The Concept of Surface Tension in Statistical Mechanics

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Resumo

RODRIGUES, J. **O Conceito de Tensão Superficial em Mecânica Estatística**. 2023.Dissertação (Mestrado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2023.

Nesta dissertação foram estudados dois artigos, [BKL83] e [GHMMS77], relacionados à existência e cotas para a tensão superficial τ_β. Em [BKL83], a teoria de Pirogov-Sinai é utilizada para obter uma cota inferior estritamente positiva para a tensão superficial, mas a existência do limite que define τ_β não é discutida. No caso especial de interações ferromagnéticas para modelos do tipo Ising, onde os acoplamentos J_A são todos não negativos, [GHMMS77] assegura a existência e uma cota superior uniforme em cada retângulo d-dimensional de lados $(L_1, ..., L_{d-1}, 2M)$ para o limite que define τ_β, por meio de um argumento de superaditividade. Seguindo o trabalho de [MG72], realizamos um estudo aprofundado das relações de dualidade entre modelos do tipo Ising, necessárias para compreender o argumento.

Palavras-chave: mecânica estatística, transformações de dualidade, modelos de contornos, tensão superficial, teoria de Pirogov-Sinai.

Abstract

RODRIGUES, J. **The Concept of Surface Tension in Statistical Mechanics**. 2023. Masters degree - Institute of Mathematics and Statistics, University of São Paulo, São Paulo, 2023.

In this thesis two papers, [BKL83] and [GHMMS77], were studied, concerning the existence and bounds for the surface tension τ_β. In [BKL83], Pirogov-Sinai theory is employed to yield a strictly positive lower bound to the surface tension, but the existence of the limit defining τ_β is not discussed. In the special case of ferromagnetic interactions for Ising-like models, where the couplings J_A are all non-negative, [GHMMS77] guarantees the existence and a uniform upper bound in each d-dimensional rectangle of sides $(L_1, ..., L_{d-1}, 2M)$ to the limit defining τ_β, by a superadditivity argument. Following the work of [MG72], we make an in-depth study of duality relations between Ising-like models, necessary to understand the argument.

Keywords: statistical mechanics, duality transformations, contour models, surface tension, Pirogov-Sinai theory.

Contents

Introduction

Rigorous Statistical Mechanics is a branch of physics that aims to provide a mathematically rigorous foundation for understanding the behavior of systems with a large number of interacting components, such as particles in a gas or atoms in a solid. Diverging from the empirical nature of thermodynamics, which predominantly addresses macroscopic phenomena, statistical mechanics provides the framework for understanding those macroscopic properties of matter starting from its microscopical constituents. Equilibrium Statistical Mechanics constitutes the branch of the theory dedicated to systems wherein macroscopic properties remain constant over time. It stands out as the most comprehensively understood aspect of the broader framework. If this is the case, then the appropriate distributions for the microscopic states ω of the system are given by the Gibbs distributions, which heuristically take the form

$$\mu_\beta(\omega) = \frac{\exp(-\beta\mathscr{H}(\omega))}{Z_{\Lambda,\beta}},$$

where $\mathscr{H}$ is the *Hamiltonian*, giving the microscopical energy of the states, $\beta = \frac{k_B}{T}$ is the inverse temperature with the Boltzmann factor k_B and Z_β is the *partition function*, a normalization factor obtained by integrating over all possibilities of states. In this thesis, however, the focus is restricted to *lattice spins*, where one may think that in each point $x \in \mathbb{Z}^d$ lies a fixed particle with spin values belonging to a given set E. Therefore, the microscopical states of the system are just the elements of $E^{\mathbb{Z}^d}$, that is, an assignment of a spin value for each point of the lattice $\mathbb{Z}^d$.

The determination of the partition function can be used in turn to derive important thermodynamic objects, such as the specific heat and free energy. The *surface tension*, the main topic of this thesis, is yet another example of a quantity related to the partiton function and is defined as the free energy per unit area of the separation interface perpendicular to a unit vector $\hat{\mathbf{n}}$ between two distinct phases of the system, like liquid and vapor. The definition adopted for the surface tension between phases, say, 1 and 2 in this thesis is discussed in section 1.2.2 and equals

$$\tau_\beta(\hat{\mathbf{n}}) \stackrel{\text{def}}{=} -\lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{1}{\beta|\Pi_{\hat{\mathbf{n}}}(\Lambda)|} \log\left(\frac{Z^{\hat{\mathbf{n}}}_{\Lambda,\beta}}{(Z^1_{\Lambda,\beta})^{\alpha(\hat{\mathbf{n}})}(Z^2_{\Lambda,\beta})^{(1-\alpha(\hat{\mathbf{n}}))}}\right),$$

where Λ is a finite region, $Z^q_{\Lambda,\beta}$ is the partition function restricted to Λ with boundary condition q, that is, one considers the spins outside Λ are all equal to q, $\Pi_{\hat{\mathbf{n}}}$ denotes the separation plane between the phases, $\alpha(\hat{\mathbf{n}})$ is the fraction of Λ in contact with phase 1 and $Z^{\hat{\mathbf{n}}}_{\Lambda,\beta}$ is the partition function with boundary condition given by

$$\eta_{\hat{\mathbf{n}}}(i) := \begin{cases} q_1, & \text{if } i\cdot\hat{\mathbf{n}} \geq 0; \\ q_2, & \text{if otherwise.} \end{cases}$$

In the special case of Ising-like models, [GHMMS77] proved that the surface tension for ferromagnetic lattice systems is well-defined, in the sense that the limit exists, and uniformly bounded above (note that the definition presented here differs from [GHMMS77] by a sign, so up to this sign a lower bound will be proved) in Λ using an approach based on group theory and duality transformations. The first objective of the thesis was to understand this approach. Next, an effort to leave the scope of the ferromagnetic Hamiltonian was made by reading and understanding the approach made by [BKL83], where the author used Pirogov-Sinai theory and estimates on the partition function to give the uniform bound to the surface tension with the less restrictive hypothesis of Pirogov-Sinai theory (although the existence of the limit is not discussed).

The thesis is divided into two chapters, each dedicated to the exploration of one of the referenced papers. The initial chapter is specifically centered on the examination of [GHMMS77] and comprises of the following parts:

- Section 1.1.1 lays the groundwork by introducing and defining the fundamental concepts of rigorous statistical mechanics, used for the entire thesis. Drawing mainly from [FV17], section 1.1 introduces the measurable spaces in which the Gibbs measures are defined upon and ends with the notion of a local function.

  Building upon the groundwork laid in the preceding section, section 1.1.2 introduces interactions and their associated Hamiltonians. Various models are provided as concrete examples, including a long-range model for greater generality. Concepts such as the finite volume Gibbs measures, partition function and pressure are defined.

  In Section 1.1.3, the focus shifts to the study of infinite volume Gibbs measures. Following the definition of a specification, we present these measures as a class that is compatible, in the sense of definition 1.10, with the specification defined by the finite volume Gibbs measures. Furthermore, critical temperatures and thermodynamical limits are discussed.

- Section 1.2.1 introduces the basic terminology of Pirogov-Sinai theory. Within this context the assumptions of the theory are discussed and perturbed Hamiltonians are introduced. Contour models are defined as probability measures on contours with some fixed boundary condition and their relation to a fixed model is given in proposition 1.18.

  In Section 1.2.2, the central theme of the thesis, surface tension, is motivated and discussed. It is conceptualized as the contribution to the free energy per unit of area arising from the presence of an interface that separates two coexisting phases. Notably, this interface is localized at sufficiently low temperatures in certain models, and the surface tension quantifies the free energy associated with this localized interface. It is expected to be zero below the spontaneous magnetization range and positive otherwise, and we point that it is indeed the case for the Ising model. A general formula for the surface tension is derived, encompassing the special case of Ising-like models, which is the one used in Chapter 2.

  The main theorem of Chapter 1 is proved in section 1.2.3 and it consists of the fact that the surface tension between two ground states is strictly positive given that the ground states are dominant, as in corollary 1.19, and given that the defining limit exists (although its existence is not discussed). This result is established through the theoretical framework developed throughout the entirety of Chapter 1.

Chapter 2 is dedicated to establishing the existence of surface tension in ferromagnetic systems. To achieve this result, the chapter is split in several sections.

- In section 2.1 we define a group structure on the configuration space Ω suitable for ferromagnetic systems and we study some of its properties, primarily referencing [GHM77]. We then proceed to rewrite the partition function with $+$ boundary conditions in two distinct ways with the help of some distinguished subgroups of Ω.

- Building on the observed similarities in the expressions of the partition function, Section 2.2 introduces the duality transformation between two models in a natural manner. Emphasis is then given to derive the relations between the ratio of the partition functions and between the correlation functions. Following the approach of [MG72], Section 2.2.1 presents a general approach to constructing dual systems for finite Λ, with three illustrative examples. In preparation for the main theorem of the chapter, Section 2.2.2 discusses the definition of duality in infinite systems.

- Finally, in section 2.3 we show that the limit defining the surface tension is uniformly bounded above by some positive constant and prove that the limit exists using a superadditivity argument.

Chapter 1

Statistical Mechanics on the Lattice

1.1 Initial Considerations

1.1.1 Configuration Space

Configurations in classical lattice statistical mechanics are defined by first specifying a *spin state space* together with a *a-priori measure* μ_0. This is a Polish probability space (E, μ_0) whose elements of E are the possible values the individual spins can attain and μ_0 represents the probability of finding each of those values when the spin is isolated, that is, assuming it does not interact with any other spins. We will give three examples.

- $E \stackrel{\text{def}}{=} \{-1, 1\}$ corresponds to the case where spins can only admit "up" or "down" states, represented by -1 and 1 respectively;
- $E \stackrel{\text{def}}{=} \{1, 2, ..., n\}$ corresponds to a generalization of the state space defined above, where spins can attain n distinct possible values;
- $E \stackrel{\text{def}}{=} \mathbb{S}^{n-1}$, which are associated with the $O(n)$ models. Note that for $n = 1$ we recover the first example, but for all $n \geq 2$ the state space is of continuum spin values. The $O(2)$ model is known as the *XY model* and $O(3)$ is known as *Heisenberg model*.

Given our initial data (E, μ_0), the configuration space of the d-dimensional lattice is $\Omega \stackrel{\text{def}}{=} E^{\mathbb{Z}^d}$. If we agree that each vertex of $\mathbb{Z}^d$ represents a particle with spin values given by the elements of E, then a configuration can be seen as a function $\sigma : \mathbb{Z}^d \to E$ assigning to each particle some spin value. Similarly we define the configurations inside some set $S \subset \mathbb{Z}^d$ as $\Omega_S \stackrel{\text{def}}{=} E^S$. If the set is finite, we adopt the notation $S \Subset \mathbb{Z}^d$. The *models* correspond to a choice of (E, μ_0) together with a *local Hamiltonian*, to be defined shortly.

Consider a finite subset $\Lambda \Subset \mathbb{Z}^d$ and some family of configurations inside Λ, say, $A \subset \Omega_\Lambda$. The event "some configuration of A is seen inside Λ" can be written as $\Pi_\Lambda^{-1}(A) = \{\omega \in \Omega : \omega_\Lambda \in A\}$, where

$$\Pi_\Lambda : \Omega \to \Omega_\Lambda$$
$$\omega \mapsto \omega_\Lambda$$

is the canonical projection. In general, we say that some collection of configurations $\Pi_\Lambda^{-1}(A)$, for $\Lambda \Subset \mathbb{Z}^d$ and $A \in \mathscr{P}(\Omega_\Lambda)$ is a cylinder with base Λ. Here, $\mathscr{P}(X)$ and $\mathscr{P}_f(X)$ will always denote the power set of X and $\mathscr{P}_f(X)$ denotes the collection of all finite subsets of X.

For example, the set $\{\omega \in \Omega : \omega_0 = -1\}$ may be written as $\Pi_{\{0\}}^{-1}(A)$, where A consists of the configuration admitting -1 at the origin, and hence is a cylinder. The collection of all cylinders with base Λ is defined by

$$\mathscr{C}_\Lambda \stackrel{\text{def}}{=} \{\Pi_\Lambda^{-1}(A) : A \in \mathscr{P}(\Omega_\Lambda)\},$$

and has a structure of an algebra of sets. Moreover, for any $S \subset \mathbb{Z}^d$ not necessarily finite, we define the family

$$\mathscr{C}_S \stackrel{\text{def}}{=} \bigcup_{\Lambda \Subset S} \mathscr{C}_\Lambda$$

of *local events inside* S. By a local event (inside S), we mean any event depending only on finitely many spins in S. This notion is better explained by proposition 1.2 below. The σ-algebra of local events in S is therefore $\mathscr{F}_S \stackrel{\text{def}}{=} \sigma(\mathscr{C}_S)$. In the special case $S = \mathbb{Z}^d$ we use the notation $\mathscr{F}_{\mathbb{Z}^d} = \mathscr{F}$ and $\mathscr{C}_{\mathbb{Z}^d} = \mathscr{C}$. We remark that, as is easily seen, $\mathscr{F}_S \subset \mathscr{F}$ for any $S \subset \mathbb{Z}^d$, and hence any $\mathscr{F}_S-$measurable function is $\mathscr{F}-$measurable.

For any $S \subset \mathbb{Z}^d$ not necessarily finite and for any $\Lambda \Subset S$, consider the projection $\tilde{\Pi}_{S,\Lambda} : \Omega_S \to \Omega_\Lambda$ and define

$$\mathscr{C}'_{S,\Lambda} \stackrel{\text{def}}{=} \{\Pi^{-1}_{S,\Lambda}(A) : A \in \mathscr{P}(\Omega_\Lambda)\},$$

$$\mathscr{C}'_S \stackrel{\text{def}}{=} \bigcup_{\Lambda \Subset S} \mathscr{C}'_{S,\Lambda}.$$

We always endow Ω_S with the σ-algebra $\mathscr{F}'_S \stackrel{\text{def}}{=} \sigma(\mathscr{C}'_S)$. Note that the only difference between the cylinder events $\mathscr{C}'_S$ and $\mathscr{C}_S$ is the base space the configurations live in: $\mathscr{F}'_S$ is a σ-algebra in Ω_S, while $\mathscr{F}_S$ is one in Ω. With respect to these $\sigma-$algebras, Π_S is measurable for every $S \subset \mathbb{Z}^d$ and so is the map $(\Omega_S \times \Omega_{S^c}, \mathscr{F}'_S \otimes \mathscr{F}'_{S^c}) \ni (\omega_S, \eta_{S^c}) \mapsto \omega_S \eta_{S^c}$.

The next lemma will we useful for the subsequent proposition:

Lemma 1.1 (Doob-Dynkin's Lemma)**.** *Let Ω_1, Ω_2 be two measurable spaces and consider maps $f : \Omega_1 \to \mathbb{R}$ and $g : \Omega_1 \to \Omega_2$. If f is measurable with respect to the σ-algebra generated by g then there is some measurable map $\varphi : \Omega_2 \to \mathbb{R}$ such that $f = \varphi \circ g$.*

Proof. In short, f contains all measurable-theoretic information about g, so it must be a measurable function of g. For a proof, see Lemma 1.13 of [Kal21]. □

The following proposition characterizes $\mathscr{F}_S-$measurability:

Proposition 1.2. *A function $g : \Omega \to \mathbb{R}$ is $\mathscr{F}_S$-measurable if and only if there is a $\mathscr{F}'_S$-measurable function $\varphi : \Omega_S \to \mathbb{R}$ such that $g(\omega) = \varphi(\omega_S)$ for every $\omega \in \Omega$.*

Proof. First, we will prove that $\mathscr{F}_S = \sigma(\Pi_S)$. Since Π_S is measurable, we already have $\sigma(\Pi_S) \subset \mathscr{F}_S$. For the other inclusion, note that any event of $\mathscr{C}_S$ is of the form $\Pi^{-1}_\Lambda(A)$, for some $A \in \mathscr{P}(\Omega_\Lambda)$ and $\Lambda \Subset S$, and we may write $\Pi^{-1}_\Lambda(A) = \Pi^{-1}_S(\Pi^{-1}_{S,\Lambda}(A)) \in \sigma(\Pi_S)$. Therefore $\mathscr{C}_S \subset \sigma(\Pi_S)$ and hence $\mathscr{F}_S \subset \sigma(\Pi_S)$, completing the proof that $\sigma(\Pi_S) = \mathscr{F}_S$. By Doob-Dynkin's Lemma, there exists a $\mathscr{F}'_S$-measurable $\varphi : \Omega_S \to \mathbb{R}$ such that $g = \varphi \circ \Pi_S$. □

The next corollary formalizes the notion that events on $\mathscr{F}_S$ depend only on finitely many spins inside S:

Corollary 1.3. *Given $\Lambda \Subset \mathbb{Z}^d$, the following two conditions are equivalent:*

1. *$g : \Omega \to \mathbb{R}$ is $\mathscr{F}_\Lambda$-measurable;*
2. *For any two configurations $\omega, \tilde{\omega} \in \Omega$, then $g(\omega) = g(\tilde{\omega})$ if $\omega_\Lambda = \tilde{\omega}_\Lambda$.*

Proof. (1) $\implies$ (2): Suppose $g : \Omega \to \mathbb{R}$ is $\mathscr{F}_\Lambda$-measurable. By Proposition 1.2, there is some $\varphi : \Omega_\Lambda \to \mathbb{R}$ such that $g(\omega) = \varphi(\omega_\Lambda)$. Hence, if $\omega_\Lambda = \tilde{\omega}_\Lambda$, we have $g(\omega) = \varphi(\omega_\Lambda) = \varphi(\tilde{\omega}_\Lambda) = g(\tilde{\omega})$.

(2) $\implies$ (1): If condition (2) is met, then the image of g is finite and consists of $\mathrm{Im}(g) = \{g(\omega_\Lambda) : \omega_\Lambda \in \Omega_\Lambda\}$. Therefore, one may write

$$g = \sum_{\omega_\Lambda \in \Omega_\Lambda} g(\omega_\Lambda)\chi_{\Pi_\Lambda^{-1}(\{\omega_\Lambda\})}.$$

Since each cylinder $\Pi_\Lambda^{-1}(\{\omega_\Lambda\})$ is $\mathscr{F}_\Lambda-$measurable, then each characteristic function $\chi : (\Omega, \mathscr{F}_\Lambda) \to \mathbb{R}$ is $\mathscr{F}_\Lambda$-measurable. Since the summation above is finite and finite sums of measurable functions is again measurable, then g is $\mathscr{F}_\Lambda-$measurable. □

A function $g : \Omega \to \mathbb{R}$ satisfying any of the two items of the previous corollary is called a Λ-*local function* or simply *local function* if Λ is clear from the context. Item (2) of the previous corollary is the main way of identifying if a given function is local, and item (1) is of importance when one wishes to integrate local functions with measures defined on the measurable space $(\Omega, \mathscr{F})$, the theory of which we will explore shortly.

