Group Representations and Real Trees

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Resumo

VINHAS BERTOLINI, M. **Representações de Grupos e Árvores Reais**. 2016. 60 f. Tese (Doutorado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2016.

Nesta tese é provado que certas seqüências de ações isométricas hiperbólicas do grupo livre em um número infinito, enumerável, de geradores, ou convergem, ou divergem para uma ação isométrica do grupo em uma árvore real. Isto aponta para uma generalização do Teorema de W. Thurston de Hiperbolização de Suspensões Compactas para monodromias pseudo-Anosov generalizadas.

Palavras-chave: Grupos Kleinianos, Árvores Reais, Teorema de Hiperbolização.

Abstract

VINHAS BERTOLINI, M. **Group Representations and Real Trees**. 2016. 60 f. Tese (Doutorado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2016.

In this thesis we stablish that certain sequences of isometric hyperbolic actions of the free group on an infinite, countable, number of generators, either converge, or diverge to an isometric action of the group on a real tree. This points towards a generalization of W. Thurston's Theorem of Hyperbolization of Compact Mapping Tori for generalized pseudo-Anosov monodromies

Keywords: Kleinian Groups, Real Trees, Hyperbolization Theorem.

Contents

Chapter 1

Introduction

This thesis partially fullfill a program that aims to generalize the following Theorem of W. Thurston. For the relevant definitions, see Chapter 4.

Theorem 1 (Thurston on Hyperbolic Mapping Tori)**.** *Let S be a compact surface, orientable, and of negative Euler characteristic. If a self-homeomorphism φ of S is isotopic to a pseudo-Anosov map, then the associated mapping torus is hyperbolic.*

In this case, the hyperbolic structure is of finite volume, by compacity, and unique up to isometry, by Mostow Rigidity. What we are after is to prove that the mapping torus is hyperbolic for punctured generalized pseudo-Anosov φ, which are defined on compact surfaces with an infinite number of points removed. In this case, if the hyperbolic structure exist, it has infinite volume. Experts on the field seems to agree that Thurston's Theorem is valid when φ is a pseudo-Anosov map of a compact surface with a finite number of punctures, but even of this we know of no written proof. In fact, the proof of Theorem above is a complicated story. A fair account should be given by more experienced researchers on the topic, but here's a brief outline.

[Ril13] [Sul81, Thu82, Mor84, Thu98, Ota96, McM14, Kap01, Hub] [BB]

Chapter 2

Kleinian Representations

This preliminary Chapter sets basic terminology and results on Kleinian representations and groups. We start with the general notions of representation and action.

Definition 2 (Representation, Action)**.** Let G be a group. A *representation of* G is a group-homomorphism ρ whose domain is G. An *action of* G *on a set* X is a representation of G in the group of self-bijections of X. The action is *conformal* if X is a Riemann surface and, for each $g \in G$, $\rho(g)$ is a conformal automorphism; and is *isometric* if X is a metric space and, for each $g \in G$, $\rho(g)$ is an isometry. For such, it suffices that it holds for every h in a generating set of G.

2.1 $\mathrm{PSL}_2\,\mathbb{C}$ Basics

Consider the group $\mathrm{PSL}_2\,\mathbb{C}$ of 2×2 matrices of complex entries with determinant 1, quotiented by the normal subgroup $\{\pm Id\}$. It is a complex manifold of dimension 3. Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the *Riemann sphere.* The formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az+b}{cz+d}, \quad z \in \hat{\mathbb{C}}, \tag{2.1}$$

define an isomorphism of $\mathrm{PSL}_2\,\mathbb{C}$ onto the group of conformal automorphisms of $\hat{\mathbb{C}}$. Then, $\mathrm{PSL}_2\,\mathbb{C}$ acts conformally on $\hat{\mathbb{C}}$. Recall that each conformal automorphism γ of $\hat{\mathbb{C}}$ is completelly determined by the images $\gamma(0)$, $\gamma(1)$ and $\gamma(\infty)$, which can be any 3 distinct points of $\hat{\mathbb{C}}$. This is in accordance with the dimension of $\mathrm{PSL}_2\,\mathbb{C}$.

Consider the 3-*dimensional hyperbolic space*, which is the complete Riemannian manifold defined by:

$$\mathbb{H}^3 = \{(z,t) \,|\, z \in \mathbb{C},\, t > 0\}, \quad ds^2 = t^{-2}(dz^2 + dt^2).$$

Its geodesics are the euclidean circles and lines orthogonal to $\mathbb{C}$. So, the endpoints of the geodesics constitute the Riemann sphere, which is the *sphere at infinity* of $\mathbb{H}^3$. Recall that each conformal map γ of $\hat{\mathbb{C}}$ is the composition of a pair of inversions on circles of $\hat{\mathbb{C}}$. Each such inversion extends to an inversion on a euclidean sphere of $\mathbb{H}^3$, which is a orientation-reversing isometry of its metric. Compositions of pairs of such inversions generate the orientation-preserving isometry group of $\mathbb{H}^3$, and the conformal automorphisms of $\hat{\mathbb{C}}$ are preciselly the values at infinity of these isometries. This stablishes an isomorphism between $\mathrm{PSL}_2\,\mathbb{C}$ and the group of orientation-preserving istrometries of $\mathbb{H}^3$. Then, $\mathrm{PSL}_2\,\mathbb{C}$ acts isometrically on $\mathbb{H}^3$.

Definition 3 (Translation Length)**.** Let (X, d) be a metric space. For each isometry φ of X, the associated *translation length* is defined by:

$$\ell(\varphi) = \inf_{x \in X} d(x, \varphi(x)).$$

In particular, each element of $\mathrm{PSL}_2\,\mathbb{C}$ have a well-defined translation length, as an isometry of $\mathbb{H}^3$. Also, recall that elements of $\mathrm{PSL}_2\,\mathbb{C}$ have well-defined squared traces tr^2.

Theorem 4 (Elements of $\mathrm{PSL}_2\,\mathbb{C}$ up to Conjugacy)**.** *Every $\gamma \in \mathrm{PSL}_2\,\mathbb{C}$ different from the identity is conjugated in $\mathrm{PSL}_2\,\mathbb{C}$ to either:*

1. *The* parabolic $z \mapsto z + 1$*. This is equivalent to $\ell(\gamma) = 0$, the infimum not being attained in $\mathbb{H}^3$.*
2. *The* hyperbolic $z \mapsto \lambda z$*, for precisely one $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. This is equivalent to $\ell(\gamma) > 0$, the infimum being attained in $\mathbb{H}^3$.*
3. *The* elliptic $z \mapsto \lambda z$*, for precisely one λ in the unit circle, $\lambda \neq 1$.*

`Finish this.`
`Figure: parabolic, hyperbolic and elliptic transformations.`

Proof. Let $\gamma \in \mathrm{PSL}_2\,\mathbb{C}$ be different from the identity, and consider the fixed-point equation $\gamma(z) = z$. It is a second order equation and, therefore, γ has either 1 or 2 fixed points in $\hat{\mathbb{C}}$. Suppose that it has only one, which gives *Case 1*: let z_0 be the fixed point, and conjugate γ by some element of $\mathrm{PSL}_2\,\mathbb{C}$ that takes z_0 to ∞, turning γ into $z \mapsto az + b$, for $a, b \in \mathbb{C} \setminus \{0\}$. A second conjugation turns it into $z \mapsto z + 1$. Notice that z_0 is the global attractor for the action of γ, both on $\hat{\mathbb{C}}$ and on $\mathbb{H}^3$. Also, notice that each horosphere centered at z_0 is invariant by γ. See Figure **??**.

Now suppose that γ fixes precisely two points $z_0, z_1 \in \hat{\mathbb{C}}$. Then, it fixes setwise the geodesic of $\mathbb{H}^3$ with endpoints z_0 and z_1, called its *axis*, which determine the pair $\{z_0, z_1\}$, and it is also determined by it. First conjugate γ to take this pair to $\{0, \infty\}$, turning γ into $z \mapsto \lambda z$, for precisely one complex number $\lambda \neq 1$. Up to conjugating with $z \mapsto 1/z$, $|\lambda| \geq 1$, with $\lambda \neq 1$ since γ is not the identity.

Suppose that $|\lambda| > 1$, which gives *Case 2*. Then, up to conjugation with inversion $z \mapsto 1/z$, 0 and ∞ are, respectivelly, global attractor and repeller for the dynamics of the conjugated γ, both on $\hat{\mathbb{C}}$ and on $\mathbb{H}^3$, and so is the pair $\{z_0, z_1\}$ for the dynamics of γ. The restriction of γ to its axis is the translation by an amount of $\log|\lambda|$ in the direction of the attracting fixed point, where the ammount is measured in the hyperbolic distance: compute the distance in $\mathbb{H}^3$ from $(0, 1)$ to $(0, |\lambda|)$.

And if $|\lambda| = 1$ (*Case 3*), the axis in $\mathbb{H}^3$ is fixed pointwise, and γ is a rotation around it. This kind of transformation will not appear in what follows. □

For formulae relating the coefficients of the complex matrix and the geometrical features of the associated isometry of $\mathbb{H}^3$, see [Bea12] and [Kou91], and [Hub].

Notice that for parabolic and hyperbolic elements of $\mathrm{PSL}_2\,\mathbb{C}$, the correspondent actions on $\mathbb{H}^3$ are topologically conjugated to the unit translation of $\mathbb{R}^3$. They correspond to different compactifications of its action, by one and by two limit-points at the infinity. Meanwhile, elliptic elements acts as a rotation. Recall that the *order* of and element g in a group G is the number of elements of the subgroup generated by g. In $\mathrm{PSL}_2\,\mathbb{C}$, every element of finite order is elliptic, and so these do not appear in torsion-free subgroups of $\mathrm{PSL}_2\,\mathbb{C}$. Also, elliptic elements with infinite order are rotations with irrational angle, and these do not appear in discrete subgroups of $\mathrm{PSL}_2\,\mathbb{C}$. So, in what follows, we'll be mainly thinking about hyperbolic and parabolic elements of $\mathrm{PSL}_2\,\mathbb{C}$.

2.2 Kleinian Representations

A subset of $\mathrm{PSL}_2\,\mathbb{C}$ is *discrete* if the only convergent sequences contained in it are the eventually constant ones. So, a subgroup Γ of $\mathrm{PSL}_2\,\mathbb{C}$ is discrete if, and only if, every sequence in Γ converging to the identity is eventually equal to the identity. Discreteness is a strong assumption over a subgroup, and its consequences can be seen as restrictions on the dynamics of the associated actions on $\hat{\mathbb{C}}$ and $\mathbb{H}^3$.

[Mar07, MT98, MSW02, Mas12, Kap01, BP12, Hub, Bea12]

Definition 5 (Kleinian Representation)**.** Let G be a group. A *Kleinian representation* of G is an injective representation $\rho : G \to \mathrm{PSL}_2\,\mathbb{C}$ whose image is a discrete subgroup of $\mathrm{PSL}_2\,\mathbb{C}$. The image $\Gamma = \rho(G)$ of a Kleinian representation ρ is called a *Kleinian group.*

It is important not to confuse a representation, that is a fixed isomorphism onto its image, with the image itself. For instance, if Γ is a Kleinian group, each group automorphism of Γ is a Kleinian representation of Γ, and any two of these give distinct representations, all of them with the same image.

Notice that the existence of a Kleinian representation of a group implies that it contains at most a countable number of elements, since the topology of $\mathrm{PSL}_2\,\mathbb{C}$ admits a countable basis of open sets. A more serious restriction, over the isomorphism type of the group, will now be explained.

Recall that if a group G acts on a set X, each $x \in X$ have an associated *orbit* under the action, which is the set $\{g \cdot x \in X \,|\, g \in G\}$. The orbits decomposes X, and the associated quotient is the *orbit space* X/G. Let Γ be a Kleinian group, and consider the projection $\pi : \mathbb{H}^3 \to \mathbb{H}^3/\Gamma$. The discreteness of Γ implies that, away from the axis of elliptic rotations, π is a Riemannian covering map. Therefore, if Γ is torsion-free, π is a universal covering map, and Γ, being the group of the covering transformations, is isomorphic to fundamental group of the 3-manifold $\mathbb{H}^3/\Gamma$. On the other direction, if M is a complete Riemannian manifold of dimension 3 and constant sectional curvature equal to -1, the classical Theorem of Hadamard states that the universal Riemannian covering is isometric to $\mathbb{H}^3$. Therefore, M is isometric to $\mathbb{H}^3/\Gamma$, for some torsion-free Kleinian group Γ isomorphic to the fundamental group of M. So, the possible isomorphism types of torsion-free Kleinian groups are precisely the ones of the fundamental groups of complete Riemannian 3-manifolds of constant sectional curvature equal to -1.

