Estimativas *a priori* para jogos de campo médio com dinâmica populacional logística

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Resumo

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Jogos de campo médio são sistemas acoplados de equações diferenciais parciais, uma equação de Hamilton-Jacobi para a *função valor* dos agentes e uma equação de Fokker-Planck para a densidade dos agentes. Tradicionalmente, a última equação é adjunta à linearização da primeira. Uma vez que a equação de Fokker-Planck modela uma dinâmica populacional, nós introduzimos características naturais como semeadura e nascimento e também taxas de mortalidade não-lineares. Neste trabalho analizamos um jogo de campo médio estacionário, ilustrando várias técnicas para obter a regularidade *a priori* das soluções nesta classe de sistemas. Sistemas estes que apresentam dinâmica logística com semeadura. O sistema de dimensão um é estudado separadamente num contexto simplificado.

Palavras-chave: Jogos de campo médio, Hamilton-Jacobi, Fokker-Planck, regularidade, método adjunto.

Abstract

RIBEIRO, R. L. ***A priori* estimates for mean field games with logistic populational dynamics**. 2013. 58f. Tese (Doutorado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2013.

A mean field game is a coupled system of partial differential equations, a Hamilton-Jacobi equation for the value function of agents and a Fokker-Planck equation for the density of agents. Traditionally, the latter equation is adjoint to the linearization of the former. Since the Fokker-Plank equation models a populational dynamic, we introduce natural features such as seeding and birth, and non-linear death rates. In this thesis we analyze a stationary mean field game, illustrating various techniques to obtain *a priori* regularity of solutions in this class of systems. This system shows logistic dynamics with seeding. The one dimensional system is studied separately in a simplified context.

Keywords: mean field games, Hamilton-Jacobi, Fokker-Planck, regularity, adjoint method.

Contents

Introduction

Mean field games is a recent and fast growing area of research. Started in the engineering setting by Peter Caines and his co-workers [HMC06, HCM07], and, independently and around the same time, by Pierre Louis Lions and Jean Michel Lasry [LL06a, LL06b, LL07a, LL07b]. The study of mean field games is done in an attempt to comprehend the effective behavior of systems comprising of large numbers of identical rational agents whose interactions are assumed to be symmetric. The rationality is encoded as a consequence of the optimization performed by each of them. In doing so, individual players handle the influence of other players through their statistical properties, such as mean geographical distribution or even mean velocity field. This results from ideas in statistical physics applied to dynamic games. One major advantage in utilizing these models is that, instead of having one equation for each player, we model the interactions between a player and the mass of other players as if there were a continuum of players, hence dealing with a system of only two equations (three in the extended case).

Literature in this field grows fast and we point out that good, now classical, references in mean field games and its applications are the survey [LLG10b] and the lecture notes [Car11] based on [Lio11]. Economic growth theory [LLG10a], design of environmental policies [LST10] and dynamics of pedestrian crowds [BDFMW13] are a few examples of areas in which applications can already be found. Numerical methods have been dealt with in the survey [Ach13] and, with more detail and specificity in [ACD10, LST10, ACCD12, AP12, CS12]. Also, systems in wich agents have finite possible states were studied in [GMS10], in the discrete time case and in [GMS11] in the continuous time case. We believe that mean field games will play an important role in economics and demographics because in many applications there is a very large number of indistinguishable agents which behave rationally and non-cooperatively.

A natural application of mean field games are problems in population dynamics (see [DFMPW11] and references therein) with, possibly non-linear, birth and death rates as well as seeding or harvesting effects, since the Fokker-Plank equation models a population dynamic through the transport of densities. The system we analyze shows logistic dynamics with seeding.

Heuristic derivation of standard mean field games

Agents are subject to a stochastic differential equation

$$\begin{cases} dX_s = \mathbf{u}_s ds + \sqrt{2} dW_s \\ X_t = x, \end{cases}$$

where $\mathbf{u}_s$ is the control of an average agent and W_s is a standard Brownian motion. Agents objective is to minimize, choosing a progressively measurable control $\mathbf{u}_s$, the cost

$$\mathbb{E}\left[\int_t^T L\left(X_s, \mathbf{u}_s, m(s)\right) ds + G(X_T, m(T))\right],$$

where $m(s)$ is the density distribution of the agents at time s and $L(x,q,m) = \sup_{p\in\mathbb{R}^d} \{-\langle p, q\rangle - H(x,p,m)\}$ is the Legendre-Fenchel transform of the Hamiltonian $H(x,p,m)$.

From standard optimization theory (see Appendix B and [FS06] for more details), the value function v satisfies the Hamilton-Jacobi equation

$$\begin{cases} -v_t + H(x, Dv, m) = \Delta v \\ v(x, T) = G(x, m(T)). \end{cases}$$

The optimal control then is given, in feedback form, by

$$\mathbf{u}_s^*(x) = -D_p H(x, Dv(s, x), m(s, x)).$$

Since (almost) all agents behave in this way, the density $m(s)$ of the solutions of

$$\left\{ dX_s = -D_p H(X_s, Dv(s, X_s), m(s))ds + \sqrt{2}dW_s X_t \quad \sim m_0, \right.$$

where $\sim$ denotes that X_t has distribution law given by m_0, satisfies the Fokker-Planck equation

$$\begin{cases} m_t - \text{div}\,(D_p H(x, Dv, m)m) = \Delta m \\ m(0) = m_0, \end{cases}$$

For the systems such as

$$\begin{cases} -v_t + H(x, Dv, m) = \Delta v \\ m_t - \text{div}\,(D_p H(x, Dv, m)m) = \Delta m \\ v(x, T) = G(x, m(T)), \quad m(0) = m_0, \end{cases}$$

the dependence in m may be local or non-local.

In the non-local case, Lasry and Lions have proved that, under fairly general conditions, there exists a unique classical solution (v, m).

With more hypothesis, together whith Cardaliaguet and Poretta, they prove the same for the local dependence on the density.

Evolution to our proposed model

It starts in the second order case. Typically, the game is a coupled system of equations on $(0, T) \times \mathbb{R}^d$, a Hamilton-Jacobi equation and a Fokker-Planck (or transport) equation as follows

$$\begin{cases} -\partial_t v + H(x, m, Dv) = \nu \Delta v \\ \partial_t m - \text{div}\,(m D_p H(x, m, Dv)) = \nu \Delta m \end{cases} \tag{1}$$

with $\nu > 0$, terminal condition $v(x, T) = \psi(x, m(x, T))$ and initial condition $m(x, 0) = m_0(x)$ for $x \in \mathbb{R}^d$. Dependence of H on m can be either local or global. The equations above describe the evolution of the value function for a player (depending on it's initial condition and players distribution), whose goal is to minimize a running cost and a final cost, and the evolution of the density of players whose dynamics is governed by a stochastic differential equation, respectively. The forward-backward characteristic of the systems allows for the interpretation that agents have a certain ability to predict the near future. It is also the key concern in the development of numerical methods fur such systems.

Naturally one is lead to consider the the stationary cases, which are interesting by themselves, but can also be thought, under appropriate conditions [CLLP], of a way to understand the asymptotic, limiting behavior of the game (1). The problem now is to find solutions $(v, m, \bar{H})$ to the system

$$\begin{cases} H(x, m, Dv) = \bar{H} + \nu \Delta v \\ -\text{div}\,(m D_p H(x, m, Dv)) = \nu \Delta m. \end{cases} \tag{2}$$

The constant $\bar{H}$, which appears in the context of homogenization theory and Aubry-Mather theory, is callet the effective Hamiltonian.

By sending ν to zero in the systems (1) and (2), one obtains a viscosity solution to the first order (deterministic) mean field game. This method for obtaining solutions for this kind of problems is known as vanishing viscosity method.

This theory meets population models, specifically pedestrian flow, through the Hughes model, [DFMPW11, BDFMW13], which can be writen as

$$\begin{cases} -v_t + f(m)\,|Dv|^2 = g(m) = \Delta v \\ m_t - \operatorname{div}\,(h(m)Dv) = \Delta m, \end{cases}$$

where f, g, h are "good" functions defined on $\mathbb{R}^+$.

A characteristic that this model shares with ours is the lack of adjoint structure present in the first examples.

Finally introducing other types of non-linearities, that will have their meanings explained in Section , we have

$$\begin{cases} H(x, Dv) = g(m) - m^\alpha v + \Delta v \\ -\operatorname{div}\,(D_p H m) = (1 - m^\alpha)m + \Delta m + \delta. \end{cases} \tag{3}$$

Methods

The methods applied in this thesis to determine the regularity of solutions to the proposed model come from a particular case. That of a Hamilton-Jacobi equation which is independent of the density function obtained as a solution to the transport equation. The fundamental aspect of the method is that it exploits the structure provided by the pair of equations. The transport equation is actually the adjoint of the linearization of the Hamilton-Jacobi equation. Such method, called adjoint method, is presented in the context of Hamilton-Jacobi equations in the works of L. C. Evans. Applications of the method include, and are not limited to,

- the vanishing viscosity problem;
- differentiability of solutions of the infinity Laplacian;
- Aubry-Mather theory in the non-convex setting;
- systems of Hamilton-Jacobi equations; and
- obstacle type problems.

The model

To avoid complications, we choose to work in the periodic setting, i.e. agents live in the d-dimensional torus $\mathbb{T}^d$. The next step towards the stationary mean field game with logistic populational dynamics model is to write the system

$$\begin{cases} H(x, Dv) = \bar{H} + g(m) - m^\alpha v + \Delta v \\ -\operatorname{div}\,(D_p H m) = (1 - m^\alpha)m + \Delta m + \delta. \end{cases} \tag{4}$$

The key difference from (4) to standard stationary mean field models (and a major source of difficulties) is that the second equation, which accounts for the population stationary regime, is not the adjoint of the linearization of the first equation (the Hamilton-Jacobi equation). Nevertheless, most of the techniques used in standard mean field games can be applied.

In the model, each agent optimizes a running cost which has two main components, a kinetic energy, corresponding to the term $H(x, Dv)$, typically a quadratic term in Dv, and a potential energy. The potential energy accounts for absolute location preferences, given by the part of the function H that depends on x and an interaction term with the population given by $g(m)$. In addition to optimizing this running cost each agent has a death rate proportional to a power of the density of players, this corresponds to the term $m^\alpha v$, where $\alpha > 0$. In addition, the agents reproduce at rate 1 and have a seeding/immigration rate, which accounts for incoming agents, which we assume spatially uniform and is given by the parameter $\delta \geq 0$.

We study the system (4) for all values of $\bar{H}$, the case $\bar{H} = 0$ is particularly interesting as in this case any solution to (4) is a stationary solution to the time-dependent equation

$$\begin{cases} -v_t + H(x, Dv) = g(m) - m^\alpha v + \Delta v \\ m_t - \operatorname{div}(D_p H m) = (1 - m^\alpha)m + \Delta m + \delta. \end{cases} \tag{5}$$

The parameter $\bar{H}$ can be seen as a long-rung average cost imposed upon all players. Without the term $m^\alpha v$, one would expect a unique value $\bar{H}$ for which (4) admits a solution. In fact, assuming $g(0) = 0$, if $\delta = 0$, we can find a solution to (4) with $m = 0$ and $\bar{H}$ the unique value for which

$$H(x, Dv) = \bar{H} + \Delta v \tag{6}$$

admits a periodic solution.

In spite of the uniqueness of $\bar{H}$ for the problem (6), as we will show in this thesis one can also find for any value $\bar{H}$ solutions (m, v), with $m > 0$.

For instance, in the case $\delta = 0$, $H = |p|^2$, it is easy to see that for any $\bar{H}$ there exists a solution to (4) with $m = 1$ and v a suitable constant.

For this reason we will assume that $\bar{H} = 0$, by adding a suitable constant to H. For the function g we choose, for simplicity, either a power-like non-linearity $g(m) = m^\gamma$, with $\gamma > 0$, as this choice illustrates most of the main points and techniques.

Organization of the thesis and Results

Put shortly, we prove that solutions to our model in dimensions one and two are *a priori* smooth, i.e. are classical solutions.

In Chapter 1, focusing on the simplified one dimensional mean field populational model, we obatin a series of *a priori* estimates, namely preliminary estimates in Section 1.1, which are used in Section 1.2 to establish H^1 bounds on v depending on the integrability of $g(m)$. Since m and its integral appear in those estimates, in Section 1.3, we produce lower bounds on m to remedy this issue. Finally, in Section 1.4, after we present new hypothesis under which $g(m)$ is integrabel, we prove the main result of the Chapter:

Theorem (1)**.** *Let (v, m) be a solution of (1.1). If either*

1. $\delta > 0$, $\gamma < \max\{1 + \alpha, \frac{1}{\alpha}\}$
2. $\alpha > 1$, $V > -\frac{\alpha}{\alpha+1}$, $\gamma \leq 1 + \alpha$

Then m is bounded by above and below. Additionally, $\|m_x\|_2$ and $\|v\|_{H^2}$ are bounded. Furthermore, in the second case the bounds are uniform in $0 \leq \delta \leq \delta_0$.

In Chapter 2 we start by following the sequence of estimates of Chapter 1 in more dimensions and more general context. The difference in dimension is felt most dramatically in Section 2.5, where we state Proposition 33, which uses Sobolev's Theorem instead of Morrey's.

In Chapter 2, after stating the Assumptions in Section 2.1 and proving elementary estimates in Section 2.2, we proceed to investigate H^1 regularity of v and $\ln m$, and further integrability of v and m in Sections 2.3, 2.4, and 2.5. This are summarized in Theorem 49 and Corollary 53. In Section 2.6, we employ the adjoint method to obtain Lipschitz bounds for v. This is used, in Section 2.7 to improve integrability and regularity of both v and m. The last result, in Section 2.8, states that m is bounded away from zero. We finish the Chapter with the proof of the main result

Theorem (26)**.** *Let (v, m) be a solution of (2.1). Suppose Assumptions 1, 2, 3, 4, 5 hold, $\delta > 0$ and $d = 2$.*

Then v and m are smooth solutions of (2.1).

Further research

In the case of time dependent systems the same kind of results should follow. One aspect we believe should facilitate the analysis is that smoothness of the terminal-initial conditions are preserved.

The proper proof of existence of solutions is indicated in the Appendix A, where a reasonable number of estimates have been developed. In Section A.1 we give a hint on what the argument for existence of solutions looks like and in Section A.2 we provide some uniform estimates for the non-local version of the system. We will pursue in this direction with the ultimate goal of determining existence and uniqueness of solutions to the mean field model of logistic populational dynamics.

Another issued that should be addressed in the future is that of the uniqueness of solutions.

Chapter 1

One Dimensional Model

We study the simplified stationary mean field game model 4 in one dimension given by

$$\begin{cases} \frac{v_x^2}{2} + V(x) = g(m) - m^\alpha v + v_{xx} \\ -(mv_x)_x = (1 - m^\alpha)m + m_{xx} + \delta, \end{cases} \tag{1.1}$$

where $H(x, v_x) = \frac{v_x^2}{2} + V(x)$, v is the value function for an agent and m is the density function of agents, both functions depend on x only.

We assume further that the potential $V : S^1 \to \mathbb{R}$ is smooth, α, δ are given constants, where $\alpha > 0$ and $\delta \geq 0$. For convenience we take $\delta < \delta_0$. Furthermore all estimates in this chapter that do not state the dependence in δ are uniform for a fixed δ_0.

We prove the following main result:

Theorem 1. *Let (v, m) be a solution of (1.1). If either*

1. $\delta > 0$, $\gamma < \max\{1 + \alpha, \frac{1}{\alpha}\}$
2. $\alpha > 1$, $V > -\frac{\alpha}{\alpha+1}$, $\gamma \leq 1 + \alpha$

Then m is bounded by above and below. Additionally, $\|m_x\|_2$ and $\|v\|_{H^2}$ are bounded. Furthermore, in the second case the bounds are uniform in $0 \leq \delta \leq \delta_0$.

Once this theorem is established one can use a bootstrapping argument to obtain regularity of solutions. Existence can be proved by a regularization argument as in

other source. [GSM11].

This chapter is organized as follows: we start in Section 1.1 by establishing various elementary bounds on the solution. Then in Section 1.2 we consider the H^1 regularity. To close the H^1 regularity proof we need various lower bounds for m. Those are studied in Section 1.3. We end the discussion of the H^1 regularity, together with further integrability properties as well as the proof of Theorem 1 in Section 1.4.

1.1 Elementary inequalities

The key idea to obtain regularity for the solutions of (1.1) is to develop a number of *a priori* estimates. In standard mean field games the density m is usually assumed to be a probability measure. This is not the case here as a balance between death, birth and seeding rates will determine the total population.

Proposition 2. *Let* (v, m) *be a solution of* (1.1)*. Then*

$$\int_{S^1} mdx \leq C, \text{ and} \tag{1.2}$$

$$\int_{S^1} m^{\alpha+1}dx \leq C. \tag{1.3}$$

Proof. Integrate the second equation in (1.1) on S^1 to get

$$\int_{S^1} m^{\alpha+1}dx = \int_{S^1} mdx + \delta.$$

The result follows from Young's inequality. □

We turn now our attention to the Hamilton-Jacobi equation to obtain further estimates.

Proposition 3. *Let* (v, m) *be a solution of* (1.1)*. Then*

$$\int_{S^1} mg(m) + \frac{v_x^2}{2}(1+m)dx \leq \int_{S^1} g(m)dx + \int_{S^1} v(m+\delta-m^\alpha)dx + C.$$

Proof. Integrate the first equation in (1.1) to get

$$\int_{S^1} \frac{{v_x}^2}{2} + V(x)dx = \int_{S^1} g(m) - m^\alpha v dx. \tag{1.4}$$

Multiply the first equation in (1.1) by m and integrate, by parts, to obtain

$$\int_{S^1} mg(m)dx = \int_{S^1} m\frac{v_x^2}{2} + mV(x) + m^{\alpha+1}v - m_{xx}v dx.$$

Using the identity

$$\int_{S^1} m^{\alpha+1}v - m_{xx}v dx = -\int_{S^1} mv_x^2 dx + \int_{S^1} vm + v\delta dx$$

we get

$$\int_{S^1} m\frac{v_x^2}{2} + mg(m)dx = \int_{S^1} vm + v\delta + Vmdx. \tag{1.5}$$

Since V is bounded, by combining (1.4) with (1.5) we obtain (1.4). □

Corollary 4. *Let* (v, m) *be a solution of* (1.1)*. Then*

$$\int_{S^1} \frac{mg(m)}{2} + \frac{v_x^2}{2}(1+m)dx \leq \int_{S^1} v(m+\delta-m^\alpha)dx + C. \tag{1.6}$$

Proof. It suffices to observe that $g(m) \leq \frac{mg(m)}{2} + C$ and use (1.4). □

Corollary 5. *Let* (v, m) *be a solution of* (1.1)*. Then*

$$\int_{S^1} m^\alpha v dx \leq C + \int_{S^1} g(m)dx. \tag{1.7}$$

and

$$\int_{S^1} v(m+\delta)dx \geq -C. \tag{1.8}$$

Proof. The estimates follow from (1.4) and (1.5). □

Corollary 6. *Let (v, m) be a solution of (1.1). Then, as $g(m) = m^\gamma$,*

$$\int_{S^1} g(m)dx \le C\left(C + 2\int_{S^1} v(m+\delta-m^\alpha)dx\right)^{\frac{\gamma}{\gamma+1}}. \tag{1.9}$$

Proof. Observe that

$$\int_{S^1} g(m)dx \le \left(\int_{S^1} mg(m)dx\right)^{\frac{\gamma}{\gamma+1}}.$$

Then (1.9) follows from (1.6). □

1.2 H^1 estimates for the Hamilton-Jacobi equation

The key terms to control in order to obtain H^1 regularity for the solution to the Hamilton-Jacobi equation are $\int_{S^1} m^\alpha v dx$ and $\int_{S^1}(m+\delta)vdx$. This is precisely the task that we will address now.