1.1.2 Interactions and Hamiltonians

Consider the set $\mathscr{M}_1(\Omega)$ of all probability measures defined on the measurable space $(\Omega, \mathscr{F})$. The expected value of any local function $f : \Omega \to \mathbb{R}$ with respect to some $\mu \in \mathscr{M}_1(\Omega)$ will be denoted by $\langle f\rangle_\mu$ and we will omit the subscript when it is implicit by context. Moreover, given a family of local functions $(\Phi_A)_{A\Subset\mathbb{Z}^d}$, called an *interaction*, we formally define the *local Hamiltonian* $\mathscr{H}_\Lambda^\eta : \Omega \to \mathbb{R}$ of this interaction with boundary condition $\eta \in \Omega$, on $\Lambda \Subset \mathbb{Z}^d$, by the quantity

$$\mathscr{H}_\Lambda^\eta(\omega) \stackrel{\text{def}}{=} \sum_{\substack{A\Subset\mathbb{Z}^d \\ A\cap\Lambda\neq\varnothing}} \Phi_A(\omega_\Lambda \eta_{\Lambda^c}). \tag{1.1}$$

Note that there is no guarantee that the sum above converges with no extra conditions on the interactions. However, we note that

$$|\mathscr{H}_\Lambda^\eta(\omega)| \leq \sum_{x\in\Lambda} \sum_{\substack{A\Subset\mathbb{Z}^d \\ A\ni x}} ||\Phi_A||, \tag{1.2}$$

where $||\Phi_A|| \stackrel{\text{def}}{=} \sup_{\sigma_A\in\Omega_A} |\Phi_A(\sigma_A)|$. An interaction $\Phi = (\Phi_A)_{A\Subset\mathbb{Z}^d}$ is called *regular* if the right-hand side of 1.2 is finite for every $x \in \mathbb{Z}^d$, which is enough to ensure convergence. As for examples, we have

- (Long-range Ising model with polynomial decay). This model is defined by the interactions $\Phi_{i,j}(\sigma) \stackrel{\text{def}}{=} -J\frac{\sigma_i\sigma_j}{|i-j|_1^\alpha}$, $i \neq j$, and zero otherwise, where $J > 0$ and $\alpha > d$. Denoting $[n] \stackrel{\text{def}}{=} \{1, 2, ..., n\}$, we note that

$$\sum_{0\neq i\in\mathbb{Z}^d} \frac{1}{(|i_1| + ... + |i_d|)^\alpha} = \sum_{j=1}^d \sum_{\substack{X\subset[d] \\ X=\{k_1,...,k_j\}}} \left(\sum_{\substack{i_{k_1}\neq 0 \\ i_{k_1}\in\mathbb{Z}}} ... \sum_{\substack{i_{k_j}\neq 0 \\ i_{k_j}\in\mathbb{Z}}} \frac{1}{(|i_{k_1}| + ... + |i_{k_j}|)^\alpha} \right)$$

$$\leq \sum_{j=1}^d \frac{2^j}{j^\alpha} \sum_{\substack{X\subset[d] \\ X=\{k_1,...,k_j\}}} \left(\sum_{i_{k_1}\geq 1} ... \sum_{i_{k_j}\geq 1} \frac{1}{(i_{k_1})^{\frac{\alpha}{d}} ... (i_{k_j})^{\frac{\alpha}{d}}} \right) = \sum_{j=1}^d \frac{2^j}{j^\alpha} \sum_{\substack{X\subset[d] \\ |X|=j}} \left(\sum_{n\geq 1} \frac{1}{n^{\frac{\alpha}{d}}} \right)^j$$

$$= \sum_{j=1}^d \frac{2^j}{j^\alpha} \binom{d}{j} \zeta\left(\frac{\alpha}{d}\right)^j < \infty,$$

where in the first line we decomposed the sum over all non-zero $i \in \mathbb{Z}^d$ into sums where each chosen j-uple $(k_1, ..., k_j)$ of coordinate indexes of i is non-zero and the others are null, then summed over all choices $X \subset [d]$, $X = \{k_1, ..., k_j\}$ of those coordinate indexes, in the second line we used the AM-GM (Arithmetic-Geometric) inequality and ζ is the Riemann zeta function.

Since $\alpha > d$, $\zeta\left(\frac{\alpha}{d}\right)$ is finite, and one can show that the regularity condition fails for all $\alpha \leq d$. Here, J controls the amount of energy the alignment (or disalignment) of spins yield and α regulates the strength of the interactions. An *external magnetic field* may be added, which is a function $h : \mathbb{Z}^d \to \mathbb{R}$. For non-zero external fields, we can consider an additional interaction $(\Phi_i)_{i\in\mathbb{Z}^d}$ given by $\Phi_i(\sigma) = -J\sigma_i h_i$. The Hamiltonian is then given by

$$\mathscr{H}^{\eta}_{\Lambda,h}(\sigma) = -J \sum_{\substack{i\in\Lambda \\ j\in\Lambda}} \frac{\sigma_i\sigma_j}{|i-j|_1^{\alpha}} - J \sum_{\substack{i\in\Lambda \\ j\in\Lambda^c}} \frac{\sigma_i\eta_j}{|i-j|_1^{\alpha}} - J\sum_{i\in\Lambda}\sigma_i h_i \tag{1.3}$$

- (Short-range multi-body Ising models). Given some fixed $A \subset \mathbb{Z}^d$, the non-zero interactions of this model are given by $\Phi_A(\sigma) \stackrel{\text{def}}{=} -J_A\sigma_A$ and its translations $\Phi_{A+x}(\sigma) = -J_A\sigma_{A+x}$ for $x \in \mathbb{Z}^d$. Moreover, we assume $J_A \geq 0$ for every A and $\sigma_A \stackrel{\text{def}}{=} \prod_{i\in A}\sigma_i$ is the product of spins inside A. External magnetic fields can also be considered, just as in the last example. Since only finitely many translates of A contain a single vertex of $\mathbb{Z}^d$, the interactions are regular. The Hamiltonian is, then,

$$\mathscr{H}^{\eta}_{\Lambda,h}(\sigma) = -J \sum_{A\cap\Lambda\neq\varnothing} J_A\sigma_{A\cap\Lambda}\eta_{A\cap\Lambda^c} - J\sum_{i\in\mathbb{Z}^d}\sigma_i h_i \tag{1.4}$$

- ($O(n)$ models). The interactions of the long-range models with polynomial decay are given by $\Phi_{i,j}(\sigma) \stackrel{\text{def}}{=} -J\frac{\sigma_i\cdot\sigma_j}{|i-j|_1^{\alpha}}$, where $J > 0$ and $\alpha > d$ as before, and $\cdot$ denotes the usual euclidean scalar product of $\mathbb{R}^d$. By the Cauchy-Schwarz inequality, $|\sigma_i \cdot \sigma_j| \leq 1$ for all $i \neq j$, so that the regularity follows in the same way as for the long-range Ising model. The first-neighbor variant is defined by $\Phi_{i,j}(\sigma) \stackrel{\text{def}}{=} -J\sigma_i\cdot\sigma_j$ for all pairs i, j with $|i-j|_1 = 1$ and zero otherwise. With an external magnetic field, the Hamiltonians are given by, respectively,

$$\mathscr{H}^{\eta}_{\Lambda,h}(\sigma) = -J \sum_{\substack{i\in\Lambda \\ j\in\Lambda}} \frac{\sigma_i\cdot\sigma_j}{|i-j|_1^{\alpha}} - J \sum_{\substack{i\in\Lambda \\ j\in\Lambda^c}} \frac{\sigma_i\cdot\eta_j}{|i-j|_1^{\alpha}} - J\sum_{i\in\Lambda}\sigma_i h_i \tag{1.5}$$

$$\mathscr{H}^{\eta}_{\Lambda,h}(\sigma) = -J \sum_{\substack{i\in\Lambda \\ j\in\Lambda}} \sigma_i\cdot\sigma_j - J \sum_{\substack{i\in\Lambda \\ j\in\Lambda^c}} \sigma_i\cdot\eta_j - J\sum_{i\in\Lambda}\sigma_i h_i. \tag{1.6}$$

One of the main objects[1] of Statistical Mechanics is the partition function $Z^{\eta}_{\Lambda,\beta}$ with inverse temperature β associated with an interaction. With respect to the a-priori measure μ_0, it is defined by

$$Z^{\eta}_{\Lambda,\beta} \stackrel{\text{def}}{=} \int_{\Omega_\Lambda} e^{-\beta\mathscr{H}^{\eta}_{\Lambda}(\omega_\Lambda\eta_{\Lambda^c})}\mu_0^{\Lambda}(d\omega_\Lambda), \tag{1.7}$$

where we define $\mu_0^{\Lambda} = \bigotimes_{i\in\Lambda}\mu_0$ as the product measure of the single spin measure μ_0. Of course, if there is an external field then one may insert it as an argument for the partition function. An important associated quantity is the *pressure*, defined by

[1] Apart from being used to derive important thermodynamic quantities like free energy and entropy, the partition function plays a central role in the expression for the Gibbs measures, which give the appropriate probabilities of finding given configurations of spins. In fact, the partition function is the normalization factor for those measures.

$$\psi(\beta, h) \stackrel{\text{def}}{=} \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log Z^{\eta}_{\Lambda,\beta,h}, \tag{1.8}$$

and it is possible to show that this quantity exists for all sequence of boxes converging to $\mathbb{Z}^d$ in the sense of van Hove (for example, the sequence $\Lambda_n \stackrel{\text{def}}{=} [-n, n]^d \cap \mathbb{Z}^d$ satisfies the van Hove convergence property), which is defined below:

Definition 1.4 (Convergence in the sense of van Hove)**.** *A sequence of subsets $(\Lambda_n)_{n\geq 1}$ of $\mathbb{Z}^d$ converges to $\mathbb{Z}^d$ in the sense of Van Hove if the conditions below are satisfied:*

- *The sequence is crescent, that is, $\Lambda_n \subset \Lambda_{n+1}$ for every $n \geq 1$;*
- *The sequence invades[2]; $\mathbb{Z}^d$ in the sense that $\bigcup_{n\geq 1} \Lambda_n = \mathbb{Z}^d$;*
- *The limit $\lim_{n\to\infty} \frac{|\partial\Lambda_n|}{|\Lambda_n|}$ is zero.*

Moreover, the limit is independent of the choice of boxes and of the boundary condition, where we also assume a regular interaction. For a proof, see [Isr79].

A class of important measures on $\mathscr{M}_1(\Omega)$ are the finite volume Gibbs measures

$$\mu^{\eta}_{\Lambda,\beta}(A) \stackrel{\text{def}}{=} \int_{\Omega_\Lambda} \mathbb{1}_A(\omega_\Lambda\eta_{\Lambda^c}) \frac{e^{-\beta \mathscr{H}^{\eta}_{\Lambda}(\omega_\Lambda\eta_{\Lambda^c})}}{Z^{\eta}_{\Lambda}} \mu^{\Lambda}_0(d\omega_\Lambda) = \int_A \frac{e^{-\beta\mathscr{H}^{\eta}_{\Lambda}(\omega)}}{Z^{\eta}_{\Lambda,\beta}} f_{\Lambda *}(\mu^{\Lambda}_0)(d\omega), \tag{1.9}$$

for all $A \in \mathscr{F}$, $f_\Lambda : \Omega_\Lambda \to \Omega$ is given by $f_\Lambda(\omega_\Lambda) \stackrel{\text{def}}{=} \omega_\Lambda\eta_{\Lambda^c}$ and $f_{\Lambda *}(\mu^{\Lambda}_0)$ is the push-forward measure of μ^{Λ}_0 by f_Λ. Note that, by definition 1.9 and the definition of the Radon-Nikodym derivative we have

$$\frac{d\mu^{\eta}_{\Lambda,\beta}}{df_{\Lambda *}(\mu^{\Lambda}_0)} = \frac{e^{-\beta\mathscr{H}^{\eta}_{\Lambda}(\omega)}}{Z^{\eta}_{\Lambda,\beta}}.$$

Given a finite volume Gibbs measure $\mu^{\eta}_{\Lambda,\beta}$, the expectation value of a given local function $g : \Omega \to \mathbb{R}$ is denoted by $\langle g\rangle^{\eta}_{\Lambda,\beta}$. Note that we have

$$\langle g\rangle^{\eta}_{\Lambda,\beta} = \int_\Omega g d\mu^{\eta}_{\Lambda,\beta} = \int_\Omega g \frac{d\mu^{\eta}_{\Lambda,\beta}}{df_{\Lambda *}(\mu^{\Lambda}_0)} df_{\Lambda *}(\mu^{\Lambda}_0) \tag{1.10}$$

$$= \int_\Omega g(\omega) \frac{e^{-\beta\mathscr{H}^{\eta}_{\Lambda}(\omega)}}{Z^{\eta}_{\Lambda,\beta}} f_{\Lambda *}(\mu^{\Lambda}_0)(d\omega) = \int_{\Omega_\Lambda} g(\omega_\Lambda\eta_{\Lambda^c}) \frac{e^{-\beta\mathscr{H}^{\eta}_{\Lambda}(\omega_\Lambda\eta_{\Lambda^c})}}{Z^{\eta}_{\Lambda,\beta}} \mu^{\Lambda}_0(d\omega_\Lambda). \tag{1.11}$$

In the special case of a finite spin values and μ_0 uniform, this yields

$$\langle g\rangle^{\eta}_{\Lambda,\beta} = \sum_{\omega_\Lambda\in\Omega_\Lambda} g(\omega_\Lambda\eta_{\Lambda^c}) \frac{e^{-\beta\mathscr{H}^{\eta}_{\Lambda}(\omega_\Lambda\eta_{\Lambda^c})}}{Z^{\eta}_{\Lambda,\beta}}. \tag{1.12}$$

The finite volume Gibbs measures give the appropriate probability distribution of configurations $\omega_\Lambda \in \Omega_\Lambda$. The infinite volume analogue of these measures should be, intuitively, be taken as some sort of *thermodynamic limit* of the finite volume measures $(\mu^{\eta}_{\Lambda_n,\beta})_{n\geq 1}$, where $\Lambda_n \nearrow \mathbb{Z}^d$ converges in the sense of van Hove, so that we get a measure capable of giving those probabilities for configurations $\omega \in \Omega$. We will provide a construction of the infinite volume Gibbs measures which reflects this desired limiting definition as a theorem.

Before the introduce the infinite volume Gibbs measures, however, we shall note some properties of the finite-volume Gibbs measures. The first one is the following compatibility condition:

[2] The notation $\Lambda_n \nearrow \mathbb{Z}^d$ will be used if the sequence $(\Lambda_n)_{n\geq 1}$ is crescent and invades $\mathbb{Z}^d$.

Lemma 1.5. *Given $\Delta, \Lambda \Subset \mathbb{Z}^d$ such that $\Delta \subset \Lambda$, then for any bounded and local function $f : \Omega \to \mathbb{R}$ one has the following compatibility condition:*

$$\langle f\rangle^{\eta}_{\Lambda,\beta} = \langle\langle f\rangle^{(\cdot)}_{\Delta,\beta}\rangle^{\eta}_{\Lambda,\beta}$$

Proof. We will start with the right-hand side and work our way to the left-hand side of the equality. By definition of expectation with respect to the finite Gibbs measures, one has

$$\langle\langle f\rangle^{(\cdot)}_{\Delta,\beta}\rangle^{\eta}_{\Lambda,\beta} = \sum_{\omega_\Lambda\in\Omega_\Lambda} \langle f\rangle^{\omega_\Lambda\eta_{\Lambda^c}}_{\Delta,\beta} \frac{e^{-\beta\mathscr{H}_\Lambda(\omega_\Lambda\eta_{\Lambda^c})}}{Z^{\eta}_{\Lambda,\beta}}, \tag{1.13}$$

$$\langle f\rangle^{\omega_\Lambda\eta_{\Lambda^c}}_{\Delta,\beta} = \sum_{\omega'_\Delta\in\Omega_\Delta} f(\omega'_\Delta\omega_{\Lambda\setminus\Delta}\eta_{\Lambda^c})\frac{e^{-\beta\mathscr{H}_\Delta(\omega'_\Delta\omega_{\Lambda\setminus\Delta}\eta_{\Lambda^c})}}{Z^{\omega_\Lambda\eta_{\Lambda^c}}_{\Delta,\beta}}. \tag{1.14}$$

On equation 1.13, we can use equation 1.14 and replace $\omega_\Lambda = \omega_\Delta\omega_{\Lambda\setminus\Delta}$ and sum over ω_Δ and $\omega_{\Lambda\setminus\Delta}$. The result is

$$\langle\langle f\rangle^{(\cdot)}_{\Delta,\beta}\rangle^{\eta}_{\Lambda,\beta} = \sum_{\omega_\Delta\in\Omega_\Delta}\sum_{\omega_{\Lambda\setminus\Delta}\in\Omega_{\Lambda\setminus\Delta}}\sum_{\omega'_\Delta\in\Omega_\Delta} f(\omega'_\Delta\omega_{\Lambda\setminus\Delta}\eta_{\Lambda^c})\frac{e^{-\beta\mathscr{H}_\Delta(\omega'_\Delta\omega_{\Lambda\setminus\Delta}\eta_{\Lambda^c})}}{Z^{\omega_\Lambda\eta_{\Lambda^c}}_{\Delta,\beta}}\frac{e^{-\beta\mathscr{H}_\Lambda(\omega_\Delta\omega_{\Lambda\setminus\Delta}\eta_{\Lambda^c})}}{Z^{\eta}_{\Lambda,\beta}}. \tag{1.15}$$

We now observe that

$$\mathscr{H}_\Lambda(\omega_\Delta\omega_{\Lambda\setminus\Delta}\eta_{\Lambda^c}) - \mathscr{H}_\Delta(\omega_\Delta\omega_{\Lambda\setminus\Delta}\eta_{\Lambda^c}) = \sum_{\substack{A\Subset\mathbb{Z}^d\\ A\cap\Lambda\neq\varnothing}} \Phi_A(\omega_\Delta\omega_{\Lambda\setminus\Delta}\eta_{\Lambda^c}) - \sum_{\substack{A\Subset\mathbb{Z}^d\\ A\cap\Delta\neq\varnothing}} \Phi_A(\omega_\Delta\omega_{\Lambda\setminus\Delta}\eta_{\Lambda^c}) \tag{1.16}$$

$$= \sum_{\substack{A\Subset\mathbb{Z}^d\\ A\cap(\Lambda\setminus\Delta)\neq\varnothing\\ A\cap\Delta=\varnothing}} \Phi_A(\omega_\Delta\omega_{\Lambda\setminus\Delta}\eta_{\Lambda^c}). \tag{1.17}$$

Note that every term Φ_A is an A-local function, and hence depends only on the spins inside A (see corollary 1.3), which is disjoint of Δ. Therefore, the values of ω_Δ are irrelevant for the left hand side in 1.16 and we may interchange ω_Δ with ω'_Δ. The end result is

$$\mathscr{H}_\Lambda(\omega_\Delta\omega_{\Lambda\setminus\Delta}\eta_{\Lambda^c}) - \mathscr{H}_\Delta(\omega_\Delta\omega_{\Lambda\setminus\Delta}\eta_{\Lambda^c}) = \mathscr{H}_\Lambda(\omega'_\Delta\omega_{\Lambda\setminus\Delta}\eta_{\Lambda^c}) - \mathscr{H}_\Delta(\omega'_\Delta\omega_{\Lambda\setminus\Delta}\eta_{\Lambda^c}).$$

Rearranging the above equation, alternatively one has

$$\mathscr{H}_\Delta(\omega'_\Delta\omega_{\Lambda\setminus\Delta}\eta_{\Lambda^c}) + \mathscr{H}_\Lambda(\omega_\Delta\omega_{\Lambda\setminus\Delta}\eta_{\Lambda^c}) = \mathscr{H}_\Delta(\omega_\Delta\omega_{\Lambda\setminus\Delta}\eta_{\Lambda^c}) + \mathscr{H}_\Lambda(\omega'_\Delta\omega_{\Lambda\setminus\Delta}\eta_{\Lambda^c}). \tag{1.18}$$

Substituting this result in the last equation of 1.15, we finally get

$$\begin{aligned}\langle\langle f\rangle^{(\cdot)}_{\Delta,\beta}\rangle^{\eta}_{\Lambda,\beta} &= \sum_{\omega_\Delta\in\Omega_\Delta}\sum_{\omega_{\Lambda\setminus\Delta}\in\Omega_{\Lambda\setminus\Delta}}\sum_{\omega'_\Delta\in\Omega_\Delta} f(\omega'_\Delta\omega_{\Lambda\setminus\Delta}\eta_{\Lambda^c})\frac{e^{-\beta\mathscr{H}_\Delta(\omega_\Delta\omega_{\Lambda\setminus\Delta}\eta_{\Lambda^c})}}{Z^{\omega_\Lambda\eta_{\Lambda^c}}_{\Delta,\beta}}\frac{e^{-\beta\mathscr{H}_\Lambda(\omega'_\Delta\omega_{\Lambda\setminus\Delta}\eta_{\Lambda^c})}}{Z^{\eta}_{\Lambda,\beta}}\\
&= \sum_{\omega_{\Lambda\setminus\Delta}\in\Omega_{\Lambda\setminus\Delta}}\sum_{\omega'_\Delta\in\Omega_\Delta}\left(\sum_{\omega_\Delta\in\Omega_\Delta}\frac{e^{-\beta\mathscr{H}_\Delta(\omega_\Delta\omega_{\Lambda\setminus\Delta}\eta_{\Lambda^c})}}{Z^{\omega_\Lambda\eta_{\Lambda^c}}_{\Delta,\beta}}\right) f(\omega'_\Delta\omega_{\Lambda\setminus\Delta}\eta_{\Lambda^c})\frac{e^{-\beta\mathscr{H}_\Lambda(\omega'_\Delta\omega_{\Lambda\setminus\Delta}\eta_{\Lambda^c})}}{Z^{\eta}_{\Lambda,\beta}}\\
&= \sum_{\omega'_\Lambda\in\Omega_\Lambda} f(\omega'_\Lambda\eta_{\Lambda^c})\frac{e^{-\beta\mathscr{H}_\Lambda(\omega'_\Lambda\eta_{\Lambda^c})}}{Z^{\eta}_{\Lambda,\beta}} = \langle f\rangle^{\eta}_{\Lambda,\beta},\end{aligned}$$

where we have made the substitution $\omega_{\Lambda\setminus\Delta}\omega'_\Delta = \omega'_\Lambda$.