From the point of view of the geometry of the orbit space, what is important is the conjugacy class in $\mathrm{PSL}_2\,\mathbb{C}$ of a Kleinian group. It can be useful to have something fixed to prove results (Definition 7), but geometric features of a Kleinian group are all invariant under conjugation in $\mathrm{PSL}_2\,\mathbb{C}$.

Proposition 6 (Conjugated Kleinian Groups)**.** *Let Γ and Γ' be torsion-free Kleinian groups. The orbit spaces $\mathbb{H}^3/\Gamma$ and $\mathbb{H}^3/\Gamma'$ are isometric if, and only if, Γ and Γ' are conjugated in* $\mathrm{PSL}_2\,\mathbb{C}$.

Proof. This is just a matter of lifting and projecting isometries. □

Definition 7. A Kleinian representation ρ of a group G is *normalized* if there are distinguished group elements $g_0, g_1, g_\infty \in G$ such that 0 is the attracting fixed point of $\rho(g_0)$, 1 is the attracting fixed point of $\rho(g_1)$, and ∞ is the attracting fixed point of $\rho(g_\infty)$. Given g_0, g_1 and g_∞, this conditions fixes ρ inside its conjugacy class in $\mathrm{PSL}_2\,\mathbb{C}$.

Example 8 (Cyclic Kleinian Groups)**.** Let Γ be generated by a single $\gamma \in \mathrm{PSL}_2\,\mathbb{C}$, assumed to be different from the identity. If γ has finite order, then γ is elliptic, and the quotient $\mathbb{H}^3/\Gamma$ is homeomorphic to $\mathbb{R}^3$. The metric of $\mathbb{H}^3$ projects to a Riemannian metric of constant sectional curvature -1 away from the projection of the axis of γ, around which the metric is conical with total angle equal to the rotation angle of γ. Suppose now that γ have infinite order. If γ is elliptic, then Γ is not discrete, and therefore is not a Kleinian group. And if γ is hyperbolic or parabolic, in any case the quotient $\mathbb{H}^3/\Gamma$ is homeomorphic to $S^1 \times \mathbb{R}^2$. Its isometry type depends on γ, in accordance with Proposition 6 and Theorem 9. If γ is hyperbolic, then free homotopy class of the S^1 factor contains an unique simple closed geodesic; and if γ is parabolic, this homotopy class contains representatives with arbitrary short lengths. See also Figure **??** for an enlightning picture in dimension 2.

Theorem 9 (Free Homotopy Classes)**.** *Let Γ be a torsion-free Kleinian group, and let $\pi : \mathbb{H}^3 \to \mathbb{H}^3/\Gamma$ be the associated universal covering map. For each non-trivial simple closed curve c in $\mathbb{H}^3/\Gamma$:*

1. *Each connected component of $\pi^{-1}(c)$ have well-defined endpoints in $\hat{\mathbb{C}}$, and those are the fixed points of some $\gamma \in \Gamma$. The conjugates $\alpha \circ \gamma \circ \alpha^{-1}$, for $\alpha \in \Gamma$, fixes precisely the endpoints of the components of $\pi^{-1}(c)$.*

2. *For any simple closed curve c' freely homotopic to c in M, the components of $\pi^{-1}(c')$ have the same endpoints in $\hat{\mathbb{C}}$ that the ones of the components of $\pi^{-1}(c)$.*

3. *If the endpoints of some component of $\pi^{-1}(c)$ coincide, then γ is a parabolic element of Γ. In this case, the free homotopy class of c contain arbitrarily short simple closed curves.*

4. *If the endpoints of some component of $\pi^{-1}(c)$ are distinct, then γ is a hyperbolic element of Γ. The axis of γ project to a simple closed geodesic that is the length-minimizer in the free homotopy class of c, with length equal to the translation length of γ.*

An important consequence of discreteness is the existence of a subset of $\hat{\mathbb{C}}$ that encodes all the possible ways of going to infinity inside a Kleinian group Γ. For sequences in Γ of the form $\gamma_k = \gamma_0^k$, for fixed $\gamma_0 \in \Gamma$ and $k \geq 0$, Theorem 4 shows that, for any $x \in \mathbb{H}^3$ and any $z \in \hat{\mathbb{C}}$, the sequences $\gamma_k(x)$ and $\gamma_k(z)$ converge to the attracting fixed point of γ. A similar thing is true for every divergent sequence in Γ.

Proposition 10. *Let Γ be a Kleinian group, and let $\gamma_k \in \Gamma$ be a sequence. Suppose that γ_k is divergent. Then, for any $x \in \mathbb{H}^3$ and any $z \in \hat{\mathbb{C}}$, possibly passing to a subsequence, the following limits are equal to a point of $\hat{\mathbb{C}}$, that is independent of x and z:*

$$\lim_{k\to\infty} \gamma_k(x) = \lim_{k\to\infty} \gamma_k(z) \in \hat{\mathbb{C}}, \quad \forall x \in \mathbb{H}^3, z \in \hat{\mathbb{C}}. \tag{2.2}$$

Definition 11 (Limit and Regular Set)**.** Let Γ be a Kleinian group. The *limit set* of Γ is the closure in $\hat{\mathbb{C}}$ of the set of all points of the form (2.2) for some divergent sequence $\gamma_k \in \Gamma$. Its complement $\Omega_\Gamma = \hat{\mathbb{C}} \setminus \Lambda_\Gamma$ is the *regular set* of Γ.

Example 12 (Elementary Kleinian Groups).

Example 13 (Schottky Groups).

Theorem 14 (Non-Elementary Kleinian Groups). *A Kleinian group Γ is non-elementary if, and only if, it has the following equivalent properties:*

1. *Λ_Γ contains more than two points. In this case, it contains an uncountable number of points.*

2. *Γ contains a subgroup free on two generators consisting only of hyperbolic elements.*

3. *Γ do not contain Abelian subgroups of finite index.*

`Comments on the Tits Alternative.`

Proposition 15. *Let Γ be a Kleinian group. If Γ' is a normal subgroup of Γ, then $\Lambda_{\Gamma'} = \Lambda_\Gamma$.*

Theorem 16 (Jorgensen's Inequality). *For any $\alpha, \beta \in \mathrm{PSL}_2\,\mathbb{C}$, let $[\alpha,\beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ be the commutator of α and β. If α and β generate a non-elementary discrete subgroup of $\mathrm{PSL}_2\,\mathbb{C}$, then*

$$\mu(\alpha,\beta) = |(\operatorname{tr}\alpha)^2 - 4| + |\operatorname{tr}[\alpha,\beta] - 2| \geq 1.$$

where tr *denotes the trace.*

Proof. This is a computation with matrices, $\mu \geq 1$ end up being the condition for a certain sequence of group elements do not acmulate on the identity. [Jør76] □

Theorem 17 (Margulis' Lemma). *There exist a constant $r_0 > 0$ such that, for any torsion-free Kleinian group Γ and any $x \in \mathbb{H}^3$, the set*

$$\{\gamma \in \Gamma \,|\, d_{\mathbb{H}^3}(x, \gamma(x)) < r_0\}$$

generate an elementary subgroup of Γ.

Proof. Consequence of Jorgensen's Inequality. □

2.3 Fuchsian Representations

Suppose that the limit set of a Kleinian group Γ is a round circle C of $\hat{\mathbb{C}}$. Suppose also that each of the two connected components of $\hat{\mathbb{C}} \setminus C$ is invariant by Γ, so they're not permuted by the action of Γ. Then, conjugate Γ by an element of $\mathrm{PSL}_2\,\mathbb{C}$ that takes C to the circle $S^1 = \mathbb{R} \cup \{\infty\}$. Each element of this conjugated Γ, as a matrix, have real entries, and belongs to $\mathrm{PSL}_2\,\mathbb{R}$. Conversely, formula (2.1) shows that the elements of $\mathrm{PSL}_2\,\mathbb{R}$ leaves invariant S^1, and the upper and the lower halfplanes, which are the components of $\hat{\mathbb{C}} \setminus S^1$. This next definition leaves out Kleinian groups that are Fuchsian in disguise, but is the one that we are going to stick with.

[Kat92, Don11, Bon09, Bor07, Bea12]

Definition 18 (Fuchsian Representation). Let G be a group. A *Fuchsian representation* of G is an injective representation $\rho : G \to \mathrm{PSL}_2\,\mathbb{R}$ whose image is a discrete subgroup of $\mathrm{PSL}_2\,\mathbb{R}$. The image $\Gamma = \rho(G)$ of a Fuchsian representation ρ is called a *Fuchsian group*.

A discrete subgroup Γ of $\mathrm{PSL}_2\,\mathbb{R}$ with $\Lambda_\Gamma = S^1$ usually is called a *Fuchsian group of the first kind*, which here we incorporate on the definition of Fuchsian group. This assumption leaves out precisely the cases where Λ_Γ is a Cantor set, called of *the second kind*, besides the elementary ones.

Conformal and isometric actions on $\mathbb{H}^2$. Hadamard Theorem in Dimension 2 and Uniformization Theorem.

Punctures and not funnels.

The lower half-plane, and the surfaces at infinity of $\mathbb{H}^3/\Gamma$.

Riemann surfaces at infinity of $\mathbb{H}^3/\Gamma$ for Kleinian Γ. Ahlfors Finiteness Theorem.

Deformations at infinity.

2.4 Sequences of Kleinian Representations

Definition 19 (Algebraic Convergence)**.** Let G be a group, not necessarily finitely generated. A sequence ρ_n of Kleinian representations of G *converges algebraically* if, for every $g \in G$, the sequence $\rho_n(g)$ converges in $\mathrm{PSL}_2\,\mathbb{C}$, with no uniformity requirements here. For such, it suffices that $\rho_n(h)$ converges in $\mathrm{PSL}_2\,\mathbb{C}$ for every h in a generating set of G.

Theorem 20 (Chuckrow's Theorem)**.** *Let G be a group, not necessarily finitely generated, and let ρ_n be a sequence of non-elementary Kleinian representations of G. If ρ_n converges algebraically, then*

$$\rho_\infty(g) = \lim_{n\to\infty} \rho_n(g), \quad g \in G,$$

define a Kleinian representation of G.

Proof. Jorgensen's Inequality. □

Example 21 (Sequences with common fixed points and limited traces)**.** Since 0 is the attractor fixed point of α_n and ∞ is the attractor fixed point of β_n,

$$\alpha_n(z) = \frac{z}{a_n z + a'_n} \quad \text{and} \quad \beta_n(z) = b_n z + b'_n, \quad z \in \hat{\mathbb{C}},$$

where a_n, a'_n, b_n, b'_n are complex numbers with $|a'_n| > 1$ and $|b_n| > 1$. Also,

$$\tau(\alpha_n) = \log|a'_n|, \quad \text{and} \quad \tau(\beta_n) = \log|b_n|,$$

where $\tau(\alpha_n)$ and $\tau(\beta_n)$ are the translation lengths in $\mathbb{H}^3$. These translation lengths coincide with $\tau(\rho_n(f^{-n}(a)))$ and $\tau(\rho_n(f^{-n}(b)))$, and those are limited by Control of Translation Lengths, giving:

$$\tau(\alpha_n) < 2\tau_0^+(a) \quad \text{and} \quad \tau(\beta_n) < 2\tau_0^+(b), \quad \forall n \geq 0.$$

The same argument shows that $\tau(\alpha_n\beta_n)$ is limited above by $\tau_0(ab)$. Also, $\alpha_n\beta_n$ fixes the point 1, which gives

$$a_n + a'_n = b_n(a_n + a'_n) + b'_n.$$

Therefore, $|a_n|$ and $|b'_n|$ are also limited above, and this concludes the proof that α_n and β_n have subsequences converging in $\mathrm{PSL}_2\,\mathbb{C}$ to α and β.

2.5 Quasiconformal and Quasi-Isometric Homeomorphisms

There are several ways of defining quasiconformality, which is a bound on the distortion `dilatation?` of a homeomorphism. But distortion of what? It is possible to consider specific metrics, quadrilaterals, annuli, cross-ratios, skews, and all these leads to interesting characterizations of quasiconformality. Probably the most acessible definition is the one for diffeomorphisms. But, for the uses we have in mind, smoothness everywhere is not there. The definition below, of analytical nature, handles with that, and is the closest to the proof of Theorem 24, which is the main reason for us to consider quasiconformal homeomoprhisms. For references on this, check [AE66, LVL73, GL00, Hub06, DD08, Väi89].

Definition 22 (Distributional Derivative, Locally Integrable)**.** Let U be an open subset of $\hat{\mathbb{C}}$. A function u defined on U have *distributional partial derivative* u_x if, for every C^∞ function ψ with compact support in U,

$$\iint u\psi_x dxdy = -\iint u_x\psi dxdy.$$

The distributional derivative u_x is *locally integrable* if every point of U have a compact neighborhood N for which $\iint_N |u_x|dxdy < \infty$. A homeomorphism $w = u+iv$ defined on U have locally integrable distributional derivatives if all four of the partial derivatives u_x,u_y,v_x, v_y exist in the distributional sense and are locally integrable. This is equivalent to the distributional derivatives $\partial w/\partial z$ and $\partial w/\partial \overline{z}$ being locally integrable.