We start by stating a Poincaré-like inequality.

Proposition 7. *For any probability density θ on S^1,*

$$\left|\int_{S^1} \theta v dx - \int_{S^1} v dx\right| \le C\,\|v_x\|_2\,.$$

Proof.

$$\left|\int_{S^1} \theta v dx - \int_{S^1} v dx\right| \le \int_{S^1} \theta\left|v(x) - \int_{S^1} v dx\right| dx \le \left\|v(x) - \int_{S^1} v dx\right\|_\infty \le C\,\|v_x\|_2\,,$$

using Hölder and Morrey's inequalities. □

Proposition 8. *Let (v, m) be a solution of (1.1). Then*

$$\int_{S^1}(m+\delta-m^\alpha)vdx \le C\left(\int_{S^1} m+\delta+m^\alpha dx\right)\|v_x\|_2 + \left(\int_{S^1}(m+\delta)dx - \int_{S^1} m^\alpha dx\right)\int_{S^1} vdx.$$

Proof. It suffices to apply Proposition 7 with $\frac{m+\delta}{\int_{S^1} m+\delta dx}$ and $\frac{m^\alpha}{\int_{S^1} m^\alpha dx}$ in the place of θ. □

Corollary 9. *Let (v, m) be a solution of (1.1). Then*

$$\frac{\|v_x\|_2^2}{2} \le C\left(\int_{S^1}(m+\delta)dx + \int_{S^1} m^\alpha dx\right)\|v_x\|_2 + \left(\int_{S^1}(m+\delta)dx - \int_{S^1} m^\alpha dx\right)\int_{S^1} vdx + C.$$

Proof. This follows by using the result in Proposition 8 in the bounds of Corollary 4. □

Corollary 10. *Let (v, m) be a solution of (1.1). Then*

$$\|v_x\|_2^2 \le C\left(\int_{S^1}(m+\delta)dx - \int_{S^1} m^\alpha dx\right)\int_{S^1} vdx + C.$$

Proof. It follows from Corollary 9 using the bounds in Proposition 2. □

Proposition 11. *Let (v, m) be a solution of (1.1). Then*

$$\left(\int_{S^1} mdx + \delta\right)\int_{S^1} vdx \ge -C - C\left(\int_{S^1} mdx + \delta\right)\|v_x\|_2,$$

and

$$\left(\int_{S^1} m^\alpha dx\right)\int_{S^1} vdx \le \int_{S^1} g(m)dx + C\left(\int_{S^1} m^\alpha dx\right)\|v_x\|_2,$$

Proof. We apply Proposition 7 with θ replaced by $\frac{m+\delta}{\int_{S^1} m+\delta dx}$ and $\frac{m^\alpha}{\int_{S^1} m^\alpha dx}$ on Equations (1.7) and (1.8), respectively. □

Corollary 12. *Let (v, m) be a solution of (1.1). Let $\lambda_1 = \int_{S^1} m + \delta dx$, and $\lambda_2 = \int_{S^1} m^\alpha dx$. Then, as long as $m > 0$,*

$$-\frac{C}{\lambda_1} - C\|v_x\|_2 \le \int_{S^1} vdx \le \frac{\int_{S^1} g(m)dx}{\lambda_2} + C\|v_x\|_2.$$

Proof. The proof is immediate from Proposition 11. □

Corollary 13. *Let (v, m) be a solution of (1.1). We have*

$$\frac{\|v_x\|_2^2}{4} \le C + \int_{S^1} g(m)dx + \frac{C}{\lambda_1}, \tag{1.10}$$

where $\lambda_1 = \int_{S^1} mdx + \delta$.

Proof. From equation (1.4) we deduce

$$\frac{\|v_x\|_2^2}{2} \le C + \int_{S^1} g(m)dx - \int_{S^1} m^\alpha vdx$$

Apply Proposition 7 to $-\int_{S^1} m^\alpha vdx$ and use Corollary 12 to obtain

$$-\int_{S^1} vdx \le \frac{C}{\lambda_1} + C\left\|v_x\right\|_2, \tag{1.11}$$

from which we get (1.10) by observing that $\int_{S^1} m^\alpha dx$ and $\int_{S^1} m + \delta dx$ are bounded by Proposition 2. □

This proposition yields the following (non-uniform estimate in terms of δ):

Corollary 14. *Let (v, m) be a solution of (1.1). We have*

$$\frac{\|v_x\|_2^2}{4} \le C_\delta + \int_{S^1} g(m)dx.$$

Proof. It suffices to observe that $\lambda_1 = \int_{S^1} mdx + \delta > \delta$, and use Corollary 13. □

Corollary 15. *Let (v, m) be a solution of (1.1). We have*

$$\left|\int_{S^1} vdx\right| \le \frac{C}{\lambda_1} + \frac{\int_{S^1} g(m)dx}{\lambda_2} + C\|v_x\|_2.$$

Proof. Observe that the quantities $\frac{C}{\lambda_1}$ and $\frac{\int_{S^1} g(m)dx}{\lambda_2}$ which appear in Corollary 12 are positive. □

From the previous corollaries it is clear that the main task is now to obtain positive lower bounds for λ_1 and λ_2, which are uniform in δ. Once these are obtained, H^1 bounds for v will be easily established.

1.3 Lower bounds on m

We now identify an additional mechanism that enables us to obtain lower bounds for m which are uniform in δ. This allows to estimate $\|v_x\|_2$ in terms of the integrability of $g(m)$. For this we need to assume that V is sufficiently large. We observe that by the example given in the introduction, lower bounds for m are not true in general. We start with an elementary but useful result:

Proposition 16. *Suppose $m > 0$ and fix $\theta, \beta > 0$. Then*

$$\left(\int_{S^1} m^\beta dx\right)^{-1} \leq \left(\int_{S^1} \frac{1}{m^\theta} dx\right)^{\frac{\beta}{\theta}}.$$

Proof. For any $b > 0$,

$$1 = \int_{S^1} \frac{m^b}{m^b} dx \leq \left(\int_{S^1} m^{bp} dx\right)^{1/p} \left(\int_{S^1} m^{-bp'} dx\right)^{1/p'}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Choose b and p so that $bp = \beta$ and $bp' = \theta$, which ends the proof. □

Proposition 17. *Let (v, m) be a solution of (1.1). Then if $V > -\frac{\alpha}{\alpha+1}$, and $m > 0$ we have*

1. *if $\gamma < \alpha$:*
$$\int_{S^1} \frac{1}{m^\alpha} dx \leq C - C \int_{S^1} v dx.$$

2. *if $\gamma \geq \alpha$*
$$\int_{S^1} \frac{1}{m^\alpha} dx \leq C - C \int_{S^1} v dx + C \left(\int_{S^1} g(m) dx\right)^{\frac{\gamma-\alpha}{\gamma}}.$$

Proof. Take the first equation in (1.1) and multiply it by $\frac{1}{m^\alpha}$. Multiply the second equation by $\frac{\alpha}{(\alpha+1)m^{\alpha+1}}$, subtract it from the first identity and observe that after integration by parts the terms that contain $m_x v_x$ cancel:

$$\int_{S^1} \frac{v_x^2}{2m^\alpha} + \alpha \frac{m_x^2}{m^{2+\alpha}} + \frac{V}{m^\alpha} + \frac{\delta\alpha}{(\alpha+1)m^{\alpha+1}} + \frac{\alpha}{(\alpha+1)m^\alpha} dx = \int_{S^1} \frac{g(m)}{m^\alpha} - v + \frac{\alpha}{\alpha+1} dx.$$

As long as $V > -\frac{\alpha}{\alpha+1}$ the term $\frac{g(m)}{m^\alpha}$ can be handled in the following way: if $\gamma < \alpha$ then it can be absorbed in the left hand side by noting that for any $\epsilon > 0$ we have

$$\frac{g(m)}{m^\alpha} \leq \frac{\epsilon}{m^\alpha} + C_\epsilon.$$

Choosing ϵ small enough the result follows. In the case $\gamma \geq \alpha$ it suffices to use Hölder inequality. □

Note that in the case $\gamma < 2\alpha + 1$ the term $\left(\int_{S^1} g(m) dx\right)^{\frac{\gamma-\alpha}{\gamma}}$ in the last proposition can be replaced by $C \int_{S^1} m^{\gamma-\alpha} dx$ which is automatically bounded.

Corollary 18. *Let (v, m) be a solution of (1.1). Suppose $\alpha > 1$. Then*

1. *if $\gamma < \alpha$:*
$$\int_{S^1} \frac{1}{m^\alpha} dx \leq C + C\|v_x\|_2.$$

2. *if $\gamma \geq \alpha$*
$$\int_{S^1} \frac{1}{m^\alpha} dx \leq C + C\|v_x\|_2 + C \left(\int_{S^1} g(m) dx\right)^{\frac{\gamma-\alpha}{\gamma}}.$$

Proof. We have, combining the previous proposition with (1.11),

$$\begin{aligned}\int_{S^1}\frac{1}{m^\alpha}dx &\leq C - C\int_{S^1} v dx\\ &\leq C + C\|v_x\|_2 + \frac{C}{\lambda_1}\\ &\leq C + C\|v_x\|_2 + C\left(\int_{S^1}\frac{1}{m^\alpha}dx\right)^{\frac{1}{\alpha}},\end{aligned}$$

where the last inequality holds by Proposition 16. Then the result follows if $\alpha > 1$. □

The following bound for $\frac{1}{m}$ will also be useful later to study the case $\delta > 0$.

Proposition 19. *Let (v, m) be a solution of (1.1). Then, if $m \neq 0$,*

$$\int_{S^1}\frac{\delta}{m} + \frac{1}{2}(\ln m)_x^2 dx \leq C + \frac{\|v_x\|_2^2}{2}.$$

Proof. Divide the second equation in (1.1) by m and integrate

$$\int_{S^1} -\frac{(mv_x)_x}{m}dx = \int_{S^1} 1 - m^\alpha + \frac{m_{xx}}{m} + \frac{\delta}{m}dx.$$

Then we have

$$\int_{S^1} -(\ln m)_x v_x dx = \int_{S^1} 1 - m^\alpha + (\ln m)_x^2 + \frac{\delta}{m}dx.$$

With Hölder's inequality and the bound given by Proposition 2,

$$\begin{aligned}\int_{S^1}\frac{\delta}{m} + (\ln m)_x^2 dx &= \int_{S^1} -(\ln m)_x v_x - 1 + m^\alpha dx\\ &\leq C + \int_{S^1}\frac{1}{2}(\ln m)_x^2 dx + \frac{\|v_x\|_2^2}{2}.\end{aligned}$$

□

1.4 Regularity for the Hamilton-Jacobi equation

The next proposition improves the result from Corollary 14 by establishing bounds for $\|v_x\|_2$ which are independent on δ.

Proposition 20. *Let (v, m) be a solution of (1.1). Suppose $\alpha > 1$. Then*

$$\|v_x\|_2^2 \leq C + C\int_{S^1} g(m)dx.$$

Proof. Using the estimate from Corollary 13 and control the term $\frac{1}{\lambda_1}$ by the bounds in Proposition 17 together with Proposition 16, we obtain

$$\begin{aligned}\|v_x\|_2^2 &\leq C + C\int_{S^1} g(m)dx + C\left(\int_{S^1}\frac{1}{m^\alpha}dx\right)^{\frac{1}{\alpha}}\\ &\leq C + C\int_{S^1} g(m)dx\\ &\quad + C\left(C + C\|v_x\|_2 + \mu\left(\int_{S^1} g(m)dx\right)^{\frac{\gamma-\alpha}{\gamma}}\right)^{\frac{1}{\alpha}},\end{aligned}$$

where in the last inequality we used Corollary 18 and $\mu = 0$ if $\gamma < \alpha$ and μ is a large enough constant otherwise. Thus if $\alpha > 1$ the result follows. □

We must now bound $\int_{S^1} v dx$ in order to get H^1 bounds for v.

Proposition 21. *Let (v, m) be a solution of (1.1). If $\delta > 0$ then*

$$\left|\int_{S^1} v dx\right| \le C_\delta + C_\delta \left(\int_{S^1} g(m) dx\right)^{1+\alpha}.$$

Proof. Use the bound $\frac{1}{\lambda_1} \le C_\delta$ and Corollary 14 on Corollary 15 to get

$$\left|\int_{S^1} v dx\right| \le C_\delta + C \left(\int_{S^1} g(m) dx\right)^{1/2} + \frac{1}{\lambda_2}\int_{S^1} g(m) dx.$$

From Proposition 19 we have

$$\int_{S^1} \frac{1}{m} dx \le C_\delta + C_\delta \|v_x\|_2^2.$$

Hence, using Proposition 16, combined with the bounds in Corollary 14 we have

$$\frac{1}{\lambda_2} \le C_\delta + \left(\int_{S^1} g(m) dx\right)^{\alpha}.$$

Consequently

$$\left|\int_{S^1} v dx\right| \le C_\delta + C_\delta \left(\int_{S^1} g(m) dx\right)^{1+\alpha}.$$

□

It is also possible to obtain bounds for $\left|\int_{S^1} v dx\right|$ which are uniform in δ, provided $\gamma < \alpha + 1$, since in this case $\int_{S^1} g(m) dx$ is bounded.

Proposition 22. *Let (v, m) be a solution of (1.1). Suppose $V > -\frac{\alpha}{1+\alpha}$, $\alpha > 1$, and $\gamma < \alpha + 1$. Then*

$$\left|\int_{S^1} v dx\right| \le C.$$

Proof. Corollary 18 gives $\frac{1}{\lambda_2} \le C + C\|v_x\|_2$. Proposition16 gives $\frac{1}{\lambda_1} \le \left(\frac{1}{\lambda_2}\right)^{\frac{1}{\alpha}}$. Since, by Proposition 20, $\|v_x\|_2 \le C$, we have

$$\frac{1}{\lambda_1} \le C, \qquad \frac{1}{\lambda_2} \le C.$$

So that Corollary 15 provides $\left|\int_{S^1} v dx\right| \le C$, as we needed to show. □

The task that remains now is to bound $\int_{S^1} g(m) dx$ if $\gamma \ge \alpha + 1$. This is not completely trivial but thanks to Corollary 6 we were able to obtain bounds that depend on δ.

Proposition 23. *Let (v, m) be a solution of (1.1). Then*

$$\int_{S^1} g(m) dx \le C_\delta,$$

if $\gamma < \frac{1}{\alpha}$.

Proof. We write Corollary 6 as

$$\left(\int_{S^1} g(m) dx\right)^{\frac{\gamma+1}{\gamma}} \le C + C\int_{S^1} v\left(m + \delta - m^\alpha\right) dx$$

and observe that Proposition 8 yields, since $\int_{S^1} m dx$ and $\int_{S^1} m^\alpha dx$ are bounded,

$$\int_{S^1} v\left(m+\delta-m^\alpha\right) dx \leq C\left\|v_x\right\|_2 + C\left|\int_{S^1} v dx\right|.$$

Using Corollary 14 and Proposition 21 we obtain

$$\left(\int_{S^1} g(m)dx\right)^{\frac{\gamma+1}{\gamma}} \leq C_\delta + C_\delta\left(\int_{S^1} g(m)dx\right)^{\frac{1}{2}} + C_\delta\left(\int_{S^1} g(m)dx\right)^{1+\alpha}.$$

Therefore, the result follows since $\gamma < \frac{1}{\alpha}$ implies $\frac{\gamma(1+\alpha)}{\gamma+1} < 1$. □

Proposition 24. *Let (v, m) be a solution of (1.1). If either*

1. $\delta > 0$, $\gamma < \max\{1+\alpha, \frac{1}{\alpha}\}$,
2. $\alpha > 1$, $V > -\frac{\alpha}{\alpha+1}$, $\gamma \leq 1+\alpha$,

then m is bounded by above and below. Additionally, $\|m_x\|_2$ and $\|v\|_{H^1}$ are bounded. Furthermore, in the second case, the bounds are uniform in $0 \leq \delta \leq \delta_0$.

Proof. By Proposition 19 we have $(\ln m)_x \in L^2$, from which we conclude by Morrey's theorem that $\ln m$ is Hölder continuous and hence m is bounded by above and by below. Then, observing that since m is bounded by above and below we have

$$\int_{S^1} m_x^2 dx \leq C\int_{S^1} (\ln m)_x^2 dx,$$

which then implies $m_x \in L^2$. □

Proposition 25. *Let (v, m) be a solution of (1.1). Then if either*

1. $\delta > 0$, $\gamma < \max\{1+\alpha, \frac{1}{\alpha}\}$
2. $\alpha > 1$, $V > -\frac{\alpha}{\alpha+1}$, $\gamma \leq 1+\alpha$

we have

$$\int_{S^1} g'(m)m_x^2 + \frac{1}{2}mv_{xx}^2 dx \leq C_\delta.$$

Furthermore, in the second case the bounds are uniform in $0 \leq \delta \leq \delta_0$.

