crossed modules.*

Moreover,

$$(\mathfrak{o}_m(Z), \Gamma_\eta(A_Z), \delta, D) \hookleftarrow (\mathfrak{o}_m(Z)^{\phi}, \Gamma_\eta(A_Z)^{\phi}, \delta, D) \xrightarrow{\phi_*} (\mathfrak{o}_m(G), \Gamma_\eta(A_G), \delta, D),$$

are quasi-isomorphisms of crossed modules. Same conclusion holds true for the weak counterpart.

Proof. First of all, the spaces $\mathfrak{o}_m(Z)^{\phi}$ and $\Gamma_\eta(A_Z)^{\phi}$ can be defined in an obvious way since ϕ is a Riemannian submersion at the levels of objects. Second, by using similar arguments as those in Lemma 2.4.1 if follows that if v is a (weak) Killing vector field on $(Z^{(0)}, \eta^{Z,0})$ then $\phi_*(v)$ is also a (weak) Killing vector field on $(G^{(0)}, \eta^{G,0})$. Therefore, the result follows by applying Proposition 7.4 and Theorem 7.3 from [100] after restricting the structure. □

Recall that two possibly different Riemannian 2-metrics on a Lie groupoid are said to be equivalent if they induce the same inner products on the normal vector spaces over groupoid orbits. Thus, the following result is clear.

Lemma 4.2.6. *If η_1 and η_2 are equivalent Riemannian metrics on $G^{(1)} \rightrightarrows G^{(0)}$ then the crossed modules $(\mathfrak{o}_m^{\mathrm{w}}(G,\eta_1),\Gamma(A_G),\delta,D)$ and $(\mathfrak{o}_m^{\mathrm{w}}(G,\eta_2),\Gamma(A_G),\delta,D)$ agree.*

In order to make the statement of the following result clearer we think it is necessary to recall some terminology introduced in [41]. Suppose that G and G' are Morita equivalent Lie groupoids so that there is a third Lie groupoid Z with Morita fibrations $Z \to G$ and $Z \to G'$. We know that if η^G is a Riemannian metric on G then there exists a Riemannian metric η^Z on Z that makes the fibration $Z \to G$ Riemannian. We can slightly modify η^Z by a cotangent averaging procedure so that we get another Riemannian metric $\tilde{\eta}^Z$ on Z which descends to G' defining a Riemannian metric $\eta^{G'}$ making of the fibration $Z \to G'$ Riemannian. It turns out that these pullback and pushforward constructions are well-defined and mutually inverse modulo equivalence of metrics. This is because η^Z and $\tilde{\eta}^Z$ turn out to be equivalent. In this case we refer to (G,η^G) and $(G',\eta^{G'})$ as being Morita equivalent Riemannian groupoids. Note that it suggests a definition for Riemannian metrics over differentiable stacks. Namely, a stacky metric on the orbit stack $[G^{(0)}/G^{(1)}]$ presented by a Lie groupoid $G^{(1)} \rightrightarrows G^{(0)}$ is defined to be an equivalence class $[\eta]$ of a Riemannian metric η on G.

Summing up, we get that:

Theorem 4.2.2. *If $(G^{(1)} \rightrightarrows G^{(0)},\eta^G)$ and $(G'^{(1)} \rightrightarrows G'^{(0)},\eta^Y)$ are Morita equivalent Riemannian groupoids then the crossed modules $(\mathfrak{o}_m^{\mathrm{w}}(G),\Gamma(A_G),\delta,D)$ and $(\mathfrak{o}_m^{\mathrm{w}}(G'),\Gamma(A_{G'}),\delta,D)$ are isomorphic in the derived category of crossed modules. In consequence, the following quotient spaces are isomorphic as Lie algebras:*

$$\mathfrak{o}_m^{\mathrm{w}}(G)/\mathrm{im}(\delta) \cong \mathfrak{o}_m^{\mathrm{w}}(G')/\mathrm{im}(\delta).$$

Proof. This result is consequence of Lemmas 4.2.5 and 4.2.6 together with Theorem 7.4 and Corollaries 7.1 and 7.2 from [100]. □

Hence, motivated by the previous result and Definition 8.1 in [100] we set up the following interesting notion.