□

Before we state the next Lemma, we'll need two definitions.

Definition 1.6 (π and λ systems)**.** *Let $\mathcal{E} \subset \mathscr{P}(\Omega)$ be any non-empty collection of subsets of a measurable space Ω. Then*

- *$\mathcal{E}$ is a π-system if is closed under finite intersections;*
- *$\mathcal{E}$ is a λ-system if it is closed under countable disjoint unions, complements and if $\Omega \in \mathcal{E}$.*

The λ-system generated by $\mathcal{E}$, that is, the intersection of all λ-systems containing $\mathcal{E}$, is denoted by $\delta(\mathcal{E})$.

Theorem 1.7 (Dynkin's $\pi - \lambda$ theorem)**.** *If $\mathcal{E} \subset \mathscr{P}(\Omega)$ is a π-system, then $\sigma(\mathcal{E}) = \delta(\mathcal{E})$.*

Proof. See theorem 1.19 of [Kle14]. □

The properties of the finite volume Gibbs measures we are going to use are following:

Lemma 1.8. *Let $\Lambda \Subset \mathbb{Z}^d$ be any. Then:*

1. *For any $\eta \in \Omega$, the map $\mathscr{F} \ni A \mapsto \mu^\eta_{\Lambda,\beta}(A) \in \mathbb{R}$ is a measure on $(\Omega, \mathscr{F})$, and moreover if $B \in \mathscr{F}_{\Lambda^c}$ then $\mu^\eta_{\Lambda,\beta}(B) = \mathbb{1}_B(\eta)$;*
2. *For any $A \in \mathscr{F}$, the map $\Omega \ni \eta \mapsto \mu^\eta_{\Lambda,\beta}(A) \in \mathbb{R}$ is $\mathscr{F}_{\Lambda^c}$-measurable;*
3. *For any two finite sets $\Lambda, \Delta \Subset \mathbb{Z}^d$ such that $\Delta \subset \Lambda$, the consistency condition given by lemma 1.5 is satisfied.*

Proof. 1. The first affirmation is straightforward. Let $\mathscr{B}$ be the set of all $B \subset \Omega$ such that $\mu^\eta_{\Lambda,\beta}(B) = \mathbb{1}_B(\eta)$. Note that $\mathscr{B}$ is a λ-system: trivially $\varnothing \in \mathscr{B}$, now if $(B_n)_{n\geq 1}$ is a collection of disjoint sets of Ω all belonging to $\mathscr{B}$, then

$$\mu^\eta_{\Lambda,\beta}\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu^\eta_{\Lambda,\beta}(B_n) = \sum_{n=1}^{\infty} \mathbb{1}_{B_n}(\eta) = \mathbb{1}_{\cup_{n=1}^\infty B_n}(\eta).$$
$$\implies \cup_{n=1}^\infty B_n \in \mathscr{B}$$

Finally, if $B \in \mathscr{B}$ then:

$$\mu^\eta_{\Lambda,\beta}(B) = \mathbb{1}_B(\eta) \implies 1 - \mu^\eta_{\Lambda,\beta}(B) = 1 - \mathbb{1}_B(\eta)$$
$$\implies \mu^\eta_{\Lambda,\beta}(B^c) = \mathbb{1}_{B^c}(\eta) \implies B^c \in \mathscr{B}.$$

Moreover, $\mathscr{B}$ contains all union of cylinders of $\mathscr{C}_{\Lambda^c}$: given $S_1, ..., S_n \Subset \Lambda^c$ and $A_1, ..., A_i, ..., A_n \in \mathscr{P}(\Omega_{S_i})$, we have $\mathbb{1}_{\cup_{i=1}^n \Pi_{S_i}^{-1}(A_i)}(\omega_\Lambda\eta_{\Lambda^c}) = \max_{1\leq i\leq n}\{\mathbb{1}_{A_i}(\Pi_{S_i}(\omega_\Lambda\eta_{\Lambda^c}))\} = \max_{1\leq i\leq n}\{\mathbb{1}_{A_i}(\eta_{S_i})\}$. Hence

$$\mu^\eta_{\Lambda,\beta}(\cup_{i=1}^n \Pi_{S_i}^{-1}(A_i)) = \sum_{\omega_\Lambda\in\Omega_\Lambda} \frac{e^{-\beta\mathscr{H}^\eta_\Lambda(\omega_\Lambda\eta_{\Lambda^c})}}{Z^\eta_\Lambda} \max_{1\leq i\leq n}\{\mathbb{1}_{A_i}(\eta_{S_i})\} = \max_{1\leq i\leq n}\{\mathbb{1}_{A_i}(\eta_{S_i})\}$$
$$= \max_{1\leq i\leq n}\{\mathbb{1}_{\Pi_{S_i}^{-1}(A_i)}(\eta)\} = \mathbb{1}_{\cup_{i=1}^n \Pi_{S_i}^{-1}(A_i)}(\eta).$$

Therefore, $\mathscr{C}_{\Lambda^c} \subset \mathscr{B}$. Since $\mathscr{C}_{\Lambda^c}$ is a π-system, the conclusion follows from Dynkin's π-λ theorem.

2. Let $A \in \mathscr{F}$ be fixed. Note that each map $\eta \mapsto \Phi_A(\omega_\Lambda \eta_{\Lambda^c})$, with $A \Subset \mathbb{Z}^d$ is $A \cap \Lambda^c$–local: if $\eta_{A\cap\Lambda^c} = \tilde{\eta}_{A\cap\Lambda^c}$ then $(\omega_\Lambda \eta_{\Lambda^c})_A = \omega_{\Lambda\cap A}\eta_{\Lambda^c \cap A} = \omega_{\Lambda\cap A}\tilde{\eta}_{A\cap\Lambda^c} = (\omega_\Lambda \tilde{\eta}_{\Lambda^c})_A$. Since Φ_A is A-local, we get $\Phi_A(\omega_\Lambda\eta_{\Lambda^c}) = \Phi_A(\omega_\Lambda\tilde{\eta}_{\Lambda^c})$, finishing the proof of $A \cap \Lambda^c$-locality. Since $|A \cap \Lambda^c| < \infty$, we may use corollary 1.3 to get that $\eta \mapsto \Phi_A(\omega_\Lambda\eta_{\Lambda^c})$ is $\mathscr{F}_{A\cap\Lambda^c}$-measurable and hence $\mathscr{F}_{\Lambda^c}$-measurable.

   Now, the map $\eta \mapsto \mathscr{H}^\eta_\Lambda(\omega_\Lambda\eta_{\Lambda^c})$ can be expressed as a convergent pointwise limit of partial sums of the corresponding maps for the Φ_A. Since finite sums of Φ_A's are again $\mathscr{F}_{\Lambda^c}$-measurable and pointwise limits of measurable functions are measurable, we get $\mathscr{F}_{\Lambda^c}$-measurability for the map above. Hence, the map $\eta \mapsto \frac{e^{-\beta\mathscr{H}^\eta_\Lambda(\omega_\Lambda\eta_{\Lambda^c})}}{Z^\eta_\Lambda}$ is $\mathscr{F}_{\Lambda^c}$-measurable. As for the map $\eta \mapsto \mathbb{1}_A(\omega_\Lambda\eta_{\Lambda^c})$ (for fixed ω_Λ), it can be written as a composition

$$\begin{gathered}\eta \mapsto \eta_{\Lambda^c} \mapsto (\omega_\Lambda, \eta_{\Lambda^c}) \mapsto \omega_\Lambda\eta_{\Lambda^c} \mapsto \chi_A(\omega_\Lambda\eta_{\Lambda^c}) \\ (\Omega, \mathscr{F}) \to (\Omega_\Lambda, \mathscr{F}'_\Lambda) \to (\Omega_\Lambda \times \Omega_{\Lambda^c}, \mathscr{F}'_\Lambda \otimes \mathscr{F}'_{\Lambda^c}) \to (\Omega, \mathscr{F}) \to \mathbb{R}\end{gathered}$$

   All the maps above are measurable. This ensures that

$$\eta \mapsto \mu^\eta_{\Lambda,\beta}(A) = \sum_{\omega_\Lambda \in \Omega_\Lambda} \frac{e^{-\beta\mathscr{H}^\eta_\Lambda(\omega_\Lambda\eta_{\Lambda^c})}}{Z^\eta_{\Lambda,\beta}} \mathbb{1}_A(\omega_\Lambda\eta_{\Lambda^c})$$

   is a sum of $\mathscr{F}_{\Lambda^c}$–measurable maps, and hence $\mathscr{F}_{\Lambda^c}$-measurable.

3. Already proven.

□

1.1.3 DLR Equations and Gibbs States

The conditions on lemma 1.8 can be generalized by the following definition:

Definition 1.9. *Let $\Lambda \Subset \mathbb{Z}^d$ be any. A map $\pi_\Lambda : \mathscr{F} \times \Omega \to [0,1]$ is called a probability kernel from $\mathscr{F}_{\Lambda^c}$ to $\mathscr{F}$ if the following properties are satisfied:*

1. *For every $\omega \in \Omega$, the map $A \mapsto \pi_\Lambda(A|\omega)$ is a probability measure on $(\Omega, \mathscr{F})$;*
2. *For every $A \in \mathscr{F}$, the map $\omega \mapsto \pi_\Lambda(A|\omega)$ is $\mathscr{F}_{\Lambda^c}$-measurable.*

If moreover $\pi_\Lambda(A|\omega) = \mathbb{1}_A(\omega)$ for all $A \in \mathscr{F}_{\Lambda^c}$ and $\omega \in \Omega$, then the kernel is called proper.

If a probability kernel π_Λ is proper, then the measure $\pi_\Lambda(\cdot|\eta)$ depends only on the configurations in $\Omega^\eta_\Lambda = \{\omega \in \Omega : \omega_{\Lambda^c} = \eta_{\Lambda^c}\}$ up to a null set. In fact, since $\Omega^\eta_\Lambda \in \mathscr{F}_{\Lambda^c}$, then $\pi_\Lambda(\Omega^\eta_\Lambda|\eta) = 1$. The composition of two probability kernels is defined by the formula

$$(\pi_\Lambda\pi_\Delta)(A|\eta) \stackrel{\text{def}}{=} \int_\Omega \pi_\Delta(A|\omega)\pi_\Lambda(d\omega|\eta) = \langle \pi_\Delta(A|\cdot)\rangle_{\pi_\Lambda(\cdot|\eta)}$$

Definition 1.10. *A specification is a family of probability kernels $\pi = (\pi_\Lambda)_{\Lambda\Subset\mathbb{Z}^d}$ satisfying the compatibility condition:*

$$\pi_\Lambda\pi_\Delta = \pi_\Lambda,$$

for every $\Lambda, \Delta \Subset \mathbb{Z}^d$ such that $\Delta \subset \Lambda$. Moreover, a probability measure $\mu \in \mathscr{M}_1(\Omega)$ is compatible with the specification π if the condition

$$\mu\pi_\Lambda = \mu$$

holds for all $\Lambda \Subset \mathbb{Z}^d$, where $\mu\pi_\Lambda$ is the measure given by $(\mu\pi_\Lambda)(A) \stackrel{def}{=} \int_\Omega \pi_\Lambda(A,\omega)\mu(d\omega)$. The set of all such measures is called $\mathscr{G}(\pi)$.

By Lemma 1.5, the family $\pi_{\Lambda,\beta}(A|\eta) \stackrel{\text{def}}{=} \mu^{\eta}_{\Lambda,\beta}(A)$ defines a specification, called *Gibbsian specification.* A measure $\mu \in \mathscr{M}_1(\Omega)$ compatible with the Gibbsian specification is called an *infinite-volume Gibbs measure.*

As we already know, the finite volume Gibbs measures giving rise to the Gibbsian specification are dependent on a Hamiltonian. This Hamiltonian is itself dependent on a choice of interactions $\Phi = (\Phi_\Lambda)_{\Lambda \Subset \mathbb{Z}^d}$. Hence, the Gibbsian specification is denoted by $\pi^{\Phi}_{\beta} = (\pi^{\Phi}_{\Lambda,\beta})_{\Lambda \Subset \mathbb{Z}^d}$, and we also put $\mathscr{G}_\beta(\Phi) \stackrel{\text{def}}{=} \mathscr{G}(\pi^{\Phi}_{\beta})$.

It can be shown (see [FV17]) that μ is compatible with a specification π if, and only if the conditional expectation of μ with respect to $\mathscr{F}_{\Lambda^c}$ equals π_Λ, that is,

$$\mu(A|\mathscr{F}_{\Lambda^c})(\cdot) = \pi_\Lambda(A|\cdot).$$

In this sense, a specification can be understood as a prescription of conditional expectations outside of every finite box Λ and the DLR (Dobrushin-Lanford-Ruelle) equations $\mu\pi_{\Lambda,\beta} = \mu$ for μ translate to the usual invariance property for conditional expectations.

An equivalent way of expressing infinite volume Gibbs measures is by the notion of *thermodynamical limit.* For this, consider the weak* limits

$$\mu^{\eta}_{\beta} \stackrel{\text{def}}{=} w^* - \lim_{n\to\infty} \mu^{\eta}_{\Lambda_n,\beta}, \tag{1.19}$$

whenever they exist, where $\eta \in \Omega$ is a boundary condition and the sequence $(\Lambda_n)_{n\geq 1}$ invades $\mathbb{Z}^d$. From now on, the fact that a sequence Λ_n invades $\mathbb{Z}^d$ will be denoted by $\Lambda_n \nearrow \mathbb{Z}^d$. The weak* limits in equation 1.19 mean, by definition, that $\lim\limits_{n\to\infty} \int_\Omega f d\mu^{\eta}_{\Lambda_n,\beta}$ exists for every continuous $f : \Omega \to \mathbb{R}$ and that

$$\int_\Omega f d\mu^{\eta}_{\beta} = \lim_{n\to\infty} \int_\Omega f d\mu^{\eta}_{\Lambda_n,\beta}$$

for all such f. These will be called *phases* associated with the boundary condition η, whenever the limit exists. It can be proven that these thermodynamical limits coincide with the DLR measures in the sense of the next theorem.

Theorem 1.11. *For all $\beta > 0$ and regular interactions $(\Phi_A)_{A\Subset\mathbb{Z}^d}$ one has*

$$\mathscr{G}_\beta(\Phi) = \overline{co}\left\{\mu^{\eta}_{\beta} : \mu^{\eta}_{\beta} = w^* - \lim_{n\to\infty} \mu^{\eta}_{\Lambda_n,\beta}, \eta \in \Omega \text{ and } \Lambda_n \nearrow \mathbb{Z}^d\right\}, \tag{1.20}$$

where the finite volume Gibbs measures $\mu^{\eta}_{\Lambda,\beta}$ are defined with respect to the interaction Φ and $\overline{co}$ means the closed convex hull.

For a proof, see [Sim93].

For Ising-like models we define the *critical inverse temperature* as $\beta_c \stackrel{\text{def}}{=} \inf\{\beta : m^*(\beta) > 0\}$, where we set $m^*(\beta) \stackrel{\text{def}}{=} \langle\sigma_0\rangle^{+}_{\beta,0}$ as the *spontaneous magnetization.* By definition, β_c is the unique value for which $m^*(\beta) = 0$ for all $\beta < \beta_c$ and $m^*(\beta) > 0$ for $\beta > \beta_c$. It can be shown that $|\mathscr{G}_\beta(\Phi)| = 1$ if, and only if $h = 0$ and $m^*(\beta) = 0$, so the spontaneous magnetization $m^*(\beta)$ can detect phase transitions.

As a consequence of this fact and restricting only to the first neighbours Ising model, since there is no phase transition for the one dimensional case, then at the critical inverse temperature β_c we have $m^*(\beta_c) = 0$. For the two dimensional case, [Yan52] proved an explicit formula for the spontaneous magnetization, which also shows $m^*(\beta_c) = 0$ for $d = 2$. Corollary 1.5 of [ADCS14] also proves this result both for the short-range case and long-range in the regularity region $\alpha > d$ for $d \geq 3$ and for the one-dimensional long-range model for $1 < \alpha < 2$.

For $d = 1$ and $\alpha = 2$, [ACCN88] proved that $m^*(\beta_c) > 0$, even though it is again zero for $\alpha > 2$ since there is no phase transition for this region (see [Rue68]).

1.2 Pirogov-Sinai Theory

S. Pirogov and Y. Sinai developed their theory (see for example [PS75] and its continuation [PS76]) as an extension to the classical Peierls argument - an argument to show phase transition for the Ising model - but not requiring any symmetries for the Hamiltonian, like the Ising model has. The theory is also robust enough to give some information regarding phase diagrams and their evolution with the temperature, which we will say more below. Although we will not make use of these specific results from Pirogov Sinai theory, their treatise of *contour models* will be of importance, which we explore in the next section. For now, we will give a basic exposition of the core concepts of the theory.

As initial data, we consider a finite spin system with single spin state space given by $E = \{1, 2, ..., n\}$, a number $1 \leq r \leq n$ labeling the ground states (to be defined just below) of the system and a model with those ground states specified by a short range interaction $\Phi = (\Phi_A)_{A \Subset \mathbb{Z}^d}$. The starting point of the theory is the determination of the periodic ground states of the system, which we now define.

Definition 1.12. *Two configurations $\omega, \tilde{\omega}$ are said to be equal at infinity if there is a finite set $\Lambda \Subset \mathbb{Z}^d$ such that $\omega_{\Lambda^c} = \tilde{\omega}_{\Lambda^c}$ and we write $\omega \overset{\infty}{=} \tilde{\omega}$ if this is the case. The relative Hamiltonian $\mathscr{H}(\omega|\tilde{\omega})$ between two configurations such that $\omega \overset{\infty}{=} \tilde{\omega}$ is defined by*

$$\mathscr{H}(\omega|\tilde{\omega}) \overset{def}{=} \sum_{A \Subset \mathbb{Z}^d} \{\Phi_A(\omega) - \Phi_A(\tilde{\omega})\}.$$

Note that the quantity above is well defined for short range systems. A ground state of the model is some configuration $\eta \in \Omega$ such that $\mathscr{H}(\omega|\eta) \geq 0$, for all $\omega \overset{\infty}{=} \eta$. A ground state η is periodic in the direction e_k if there is a number l_k such that $\eta_{i+\ell_k e_k} = \eta_i$, for every $i \in \mathbb{Z}^d$, and the period in this direction is the smallest such ℓ_k. Finally, a configuration η is periodic if it is periodic in every direction, and its period is the smallest coordinate-wise vector $(\ell_1, ..., \ell_d)$ making the configuration periodic. The set of all periodic configurations is denoted by Ω^{per} and the set of ground states (resp. periodic ground states) is denoted by $g(\Phi)$ (resp. $g^{per}(\Phi)$).