Definition 23 (Quasiconformal Map)**.** Let U be an open subset of $\hat{\mathbb{C}}$. A homeomorphism w of U onto $w(U)$ is *quasiconformal* if w has locally integrable distributional derivatives $\partial w/\partial z$ and $\partial w/\partial_{\overline{z}}$ on U; and there exist $k < 1$ such that

$$\left|\frac{\partial w}{\partial \overline{z}}\right| \leq k \left|\frac{\partial w}{\partial z}\right|.$$

Such an w is called K-quasiconformal, for $K = (1+k)/(1-k)$. The function

$$\mu(z) = \frac{|\partial w/\partial \overline{z}|}{|\partial w/\partial z|}$$

is the *Beltrami coefficient* of φ.

Theorem 24 (Measurable Riemann Mapping Theorem)**.** *Let U be a domain of the Riemann sphere. Given a measurable function $\mu : U \to \mathbb{D}$ with $||\mu||_\infty < 1$, there exist a quasiconformal homeomorphism $w : U \to w(U)$ that solves the* Beltrami equation

$$\frac{\partial w}{\partial \overline{z}} = \mu\frac{\partial w}{\partial z}.$$

Any other solution of this Beltrami equation is w post-composed with a conformal map. In case $U = \hat{\mathbb{C}}$, w is said to be normalized *if it is the solution that fixes the points* 0, 1 *and* ∞.

Theorem 25 (Uniformly Quasiconformal Maps)**.** *Let φ_n be a sequence of K-quasiconformal homeomorphisms of $\hat{\mathbb{C}}$. Assume that the sequences of points $\varphi_n(0)$, $\varphi_n(1)$ and $\varphi_n(\infty)$ are uniformly bounded away one from each other. Then, φ_n have a subsequence that converges to a K-quasiconformal homeomorphism φ_∞ of $\hat{\mathbb{C}}$.*

Definition 26 (Quasi-Isometric Homeomorphism)**.** Let (X, d_X) and (Y, d_Y) be metric spaces, and let $\varphi : X \to Y$ be a homeomorphism. The *dilatation of* φ is defined by:

$$\operatorname{dil}\varphi = \sup \frac{d_Y(\varphi(x_1), \varphi(x_2))}{d_X(x_1, x_2)}, \tag{2.3}$$

where the supremum is taken over all distinct $x_1, x_2 \in X$. And the *Lipschitz constant of* φ is defined by:

$$L(\varphi) = \max\{\operatorname{dil}\varphi, \operatorname{dil}\varphi^{-1}\}. \tag{2.4}$$

The homeomorphism φ is *quasi-isometric* if $L(\varphi) < \infty$. Notice that

$$L(\varphi_1 \circ \cdots \circ \varphi_N) \leq L(\varphi_1) \cdots L(\varphi_N). \tag{2.5}$$

Quasi-isometric self-homeomorphisms of $\mathbb{H}^3$ are quasiconformal at infinity. Converselly:

Theorem 27 (Visual Extension, McMullen B.23)**.** *Let Γ and Γ' be torsion-free Kleinian groups, and let φ be a K-quasiconformal conjugacy between Γ and Γ'. Then, φ extends to an equivariant quasi-isometric homemorphism of $\mathbb{H}^3$, with bi-Lipschitz constant bounded by $L = K^{3/2}$. It is called the* visual extension *of φ, and induces the same group isomorphism than φ.*

Chapter 3

Diverging to Isometric Actions on $\mathbb{R}$-trees

The purpose of this chapter is to understand the divergence of certain sequences of isometric group actions on a metric space. Supposing that the metric space is negativelly curved in the sense of Section 3.2, limiting isometric actions of the group on $\mathbb{R}$-trees will be constructed in Section 3.3. The central result is Theorem 39. For finitely generated groups, this and the related results of Section 3.5 were first proved by J. Morgan and P. Shalen [MS84] in algebraic-geometrical terms. Later, M. Bestvina [Bes88] and F. Paulin [Pau88] proved, independently, the same result, by giving the same geometric argument, in slightly different conceptual settings. See also [BS94]. This geometric argument was reorganized by J.-P. Otal [Ota96], where the $\mathbb{R}$-tree is constructed in terms of "Chiswell Functions". Here, following [Bes02], the construction of the $\mathbb{R}$-tree relies on the Connecting the Dots Lemma 35, instead of Chiswell's formalism. In fact, for isometric actions, both options end up being equivalent, and the main argument is the same. But the present approach points towards possible further generalizations for quasi-isometric actions.

The basic phenomena behind "diverging to an $\mathbb{R}$-tree" is the so-called "degeneration" of hyperbolic spaces when their metrics are re-scaled by a factor tending to zero. For instance, recall that if a negativelly-curved metric is re-scaled as such, then the curvature tends to $-\infty$. This can be seen as geodesic triangles becoming infinitely thin, as the ones in an $\mathbb{R}$-tree. On the other hand, under these re-scaling, the metric is expected to collapse to a single point. The trick is to re-scale in accordance with a given divergent sequence of isometric actions, so the "sizes" of the actions stays approximatelly constant, giving the desired limiting action.

`Figure: degeneration of hyperbolic space.`

So, the main object being considered here is:

Definition 28 (Space of Isometric Actions)**.** Let G be a group, and let X be a metric space. The associated *space of isometric actions* is the set $\mathrm{Rep}(G, X)$ of representations of G in the isometry group of X, with the topology of *algebraic convergence*: a sequence $\rho_n \in \mathrm{Rep}(G, X)$ is *convergent* if, for each $g \in G$, $\rho_n(g)$ converges, uniformly on compact subsets of X, to an isometry $\rho_\infty(g)$ of X. For such, it suffices that it holds for every h in a generating set of G and, in this case, $\rho_\infty \in \mathrm{Rep}(G, X)$. Recall that being conjugated in $\mathrm{Isom}\, X$ is an equivalence relation, and denote by $[\mathrm{Rep}(G, X)]$ the associated quotient space. Finally, call an isometric action ρ *non-trivial* if there exist $x \in X$ and $g \in G$ such that $\rho(g)(x) \neq x$.

Notice that, for $X = \mathbb{H}^3$, this coincides with Definition 19.

3.1 Convergent Sequences

If $\rho_n \in \mathrm{Rep}(G,X)$ is convergent, then the sequence of displacements $d(x_0, \rho_n(g)(x_0))$ is convergent, by continuity of d, for any $x_0 \in X$. On the other direction, recall the classical technique to extract a convergent subsequence of sequence of isometries:

Theorem 29 (Arzelà-Ascoli)**.** *Let (X,d) be a complete and separable metric space, and let φ_n be a sequence of isometries of X. Suppose that there exist a point $x_0 \in X$ and a constant $M > 0$ such that, for every $n \geq 0$, $d(x_0, \varphi_n(x_0)) \leq M$. Then, φ_n contains a subsequence that converges, uniformly on compact subsets of X, to an isometry of X.*

Notice that, for any $\rho \in \mathrm{Rep}(G,X)$, with no assumptions on X, the displacement of a basepoint x_0 by a word $g = h_1 \cdots h_N$ can be estimated, from above, in terms of the number N of letters, and the displacement of the basepoint by each of the letters h_j. This is a consequence of the triangle inequality and the fact the ρ is isometric:

$$d(x_0, \rho(g)(x_0)) \leq \sum_{j=1}^{N} d(x_0, \rho(h_j)(x_0)). \tag{3.1}$$

Corollary 30. *Let $\rho_n \in \mathrm{Rep}(G,X)$, and assume that X is complete and separable. Suppose that there exist a countable generating set $\mathcal{G}$ of G and a basepoint $x_0 \in X$, such that, for each $h \in \mathcal{G}$, $d(x_0, \rho_n(h)(x_0)) \leq C_h$ for some constant C_h. Then, ρ_n contains a convergent subsequence.*

Proof. This is 3.1 and Cantor's Diagonal Argument. `Fill in details.`

□

In the result above it is important to keep the basepoint x_0 fixed, or at least inside a compact of X. For instance, suppose that ρ_n is a convergent sequence of Kleinian representations, and let Ω be the regular set of its limit Kleinian representation. Suppose that $\Omega \neq \emptyset$, and take a sequence $x_n \in \mathbb{H}^3$ converging to some point of Ω fast enough, so $d(x_n, \rho_n(g)(x_n)) \to \infty$. This situation is also covered by Theorem 39.

Anyway, if $\mathrm{Isom}\,X$ is big enough, it is possible to consider the displacements of a sequence $x_n \in X$ of basepoints and, if those are limited, guarantee convergence in $[\mathrm{Rep}(G,X)]$. Understand by "big enough" that the action of $\mathrm{Isom}\,X$ on X is *transitive*: for any $x_1, x_2 \in X$, there exist $\varphi \in \mathrm{Isom}\,X$ such that $\varphi(x_1) = x_2$. This can be replaced by the weaker assumption that there exist a compact $K \subset X$ such that every x can be taken to K by some isometry, but transitivity is enough to us.

Corollary 31. *Let $\rho_n \in \mathrm{Rep}(G,X)$, and assume that X is complete and separable, and that the action of $\mathrm{Isom}\,X$ on X is transitive. Suppose that there exist a countable generating set $\mathcal{G}$ of G, and a sequence of basepoints $x_n \in X$, such that, for each $g \in \mathcal{G}$ and $n \geq 0$, $d(x_n, \rho_n(g)(x_n)) \leq C_g$ for some constant C_g. Then, $[\rho_n]$ contains a subsequence that is convergent in $[\mathrm{Rep}(G,X)]$.*

Proof. Fix a point $x_0 \in X$, and take $\alpha_n \in \mathrm{Isom}\,X$ such that $\alpha_n(x_n) = x_0$, and define the isometric action $\rho'_n(g) = \alpha_n \circ \rho_n(h) \circ \alpha_n^{-1}$. It belongs to $[\rho_n]$ and, for each $g \in G$, $\rho'_n(g)$ moves x_0 the same limited amount that $\rho_n(g)$ moves x_n. So, Corollary 30 applies to ρ'_n, and this finishes the proof. □

3.2 δ-Hyperbolicity and $\mathbb{R}$-trees

[Gro87, GdlH90]

Definition 32 (δ-Hyperbolicity)**.** Let (X, d) be a metric space. For each $x_0 \in X$, the associated *Gromov product* is, for each $x, y \in X$, equal to half the triangle difference of x_0, x and y:

$$(x \cdot y)_{x_0} = \frac{1}{2}(d(x_0, x) + d(x_0, y) - d(x, y)). \tag{3.2}$$

For $\delta \geq 0$, the space X is δ*-hyperbolic* if there exist $x_0 \in X$ such that, for any $x, y, z \in X$:

$$(x \cdot y)_{x_0} \geq \min\{(x \cdot z)_{x_0}, (y \cdot z)_{x_0}\} - \delta. \tag{3.3}$$

Accordingly, a semi-metric on X is δ*-hyperbolic* if the corresponding quotient metric is δ-hyperbolic.

If X is δ-hyperbolic with respect to the basepoint x_0, then it is 2δ-hyperbolic with respect to any other $x \in X$. So, "hyperbolicity',' and "0-hyperbolicity", are properties of the metric space itself, independent of the choice of a basepoint. Also, if d is δ-hyperbolic, then the re-scaled λd, for $\lambda > 0$, is $\lambda\delta$-hyperbolic.

As examples of δ-hyperbolic spaces, consider, in ascending degree of generality, the hyperbolic spaces $\mathbb{H}^n$, and simply connected geodesic metric spaces of negative curvature. In geodesic metric spaces as such, δ-hyperbolicity is equivalent to the existence of a radius $r = r(\delta)$ such that, for every geodesic triangle, the r-neighbourhood of any two of the sides contains the third one. See [BBI01, BH99, Gro07] for the geometric theory on these spaces. Regarding 0-hyperbolic spaces,

Definition 33 ($\mathbb{R}$-tree)**.** An $\mathbb{R}$*-tree* is a metric space (T, d) such that, for every $t_0, t_1 \in T$, the intersection of all connected sets containing t_0 and t_1 is isometric to the interval of length equal to $d(t_0, t_1)$. It is called the *segment* $[t_0, t_1]$.

Every $\mathbb{R}$-tree is a geodesic metric space in which, given two points, there is an unique shortest path in T joining t_0 and t_1. Also, every $\mathbb{R}$-tree is 0-hyperbolic, in accordance with all its geodesic triangles being infinitely thin, as in Figure **??**. For points t_0, t_1 and t_2 in an $\mathbb{R}$-tree, the Gromov product $(t_1 \cdot t_2)_{t_0}$ is equal to the length of $[t_0, t_1] \cap [t_0, t_2]$. So, in particular $(t_1 \cdot t_2)_{t_0} = 0$ if t_1 and t_2 are in opposite sides of t_0. Condition (3.3) is illustrated in Figure **??**.