Definition 4.2.8. Let $(G^{(1)} \rightrightarrows G^{(0)},\eta)$ be a Riemannian groupoid. A **geometric Killing vector field** on the quotient Riemannian stack $([G^{(0)}/G^{(1)}],[\eta])$ is defined to be an element of the quotient

$$\mathfrak{o}([G^{(0)}/G^{(1)}],[\eta]) := \mathfrak{o}_m^{\mathrm{w}}(G)/\mathrm{im}(\delta).$$

Let us illustrate this notion with some examples.

Example 4.2.4. The Riemannian stack $([M],[\eta])$ associated to a Riemannian manifold (M,η) is presented by the unit Riemannian groupoid $(M,\eta) \rightrightarrows (M,\eta)$. In this case, the tangent groupoid is also a unit groupoid $TM \rightrightarrows TM$ and the Lie algebroid is $A = 0$. A straightforward computation shows that a weak multiplicative Killing vector field is just a Killing vector field on M, so that the complex of weak multiplicative Killing vector fields is $0 \to \mathfrak{o}(M,\eta)$ and hence $\mathfrak{o}([M],[\eta]) = \mathfrak{o}(M,\eta)$.

Example 4.2.5. Recall that Riemannian orbifolds can be seen as a certain class of Riemannian stacks. More precisely, Riemannian orbifolds are presented by proper étale Riemannian Lie groupoids. Assume that $(G^{(1)} \rightrightarrows G^{(0)},\eta)$ is a proper étale Riemannian Lie groupoid. The Lie algebroid of $G^{(1)}$ is $A = G^{(0)} \times 0$ and the space of weak multiplicative Killing vector fields $\mathfrak{o}_m^{\mathrm{w}}(G)$ is isomorphic to the space $\mathfrak{o}(G^{(0)}, \eta)^G$ of G-invariant Killing vector fields on $G^{(0)}$. Therefore, the complex of weak multiplicative Killing vector fields on $G^{(1)}$ is $0 \to \mathfrak{o}(G^{(0)}, \eta)^G$, meaning that the space of geometric Killing vector fields on the quotient stack/orbifold $[G^{(0)}/G^{(1)}]$ is given by $\mathfrak{o}([G^{(0)}/G^{(1)}], [\eta]) = \mathfrak{o}(G^{(0)}, \eta)^G$.

In other words, the previous examples show that if we consider proper étale Riemannian groupoids then geometric Killing vector fields recover the classical notions of Killing vector fields on both Riemannian manifolds and Riemannian orbifolds as defined for instance in [11].

Example 4.2.6. Let $G^{(1)} \rightrightarrows G^{(0)}$ be a regular Lie groupoid. It follows that the tangent groupoid $TG^{(1)} \rightrightarrows TG^{(0)}$ is also regular and the anchor map $\rho : A_G \to TG^{(0)}$ of the associated Lie algebroid has constant rank. In this case, the choice of a splitting of $TG^{(1)} \rightrightarrows TG^{(0)}$ induces an isomorphism of complexes

$$(\Gamma(A_G) \to \mathfrak{X}_m(G)) \cong (\Gamma(K) \oplus \Gamma(F) \to \Gamma_m(t^*K \oplus s^*\vartheta) \oplus \Gamma(F)),$$

where $K = \ker(\rho)$, $F = \mathrm{im}(\rho)$ and $\vartheta = TM/F$, see [100, Ex. 3.10]. By Corollary 5.2 in [100] it holds that the complex of multiplicative vector fields on $G^{(1)}$ is quasi-isomorphic to the 2-term complex $\Gamma(K) \to \Gamma_m(t^*K \oplus s^*\vartheta)$. Furthermore, if $G^{(1)} \rightrightarrows G^{(0)}$ is a foliation groupoid then the complex of multiplicative vector fields on $G^{(1)}$ is quasi-isomorphic to $0 \to \Gamma_m(s^*\vartheta)$ or $0 \to \Gamma(\vartheta)^G$ since $\rho : A_G \to TG^{(0)}$ is injective.