The relative Hamiltonian measures the difference of global energy (that is, summing over all interactions not necessarily intersecting some finite box Λ) between two configurations. Of course, there is no hope for this energy difference to converge to a finite number in the general case, but it does converge if both configurations are equal at infinity and the interactions are of short-range type. If this is the case, then referring to the energy difference intuition we get that a ground state is a configuration that minimizes the energy of the system if local changes (i.e, changes made in a finite region) are made to it. In other words, η is a ground state if the energy of η is less or equal than the energy of ω for every configuration ω differing from η only in a finite region.

We define the energy density $e : \Omega^{\text{per}} \to \mathbb{R}$, by

$$e(\omega) \overset{\text{def}}{=} \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \mathscr{H}_\Lambda(\omega). \tag{1.21}$$

It is then a result (see [FV17], chapter 7) that a periodic configuration η is a periodic ground state if, and only if $e(\eta) = \inf_{\omega \in \Omega^{\text{per}}} e(\omega)$. This means that a periodic configuration is a ground state if, and only if it minimizes the energy density of the system. This result is compatible and should be compared with the local minimizing configuration intuition for the ground states above.

For example, for the two dimensional Ising model with no external field, if a configuration ω is not identically equal to $+$ or $-$, then there is at least two neighboring points i_0, j_0 such that

$\omega_{i_0} \neq \omega_{j_0}$ and hence $\omega_{i_0}\omega_{j_0} = -1$. Let Λ_1 denote the box of sizes (ℓ_1, ℓ_2), where ℓ_i are periods of ω chosen big enough so that $i_0, j_0 \in \Lambda_1$. This defines a tessellation of $\mathbb{Z}^2$ with tilings given by translates of Λ_1, where the configuration repeats itself inside each copy of Λ_1. We define a sequence $(\Lambda_n)_{n\geq 1}$ of boxes invading $\mathbb{Z}^2$ by the procedure shown in the next figure:

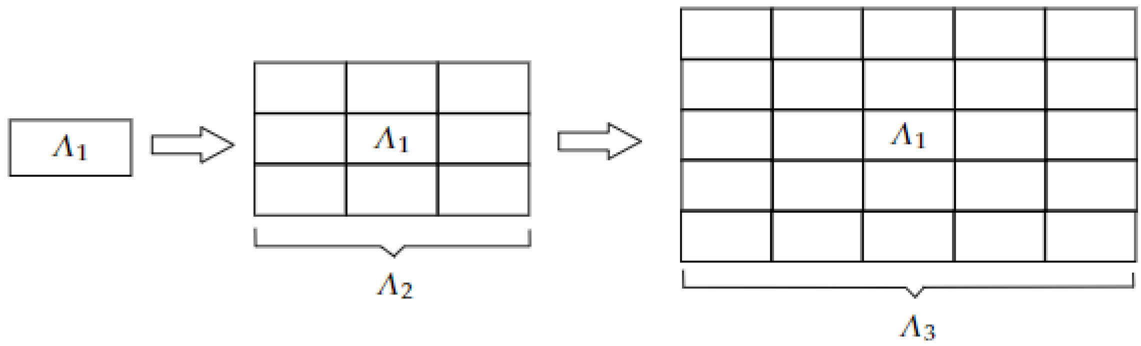


Figure 1.1: *Construction of the sequence Λ_n. Note that Λ_2 consists of 8 neighbouring copies of Λ_1 surrounding it, Λ_3 is given by gluing extra translations of Λ_1 on the boundary of Λ_2 and so on.*

In this way, there are $(2i-1)^2$ copies of Λ_1 in Λ_i, and in special at least $(2i-1)^2$ first neighbors with different spins in Λ_i. Since $|\Lambda_i| = (2i-1)^2|\Lambda_1|$, then:

$$\begin{aligned} e(\omega) = \lim_{i\to\infty} \frac{1}{|\Lambda_i|}\mathscr{H}_{\Lambda_i}(\omega) = -J \lim_{i\to\infty} \frac{1}{|\Lambda_i|} \sum_{x,y\in\Lambda_i} \sigma_x\sigma_x = -J \lim_{i\to\infty} \frac{1}{|\Lambda_i|} \left(-(2i-1)^2 + {\sum_{x,y\in\Lambda_i}}' \sigma_x\sigma_x \right) \\ \geq \frac{J}{|\Lambda_1|} - J \lim_{i\to\infty} \frac{1}{|\Lambda_i|} {\sum_{x,y\in\Lambda_i}}' 1 \geq \frac{J}{|\Lambda_1|} - J \lim_{i\to\infty} \frac{1}{|\Lambda_i|} \sum_{x,y\in\Lambda_i} 1 = \frac{J}{|\Lambda_1|} + e(\pm 1), \end{aligned} \tag{1.22}$$

where $\sum'$ means that the sum is over all first neighbors i, j minus the translations of i_0, j_0 present in each tiling of Λ_1 inside of Λ_i. This implies that $e(\omega) > e(\pm 1)$, so the only possibilities for periodic ground states are the configurations either equal to $+1$ or -1. These are indeed ground states, since for every $\omega \overset{\infty}{=} \pm 1$ one has

$$\mathscr{H}(\omega|\pm 1) = \sum_{\{i,j\}\subset\mathbb{Z}^2} (-\omega_i\omega_j + 1) = \sum_{\{i,j\}\subset\mathbb{Z}^2} (1 - \omega_i\omega_j) \geq 0,$$

so in this case $g^{\text{per}}(\Phi) = \{+1, -1\}$. The condition of a model having only finitely many periodic ground states is one of the requirements of Pirogov-Sinai theory. The full requirements are:

1. $\Phi_{A+x}(T_x(\omega)) = \Phi_A(\omega)$, for all $x \in \mathbb{Z}^d$, where $A + x \overset{\text{def}}{=} \{a + x : a \in A\}$ and $(T_x(\omega))(i) \overset{\text{def}}{=} \omega_{i-x}$;

2. The interactions are of short range type;

3. $0 < |g^{\text{per}}(\Phi)| < \infty$, i.e, there are finitely many ground states.

The first condition is usually expressed by saying that the interactions are *translation invariant*. Note that we can always normalize the interactions by some ground state by defining new interactions $\tilde{\Phi}_A(\omega) \overset{\text{def}}{=} \Phi_A(\omega) - \Phi_A(\eta)$, with $\eta \in g^{\text{per}}(\Phi)$. Since the definition of ground state takes only into account differences between interactions, the ground states of the model $\tilde{\Phi}$ are the same as of the model Φ. In this new model we have $\tilde{\Phi}_A(\eta) = 0$, for every $A \Subset \mathbb{Z}^d$, implying that $e_{\tilde{\Phi}}(\eta) = 0$ and hence all periodic ground states also have zero energy density. To summarize, one can always assume in Pirogov-Sinai theory that all periodic ground state have zero energy density.

If the conditions above are satisfied and the interactions are all non-negative, then $\eta \in \Omega^{\text{per}}$ is a ground state if, and only if $\Phi_A(\eta) = 0$, for all $A \Subset \mathbb{Z}^d$. To see this, assume for simplicity that $d = 2$ and that η is a ground state. Choose the sequence of boxes Λ_i defined by neighboring translates of $\Lambda_1 = (\ell_1, \ell_2)$, where ℓ_i are the periods of η, just like in the development leading to equation 1.22. Now, if there was some $A \Subset \mathbb{Z}^2$ such that $\Phi_A(\eta) > 0$, then by translation invariance and the periodicity of η there would be at least $\frac{|\Lambda_i|}{|\Lambda_1|}$ other sets $A \Subset \Lambda$ such that $\Phi_A(\eta) > 0$, where i is large enough to contain A. Moreover, by Lemma 1.13 the positivity of those $\Phi_A(\eta)$ yields a uniform bound $\Phi_A(\eta) > c$ over all $A \Subset \mathbb{Z}^d$ such that $\Phi_A(\eta) > 0$. Hence

$$\sum_{A \Subset \Lambda_i} \Phi_A(\eta) > \frac{c|\Lambda_i|}{|\Lambda_1|}.$$

However, recall that there is no loss of generality in assuming that all periodic ground states have zero energy density. By definition of the energy density, this implies that for all large enough i

$$\sum_{A \Subset \Lambda_i} \Phi_A(\eta) \leq \frac{c|\Lambda_i|}{|\Lambda_1|},$$

a contradiction. Therefore, the only possibility is to have $\Phi_A(\eta) = 0$, for all $A \Subset \mathbb{Z}^2$. If now $\Phi_A(\eta) = 0$ for all $A \Subset \mathbb{Z}^d$, then for all $\omega \overset{\infty}{=} \eta$ we have

$$\mathscr{H}(\omega|\eta) = \sum_{A \Subset \mathbb{Z}^d} \Phi_A(\omega) \geq 0,$$

since the interactions are all non-negative. Note that the expression above is also finite, since $\omega \overset{\infty}{=} \eta$ and $\Phi_A(\eta) = 0$ for all $A \Subset \mathbb{Z}^d$. Therefore, by re-defining the interactions, we see that there is no loss of generality to assume the following conditions in Pirogov-Sinai theory:

- $\Phi_A \geq 0$, for all $A \Subset \mathbb{Z}^d$;
- The interactions are translation invariant;
- The interactions are of short range type;
- $0 < |g^{\text{per}}(\Phi)| < \infty$, i.e, there are finitely many ground states;
- $\eta \in \Omega^{\text{per}}$ is a ground state if, and only if $\Phi_A(\eta) = 0$, for all $A \Subset \mathbb{Z}^d$.

These hypothesis are enough to prove, for example, the following result:

Lemma 1.13. *If the conditions of Pirogov-Sinai theory above are satisfied, then there is a constant $c > 0$ such that all configurations ω and interactions Φ_A satisfying $\Phi_A(\omega) > 0$ also satisfy $\Phi_A(\omega) \geq c$.*

Proof. Let $R > 0$ be the radius of the short-range interactions. If we consider the box $\Lambda_R := [0, R]^2$, then every $A \Subset \mathbb{Z}^2$ such that $|A| \leq R$ is a translation of some subset of Λ_R.

Fixing $c := \min_{\substack{A \subset \Lambda_R: \\ \Phi_A > 0}} \min_{\omega_A \in \Omega_A} \Phi_A(\omega_A)$, then c is a minimum of finitely many positive real numbers and therefore is positive. By the last observation, any non-zero potential ϕ_A can be translated to another potential $\Phi_{A'}$ with $A' \subset \Lambda_R$. Then $\Phi_A(\omega) = \Phi_{A'}(\omega_{A'}) \geq c$. □

Let us now show how the re-definition of the interactions take place in the nearest neighbour Ising model (with empty boundary conditions) and with external field h. First, since we are working with translation invariant interactions, the external field must be constant everywhere and equal to some real number h. To get the Hamiltonian in the desired form, we will normalize it in the $\sigma = 1$ ground state, as follows:

$$\begin{aligned}\mathscr{H}_{\Lambda,h}(\sigma) = -\sum_{\substack{\{x,y\}\subset\Lambda\\|x-y|=1}} J\sigma_x\sigma_y - h\sum_{x\in\Lambda}\sigma_x &= \mathscr{H}_{\Lambda,h}(\sigma_+) - \mathscr{H}_{\Lambda,h}(\sigma_+) - \sum_{\substack{\{x,y\}\subset\Lambda\\|x-y|=1}} J\sigma_x\sigma_y - h\sum_{x\in\Lambda}\sigma_x\\
&= \mathscr{H}_{\Lambda,h}(\sigma_+) + J\sum_{\substack{\{x,y\}\subset\Lambda\\|x-y|=1}}\{1-\sigma_x\sigma_y\} + h\sum_{x\in\Lambda}\{1-\sigma_x\}\\
&= \mathscr{H}_{\Lambda,h}(\sigma_+) + 2J\sum_{\substack{\{x,y\}\subset\Lambda\\|x-y|=1}}\mathbb{1}_{\{\sigma_x\neq\sigma_y\}} + 2h\sum_{x\in\Lambda}\mathbb{1}_{\{\sigma_x\neq 1\}}.\end{aligned}$$

Note that the new interactions are now non-negative. To simplify the discussions up to the end of this section, we will assume that the ground states of the unperturbed Hamiltonian are the constant configurations $\sigma_x \equiv i$, for $1 \le i \le r$ and that the support A of every non-zero interaction Φ_A has cardinality bounded by p. As we argued before, there is a constant $c > 0$ such that, uniformly in A and in ω, $\Phi_A(\omega) \ge c$ whenever Φ_A is strictly positive. Since this happens if and only if ω is not equal to some ground state $1, 2, ..., r$, then $\Phi_A(\omega) \ge c$ if ω is not equal to some ground state in A.

Let n be such that the single spin state space is given by $\{1, 2, ..., n\}$ and let $E : \{1, 2, ..., n\} \to \mathbb{R}$ be a function satisfying

$$E(\ell) - \min_{1\le k\le n} E(k) \le \frac{c}{2p} \text{ and } \min_{1\le k\le r} E(k) = 0, \tag{1.23}$$

for $\ell = 1, 2, ..., r$. Of course, every $r-$uple $\mu = (\mu_1, ..., \mu_r)$ is sufficient to define some such function by setting $E(i) \overset{\text{def}}{=} \mu_i$ (and the other values can be chosen as to satisfy condition 1.23), from which we define the perturbed Hamiltonian by

$$\mathscr{H}_{\Lambda,\mu}(\omega) \overset{\text{def}}{=} \mathscr{H}_{\Lambda}(\omega) + \sum_{x\in\Lambda} E(\omega_x). \tag{1.24}$$

This new perturbed Hamiltonian has as new interactions the ones from the unperturbed Hamiltonian plus the new external fields $\Phi_x(\omega) \overset{\text{def}}{=} E(\omega_x)$, and we denote this new set of interactions by Φ_μ. For every ground state i and any $\omega \overset{\infty}{=} i$ different from i we let B denote the smallest region where $\omega_{B^c} = i_{B^c}$. Then

$$\begin{aligned}\mathscr{H}_\mu(\omega|i) &= \sum_{A\Subset B}\Phi_A(\omega) + \sum_{x\in B}\{E(\omega_x) - E(i)\}\\
&\ge c|\{A \Subset B : \omega_A \neq i\}| - \frac{c}{2p}|\{x \in B : \omega_x \neq i\}|.\end{aligned}$$

Note that

$$\begin{aligned}\sum_{x\in B}\mathbb{1}_{\{\omega_x\neq i\}} &\le \sum_{A\Subset B}\sum_{x\in A}\mathbb{1}_{\{\omega_x\neq i\}} \le \sum_{A\Subset B}\sum_{x\in A}\mathbb{1}_{\{\omega_A\neq i\}}\\
&= \sum_{A\Subset B}\mathbb{1}_{\{\omega_A\neq i\}}\sum_{x\in A}1 = \sum_{A\Subset B}\mathbb{1}_{\{\omega_A\neq i\}}|A| \le p\sum_{A\Subset B}\mathbb{1}_{\{\omega_A\neq i\}},\end{aligned}$$

where, for clarification, $\mathbb{1}_{\{\omega_A\neq i\}}$ equals 1 if all spins inside A are different from i and zero otherwise. This implies that $|\{A \Subset B : \omega_A \neq i\}| \ge \frac{|\{x\in B:\omega_x\neq i\}|}{p}$ and, in special,

$$\mathscr{H}_\mu(\omega|i) \ge \frac{c}{2p}|\{x \in B : \omega_x \neq i\}| \ge 0. \tag{1.25}$$

Hence, the relative energy of a periodic configuration ω in relation to a ground state i grows with the amount of points different to the ground state. Moreover, this result implies that the periodic ground states of the unperturbed Hamiltonian are still periodic ground states of the perturbed Hamiltonian, i.e, $g^{\text{per}}(\Phi) \subset g^{\text{per}}(\Phi_\mu)$. Therefore, one can add a small perturbation $\mu = (\mu_1, ..., \mu_n)$ of the magnetic field without removing any unperturbed ground states from the new system. One of the main results of Pirogov-Sinai theory is that the *phase diagram* - that is, a plot describing the phase distribution for the perturbed system- has a particular shape at low temperatures. To better explain the phase diagram, let us define the *parameter space* G_ϵ by

$$G_\epsilon \stackrel{\text{def}}{=} \{\mu = (\mu_1, ..., \mu_r) : \min_{1\le i\le r} \mu_i = 0 \text{ and } |\mu| = \max_{1\le i\le r} |\mu_i| < \epsilon\}. \tag{1.26}$$

Then, for all sufficiently small $\epsilon > 0$ and all big enough $\beta > 0$:

1. There is an r-dimensional bounded hypersurface $\gamma_{q_1} \subset G_\epsilon$ for each $q_1 \in \{1, ..., r\}$, where all phases[3] μ_ν^q with $\nu \in \gamma_{q_1}$ satisfy $q = q_1$;

2. There is an $r-1$-dimensional bounded hypersurface $\gamma_{q_1,q_2} \subset G_\epsilon$ for each pair $\{q_1, q_2\} \subset \{1, ..., r\}$, where all phases μ_ν^q with $q \in \{q_1, q_2\}$ and $\nu \in \gamma_{q_1,q_2}$ are distinct, i.e, the phases q_1, q_2 coexist inside γ_{q_1,q_2};

   In general,

3. There exists a $r-k$-dimensional bounded hypersurface $\gamma_A \subset G_\epsilon$, $k \le r-1$ for each $A \subset \{1, ..., r\}$ and $|A| = k+1$, where all phases μ_ν^q with $q \in \{q_1, ..., q_{k+1}\}$ and $\nu \in \gamma_A$ are distinct, i.e, the phases $q_1, ..., q_{k+1}$ coexist inside γ_A;

4. There exists a point $\overline{\nu}$ such that all phases $\mu_{\overline{\nu}}^q$ with $q \in \{1, 2, ..., r\}$ are distinct;

5. Finally, $G_\epsilon = \bigcup_{A\subset\{1,2,...,r\}} \gamma_A$.

Moreover, it can be shown that the boundary of some γ_A consists of certain hypersurfaces $\gamma_{B_1}, ..., \gamma_{B_i}$ where the dimension of each γ_{B_i} is one less than γ_A. The hypersurfaces γ_A are known as the coexistence hypersurfaces. The next figure illustrates the objects 1 - 5 given above.

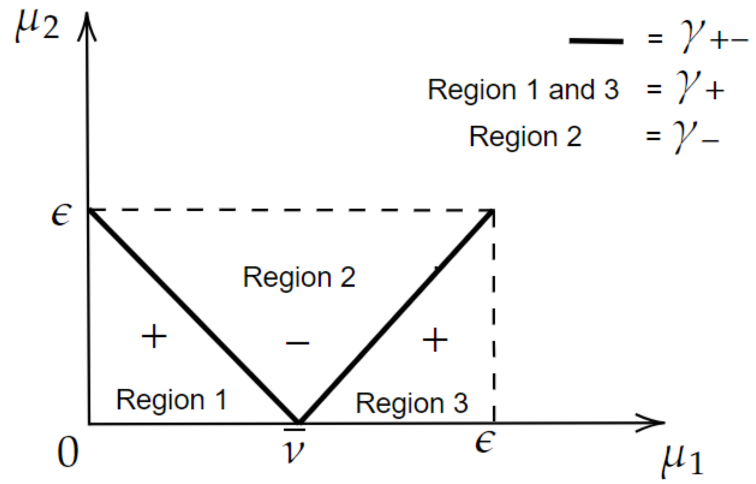


Figure 1.2: *An example of a hypothetical model with two ground states ($r = 2$), + and −. The regions 1, 2 and 3 describe the points where only one phase is present and the thick line represents the coexistence line between both phases, where both phases are distinct.*

[3] A phase in this context means an infinite volume Gibbs measure. Note that the choice of ν is a choice of a perturbative external magnetic field $\nu = (\mu_1, ..., \mu_r)$, which fixes a choice of local Hamiltonians (see 1.24).