Proof. Apply the Laplacian to the first equation in (1.1)

$$v_{xx}^2 + v_x v_{xxx} + V_{xx} = \left(g'(m)m_x\right)_x - \left(\alpha m^{\alpha-1}m_x v + m^\alpha v_x\right)_x + v_{xxxx}.$$

Multiply by m and integrate to obtain

$$\int_{S^1} mv_{xx}^2 + mv_x v_{xxx} + mV_{xx}dx = \int_{S^1} m\left(g'(m)m_x\right)_x - m(\alpha m^{\alpha-1}m_x v + m^\alpha v_x)_x + mv_{xxxx}dx.$$

Integrating by parts

$$\int_{S^1} mv_{xx}^2 - (mv_x)_x v_{xx} + mV_{xx}dx = \int_{S^1} -g'(m)m_x^2 + \alpha m^{\alpha-1}m_x^2 v + m_x m^\alpha v_x + m_{xx}v_{xx}dx.$$

Observing that m is a solution to the second equation in (1.1)

$$\begin{aligned}
\int_{S^1} g'(m)m_x^2 + mv_{xx}^2 dx &= \int_{S^1} -mV_{xx} + \alpha m^{\alpha-1}m_x^2 v - \alpha m^\alpha m_x v_x + v_x m_x dx \\
&= \int_{S^1} -mV_{xx} + \alpha m^{\alpha-1}m_x^2 v - \frac{\alpha}{\alpha+1}(m^{\alpha+1})_x v_x + v_x m_x dx \\
&= \int_{S^1} -mV_{xx} + \alpha m^{\alpha-1}m_x^2 v + \left(\frac{\alpha}{\alpha+1}m^{\alpha+1} - m\right) v_{xx} dx \\
&\leq C_\delta + \int_{S^1} \alpha m^{\alpha-1}m_x^2 v dx + \frac{1}{2}\int_{S^1} m v_{xx}^2 dx + \frac{1}{2}\int_{S^1}\left(1 - \frac{\alpha}{\alpha+1}m^\alpha\right)^2 m dx
\end{aligned}$$

The result follows from the bounds in Proposition 24. □

We now can present the

Proof of Theorem 1. The statement of the theorem follows by combining the results in Proposition 24 with Proposition 25. In particular, because m is bounded by above and below it follows that $v_{xx} \in L^2$. □

Chapter 2

Model problem in more dimensions

In this chapter we analyze

$$\begin{cases} H(x, Dv) = g(m) - m^{\alpha} v + \Delta v \\ -\mathrm{div}\,(m D_p H(x, Dv)) = (1 - m^{\alpha}) m + \Delta m + \delta, \end{cases} \tag{2.1}$$

where v and m are functions of $x \in \mathbb{T}^d$ only, H is a smooth function, and α and δ are non-negative constants.

After stating the Assumptions in Section 2.1 and proving elementary estimates in Section 2.2, we proceed to investigate H^1 regularity of v and $\ln m$, and further integrability of v and m in Sections 2.3, 2.4, and 2.5. This are summarized in Theorem 49 and Corollary 53. In Section 2.6, we employ the adjoint method to obtain Lipschitz bounds for v. This is used, in Section 2.7 to improve integrability and regularity of both v and m. The last result, in Section 2.8, states that m is bounded away from zero. We finish the Chapter with the proof of the main result

Theorem 26 (*A priori* smooothness). *Let (v, m) be a solution of (2.1). Suppose Assumptions 1, 2, 3, 4, 5 hold, $\delta > 0$ and $d = 2$.*

Then v and m are smooth solutions of (2.1).

2.1 Basic Assumptions

1. There exist c_1 and C_1 positive constants such that

$$D_p H(x, p) p - H(x, p) =: \hat{L}(x, p) \geq c_1 H(x, p) - C_1;$$

2. There exist c_2 and C_2 positive constants such that

$$H(x, p) \geq c_2 |p|^2 - C_2;$$

3. $d = 2$ or $2 < d \leq 6$ and $0 < \alpha \leq \frac{d+2}{2d-4}$, i.e. $m, m^{\alpha} \in L^{\frac{2d}{d+2}}(\mathbb{T}^d)$. See Corollary 53;

4. H is uniformly convex, for some $\xi > 0$, $D_p^2 H \geq \xi$.

5. There exist C_3 and $\hat{C}_3$ positive constants such that

$$|D_p H|^2 \leq C_3 |p|^2 + \hat{C}_3.$$

6. $g(m) = m^{\gamma}$, with $\gamma > 0$;

7. $g(m) = \ln m$;

2.2 Elementary inequalities

As in the previous chapter, the key idea to obtain regularity for solutions of (2.1) is to develop a number of *a priori* estimates.

Proposition 27. *Let* (v, m) *be a solution of* (2.1)*. Then*

$$\int_{\mathbb{T}^d} m dx \leq C,$$

and

$$\int_{\mathbb{T}^d} m^{\alpha+1} dx \leq C.$$

Proof. Integrate the second identity in (2.1) to get

$$\int_{\mathbb{T}^d} m^{\alpha+1} dx = \int_{\mathbb{T}^d} (m+\delta) dx$$

then use Young's inequality. □

Proposition 28. *Let* (v, m) *be a solution of* (2.1)*. Suppose Assumption 1 holds, then*

$$\int_{\mathbb{T}^d} H(1+cm) + mg(m) dx \leq C + \int_{\mathbb{T}^d} g(m) + v(m+\delta-m^{\alpha}) dx.$$

Proof. Integrating the first identity in (2.1) gives

$$\int_{\mathbb{T}^d} H dx = \int_{\mathbb{T}^d} g(m) - m^{\alpha} v dx. \tag{2.2}$$

We integrate the first identity in (2.1) against m to obtain

$$\int_{\mathbb{T}^d} H m dx = \int_{\mathbb{T}^d} mg(m) - m^{\alpha+1} v - Dm \cdot Dv dx \tag{2.3}$$

and use the integral of the second identity in (2.1) multiplied by v,

$$\int_{\mathbb{T}^d} m D_p H Dv dx = \int_{\mathbb{T}^d} v(m+\delta) - m^{\alpha+1} v - Dm \cdot Dv dx \tag{2.4}$$

to get, by subtracting (2.3) from (2.4):

$$\int_{\mathbb{T}^d} cmH + mg(m) dx \leq C + \int_{\mathbb{T}^d} v(m+\delta) dx, \tag{2.5}$$

where we used Assumption 1.

We sum this last inequality with (2.2) to finish the proof. □

Proposition 29. *Let* (v, m) *be a solution of* (2.1)*. Suppose Assumptions 1 and either 6 or 7 hold.*

$$\int_{\mathbb{T}^d} \frac{mg(m)}{2} + H(1+cm) dx \leq C + \int_{\mathbb{T}^d} v(m+\delta-m^{\alpha}) dx.$$

Proof. We observe that, for both Assumptions 6 and 7, there exists $C > 0$ such that

$$g(m) \leq \frac{mg(m)}{2} + C.$$

□

Proposition 30. *Let (v, m) be a solution of (2.1). Suppose Assumptions 1, 2, and either 6 or 7 hold, then*

$$\int_{\mathbb{T}^d} m^\alpha v dx \leq C + \int_{\mathbb{T}^d} g(m) dx,$$

and

$$\int_{\mathbb{T}^d} v(m+\delta) dx \geq -C.$$

Proof. Assumption 2 applied to the identity (2.2) implies

$$-C \leq \int_{\mathbb{T}^d} H dx = \int_{\mathbb{T}^d} g(m) - m^\alpha v dx.$$

The same Assumption applied to estimate (2.5), together with $mg(m)$ bounded by below, yields

$$-C \leq \int_{\mathbb{T}^d} -Cm dx \leq \int_{\mathbb{T}^d} cmH dx \leq C + \int_{\mathbb{T}^d} v(m+\delta) dx.$$

□

Corollary 31. *Let (v, m) be a solution of (2.1). Suppose Assumptions 1, 2, and 6 ($g(m) = m^\gamma$) hold. We have*

$$\int_{\mathbb{T}^d} g(m) dx \leq \left(C + 2\int_{\mathbb{T}^d} v(m + \delta - m^\alpha) dx \right)^{\frac{\gamma}{\gamma+1}}.$$

Proof. From Hölder inequality,

$$\int_{\mathbb{T}^d} g(m) dx \leq \left(\int_{\mathbb{T}^d} mg(m) dx \right)^{\frac{\gamma}{\gamma+1}}.$$

then, apply Proposition 29 inside the parentheses. □

Corollary 32. *Let (v, m) be a solution of (2.1). Suppose Assumptions 1, 2, and 7 ($g(m) = \ln m$) hold. For every $\theta > 0$ there exists C_θ such that*

$$\int_{\mathbb{T}^d} g(m) dx \leq \left(C_\theta + 2\int_{\mathbb{T}^d} v(m + \delta - m^\alpha) dx \right)^{\theta}.$$

Proof. Note that $\lim_{x\to\infty} (C + x\ln x)^\theta / \ln x = \infty$ means that $(C + x\ln x)^\theta$ grows faster than $\ln x$.

To make sure that the minimum of the function $(C + x\ln x)^\theta - \ln x$ is positive, we need, after fixing θ, to choose a large enough C. With this in mind we have

$$\int_{\mathbb{T}^d} g(m) dx \leq \left(C_\theta + \int_{\mathbb{T}^d} mg(m) dx \right)^{\theta},$$

where apply Proposition 29 inside the parentheses. □

2.3 H^1 estimates for the Hamilton-Jacobi equation

In order to obtain H^1 regularity for the solution to the first equation of (2.1), we need to control the terms $\int_{S^1} m^\alpha v dx$ and $\int_{S^1} (m+\delta) v dx$.

We start by stating a Poincaré-like inequality.

Proposition 33. *For any probability density θ on $\mathbb{T}^d$, we have*

$$\left| \int_{\mathbb{T}^d} \theta v dx - \int_{\mathbb{T}^d} v dx \right| \leq C \|\theta\|_p \|Dv\|_2.$$

1. *If* $d = 2$ *then* $1 < p < \infty$.
2. *If* $d > 2$ *then* $p \geq \frac{2d}{d+2}$.

Before proving this, we recall:

Theorem 34 (Evans, pp. 279 Theorem 3)**.** *Assume* U *is a bounded, open subset of* $\mathbb{R}^d$*. Suppose* $u \in W_0^{1,p}(U)$ *for some* $1 \leq p < d$*. Let the Sobolev conjugate of* p *be* $p^* = \frac{pd}{d-p}$*. Then we have the estimate*

$$\|u\|_{L^q(U)} \leq C \|Du\|_{L^p(U)},$$

for each $q \in [1, p^*]$*, the constant* C *depending only on* p, q, n *and* U*.*

In particular, for all $1 \leq p \leq \infty$*,*

$$\|u\|_{L^p(U)} \leq C \|Du\|_{L^p(U)}.$$

Proof (of Proposition 33).

$$\begin{aligned} \left| \int_{\mathbb{T}^d} \theta v dx - \int_{\mathbb{T}^d} v dx \right| &\leq \int_{\mathbb{T}^d} \theta \left| v(x) - \int_{\mathbb{T}^d} v dx \right| dx \\ \text{(Hölder's inequality)} &\leq \|\theta\|_p \left\| v(x) - \int_{\mathbb{T}^d} v dx \right\|_{p'} \\ \text{(Sobolev's theorem)} &\leq C \|\theta\|_p \|Dv\|_2, \end{aligned}$$

provided $d = 2$, in which case Sobolev's theorem gives $\left\| v - \int v \right\|_{p'} \leq C \|Dv\|_2$ for any $1 \leq p' < \infty$. So that $1 < p < \infty$. Or, if $d > 2$, $p' \leq 2^*$, which gives the relation

$$\frac{p}{p-1} \leq \frac{2d}{d-2}.$$

□

Proposition 35. *Suppose Assumption 3 holds. Then*

$$\int_{\mathbb{T}^d} v(m + \delta - m^\alpha) dx \leq C \int_{\mathbb{T}^d} v dx + C \|Dv\|_2.$$

Proof. Apply the Poincaré-like Proposition 33 to obtain

$$\int_{\mathbb{T}^d} v(m+\delta) dx \leq \left(\int_{\mathbb{T}^d} v dx \right) \left(\int_{\mathbb{T}^d} m + \delta dx \right) + C \|m + \delta\|_p \|Dv\|_2,$$

and

$$-\int_{\mathbb{T}^d} v m^\alpha dx \leq -\left(\int_{\mathbb{T}^d} v dx \right) \left(\int_{\mathbb{T}^d} m^\alpha dx \right) + C \|m^\alpha\|_p \|Dv\|_2.$$

Sum these inequalities and use Assumption 3, together with Proposition 27. □

Proposition 36. *Let* (v, m) *be a solution of* (2.1)*. Suppose Assumptions 1, 2, and 3 hold. Then*

$$\|Dv\|_2^2 \leq C + C \int_{\mathbb{T}^d} v dx.$$

Proof. It follows from applying Assumption 2 to Proposition 29 and then using the result from

Proposition 35.

$$\int_{\mathbb{T}^d} c\,|Dv|^2\,dx - C \le C + \int_{\mathbb{T}^d} v(m+\delta-m^\alpha)dx$$
$$\le C + C\int_{\mathbb{T}^d} vdx + C\,\|Dv\|_2\,.$$

□

Proposition 37. *Let (v,m) be a solution of (2.1). Suppose Assumptions 1, 2, and 3 hold. We have*

$$\left(\int_{\mathbb{T}^d} m^\alpha dx\right)\left(\int_{\mathbb{T}^d} vdx\right) \le C + \int_{\mathbb{T}^d} g(m)dx + C\,\|Dv\|_2$$

and

$$\left(\int_{\mathbb{T}^d} m+\delta dx\right)\left(\int_{\mathbb{T}^d} vdx\right) \ge -C - C\,\|Dv\|_2\,.$$

Proof. We apply Proposition 33 on the estimates of Proposition 30. □

Corollary 38. *Let (v,m) be a solution of (2.1). Define $\lambda_1 = \int_{\mathbb{T}^d} m+\delta dx$ and $\lambda_2 = \int_{\mathbb{T}^d} m^\alpha dx$. Suppose Assumptions 1, 2, and 3 hold, $\lambda_1 > 0$, and $\lambda_2 > 0$. Then we get*

$$-\frac{C}{\lambda_1} - \frac{C}{\lambda_1}\,\|Dv\|_2 \le \int_{\mathbb{T}^d} vdx \le \frac{C}{\lambda_2} + \int_{\mathbb{T}^d} \frac{g(m)}{\lambda_2}dx + \frac{C}{\lambda_2}\,\|Dv\|_2\,.$$

Proof. Since the integrals λ_1 and λ_2 are positive, we can divide the inequalities obtained in Proposition 37 by them. □

Corollary 39. *Let (v,m) be a solution of (2.1). Suppose Assumptions 1 and 2 hold, and $\lambda_1 > 0$. We have*

$$c\,\|Dv\|_2^2 \le \frac{C}{\lambda_1} + \int_{\mathbb{T}^d} g(m)dx + \frac{C}{\lambda_1}\,\|Dv\|_2\,.$$

Proof. We have, from (2.2) and Assumption 2,

$$c\,\|Dv\|_2^2 \le C + \int_{\mathbb{T}^d} g(m) - m^\alpha vdx.$$

We then apply Proposition 33 and Corollary 38 to get

$$c\,\|Dv\|_2^2 \le C + \int_{\mathbb{T}^d} g(m)dx - \left(\int_{\mathbb{T}^d} m^\alpha dx\right)\left(\int_{\mathbb{T}^d} vdx\right) + C\,\|m^\alpha\|_p\,\|Dv\|_2$$
$$\le C + \int_{\mathbb{T}^d} g(m)dx + \frac{C}{\lambda_1}\left(1 + \|m+\delta\|_p\,\|Dv\|_2\right) + C\,\|m^\alpha\|_p\,\|Dv\|_2\,.$$

□

Corollary 40. *Let (v,m) be a solution of (2.1). Suppose Assumptions 1, 2, and 3 hold, $\lambda_1 > 0$ and $\lambda_2 > 0$. Then*

$$\left|\int_{\mathbb{T}^d} vdx\right| \le \frac{C}{\lambda_1} + \frac{C}{\lambda_2} + \int_{\mathbb{T}^d} \frac{g(m)}{\lambda_2}dx + \left(\frac{C}{\lambda_1} + \frac{C}{\lambda_2}\right)\|Dv\|_2\,.$$

Proof. This simple fact comes from the observation that the terms $\frac{C}{\lambda_1}$, $\frac{C}{\lambda_2}$, $\int_{\mathbb{T}^d} \frac{g(m)}{\lambda_2}dx$, $\frac{C}{\lambda_2}\,\|Dv\|_2$ and $\frac{C}{\lambda_1}\,\|Dv\|_2$ are non-negative. □

2.4 Lower bounds on m

Proposition 41. *Let (v, m) be a solution of (2.1). Suppose $m > 0$ and fix $\theta, \beta > 0$. Then*

$$\left(\int_{\mathbb{T}^d} m^\beta dx\right)^{-1} \leq \left(\int_{\mathbb{T}^d} \frac{1}{m^\theta} dx\right)^{\frac{\beta}{\theta}}.$$

For completion we give the proof, which is the same as in the one dimensional case.

Proof. Let $b > 0$,

$$1 = \int_{\mathbb{T}^d} \frac{m^b}{m^b} dx \leq \left(\int_{\mathbb{T}^d} m^{bp} dx\right)^{1/p} \left(\int_{\mathbb{T}^d} m^{-bp'} dx\right)^{1/p'}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Choose b and p so that $bp = \beta$ and $bp' = \theta$, which ends the proof. □

Lemma 42. *Suppose Assumptions 1, 2, and 5. Then there exists κ_1 and κ_2 positive constants such that*

$$|D_p H - p|^2 \leq \kappa_1 |p|^2 + \kappa_2.$$

Proof. Indeed, 1, 2, and 5 imply

$$-D_p H \cdot p \leq -(1 + c_1) H + C_1,$$
$$-H \leq -c_2 |p|^2 + C_2,$$

and

$$|D_p H|^2 \leq C_3 |p|^2 + \hat{C}_3.$$

Putting the above together,

$$\begin{aligned} |D_p H - p|^2 &\leq |D_p H|^2 + |p|^2 - 2 D_p H \cdot p \\ &\leq (C_3 + 1) |p|^2 + \hat{C}_3 - 2(1 + c_1) H + 2C_1 \\ &\leq (C_3 + 1 - 2c_2 - 2c_1 c_2) |p|^2 + 2C_1 + 2C_2 + \hat{C}_3 + 2c_1 C_2. \end{aligned}$$

The constant $(C_3 + 1 - 2c_2 - 2c_1 c_2)$ can be made positive by increasing C_3 or decreasing c_1 and c_2. □

Proposition 43. *Let (v, m) be a solution of (2.1). Suppose Assumptions 1, 2, and 5 hold, and $m > 0$. If the constants given by Lemma 42 satisfy $\kappa_1 < \frac{4c_2}{\alpha}$ and $\kappa_2 < \frac{4}{\alpha}\left(\frac{\alpha}{\alpha+1} - C_2\right)$, then*

1. *if Assumption 6 holds with $\gamma < \alpha$ or Assumption 7 holds we have*

$$\int_{\mathbb{T}^d} \frac{1}{m^\alpha} + \left|D\left(m^{-\frac{\alpha}{2}}\right)\right|^2 dx \leq C - C\int_{\mathbb{T}^d} v dx.$$

2. *if Assumption 6 holds with $\gamma \geq \alpha$ we have*

$$\int_{\mathbb{T}^d} \frac{1}{m^\alpha} + \left|D\left(m^{-\frac{\alpha}{2}}\right)\right|^2 dx \leq C - C\int_{\mathbb{T}^d} v dx + C\left(\int_{\mathbb{T}^d} g(m) dx\right)^{\frac{\gamma-\alpha}{\gamma}}.$$

Proof. Take the first identity in (2.1) and multiply it by $\frac{1}{m^\alpha}$. Multiply the second identity by

$\frac{\alpha}{(\alpha+1)m^{\alpha+1}}$, subtract it from the first, and get

$$\int_{\mathbb{T}^d} \alpha \frac{|Dm|^2}{m^{\alpha+2}} + \frac{\delta\alpha}{(\alpha+1)m^{\alpha+1}} + \frac{H}{m^\alpha} + \frac{\alpha}{(\alpha+1)m^\alpha} dx = \int_{\mathbb{T}^d} \frac{g(m)}{m^\alpha} - v + \frac{\alpha}{\alpha+1} + \frac{\alpha}{m^{\alpha+1}} (Dv - D_pH) \cdot Dm dx.$$

Thus, observing that $\delta \geq 0$ and using Assumption 2,

$$\int_{\mathbb{T}^d} \alpha \frac{|Dm|^2}{m^{\alpha+2}} + c_2 \frac{|Dv|^2}{m^\alpha} + \left(\frac{\alpha}{\alpha+1} - C_2 \right) \frac{1}{m^\alpha} dx \leq C + \int_{\mathbb{T}^d} \frac{g(m)}{m^\alpha} - v + \frac{\alpha}{m^{\alpha+1}} (Dv - D_pH) \cdot Dm dx.$$

From Cauchy's inequality with weights $\frac{1}{4(1-\epsilon)}$ and $(1-\epsilon)$,

$$\begin{aligned} \int_{\mathbb{T}^d} \alpha \frac{|Dm|^2}{m^{\alpha+2}} + c_2 \frac{|Dv|^2}{m^\alpha} + \left(\frac{\alpha}{\alpha+1} - C_2 \right) \frac{1}{m^\alpha} dx \\ \leq C + \int_{\mathbb{T}^d} \frac{g(m)}{m^\alpha} - v + \alpha \left(\frac{|Dv - D_pH|^2}{4(1-\epsilon)m^\alpha} + (1-\epsilon) \frac{|Dm|^2}{m^{\alpha+2}} \right) dx \\ \leq C + \int_{\mathbb{T}^d} \frac{g(m)}{m^\alpha} - v + \frac{\alpha\kappa_1}{4(1-\epsilon)} \frac{|Dv|^2}{m^\alpha} + \frac{\alpha\kappa_2}{4(1-\epsilon)} \frac{1}{m^\alpha} + \alpha(1-\epsilon) \frac{|Dm|^2}{m^{\alpha+2}} dx. \end{aligned}$$

As long as $C_2 < \frac{\alpha}{\alpha+1}$, the term $\frac{g(m)}{m^\alpha}$ can be handled in the following way: if Assumption 7 or 6 with $0 < \gamma < \alpha$ hold then it can be absorbed in the left hand side by noting that for any $\epsilon > 0$ we have

$$\frac{g(m)}{m^\alpha} \leq \frac{\epsilon}{m^\alpha} + C_\epsilon.$$

Choosing ϵ small enough the result follows. In the case $\gamma \geq \alpha$ it suffices to use Hölder inequality.