`Figures: possible relative positions of points in an ℝ-tree. Unusual metric on ℝ².`

First examples of $\mathbb{R}$-trees are simplicial trees. On the other extreme, consider the set $\mathbb{R}^2$ with the metric whose geodesics are depicted on Figure **??**. In particular, it is not clear what a "vertex" of an $\mathbb{R}$-tree should be, unless one admits that these "vertices" can accumulate on each other. A guiding example of $\mathbb{R}$-tree, after Section 4.2, is:

Example 34 ($\mathbb{R}$-trees From Measured Foliations)**.** Let Γ be a Fuchsian group such that $\mathbb{H}^2/\Gamma$ is a compact surface, and let $\mathcal{F}$ be a measured foliation on $\mathbb{H}^2/\Gamma$, in the sense of Section 4.2. By compacity, $\mathcal{F}$ have a finite number of singularities. Assume that none of them is 1-pronged. Lift $\mathcal{F}$ to a measured foliation $\tilde{\mathcal{F}}$ of $\mathbb{H}^2$, and let $d_{\tilde{\mathcal{F}}}$ be the semi-metric associated to the transverse measure of $\mathcal{F}$. Then, the metric quotient $\mathbb{H}^2/d_{\tilde{\mathcal{F}}}$ is an $\mathbb{R}$-tree, and Γ acts on it isometrically. See [Ota96] for a proof.

Lemma 35 ("Connecting The Dots")**.** *Given a 0-hyperbolic semi-metric space (X, d) with at most a countable number of elements, and a fixed basepoint $x_0 \in X$, there exist an $\mathbb{R}$-tree (T, d_T) and an isometric embedding $i : X \to T$ such that:*

1. *No proper subtree of T contains $i(X)$.*
2. *If $j : X \to T'$ is an isometric embedding of X into an $\mathbb{R}$-tree T', then there is a unique isometric embedding $k : T \to T'$ such that $k \circ i = j$. Consequently, T is unique up to isometry.*
3. *If ρ is an isometric action of a group G on X, then there exist an isometric action ρ' of G on T such that, for every $g \in G$ and $x \in X$, $i(\rho(g)(x)) = \rho'(g)(i(x))$.*

Proof. `Fill in details here.`

For each $x \in X$, let I_x be the interval $[0, d(x_0, x)]$ with its standard metric. Then, glue isometrically I_{x_1} to I_{x_2} through their sub-intervals $[0, (x_1 \cdot x_2)_{x_0}]$ (possibly degenerated to the points 0). See Figure **??**. The points $0 \in I_x$ all project to a same $t_0 \in T$, and denote by $x \cdot t_0$ the projection on T of $d(x_0, x) \in I_x$. Notice that, for any $g \in G$, and any $x, x' \in X$, the segments $[x \cdot t_0, x' \cdot t_0]$ and $[\rho(g)(x) \cdot t_0, \rho(g)(x') \cdot t_0]$ have the same lengths, and this suffices to define ρ'.

`Figure: gluing segments.` □

The 0-hyperbolic metrics of interest will appear as limits of sequences of semi-metrics.

Definition 36 (Sequences of Semi-Metrics)**.** A sequence d_n of semi-metrics on a set X is *convergent* if, for any $x_1, x_2 \in X$, the limit $d_\infty(x_1, x_2) = \lim_{n\to\infty} d_n(x_1, x_2)$ is finite. In this case, d_∞ define a semi-metric on X.

Proposition 37. *Let d_n be a sequence of δ_n-hyperbolic semi-metrics on a set X. Suppose that d_n converges to a semi-metric d_∞ and that δ_n converges to $\delta_\infty < \infty$. Then, d_∞ is δ_∞-hyperbolic.*

Proof. Clear from the definition. □

3.3 Divergent Sequences

In view of the results of Section 3.1, now it will be considered sequences $\rho_n \in \mathrm{Rep}(G, X)$ such that, for some sequence of basepoints $x_n \in X$ and some $g \in G$, the displacements $d(x_n, \rho_n(g)(x_n))$ are unlimited. For instance, this is so if ρ_n is not convergent and $x_n = x_0$, for any fixed $x_0 \in X$. Notice that, in this case, by the estimate (3.1), every generating set of G contains some h with this property.

Definition 38 (Arboreal Sequences)**.** A sequence $\rho_n \in \mathrm{Rep}(G, X)$ is *arboreal* if:

1. G has at most a countable number of elements and X is a δ-hyperbolic metric space.
2. There exist sequences $x_n \in X$ and $M_n \to \infty$ such that

$$d_\infty(g_1, g_2) = \lim_{n\to\infty} \frac{d(\rho_n(g_1)(x_n), \rho_n(g_2)(x_n))}{M_n} \tag{3.4}$$

is a non-trivial semi-metric on G. Here, *non-trivial* means that there exist $g_1, g_2 \in G$ such that $d(g_1, g_2) \neq 0$. In particular, the x_n are not globally fixed by the action.

In particular, if ρ_n is convergent, then it is not arboreal with respect to $x_n = x_0$, because $M_n \to \infty$ imply, in this case, that d_∞ is trivial. But a convergent ρ_n can be arboreal for non-constant x_n.

Theorem 39. *If ρ_n is an arboreal sequence, then there exist an $\mathbb{R}$-tree (T, d_T), a basepoint $t_0 \in T$, and a non-trivial isometric action ρ_∞ of G on T, such that*

$$d_\infty(g_1, g_2) = d_T(\rho_\infty(g_1)(t_0), \rho_\infty(g_2)(t_0)), \quad \forall g_1, g_2 \in G, \tag{3.5}$$

where d_∞ is defined by (3.4).

Proof. Let x_n and M_n be as in Definition 38. The following define a sequence of semi-metrics on G:

$$d_n(g_1, g_2) = d(\rho_n(g_1)(x_n), \rho_n(g_2)(x_n)), \quad g_1, g_2 \in G. \tag{3.6}$$

Since d_n/M_n is (δ/M_n)-hyperbolic and $M_n \to \infty$, d_∞ is 0-hyperbolic. Now, for every $g \in G$, $d_\infty(gg_1, gg_2) = d_\infty(g_1, g_2)$, since this is so for each d_n, because each ρ_n is isometric. So, G acts, by multiplication on the left, isometrically on (G, d_∞). The "Connecting The Dots" Lemma 35, then, concludes the proof. □

It is possible that a part of G disappears in the limiting action given by Theorem 39. For instance, if $g \in G$ is such that $d(x_0, \rho_n(g)(x_0))/M_n \to 0$, then the subgroup generated by g acts trivially on the $\mathbb{R}$-tree. In this sense, the worst that can happen is that the limiting action degenerate to a $\mathbb{Z}$-action, for $\mathbb{Z}$ being generated by b_1.

Now, the question is, given ρ_n, wheter or not there exist x_n and M_n as in Definition 38. The answer is "yes", up to passing to a subsequence, for every divergent ρ_n if G is finitely generated (Proposition 41), and for certain divergent ρ_n if G is generated by an infinite, countable, number of elements (Theorem 46). The idea is to use the growth, in the sense of displacements, of a finite set $F \subset G$ along ρ_n to control the growth of the whole group, in order to properly define M_n.

Definition 40 (Maximum Displacement Function)**.** For each $\rho \in \mathrm{Rep}(G, X)$, finite non-empty set $F \subset G$, and $x \in X$, the associated *maximum displacement* is the quantity

$$M(\rho, F, x) = \max_{b \in F} d(x, \rho(b)(x)). \tag{3.7}$$

It is the smallest radius of a ball centered at x that contains its images by the $\rho(b)$, with $b \in F$. The pair (F, x) should be thought as a referential from which the representation ρ will be measured. See Figure **??**. If ρ_n is divergent, then, for some F and any x_0, the sequence $M(\rho_n, F, x_0)$ is unlimited.

`Figure: basepoint and geodesics connecting it to its images by the elements of` F`.`

Proposition 41. *If G is a finitely generated group and X is a δ-hyperbolic metric space, every divergent sequence $\rho_n \in \mathrm{Rep}(G, X)$ contains an arboreal subsequence.*

Proof. `Fill in details.` Let $\mathcal{G}$ be any finite generating set of G, and take $M_n = M(\rho_n, \mathcal{G}, x_0)$ for some fixed x_0. The result follows from the triangle inequality, estimate (3.1), and Cantor's Diagonal Argument. □

So, in order to get arboreal subsequences, it suffices that, for some finite set $F \subset G$ and choice of basepoints $x_n \in X$, for every h in a generating set of G, there exist a non-negative constant C_h, and a sequence of non-negative numbers $C'_{h,n}$, such that

$$d(x_n, \rho_n(h)(x_n)) \le C_h M(\rho_n, F, x_n) + C'_{h,n}, \quad \forall n \ge 0, \tag{3.8}$$

and such that $C'_{h,n}/M(\rho_n, F, x_n)$ is limited above. Proposition 41 is the particular case in which $C_h = 1$ and $C_{h,n} = 0$.

The results above are due to some generalization. Definition 38 and Theorem 39 works, with the same argument, if X is replaced by a sequence X_n of δ_n-hyperbolic spaces such that $\delta_n/M_n \to 0$. The convergent case of Section 3.1 would require the understanding of some limiting space for the group to act on.

And finally, instead of considering isometric actions, one could consider "quasi-isometric" actions, supposing that each $\rho(g)$ is a only a quasi-isometric homeomorphism of X. Assuming uniformly bounded bi-Lipschitz constants over the sequence, the convergent case, and the argument above on the divergent case goes well. The $\mathbb{R}$-tree can be constructed, but it is not clear how to define a quasi-isometric action of G on it. More precisely, we don't know if item 3 of the Connecting The Dots Lemma 35 holds. If so, Theorem 39 holds, and Proposition 41 can be proved with a small adaptation on the estimate of the displacement of the basepoint by a word of the group. `Write estimate here.`

3.4 The Modular Action

`Maybe merge this into the next chapter.`

The results of the previous Sections apply to an interesting class of sequences of isometric actions of a group isomorphic to the free group on an infinite, countable, number of generators. These are obtained by keeping fixed the image of an initial representation of the group, but changing the representation by pre-composing with iterates a group automorphism. The automorphism is supposed to be "realizable" under the representation (Definition 44). Also, the automorphism is supposed to be *generating* (Definition 45). Together, these two assumptions ensures that a finite number of group elements controls the growth of the group over the sequence of representations, and this gives Theorem 46. The concepts are well-behaved under conjugation, so Theorem 48 is obtained.

Definition 42 (Inner and Outer Automorphisms)**.** Let G be a group, and denote by $\operatorname{Aut} G$ the group of automorphisms of G. Automorphisms of the form $g \mapsto hgh^{-1}$ for some $h \in G$ are called *inner automorphisms*, and constitute the normal subgroup $\operatorname{Inn} G$ of $\operatorname{Aut} G$. The associated quotient is the *outer automorphisms* group of G:

$$\operatorname{Out} G = \operatorname{Aut} G / \operatorname{Inn} G. \tag{3.9}$$

So, two automorphisms represents the same outer automorphism if, and only if, they are conjugated by some element of G.

Definition 43 (Modular Action)**.** The formula $f \cdot \rho = \rho \circ f^{-1}$ define an action of $\operatorname{Aut} G$ on $\operatorname{Rep}(G, X)$, which quotient to the *modular action* of $\operatorname{Out} G$ on $[\operatorname{Rep}(G, X)]$.

Definition 44 (Quasi-Isometric Outer Automorphism)**.** An outer automorphism $[f]$ is *quasi-isometric* with respect to $[\rho]$ if there exist a quasi-isometric homeomorphism φ of X such that

$$\rho(f(g)) = \varphi \circ \rho(g) \circ \varphi^{-1}, \quad \forall g \in G. \tag{3.10}$$

If $X = \mathbb{H}^2$ or $X = \mathbb{H}^3$: being quasi-isometric with respect to a Kleinian representation is the same that being induced in homotopy by a quasi-isometric homeomorphism of the orbit space. In this case, φ in equation (3.10) is given by a lift of the homeomorphism. Every quasi-isometric automorphism is type-preserving. And, by the classical Dehn-Nielsen Isomorphism Theorem, if Γ is a finitely generated Fuchsian group, then every type-preserving automorphism of Γ is quasi-isometric.

Definition 45 (Generating Outer Automorphism)**.** Let G be isomorphic to the free group on an infinite, countable, number of generators. A group automorphism $f \in \operatorname{Aut}\Gamma$ is *generating* if there exist a generating set $\mathcal{G}$ of G, and a finite set $F \subset \mathcal{G}$, such that, for each $h \in \mathcal{G}$, $h = f^{\circ k}(b)$ for some $b \in F$ and $k \in \mathbb{Z}$. Notice that f is generating if, and only if, f^{-1} is generating, and that to generate G is a property of $[f] \in \operatorname{Out}\Gamma$. Also, f is generating if, and only if, the mapping torus group $\Gamma \ltimes_f \mathbb{Z}$ is finitely generated.