1.2.1 Contour Models

Here, we will work only on $d = 2$, but all the definitions and results naturally extend to higher dimensions.

Definition 1.14. *A contour is a pair $\gamma = (\overline{\gamma}, \sigma_{\overline{\gamma}})$, where $\overline{\gamma}$ is a finite connected subset of $\mathbb{Z}^2$ and $\sigma_{\overline{\gamma}}$ is a configuration with support equal to $\overline{\gamma}$. Moreover, we denote $\overline{\gamma}$ by $sp(\gamma)$.*

Given any connected $A \Subset \mathbb{Z}^2$, there is exactly one unbounded connected component $\mathrm{Ext}(A)$ of A^c, which we call the *exterior* of A, and a finite number of connected components $\mathrm{I}_1(A), ..., \mathrm{I}_n(A)$, called the *interiors* of A, such that

$$A^c = \mathrm{Ext}(A) \cup \mathrm{I}_1(A) \cup ... \cup \mathrm{I}_n(A).$$

The next figure gives a visual representation of the exterior and the interior of a contour γ:

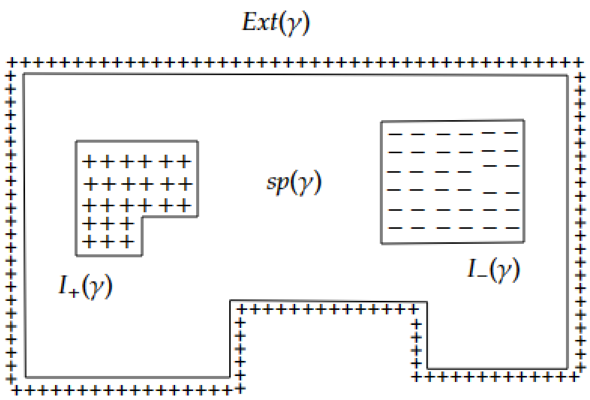


Figure 1.3: *In the figure, the interior of γ is the union of $I_+(\gamma)$ and $I_-(\gamma)$. The support is the blank region surrounded by the spins on the boundary and the exterior is the unbounded connected component encircling the + strip.*

Now, let $\Lambda \Subset \mathbb{Z}^2$ be any. We say that a point $x \in \Lambda$ is *q-correct* for a configuration $\sigma \in \Omega_\Lambda$ if $q \in \{1, 2, ..., r\}$ and $\sigma_y = q$, for all $y \in \overline{B_1(x)}$ (where the ball is taken in the ℓ_1-norm). A point $x \in \Lambda$ is *incorrect* if it is not q-correct for some $q = 1, 2, ..., r$.

Given a configuration $\sigma \in \Omega_\Lambda$, we can associate to it a family of contours in the following way: first, denote by $\Gamma(\sigma)$ the collection of all incorrect points of Λ. Then, split this set in its connected components, which we call $\overline{\gamma}_1, ..., \overline{\gamma}_n$. The contours are then $\gamma_i \stackrel{\text{def}}{=} (\overline{\gamma}_i, \sigma_{\overline{\gamma_i}})$. Then,

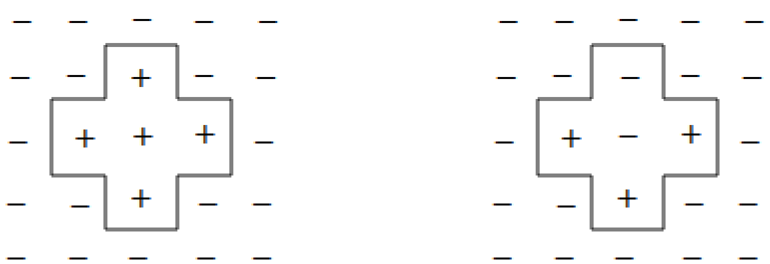


Figure 1.4: *The ℓ_1-norm closed ball on the left has as center a $+$ correct point, but all other points in the ball are incorrect. As for the closed ball on the right, the center is now an incorrect point, but the point just above it is now $-$ correct.*

Lemma 1.15. *The map* $\Omega_\Lambda \ni \sigma \mapsto \Gamma(\sigma)$ *is injective.*

Proof. Since we already know the configuration in each $\mathrm{sp}(\gamma_i)$, it is enough to prove that the configurations in each support uniquely specifies the configuration in $\Gamma(\sigma)^c$. In fact, since $\Gamma(\sigma)$ contains all irregular points in Λ, then $\Gamma(\sigma)^c$ contains all q-regular points, with q varying over $\{1, 2, ..., r\}$. After splitting $\Gamma(\sigma)^c$ in disjoint connected components, it follows that in each of these connected components the spin is identically equal to q, for some $q = 1, 2, ..., r$. Indeed, given any two points x_1, x_2 in one of those connected components, there is a path $c = (c_1, c_2, ..., c_n)$ joining the points, where $c_1 = x_1$ and $c_n = x_2$. We note that x_1 is equal to some q since it is q-correct and that $c_2 \in \overline{B_1(x_1)}$, so that we also have $c_2 = q$. After iterating this argument, we see that all points of the path c must carry a spin of q, including x_n.

Now, we note that knowledge of the (inner) boundary points of $\mathrm{sp}(\gamma_i)$ uniquely determine the configurations of the connected components of $\Gamma(\sigma)^c$. In fact, first note that $\partial^{\mathrm{in}}\mathrm{sp}(\gamma_i)$ can be decomposed in a disjoint union of arcs, and the configuration is constant in any of these arcs. See figure 1.3 for an example (in that figure, there are three arcs, one surrounding each interior $I_+(\gamma), I_-(\gamma)$ and one surrounding the support of the contour).

To justify this fact, pick any two points x, y in some connected arc of $\partial^{\mathrm{in}}\mathrm{sp}(\gamma_i)$ and choose points x_1, y_1 both in some connected component, say, A, of $\Gamma(\sigma)^c$ such that $|x_1 - x| = 1$ and $|y_1 - y| = 1$. Now, the configuration in A is identically equal to some $q \in \{1, 2, ..., r\}$ and since $x \in \overline{B_1(x_1)}$ and x_1 is q-correct, we must have $\sigma_x = q$. Analogously we have $\sigma_y = q$, finishing this argument. Hence, we see that the constant configuration in each connected arc of $\partial^{\mathrm{in}}\mathrm{sp}(\gamma_i)$ matches the configuration of the corresponding neighboring connected component of $\Gamma(\sigma)^c$, so the connected components of $\Gamma(\sigma)^c$ are determined by the configuration in $\sigma_{\Gamma(\sigma)}$ and the injectivity follows.

□

Note that there can be contours inside the interior of other contours. In any case, given some contour γ, the proof of the last lemma reveals that the boundary of $\mathrm{Ext}(\gamma)$ neighboring $\mathrm{sp}(\gamma)$ has constant spin values $q \in \{1, 2, ..., r\}$. We then say that γ has q-boundary conditions and we write γ^q to indicate this fact. In the same way, the connected components of the interior of γ are encircled by arcs of constant spin values, so we label the interiors in components with constant neighboring spins q and denote them by $I_q^{(1)}(\gamma), ..., I_q^{(n)}(\gamma)$. For example, the contour shown in 1.3 has $+$ boundary conditions and there are two connected components of the interior, which we denoted by $I_-(\gamma)$ and $I_+(\gamma)$.

The following picture illustrates a configuration $\sigma \in \Omega_\Lambda$ and the corresponding family of contours $\Gamma(\sigma)$, together with the labels for the interiors and the individual contours.

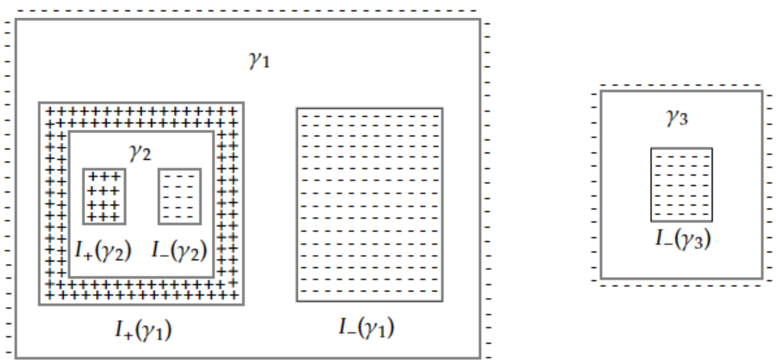


Figure 1.5: *An example of a family of contours $\Gamma(\sigma) = \{\gamma_1, \gamma_2, \gamma_3\}$, for some configuration σ. Here, γ_1 and γ_3 have boundary condition $-$, while γ_2 has boundary condition $+$. Note that it is possible, as is shown here, for a contour to be inside the interior of some other contour.*

If all contours in some family of contours $\Gamma(\sigma)$ have the same label q, we indicate this family by $\Gamma^q(\sigma)$ or simply Γ^q if the configuration is already known.

Now, given a family of contours $\Gamma \subset \Lambda$ not intersecting the boundary, we say that Γ is *compatible* if there is some configuration $\sigma \in \Omega_\Lambda$ such that $\Gamma = \Gamma(\sigma)$. Moreover, we denote by $D(\Lambda)$ the family of all compatible families of contours in Λ and $D_q(\Lambda)$ the subset of $D(\Lambda)$ where every family of contours has q-boundary conditions. Note that we have a bijection between Ω_Λ and $D(\Lambda)$ given by $\sigma \mapsto \Gamma(\sigma)$.

As figure 1.5 suggests, not every configuration gives rise to a family of contours with uniform boundary condition, as two distinct contours in the family can have different boundary conditions. We are now ready to define contour models.

Definition 1.16. *Given $q \in \{1, 2, ..., r\}$ and a positive real number τ, a function $F_q : D_q(\Lambda) \to [0, +\infty)$ is called a τ-functional if both conditions below are satisfied:*

- $F_q(\Gamma^q) \geq \tau|\Gamma^q|$;
- $F_q(\Gamma^q + a) = F_q(\Gamma^q)$, *for every $\Gamma^q \in D_q(\Lambda)$ and every $a \in \mathbb{Z}^2$ such that $\Gamma^q + a \in D_q(\Lambda)$ (i.e, F_q is translation-invariant).*

The contour model associated to a τ-functional F_q is the probability measure $\mathbb{P}_\Lambda$ given on collections $\underline{\Gamma^q} = \{\Gamma^q_1, ..., \Gamma^q_n\} \subset D_q(\Lambda)$ by

$$\mathbb{P}_\Lambda(\underline{\Gamma^q}) \stackrel{def}{=} \frac{\exp\left(-\sum_{k=1}^{n} F_q(\Gamma^q_k)\right)}{\sum_{\underline{\Gamma^q} \subset D_q(\Lambda)} \exp\left(-\sum_{\Gamma^q \in \underline{\Gamma^q}} F_q(\Gamma^q)\right)}. \tag{1.27}$$

The normalization factor of the last definition is denoted by

$$\Omega^0(\Lambda : F_q) \stackrel{def}{=} \sum_{\underline{\Gamma^q} \subset D_q(\Lambda)} \exp\left(-\sum_{\Gamma^q \in \underline{\Gamma^q}} F_q(\Gamma^q)\right).$$

Now, let $\mathcal{C}$ denote the collection of all maps $x : D_q(\Lambda) \to \mathbb{N}$ and let $\mathscr{F}$ denote the collection of all maps $\psi : \mathcal{C} \to \mathbb{R}$. We formally define their product as the convolution

$$(\psi_1 \cdot \psi_2)(x) \stackrel{\text{def}}{=} \sum_{\substack{(x_1,x_2) \subset \mathscr{C} \\ x_1 + x_2 = x}} \psi_1(x_1)\psi_2(x_2), \tag{1.28}$$

where addition in $\mathcal{C}$ is defined pointwise. For each $\psi \in \mathscr{F}$, the product above lets us quickly define their logarithm as

$$\psi^T(x) = (\log\psi)(x) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \psi_0^n(x), \tag{1.29}$$

with $\psi_0(x) = \psi(x)$ if $x \neq 0$ and zero otherwise. Moreover, we define $\varphi \in \mathscr{F}$ as $\varphi\left(\mathbb{1}_{\{\Gamma \in \underline{\Gamma^q}\}}\right) \stackrel{\text{def}}{=} \exp(-\sum_{\Gamma^q \in \underline{\Gamma^q}} F_q(\Gamma^q))$ for compatible families $\underline{\Gamma^q}$ and zero for all other $x \in \mathscr{F}$.

We introduce the notation $x \ni 0$ to indicate that $x(\Gamma) \neq 0$ for some Γ containing zero and we agree that $x \subset \Lambda$ (resp. $x \cap A \neq \varnothing$, for some $A \subset \Lambda$) means that $x(\Gamma) = 0$ unless $\Gamma \subset \Lambda$ (resp. $\Gamma \cap A \neq \varnothing$). Then, we have the following.

Lemma 1.17. *For all sufficiently large τ, we have*

$$\sum_{x \ni 0} |\varphi^T(x)| \leq \exp(-c\tau) \tag{1.30}$$

for some positive constant c. In addition, the expression $\Omega^0(\Lambda : F_q) = \exp\left(\sum_{x \subset \Lambda} \varphi^T(x)\right)$ holds, and

$$S(F_q) = \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \Omega^0(\Lambda, F_q) \tag{1.31}$$

converges in the sense of van Hove. Finally, if $\Delta(\Lambda : F_q) \stackrel{def}{=} \log \Omega^0(\Lambda : F_q) - S(F_q)|\Lambda|$, *then* $|\Delta(\Lambda, F_q)| \leq \exp(-c\tau)|\partial\Lambda|$.

For a proof, see [GMS]. We introduce the parametric contour statistical sum as

$$\Omega^a(\Lambda : F_q) \stackrel{\text{def}}{=} \sum_{\underline{\Gamma^q} \subset D_q(\Lambda)} \exp\left(-\sum_{\Gamma^q \in \underline{\Gamma^q}} F_q(\Gamma^q)\right) \exp\left(a \left| \bigcup_{\Gamma^q \in \underline{\Gamma^q}} I(\Gamma^q) \right| \right), \tag{1.32}$$

where $I(\Gamma^q)$ is the union of all interiors of the family Γ^q. Bounding $\left| \bigcup_{\Gamma^q \in \underline{\Gamma^q}} I(\Gamma^q) \right| \leq |\Lambda|$, we see that $\Omega^a(\Lambda : F_q) \leq \exp(a|\Lambda|)\Omega^0(\Lambda : F_q)$. Defining

$$\widetilde{Z}^q_\Lambda \stackrel{\text{def}}{=} \exp(\beta\mu_q|\Lambda|)Z^q_\Lambda, \tag{1.33}$$

then we have (see Lemma 4.1 of [PS75])

Proposition 1.18. *There exists* $\epsilon > 0$ *and* $\beta_0 < \infty$ *such that for all* μ *and* β *such that* $|\mu| < \epsilon$ *and* $\beta > \beta_0$ *there is a family of* $\tau-$*functionals* $\{F_1, ..., F_r\}$ *with* τ *proportional to* β *and a constant* $\alpha \in \mathbb{R}$ *such that for all* $q \in \{1, 2, ..., r\}$

$$\widetilde{Z}^q_\Lambda = \Omega^{a^q}(\Lambda : F_q), \ \textit{where} \tag{1.34}$$

$$a^q(F_q)^4 = \beta\mu_q - S(F_q) + \alpha. \tag{1.35}$$

This proposition relates the original partition function to a partition function of a parametric contour model. As a corollary, we have

Corollary 1.19. *If* $a^q(F_q) = 0$, *then for all* $p \in \{1, 2, ..., r\}$

$$\frac{Z^p_\Lambda}{Z^q_\Lambda} \leq \exp(2e^{-c\tau}|\partial\Lambda|), \tag{1.36}$$

where c *is a constant not depending on* β.

Proof. Using the previous proposition and the bound after equation 1.32, we have

$$\frac{Z^p_\Lambda}{Z^q_\Lambda} = \exp(-\beta|\Lambda|\{\mu_p - \mu_q\})\frac{\widetilde{Z}^p_\Lambda}{\widetilde{Z}^q_\Lambda} = \exp(-\beta|\Lambda|\{\mu_p - \mu_q\})\frac{\Omega^{a^p}(\Lambda : F_p)}{\Omega^0(\Lambda : F_q)}$$
$$\leq \exp(-\beta|\Lambda|\{\mu_p - \mu_q\}) \exp(a^p(F_p)|\Lambda|)\frac{\Omega^0(\Lambda : F_p)}{\Omega^0(\Lambda : F_q)}.$$

Now, since $a^q(F_q) = 0$, we have $\beta\mu_q = S(F_q) - \alpha$, so that

$$\frac{Z^p_\Lambda}{Z^q_\Lambda} \leq \exp(|\Lambda|\{-\beta\mu_p - \alpha + S(F_q) + a^p(F_p)\})\frac{\Omega^0(\Lambda : F_p)}{\Omega^0(\Lambda : F_q)}$$
$$= \exp(|\Lambda|\{S(F_q) - S(F_p)\})\frac{\Omega^0(\Lambda : F_p)}{\Omega^0(\Lambda : F_q)}.$$

Now, by definition of $\Delta(\Lambda : F_q)$, we have $\Omega^0(\Lambda : F_q) = \exp(\Delta(\Lambda : F_q)) \exp(S(F_q)|\Lambda|)$, so that

[4]Remember that a choice of τ-functionals fixes the contour model, and hence a^q (the parameter of the parametric contour statistical sum) depends on the choice of $\tau-$functionals $\{F_1, ..., F_r\}$.

$$\frac{Z_\Lambda^p}{Z_\Lambda^q} \leq \exp(\Delta(\Lambda : F_q) - \Delta(\Lambda : F_p)) \leq \exp(|\Delta(\Lambda : F_q)| + |\Delta(\Lambda : F_p)|) \leq \exp(2e^{-c\tau}|\partial\Lambda|),$$

as we wanted.

□

Ground states q satisfying the equality $a^q(F_q) = 0$ are called *dominant ground states.*

1.2.2 Surface Tension: Heuristics

Let us consider again the general setup of Pirogov-Sinai theory for the rest of this section. To better understand the importance of the surface tension and the form of its definition in the general case, we need to study boundary and interface effects induced by some specific boundary conditions in our spin systems.

In fact, if we consider boundary conditions consisting of two phases sharing a common boundary, then the appearance of *interfaces* happen. To simplify the notation, we will fix two boundary conditions q_1, q_2 from now on and denote them by 1 and 2 respectively. The $1-$boundary (resp. 2-boundary) of the box is the part of Λ having 1 (resp. 2) as boundary condition. A more practical way to denote this boundary condition is to consider a family $(\eta_{\hat{\mathbf{n}}})$ with $\hat{\mathbf{n}}$ a unit vector of $\mathbb{R}^d$, given by:

$$\eta_{\hat{\mathbf{n}}}(i) := \begin{cases} q_1, & \text{if } i \cdot \hat{\mathbf{n}} \geq 0; \\ q_2, & \text{if otherwise.} \end{cases}$$

Note that there are always two incorrect points $x_l(\sigma), x_r(\sigma)$ for any configuration $\sigma \in \Omega_\Lambda^{\eta_{\hat{\mathbf{n}}}}$ with respect to this boundary condition, corresponding to the line separating the different ground states.