We finally get

$$\int_{\mathbb{T}^d} \frac{\epsilon}{m^\alpha} + \epsilon \frac{4}{\alpha^2} \frac{|Dm|^2}{m^{\alpha+2}} dx \leq C - \int_{\mathbb{T}^d} v dx + \mu \left(\int_{\mathbb{T}^d} g(m) dx \right)^{\frac{\gamma-\alpha}{\gamma}},$$

where μ is a large enough constant if $\gamma \geq \alpha$ or zero otherwise. □

Corollary 44. *Let (v, m) be a solution of (2.1). Suppose Assumptions 1, 2, and 5 hold, $m > 0$, $\delta > 0$, and $\alpha > 1$. If the constants given by Lemma 42 satisfy $\kappa_1 \leq \frac{4c_2}{\alpha}$ and $\kappa_2 < \frac{4}{\alpha}\left(\frac{\alpha}{\alpha+1} - C_2\right)$, then*

1. *if Assumption 6 holds with $\gamma < \alpha$ or Assumption 7 holds we have*

$$\int_{\mathbb{T}^d} \frac{1}{m^\alpha} + \left| D \left(m^{-\frac{\alpha}{2}} \right) \right|^2 dx \leq C + C_\delta \|Dv\|_2 .$$

2. *if Assumption 6 holds with $\gamma \geq \alpha$ we have*

$$\int_{\mathbb{T}^d} \frac{1}{m^\alpha} + \left| D \left(m^{-\frac{\alpha}{2}} \right) \right|^2 dx \leq C + C_\delta \|Dv\|_2 + C \left(\int_{\mathbb{T}^d} g(m) dx \right)^{\frac{\gamma-\alpha}{\gamma}}.$$

Proof. We combine the bound on $-\int_{\mathbb{T}^d} v dx$ obtained from Corollary 38 with Proposition 43,

$$\begin{aligned}\int_{\mathbb{T}^d} \frac{1}{m^\alpha} + \left|D\left(m^{-\frac{\alpha}{2}}\right)\right|^2 dx &\le C - C\int_{\mathbb{T}^d} vdx \\ &\le C + C\left(\frac{C}{\lambda_1} + \frac{C}{\lambda_1}\|Dv\|_2\right) \\ &\le C + C\left(\int_{\mathbb{T}^d} \frac{1}{m^\alpha}dx\right)^{\frac{1}{\alpha}} + C_\delta \|Dv\|_2 \\ &\le C + C_\delta\|Dv\|_2.\end{aligned}$$

□

Proposition 45. *Let* (v, m) *be a solution of* (2.1)*. Suppose Assumption 5 holds and* $m > 0$ *then*

$$\int_{\mathbb{T}^d} \frac{\delta}{m}dx + \frac{1}{2}\|D\ln m\|_2^2 \le C + C\|Dv\|_2^2.$$

Proof. Divide the second identity in (2.1) by m and integrate

$$\int_{\mathbb{T}^d} -\frac{\operatorname{div}(mD_pH)}{m}dx = \int_{\mathbb{T}^d} 1 - m^\alpha + \frac{\Delta m}{m} + \frac{\delta}{m}dx$$

then we have

$$\int_{\mathbb{T}^d} -D\ln m \cdot D_pH dx = \int_{\mathbb{T}^d} 1 - m^\alpha + \|D\ln m\|^2 + \frac{\delta}{m}dx.$$

With Hölder's inequality,

$$\begin{aligned}\int_{\mathbb{T}^d} \frac{\delta}{m} + \|D\ln m\|^2 dx &= \int_{\mathbb{T}^d} -D\ln m \cdot D_pH - 1 + m^\alpha dx \\ &\le C + \int_{\mathbb{T}^d} \frac{1}{2}\|D\ln m\|^2 + \frac{\|D_pH\|^2}{2}dx,\end{aligned}$$

where we use Assumption 5 to finish. □

2.5 Regularity for the Hamilton-Jacobi equation

We must now bound $\int_{\mathbb{T}^d} vdx$ in order to get H^1 bounds for v.

Proposition 46. *Let* (v, m) *be a solution of* (2.1)*. Suppose Assumptions 1, 2, 3, and 5 hold,* $m > 0$*, and* $\delta > 0$ *then*

$$\left|\int_{\mathbb{T}^d} vdx\right| \le C_\delta + C_\delta\left(\int_{\mathbb{T}^d} g(m)dx\right)^{1+\alpha}.$$

Proof. We use $\frac{1}{\lambda_1} \le C_\delta$ on Corollary 39 to get

$$\|Dv\|_2^2 \le C_\delta + C_\delta\int_{\mathbb{T}^d} g(m)dx$$

which, together with Proposition 45 and Proposition 41, gives

$$\frac{1}{\lambda_2} \le \left(\int_{\mathbb{T}^d} \frac{1}{m}dx\right)^\alpha \le C_\delta + C_\delta\left(\int_{\mathbb{T}^d} g(m)dx\right)^\alpha.$$

Plugging this into Corollary 40 gives

$$\left|\int_{\mathbb{T}^d} v dx\right| \leq C_\delta + C_\delta \left(\int_{\mathbb{T}^d} g(m) dx\right)^{\alpha+1}.$$

The power $\alpha+1$ being the largest of the ones present in the estimate. ☐

The task that remains now is to bound $\int_{\mathbb{T}^d} g(m)dx$ if $\gamma > \alpha+1$. This is not completely trivial but thanks to Corollaries 31 and 32 we were able to obtain bounds that depend on δ.

Proposition 47. *Let (v, m) be a solution of (2.1). Suppose Assumptions 1, 2, 3, and 5 hold, $m > 0$, and $\delta > 0$. If either Assumption 6 holds with $\gamma < \max\{\alpha+1, \frac{1}{\alpha}\}$ or Assumption 7 holds, then*

$$\int_{\mathbb{T}^d} g(m) dx \leq C_\delta.$$

Proof. We write Corollary 31 as

$$\left(\int_{\mathbb{T}^d} g(m) dx\right)^{1+\frac{1}{\gamma}} \leq C + C\int_{\mathbb{T}^d} v\left(m + \delta - m^\alpha\right) dx$$

and observe that Proposition 35 yields, since $\int_{\mathbb{T}^d} m dx$ and $\int_{\mathbb{T}^d} m^\alpha dx$ are bounded,

$$\int_{\mathbb{T}^d} v\left(m+\delta - m^\alpha\right) dx \leq C\left\|Dv\right\|_2 + C\left|\int_{\mathbb{T}^d} v dx\right|.$$

Using Corollary 39 and Proposition 46 we obtain

$$\begin{aligned}\left(\int_{\mathbb{T}^d} g(m) dx\right)^{1+\frac{1}{\gamma}} &\leq C_\delta + C_\delta\left(\int_{\mathbb{T}^d} g(m) dx\right)^{\frac{1}{2}} \\ &+ C_\delta\left(\int_{\mathbb{T}^d} g(m) dx\right)^{1+\alpha}.\end{aligned}$$

Therefore, the result follows since $\gamma < \frac{1}{\alpha}$ implies that the power on the right is larger than those on the left.

The same reasoning applies to Corollary 32. ☐

Proposition 48. *Let (v, m) be a solution of (2.1). Suppose Assumptions 1, 2, 3, and 5 hold, $m > 0$, and $\delta > 0$. If either Assumption 6 holds with $\gamma \leq \max\{\alpha+1, \frac{1}{\alpha}\}$ or Assumption 7 holds, then*

$$\left|\int_{\mathbb{T}^d} v dx\right| \leq C_\delta.$$

Proof. Observe that the integrability of $g(m)$ with Proposition 46 yield the result.

☐

Theorem 49. *Let (v, m) be a solution of (2.1). Suppose Assumptions 1, 2, 3, and 5 hold, $m > 0$, and $\delta > 0$. If either Assumption 6 holds with $\gamma \leq \max\{\alpha+1, \frac{1}{\alpha}\}$ or Assumption 7 holds.*

Then

$$\|\ln m\|_{H^1} \quad \text{and} \quad \|v\|_{H^1}$$

are bounded.

Proof. We have $Dv \in L^2(\mathbb{T}^d)$, and $v \in L^1(\mathbb{T}^d)$. Poincaré's theorem then can be used to get $v \in L^2(\mathbb{T}^d)$.

By Proposition 45 we have $D(\ln m) \in L^2(\mathbb{T}^d)$, from which we conclude by Sobolev's theorem 34 that $\ln m \in L^{2^*}(\mathbb{T}^d)$. ☐

Corollary 50. *Let (v, m) be a solution of (2.1). Suppose Assumptions 1, 2, 3, and 5 hold, $m > 0$, and $\delta > 0$. If either Assumption 6 holds with $\gamma \leq \max\{\alpha+1, \frac{1}{\alpha}\}$ or Assumption 7 holds. If $d = 2$, then*

$$\frac{1}{m} \in L^p(\mathbb{T}^d)$$

for any $p \geq 1$.

Proof. Corollary 44 and Theorem 49 imply

$$m^{-\frac{\alpha}{2}}, D\left(m^{-\frac{\alpha}{2}}\right) \in L^2(\mathbb{T}^d), \text{ that is } m^{-\frac{\alpha}{2}} \in W^{1,2}(\mathbb{T}^d).$$

Sobolev's Theorem then gives

$$m^{-\frac{\alpha}{2}} \in L^{2^*}(\mathbb{T}^d).$$

If $d = 2$ then $\frac{1}{m} \in L^p(\mathbb{T}^d)$ for any $p \geq 1$. If $d = 3$ then $\frac{1}{m} \in L^{3\alpha}(\mathbb{T}^d)$.

□

Proposition 51. *Let (v, m) be a solution of (2.1). Suppose Assumptions 1, 2, 3, and 5 hold, $m > 0$, and $\delta > 0$. If either Assumption 6 holds with $\gamma < \max\{\alpha+1, \frac{1}{\alpha}\}$ or Assumption 7 holds, then*

$$\int_{\mathbb{T}^d} m\,|Dv|^2\,dx \leq C_\delta.$$

Proof. Propositions 29, 35, 36, and Assumption 2 imply

$$\begin{aligned}\int_{\mathbb{T}^d} m\,|Dv|^2\,dx &\leq \left(\int_{\mathbb{T}^d} v dx\right)\left(\int_{\mathbb{T}^d} m + \delta - m^\alpha dx\right) + C\,\|Dv\|_2 \\ &\leq C + C\left|\int_{\mathbb{T}^d} v dx\right|.\end{aligned}$$

The proof is finished with the application of Proposition 48. □

Proposition 52. *Let (v, m) be a solution of (2.1). Suppose Assumptions 1, 2, 3, and 5 hold, $m > 0$, and $\delta > 0$. If either Assumption 6 holds with $\gamma < \max\{\alpha+1, \frac{1}{\alpha}\}$ or Assumption 7 holds, then*

$$\int_{\mathbb{T}^d} \left|D\left(m^{\frac{1}{2}}\right)\right|^2 dx \leq C_\delta.$$

Proof. Integrate the second identity in (2.1) against $\ln m$ to obtain

$$\begin{aligned}4\int_{\mathbb{T}^d} \left|D\left(m^{\frac{1}{2}}\right)\right|^2 dx &= \int_{\mathbb{T}^d} (m - m^{\alpha+1} + \delta)\ln m - D_pHDmdx \\ &\leq \int_{\mathbb{T}^d} (m - m^{\alpha+1} + \delta)\ln m + |D_pH|^2 m + \frac{1}{4m}|Dm|^2 dx \\ &\leq C + \int_{\mathbb{T}^d} (m+\delta)\ln m + C\,|Dv|^2 m + \left|D\left(m^{\frac{1}{2}}\right)\right|^2 dx \\ &\leq C + \int_{\mathbb{T}^d} Cm^{\frac{2d}{d+2}} + C\,|\ln m|^{2^*} + C_\delta + \left|D\left(m^{\frac{1}{2}}\right)\right|^2 dx.\end{aligned}$$

We apply Assumption 3 and Theorem 49 explicitly in the final argument. □

Corollary 53. *Let (v, m) be a solution of (2.1). Suppose Assumptions 1, 2, 3, and 5 hold, $m > 0$, and $\delta > 0$. If either Assumption 6 holds with $\gamma < \max\{\alpha+1, \frac{1}{\alpha}\}$ or Assumption 7 holds, then*

$$\|m\|_{L^{\frac{2^*}{2}}(\mathbb{T}^d)} \leq C_\delta \text{ and } \|v\|_{L^{2^*}(\mathbb{T}^d)} \leq C_\delta.$$

Proof. These are consequences of Sobolev's inequality. □

Proposition 54. *Let* (v, m) *be a solution of* (2.1)*. Suppose Assumptions 1, 2, 3, 4, and 5 hold,* $m > 0$*,* $\delta > 0$*, and* $\alpha < \frac{1}{d-2}$*.*

Then we have

$$\int_{\mathbb{T}^d} g'(m)\,|Dm|^2 + \frac{\xi}{2} m \left|D^2 v\right|^2 dx \leq C + C\int_{\mathbb{T}^d} v \left|D\left(m^{\frac{\alpha+1}{2}}\right)\right|^2 dx.$$

Proof. Apply the Laplacian to the first equation in (1.1)

$$\Delta_x H + 2\mathrm{Tr}\left(D_{px}HD^2v\right) + \mathrm{Tr}\left(D_p^2H\left(D^2v\right)^2\right) + D_pHD\Delta v$$
$$= \mathrm{div}\left(g'(m)Dm\right) - \mathrm{div}\left(\alpha m^{\alpha-1}vDm + m^\alpha Dv\right) + \Delta\Delta v.$$

Multiply by m and integrate to obtain

$$\int_{\mathbb{T}^d} m\Delta_x H + 2m\mathrm{Tr}\left(D_{px}HD^2v\right) + m\mathrm{Tr}\left(D_p^2H\left(D^2v\right)^2\right) + mD_pHD\Delta v dx$$
$$= \int_{\mathbb{T}^d} m\mathrm{div}\left(g'(m)Dm\right) - m\mathrm{div}\left(\alpha m^{\alpha-1}vDm + m^\alpha Dv\right) + m\Delta\Delta v dx.$$

Integrating by parts

$$\int_{\mathbb{T}^d} g'(m)\,|Dm|^2 + 2m\mathrm{Tr}\left(D_{px}HD^2v\right) + m\mathrm{Tr}\left(D_p^2H\left(D^2v\right)^2\right) dx$$
$$= \int_{\mathbb{T}^d} -m\Delta_x H + \alpha m^{\alpha-1}v\,|Dm|^2 + m^\alpha Dv\cdot Dm + \Delta v\left(\Delta m + \mathrm{div}\left(mD_pH\right)\right) dx.$$

Observing that m is a solution to the second equation in (2.1)

$$\int_{\mathbb{T}^d} g'(m)\,|Dm|^2 + 2m\mathrm{Tr}\left(D_{px}HD^2v\right) + m\mathrm{Tr}\left(D_p^2H\left(D^2v\right)^2\right) dx$$
$$= \int_{\mathbb{T}^d} -m\Delta_x H + \alpha m^{\alpha-1}v\,|Dm|^2 + \frac{D\left(m^{\alpha+1}\right)}{\alpha+1}\cdot Dv + \Delta v\left(m^{\alpha+1} - m - \delta\right) dx$$
$$= \int_{\mathbb{T}^d} -m\Delta_x H + \alpha m^{\alpha-1}v\,|Dm|^2 + \Delta v\left(\frac{\alpha}{\alpha+1}m^{\alpha+1} - m\right) dx.$$

Now, use some hypothesis to simplify the expression.

$$\int_{\mathbb{T}^d} g'(m)\,|Dm|^2 + \xi m\left|D^2v\right|^2 \leq \int_{\mathbb{T}^d} g'(m)\,|Dm|^2 + m\mathrm{Tr}\left(D_p^2H\left(D^2v\right)^2\right) dx$$
$$\leq \int_{\mathbb{T}^d} m\left(\left|D_x^2H\right| + C\left|D_{px}H\right|^2 + \frac{\xi}{4}\left|D^2v\right|^2\right) + \alpha m^{\alpha-1}v\,|Dm|^2 + \frac{\xi}{4}m\left|D^2v\right|^2$$
$$+ Cm\left(\frac{\alpha}{\alpha+1}m^\alpha - 1\right)^2 dx$$
$$\leq C + \int_{\mathbb{T}^d} m\frac{\xi}{2}\left|D^2v\right|^2 + \alpha m^{\alpha-1}v\,|Dm|^2 + Cm\left(\frac{\alpha}{\alpha+1}m^\alpha - 1\right)^2 dx.$$

□

2.6 Regularity by the adjoint method

Consider the time-dependent system

$$\begin{cases} H(x, Dv) = g(m) - m^\alpha v + \Delta v, \text{ on } \mathbb{T}^d \\ \rho_t - \operatorname{div}(D_p H \rho) = -m^\alpha \rho + \Delta \rho, \text{ on } \mathbb{T}^d \times \mathbb{R}^+. \end{cases} \tag{2.6}$$

with initial condition $\rho(x,t) = \delta_{x_0}$, for some $x_0 \in \mathbb{T}^d$. We assume m is a smooth, non-negative given function.