`We could also not require that` $F \subset \mathcal{G}$.

Theorem 46. *Let G be a group isomorphic to the free group on an infinite, countable, number of generators, and let X be a δ-hyperbolic metric space. Suppose that $f \in \operatorname{Aut} G$ is generating, and quasi-isometric with respect $\rho \in \operatorname{Rep}(G, X)$. If the iterated sequence $\rho_n = \rho \circ f^{-n}$ is divergent, then it contains an arboreal subsequence.*

Besides being quite simple, the next Proposition will be stated and proved separatedly, so we can emphasize its relation to its analogous, Proposition 66, in the context of quasi-Fuchsian representations.

Proposition 47. *If $[f]$ is quasi-isometric with respect to $[\rho]$, then it is quasi-isometric, with the same bi-Lipschitz constant, with respect to every $[\rho_n] = [f]^n \cdot [\rho]$.*

Proof. This is because the automorphism f commutes with itself, and by the equivariance equation (3.10):

$$\rho_n(f(g)) = \rho(f^{-n} \circ f(g)) = \rho(f \circ f^{-n}(g)) = \varphi \circ \rho(f^{-n}(g)) \circ \varphi^{-1} = \varphi \circ \rho_n(g) \circ \varphi^{-1}. \tag{3.11}$$

□

Proof of Theorem 46. Let φ be a quasi-isometric homeomorphism of X such that (3.10) holds. So, for any $k, n \geq 0$, $x_n \in X$, and $g \in G$, equation (3.11) gives:

$$d(x_n, \rho_n(f^{\circ k}(g)(x_n)) = d(x_n, \varphi^k \circ \rho_n(g) \circ \varphi^{-k}(x_n)).$$

Then, denoting by L the bi-Lipschitz constant of φ, using the triangle inequality, and that each

$\rho_n(g)$ is an isometry:

$$\begin{aligned} d(x_n, \rho_n(f^{\circ k}(g)(x_n)) &= d(\varphi^k \circ \varphi^{-k}(x_n), \varphi^k \circ \rho_n(g) \circ \varphi^{-k}(x_n)) & (3.12)\\ &\leq L^k d(\varphi^{-k}(x_n), \rho_n(g) \circ \varphi^{-k}(x_n)) & (3.13)\\ &\leq L^k[d(\varphi^{-k}(x_n), x_n) + d(x_n, \rho_n(g)(x_n)) & (3.14)\\ &\quad +d(\rho_n(g)(x_n), \rho_n(g) \circ \varphi^{-k}(x_n))] & (3.15)\\ &= L^k[d(x_n, \rho_n(g)(x_n)) + 2d(x_n, \varphi^{-k}(x_n))]. & (3.16) \end{aligned}$$

Let $\mathcal{G}$ be a generating set as in Definition 45. Since ρ_n is divergent, there exist $h \in \mathcal{G}$ such that, for any $x_0 \in X$, the sequence $d(x_0, \rho_n(h)(x_0))$ is unlimited. Since $h = f^{\circ k}(b)$ for some $k \in \mathbb{Z}$ and $b \in F$, the estimate above guarantee that $d(x_0, \rho_n(b)(x_0)) \to \infty$. Then, $M(\rho_n, F, x_0) \to \infty$. Also, it follows from the estimate that

$$d(x_0, \rho_n(h)(x_0)) \leq C_h M(\rho_n, F, x_0) + C'_h,$$

for $C_h = L^k$ and $C'_h = 2L^k d(x_0, \varphi^{-k}(x_0))$. Theorem 39, then, stablishes the result. □

Theorem 48. *Let G be a group isomorphic to the free group on an infinite, countable, number of generators, and let X be a complete and separable δ-hyperbolic metric space such that* $\operatorname{Isom} X$ *acts transitivelly on X. Suppose that $[f] \in \operatorname{Out} G$ is generating, and quasi-isometric with respect to $[\rho] \in [\operatorname{Rep}(G, X)]$. Then, either $[\rho]_n = [f]^n \cdot [\rho]$ contains a convergent subsequence, or it contains an arboreal subsequence of representatives.*

Question 49. Which conditions guarantee that an iterated sequence is divergent?

See also the discussion in the end of Section 4.2.

3.5 Further Properties

Now the discussion will be restricted to sequences of Kleinian representations, and correspondent isometric actions on $\mathbb{H}^3$. If a sequence converges, then the limit define a Kleinian representation, by Chuckrow's Theorem 20. On the other hand, if a sequence is arboreal, we'll be able to:

1. Pass the Margulis Lemma 17 to the limiting actions on ℝ-trees, in Proposition 52.
2. Use a bit of the knowledge about how isometries behave when $x \to \infty$, in Proposition 54, and the fact that $\operatorname{Isom} \mathbb{H}^3$ is big enough, to obtain "tight" limiting actions on ℝ-trees in Proposition 55.
3. Understand how translation lengths pass to the limit, in Proposition 55.

These items are related to Example 34, and to the question of which isometric actions of a Fuchsian group on an ℝ-tree are of that form. More precisely, we are thinking about:

Theorem 50 (Skora)**.** *Let Γ be a finitelly generated Fuchsian group, and let ρ be an isometric action of Γ on an ℝ-tree T. Suppose that: ρ is non-trivial, have small arc stabilizers, is minimal, and is such that the translation length of $\rho(\gamma)$ is zero for every parabolic element $\gamma \in \Gamma$. Then, T is isometric to an ℝ-tree of the form discussed in Example 34.*

`This needs fixing: either the example permits punctures, or this statement avoids it.`

The proof of the next results relies on the following technique for approximating distances on an $\mathbb{R}$-tree T associated to an arboreal sequence. For any $t_1, t_2 \in T$, take any $g_1, g_2 \in G$ such that t_1 and t_2 are in the segment $I = [\rho_\infty(g_1)(t_0), \rho_\infty(g_2)(t_0)]$, and let $x_{1,n}$ and $x_{2,n}$ be the points of the geodesic segment $I_n = [\rho_n(g_1)(x_n), \rho_n(g_2)(x_n)]$ that divide it with the same proportions that t_1 and t_2 divide I. Then,

$$d_T(t_1, t_2) = \lim_{n\to\infty} \frac{d(x_{1,n}, x_{2,n})}{M_n}.$$

See [Bes02] for more details.

We emphasize that the next results holds for divergent sequences of Kleinian iterated sequences associated to generating and quasi-isometric automorphisms. In neither of them the group is assumed to be finitely generated. For the proofs, see [Ota96, Bes88, Bes02]. `Write them here.`

Definition 51 (Small Arc Stabilizer)**.** An isometric action of a group G on an $\mathbb{R}$-tree T have *small arc stabilizers* if the stabilizer of any arc in T under the action of G do not contain Abelian subgroups of finite index.

Proposition 52. *If ρ_n is an arboreal sequence of non-elementary Kleinian representations of a group G, then the ρ_∞ given by Theorem 39 have small edge stabilizers.*

Definition 53 (Minimality)**.** An isometric action on an $\mathbb{R}$-tree is *minimal* if every invariant subtree is either a single point or the whole $\mathbb{R}$-tree.

Proposition 54. *Let ρ be a Kleinian representation of a group G. Suppose that a finite set $F \subset G$ contains at least two elements whose images by ρ fix different points at infinity. Then, the function $x \mapsto M(\rho, F, x)$ assumes a global minimum at some $x = x(\rho, F) \in \mathbb{H}^n$.*

Proof. For each $b \in F$, moving x to infinity makes $d(x, \rho(b)(x))$ goes to infinity, except if x tends to a fixed point of $\rho(b)$. In this case, for $b' \in F$ with other fixed points, $d(x, \rho(b')(x))$ goes to infinity, and the maximum in question goes to infinity whenever $x \to \infty$. By continuity, this suffices to prove that the minimum is attained. ☐

Proposition 55. *Let ρ_n be a sequence of non-elementary Kleinian representations of a group G, let $F \subset G$ be a finite set containing elements h_1 and h_2 such that $\rho_n(h_1)$ and $\rho_n(h_2)$ fixes distinct points at infinity, and let $x_n \in \mathbb{H}^3$ be the minimum of $x \mapsto M(\rho_n, F, x)$ given by Proposition 54. If ρ_n is arboreal with respect to x_n and $M_n = M(\rho_n, F, x_n)$, then the ρ_∞ given by Theorem 39 is minimal, and*

$$\ell_T(\rho_\infty(g)) = \lim_{n\to\infty} \frac{\ell(\rho_n(g))}{M_n}, \quad \forall g \in G.$$

Chapter 4

Hyperbolization of Mapping Tori

4.1 Mapping Tori

Definition 56 (Mapping Torus)**.** Let S be a topological surface, oriented, and let φ be an orientation-preserving self-homeomorphism of S. The *associated mapping torus* is the oriented topological 3-manifold M_φ defined as the quotient:

$$M_\varphi = S \times [0,1]/(x,1) \sim (\varphi(x),0). \tag{4.1}$$

The homeomorphism φ is called the *monodromy* of the mapping torus. The topological type of M_φ depends only on the isotopy class of φ: if φ' is isotopic to φ, then $M_{\varphi'}$ is homeomorphic to M_φ.

`Figure: example on a punctured surface, figure eight braid, compact fiber.`

The inclusion $S \to S \times \{0\}$ and the projection $S \times [0,1] \to [0,1]$ pass to the quotient as a fibration

$$S \to M_\varphi \to S^1. \tag{4.2}$$

And every oriented 3-manifold M that fibers over S^1 with an oriented surface S as the fiber is of this form for some φ: cut out a fiber of M to get a $S \times [0,1]$ and define φ as the homeomorphism that glues back $S \times \{0\}$ to $S \times \{1\}$ to produce M.

Question 57 (Hyperbolization of Mapping Tori)**.** Given S and φ, we are interested in finding a Kleinian group Γ such that M_φ is homeomorphic to $\mathbb{H}^3/\Gamma$. This is equivalent to defining on M_φ a complete Riemannian metric of constant sectional curvature equal to -1. In case it exists, the mapping torus M_φ is said to be *hyperbolic*.

Our approach to this Question is to follow the proof of Thurston's celebrated Theorem below for compact surfaces and pseudo-Anosov monodromy, trying to relax the hypothesis and adapt the argument for the case of the infinitely punctured surfaces associated to generalized pseudo-Anosov mondromies. For references and a bit on the complicated development of the proof of this Theorem, see the Introduction.

Theorem 58 (Thurston on Compact Mapping Tori)**.** *Let S be a compact surface, orientable, and of negative Euler characteristic. For each self-homeomorphism φ of S, M_φ is hyperbolic if, and only if, φ is isotopic to a pseudo-Anosov map of S.*

If Γ as in Question 57 exist, it is isomorphic to the fundamental group of M_φ. So, to begin, this fundamental group will be described. Here, S do not need to be compact. Denote $G = \pi_1(S,*)$ for a fixed basepoint $* \in S$, and fix a generator t of $\pi_1(S^1,[0])$, identifying it with $\mathbb{Z}$. Also, denote

$\hat{G} = \pi_1(M_\varphi, [*, 0])$. The short exact sequence of fundamental groups corresponding to the fibration (4.2) is, then:

$$\{e\} \to G \to \hat{G} \to \mathbb{Z} \to \{e\}. \tag{4.3}$$

This sequence, or the van Kampen Theorem, gives the presentation

$$\hat{G} = G \ltimes_f \mathbb{Z} = \langle G, t \mid f(g) = t^{-1} g t \text{ for each } g \in G \rangle, \tag{4.4}$$

where $f = \varphi_\# \in \operatorname{Aut} G$ is induced by φ. Therefore, if M_φ is homeomorphic to $\mathbb{H}^3/\Gamma$ for a Kleinian group Γ, then Γ is isomorphic to $\hat{G}$.

Given G and f, we ask if there exist a Kleinian representation $\hat{\rho}$ of $\hat{G}$. Notice that, in this case, $\hat{\rho}$ restricts to a Kleinian representation of G. Suppose for a moment that such $\hat{\rho}$ exist. Since G is normal in $\hat{G}$, the limit set of $\hat{\rho}(G)$ coincide with the limit set of $\hat{\rho}(\hat{G})$, by Proposition 15. In particular, if S is compact, and so M_φ is compact, the limit set of $\hat{\rho}(G)$ is the whole sphere $\hat{\mathbb{C}}$. This is a very interesting property of $\hat{\rho}$. It implies, for instance, that a connected component of the pre-image in $\mathbb{H}^3$ of a fiber of (4.2) acumulate on every point of $\hat{\mathbb{C}}$. Also, $\hat{\rho}$ gives G a boundary at infinity which is topologically a 2-sphere, instead of the circle given by any Fuchsian representations of G. It is possible to show that there exist a continuous and surjective $\rho(G)$-equivariant map $S^1 \to S^2$ relating both, and this is known as the *Cannon-Thurston Map* [CT07]. See Figure **??** and the discussion before it. We are not going to give precise statements and proofs of this, but it is clarifying to keep this fact in mind along the following constructions.