Definition 1.20. *An interface λ is a connected contour, for some configuration $\sigma \in \Omega_\Lambda^{\eta_{\hat{\mathbf{n}}}}$, containing both incorrect points $x_l(\sigma)$ and $x_r(\sigma)$ described above.*

Every configuration with the prescribed boundary condition admits an interface. To simplify the proof, we will assume that $\hat{\mathbf{n}} = 0$. Note that it is enough to show that there is a connected path of incorrect points connecting $x_l(\sigma)$ to $x_r(\sigma)$. If this was not the case, then there would be a family of correct points $\rho = (x_1, x_2, ..., x_n)$ such that $x_{i-1}, x_{i+1} \in \overline{B_1^{||\cdot||_{\max}}(x_i)}$ for every i and connecting the 1 and 2-boundaries of Λ. This family may not be a path, however x_i lies at maximum diagonally to x_{i+1}. Even in this case, it is easy to see that any such family ρ has constant sign, which is impossible since $x_1 = 1$ and $x_n = 2$ by the boundary condition.

We should expect three boundary-effect contributions to the finite volume free energy: two corresponding to the interaction of the phase i with the i-boundary and one term corresponding to the interactions of both phases with the interface. Since there are different interfaces for each configuration, we account this term by considering a linear interface, as the next figure shows.

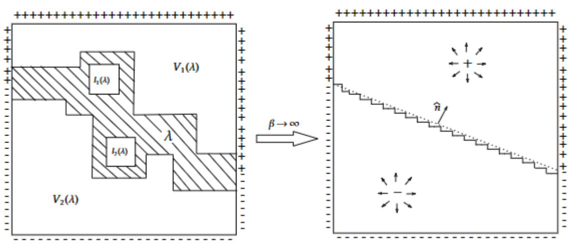


Figure 1.6: *The figure on the left represents a typical interface. Here, $V_i(\lambda)$ means the portion outside the interface in contact with the i'th phase and $I_k(\lambda)$ are the interiors of the interface λ. In some models, as the temperature diminishes the typical configurations contain localized interfaces, like the ones on the right.*

This choice of linear interface makes sense in models where this interface is localized, at least in low enough temperatures. For the Ising model, one example of this localization is due to the following theorem from [Dob73].

Theorem 1.21. *For $d \geq 3$ there is a constant $\beta_d > 0$ such that for all $\beta \geq \beta_d$ and all choices of $j \in \{1, 2, ..., d\}$, $a \in \mathbb{Z}$ and $l = 0, 1$ there are Gibbs states $\mu^l_{j,a,\beta}$ such that*

- $\mu^0_{j,a,\beta}(\sigma_x = -1) \geq 1 - g(\beta)$, *for all $x \in \mathbb{Z}^d$ with $x_j < a$, and*
- $\mu^0_{j,a,\beta}(\sigma_x = 1) \leq g(\beta)$, *for all $x \in \mathbb{Z}^d$ with $x_j \geq a$,*

and

- $\mu^1_{j,a,\beta}(\sigma_x = -1) \geq 1 - g(\beta)$, *for all $x \in \mathbb{Z}^d$ with $x_j \geq a$, and*
- $\mu^1_{j,a,\beta}(\sigma_x = 1) \leq g(\beta)$, *for all $x \in \mathbb{Z}^d$ with $x_j < a$,*

where $g(\beta) \to 0$ as $\beta \to \infty$.

The last theorem implies that in dimensions $d \geq 3$ there can be a phase separation phenomenon in the interface $x_j = a$, where positive spins are located at one side of the plane and negative spins are in the other half-space, apart from small defects. The choice of $l = 0, 1$ simply reflects the spins with respect to the interface.

As for $d = 2$, the interface is not localized. As in [FV17], consider $\Lambda_n = [-n, n]^2 \cap \mathbb{Z}^2$ and consider the Dobrushin boundary condition η_D, where the spins are $+1$ for (i, j) with $j > 0$ and -1 otherwise. For any configuration ω, let λ be the associated interface. There may be other contours in Λ_n, so we let $\omega(\lambda)$ be the configuration whose only contour is λ. Then, for any $i \in \mathbb{Z}$ define the envelopes

$$\lambda^+_n(i) \stackrel{\text{def}}{=} \max\{j \in \mathbb{Z} : \omega_{(i,j)}(\lambda) = -1\} + 1$$
$$\lambda^-_n(i) \stackrel{\text{def}}{=} \min\{j \in \mathbb{Z} : \omega_{(i,j)}(\lambda) = -1\} - 1.$$

These functions depend on n, since the configurations have boundary condition η_D outside of Λ_n, and of course they also depend on the configurations $\omega \in \Omega^{\eta_D}_{\Lambda_n}$, so they can be seen an random variables. It can be shown that, relative to the scaling of Λ_n, λ^-_n and λ^+_n are close to each other for large n in the sense that

$$\max_{i \in \mathbb{Z}} \left|\lambda^+_n(i) - \lambda^-_n(i)\right| \leq K \log(n),$$

for some constant K depending only on the inverse temperature at hand (see [CIV03]). Note that the width of Λ_n grows with n and the bound above grows with $\log(n)$. Therefore, with large values of n the interface is squeezed between the boundaries defined by λ^+_n and λ^-_n, and so are the rescaled functions $\hat{\lambda}^\pm_n : [-1, 1] \to \mathbb{R}$ given by

$$\hat{\lambda}^\pm_n(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{n}} \lambda^\pm_n(\lfloor nx \rfloor).$$

[GI05] showed that these functions converge in distribution to a Brownian motion with fixed endpoints at zero. In special, the interface in this case is not rigid.

In the cases where the interface is localized, we can write the free energies $\psi^1_\Lambda(\beta)$ and $\psi^2_\Lambda(\beta)$ of definition[5] 1.8 of the isolated phases 1 and 2 as

[5] Remember that the free energy after the thermodynamic limit does not depend of the boundary condition, but the finite volume free energies depend of them, so we keep track of the boundary conditions as a superscript.

$$\psi^1_\Lambda(\beta) = \frac{-\alpha(\hat{\mathbf{n}})}{\beta} \log\left(Z^1_{\Lambda,\beta}\right) \text{ and}$$

$$\psi^2_\Lambda(\beta) = \frac{-(1-\alpha(\hat{\mathbf{n}}))}{\beta} \log\left(Z^2_{\Lambda,\beta}\right),$$

where $\alpha(\hat{\mathbf{n}})$ denotes the fraction of the box Λ consisting of the phase 1. The total contribution to the free energy is given by

$$\frac{-1}{\beta} \log\left(Z^{\hat{\mathbf{n}}}_{\Lambda,\beta}\right).$$

Note that the term $\frac{-1}{\beta}$ is not present in the definition 1.8, but it is present in physics textbooks. We put the term here so that the surface tension, and the free energy as well, have the correct dimensions. By subtracting the free energies from the isolated phases from the total free energy, the remaining term is the free energy corresponding to the interaction of both phases with the interface. Computing the difference explicitly, it is given by

$$-\frac{1}{\beta} \log\left(\frac{Z^{\hat{\mathbf{n}}}_{\Lambda,\beta}}{(Z^1_{\Lambda,\beta})^{\alpha(\hat{\mathbf{n}})}(Z^2_{\Lambda,\beta})^{(1-\alpha(\hat{\mathbf{n}}))}}\right). \tag{1.37}$$

Of course, this free energy may grow arbitrarily large as $\Lambda \nearrow \mathbb{Z}^d$, so we compute the free energy density instead of the total free energy. Letting $\Pi_{\hat{\mathbf{n}}}(\Lambda)$ denote the interface (the support of the interface, seen as a contour), then a very natural definition of surface tension can be given by the limit

$$\tau_\beta(\hat{\mathbf{n}}) := -\lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{1}{\beta|\Pi_{\hat{\mathbf{n}}}(\Lambda)|} \log\left(\frac{Z^{\hat{\mathbf{n}}}_{\Lambda,\beta}}{(Z^1_{\Lambda,\beta})^{\alpha(\hat{\mathbf{n}})}(Z^2_{\Lambda,\beta})^{(1-\alpha(\hat{\mathbf{n}}))}}\right), \tag{1.38}$$

whenever it exists. We show that the limit exists and is finite for Ising-like ferromagnetic models in chapter two, and in the upcoming section we show that that the limit is finite whenever it exists for every model satisfying the conditions of Pirogov-Sinai theory. Note that for Ising-like models the Hamiltonians with $\pm$ boundary conditions can be given by

$$\mathscr{H}^+_\Lambda(\sigma) = \sum_{A\cap\Lambda\neq\varnothing} J_A\sigma_{A\cap\Lambda} \text{ and}$$

$$\mathscr{H}^-_\Lambda(\sigma) = \sum_{A\cap\Lambda\neq\varnothing} (-1)^{|A\cap\Lambda^c|} J_A\sigma_{A\cap\Lambda}$$

for $\sigma \in \Omega_\Lambda$. Note that for any finite set A such that $A\cap\Lambda \neq \varnothing$ we have, by flipping all the spins in $A\cap\Lambda$,

$$Z^-_{\Lambda,\beta} = \sum_{\sigma\in\Omega_\Lambda} \exp\left(-\sum_{A\cap\Lambda\neq\varnothing} (-1)^{|A\cap\Lambda^c|} J_A\sigma_{A\cap\Lambda}\right) = \sum_{\sigma\in\Omega_\Lambda} \exp\left(-\sum_{A\cap\Lambda\neq\varnothing} (-1)^{|A|} J_A\sigma_{A\cap\Lambda}\right),$$

since the spin flipping yields a symmetry transformation $\sigma_{A\cap\Lambda} \to (-1)^{|A\cap\Lambda|}\sigma_{A\cap\Lambda}$. Noting that every A such that $J_A \neq 0$ are translates of each other by the definition of these models, then these sets have all the same cardinality. If this cardinality is even, then the computation above already yields $Z^+_{\Lambda,\beta} = Z^-_{\Lambda,\beta}$. If it is odd, then $(-1)^{|A|} = -1$ for every A, and by flipping only one spin in $A\cap\Lambda$ at a time instead of all the spins simultaneously we induce another symmetry transformation $\sigma_{A\cap\Lambda} \to -\sigma_{A\cap\Lambda}$, which cancels the other negative sign. In any case, we have established $Z^+_{\Lambda,\beta} = Z^-_{\Lambda,\beta}$ for Ising-like models.

In terms of the surface tension, for these models we have

$$\tau_\beta(0,1) := -\lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{\beta(2L+1)^{d-1}} \log\left(\frac{Z^{\pm}_{\Lambda,\beta}}{Z^{+}_{\Lambda,\beta}}\right), \tag{1.39}$$

for a box Λ of length $2L+1$ in d dimensions. Moreover, it can be shown (see [BLP80]) that $\tau_\beta(0,1)$ is increasing in the couplings J_B. This, in particular, implies that the surface tension of these models is increasing in the dimensions, since one can obtain a d-dimensional model by turning off sufficiently many couplings (which are non-negative by hypothesis). Moreover, by [BLP80] and [LP81], the inequalities

$$\tau_\beta(0,1) \leq 2(m^*(\beta))^2 \tag{1.40}$$

$$\beta\frac{d\tau_\beta(0,1)}{d\beta} \geq 2(m^*(\beta))^2 \tag{1.41}$$

hold. Hence, if there is no spontaneous magnetization, i.e $\beta < \beta_c$, then the first inequality implies that $\tau_\beta(0,1) = 0$, and for all β such that $m^*(\beta) > 0$ one has that $\beta\tau_\beta(0,1)$ is increasing in β. In special, $\beta\tau_\beta(0,1) > 0$ and hence $\tau_\beta(0,1) > 0$ for all $\beta > \beta_c$ and $\tau_\beta(0,1) = 0$ for all $\beta < \beta_c$. This is the expected behaviour: for $\beta > \beta_c$ the two phases in the same system induce the appearance of interfaces, and for $\beta < \beta_c$ the randomness of the system blocks most interfaces from appearing.

Finally, we show an application of the surface tension in the context of phase separation. For this, consider the functional

$$\mathscr{F}_\tau(V) \overset{\text{def}}{=} \int_{\partial V} \tau_\beta(\hat{\mathbf{n}}(x))\lambda_{d-1}(dx), \tag{1.42}$$

defined for all smooth enough subsets $V \subset \mathbb{R}^d$ so that they admit an unit normal vector field pointing outwards and λ_d is the d-dimensional Lebesgue measure. The subset V_* minimizing $\mathscr{F}_\tau$ is unique up to translations and can be given explicitly by *Wulff shape*[6]

$$V_* \overset{\text{def}}{=} \{x \in \mathbb{R}^n : \tau_\beta(\hat{\mathbf{n}}) \geq x \cdot \hat{\mathbf{n}} \text{ for every unit vector } \hat{\mathbf{n}}\}. \tag{1.43}$$

The model of interest for this result is the nearest neighbor lattice gas, whose Hamiltonian is given by

$$\mathscr{H}_\Lambda(\omega) \overset{\text{def}}{=} -\sum_{\{i,j\}\subset\Lambda} J_{ij}\omega_i\omega_j,$$

where the only difference from the Ising model is that the spin variables are such that the single spin space is $E = \{0,1\}$ instead of $\{-1,1\}$. This is to represent occupation numbers, so that 0 means absence of particles and 1 means the presence of one. Here, we consider the canonical ensemble, where the total number of particles is fixed and equal to a prescribed number N. The number of particles, the partition function and the finite volume Gibbs measure for the model in this ensemble are given by

$$N_\Lambda(\omega) \overset{\text{def}}{=} \sum_{i\in\Lambda} \omega_i, \tag{1.44}$$

$$Z_{\Lambda,N,\beta} \overset{\text{def}}{=} \sum_{\substack{\omega_\Lambda\in\Omega_\Lambda \\ N_\Lambda(\omega_\Lambda)=N}} e^{-\beta\mathscr{H}_\Lambda(\omega_\Lambda)}, \tag{1.45}$$

$$\mu_{\Lambda,N,\beta}(\omega) \overset{\text{def}}{=} \frac{\exp(-\beta\mathscr{H}_\Lambda(\omega))}{Z_{\Lambda,N,\beta}}. \tag{1.46}$$

[6] Other names for this object in the literature are *equilibrium shape* or *equilibrium crystal shape.*

In the grand canonical ensemble, the number of particles is allowed to change. The grand canonical partition function depends on a parameter, the *chemical potential* $\mu \in \mathbb{R}$, and is given by

$$\Theta_{\Lambda,\mu,\beta} \stackrel{\text{def}}{=} \sum_{\omega_\Lambda \in \Omega_\Lambda} \exp(-\beta(\mathscr{H}_\Lambda(\omega_\Lambda) - \mu N_\Lambda(\omega_\Lambda))). \tag{1.47}$$

The finite volume *pressure* is the function $p_{\Lambda,\beta}(\mu) = \frac{1}{\beta|\Lambda|} \log \Theta_{\Lambda,\mu,\beta}$ and its thermodynamical limit $p_\beta(\mu)$ always exists for all μ. In the Ising model, the system can only exhibit phase transition for zero external fields, and in the lattice gas the same logic holds, as there is some chemical potential μ_* where phase transition can only occur for $\mu \neq \mu_*$. The average densities $\frac{\partial p_\beta}{\partial \mu^+}$ and $\frac{\partial p_\beta}{\partial \mu^-}$ are well-defined for every μ, including $\mu = \mu_*$ (this property follows from the convexity of the pressure) and when they coincide the common value is defined to be the grand canonical density ρ_β. It can be shown that the pressure in this model is actually analytic for every $\mu \neq \mu_*$, but the pressure is non-differentiable at μ_*. The gas and liquid densities ρ_g, ρ_l are hence defined by

$$\rho_g \stackrel{\text{def}}{=} \left.\frac{\partial p_\beta}{\partial \mu^-}\right|_{\mu_*} \quad \text{and}$$

$$\rho_l \stackrel{\text{def}}{=} \left.\frac{\partial p_\beta}{\partial \mu^+}\right|_{\mu_*}$$

respectively. If we let $\beta_c^{l.g}$ denote the critical inverse temperature for this model, then the phase separation result relating to the Wulff shape is given in the next result.

Theorem 1.22. *For the two dimensional lattice gas in a square box Λ_n of side length n, for $\beta > \beta_c^{l.g}$, $\rho \in (\rho_g, \rho_l)$ and $N_n = \rho|\Lambda_n|$ one has*

$$\lim_{n\to\infty} \mu^0_{\Lambda_n, N_n, \beta}(\mathscr{D}) = 1, \tag{1.48}$$

where the event $\mathscr{D}$ is defined as: there are constants $c_1 = c_1(\beta)$ and $c_2 = c_2(\beta)$ such that

- *there exists a contour γ_0 such that all other contours γ satisfy diam$(\gamma) \leq c_1 \log n$,*
- *the contour γ_0 is closely approximated by a dilatation and translation of the Wulff shape, that is,*

$$\min_{x \in [-1,1]^2} \frac{1}{n} d_{\mathbb{H}}(\gamma_0, x + \partial V_*) \leq \frac{c_2}{n^4} \sqrt{\log n}, \tag{1.49}$$

where $d_{\mathbb{H}}$ denotes the Hausdorff distances between sets.[7]

Moreover, it can be shown that there is a predominance of the phase 1 inside γ_0 and 0 outside of it, so that the contour separating the phases is given by the Wulff shape.

More information about the surface tension can be found in [Pfi09].

1.2.3 Surface Tension: Pirogov-Sinai approach

Consider two dominant ground states $q_1, q_2 \leq r$ and finite box $\Lambda \subset \mathbb{Z}^2$. For the rest of this section, we will fix the boundary condition (q_1, q_2) given by $\sigma_{(x,y)} = q_1$ for all $(x,y) \in \mathbb{Z}^2$ with $y \geq 0$ and $\sigma_{(x,y)} = q_2$ otherwise. We will abbreviate the boundary conditions q_1 and q_2 by 1 and 2 respectively.

[7] With respect to the distance induced by the ℓ_1 norm on $\mathbb{Z}^d$, the Hausdorff distance is given by $d_{\mathbb{H}}(X,Y) \stackrel{\text{def}}{=} \max\left\{\sup_{x\in X} d(x,Y), \sup_{y \in Y} d(X,y)\right\}$, where $d(x,Y) \stackrel{\text{def}}{=} \inf_{y\in Y} d(x,y)$.

For any interface λ as in definition 1.20, let $\overline{\lambda}$ denote the set of all points $x \in \lambda$ together with interaction supports $A \subset \mathbb{Z}^2$ with $x \ni A$. We split Λ as the union of $\overline{\lambda}$, the interiors $I_i(\lambda)$ of λ, each with a boundary condition m_i and the remaining volumes $V_1(\lambda), V_2(\lambda)$ neighboring the 1 and 2-boundaries of Λ, respectively (see image 1.6). If we denote

$$E(\overline{\lambda}) := \sum_{A \cap \lambda \neq \varnothing} \Phi_A(\sigma_{\overline{\lambda}}) + \sum_{x \in \overline{\lambda}} E_x(\sigma_\lambda), \tag{1.50}$$

where $E_x(\sigma) = E(\sigma_x)$, then one can recover the partition function by summing over all possibilities of interfaces, volumes $V_1(\lambda), V_2(\lambda)$ and the interiors $I_i(\lambda)$,

$$Z^{1,2}_{\Lambda,\beta} = \sum_{\lambda} \exp\left(-\beta E(\overline{\lambda})\right) Z^1_{V_1(\lambda)} Z^2_{V_2(\lambda)} \prod_{i=1}^{k} Z^{m_i}_{I_i(\lambda)}. \tag{1.51}$$

It is at this point that an important remark about Pirogov-Sinai theory should be made. Note that in definition 1.50 we assumed implicitly that the configuration in $\overline{\lambda}$ is determined by the configuration only in λ. This is possible to assume given that we change the definition of q-correct points x to denote those points such that $\sigma_i = q$, for all $i \in B_R(x)$ and we take R big enough. The resulting effect is that the contours become far away from each other, since there are more q-correct points separating them. If the interactions are of short-range type, one can then always choose R big enough so that $A \cap \lambda \neq \varnothing$ implies that A intersects at most λ and those q-correct points determined by the boundary of λ.