The second equation in (2.6) is the time dependent equation for the adjoint variable of v that also safisfies $-v_t + H(x, Dv) = g(m) - m^\alpha v + \Delta v$.

By the maximum principle, $\rho \geq 0$. Integrating the second equation of (2.6) we get

$$\frac{d}{dt} \int_{\mathbb{T}^d} \rho dx = - \int_{\mathbb{T}^d} m^\alpha \rho dx,$$

which gives $\rho(x,t) \leq 1$ for all $t > 0$.

Further set of Assumptions:

8. $g(m) \in L^r(\mathbb{T}^d)$ for some $r > d$. We prove in Proposition 64 that $g(m) \in L^r(\mathbb{T}^d)$ for any $r > 1$.
9. $|D_x H| \leq C + \psi(x) |p|^\beta$, where $0 \leq \beta < 2$, r is the same as above, and $\psi \in L^{\frac{2r}{2-\beta}}(\mathbb{T}^d)$.
10. v solution to the first equation in (2.1) satisfies the *a priori* bound $v \in L^2(\mathbb{T}^d)$.

Proposition 55. *Let (v, ρ) be a solution to (2.6). Then, for any $T > 0$*

$$v(x_0) = \int_{\mathbb{T}^d} v(x)\rho(x,0)dx = \int_{\mathbb{T}^d} v(x)\rho(x,T)dx + \int_0^T \int_{\mathbb{T}^d} (D_p H \cdot Dv - H + g(m))\, \rho dx dt.$$

Proof. Integrate the first identity in (2.6) against ρ and use the identity and initial condition for ρ. □

For fixed $T > 0$, define $\|\rho\|_{L^1(L^q(dx),dt)} = \int_0^T \|\rho(\cdot,t)\|_{L^q(\mathbb{T}^d)} dt$. Denote $\mathbf{osc}\,(f) = \sup_x f - \inf_x f$, for any bounded function $f : \mathbb{T}^d \to \mathbb{R}$.

Corollary 56. *Let (v, m) be a solution to (2.1) and (v, ρ) be a solution to (2.6). Suppose Assumptions 1, 2, 3, 5, and 8 hold, $m > 0$, and $\delta > 0$. Then, for any $T > 0$*

$$\int_0^T \int_{\mathbb{T}^d} \rho |Dv|^2 dx dt \leq C_\delta + C \; Lip\,(v) + C \|\rho\|_{L^1(L^q(dx),dt)},$$

where q satisfies $\frac{1}{q} + \frac{1}{r} = 1$.

Proof. We use Assumption 1 in Proposition 55

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^d} \left(c_1 |Dv|^2 + C_1 + g(m) \right) \rho dx dt &\leq v(x_0) + \int_{\mathbb{T}^d} -v(x)\rho(x,T)dx \\ &\leq \sup_x v - \inf_x v \int_{\mathbb{T}^d} \rho(x,T)dx \\ &\leq \sup_x v \left(1 - \int_{\mathbb{T}^d} \rho(x,T)dx \right) + \left(\sup_x v - \inf_x v \right) \int_{\mathbb{T}^d} \rho(x,T)dx \\ &\leq C \int_{\mathbb{T}^d} v(x)dx + C \text{ Lip}\,(v), \end{aligned}$$

where we use $v(y) \leq \int_{\mathbb{T}^d} v(x)dx + \text{ Lip}\,(v)$ for every $y \in \mathbb{T}^d$, and $\mathbf{osc}\,(v) \leq C \text{ Lip}\,(v)$.

Noting that, by Hölder inequality,

$$\int_0^T \int_{\mathbb{T}^d} \rho dxdt \leq \|\rho\|_{L^1(L^q(dx),dt)},$$

and

$$\int_0^T \int_{\mathbb{T}^d} |g(m)| \rho dxdt \leq \|g(m)\|_{L^r(\mathbb{T}^d)} \|\rho\|_{L^1(L^q(dx),dt)},$$

ends the proof. □

Proposition 57. *Let (v, ρ) be a solution to (2.6). Then, for any $T > 0$, $0 < \nu < 1$ such that $m^\alpha \in L^{\frac{q}{q-\nu}}(\mathbb{T}^d)$, and $\epsilon_1 > 0$ there exists C_{ϵ_1} such that*

$$\int_0^T \int_{\mathbb{T}^d} \left|D\left(\rho^{\frac{\nu}{2}}\right)\right|^2 dxdt \leq C_{\epsilon_1} + \epsilon_1 \int_0^T \int_{\mathbb{T}^d} \rho |Dv|^2 dxdt + C \|\rho\|^\nu_{L^1(L^q(dx),dt)},$$

where q satisfies $\frac{1}{q} + \frac{1}{r} = 1$.

Proof. We obtain, by multiplying the second identity in (2.6) by $\rho^{\nu-1}$, integrating by parts, and using $\int_{\mathbb{T}^d} \rho^\nu(x,t)dx \leq 1$,

$$\begin{aligned}
4\frac{(1-\nu)}{\nu^2} \int_0^T \int_{\mathbb{T}^d} \left|D\left(\rho^{\frac{\nu}{2}}\right)\right|^2 dxdt &= \frac{1}{\nu} \int_{\mathbb{T}^d} \rho^\nu(x,T) - \rho^\nu(x,0)dx \\
&\quad + \int_0^T \int_{\mathbb{T}^d} m^\alpha \rho^\nu + (\nu-1)\rho^{\nu-1} D_pH D\rho dxdt \\
&\leq C + \int_0^T \int_{\mathbb{T}^d} m^\alpha \rho^\nu dxdt + \epsilon \int_0^T \int_{\mathbb{T}^d} \left|D\left(\rho^{\frac{\nu}{2}}\right)\right|^2 dxdt \\
&\quad + C_\epsilon \int_0^T \int_{\mathbb{T}^d} |D_pH|^2 \rho^\nu dxdt.
\end{aligned}$$

Now, given $\epsilon_1 > 0$ there exists $C_{\epsilon_1} > 0$ such that $\rho^\nu \leq \epsilon_1 \rho + C_{\epsilon_1}$. This implies, after fixing ϵ sufficiently small,

$$C_\epsilon \int_0^T \int_{\mathbb{T}^d} |D_pH|^2 \rho^\nu dxdt \leq C_{\epsilon_1} + \epsilon_1 \int_0^T \int_{\mathbb{T}^d} \rho |Dv|^2 dxdt.$$

Using Proposition 27, Hölder's and Jensen's inequalities we achieve the bound

$$\begin{aligned}
\int_0^T \int_{\mathbb{T}^d} m^\alpha \rho^\nu dxdt &\leq \int_0^T \|m^\alpha\|_{L^{\frac{q}{q-\nu}}(\mathbb{T}^d)} \|\rho^\nu(\cdot,t)\|_{L^{\frac{q}{\nu}}(\mathbb{T}^d)} dt \\
&\leq C \int_0^T \|\rho(\cdot,t)\|^\nu_{L^q(\mathbb{T}^d)} dt \\
&\leq C \|\rho\|^\nu_{L^1(L^q(dx),dt)}.
\end{aligned}$$

□

Since ρ^ν does not make sense for $\rho = \delta_{x_0}$, we consider ρ^ϵ the solution to the second equation in (2.6) with $\rho^\epsilon(x,0) = \eta_\epsilon(x)$, where η_ϵ satisfies $\int_{\mathbb{T}^d} \eta_\epsilon dx = 1$ and $\eta_\epsilon \rightharpoonup \delta_{x_0}$ with $\epsilon \to 0$. We send $\epsilon \to 0$ after the estimates are finished.

Corollary 58. *Let (v, m) be a solution to (2.1) and (v, ρ) be a solution to (2.6). Suppose Assumptions 1, 2, 3, 5, and 8 hold, $m > 0$, and $\delta > 0$. Then, for any $T > 0$, $0 < \nu < 1$ such that*

$m^\alpha \in L^{\frac{q}{q-\nu}}(\mathbb{T}^d)$, and $\epsilon_1 > 0$ there exists C_{ϵ_1} such that

$$\int_0^T \int_{\mathbb{T}^d} \left| D\left(\rho^{\frac{\nu}{2}}\right)\right|^2 dxdt \le C_{\epsilon_1} + C_\delta + \epsilon_1 C \; Lip\,(v) + C \left\|\rho\right\|_{L^1(L^q(dx),dt)},$$

where q satisfies $\frac{1}{q} + \frac{1}{r} = 1$.

Proof. Combine Proposition 57 with Corollary 56 □

Define $\nu_{rd} = 1 + \frac{1}{r} - \frac{2}{d}$. We have $\nu_{rd} < 1$ if $r > d$.

Proposition 59. *Let (v, ρ) be a solution to (2.6). Then, for any $T > 0$, and $\nu > \nu_{rd}$ there exists $0 < \mu < 1$ such that*

$$\left\|\rho\right\|_{L^1(L^q(dx),dt)} \le C + C\left(\int_0^T \int_{\mathbb{T}^d} \left| D\left(\rho^{\frac{\nu}{2}}\right)\right|^2 dxdt\right)^\mu,$$

where q satisfies $\frac{1}{q} + \frac{1}{r} = 1$.

Proof. For $1 \le p_0 < p_1 < \infty$ and $0 < \theta < 1$ we have the interpolation inequality

$$\left\|f\right\|_{L^{p_\theta}} \le \left\|f\right\|_{L^{p_1}}^\theta \left\|f\right\|_{L^{p_0}}^{1-\theta},$$

with

$$\frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_0}.$$

Let $p = 2^*$ or sufficiently large, if $d > 2$ or $d = 2$ respectively. Take $p_0 = 1, \;\; p_1 = \frac{\nu p}{2}$.

If $d = 2$ and $\nu > \frac{1}{r}$ then $q < p_1$. Indeed, the former inequality is equivalent, using that q is the exponent conjugate of r, to $q < \frac{1}{1-\nu}$. Given $\frac{1}{r} < \nu < 1$, there exists p large enough so that $\frac{1}{1-\nu} < \frac{\nu p}{2}$.

If $d > 2$ and $\nu > 1 + \frac{1}{r} - \frac{2}{d}$ then $q < p_1 = \frac{\nu d}{d-2}$. Indeed, $\nu > 1 + \frac{1}{r} - \frac{2}{d}$ is equivalent to $q < \frac{d}{d(2-\nu)-2}$, and $\nu > 1 - \frac{2}{d}$ is equivalent to $\frac{d}{d(2-\nu)-2} < \frac{\nu d}{d-2}$.

Setting $p_\theta = q$ we have

$$\theta = \frac{1 - \frac{1}{q}}{1 - \frac{1}{p_1}} = \frac{\nu p}{r(\nu p - 2)},$$

which we note is smaller than ν if $\nu > \frac{1}{r} + \frac{2}{p}$.

Sobolev's inequality gives

$$\left(\int_{\mathbb{T}^d} \rho^{\frac{\nu p}{2}}(x,t)dx\right)^{\frac{1}{p}} \le C + C\left(\int_{\mathbb{T}^d} \left| D\left(\rho^{\frac{\nu}{2}}\right)\right|^2 dx\right)^{\frac{1}{2}},$$

and so

$$\left\|\rho(\cdot,t)\right\|_{L^{\frac{\nu p}{2}}(\mathbb{T}^d)} \le C + C\left(\int_{\mathbb{T}^d} \left| D\left(\rho^{\frac{\nu}{2}}\right)\right|^2 dx\right)^{\frac{1}{\nu}}.$$

As $\left\|\rho(\cdot,t)\right\|_{L^1} \le 1$, we get, using the interpolation inequality that

$$\left\|\rho\right\|_{L^1(L^q(dx),dt)} \le C + C\int_0^T \left(\int_{\mathbb{T}^d} \left| D\left(\rho^{\frac{\nu}{2}}\right)\right|^2 dx\right)^\mu dt,$$

where $\mu = \frac{\theta}{\nu} < 1$ which allows us to use Jensen's inequality

$$\left\|\rho\right\|_{L^1(L^q(dx),dt)} \le C + C\left(\int_0^T \int_{\mathbb{T}^d} \left| D\left(\rho^{\frac{\nu}{2}}\right)\right|^2 dxdt\right)^\mu.$$

□

Corollary 60. *Let (v,m) be a solution to (2.1) and (v,ρ) be a solution to (2.6). Suppose Assumptions 1, 2, 3, 5, and 8 hold, $m>0$, and $\delta>0$. Then, for any $T>0$, $\nu>\nu_{rd}$, and $\epsilon_1>0$ there exists C_{ϵ_1} such that*

$$\int_0^T\int_{\mathbb{T}^d}\left|D\left(\rho^{\frac{\nu}{2}}\right)\right|^2 dxdt \leq C_{\epsilon_1}+C_\delta+\epsilon_1 C\ Lip\,(v)\,,$$

where q satisfies $\frac{1}{q}+\frac{1}{r}=1$.

Proof. This is one result of the combination of Corollary 58 and Proposition 59. □

Corollary 61. *Let (v,m) be a solution to (2.1) and (v,ρ) be a solution to (2.6). Suppose Assumptions 1, 2, 3, 5, and 8 hold, $m>0$, and $\delta>0$. Then, for any $T>0$, and $\nu>\nu_{rd}$ there exists $0<\mu<1$ such that*

$$\|\rho\|_{L^1(L^q(dx),dt)} \leq C_{\epsilon_1}+C_\delta+C\left(\epsilon_1\ Lip\,(v)\right)^{\mu},$$

where q satisfies $\frac{1}{q}+\frac{1}{r}=1$.

Proof. Apply Corollary 58 to the estimate obtained in Proposition 59. □

Corollary 62. *Let (v,m) be a solution to (2.1) and (v,ρ) be a solution to (2.6). Suppose Assumptions 1, 2, 3, 5, and 8 hold, $m>0$, $\delta>0$, and $\nu>\nu_{rd}$. Then, for any $T>0$*

$$\int_0^T\int_{\mathbb{T}^d}\rho\,|Dv|^2\,dxdt \leq C_{\epsilon_1}+C_\delta+C\ Lip\,(v)\,,$$

where q satisfies $\frac{1}{q}+\frac{1}{r}=1$.

Proof. Corollaries 56 and 61 provide the right estimates. □

Proposition 63. *Let (v,m) be a solution to (2.1) and (v,ρ) be a solution to (2.6). Suppose Assumptions 1, 2, 3, 5, and 8 hold, $m>0$, $\delta>0$, and d is either 2 or 3. Then*

$$Lip\,(v)\leq C_\delta.$$

Proof. Let $\eta=D_{x_i}v$, then it solves

$$D_{x_i}H(x,Dv)+D_pH(x,Dv)D\eta = D_{x_i}g(m)-D_{x_i}\left(m^\alpha\right)v-m^\alpha\eta+\Delta\eta \tag{2.7}$$

Choose a smooth[1] $\phi(t)$ such that $\phi(0)=1$ and $\phi(T)=0$ and define $w(x,t)=\eta(x)\phi(t)$. We integrate (2.7) against $\rho(x,t)\phi(t)$ with respect to both variables

$$\int_0^T\int_{\mathbb{T}^d}\rho\phi D_{x_i}H(x,Dv)+\rho D_pH(x,Dv)Dwdxdt$$
$$=\int_0^T\int_{\mathbb{T}^d}\rho\phi D_{x_i}g(m)-\rho\phi D_{x_i}\left(m^\alpha\right)v-\rho m^\alpha w+\rho\Delta wdxdt.$$

Using the second identity in (2.6) integrated against w

$$\int_0^T\int -wm^\alpha\rho+\rho\Delta wdxdt=\int_0^T\int w\rho_t+\rho D_pHDwdxdt,$$

and the fact that $\int_0^T w\rho_t dt=-w(x,0)\rho(x,0)-\int_0^T w_t\rho dt$, $w_t\rho=\phi'\eta\rho$, $w(x,T)=0$, and $w(x,0)=\eta(x)$. We obtain

[1]This implies that $|\phi|$ and $|\phi'|$ are bounded.

$$-w(x_0,0)-\int_0^T\int_{\mathbb{T}^d}\phi'\eta\rho dxdt+\int_0^T\int_{\mathbb{T}^d}\rho D_pHDwdxdt=\int_0^T\int_{\mathbb{T}^d}-wm^\alpha\rho+\rho\Delta wdxdt,$$

and finally

$$-w(x_0,0)=\int_0^T\int_{\mathbb{T}^d}\rho\phi D_{x_i}H(x,Dv)-\rho\phi D_{x_i}g(m)+\rho\phi D_{x_i}\left(m^\alpha\right)v+\phi'\eta\rho dxdt$$

We observe that, for any $\epsilon>0$ we have $|\eta|\leq|Dv|\leq\epsilon\left|Dv\right|^2+C_\epsilon$. With $0\leq\phi\leq1$ we have

$$\begin{aligned}|w(x_0,0)|&\leq\int_0^T\int_{\mathbb{T}^d}\rho\left|\phi D_{x_i}H(x,Dv)+\phi D_{x_i}\left(m^\alpha\right)v+\phi'\eta\right|dxdt+C\int_0^T\left|\int_{\mathbb{T}^d}\phi D_{x_i}g(m)dx\right|dt\\&\leq\int_0^T\int_{\mathbb{T}^d}\rho\left|\phi\right|\left|D_{x_i}H(x,Dv)\right|+\rho\left|\phi\right|\left|D_{x_i}\left(m^\alpha\right)v\right|+\rho\left|\phi'\right|\left|\eta\right|dxdt\\&\quad+C\int_0^T\left|\int_{\mathbb{T}^d}\phi D_{x_i}g(m)dx\right|dt\\&\leq C\int_0^T\int_{\mathbb{T}^d}\rho\left|D_{x_i}H(x,Dv)\right|+\rho\left|D_{x_i}\left(m^\alpha\right)v\right|+\epsilon\rho\left|Dv\right|^2+C_\epsilon\rho dxdt\\&\quad+C\int_0^T\left|\int_{\mathbb{T}^d}\phi D_{x_i}g(m)dx\right|dt\end{aligned}\tag{2.8}$$

The first term on the right-hand side of (2.8) is estimated using Assumption 9 and Corollary 61