Back to the general case, recall that $f = \varphi_\#$ is determined by the choice of a homotopy class of paths from $*$ to $\varphi(*)$ (rel. extremities). This is so even if $*$ is a fixed-point of φ. Different choices results in automorphisms conjugated in G, and the homeomorphism φ itself determine $[f] \in \operatorname{Out} G$ (42). Any homeomorphism of S isotopic to φ determine the same outer automorphism, and reciprocally, provided that the correct identification between fundamental groups with different basepoints are made. This is in accordance with the dependence of the topological type of M_φ being only on the isotopy class of φ. Understanding that the *mapping class group* of S is the group of isotopy classes of homeomorphisms of S, an injective group homomorphism of the mapping class group of S on $\operatorname{Out} G$ is well-defined. See [FM12, Hat02] for details.

To assume that S is orientable and have negative Euler characteristic, possibly $-\infty$, ensures that we can assume, without loss of generality, that $S = \mathbb{H}^2/\Gamma$, where Γ is a Fuchsian group, well-defined up to conjugacy in $\mathrm{PSL}_2\,\mathbb{R}$, and isomorphic to G. So, we'll also assume $G = \Gamma$. But recall that our definition of Fuchsian group assumes that $\Lambda_\Gamma = S^1$. If S is compact, this is always the case. But for non-compact S, this imposes a restriction, namelly that $\mathbb{H}^2/\Gamma$ have punctures, but no funnels – see Figure **??**. We require that φ respect this type of non-compacity, and do not open punctures into funnels. For such, it suffices to assume that φ is a quasiconformal, or quasi-isometric, homeomorphism of $\mathbb{H}^2/\Gamma$. For Fuchsian groups arising from generalized pseudo-Anosov maps this is always the case.

Take a lift $\tilde{\varphi}$ of φ, which is a quasiconformal, or quasi-isometric, homeomorphism of $\mathbb{H}^2$. It determines the choice of automorphism f inside its class in $\operatorname{Out}\Gamma$ and, reciprocally, the choice of the automorphism determines a lift, in the sense that they're related by the following *equivariance equation*:

$$f(\gamma)(z) = \tilde{\varphi} \circ \gamma \circ \tilde{\varphi}^{-1}(z), \quad \forall z \in \mathbb{H}^2,\ \gamma \in \Gamma. \tag{4.5}$$

In the perspective of Definition 44, this is to say that $f \in \operatorname{Aut}\Gamma$, and its class $[f] \in \operatorname{Out}\Gamma$, are quasi-isometric, or *quasiconformal*, with respect to the Fuchsian representation id_Γ. In particular, they're *topological*. For finitely generated Γ, the Dehn-Nielsen Isomorphism Theorem stablishes that every type-preserving group automorphism of Γ is topological (see [FM12]). This is known to not be true for infinitely generated Γ. Anyway, thinking about Question 57, the involved group automorphisms are topological a priori.

Notice that equation (4.5) ressembles the relation that define the group $\hat{G}$. For a moment, define $\hat{\rho}(\gamma) = \gamma$, for $\gamma \in \Gamma$, and $\hat{\rho}(t) = \tilde{\varphi}$. Then, by (4.5), $\hat{\rho}$ is a "representation" of $\hat{G}$. But it is not the one we want. A first issue is that $\hat{\rho}$ is an action of $\hat{G}$ on $\mathbb{H}^2$, and not on $\hat{\mathbb{C}}$. But this is no big deal, since the γ's are defined on $\hat{\mathbb{C}}$, and $\tilde{\varphi}$ can be extended so equation (4.5) is true on the whole sphere – see equation (4.8). Now, the γ's are conformal on $\hat{\mathbb{C}}$, but $\tilde{\varphi}$ is not. This points towards looking to this wrong $\hat{\rho}$ as a quasiconformal, or quasi-isometric, representation – see the comments in the end of Section 3.3. This approach have not been pursued. Notice that this kind of representation should not be confused with the "quasiconformal groups" in the sense of Tukia [Tuk86], since the quasiconformal dilatations of iterates of $\tilde{\varphi}$ are growing to infinity.

The strategy that works to tackle Question 57 is to produce convergent sequences Γ_n and $\tilde{\varphi}_n$ of deformations of Γ and $\tilde{\varphi}$, keeping equivariance valid along the sequences, and also for the limits Γ_∞ and $\tilde{\varphi}_\infty$. This is done in a way that $\tilde{\varphi}_\infty$ end up being conformal, and not only quasiconformal, so the attempt to define $\hat{\rho}$ of the last paragraph becomes right. This will be explained in Sections 4.3 and 4.4. Before that, the meaning of "pseudo-Anosov" is in order.

4.2 Generalized pseudo-Anosov Maps

Classical pseudo-Anosov maps were introduced by W. Thurston as canonical representatives of aperiodic and irreducible isotopy classes of homeomorphisms of a surface of finite topological type. A lot is known about them nowadays, and standard material on the Nielsen-Thurston Classification include [Thu88, Ber78, FLP79, CB88, BH95]; while [Hub06, FM12] are more recent references, where an extended bibliography can be found. Their property that passes to their generalized version, in the sense introduced by A. de Carvalho [dC05, dCH04], is the simulteneous expanding and contracting dynamical behaviour, in the fashion of Smale's Horseshoe and Markov Partitions. See Figure **??**. They should not be thought as canonical representatives of homotopy classes and, in fact, a lot of examples are defined on topological 2-spheres. Anyway, puncturing orbits brings back some topology, as we shall see.

`Figure: simplified picture showing the action of a pA.`

`Define measured foliation.`

Definition 59 (Generalized pseudo-Anosov Map)**.** A *generalized pseudo-Anosov map* is a homeomorphism ϕ of a surface S such that:

1. There exist a finite ϕ-invariant set $\Sigma \subset S$, and a pair of ϕ-invariant transverse measured foliations on $S \setminus \Sigma$, such that ϕ expands one of the foliations by a *multiplier* $\lambda > 1$, and contracts the other by $1/\lambda$ (see Figure **??**).

2. The foliations can have singularities of the types on Figure 4.1, and an infinite number of singularities is allowed, provided that they accumulate only on the finite set Σ (see Figure 4.2).

If $\Sigma = \emptyset$ and there are no 1-pronged singularities, ϕ is a non-generalized, or classical, pseudo-Anosov map.

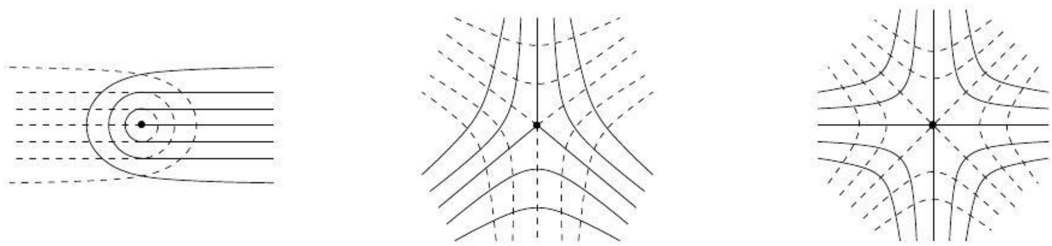

Figure 4.1: *Singularities admited in the foliations associated to a generalized pseudo-Anosov map. One of the foliations is drawn in full lines, and the other on dotted lines. Any finite number $\neq 2$ of leafs meeting in a singularity is permited, and this number is called the* number of prongs *of the singularity.*

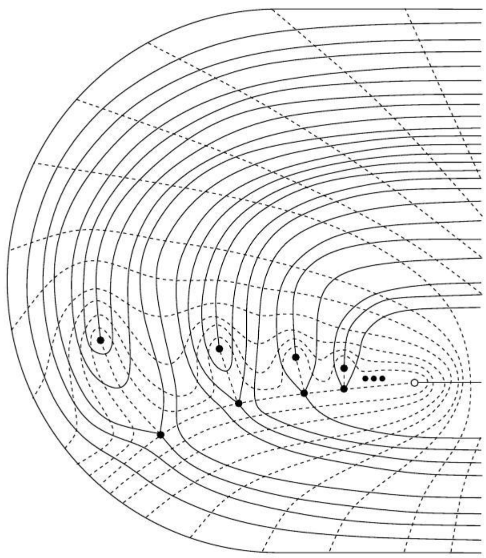

Figure 4.2: 1 *and* 3*-pronged singularities accumulating in one point.*

Example 60 (The Tight Horseshoe)**.** Following Figure 4.3, take the unit square, and transform it linearly in a rectangular stripe with height $1/2$ and width 2. Then, cut along the central vertical of the stripe, and glue isometrically the top sides of the obtained rectangles as indicated. In order that this define a homeomorphism of a Hausdorff topological space, points of the square's frontier need to be identified as in Figure 4.4. Dotted lines outside the square connect points of its frontier that are identified to each other, and an infinite number of foldings centered at the $\bullet$ points accumulate on the inferior vertex on the left. The equivalence class of this vertex are the points marked with $\times$. By a classical theorem of R. Moore on monotone and upper semi-continuous decompositions of the 2-sphere, the quotient S is topologically a 2-sphere (see [dCH10, dCH12, dCH11] for details). The *tight horseshoe* is the self-homeomorphism of S obtained by this construction, and is a generalized pseudo-Anosov map. The horizontal and vertical foliations of the square, with the euclidean length, project to the invariant transverse measured foliations on $S \setminus \{\times\}$. The horizontal one is expanded, while the vertical is contracted. The multiplier is equal to 2. The points $\bullet$ are 1-pronged singularities accumulating on the point $\times$.

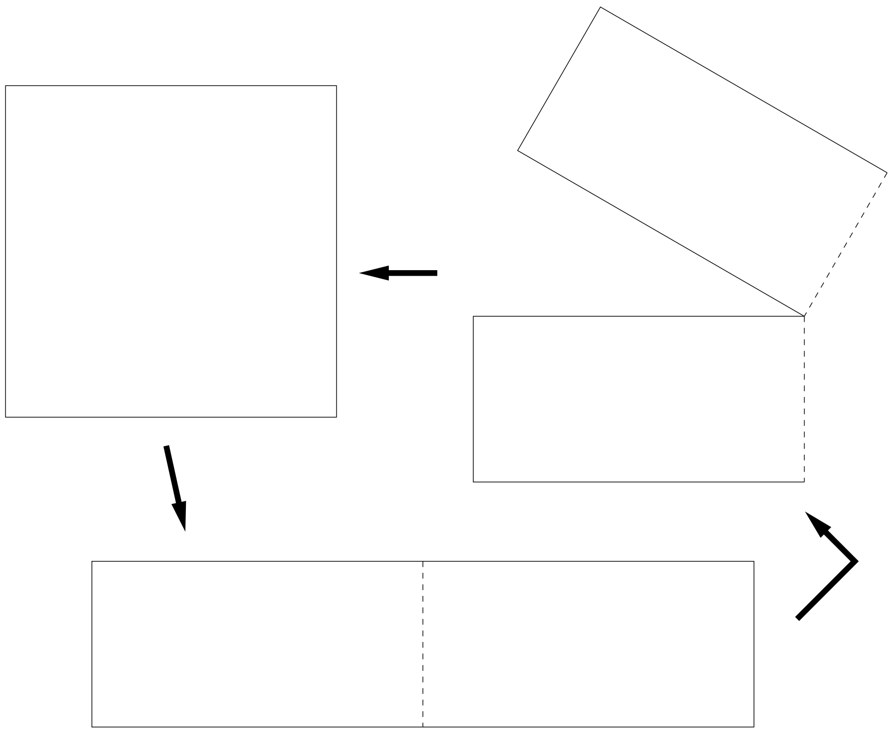

Figure 4.3: *The definition of the Tight Horseshoe map.*

Let ϕ be the tight horseshoe constructed above. The euclidean metric of the square project to a complete geodesic metric on S, called the *paper model of* ϕ. This metric is conic-flat on $S\setminus\{\times\}$, with conical angle π around each $\bullet$ point. In view of the Gauss-Bonnet Theorem on Compact Surfaces, applied to S, the point $\times$ must carry infinite negative curvature. A finer understanding of how S compactifies $S\setminus\{\times\}$, in the metric sense, should appear in the future. For instance, notice that no geodesic ray of the square starting at the inferior vertex on the left projects to a geodesic of S. In fact, the projections do not define directions of $\times$. In [Ber12], a certain *metric space of directions* of $\times$ is recognized as being the real line, which goes along $\times$ having an infinite total angle around it.