With this decomposition of the partition function, one gets

$$\frac{Z^{1,2}_{\Lambda,\beta}}{\sqrt{Z^1_{\Lambda,\beta} Z^2_{\Lambda,\beta}}} = \sum_{\lambda} \left\{-\beta\left(E(\overline{\lambda}) - \frac{\mu_1+\mu_2}{2}|\overline{\lambda}|\right)\right\} \sqrt{W_1(\lambda) W_2(\lambda) W_3(\lambda)}, \tag{1.52}$$

where

$$W_1(\lambda) = \frac{Z^1_{V_1(\lambda)} Z^1_{V_2(\lambda)} \prod_{i=1}^k Z^1_{I_i(\lambda)}}{Z^1_\Lambda} \exp\left(-\beta\mu_1|\overline{\lambda}|\right) \prod_{i=1}^{k} \frac{Z^{m_i}_{I_i(\lambda)}}{Z^1_{I_i(\lambda)}}, \tag{1.53}$$

$$W_2(\lambda) = \frac{Z^2_{V_1(\lambda)} Z^2_{V_2(\lambda)} \prod_{i=1}^k Z^2_{I_i(\lambda)}}{Z^2_\Lambda} \exp\left(-\beta\mu_2|\overline{\lambda}|\right) \prod_{i=1}^{k} \frac{Z^{m_i}_{I_i(\lambda)}}{Z^2_{I_i(\lambda)}}, \tag{1.54}$$

$$W_3(\lambda) = \frac{Z^1_{V_1(\lambda)} Z^2_{V_2(\lambda)}}{Z^1_{V_2(\lambda)} Z^2_{V_1(\lambda)}}, \tag{1.55}$$

and we have omitted the inverse temperature for each term, for simplicity. Bounds for the relevant terms in equation 1.52 are given in terms of $|\lambda|$ and $B_\lambda := \{A : A \cap \lambda \neq \varnothing \text{ and } \Phi_A \neq 0\}$.

Lemma 1.23. *There exists a constant $C_1 > 0$ such that for all sufficiently small $\epsilon > 0$ and each $|\mu| < \epsilon$ the bound $E(\overline{\lambda}) - \frac{1}{2}(\mu_1+\mu_2)|\overline{\lambda}| \geq C_1|B_\lambda|$ holds.*

Proof. By lemma 1.13, we have $E(\overline{\lambda}) \geq c|B_\lambda| + |\overline{\lambda}| \min_{1\leq i\leq n} \mu_i \geq c|B_\lambda| + |\lambda| \min_{1\leq i\leq n}\mu_i$. By splitting $|\overline{\lambda}| = |\overline{\lambda} \setminus \lambda| + |\lambda|$, one gets

$$E(\overline{\lambda}) - \frac{1}{2}(\mu_1+\mu_2)|\overline{\lambda}| \geq cB_\lambda - \frac{1}{2}|\overline{\lambda} \setminus \lambda|(\mu_1+\mu_2) - \frac{1}{2}|\lambda|(\mu_1+\mu_2 - 2\min_{1\leq i\leq n}\mu_i).$$

Now we will show that there is a constant $c_1 > 0$ such that $|\overline{\lambda} \setminus \lambda| \leq c_1|B_\lambda|$ and $|\lambda| \leq c_1|B_\lambda|$. In fact, note that by definition of $\overline{\lambda}$, one has $|\overline{\lambda} \setminus \lambda| \leq p|\partial\lambda| \leq p|\lambda|$, where we remember that $p = \max\{|A| : \phi_A \neq \varnothing\}$. Hence, it is enough to find a bound of the form $|\lambda| \leq c_1|B_\lambda|$. Now, for $x \in \mathbb{Z}^2$ we let $I(x) := |\{A : \phi_A \neq 0 \text{ and } A \ni x\}|$. By translation invariance, $I(x)$ is independent of x. Now,

$$\sum_{x\in\lambda} I(x) = \sum_{x\in\lambda}\sum_{A\Subset\mathbb{Z}^d} \mathbb{1}_{\left\{\substack{A\ni x\\ \phi_A\neq\varnothing}\right\}} = \sum_{x\in\lambda}\sum_{A\cap\lambda\neq\varnothing} \mathbb{1}_{\left\{\substack{A\ni x\\ \phi_A\neq\varnothing}\right\}} = \sum_{A\cap\lambda\neq\varnothing}\sum_{x\in\lambda} \mathbb{1}_{\left\{\substack{A\ni x\\ \phi_A\neq\varnothing}\right\}}$$
$$\leq \sum_{A\cap\lambda\neq\varnothing} |\{x\in\lambda : x\in A\}| \leq p|B_\lambda|.$$

Therefore $I(0)|\lambda| = \sum_{x\in\overline{\lambda}} I(x) \leq p|B_\lambda|$, so that $|\lambda| \leq \frac{p}{I(0)}|B_\lambda|$ and we see that one can take $c_1 = \frac{p}{I(0)}$. Let A_0 be any interaction support with cardinality p containing 0. One can translate this set p times in such a way that it still contains 0, so that $I(0) \geq p$ and hence $c_1 \leq 1$.

We now note that for $|\mu| < \epsilon$ we get $-\frac{1}{2}|\overline{\lambda}\setminus\lambda|(\mu_1+\mu_2) > -\epsilon|B_\lambda|$ and, since $\mu_j - \min_{1\leq i\leq n}\mu_i < \frac{c}{2p}$ for all $1 \leq j \leq r$, then

$$-\frac{1}{2}|\lambda|(\mu_1+\mu_2 - 2\min_{1\leq i\leq n}\mu_i) = -\frac{1}{2}|\lambda|(\{\mu_1 - \min_{1\leq i\leq n}\mu_i\} + \{\mu_2 - \min_{1\leq i\leq n}\mu_i\}) > -\frac{c}{4p}|B_\lambda|.$$

After regrouping all the terms, this yields

$$E(\overline{\lambda}) - \frac{1}{2}(\mu_1+\mu_2)|\overline{\lambda}| > (c - \epsilon - \frac{1}{4p}c)|B_\lambda|,$$

and we note that the overall constant is $c(1-\frac{1}{2p}) > 0$ for $\epsilon < \frac{c}{4p}$, as we wanted.

□

Lemma 1.24. *There is a function $\delta_1 = \delta_1(\beta)$ decaying exponentially on β such that*

$$\max\{W_1(\lambda), W_2(\lambda)\} \leq \exp(\delta_1(\beta)|B_\lambda|).$$

Proof. To start, we rewrite the partition function in Λ by summing in each region $V_1(\lambda), V_2(\lambda)$ and the interiors $I_i(\lambda)$ separately, that is,

$$Z_\Lambda^1 = \sum_{\sigma_{\overline{\lambda}}\in\Omega_{\overline{\lambda}}}\ \sum_{\sigma_{V_1(\lambda)}\in\Omega^1_{V_1(\lambda)}}\ \sum_{\sigma_{V_2(\lambda)}\in\Omega^1_{V_2(\lambda)}}\ \sum_{\sigma_{I_1(\lambda)}\in\Omega^{m_1}_{I_1(\lambda)}} \cdots \sum_{\sigma_{I_k(\lambda)}\in\Omega^{m_k}_{I_k(\lambda)}} \exp\left(-\beta\sum_{A\cap\Lambda\neq\varnothing}\phi_A(\sigma) - \beta\sum_{x\in\Lambda}E_x(\sigma)\right),$$

with $\sigma = \sigma_{\overline{\lambda}}\sigma_{V_1(\lambda)}\sigma_{V_2(\lambda)}\sigma_{I_1(\lambda)}\ldots\sigma_{I_k(\lambda)}$. We now split the sums $\sum_{A\cap\Lambda\neq\varnothing}\phi_A(\sigma)$ and $\sum_{x\in\Lambda}E_x(\sigma)$ into sums over those A intersecting λ and at most Λ^c, but not $V_1(\lambda)$ and $V_2(\lambda)$ (we denote this family by $A_0(\lambda)$), $\{A : A\cap V_1(\lambda)\neq\varnothing\}$, $\{A : A\cap V_2(\lambda)\neq\varnothing\}$ and in $\{A : A\subset I_k(\lambda)\}$. We can then redistribute these terms inside each sum in the above, since the potentials ϕ_A depend on the configurations only inside A. The result is

$$Z_\Lambda^1 = \sum_{\sigma_{\overline{\lambda}}\in\Omega_{\overline{\lambda}}} \exp\left(-\beta\sum_{A\in A_0(\lambda)}\phi_A(\sigma_{\overline{\lambda}}) - \beta\sum_{x\in\overline{\lambda}}E_x(\sigma_{\overline{\lambda}})\right)\cdots$$
$$\sum_{\sigma_{I_k(\lambda)}\in\Omega^{m_k}_{I_k(\lambda)}} \exp\left(-\beta\sum_{A\subset I_k(\lambda)}\phi_A(\sigma_{I_k(\lambda)}) - \beta\sum_{x\in I_k(\lambda)}E_x(\sigma_{I_k(\lambda)})\right)$$
$$= Z^1_{V_1(\lambda)}Z^1_{V_2(\lambda)}\prod_{i=1}^k Z^1_{I_i(\lambda)} \sum_{\sigma_{\overline{\lambda}}\in\Omega_{\overline{\lambda}}} \exp\left(-\beta\sum_{A\in A_0(\lambda)}\phi_A(\sigma_{\overline{\lambda}}) - \beta\sum_{x\in\overline{\lambda}}E_x(\sigma_{\overline{\lambda}})\right).$$

We now bound below the sum on the RHS by its summand on the particular configuration where $\sigma_{\overline{\lambda}}$ equals the ground state 1 everywhere. Since this is a ground state, all potentials $\phi_A(1)$ are zero and $\sum_{x\in\overline{\lambda}}E_x(1) = \mu_1|\overline{\lambda}|$. This implies

$$Z^1_\Lambda \geq Z^1_{V_1(\lambda)} Z^1_{V_2(\lambda)} \prod_{i=1}^{k} Z^1_{I_i(\lambda)} \exp(-\beta\mu_1|\overline{\lambda}|), \tag{1.56}$$

and hence

$$\frac{Z^1_{V_1(\lambda)} Z^1_{V_2(\lambda)} \prod_{i=1}^{k} Z^1_{I_i(\lambda)}}{Z^1_\Lambda} \exp\left(-\beta\mu_1|\overline{\lambda}|\right) \leq 1. \tag{1.57}$$

By corollary 1.19, the remaining term is bounded by

$$\prod_{i=1}^{k} \frac{Z^{m_i}_{I_i(\lambda)}}{Z^1_{I_i(\lambda)}} \leq \exp(2e^{-c\tau}\{|\partial I_1(\lambda)| + ... + |\partial I_k(\lambda)|\}) \leq \exp(2e^{-c\tau}|\lambda|).$$

As in the proof of the last lemma, we have $|\lambda| \leq c_1|B_\lambda|$ for some positive constant c_1 and, since $\tau = \tau(\beta)$ is proportional to β, we get the result by setting $\delta_1(\beta) := 2c_1e^{-c\tau(\beta)}$. □

Lemma 1.25. *There is a function $\delta_2 = \delta_2(\beta)$ decaying exponentially on β such that*

$$W_3(\lambda) \leq \exp(\delta_2(\beta)|B_\lambda|) \tag{1.58}$$

Proof. By the proof of corollary 1.19, we have

$$\begin{aligned} W_3(\lambda) &= \frac{Z^1_{V_1(\lambda)}}{Z^2_{V_1(\lambda)}} \frac{Z^2_{V_2(\lambda)}}{Z^1_{V_1(\lambda)}} \\ &\leq \exp(-\beta|V_1(\lambda)|\{\mu_1 - \mu_2\}) \exp(-\beta|V_2(\lambda)|\{\mu_2 - \mu_1\}) \frac{\Omega^0(V_1(\lambda) : F_1)\Omega^0(V_2(\lambda) : F_2)}{\Omega^0(V_2(\lambda) : F_1)\Omega^0(V_1(\lambda) : F_2)} \\ &= \exp(-\beta\mu_1(|V_1(\lambda)| - |V_2(\lambda)|) - \beta\mu_2(|V_2(\lambda)| - |V_1(\lambda)|)) \frac{\Omega^0(V_1(\lambda) : F_1)\Omega^0(V_2(\lambda) : F_2)}{\Omega^0(V_2(\lambda) : F_1)\Omega^0(V_1(\lambda) : F_2)} \end{aligned} \tag{1.59}$$

where $\{F_1, ..., F_r\}$ denotes the corresponding contour model. By lemma 1.17, the right-most term equals

$$\exp\left(\sum_{x \subset V_1(\lambda)} \varphi_1^T(x) - \sum_{x \subset V_2(\lambda)} \varphi_1^T(x) + \sum_{x \subset V_2(\lambda)} \varphi_2^T(x) - \sum_{x \subset V_1(\lambda)} \varphi_2^T(x) \right). \tag{1.60}$$

From here, we will consider the reflection map $R(x, y) = (x, -y)$. Given any interface λ, let R_λ denote $R(V(\lambda)) \cup V(\lambda)$ and define $V_1^0(\lambda) = (\mathbb{Z}^2_u \cap \Lambda) \setminus R_\lambda$, $V_1^1(\lambda) = R_\lambda \setminus \lambda_u$ and analogously for $V_2^0(\lambda)$ and $V_2^1(\lambda)$.

Note that we have $V_1(\lambda) = V_1^0(\lambda) \cup V_1^1(\lambda)$ and $V_2(\lambda) = V_2^0(\lambda) \cup V_2^1(\lambda)$, see the figure 1.7.

In general, for disjoint sets A, B and any set function f one has

$$\sum_{x \subset A \cup B} f(x) = \sum_{x \subset A} f(x) + \sum_{x \subset B} f(x) + \sum_{\substack{x \cap A \neq \varnothing \\ x \cap B \neq \varnothing}} f(x). \tag{1.61}$$

Applying this for $V_i = V_i^0 \cup V_i^1$, the argument of the exponential in equation 1.60 equals

$$\sum_{x \subset V_1^0(\lambda)} \varphi_1^T(x) + \sum_{x \subset V_1^1(\lambda)} \varphi_1^T(x) + \sum_{\substack{x \cap V_1^0 \neq \varnothing \\ x \cap V_1^1 \neq \varnothing}} \varphi_1^T(x)$$

$$- \sum_{x \subset V_2^0(\lambda)} \varphi_1^T(x) - \sum_{x \subset V_2^1(\lambda)} \varphi_1^T(x) - \sum_{\substack{x \cap V_2^0 \neq \varnothing \\ x \cap V_2^1 \neq \varnothing}} \varphi_1^T(x)$$

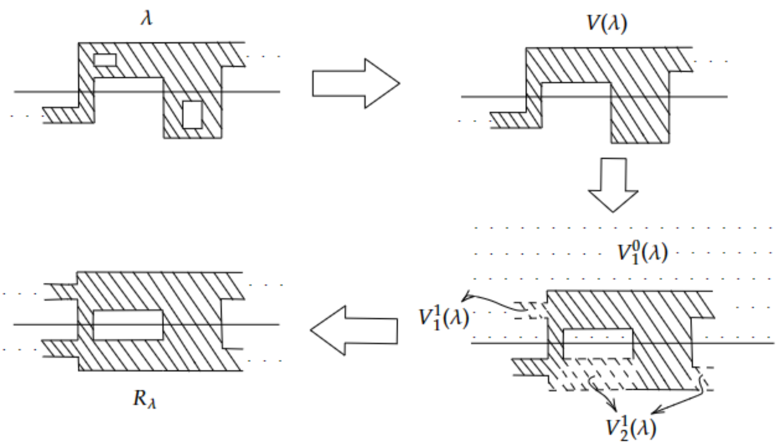


Figure 1.7: *The detailed construction of R_λ and $V_i^j(\lambda)$. After the second arrow, we see that $V_1^0(\lambda)$ is the dotted region and $V_1^1(\lambda)$ is the dashed region inside the upper part of the interface λ. Note that this region corresponds to added mass to the interface given by the reflection of the lower part. Analogously, V_2^1 is the added part to the interface given by the reflection of the upper part, and the remaining blank region is V_2^0. The set R_λ consists of the original interface and its reflection.*

$$\sum_{x\subset V_2^0(\lambda)} \varphi_2^T(x) + \sum_{x\subset V_2^1(\lambda)} \varphi_2^T(x) + \sum_{\substack{x\cap V_2^0\neq\varnothing \\ x\cap V_2^1\neq\varnothing}} \varphi_2^T(x)$$
$$- \sum_{x\subset V_1^0(\lambda)} \varphi_2^T(x) - \sum_{x\subset V_1^1(\lambda)} \varphi_2^T(x) - \sum_{\substack{x\cap V_1^0\neq\varnothing \\ x\cap V_1^1\neq\varnothing}} \varphi_2^T(x)$$

Now, since $V_1^0(\lambda) = R(V_2^0)$, by symmetry we have

$$\sum_{x\subset V_1^0(\lambda)} \varphi_1^T(x) = \sum_{x\subset V_2^0(\lambda)} \varphi_1^T(x),$$

This implies that all terms involving $V_1^0(\lambda)$ and $V_2^0(\lambda)$ cancel out. Moreover, we can use the identity valid for a translation invariant function f

$$\sum_{x\cap A\neq\varnothing} f(x) \leq \sum_{a\in A}\sum_{x\ni a} f(x) = \sum_{a\in A}\sum_{x\ni 0} f(x) = |A|\sum_{x\ni 0} f(x)$$

to get the estimate

$$\sum_{\substack{x\cap V_1^0\neq\varnothing \\ x\cap V_1^1\neq\varnothing}} \varphi_1^T(x) \leq \sum_{x\cap\partial V_1^0\neq\varnothing} \varphi_1^T(x) \leq |\lambda|\sum_{x\ni 0}\varphi_1^T(x) \leq |\lambda| e^{-c\tau(\beta)},$$

by lemma 1.17 and we noted that $|\partial V_1^0| \leq |\lambda|$. Since $|V_i^1| \leq |\lambda|$, the same bound holds for those terms. Therefore, after the cancelation described above and this estimate, we get

$$\frac{\Omega^0(V_1(\lambda):F_1)\Omega^0(V_2(\lambda):F_2)}{\Omega^0(V_2(\lambda):F_1)\Omega^0(V_1(\lambda):F_2)} \leq \exp(4|\lambda|e^{-c\tau(\beta)}) \tag{1.62}$$

$$\times \exp\left(\sum_{x\subset V_1^1(\lambda)} \varphi_1^T(x) - \sum_{x\subset V_2^1(\lambda)} \varphi_1^T(x) + \sum_{x\subset V_1^1(\lambda)} \varphi_2^T(x) - \sum_{x\subset V_2^1(\lambda)} \varphi_2^T(x)\right) \tag{1.63}$$

$$= \exp(4|\lambda|e^{-c\tau(\beta)}) \frac{\Omega^0(V_1^1(\lambda):F_1)\Omega^0(V_2^1(\lambda):F_2)}{\Omega^0(V_2^1(\lambda):F_1)\Omega^0(V_1^1(\lambda):F_2)}. \tag{1.64}$$

Remembering the definition $\Delta(\Lambda : F_q) = \log \Omega^0(\Lambda : F_q) - S(F_q)|\Lambda|$ and the result $|\Delta(\Lambda, F_q)| \leq \exp(-c\tau)|\partial\Lambda|$, we have that the term in 1.64 equals to

$$\exp(4|\lambda|e^{-c\tau(\beta)}) \exp\left(\log \Omega^0(V_1^1(\lambda) : F_1) + \log \Omega^0(V_2^1(\lambda) : F_2) - \log \Omega^0(V_2^1(\lambda) : F_1) - \log \Omega^0(V_1^1(\lambda) : F_2)\right)$$

$$\leq \exp(4|\lambda|e^{-c\tau(\beta)}) \exp(|V_1^1(\lambda)|S(F_1) + e^{-c\tau(\beta)}|\partial V_1^1(\lambda)| - S(F_2)|V_1^1(\lambda)| + e^{-c\tau(\beta)}|\partial V_1^1(\lambda)|)$$
$$\times \exp(|V_2^1(\lambda)|S(F_2) + e^{-c\tau(\beta)}|\partial V_2^1(\lambda)| - S(F_1)|V_2^1(\lambda)| + e^{-c\tau(\beta)}|\partial V_2^1(\lambda)|).$$

Noting that $|\partial V_1^1(\lambda)| + |\partial V_2^1(\lambda)| \leq |\partial\lambda| \leq |\lambda|$, this yields the result

$$\exp(6|\lambda|e^{-c\tau(\beta)}) \exp(|V_1^1(\lambda)|\{S(F_1) - S(F_2)\} + |V_2^1(\lambda)|\{S(F_2) - S(F_1)\})$$
$$= \exp(6|\lambda|e^{-c\tau(\beta)}) \exp((|V_1(\lambda)| - |V_1^0(\lambda)|)\{S(F_1) - S(F_2)\} + (|V_2(\lambda)| - |V_2^0(\lambda)|)\{S(F_2) - S(F_1)\})$$
$$= \exp(6|\lambda|e^{-c\tau(\beta)}) \exp(|V_1(\lambda)|\{S(F_1) - S(F_2)\} + |V_2(\lambda)|\{S(F_2) - S(F_1)\}),$$

Where we have used the fact that R is a bijection between $V_1^0(\lambda)$ and $V_2^0(\lambda)$, and hence they have the same volume. Inserting this back in equation 1.59, we get

$$W_3(\lambda) \leq \exp(6|\lambda|e^{-c\tau(\beta)}) \exp\left(|V_1(\lambda)(\{S(F_1) - \beta\mu_1)) - (S(F_2) - \beta\mu_2)\}\right)$$
$$\times \exp\left(|V_2(\lambda)|\,\{(S(F_2) - \beta\mu_2) - (S(F_1) - \beta\mu_1)\}\right)$$

We now use the fact that $a^1(F_1) = a^2(F_2) = 0$, implying that $S(F_1) - \beta\mu_1 = -\alpha = S(F_2) - \beta\mu_2$ (see equation 1.35). This implies that $W_3(\lambda) \leq \exp(\delta_2(\beta)|\lambda|)$ with $\delta_2(\beta) := 6e^{-c\tau(\beta)}$, as wanted. □

With these bounds, we are now ready to show the main result. Here, we will assume that the box Λ is placed symmetrically with respect to the boundary condition, so that $\alpha(\hat{\mathbf{n}}) = \frac{1}{2}$ for all unit vectors $\hat{\mathbf{n}}$.