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^d}\rho\left|D_{x_i}H(x,Dv)\right|dxdt&\leq\int_0^T\int_{\mathbb{T}^d}C\rho+\psi\left|Dv\right|^\beta dxdt\\&\leq C+\int_0^T\int_{\mathbb{T}^d}\epsilon\rho\left|Dv\right|^2+C_\epsilon\psi^{\frac{2}{2-\beta}}\rho dxdt\\&\leq C+\int_0^T\int_{\mathbb{T}^d}\epsilon\rho\left|Dv\right|^2dxdt+C_\epsilon\int_0^T\left\|\psi^{\frac{2}{2-\beta}}\right\|_r\left\|\rho\right\|_{L^q(\mathbb{T}^d)}dt\\&\leq\epsilon\left(C_{\epsilon_1}+C_\delta+C\text{ Lip}\left(v\right)\right)+C_\epsilon\left\|\rho\right\|_{L^1(L^q(dx),dt)}\int_0^T\left\|\psi\right\|^{\frac{2}{2-\beta}}_{L^{\frac{2r}{2-\beta}}}dt\\&\leq C_{\epsilon_1}+C_\delta+\epsilon C\text{ Lip}\left(v\right)+C\left(\epsilon_1\text{ Lip}\left(v\right)\right)^\mu.\\&\leq C_{\epsilon_1}+C_\delta+\epsilon C\text{ Lip}\left(v\right).\end{aligned}\tag{2.9}$$

Now we estimate the last term on the right-hand side of (2.8). Noting

$$-\int_{\mathbb{T}^d}\rho D_{x_i}g(m)dx=\int_{\mathbb{T}^d}g(m)D_{x_i}\rho dx=\frac{2}{\nu}\int_{\mathbb{T}^d}g(m)\rho^{1-\frac{\nu}{2}}D_{x_i}\left(\rho^{\frac{\nu}{2}}\right)dx,$$

we get

$$\int_0^T\left|\int_{\mathbb{T}^d}\rho D_{x_i}g(m)dx\right|xdt\leq C\int_0^T\int_{\mathbb{T}^d}g(m)^2\rho^{2-\nu}+\left|D\left(\rho^{\frac{\nu}{2}}\right)\right|^2dxdt.$$

Note that

$$\int_{\mathbb{T}^d}g(m)^2\rho^{2-\nu}dx\leq\left\|g(m)^2\right\|_{L^{\frac{r}{2}}(\mathbb{T}^d)}\left\|\rho^{2-\nu}\right\|_{L^{\frac{r}{r-2}}(\mathbb{T}^d)}=\left\|g(m)\right\|^2_{L^r(\mathbb{T}^d)}\left\|\rho\right\|^{2-\nu}_{L^{\frac{r(2-\nu)}{r-2}}(\mathbb{T}^d)}.$$

Sobolev's inequality,

$$\left\|\rho(\cdot,t)\right\|^{\frac{\nu}{2}}_{L^{\frac{2^*\nu}{2}}}=\left\|\rho^{\frac{\nu}{2}}(\cdot,t)\right\|_{2^*}\leq C+\left\|D\left(\rho^{\frac{\nu}{2}}\right)(\cdot,t)\right\|_2,$$

gives

$$\|\rho(\cdot,t)\|_{L^{\frac{2^*\nu}{2}}} \leq C + \left\|D\left(\rho^{\frac{\nu}{2}}\right)(\cdot,t)\right\|_2^{\frac{2}{\nu}}.$$

Now we use the interpolation inequality with $p_0 = 1$, $p_{\theta_1} = \frac{r(2-\nu)}{r-2}$, and $p_1 = \frac{\nu 2^*}{2}$. This defines $\theta_1 = \frac{1-\frac{1}{p_{\theta_1}}}{1-\frac{1}{p_1}} = \frac{r-r\nu+2}{r(2-\nu)}\frac{\nu 2^*}{\nu 2^*-2}$, which we prove to satisfy $0<\theta_1<1$ for $\nu>\nu_{rd}$ sufficiently close to 1.

First,

$$0<\theta_1 \iff \left\{\frac{2}{p}<\nu<1+\frac{2}{r}\right\},$$

and then, as $\nu \to 1$, we observe $\theta_1 \to \frac{22^*}{r(2^*-2)}$. For $d=2$, substitute 2^* by a large enough constant. Otherwise, $\theta_1 \to \frac{d}{r}$.

Continuing with the interpolation,

$$\|\rho(\cdot,t)\|_{\frac{r(2-\nu)}{r-2}} \leq \|\rho(\cdot,t)\|_1^{1-\theta_1}\|\rho(\cdot,t)\|_{\frac{\nu 2^*}{2}}^{\theta_1} \leq C + \left\|D\left(\rho^{\frac{\nu}{2}}\right)(\cdot,t)\right\|_2^{\frac{2\theta_1}{\nu}}.$$

From[2] $\nu>\nu_{rd}$ sufficiently close to 1 implies $\frac{\theta_1(2-\nu)}{\nu}<1$, Jensen's inequality, and Corollary 60,

$$\begin{aligned}\int_0^T\left|\int_{\mathbb{T}^d}\rho D_{x_i}g(m)dx\right|xdt &\leq C\int_0^T\int_{\mathbb{T}^d} g(m)^2\rho^{2-\nu}+\left|D\left(\rho^{\frac{\nu}{2}}\right)\right|^2 dxdt\\
&\leq C\int_0^T\|\rho(\cdot,t)\|^{2-\nu}_{L^{\frac{r(2-\nu)}{r-2}}(\mathbb{T}^d)}dt + C\int_0^T\int_{\mathbb{T}^d}\left|D\left(\rho^{\frac{\nu}{2}}\right)\right|^2dxdt\\
&\leq C + C\int_0^T\left\|D\left(\rho^{\frac{\nu}{2}}\right)(\cdot,t)\right\|_2^{2\frac{\theta_1(2-\nu)}{\nu}}dt + C\int_0^T\int_{\mathbb{T}^d}\left|D\left(\rho^{\frac{\nu}{2}}\right)\right|^2dxdt\\
&\leq C + C\int_0^T\left(\int_{\mathbb{T}^d}\left|D\left(\rho^{\frac{\nu}{2}}\right)\right|^2dx\right)^{\frac{\theta_1(2-\nu)}{\nu}}dt + C\int_0^T\int_{\mathbb{T}^d}\left|D\left(\rho^{\frac{\nu}{2}}\right)\right|^2dxdt\\
&\leq C + C\int_0^T\int_{\mathbb{T}^d}\left|D\left(\rho^{\frac{\nu}{2}}\right)\right|^2dxdt\\
&\leq C_{\epsilon_1}+C_\delta+\epsilon_1 C\,\mathrm{Lip}\,(v)\,. \qquad (2.10)\end{aligned}$$

The next term we need to control is

$$\begin{aligned}\int_{\mathbb{T}^d}|\rho v D_{x_i}(m^\alpha)|\,dx &= \int_{\mathbb{T}^d}|D_{x_i}\rho v m^\alpha + \rho D_{x_i} v m^\alpha|\,dx\\
&\leq \int_{\mathbb{T}^d}\left|\frac{2}{\nu}D_{x_i}\left(\rho^{\frac{\nu}{2}}\right)\rho^{1-\frac{\nu}{2}}vm^\alpha + \rho\eta m^\alpha\right|dx, \qquad (2.11)\end{aligned}$$

where, integrating from 0 to T,

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^d}D_{x_i}\left(\rho^{\frac{\nu}{2}}\right)\rho^{1-\frac{\nu}{2}}vm^\alpha dxdt &\leq \int_0^T\int_{\mathbb{T}^d}\left|D\left(\rho^{\frac{\nu}{2}}\right)\right|^2+\rho^{2-\nu}v^2m^{2\alpha}dxdt\\
&\leq C_{\epsilon_1}+C_\delta+\epsilon_1 C\,\mathrm{Lip}\,(v) + \int_0^T\int_{\mathbb{T}^d}\rho^{2-\nu}v^2m^{2\alpha}dxdt\\
&\leq C_{\epsilon_1}+C_\delta+\epsilon_1 C\,\mathrm{Lip}\,(v) + \int_0^T\int_{\mathbb{T}^d}v^r m^{\alpha r}dxdt \qquad (2.12)\end{aligned}$$

[2]It is easy to see that this is equivalent to $\nu>\frac{1}{r}+\frac{1}{2}+\frac{1}{p}$, which is smaller than 1.

If $d = 2$ Corollary 53 gives

$$\int_0^T \int_{\mathbb{T}^d} v^r m^{\alpha r} dxdt \le C \int_0^T \int_{\mathbb{T}^d} v^{2r} + m^{2\alpha r} dxdt \le C. \tag{2.13}$$

If $d = 3$ Corollary 53 gives

$$\int_0^T \int_{\mathbb{T}^d} v^r m^{\alpha r} dxdt \le C \int_0^T \int_{\mathbb{T}^d} v^{\frac{3r}{3-\alpha r}} + m^3 dxdt \le C_\delta + C \int_0^T \int_{\mathbb{T}^d} v^{\frac{3r}{3-\alpha r}} dxdt, \tag{2.14}$$

which is bounded as long as $\alpha < \frac{3}{r} - \frac{1}{2}$ is sufficiently small. This assures that $\frac{3r}{3-\alpha r} < 2^*$ if $3 < r < 6$ (this implies $1 < q < \frac{6}{5}$).

The two strategies above will not work for $d > 3$ because $\frac{2^*}{r} > 1$, with $r > d$, is equivalent to $d < 4$.

Now, for $d > 3$ we could do the following,

$$\begin{aligned}
\int_0^T \int_{\mathbb{T}^d} v^r m^{\alpha r} dxdt &\le C \int_0^T \int_{\mathbb{T}^d} v^R + m^{\frac{2^*}{2}} dxdt \\
&\le C_\delta + C \int_0^T \int_{\mathbb{T}^d} v^{R-2^*} v^{2^*} dxdt \\
&\le C_\delta + C \int_0^T \left\| v^{R-2^*} \right\|_\infty \int_{\mathbb{T}^d} v^{2^*} dxdt \\
&\le C_\delta + C \left(\|v\|_d + \|Dv\|_d \right)^{R-2^*} \\
&\le C_\delta + C \|v\|_d^{R-2^*} + C \|Dv\|_2^{\frac{2(R-2^*)}{d}} \|Dv\|_\infty^{\frac{(d-2)(R-2^*)}{d}} \\
&\le C_\delta + C \|v\|_d^{R-2^*} + C_\delta \operatorname{Lip}(v)^{\frac{(d-2)(R-2^*)}{d}} \\
&\le C_\delta + C \|v\|_d^{R-2^*} + \epsilon_\delta \operatorname{Lip}(v),
\end{aligned} \tag{2.15}$$

where $R = \frac{2^* r}{2^* - 2\alpha r}$ and $R > 2^*$ which is equivalent to $\frac{2^*}{2r} - \frac{1}{2} < \alpha < \frac{2^*}{2r}$. Also, $R - 2^* < 1$ is equivalent to $\frac{2^*}{2r} - \frac{1}{2} < \alpha < \frac{2^*}{2r} - \frac{2^*}{2(1+2^*)}$. (the latter is actually grater than the former). Since $\frac{d-2}{d} < 1$, we have $\frac{(d-2)(R-2^*)}{d} < 1$. So that the last inequality holds. In order for the d-norm of v to be bounded, we need $d < 2^*$. But this implies $d < 4$.

The last term to estimate is

$$\begin{aligned}
\int_{\mathbb{T}^d} \rho \eta m^\alpha dx &\le C \int_{\mathbb{T}^d} \rho \eta^2 + \rho m^{2\alpha} dx \\
&\le C \int_{\mathbb{T}^d} \rho |Dv|^2 + \rho^{\frac{2^*}{2^*-4\alpha}} + m^{\frac{2^*}{2}} dx \\
&\le C_\delta + C_{\epsilon_1} + \epsilon_1 C \operatorname{Lip}(v).
\end{aligned} \tag{2.16}$$

The limit as α goes to 0 in $\frac{2^*}{2^*-4\alpha}$ is 1 and as ν goes to 1 in $\frac{\nu 2^*}{2}$ is $\frac{2^*}{2}$. It shoud not be a problem to find α and ν accordingly.

To wrap up, we choose i and x_0 such that

$$\operatorname{Lip}(v) = |D_{x_i}(v(x_0)|,$$

and gather the estimates for (2.8): (2.9), (2.10), (2.11), (2.12), (2.13) for $d = 2$, (2.14) for $d = 3$, and (2.16).

$$\operatorname{Lip}(v) \le C_{\epsilon_1} + C_\delta + (\epsilon + \epsilon_1) C \operatorname{Lip}(v).$$

Choosing ϵ and ϵ_1 sufficiently small finishes the proof. □

2.7 Sobolev and Hölder regularity of solutions

In this section, we determine the regularity of both v and m by iterating on the estimates.

Proposition 64. *Let (v, m) be a solution to (2.1) and (v, ρ) be a solution to (2.6). Suppose Assumptions 1, 2, 3, 5, and 8 hold, $m > 0$, and $\delta > 0$. If either Assumption 6 holds with $\gamma < \max\{\alpha+1, \frac{1}{\alpha}\}$ or Assumption 7 holds, then*

$$m \in L^{\beta+1}(\mathbb{T}^d), \quad D\left(m^\beta\right) \in L^2,$$

for any $\beta > 0$.

Proof. Integrate the second identity in (2.1) against $\beta m^{\beta-1}$ for $\beta > 1$,

$$(\beta-1)\,\beta \int_{\mathbb{T}^d} m^{\beta-2}\,|Dm|^2\,dx = \int_{\mathbb{T}^d} -(\beta-1)\,\beta m^{\beta-1} D_pH\cdot Dm + \beta\left(m^\beta(1-m^\alpha) + \delta m^{\beta-1}\right)dx.$$

Taking the modulus and using Cauchy's inequality

$$\frac{2\,(\beta-1)}{\beta}\int_{\mathbb{T}^d}\left|D\left(m^{\frac{\beta}{2}}\right)\right|^2 dx \le \int_{\mathbb{T}^d}\frac{(\beta-1)\,\beta}{2} m^\beta\,|D_pH|^2 + \beta\left(m^\beta - m^{\alpha+\beta} + \delta m^{\beta-1}\right)dx.$$

Consequently, from Propositions 27 and 63, we get

$$\int_{\mathbb{T}^d}\frac{2\,(\beta-1)}{\beta}\left|D\left(m^{\frac{\beta}{2}}\right)\right|^2 + \beta m^{\alpha+\beta} dx \le C_\delta,$$

which assures $m \in L^{\alpha+\beta}(\mathbb{T}^d)$, $D\left(m^{\frac{\beta}{2}}\right) \in L^2(\mathbb{T}^d)$, and $m^{\frac{\beta}{2}} \in L^{2^*}(\mathbb{T}^d)$.

Let $0 < \beta \le 1$ and p be the conjugate exponent of $q > \frac{1}{\beta}$. Since, now, $\frac{\beta}{2} - 1$ can be written as $\frac{q\beta-2}{2q} - \frac{1}{p}$, we have

$$\left|D\left(m^{\frac{\beta}{2}}\right)\right| = \frac{\beta}{2} m^{\frac{\beta}{2}-1}\,|Dm| = \left(\frac{1}{q}\left|D\left(m^{\frac{q\beta}{2}}\right)\right|\right)^{\frac{1}{q}}\left(\frac{\beta}{2}\,|D(\ln m)|\right)^{\frac{1}{p}}.$$

Observe that one of the conclusions above and Proposition 49 then gives

$$\left(\frac{1}{q}\left|D\left(m^{\frac{q\beta}{2}}\right)\right|\right)^{\frac{1}{q}} \in L^{2q}(\mathbb{T}^d), \quad \left(\frac{\beta}{2}\,|D(\ln m)|\right)^{\frac{1}{p}} \in L^{2p}(\mathbb{T}^d)$$

which allow us to conclude $D\left(m^\beta\right) \in L^2(\mathbb{T}^d)$ for any $\beta > 0$. □

Now, from Proposition 54 we infer

Proposition 65. *Let (v, m) be a solution of (2.1). Suppose Assumptions 1, 2, 3, 4, and 5 hold, $m > 0$, and $\delta > 0$.*

Then

$$v \in W^{1,\infty}(\mathbb{T}^d) \text{ and } v \in W^{2,p}(\mathbb{T}^d)$$

for any $p \ge 1$.

Proof. From the first identity in (2.1), we get

$$\int |\Delta v|^p\,dx \le C + \int |H|^p + |m^\alpha v|^p + |g(m)|^p\,dx.$$

Since $Dv \in L^\infty(\mathbb{T}^d)$, $m \in L^\beta(\mathbb{T}^d)$ for any $\beta \geq 1$, and $v \in L^\infty(\mathbb{T}^d)$ (this is a consequence of $\operatorname{Lip}(v)$ bounded, v has bounded mean, and the compactness of the domain), we have $|H(x, Dv)| \leq C$, $|m^\alpha v| \leq C$, and $|g(m)| \leq C$. This implies $v \in W^{2,p}$ for any $p \geq 1$. □

Proposition 66. *Let (v, m) be a solution of (2.1). Suppose Assumptions 1, 2, 3, 4, 5, and 6 hold, $m > 0$, $\delta > 0$, and $d < 4$.*

Then

$$Dm \in C^{0,1-\frac{d}{2^*}}(\mathbb{T}^d).$$

Furthermore, Dm is continuous and bounded.

Proof. We use the second identity in (1.1) to get

$$|\Delta m|^p \leq C + |\operatorname{div}(D_pHm)|^p + |(1 - m^\alpha)m|^p . \tag{2.17}$$

Since

$$\operatorname{div}(D_pHm) = m\operatorname{div}(D_pH) + D_pH \cdot Dm.$$

we use Propositions 64 and 65, to conclude that the first term on the right hand side of (2.17) is bounded and in $L^p(\mathbb{T}^d)$, bur the second term has Dm, which is in $L^2(\mathbb{T}^d)$, so that $m \in W^{2,2}(\mathbb{T}^d)$.

From this we get $Dm \in W^{1,2}(\mathbb{T}^d)$ which in turn gives $Dm \in L^{2^*}(\mathbb{T}^d)$. Iterating the proof with this new information gives $Dm \in W^{1,2^*}(\mathbb{T}^d)$ and applying Morrey's theorem, we conclude the $\left(1 - \frac{d}{2^*}\right)$-Hölder continuity of Dm. □

Corollary 67. *Let (v, m) be a solution of (2.1). Suppose Assumptions 1, 2, 3, 4, 5, and 9 hold, $m > 0$, and $\delta > 0$. If either Assumption 6 or 7 hold, then*

$$v \in W^{3,2}(\mathbb{T}^d).$$

Proof. Differentiating the first identity in (2.1) with respect to x_i

$$\Delta D_{x_i} v = D_{x_i} H + D_pHDD_{x_i}v + m^\alpha D_{x_i}v + vD_{x_i}(m^\alpha) - D_{x_i}g(m).$$

Assumption 9, Propositions 63, and 65 are used to show the L^p integrability of the first three terms in the right hand side. Regardless of the type of $g(m)$, the term that determines the integrability is Dm, which appear in the last two terms and is in $L^2(\mathbb{T}^d)$. Hence $\|D\Delta v\|_2 \leq C$ and therefore $v \in W^{3,2}(\mathbb{T}^d)$. □

Corollary 68. *Let (v, m) be a solution of (2.1). Suppose Assumptions 1, 2, 3, 4, 5, and 6 hold, $m > 0$, $\delta > 0$, and $d < 4$.*

Then

$$m \in W^{3,2}(\mathbb{T}^d).$$

Furthermore, m is continuous and bounded.