Suppose now that $\mathrm{Hol}(M, \mathcal{F}) \rightrightarrows M$ is the holonomy groupoid associated to a regular foliation $\mathcal{F}$ on M. Let $\iota : T \hookrightarrow M$ be a complete transversal submanifold to $\mathcal{F}$ and consider its restricted groupoid $\mathrm{Hol}(M, \mathcal{F})_T \rightrightarrows T$. As $\mathrm{Hol}(M, \mathcal{F})_T \rightrightarrows T$ is étale and Morita equivalent to $\mathrm{Hol}(M, \mathcal{F}) \rightrightarrows M$ (see [92, p. 136]), we get that the Lie algebra of geometric vector fields of the stack $[M/\mathrm{Hol}(M, \mathcal{F})]$ is isomorphic to the Lie algebra $\mathfrak{X}(T)^{\mathcal{F}}$ of transversal vector field that are invariant by the normal action, compare [100, Ex. 8.16]. Hence, as consequence of Theorem 4.2.2, if M is a Riemannian manifold and $\mathcal{F}$ is a regular Riemannian foliation then the geometric Killing vector fields on the Riemannian stack presented by $\mathrm{Hol}(M, \mathcal{F}) \rightrightarrows M$ are precisely the so-called transverse Killing vector fields to $\mathcal{F}$ with respect to T, as defined in [95, p. 84].

Each of the particular cases presented in the previous examples has the property that the obtained algebra of geometric Killing vector fields is finite dimensional. We carry on our exposition by proving that this is always the case if our quotient Riemannian stack is separated, i.e. it is presented by a proper Riemannian groupoid. Due to our purposes, let us start by analyzing the Riemannian foliation groupoid case which was more or less described in Example 4.2.6. Recall that if $G^{(1)} \rightrightarrows G^{(0)}$ is a foliation groupoid then the algebroid anchor map $\rho : A_G \to TG^{(0)}$ is injective, so that the manifold $G^{(0)}$ comes with a regular foliation $\mathcal{F}$ tangent to the leaves of $\mathrm{im}(\rho) \subseteq TG^{(0)}$. Moreover, if $G^{(1)} \rightrightarrows G^{(0)}$ is source-connected then the leaves of $\mathrm{im}(\rho) \subseteq TG^{(0)}$ coincide with the groupoid orbits.

Lemma 4.2.7. *If $(G^{(1)} \rightrightarrows G^{(0)}, \eta)$ is a Riemannian foliation groupoid with compact orbit space $G^{(0)}/G^{(1)}$ then the algebra of geometric Killing vector fields on $([G^{(0)}/G^{(1)}], [\eta])$ has finite dimension.*

Proof. Let $\iota : T \hookrightarrow G^{(0)}$ be a complete transversal submanifold to the orbit foliation $\mathcal{F}$ of $G^{(1)} \rightrightarrows G^{(0)}$ and consider its restricted groupoid $G_T \rightrightarrows T$. Again, as $G_T \rightrightarrows T$ is étale and Morita equivalent to $G^{(1)} \rightrightarrows G^{(0)}$, from Theorem 4.2.2 it follows that

$$\mathfrak{o}([G^{(0)}/G^{(1)}], [\eta]) = \mathfrak{o}_m^{\mathrm{w}}(G_T, \iota^*\eta)/\mathrm{im}(\delta) \cong \mathfrak{o}(T)^{\mathcal{F}}.$$

Here $\mathfrak{o}(T)^{\mathcal{F}}$ denotes the transversal Killing vector fields that are invariant by the normal action. Since $G^{(0)}/G^{(1)}$ is compact we have that T has a finite number of connected components (see [92, p. 135]), so that the dimension of $\mathfrak{o}(T)^{\mathcal{F}}$ is finite by Theorem 3.3 from [67, p. 238] (consult also [95, p. 85]). That is, $\mathfrak{o}([G^{(0)}/G^{(1)}],[\eta])$ is finite dimensional. □