From the conformal point of view, the paper model define a conformal structure on $S\setminus\{\times\}$, by correcting to 2π the total angle around each $\bullet$, using $z\mapsto z^2$ to define coordinate charts. Then, ϕ is a λ^2-quasiconformal map of this structure, for $\lambda = 2$. In [dC05, dCH04, dCH12, dCH11] it is proved that this conformal structure on $S\setminus\{\times\}$ extends uniquely to a conformal structure on S. So, after uniformizing, S is the Riemann sphere $\hat{\mathbb{C}}$, of which ϕ is a λ^2-quasiconformal homeomorphism.

Define S^* as the open subsurface of S obtained by removing $\times$ and the $\bullet$ points. Since they constitute a ϕ-invariant set, ϕ is a homeomorphism of S^*. For a fixed basepoint $*\in S^*$, the loops based at $*$ that go one time around each $\bullet$ generate the fundamental group $G=\pi_1(S^*,*)$, fixing an isomorphism of G onto the free group on an infinite, countable, number of generators. Fix an automorphism $f=\phi_\#$ of G induced by ϕ, and name the generators h_i in a way that $f(h_i)=h_{i+1}$ except for a finite number of h_i. See Figure **??**.

`Figure: homotopy of the tight horseshoe - on the square, topologically`

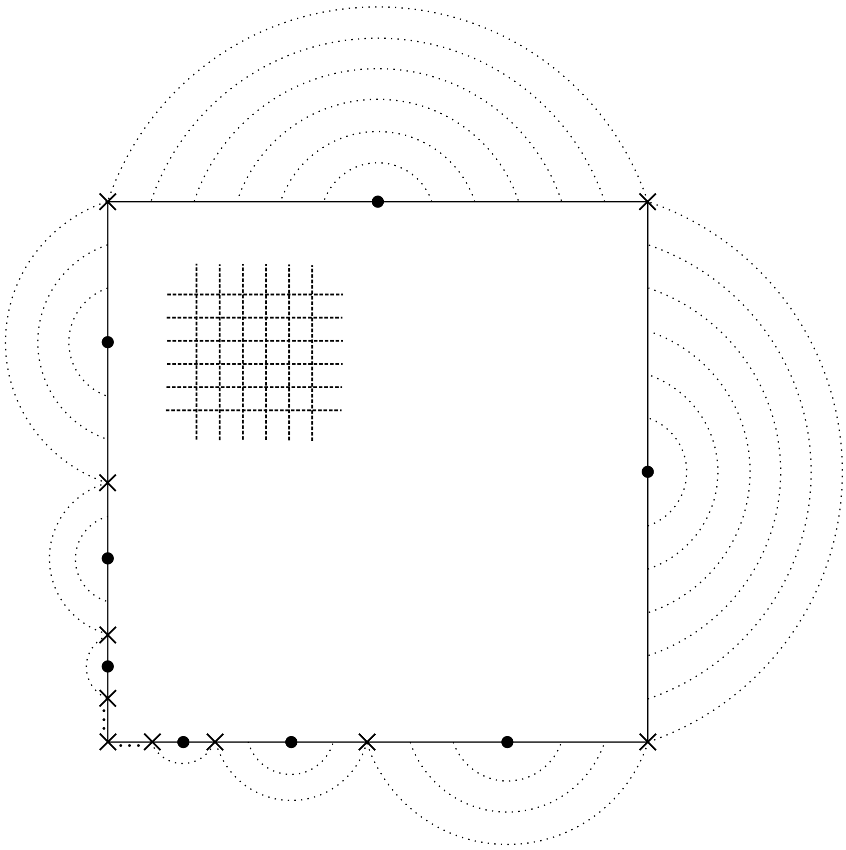

Figure 4.4: *The domain of the Tight Horseshoe map, and its transverse measured foliations. The • points are* 1*-pronged singularities that accumulate on the* × *point.*

on a sphere, and the braid.

So, there exist a finite set $F = \{h_1, \ldots, h_N\}$ such that every generator is equal to $f^k(h_i)$ for some $i = 1, \ldots, N$ and $k \in \mathbb{Z}$. This is preciselly to say that f is generating, in the sense of Definition 45. Notice that, besides G being free on an infinite, countable, number of generators, the finite set $F \cup \{t\}$ generate the associated mapping torus group $\hat{G} = G \ltimes_f \mathbb{Z}$. For future statements, the following will be usefull:

Definition 61 (Generating Homeomorphism)**.** Let S be a surface with fundamental group isomorphic to the free group on an infinite, countable, number of generators. A homeomorphism of S is *generating* if the automorphism it induces in homotopy is generating in the sense of Definition 45.

As an open Riemann subsurface of $\hat{\mathbb{C}}$, S^* is also a Riemann surface, of which ϕ is a λ^2-quasiconformal homeomorphism. Since the number of points removed is greater than 2, after uniformizing, S^* determine the conjugacy class of Fuchsian group Γ isomorphic to G. Identifying $S^* = \mathbb{H}^2/\Gamma$, the tight horseshoe ϕ is in the context of the last section. Figure **??** gives a sketch of a fundamental domain of the associated action on $\mathbb{H}^2$. The generators of G defined above correspond to parabolic elements γ_i of Γ, since their free homotopy classes contain arbitrarily short representatives, namelly the ones that shrinks to the corresponding puncture. Any lift $\tilde{\phi}$ of ϕ takes the conjugacy class $[\gamma_i]$ to $[\gamma_{i+1}]$.

Not much is known about the length spectrum of Γ, but this should appear in the future. For instance, changing to hyperbolic generators related to each other as the parabolic above, the equivariance equation (4.5) opens the possibility of relating their sizes by the bi-Lipschitz constant of $\tilde{\phi}$, and maybe this end-up being a good first step on understanding the full length spectrum of Γ. Anyway, this has not been pursued yet.

`Figure: a fundamental domain and the "flute surface" of the tight horseshoe.`

In [dCH04], a big family of generalized pseudo-Anosov maps is constructed from graph endomorphisms using Peron-Frobenius Theory. The tight horseshoe is the particular case that originates from the graph endomorphism of Figure **??**, which is same that the tent map of $[0,1]$ with slope ± 2 and critical point $1/2$. For another example, consider Figures **??** and **??**. The description above for the tight horseshoe works as a program on every known generalized pseudo-Anosov map considered in [dC05, dCH04, dCH12, dCH11]. This includes realization as quasiconformal homeomorphisms of Riemann surfaces, and being puncturable in a way that produces generating automorphisms in homotopy.

`Figures: other gpA.`

Let's put things in the terms of Chapter 3. Consider a classical pseudo-Anosov homeomorphism ϕ, realized as a quasiconformal homeomorphism of a compact $\mathbb{H}^2/\Gamma$, and let $f \in \operatorname{Aut}\Gamma$ be induced in homotopy by ϕ. The iterated sequence of Fuchsian representations of Γ defined by $\rho_n(\gamma) = f^{-n}(\gamma)$ is known to be divergent and, since Γ is finitely generated, it is arboreal. Therefore, possibly passing to a subsequence, it determine an isometric action of Γ on an $\mathbb{R}$-tree. This can be shown to be the same that the one obtained, as in Example 34, from the contracting foliation of ϕ. The same reasoning applied to ϕ^{-1} gives the expanding foliation. For generalized pseudo-Anosov maps, a first question is: does the iterated sequences associated to a generalized pseudo-Anosov and its inverse diverge? See also Question **??**. If so, by the generating property, Proposition 46 establishes that the sequences are arboreal, and therefore determine a pair of isometric actions on $\mathbb{R}$-trees. A second question is, then, if these coincide with the ones arising from the foliations.

Starting from an arbitrary quasiconformal map of $\mathbb{H}^2/\Gamma$, and looking to the iterated sequences of Fuchsian representations associated to the map and its inverse, gives an approach to the Nielsen-Thurston Classification mentioned in the begining of the Section. Suppose that these sequences diverge, and assume that $\mathbb{H}^2/\Gamma$ is finitely generated. Then, the sequences are arboreal, and determine isometric actions on $\mathbb{R}$-trees that are in the conditions of Skora's Theorem 50. Therefore, they determine a pair of measured foliations on $\mathbb{H}^2/\Gamma$. This, in fact, characterizes pseudo-Anosov homeomorphisms in terms of their modular actions, and is a glimpse of Thurston's compactification of the modular action on the Teichmüller space of Γ, which also provides lots of other information, including the Nielsen-Thurston Classification. A generalized Skora's Theorem is still missing but, when proved, should make possible to reproduce this scenario for generalized pseudo-Anosov.

In the next section the sequences of deformations of Γ and ϕ mentioned in the end of Section 4.1 will be defined. They are obtained by lifting to the upper half-plane $\mathbb{H}^2_+$ the contracting invariant foliation of ϕ, lifting to the lower half-plane $\mathbb{H}^2_-$ the expanding, and then bending simultenously both by the iterates ϕ^n and ϕ^{-n}. See Figures **??** and **??**. This is achieved by a sequence of quasiconformal maps w_n of $\hat{\mathbb{C}}$, that deform S^1 quite strongly into quasicircles (see Figures **??**). For a topological picture of what should be expected as a limit of this, when the lifts of the foliations have been "totally" contracted, take the quotient of the sphere by the decomposition whose elements are the leafs of the lifts. By the same theorem of R. Moore mentioned above, the quotient is a topological sphere. The circle S^1 is a fundamental domain of this decomposition, and its projection onto the quotient is the Cannon-Thurston Map mentioned in last section. This have been fully proved in

[CT07] for classical pseudo-Anosov maps in compact surfaces, and have been proved for punctured surfaces much later, in [Bow07]. It is not known wheter or not this should be true for generalized pseudo-Anosov maps.

`Figure: The Cannon-Thurston map.`

4.3 Quasi-Fuchsian Reresentations

Definition 62 (Quasi-Fuchsian Representation)**.** Let Γ be a Fuchsian group, and let $\mathrm{QF}(\Gamma)$ be the set of *quasi-Fuchsian representations of* Γ, which are the Kleinian representations ρ of Γ of the form

$$\rho(\gamma) = w \circ \gamma \circ w^{-1}, \quad \gamma \in \Gamma, \tag{4.6}$$

for some quasiconformal map w of $\hat{\mathbb{C}}$.

Our definition of Fuchsian group assumes that $\Lambda_\Gamma = S^1$. Therefore, for each $\rho \in \mathrm{QF}(\Gamma)$, as in (4.6), the limit set Λ of $\rho(\Gamma)$ is equal to $w(S^1)$, which is the homeomorphic image of S^1 by a quasiconformal map. This is, by definition, a *quasicircle* – see Figure **??**. Back on the context of Section 4.1, and assuming that $\mathbb{H}^2/\Gamma$ is compact, if $\hat{\rho}$ is a Kleinian representation of $\hat{G}$, then the restriction of $\hat{\rho}$ to the surface group G is not quasi-Fuchsian, since $\Lambda \neq \hat{\mathbb{C}}$, no matter the amount a quasicircle can fill up the sphere. But, still, this restricted representation can be obtained as the limit of a sequence of quasi-Fuchsian representations. Since Λ contains more than 2 points, every quasi-Fuchsian representation is non-elementary, so Chucrow's Theorem 20 applies, and convergent sequences of quasi-Fuchsian representations define limiting Kleinian representations. Those are not necessarilly quasi-Fuchsian, and this is what opens the possibility of finding as such limit the restriction to G of the wanted $\hat{\rho}$. This is provided, for compact $\mathbb{H}^2/\Gamma$ and classical pseudo-Anosov monodromy, by Thurston's Double Limit Theorem 68, which is one of the main cores of Theorem 58. This is the way to go.