Proof. Observe that

$$D(D_pHm) = D(D_pH)m + D_pH \cdot Dm.$$

From the discussion in the previous proof we have $D(D_pHm) \in L^2(\mathbb{T}^d)$. Differentiating the expression observed above gives

$$D^2(D_pHm) = D^2(D_pH)m + 2D(D_pH) \cdot Dm + D_pH \cdot D^2m.$$

Note that the first term in the right hand side belongs to $L^2(\mathbb{T}^d)$. Indeed, using Corollary 67 with Assumption 6, and Propositions 63 and 66, we obtain the result.

The other terms are also in $L^2(\mathbb{T}^d)$ for the same reasons. We finish with the observation that

$$\Delta D_{x_i} m = -D_{x_i}\operatorname{div}(D_pHm) - D_{x_i}m + (\alpha + 1)m^\alpha Dm$$

is L^2 integrable. So that $m \in W^{3,2}(\mathbb{T}^d)$

□

2.8 Hopf-Cole transform

Proposition 69. *Let* (v, m) *be a solution of* (2.1). *Suppose Assumptions 1, 2, 3, 4, 5, and 6 hold,* $m > 0$, $\delta > 0$, *and* $d = 2$.

Then there exists $\bar{m} > 0$ *such that* $m > \bar{m}$.

Proof. Let $w = -\ln m$, then it solves

$$-\mathrm{div}\,(D_pH(x, Dv)) + D_pH(x, Dv)Dw = (1 - m^\alpha) + \frac{\delta}{m} + |Dw|^2 - \Delta w.$$

Put $|Dw|^2 - D_pH(x, Dv)Dw + \mathrm{div}\,(D_pH(x, Dv))$ in place of H and $(1 - m^\alpha) + \frac{\delta}{m}$ in the place of $g(m)$. The same thechniques of Section 2.6 apply to determine that $\mathrm{Lip}\,(w)$ is bounded. Since this gives $\|\ln m\|_\infty$ bounded, we conclude the existence of $\bar{m} > 0$ such that $m \leq \bar{m}$.

□

The above Proposition allows the conclusion that both, v and m belong to $W^{k,q}(\mathbb{T}^d)$ for $k \leq k_0$ and to $C^r(\mathbb{T}^d)$ for every $r < \infty$, by iterating the procedures of Corollaries 67 and 68.

Appendix A

Regularization

Consider the ϵ-system solved by (v^ϵ, m^ϵ)

$$\begin{cases} H(x, Dv^\epsilon) = g_\epsilon(m^\epsilon) - (\eta_\epsilon * (m_\epsilon^\epsilon)^\alpha)\, v^\epsilon + \Delta v^\epsilon \\ -\text{div}\,(m^\epsilon D_p H(x, Dv^\epsilon)) = (1 - \eta_\epsilon * (m_\epsilon^\epsilon)^\alpha)\, m^\epsilon + \Delta m^\epsilon + \delta, \end{cases} \tag{A.1}$$

where $\eta : \mathbb{R}^+ \times \mathbb{R}^d$ is the fundamental solution to the heat equation on the d-dimensional Euclidean space, $\frac{d}{d\epsilon}\eta - \Delta_z \eta = 0$ with $\eta(0, z) = \delta_0$ the Dirac delta at the origin, $g_\epsilon(f) = \eta_\epsilon * g(\eta_\epsilon * f)$ is a non-local operator for $f : \mathbb{T}^d \to \mathbb{R}$, $\eta_\epsilon(z) := \eta(\epsilon, z)$, and $m_\epsilon^\epsilon(x) := [\eta_\epsilon * m^\epsilon](x) := \int_{\mathbb{R}^d} \eta_\epsilon(x - y) m^\epsilon(y) dy$. In the same way, we define $v_\epsilon^\epsilon = \eta_\epsilon * v$ and $(\eta_\epsilon * m^\epsilon)^\alpha = (m_\epsilon^\epsilon)^\alpha$.

A.1 Existence of solutions to the regularized system

To prove existence, we construct a mapping $\Phi : \mathcal{F}(\mathbb{T}^d) \to \mathcal{F}(\mathbb{T}^d)$ that takes m_0^ϵ to $\Phi(m_0^\epsilon) = m^\epsilon$. Given m_0^ϵ, solve the first equation in (A.1) for v^ϵ. Then solve the secont equation in (A.1) for m^ϵ.

We can prove that this mapping is contiunous and compact.

A.2 Uniform estimates

To make most of the estimates in Chapter 2 rigorous, we verify that some *a priori* estimates are uniform with respect to ϵ, the new variable introduced above.

Note that, as the convolution is a contraction in any L^p space, $f \in L^p$ implies $\eta_\epsilon * f \in L^p$ and $f^\alpha \in L^p$ implies $\eta_\epsilon * (\eta_\epsilon * f)^\alpha \in L^p$. The bounds do not depend on ϵ.

A.2.1 Regularity for the transport equation

Proposition 70. *Let (v^ϵ, m^ϵ) be a solution to (A.1).*

$$\int_{\mathbb{T}^d} m_\epsilon^\epsilon dx \le C,$$

and

$$\int_{\mathbb{T}^d} (m_\epsilon^\epsilon)^{\alpha+1}\, dx \le C.$$

Proof. Integrating the second identity of (A.1), we get

$$\int_{\mathbb{T}^d} m_\epsilon^\epsilon + \delta dx = \int_{\mathbb{T}^d} m^\epsilon + \delta dx = \int_{\mathbb{T}^d} m^\epsilon \eta_\epsilon * (m_\epsilon^\epsilon)^\alpha dx = \int_{\mathbb{T}^d} (m_\epsilon^\epsilon)^{\alpha+1} dx,$$

which proves the Proposition. □

Proposition 71. *Let (v^ϵ, m^ϵ) be a solution to (A.1). Suppose Assumptions 1 and 6 or 7 hold. Then*

$$\int_{\mathbb{T}^d} \frac{1}{2} m_\epsilon^\epsilon g(m_\epsilon^\epsilon) + H(1+cm^\epsilon)dx \leq C + \int_{\mathbb{T}^d} v^\epsilon(m^\epsilon+\delta) - (m_\epsilon^\epsilon)^\alpha v_\epsilon^\epsilon dx.$$

Proof. Integrating the first identity of (A.1),

$$\int_{\mathbb{T}^d} H(x, Dv^\epsilon)dx = \int_{\mathbb{T}^d} g_\epsilon(m^\epsilon) - (m_\epsilon^\epsilon)^\alpha v_\epsilon^\epsilon dx = \int_{\mathbb{T}^d} g(m_\epsilon^\epsilon) - (m_\epsilon^\epsilon)^\alpha v_\epsilon^\epsilon dx.$$

Integrating the second identity of (A.1) multiplied by v^ϵ,

$$\int_{\mathbb{T}^d} D_pH \cdot Dv^\epsilon m^\epsilon dx = \int_{\mathbb{T}^d} \left(1 - \eta_\epsilon * (m_\epsilon^\epsilon)^\alpha\right) m^\epsilon v^\epsilon + v^\epsilon\delta + v^\epsilon \Delta m^\epsilon dx.$$

Integrating the first identity of (A.1) multiplied by m^ϵ,

$$\int_{\mathbb{T}^d} H(x, Dv^\epsilon)m^\epsilon dx = \int_{\mathbb{T}^d} m^\epsilon g_\epsilon(m^\epsilon) - \left(\eta_\epsilon * (m_\epsilon^\epsilon)^\alpha\right) v^\epsilon m^\epsilon + m^\epsilon \Delta v^\epsilon dx.$$

We get

$$\int_{\mathbb{T}^d} m^\epsilon g_\epsilon(m^\epsilon) + m^\epsilon\left(D_pH \cdot Dv^\epsilon - H(x, Dv^\epsilon)\right) dx = \int_{\mathbb{T}^d} v^\epsilon(m^\epsilon+\delta)dx,$$

and then, with Assumption 1,

$$\int_{\mathbb{T}^d} m_\epsilon^\epsilon g(m_\epsilon^\epsilon) + cm^\epsilon H(x, Dv^\epsilon)dx \leq C\int_{\mathbb{T}^d} m^\epsilon dx + \int_{\mathbb{T}^d} v^\epsilon(m^\epsilon+\delta)dx.$$

From this we get

$$\int_{\mathbb{T}^d} m_\epsilon^\epsilon g(m_\epsilon^\epsilon) + H(1+cm^\epsilon)dx \leq C + \int_{\mathbb{T}^d} v^\epsilon(m^\epsilon+\delta) + g(m_\epsilon^\epsilon) - (m_\epsilon^\epsilon)^\alpha v_\epsilon^\epsilon dx.$$

Finally, note that $g(f) \leq \frac{1}{2} f g(f) + C$, for any $f \geq 0$. □

Observing the boundedness from below of $m_\epsilon^\epsilon g(m_\epsilon^\epsilon)$ and $H(1+cm^\epsilon)$ given by $m^\epsilon > 0$ and Assumption 2 we get

Proposition 72. *Let (v^ϵ, m^ϵ) be a solution to (A.1). Suppose Assumptions 1, 2, and 6 or 7 hold. Then*

$$c\|Dv^\epsilon\|_2^2 + \int_{\mathbb{T}^d} v_\epsilon^\epsilon (m_\epsilon^\epsilon)^\alpha dx \leq C + \int_{\mathbb{T}^d} g(m_\epsilon^\epsilon)dx,$$

and

$$\int_{\mathbb{T}^d} v^\epsilon(m^\epsilon+\delta)dx \geq -C.$$

Corollary 73. *Let (v^ϵ, m^ϵ) be a solution to (A.1). Then*

$$\int_{\mathbb{T}^d} g(m_\epsilon^\epsilon)dx \leq \left(C + 2\int_{\mathbb{T}^d} v^\epsilon(m^\epsilon+\delta) - (m_\epsilon^\epsilon)^\alpha v_\epsilon^\epsilon dx\right)^{\frac{\gamma}{\gamma+1}}.$$

if Assumption 6 holds and for any $\theta > 0$

$$\int_{\mathbb{T}^d} g(m_\epsilon^\epsilon)dx \leq \left(C_\theta + 2\int_{\mathbb{T}^d} v^\epsilon(m^\epsilon+\delta) - (m_\epsilon^\epsilon)^\alpha v_\epsilon^\epsilon dx\right)^{\theta}.$$

if Assumption 7 holds.

A.2.2 Estimates for the Hamilton-Jacobi equation

We now turn to the rigorous estimates on Dv^ϵ and v^ϵ.

Proposition 74. *Let (v^ϵ, m^ϵ) be a solution to (A.1).*
Then

$$\|Dv^\epsilon\|_2^2 + \int_{\mathbb{T}^d} |Dv^\epsilon|^2 m^\epsilon dx \leq C + C\int_{\mathbb{T}^d} v^\epsilon dx.$$

Proof. Using Proposition 33, under the right Assumptions, $\|m^\epsilon + \delta\|_p \leq C$ and $\|(m^\epsilon_\epsilon)^\alpha\|_p \leq C$

$$-\int_{\mathbb{T}^d} v^\epsilon_\epsilon (m^\epsilon_\epsilon)^\alpha dx \leq -\int_{\mathbb{T}^d} (m^\epsilon_\epsilon)^\alpha dx \int_{\mathbb{T}^d} v^\epsilon dx + C\|Dv^\epsilon\|_2$$

$$\int_{\mathbb{T}^d} v^\epsilon_\epsilon (m^\epsilon + \delta) - v^\epsilon_\epsilon (m^\epsilon_\epsilon)^\alpha dx \leq \int_{\mathbb{T}^d} m^\epsilon + \delta - (m^\epsilon_\epsilon)^\alpha dx \int_{\mathbb{T}^d} v^\epsilon dx + C\|Dv^\epsilon\|_2 .$$

$$\begin{aligned}\int_{\mathbb{T}^d} c|Dv^\epsilon|^2 (1 + m^\epsilon) &\leq C + \int_{\mathbb{T}^d} v^\epsilon_\epsilon (m^\epsilon + \delta) - v^\epsilon_\epsilon (m^\epsilon_\epsilon)^\alpha dx \\ &\leq C + C\int_{\mathbb{T}^d} v^\epsilon dx + C\|Dv^\epsilon\|_2 .\end{aligned}$$

□

Define $\lambda_1^\epsilon = \int_{\mathbb{T}^d} m^\epsilon + \delta dx$ and $\lambda_2^\epsilon = \int_{\mathbb{T}^d} (m^\epsilon_\epsilon)^\alpha dx$.

Proposition 75. *Let (v^ϵ, m^ϵ) be a solution to (A.1).*

$$\lambda_2^\epsilon \int_{\mathbb{T}^d} v^\epsilon dx \leq C + \int_{\mathbb{T}^d} g(m^\epsilon_\epsilon) dx + C\|Dv^\epsilon\|_2 ,$$

$$\lambda_1^\epsilon \int_{\mathbb{T}^d} v^\epsilon dx \geq -C - C\|Dv^\epsilon\|_2 ,$$

and

$$\|Dv^\epsilon\|_2^2 \leq C_\delta + C_\delta \int_{\mathbb{T}^d} g(m^\epsilon_\epsilon) dx.$$

Proof. This is a consequence of the Proposition before the Corollary and the new estimates above. The last estimate uses the fact $\frac{1}{\lambda_1^\epsilon} \leq C_\delta$. □

Corollary 76. *Let (v^ϵ, m^ϵ) be a solution to (A.1).*

$$-\int_{\mathbb{T}^d} v^\epsilon dx \leq \frac{C}{\lambda_1^\epsilon} + \frac{C}{\lambda_1^\epsilon}\|Dv^\epsilon\|_2 ,$$

and

$$\left|\int_{\mathbb{T}^d} v^\epsilon dx\right| \leq \frac{C}{\lambda_1^\epsilon} + \frac{C}{\lambda_2^\epsilon} + \frac{C}{\lambda_2^\epsilon}\int_{\mathbb{T}^d} g(m^\epsilon_\epsilon) dx + \left(\frac{C}{\lambda_1^\epsilon} + \frac{C}{\lambda_2^\epsilon}\right)\|Dv^\epsilon\|_2 ,$$

Proof. It is straightforward from the Proposition above. □

A.2.3 Lower bounds for the density

Proposition 77. *Let (v^ϵ, m^ϵ) be a solution to (A.1). Suppose Assumption 5 holds.*
Then

$$\int_{\mathbb{T}^d} \frac{\delta}{m^\epsilon} dx + \frac{1}{2}\|D(\ln m^\epsilon)\|_2^2 \leq C + C\|Dv^\epsilon\|_2^2 .$$

Proof. Divide the second identity in (A.1) by m^ϵ and integrate by parts

$$\int_{\mathbb{T}^d} -D_pH(x, Dv^\epsilon)\cdot D\left(\ln m^\epsilon\right)dx = \int_{\mathbb{T}^d} 1-(m^\epsilon_\epsilon)^\alpha + \frac{\delta}{m^\epsilon} + \left|D(\ln m^\epsilon)\right|^2 dx$$

□

Proposition 78. *Let (v^ϵ, m^ϵ) be a solution to (A.1). Suppose Assumptions 1, 2, 3, and 5 hold, $m^\epsilon > 0$, and $\delta > 0$. If either Assumption 6 holds with $\gamma < \alpha+1$ or Assumption 7 holds*

Then

$$\int_{\mathbb{T}^d} \left|D(m^\epsilon)^{\frac{1}{2}}\right|^2 dx \leq C + C\int_{\mathbb{T}^d} v^\epsilon dx.$$

Proof. Integrate the second identity in (A.1) against $\ln m^\epsilon$

$$\begin{aligned}\int_{\mathbb{T}^d} \frac{|Dm^\epsilon|^2}{m^\epsilon}dx &= \int_{\mathbb{T}^d} \ln m^\epsilon\left(m^\epsilon+\delta-\eta_\epsilon*(m^\epsilon_\epsilon)^\alpha m^\epsilon\right) - D_pH(x,Dv^\epsilon)\cdot Dm^\epsilon dx\\ &\leq C + \int_{\mathbb{T}^d} C\left|Dv^\epsilon\right|^2 m^\epsilon + \frac{|Dm^\epsilon|^2}{4m^\epsilon}dx\\ &\leq C + C\int_{\mathbb{T}^d} v^\epsilon dx + \int_{\mathbb{T}^d}\frac{|Dm^\epsilon|^2}{4m^\epsilon}dx.\end{aligned}$$

To bound the terms $\ln m^\epsilon\left(m^\epsilon+\delta-\eta_\epsilon*(m^\epsilon_\epsilon)^\alpha m^\epsilon\right)$ we used that $m^\epsilon \ln m^\epsilon \leq C(m^\epsilon)^{\alpha+1}+C$, $\ln m^\epsilon \leq m^\epsilon$, which are integrable, and $\eta_\epsilon*(m^\epsilon_\epsilon)^\alpha m^\epsilon \ln m^\epsilon$ is bounded below. □

Proposition 79. *Let (v^ϵ, m^ϵ) be a solution to (A.1). Suppose Assumptions 1, 2, and 5 hold, $\delta > 0$, and $m^\epsilon > 0$. If the constants given by Lemma 42 satisfy $\kappa_1 < \frac{4c_2}{\alpha}$ and $\kappa_2 < \frac{4}{\alpha}\left(\frac{\alpha}{\alpha+1} - C_2\right)$.*

Then

1. *if Assumption 6 holds with $\gamma < \alpha$ or Assumption 7 holds we have*

$$\int_{\mathbb{T}^d} \frac{1}{(m^\epsilon)^\alpha} + \left|D\left((m^\epsilon)^{-\frac{\alpha}{2}}\right)\right|^2 dx \leq C + C\left(\int_{\mathbb{T}^d} v^\epsilon dx\right)^{2^*}.$$

2. *if Assumption 6 holds with $\gamma \geq \alpha$ we have*

$$\int_{\mathbb{T}^d} \frac{1}{(m^\epsilon)^\alpha} + \left|D\left((m^\epsilon)^{-\frac{\alpha}{2}}\right)\right|^2 dx \leq C + C\left(\int_{\mathbb{T}^d} v^\epsilon dx\right)^{2^*} + C\left(\int_{\mathbb{T}^d} g(m^\epsilon)dx\right)^{\frac{\gamma-\alpha}{\gamma}}.$$