By following the ideas in Example 4.2.6, it is simple to see that similar arguments to those used in the proof of the previous lemma work if we consider regular Riemannian groupoids instead of the Riemannian foliation ones. Therefore, as every proper groupoid is regular over a dense and open subset then one would expect that a similar result can be proven if we consider proper Riemannian groupoids. We show below that such a finite dimensional result is true by using the desingularization theorem for proper Riemannian groupoids proved in [105, s. 6], see also Section 3.2 for details. Namely:

Theorem 4.2.3. *Let $(G^{(1)} \rightrightarrows G^{(0)}, \eta)$ be a proper Riemannian groupoid with compact orbit space $G^{(0)}/G^{(1)}$. Then the algebra of geometric Killing vector fields on $([G^{(0)}/G^{(1)}],[\eta])$ has finite dimension.*

Proof. Let us denote by $(\tilde{G^{(1)}} \rightrightarrows \tilde{G^{(0)}}, \tilde{\eta}, \pi)$ the Riemannian desingularization of $(G^{(1)} \rightrightarrows G^{(0)}, \eta)$, see Theorem 6.10 in [105]. Recall that $(\tilde{G^{(1)}} \rightrightarrows \tilde{G^{(0)}}, \tilde{\eta})$ is a proper regular Riemannian groupoid and $\pi : \tilde{G^{(1)}} \to G^{(1)}$ is a proper Riemannian fibration which is an isometry almost-everywhere. As consequence of the functoriality properties described in [105, s. 5.2] it follows that any Lie groupoid automorphism of $G^{(1)}$ preserves its co-dimensional stratum data (compare [105, s. 3]), so that they can be lifted to Lie groupoid automorphism of $\tilde{G^{(1)}}$. That is, we can lift multiplicative vector fields on $G^{(1)}$ to multiplicative vector fields on $\tilde{G^{(1)}}$ by lifting their 1-parametric (local) families of Lie groupoid automorphisms determined by their flows. In particular, by using both Proposition 6.13 and Theorem 6.14 from [105] we get that weak multiplicative Killing vector fields on $(G^{(1)} \rightrightarrows G^{(0)}, \eta)$ can be lifted to weak multiplicative Killing vector fields on its desingularization $(\tilde{G^{(1)}} \rightrightarrows \tilde{G^{(0)}}, \tilde{\eta})$, thus obtaining an surjective algebra homomorphism

$$[\pi] : \mathfrak{o}([\tilde{G^{(0)}}/\tilde{G^{(1)}}],[\tilde{\eta}]) \to \mathfrak{o}([G^{(0)}/G^{(1)}],[\eta]).$$

By arguing as in Proposition 6.3.2 from [41] and by using the classification of regular Lie groupoids given in [91], since the groupoid $\tilde{G^{(1)}}$ is regular we may assume that it fits into an extension of Riemannian groupoids $(K, \iota^*\tilde{\eta}) \xrightarrow{\iota} (\tilde{G^{(1)}}, \tilde{\eta}) \xrightarrow{q} (E, q_*\tilde{\eta})$ over $\tilde{G^{(0)}}$ where K is a bundle of connected Lie groups (i.e. a Lie groupoid whose source and target maps agree), E is a foliation groupoid, ι is a groupoid Riemannian embedding and π is a groupoid Riemannian submersion with connected fibers ($\tilde{\eta}$ possibly needs to be averaged in order to define an equivalent groupoid metric which descends to the quotient). This in turn induces an extension of algebras

$$0 \to \mathfrak{o}(\tilde{G^{(0)}}, \iota^*\tilde{\eta}) \to \mathfrak{o}([\tilde{G^{(0)}}/\tilde{G^{(1)}}],[\tilde{\eta}]) \to \mathfrak{o}([\tilde{G^{(0)}}/E],[q_*\tilde{\eta}]) \to 0.$$