`Figure: quasi-circles`

Quasi-Fuchsian representations can be written on their *Ahlfors-Bers Coordinates*. We emphasize that here it is not needed to assume that Γ is finitely generated. Let $\rho \in \mathrm{QF}(\Gamma)$ be of the form (4.6). The regular set Ω of $\rho(\Gamma)$ have two connected components, namelly $\Omega^+ = w(\mathbb{H}^2_+)$ and $\Omega^- = w(\mathbb{H}^2_-)$. Each is invariant by the action of $\rho(\Gamma)$. Also, they're open and simply-connected subsets of $\hat{\mathbb{C}}$ and, by the Riemann Mapping Theorem, there exist conformal homeomorphisms $u^+ : \Omega^+ \to \mathbb{H}^2_+$ and $u^- : \Omega^- \to \mathbb{H}^2_-$. By conjugation with them, a pair (ρ^+, ρ^-) of Fuchsian representations of Γ is defined:

$$\rho^+(\gamma) = u^+ \circ \rho(\gamma) \circ (u^+)^{-1} \text{ and } \rho^-(\gamma) = u^- \circ \rho(\gamma) \circ (u^-)^{-1}, \quad \gamma \in \Gamma.$$

The pair (ρ^+, ρ^-) is called *Ahlfors-Bers Coordinates of* ρ. `Discreteness?`

On the other direction, take a pair (ρ^+, ρ^-) of Fuchsian representations of Γ of the form

$$\rho^+(\gamma) = w^+ \circ \gamma \circ (w^+)^{-1} \text{ and } \rho^-(\gamma) = w^- \circ \gamma \circ (w^-)^{-1}, \quad \gamma \in \Gamma, \tag{4.7}$$

for quasiconformal maps w^+ of $\mathbb{H}^2_+$ and w^- of $\mathbb{H}^2_-$. Representations ρ^+ and ρ^- of this form should be thought as conformal actions of Γ on $\mathbb{H}^2_+$ and $\mathbb{H}^2_-$. Define the Beltrami form μ as the Beltrami form of w^+ on $\mathbb{H}^2_+$, and as the Beltrami form of w^- on $\mathbb{H}^2_-$. It's defined on the full-measure set $\hat{\mathbb{C}} \setminus S^1$, and $||\mu|| < 1$. Therefore, the Measurable Riemann Mapping Theorem 24 applies, and there exist a quasiconformal map w of $\hat{\mathbb{C}}$ that solves the Beltrami equation associated to μ. This w

determine, by (4.6), a quasi-Fuchsian representation of Γ whose Ahlfors-Bers coordinates are ρ^+ and ρ^-. `Discreteness? Conformality?`

Both the Riemann Mapping Theorem, and the Measurable Riemann Mapping Theorem, give maps well-defined up to post-composition with conformal maps. Taking this into account, one gets a bijection between $[\mathrm{QF}(\Gamma)]$ and the two-fold product of the space of conjugacy classes of Fuchsian representations of Γ of the form 4.7. Here we're not interested in holomorphic properties of this bijection, but they can be obtained by taking, instead of these conjugacy classes, the sets of representatives normalized by fixed group elements, in the sense of Definition 7. Using holomorphic dependence on parameters on the Measurable Riemann Mapping Theorem, making it be called the Ahlfors-Bers Theorem, the *Ahlfors-Bers Isomorphism* is defined.

For a quasi-Fuchsian representation $\rho = (\rho^+, \rho^-)$, the following result bounds the geometry of $\mathbb{H}^3/\rho(\Gamma)$ by the geometries of $\mathbb{H}^2_+/\rho^+(\Gamma)$ and $\mathbb{H}^2_-/\rho^-(\Gamma)$. It's valid for every quasi-Fuchsian group, not necessarily finitely generated. The proof is by a clever argument on the modulus of annuli, see – [Ota96, MT98].

Theorem 63 (Bers' Inequality)**.** *Let Γ be any Fuchsian group. For any quasi-Fuchsian representation $\rho = (\rho^+, \rho^-)$ of Γ, and any hyperbolic $\gamma \in \Gamma$,*

$$\frac{1}{\ell(\rho^+(\gamma))} + \frac{1}{\ell(\rho^-(\gamma))} \leq \frac{2}{\ell(\rho(\gamma))},$$

where ℓ denote the translation length. Therefore, $\ell(\rho(\gamma)) < 2\min\{\ell(\rho^+(\gamma)), \ell(\rho^-(\gamma))\}$.

Corollary 64. *If a sequence $\rho_n \in \mathrm{QF}(\Gamma)$ is divergent, then its Ahlfors-Bers Coordinates ρ_n^+ and ρ_n^- are also divergent.*

On the contrary, the Ahlfors-Bers Coordinates can diverge with ρ_n being convergent, and this is precicelly what happens for certain sequences in $\mathrm{QF}(\Gamma)$, due to Thurston's Double Limit Theorem 68. These "certain sequences" include the ones that we are interested, that will now finally be defined.

Definition 65 (Doubly Iterated Sequence)**.** Let Γ be any Fuchsian group, and let $f \in \mathrm{Aut}\,\Gamma$. The *doubly iterated sequence associated to f* is the sequence ρ_n of quasi-Fuchsian representations of Γ defined on Ahlfors-Bers Coordinates by $\rho_n = (f^n, f^{-n})$. If f is induced in homotopy by a homeomorphism φ of $\mathbb{H}^2/\Gamma$, ρ_n is called *associated to φ*. Notice that, in this definition, instead of f we could consider a pair of automorphisms of Γ, but things will not be considered in such generality.

`Figure: the doubly-iterated sequence.`

Consider the doubly iterated sequence ρ_n associated to a K-quasiconformal homeomorphism φ of $\mathbb{H}^2/\Gamma$. The involved objects are summarized on Figure **??**. Lift φ to $\tilde{\varphi}$, which is a K-quasiconformal homeomorphism of $\mathbb{H}^2$, and take $f = \tilde{\varphi}_\#$. The equivariance equation (4.5) is satisfied. Extend $\tilde{\varphi}$ to S^1, by continuity, and to $\mathbb{H}^2_-$ by conjugation with complex conjugation $z \mapsto \overline{z}$, turning $\tilde{\varphi}$ into a K-quasiconformal map of $\hat{\mathbb{C}}$. The extended $\tilde{\varphi}$ satisfies the same equivariance equation, but now on the whole $\hat{\mathbb{C}}$:

$$f(\gamma)(z) = \tilde{\varphi} \circ \gamma \circ \tilde{\varphi}^{-1}(z), \quad \forall z \in \hat{\mathbb{C}},\, \gamma \in \Gamma. \tag{4.8}$$

Denoting the Ahlfors-Bers Coordinates of ρ_n by (ρ_n^+, ρ_n^-):

$$\rho_n^+(\gamma) = f^n(\gamma) = \tilde{\varphi}^n \circ \gamma \circ \tilde{\varphi}^{-n} \;\text{ and }\; \rho_n^-(\gamma) = f^{-n}(\gamma) = \tilde{\varphi}^n \circ \gamma \circ \tilde{\varphi}^{-n}, \quad \gamma \in \Gamma.$$

Then, by the construction above,

$$\rho_n(\gamma) = w_n \circ \gamma \circ w_n^{-1}, \quad \gamma \in \Gamma, \tag{4.9}$$

where w_n is a quasiconformal map of $\hat{\mathbb{C}}$ whose Beltrami form is the one of $\tilde{\varphi}^n$ on $\mathbb{H}^2_+$, and of $\tilde{\varphi}^{-n}$ on $\mathbb{H}^2_-$. If φ is a pseudo-Anosov homeomorphism (classical or generalized), these Beltrami forms are aligned with the lifts of leafs of the the invariant measured foliations, and ρ_n is the sequence of deformations obtained by bending them, as claimed in the ends of Sections 4.1 and 4.2. Regarding the correspondent sequence of deformations of $\tilde{\varphi}$, define

$$\tilde{\varphi}_n = w_n \circ \tilde{\varphi} \circ w_n^{-1}. \tag{4.10}$$

And notice that, as claimed, the equivariance equation (4.8) is valid along the sequence:

$$\rho_n(f(\gamma)) = \tilde{\varphi}_n \circ \gamma \circ \tilde{\varphi}_n^{-1}, \quad \gamma \in \Gamma. \tag{4.11}$$

The next result should be compared to Proposition 47.

Proposition 66. *Let Γ be any Fuchsian group, and let φ be a K-quasiconformal homeomorphism of $\mathbb{H}^2/\Gamma$. If $\tilde{\varphi}_n$ is defined, as above, by (4.10), then $\tilde{\varphi}_n$ is K-quasiconformal for every $n \geq 0$.*

Proof. `Fill in details.` This can also be checked by a computation with Beltrami forms, but the following argument on which ellipses are taken to circles is enlightening. See Figure **??**. □

As a consequence of Proposition 66 and the usual technique for extracting convergent subsequences of sequences of quasiconformal maps, Theorem 25, one gets:

Proposition 67. *Let Γ be a Fuchsian group, and let $\rho_n \in \mathrm{QF}(\Gamma)$ be the iterated sequence associated to a K-quasiconformal homeomorphism φ of $\mathbb{H}^2/\Gamma$. Suppose ρ_n is normalized, and that $\tilde{\varphi}_n$ is defined, as above, by (4.10). If ρ_n is convergent, then $\tilde{\varphi}_n$ contains a subsequence that converges to a K-quasiconformal homeomorphism $\tilde{\varphi}_\infty$ of $\hat{\mathbb{C}}$.*

Proof. Recall what "normalized" means (Definition 7). To say that $\rho_0 = \mathrm{id}_\Gamma$ is normalized is to say that Γ contains elements γ_0, γ_1 and γ_∞ whose attracting fixed points are 0, 1 and ∞. This is a matter of chosing Γ inside its conjugation class in $\mathrm{PSL}_2\,\mathbb{R}$. And to say that ρ_n is normalized, for $n \geq 1$, is to say that $\rho_n(\gamma_0)$, $\rho_n(\gamma_1)$ and $\rho_n(\gamma_\infty)$ also have 0, 1 and ∞ as its attracting fixed points. This is obtained by taking w_n in (4.9) as the solution of the correspondent Beltrami equation that fixes 0, 1 and ∞.

By the equivariance equation (4.11), $\tilde{\varphi}_n(0)$, $\tilde{\varphi}_n(1)$ and $\tilde{\varphi}_n(\infty)$ are the attracting fixed points of $\rho_n(f(\gamma_0))$, $\rho_n(f(\gamma_1))$ and $\rho_n(f(\gamma_\infty))$, for every $n \geq 0$. As $n \to \infty$, these points converge to the atrracting fixed points of $\rho_\infty(f(\gamma_0))$, $\rho_\infty(f(\gamma_1))$ and $\rho_\infty(f(\gamma_\infty))$. By Chuckrow's Theorem 20, these are distinct, since the limiting representation is non-elementary. The result follows, then, from Theorem 25. □

By continuity, if the limits of Proposition 67 exist, then

$$\rho_\infty(f(\gamma)) = \tilde{\varphi}_\infty \circ \rho_\infty(\gamma) \circ \tilde{\varphi}_\infty^{-1}. \tag{4.12}$$

Recall, from the end of Section 4.1, that the main goal is to define a Kleinian representation $\hat{\rho}$ of $\hat{G} = \Gamma \ltimes_f \mathbb{Z}$ by setting $\hat{\rho}(\gamma) = \rho_\infty(\gamma)$ and $\hat{\rho}(t) = \tilde{\varphi}_\infty$. Equation (4.12) is a sign that this will be possible and, from here, the discussion splits in: proving the existence of ρ_∞ and, provided this, proving that $\tilde{\varphi}$ is conformal, and also that our candidate $\hat{\rho}$ is really Kleinian, namelly that it is discrete. The second part will be briefly discussed on Section 4.4. Regarding the first one,

Theorem 68 (Thurston's Double Limit – Particular Case)**.** *Let Γ be a Fuchsian group, and let ρ_n be the doubly iterated sequence associated to a quasiconformal homeomorphism ϕ of $\mathbb{H}^2/\Gamma$. Suppose that ρ_n is normalized, that $\mathbb{H}^2/\Gamma$ is compact, and that ϕ is a classical pseudo-Anosov homeomorphism. Then, ρ_n contains a convergent subsequence.*

The proof of this Theorem, that we are not going to give, is by contradiction. Suppose that ρ_n diverges. Then, by Corollary 64, its Ahlfors-Bers Coordinates ρ_n^+ and ρ_n^- also diverge. In the conditions of Theorem 68, these sequences can then be looked from the perspective of Thurston's compactification of the modular action on the Teichmüller space, which provides information on how the divergence happens, namelly on how these sequences "converge" to the pair of measured foliations of ϕ. Also, Γ is finitely generated and, as a sequence of isometric actions of Γ on $\mathbb{H}^3$, ρ_n is arboreal (Section 3.3. Therefore, it determine an isometric action on an $\mathbb{R}$-tree. This action is in the conditions of Skora's Theorem, that gives third measured foliation on $\mathbb{H}^2/\Gamma$. A cautious look to these three measured foliations, with the fact that two of them "fills" $\mathbb{H}^2/\Gamma$, provides the contradiction.

Now, what we want is to get Theorem 68 for Γ associated to punctured generalized pseudo-Anosov maps, using the generating hypothesis. In this direction, Proposition 46 stablishes that ρ_n^+ and ρ_n^-, as isometric actions of Γ on $\mathbb{H}^2_+$ and $\mathbb{H}^2_-$, are arboreal. Regarding ρ_n,

Proposition 69. *Let Γ be a Fuchsian group isomorphic to the free group on an infinite, countable, number of generators, and let ρ_n be the doubly iterated sequence associated to a quasiconformal homeomorphism φ of $\mathbb{H}^2/\Gamma$. Suppose that ρ_n is divergent, and that φ is generating. Then, as a sequence of isometric actions of Γ on $\mathbb{H}^3$, ρ_n contains an arboreal subsequence.*

4.4 How to Finish the Hyperbolization

Theorem 70 (Sullivan Rigidity)**.** *Let Γ be a Kleinian group. Suppose that $\Lambda_\Gamma = \hat{\mathbb{C}}$, and let μ be a Γ-invariant Beltrami form defined on a positive-measure borelian of $\hat{\mathbb{C}}$. Then, $\mu = 0$.*

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