Proof. Following the steps in the proof of Proposition 43 multiplying instead by $\frac{1}{(m^\epsilon)^\alpha}$ and $\frac{\alpha}{\alpha+1}\frac{1}{(m^\epsilon)^{\alpha+1}}$,

$$\begin{aligned}&\int_{\mathbb{T}^d} \alpha\frac{|D(m^\epsilon)|^2}{(m^\epsilon)^{\alpha+2}} + \frac{\delta\alpha}{(\alpha+1)(m^\epsilon)^{\alpha+1}} + \frac{H}{(m^\epsilon)^\alpha} + \frac{\alpha}{(\alpha+1)(m^\epsilon)^\alpha}dx\\ &\quad= \int_{\mathbb{T}^d} \frac{g(m^\epsilon)}{(m^\epsilon)^\alpha} + \frac{\eta_\epsilon*(m^\epsilon_\epsilon)^\alpha}{(m^\epsilon)^\alpha}\left(\frac{\alpha}{\alpha+1} - v^\epsilon\right) + \frac{\alpha}{(m^\epsilon)^{\alpha+1}}\left(Dv^\epsilon - D_pH\right)\cdot D(m^\epsilon)dx.\end{aligned}$$

Thus, using Assumption 2, applying Cauchy's inequality with weights $\frac{1}{4(1-\sigma)}$ and $(1-\sigma)$, for

$\sigma > 0$, and the Lemma 42

$$\int_{\mathbb{T}^d} \alpha \frac{|D(m^\epsilon)|^2}{(m^\epsilon)^{\alpha+2}} + \frac{\delta\alpha}{(\alpha+1)(m^\epsilon)^{\alpha+1}} + c_2 \frac{|Dv^\epsilon|^2}{(m^\epsilon)^\alpha} + \left(\frac{\alpha}{\alpha+1} - C_2\right) \frac{1}{(m^\epsilon)^\alpha} dx$$
$$\leq C + \int_{\mathbb{T}^d} \frac{g(m^\epsilon)}{(m^\epsilon)^\alpha} + \frac{\eta_\epsilon * (m^\epsilon_\epsilon)^\alpha}{(m^\epsilon)^\alpha} \left(\frac{\alpha}{\alpha+1} - v^\epsilon\right) + \alpha \left(\frac{|Dv^\epsilon - D_pH|^2}{4(1-\sigma)(m^\epsilon)^\alpha} + (1-\sigma)\frac{|D(m^\epsilon)|^2}{(m^\epsilon)^{\alpha+2}}\right) dx$$
$$\leq C + \int_{\mathbb{T}^d} \frac{g(m^\epsilon)}{(m^\epsilon)^\alpha} + \frac{\eta_\epsilon * (m^\epsilon_\epsilon)^\alpha}{(m^\epsilon)^\alpha} \left(\frac{\alpha}{\alpha+1} - v^\epsilon\right) + \frac{\alpha\kappa_1}{4(1-\epsilon)} \frac{|Dv^\epsilon|^2}{(m^\epsilon)^\alpha} + \frac{\alpha\kappa_2}{4(1-\sigma)} \frac{1}{(m^\epsilon)^\alpha} + \alpha(1-\sigma) \frac{|D(m^\epsilon)|^2}{(m^\epsilon)^{\alpha+2}} dx.$$

The term $\int_{\mathbb{T}^d} \frac{\eta_\epsilon * (m^\epsilon_\epsilon)^\alpha}{(m^\epsilon)^\alpha} \left(\frac{\alpha}{\alpha+1} - v^\epsilon\right) dx$, using Young's inequality, is bounded by

$$\int_{\mathbb{T}^d} C_\sigma \left|\eta_\epsilon * (m^\epsilon_\epsilon)^\alpha\right|^p + \sigma \left|\frac{1}{(m^\epsilon)^\alpha}\right|^q + C_\sigma \left|\frac{\alpha}{\alpha+1} - v^\epsilon\right|^r dx \leq C_\sigma + \int_{\mathbb{T}^d} \frac{\sigma}{(m^\epsilon_\epsilon)^{\alpha+1}} dx + C_\sigma \|v^\epsilon\|_{2^*}^{2^*}.$$

by setting $p = \frac{\alpha+1}{\alpha}$, $r = 2^*$ and $q = \frac{2d(\alpha+1)}{d+2\alpha+2}$. If $\alpha \leq \frac{1}{2}$ then $\alpha q \leq \alpha + 1$. So, imposing $\delta > 0$, we can choose $\sigma < \frac{\alpha\delta}{\alpha+1}$.

Sobolev and Poincaré's inequalities imply

$$\|v^\epsilon\|_{2^*}^{2^*} \leq C \left(\|v^\epsilon\|_2 + \|Dv^\epsilon\|_2\right)^{2^*}.$$

$$\|v^\epsilon\|_{2^*}^{2^*} \leq C \left(C \int_{\mathbb{T}^d} v^\epsilon dx + C \|Dv^\epsilon\|_2\right)^{2^*} \leq C \left(\int_{\mathbb{T}^d} v^\epsilon dx\right)^{2^*} + C \|Dv^\epsilon\|_2^{2^*}.$$

As long as $C_2 < \frac{\alpha}{\alpha+1}$, the term $\frac{g(m^\epsilon)}{(m^\epsilon)^\alpha}$ can be handled in the following way: if Assumption 7 or 6 with $0 < \gamma < \alpha$ hold then it can be absorbed in the left hand side by noting that for any $\sigma > 0$ we have

$$\frac{g(m^\epsilon)}{(m^\epsilon)^\alpha} \leq \frac{\sigma}{(m^\epsilon)^\alpha} + C_\sigma.$$

Choosing σ small enough the result follows. In the case $\gamma \geq \alpha$ it suffices to use Hölder inequality. Since $\|Dv^\epsilon\|_2^2 \leq C + C\int_{\mathbb{T}^d} v^\epsilon dx$, we finally get

$$\int_{\mathbb{T}^d} \frac{\sigma}{(m^\epsilon)^\alpha} + \sigma \frac{4}{\alpha^2} \frac{|D(m^\epsilon)|^2}{(m^\epsilon)^{\alpha+2}} dx \leq C + C \left(\int_{\mathbb{T}^d} v^\epsilon dx\right)^{2^*} + \mu \left(\int_{\mathbb{T}^d} g(m^\epsilon) dx\right)^{\frac{\gamma-\alpha}{\gamma}},$$

where μ is a large constant if $\gamma \geq \alpha$ or zero otherwise. □

Proposition 80. *Let (v^ϵ, m^ϵ) be a solution to (A.1). Suppose Assumptions 1, 2, 5 hold, $m^\epsilon > 0$ and $\delta > 0$. Aditionally, if either Assumption 6 or 7 holds, with $\gamma \leq \max\left[\alpha+1, \frac{1}{\alpha}\right]$.*

Then

$$\int_{\mathbb{T}^d} g(m^\epsilon_\epsilon) dx \leq C_\delta, \qquad \left|\int_{\mathbb{T}^d} v^\epsilon dx\right| \leq C_\delta, \qquad \text{and } \|Dv^\epsilon\|_2 \leq C_\delta.$$

Proof. From Corollary 76 and the estimates

$$\frac{C}{\lambda_1^\epsilon} \leq C_\delta,$$

and, from Proposition 77,

$$\frac{C}{\lambda_2^\epsilon} \leq \left(\int_{\mathbb{T}^d} \frac{1}{m^\epsilon} dx\right)^\alpha \leq C_\delta + C_\delta \left(\int_{\mathbb{T}^d} g(m^\epsilon_\epsilon) dx\right)^\alpha,$$

we get

$$\left|\int_{\mathbb{T}^d} v^\epsilon dx\right| \leq C_\delta + C_\delta \left(\int_{\mathbb{T}^d} g(m_\epsilon^\epsilon)dx\right)^{\alpha+1}.$$

The last estimate is an application of the first to Proposition 74. The first estimate is trivial if $\gamma \leq \alpha+1$. Otherwise we need to observe Corollary 73 and Proposition 72 with the inequality above, noting that $\alpha < \frac{1}{\gamma}$. □

Corollary 81. *Let (v^ϵ, m^ϵ) be a solution to (A.1). Suppose Assumptions 1, 2, 5 hold, $m^\epsilon > 0$ and $\delta > 0$. Aditionally, if either Assumption 6 or 7 holds, with $\gamma \leq \max\left[\alpha+1, \frac{1}{\alpha}\right]$.*

Then

$$\int_{\mathbb{T}^d} |Dv^\epsilon|^2 m^\epsilon dx, \quad \int_{\mathbb{T}^d} \frac{1}{m^\epsilon} dx, \quad \int_{\mathbb{T}^d} \frac{1}{(m^\epsilon)\alpha} dx, \quad \|D(\ln m^\epsilon)\|_2, \quad \text{and } \left\|D\left((m^\epsilon)^{\frac{1}{2}}\right)\right\|_2$$

are bounded by C_δ.

Furthermore, $\int_{\mathbb{T}^d} m^\beta dx \geq \frac{1}{C_\delta}$ for any $\beta \geq 0$.

Proof. See where the previous Proposition applies. □

Theorem 82. *For $d > 2$,*

$$v^\epsilon \in L^{2^*}(\mathbb{T}^d), \quad \ln m^\epsilon \in L^{2^*}(\mathbb{T}^d), \quad m^\epsilon \in L^{\frac{2^*}{2}}(\mathbb{T}^d).$$

For $d = 2$,

$$v^\epsilon \in L^p(\mathbb{T}^d), \quad \ln m^\epsilon \in L^p(\mathbb{T}^d), \quad m^\epsilon \in L^p(\mathbb{T}^d),$$

for any $p \geq 1$.

For $d = 1$,

$$v^\epsilon \in C^{0,\frac{1}{2}}(\mathbb{T}^d), \quad \ln m^\epsilon \in C^{0,\frac{1}{2}}(\mathbb{T}^d), \quad (m^\epsilon)^{\frac{1}{2}} \in C^{0,\frac{1}{2}}(\mathbb{T}^d),$$

hence, v^ϵ and m^ϵ are continuous.

With these results in mind, we follow the same reasoning as in Section 2.6 to conclude that $\mathrm{Lip}(v^\epsilon) \leq C$.

A.2.4 Improving the regularity for the Fokker-Planck equation

Consider $\sigma \in (0, \epsilon)$ and the convolution $\eta_\sigma * m^\epsilon = m_\sigma^\epsilon$, with the convention that $\eta_0 * m^\epsilon = m^\epsilon$. Proposition 64 can be adapted to the regularized case to obtain

Theorem 83. *Let (v^ϵ, m^ϵ) be a solution to (A.1).*

$$D(m_\sigma^\epsilon) \in L^2(\mathbb{T}^d).$$

We also have the estimate

$$\int_{\mathbb{T}^d} \eta_\sigma * (m_\sigma^\epsilon)^{\beta-1} \eta_\epsilon * (m_\epsilon^\epsilon)^\alpha m^\epsilon dx \leq C.$$

A.3 Existence of smooth solutions to the original system

After obtaining sufficient regularity, we pass to the limit in ϵ. The family $m^\epsilon > \bar{m} > 0$ should be equicontinuous and bounded uniformly in ϵ. In particular, the regularity of g_ϵ should be the same as that of m^ϵ. Smoothness is obtained by standard regularity.

Appendix B

Optimal Control

B.1 Deterministic optimal control

A deterministic problem of optimal control in the calculus of variations setting is given by an initial value problem

$$\dot{x}(s) = u(s) \text{ , for } s \in [t, T] \tag{B.1}$$

$$x(t) = x \tag{B.2}$$

and a functional

$$J(t, x; u) = \underbrace{\int_t^T e^{-\int_t^s a(r,x(r))dr} L(s, x(s), u(s))ds}_{\text{running cost}} + \underbrace{e^{-\int_t^T a(r,x(r))dr} \psi(x(T))}_{\text{final cost}}$$

that needs to be minimized among the controls $u : [t, T] \to \mathbb{R}$ in a convenient function space.

We define the value function

$$v(t, x) = \inf_u J(t, x; u)$$

which satisfies the dynamic programming principle

Theorem 84. *For every $t' \in [t, T]$, the value function $v(t, x)$ defined above is equal to the infimum, over admissible controls u, of*

$$\int_t^{t'} e^{-\int_t^s a(r,x(r))dr} L(s, x(s), u(s))ds + e^{-\int_t^{t'} a(r,x(r))dr} v(t', x(t')).$$

Proof. We show first that $v(t, x) \geq \inf_u J(t, x; u)$. This is done by means of a δ-optimal control u_δ, where $\delta > 0$ is a small real number, that satisfies $J(t, x; u_\delta) \leq v(t, x) + \delta$. Since this is possible for any δ, we get the inequality.

Now we prove that $v(t, x) \leq \inf_u J(t, x; u)$. Use any admissible control u up to time $t' \in [t, T]$ and then change the control to a δ-optimal control

$u_\delta : [t', T] \to \mathbb{R}$, call the whole control $\tilde{u}$. We have

$$\begin{aligned} v(t,x) \leq J(t,x;\tilde{u}) = \int_t^{t'} & e^{-\int_t^s a(r,x(r))dr} L(s,x(s),u(s))ds \\ + \int_{t'}^{T} & e^{-\int_t^s a(r,x_\delta(r))dr} L(s,x_\delta(s),u_\delta(s))ds + e^{-\int_t^T a(r,x_\delta(r))dr}\psi(x_\delta(T)) \\ & \leq \int_t^{t'} e^{-\int_t^s a(r,x(r))dr} L(s,x(s),u(s))ds \\ & + e^{-\int_t^{t'} a(r,x(r))dr} \int_{t'}^{t} e^{-\int_{t'}^s a(r,x_\delta(r))dr} L(s,x_\delta(s),u_\delta(s))ds \\ & \qquad + e^{-\int_t^{t'} a(r,x_\delta(r))dr}\psi(x_\delta(T)) \\ \leq \int_t^{t'} & e^{-\int_t^s a(r,x(r))dr} L(s,x(s),u(s))ds + e^{-\int_t^{t'} a(r,x(r))dr} \left(v(t',x(t')) + \delta\right) \end{aligned} \tag{B.3}$$

since $J(t',x(t');u_\delta) \leq v(t',x(t')) + \delta$. □

The Legendre-Fenchel transform of $L(t,x,u)$ is given by

$$\sup_u \{-pu - L(t,x,u)\}$$

and it is denoted by $H(t,x,p)$, called the Hamiltonian.

There are properties of H that arise from properties of L. For instance, super-linear growth and convexity in u for L imply the same for H in terms of p.

Theorem 85. *Associated to the value function v, there is an equation, the Hamilton-Jacobi equation*

$$-v_t(t,x) + H(t,x,v_x) + a(t,x)v = 0$$

with terminal condition $v(T,x) = \psi(x(T))$. If v is differentiable enough, then it solves the above equation.

Proof. From the dynamic programming principle, the infimum over u of

$$\frac{1}{h}\int_t^{t+h} e^{-\int_t^s a(r,x(r))dr} L(s,x(s),u(s))ds \; + \; \frac{1}{h}\left[e^{-\int_t^{t+h} a(r,x(r))dr} v(t+h,x(t+h)) - v(t,x)\right] \tag{B.4}$$

is equal to 0 for all $h > 0$. Taking, formally, $h \to 0$, and assuming $\lim_{s\to t^+} u(s) = w \in \mathbb{R}$

$$0 = \inf_w \{L(t,x,w) - a(t,x)v(t,x) + v_x(t,x)w + v_t(t,x)\}$$

which gives the Hamilton-Jacobi equation wherever H is defined. □

Everything that was done in this section for $\mathbb{R}$ can be done, without significant changes, in $\mathbb{R}^n$.

B.2 A stochastic optimal control problem

A stochastic problem of optimal control in the calculus of variations setting is given by an initial value problem in $\mathbb{R}^d$

$$dx(s) = u(s)ds + \sqrt{2}dW(s) \text{ , for } s \in [t,T] \tag{B.5}$$

$$x(t) = x, \tag{B.6}$$

where $W(s)$ is a d-dimensional Brownian motion, and a functional

$$J(t,x;u) = \mathbb{E}\left[\underbrace{\int_t^T e^{-\int_t^s a(r,x(r))dr} L(s,x(s),u(s))ds}_{\text{running cost}} + \underbrace{e^{-\int_t^T a(r,x(r))dr}\psi(x(T))}_{\text{final cost}}\right]$$

that needs to be minimized among the bounded and progressively measurable controls u.

We define the value function

$$v(t,x) = \inf_u J(t,x;u)$$

which intuitively satisfies the dynamic programming principle

Proposition 86. *For every $t' \in [t,T]$, the value function $v(t,x)$ defined above is equal to the infimum, over progressively measurable controls u, of*

$$\mathbb{E}\left[\int_t^{t'} e^{-\int_t^s a(r,x(r))dr} L(s,x(s),u(s))ds + e^{-\int_t^{t'} a(r,x(r))dr} v(t',x(t'))\right].$$

Theorem 87. *Associated to the value function v, there is an equation, the Hamilton-Jacobi equation*

$$-v_t(t,x) + H(t,x,v_x) = -a(t,x)v + \Delta v$$

with terminal condition $v(T,x) = \psi(x(T))$. If v is differentiable enough, then it solves the above equation.

Proof. Use any constant control u up to time $t' \in [t,T]$, then

$$v(t,x) \le \mathbb{E}\left[\int_t^{t'} e^{-\int_t^s a(r,x(r))dr} L(s,x(s),u(s))ds\right] + \mathbb{E}\left[e^{-\int_t^{t'} a(r,x(r))dr} v(t',x(t'))\right] \tag{B.7}$$

Itô's differentiation rule under (B.5) gives, for any $\varphi : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$

$$d\varphi(s,x(s)) = \left(\frac{\partial \varphi}{\partial s} + D_x\varphi \cdot u(s) + \operatorname{Tr}\left(D_x^2\varphi\right)\right) ds + \sqrt{2} D_x\varphi dW(s)$$

where the last term is a Martingale, i.e. has incremental expectancy equal to zero. Subtracting $v(t,x)$, dividing by $t'-t$, taking the above into account, and leting $t' \to t$, we get

$$0 \le L(t,x) - a(t,x)v(t,x) + v_t(t,x) + D_x v(t,x)\cdot w + \Delta v.$$

□

In the infinite horizon discounted cost problem, we get the equation

$$H(t,x,v_x) = \bar{H} - a(t,x)v + \Delta v$$

for the minimization of

$$J(x;u) = \mathbb{E}\left[\int_0^\infty e^{-\int_0^s a(r,x(r))dr} L(s,x(s),u(s))ds\right],$$

under

$$\begin{aligned} dx(s) =& u(s)ds + \sqrt{2}dW(s) \text{ , for } s \in [0,\infty) \\ x(0) =& x, \end{aligned}$$

This shows that the problem we deal with in the thesis can be seen as an optimization of each player with the assumption that he uses $a(x) = m^{\alpha}(x)$ as a parameter.

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