As $G^{(0)}/G^{(1)}$ is compact and π is proper and surjective it follows $\tilde{G^{(0)}}/\tilde{G^{(1)}}$ is also compact and from Lemma 4.2.7 together with Theorem 3.3 from [67, p. 238] (see also [95, p. 85]) it follows that both $\mathfrak{o}([\tilde{G^{(0)}}/E],[q_*\tilde{\eta}])$ and $\mathfrak{o}(\tilde{G^{(0)}}, \iota^*\tilde{\eta})$ have finite dimension so that $\mathfrak{o}([\tilde{G^{(0)}}/\tilde{G^{(1)}}],[\tilde{\eta}])$ has also finite dimension. That is, $\mathfrak{o}([G^{(0)}/G^{(1)}],[\eta])$ is finite dimensional since $[\pi]$ is surjective. □

We end this chapter by describing some features about our notion of geometric Killing vector field which are motivated by some results proved in [35]. To do so, we need to introduce a notion of projectable vector field of Killing type by mimicking Definition 4.6 in [35]. Let $(G^{(1)} \rightrightarrows G^{(0)}, \eta)$ be a Riemannian groupoid.

Definition 4.2.9. A vector field ξ_1 on $G^{(1)}$ is said to be **projectable of Killing type** if there exists a vector field v_1 on $(G^{(0)}, \eta^{(0)})$ such that the following conditions are satisfied:

i. ξ_1 and v_1 are both s_G-related and t_G-related, and

ii. if $\varPhi_\tau$ and φ_τ respectively denote the (local) flows of ξ_1 and v_1 then the induced map $\overline{d\varphi_\tau} : \nu(\mathcal{O}_x) \to \nu(\mathcal{O}_{\varphi_\tau(x)})$ is a linear isometry for each groupoid orbit $\mathcal{O}_x$ for which $\varphi_\tau(x)$ is defined.

Sometimes, we shall also refer to v_1 as a **weak Killing vector field** on $(G^{(0)}, \eta^{(0)})$. The space of vector fields of Killing type on $G^{(1)}$ will be denoted by $\Gamma_\eta^{\mathrm{proj}}(G^{(1)})$. Firstly, note that Condition ii. makes sense because the local flows $\varPhi_\tau$ and φ_τ commute with both s_G and t_G as consequence of Condition i. Secondly, it follows that ξ_1 is also a weak Killing vector field on $(G^{(1)}, \eta^{(1)})$ in the sense that $\overline{d\varPhi_\tau} : \nu(G_{\mathcal{O}_x}) \to \nu(G_{\mathcal{O}_{\varphi_\tau(x)}})$ is a linear isometry since s_G and t_G are Riemannian submersions. Actually, it is simple to see that a similar property holds true for the vector field $\xi_2(g, g') = (\xi_1(g), \xi_1(g'))$ on $(G^{(2)}, \eta^{(2)})$ induced by ξ_1.

We explain below how proper Haar measure systems on proper Riemannian groupoids determine projections from the space of projectable vector fields of Killing type on $(G^{(1)}, \eta^{(1)})$ to the space of weak multiplicative Killing vector fields on $(G^{(1)} \rightrightarrows G^{(0)}, \eta)$. This can be done by following [36, s. 2.5] closely.

Theorem 4.2.4. *Let $(G^{(1)} \rightrightarrows G^{(0)}, \eta)$ be a proper Riemannian groupoid. Then any proper Haar measure system $\{\mu^x\}_{x\in G^{(0)}}$ for $G^{(1)} \rightrightarrows G^{(0)}$ induces a linear map from $\Gamma_\eta^{\mathrm{proj}}(G^{(1)})$ to $\mathfrak{o}_m^{\mathrm{w}}(G, \eta)$.*

Proof. Let us pick a projectable vector field of Killing type ξ_1 on $(G^{(1)}, \eta^{(1)})$. From [36, s. 2.5] we know that we can construct a multiplicative vector field $\xi : G^{(1)} \to TG^{(1)}$ by taking the average with respect to $\{\mu^x\}_{x\in G^{(0)}}$:

$$\xi(g) = \int_{a\in t_G^{-1}(s(g))} dm_{(ga,a^{-1})}(\xi_1(ag), di_a(\xi_1(a))\mu(a).$$

This vector field is s_G-related with the vector field v on $G^{(0)}$ defined as

$$v(x) = \int_{a\in t_G^{-1}(x)} dt_a(\xi_1(a))\mu(a).$$

Therefore, our result will follow once we prove that the flow of v is a local transversal isometry of $(G^{(0)}, \eta)$ since ξ and v are s_G-related. However, as ξ_1 is a weak Killing vector field on $(G^{(1)}, \eta^{(1)})$, this follows from a straightforward computation after noting that the flow of v is given by $\varphi_r^v(x) = \int_{a\in t_G^{-1}(x)}(t \circ \varphi_r^{\xi_1})(a)\mu(a)$ and $\overline{dt_G} : \nu(G_{\mathcal{O}}) \to \nu(\mathcal{O})$ is a fiberwise isometry since t_G is a Riemannian submersion. Hence, the assignment $\xi_1 \mapsto (\xi, v)$ establishes the desired projection. $\square$

Remark 4.2.2. It is worth commenting that as consequence of Remark 4.2.1 we may conclude that the previous result is actually a fact that can be proven for any n-metric $\eta^{(n)}$ on $G^{(n)}$.

In [35, s. 4.4] it was studied the "normal" bundle $\vartheta := TG^{(0)}/\rho(A_G)$ where $\rho : A_G \to TG^{(0)}$ is the anchor map of the Lie algebroid A_G of $G^{(1)} \rightrightarrows G^{(0)}$. This is a smooth vector bundle only in the regular case, compare Example 4.2.6. Its space of sections is defined to be the quotient $\Gamma(\vartheta) := \mathfrak{X}(G^{(0)})/\mathrm{im}(\rho)$. A section $[v] \in \Gamma(\vartheta)$ is called **invariant** if there exists a vector field ξ on $G^{(1)}$ which is both s_G-related and t_G-related to v. The resulting space of invariant elements is denoted by $\Gamma(\vartheta)^{\mathrm{inv}}$. Recall that for each section $\alpha \in \Gamma(A_G)$ we have an associated multiplicative vector field $\delta(\alpha) = (\alpha^r - \alpha^l, \rho(\alpha))$ on $G^{(1)} \rightrightarrows G^{(0)}$. From Lemma 4.7 in [35] it follows that there is a natural linear map from $\mathfrak{X}_m(G)/\mathrm{im}(\delta)$ to $\Gamma(\vartheta)^{\mathrm{inv}}$ which associates to a multiplicative vector field ξ on $G^{(1)}$ the class modulo $\mathrm{im}(\rho)$ of the vector field v on $G^{(0)}$ associated with ξ. Furthermore, if $G^{(1)} \rightrightarrows G^{(0)}$ is proper then the latter map induces an isomorphism $\mathfrak{X}_m(G)/\mathrm{im}(\delta) \cong \Gamma(\vartheta)^{\mathrm{inv}}$, see Theorem 6.1 in [35].

Let $(G^{(1)} \rightrightarrows G^{(0)}, \eta)$ be a proper Riemannian groupoid. Motivated by the previous facts we define the space of **Killing invariant sections** $\Gamma_\eta(\vartheta)^{\mathrm{inv}}$ as the set of sections $[v] \in \Gamma(\vartheta)$ for which there exists a projectable vector field of Killing type ξ on $G^{(1)}$ over v. That is, Conditions i. and ii. from Definition 4.2.9 for projectable vector field of Killing type are satisfied with ξ and v. Therefore, as consequence of Lemma 4.7 and Theorem 6.1 in [35] we immediately get that:

Proposition 4.2.5. *There exists a natural isomorphism between* $\mathfrak{o}_m^{\mathrm{w}}(G, \eta)/\mathrm{im}(\delta)$ *and* $\Gamma_\eta(\vartheta)^{\mathrm{inv}}$.

In particular, Theorem 4.2.4 provides us with a method to construct geometric Killing vector fields over the quotient Riemannian stack $([G^{(0)}/G^{(1)}], [\eta])$.

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