Theorem 1.26. *The surface tension in equation 1.38 is strictly positive for all dominant ground states q_1, q_2.*

Proof. Using all the bounds gathered so far, we have

$$\frac{Z_{\Lambda,\beta}^{1,2}}{\sqrt{Z_{\Lambda,\beta}^1 Z_{\Lambda,\beta}^2}} \leq \sum_\lambda \exp\left(\frac{1}{2}(-2C_1\beta + \delta_1(\beta) + \delta_2(\beta))|B_\lambda|\right). \tag{1.65}$$

For all high enough β we have $\delta_1(\beta), \delta_2(\beta) \leq \frac{1}{2}C_1\beta$, since they decay exponentially on β. Therefore the sum above is bounded by

$$\sum_\lambda \exp\left(-K\beta|\lambda|\right) \leq \sum_{n=2L+1}^{\infty} |\{\lambda : |\lambda| = n\}| \exp\left(-\beta K n\right) = \sum_{n=2L+1} \exp((-\beta K + C_0)n), \tag{1.66}$$

for some constant K, where we have used the bound $|B_\lambda| \geq \frac{1}{c_1}|\lambda|$ obtained in lemma 1.23, the known result that the number of Pirogov-Sinai contours with fixed size n and containing a fixed point is less than $\exp(C_0 n)$ for some constant C_0 and the fact that all interfaces λ have at least $2L+1$ points, since they travel from one side of Λ to the other. Since $-\beta K + C_0 \leq -\frac{K}{2}\beta \iff C_0$ for $\beta \geq \frac{2C_0}{K}$, then the sum above is bounded by

$$\left(\frac{1}{e}\right)^{\frac{K}{2}\beta(2L+1)} \frac{1}{1 - e^{-1}}.$$

Applying the logarithm and then the limit in L, we get, finally,

$$\lim_{L\to\infty} \frac{1}{2L+1} \log \left(\frac{Z^{1,2}_{\Lambda,\beta}}{\sqrt{Z^{1}_{\Lambda,\beta} Z^{2}_{\Lambda,\beta}}} \right) \leq -\beta \frac{K}{2}. \tag{1.67}$$

□

Chapter 2

Duality Transformations for 1/2-spin Systems

In the present chapter we introduce the notion of duality transformations in half-spin ferromagnetic systems. This concept will be then used to prove the existence of the surface tension, as in [GHMMS77].

Duality transformations are relations satisfied by the Ising model, relating the partition and correlation functions of the model at high temperatures with the ones of a dual model in low temperatures. A set of sufficient conditions for the existence of such dual Ising models was given by [Weg71], requiring positive spin interactions and the existence of solutions for a system of linear equations. A year later, [MG72] gave an alternative construction of such duality transformations - much simpler, elegant and less restrictive - based on the intuition that such duality relations reflect on some symmetry of the Ising model and hence can be described by the usual group-theoretic structure of 1/2 spin lattice systems, to be described below. For the rest of the chapter, we will follow closely the notation and the work of [MG72].

Given any abstract finite set Λ with cardinality $|\Lambda| < \infty$, its power set $\mathscr{P}(\Lambda)$ has $2^{|\Lambda|}$ elements and inherits a finite group structure with product given by the symmetric difference: given any two elements $A, B \in \mathscr{P}(\Lambda)$, we set $A \cdot B \overset{\text{def}}{=} A \Delta B$, where $A \Delta B = A \cup B \backslash A \cap B$. It is not hard to see that $(\mathscr{P}(\Lambda), \cdot)$ is an abelian group, with the empty set serving as the identity element and every element being its own inverse. We will typically denote the elements of Λ by lowercase letters like $x, y, z, ...$ and the elements of $\mathscr{P}(\Lambda)$ are denoted by the uppercase letters $A, B, C....$

The first remarkable property of $\mathscr{P}(\Lambda)$ is that it is a vector field over $\mathbb{Z}_2$. This property allows us to transfer useful theorems valid for vector spaces over to the group $\mathscr{P}(\Lambda)$.

Lemma 2.1. *The group $\mathscr{P}(\Lambda)$ is a $\mathbb{Z}_2-$vector space with vector addition being the group operation and scalar multiplication given by $0X \mapsto e$ and $1X \mapsto X$.*

Proof. Associativity follows from the group axioms. Commutativity is already given since the group is abelian, the inverse element for every vector is the vector itself, the identity element of scalar multiplication is 1, and the distributivity of the scalar multiplication with respect to vector addition is straightforward.

The only property left to prove is the distribution of scalar multiplication with respect to scalar addition, i.e, $(a + b)X = aX + bX$. We will split the proof into 4 cases:

- a = b = 1: $(1 + 1)X = 0X = \varnothing$, and $1X + 1X = X + X = X^2 = \varnothing$;
- a = b = 0: $(0 + 0)X = 0X = \varnothing$, and $0X + 0X = \varnothing^2 = \varnothing$;
- a = 1 and b = 0: $(1 + 0)X = 1X = X$, and $1X + 0X = X + \varnothing = X \cdot e = X$;
- a = 0 and b = 1: same as before.

Therefore $\mathscr{P}(\Lambda)$ is a $\mathbb{Z}_2-$vector space.

$\square$

An elementary but useful observation is that a subset $W \subset \mathscr{P}(\Lambda)$ is a vector subspace if, and only if it is a subgroup. This is valid since the vector addition corresponds to the group multiplication. Moreover, we define the maps $\sigma_A : \mathscr{P}(\Lambda) \to \{-1, 1\}$ with $A \in \mathscr{P}(\Lambda)$ by

$$\sigma_A(R) \stackrel{\text{def}}{=} (-1)^{|A \cap R|}. \tag{2.1}$$

These maps have the following properties:

$$\begin{aligned} &(1) \ \ \sigma_A(R) = \sigma_R(A); \\ &(2) \ \ \sigma_A(R)\sigma_A(S) = \sigma_A(R \cdot S); \\ &(3) \ \ \sigma_A(R)\sigma_B(R) = \sigma_{A \cdot B}(R). \end{aligned} \tag{2.2}$$

For example, (2) is proved by noticing that $|A \cap (R \Delta S)| = |(A \cap R)\Delta(A \cap S)| = |A \cap R| + |A \cap S| - 2|A \cap R \cap S|$ and similarly for (3). Moreover, property (2) implies that each σ_A is a character[1] of $\mathscr{P}(\Lambda)$.

Note that if $\sigma_A = \sigma_B$, then $A = B$. In fact, if $x \in A$ is arbitrary and $x \notin B$ was the case, then $\sigma_A(\{x\}) = -1$ and $\sigma_B(\{x\}) = (-1)^0 = 1$, violating the equality $\sigma_A = \sigma_B$. Hence, we have $|\mathscr{P}(\Lambda)|$ distinct functions σ_A, one for each $A \subset \Lambda$ and therefore the collection $(\sigma_A)_{A \in \mathscr{P}(\Lambda)}$ defines a family of $|\mathscr{P}(\Lambda)|$ distinct characters of $\mathscr{P}(\Lambda)$. Since it is a well known result that a finite abelian group G has exactly $|G|$ characters, we find that the collection of all σ_A are precisely the characters of $\mathscr{P}(\Lambda)$. By property (3), we also have an explicit group isomorphism $A \mapsto \sigma_A$ from $\mathscr{P}(\Lambda)$ to its character group $\widehat{\mathscr{P}(\Lambda)}$.

Given a fixed $\mathscr{B} \subset \mathscr{P}(\Lambda)$, define the set $\mathscr{S}_\Lambda \stackrel{\text{def}}{=} \{S : \sigma_S(B) = 1, \forall B \in \mathscr{B}\}$. Later, we will show that there is a bijection between configurations in Λ and finite sets $A \in \mathscr{P}(\Lambda)$, where to each set we associate the unique configuration having all spins inside of it equal to -1 and $+1$ outside. In this sense:

- The maps σ_A take as input a configuration and returns the product of the spins inside the set of sites in A;
- $\mathscr{B}$ will be given as the support of the interactions;
- The elements S of $\mathscr{S}_\Lambda$ are configurations such that, for any $B \in \mathscr{B}$, the product of the spins inside B is always 1.

Give any group G, the orthogonal complement of a subset $H \leq G$ is $H^\perp \stackrel{\text{def}}{=} \{\chi \in \widehat{G} : \chi(h) = 1, \forall h \in H\}$. We have the following lemmas.

Lemma 2.2. *Let G be any finite abelian group. Then*

- $H^\perp \cong \left(G/\overline{H}\right)^\wedge$;
- $\overline{H} \cong \left(G^\wedge/H^\perp\right)^\wedge$.

Proof. To prove the first identity, consider the map $T : H^\perp \to \left(G/\overline{H}\right)^\wedge$ defined by

$$T(\chi)([g]) \stackrel{\text{def}}{=} \chi(g),$$

[1]Remember that a character of a group G is a homomorphism $\rho : G \to \mathbb{C}^\times \backslash \{0\}$ from G to the multiplicative group of the complex numbers. The set of all characters form a group under pointwise multiplication, denoted by $\widehat{G}$ or $G^\wedge$.

where we remember that the quotient group $G/\overline{H}$ consists of equivalence classes $[g]$ where $g \sim h$ iff $gh^{-1} \in \overline{H}$. Then T is well-defined, since $[g] = [h]$ implies that $g \sim h$ and hence $gh^{-1} \in \overline{H}$. As such, gh^{-1} can be decomposed as a finite product $gh^{-1} = a_1 a_2 ... a_n$ where each a_i belongs to H. In special, since $\chi \in H^{\perp}$ we have $\chi(gh^{-1}) = \chi(a_1)...\chi(a_n) = e$ and therefore

$$\chi(g) = \chi(gh^{-1}h) = \chi(gh^{-1})\chi(h) = \chi(h).$$

The map is injective, since $T(\chi_1) = T(\chi_2)$ implies, after computing both sides of the equality in a general element $[g]$, that

$$\chi_1(g) = T(\chi_1)([g]) = T(\chi_2)([g]) = \chi_2(g),$$

so that $\chi_1 = \chi_2$.

As for surjectivity, given any $\Phi \in \left(G/\overline{H}\right)^{\wedge}$ we define $\chi \in \widehat{G}$ by $\chi(g) = \Phi([g])$. It follows that χ is a homomorphism and that for any $h \in H$ one has $\chi(h) = \Phi([h]) = \Phi([e]) = 1$, since $h \sim e$ for any $h \in H$ by definition of the quotient subgroup. Finally, T is also a homomorphism, since

$$T(\chi_1 \cdot \chi_2)([g]) = (\chi_1 \cdot \chi_2)(g) = \chi_1(g) \cdot \chi_2(g) = (T(\chi_1) \cdot T(\chi_2))([g]),$$

implying $T(\chi_1 \cdot \chi_2) = T(\chi_1) \cdot T(\chi_2)$.

For the second item, define the map $F : \overline{H} \to \left(G^{\wedge}/H^{\perp}\right)^{\wedge}$ by $F(h) \stackrel{\text{def}}{=} Q_h$, where $Q_h : G^{\wedge}/H^{\perp} \to \mathbb{C} \setminus \{0\}$ is given by $Q_h([\chi]) \stackrel{\text{def}}{=} \chi(h)$. Note that each Q_h is well-defined, since $[\chi_1] = [\chi_2]$ implies that $\chi_1 \cdot \chi_2^{-1} \in H^{\perp}$. In special, since $h \in \overline{H}$, then $1 = (\chi_1 \cdot \chi_2^{-1})(h)$, implying $\chi_1(h) = \chi_2(h)$. Since Q_h is well-defined, so is F.

F is also injective, since $F(h_1) = F(h_2)$ implies that $Q_{h_1} = Q_{h_2}$, which implies that $\chi(h_1) = \chi(h_2)$ for every character χ. Since locally compact topological groups G (which includes finite abelian groups with the discrete topology, as in our case) are such that $\widehat{G}$ separates points (see [Pon39]), this implies that $h_1 = h_2$. It is straightforward that F is a homomorphism and, for surjectivity, since by the first item we have by Lagrange's Theorem[2] $|H^{\perp}| = \frac{|G|}{|\overline{H}|}$, which implies that $|\overline{H}| = \frac{|G|}{|H^{\perp}|} = \frac{|G^{\wedge}|}{|H^{\perp}|} = \left|\left(\frac{|G^{\wedge}|}{|H^{\perp}|}\right)^{\wedge}\right|$. Thus, since F is an injection between two sets of equal finite cardinality, F is an isomorphism.

□

Lemma 2.3. *If $\overline{\mathscr{B}}$ is the subgroup generated by $\mathscr{B}$, then:*

$$\mathscr{S}_{\Lambda} \cong \mathscr{P}(\Lambda)/\overline{\mathscr{B}}$$

Moreover, $\overline{\mathscr{B}} = \{A : \sigma_A(S) = 1, \forall S \in \mathscr{S}_{\Lambda}\}$.

Proof. Using the last lemma, we have $\mathscr{B}^{\perp} = \{\chi \in \widehat{P(\Lambda)} : \chi(B) = 1, \forall B \in \mathscr{B}\}$ and since every character of $\mathscr{P}(\Lambda)$ is of the form σ_A, for some $A \subset \Lambda$, then we have

$$\mathscr{B}^{\perp} = \{\sigma_A : \sigma_A(B) = 1, \forall B \in \mathscr{B}\} \cong \mathscr{S}_{\Lambda}.$$

Hence, the above equations with $G = \mathscr{P}(\Lambda)$ and $H = \mathscr{B}$ yield

$$\mathscr{S}_{\Lambda} = \mathscr{B}^{\perp} \cong \left(\mathscr{P}(\Lambda)/\overline{\mathscr{B}}\right)^{\wedge} \cong \mathscr{P}(\Lambda)/\overline{\mathscr{B}},$$

where we remember that the character group is isomorphic to the group itself for finite abelian groups. Finally, since $(H^{\perp})^{\perp} = \overline{H}$, then

$$\overline{\mathscr{B}} = \mathscr{S}_{\Lambda}^{\perp} = \{\sigma_A : \sigma_A(S) = 1, \forall S \in \mathscr{S}_{\Lambda}\}$$

[2] Here, we are using the implication that $|G/H| = \frac{|G|}{|H|}$ and that for finite abelian groups $|\widehat{G}| = |G|$.

$$\cong \{A : \sigma_A(S) = 1, \forall S \in \mathscr{S}_\Lambda\}.$$

□

Moreover, the order of every subgroup $H \leq \mathscr{P}(\Lambda)$ can be found in terms of its minimal generators. We will enunciate this fact in the next lemma:

Lemma 2.4. *Let $H \leq \mathscr{P}(\Lambda)$ be any subgroup. If H is generated by n minimal elements, then $|H| = 2^n$.*

Proof. Recall that our group can be endowed with a structure of a vector space over $\mathbb{Z}_2$. With this correspondence in mind, subgroups correspond to vector subspaces.

First, note that $H = \overline{\{h_1, ..., h_n\}}$ is the same as $H = \text{span}\{h_1, ..., h_n\}$. By minimality, this implies that $\{h_1, ..., h_n\}$ is a basis of H, so every element of H can be written uniquely as products of $h_1, h_2, ..., h_n$. The total number of elements of H can then be found by counting the possible ways of grouping $h_1, ..., h_n$, which is just 2^n, the amount of subsets of $\{h_1, ..., h_n\}$.

□

By the last lemma, we write $|\mathscr{S}_\Lambda| = 2^{N_S}$ and $|\overline{\mathscr{B}}| = 2^{N_i}$ where N_S is the minimal number of generators of $\mathscr{S}_\Lambda$ and N_i is the minimal number of generators of $\overline{\mathscr{B}}$.

2.1 Connection with Statistical Mechanics

Consider any possibly infinite subset $\Lambda \subset \mathbb{Z}^d$, a set of *bonds* $\mathscr{B} \subset \mathscr{P}_f(\mathbb{Z}^d)$ and a real or complex function $J : \mathscr{B} \to \mathbb{C}$. We call the triple $(\Lambda, \mathscr{B}, J)$ a *general lattice system*. To establish the connection with the previous section, we identify each $X \in \mathscr{P}(\mathbb{Z}^d)$ with the configuration σ given by $\sigma(x) = -1$, for all $x \in X$ and $\sigma(x) = 1$, for all $x \in X^c$. In this way, the configuration space is identified with $\mathscr{P}(\mathbb{Z}^d)$.

As before, we think of a box Λ as some set of interacting spins and the map J represents their interactions. Depending on the definition of $\mathscr{B}$, the spins inside Λ can interact with the outside of Λ or the interaction can be restricted only to the inside of the box. In this new language a boundary condition is just a subset $Y \subset \mathbb{Z}^d$, for example $Y = \varnothing$ and $Y = \mathbb{Z}^d$ correspond to $+$ and $-$ boundary conditions, respectively.

Given a general lattice system $(\Lambda, \mathscr{B}, J)$ and a boundary condition Y, the Hamiltonian $\mathscr{H}_\Lambda : \mathscr{P}(\Lambda) \to \mathbb{C}$ of the system with boundary condition Y is defined by

$$\mathscr{H}_\Lambda^Y(X) \stackrel{\text{def}}{=} -\sum_{B \in \mathscr{B}_\Lambda} J_B \sigma_B(X \cdot (Y \cap \Lambda^c)), \tag{2.3}$$

with $\mathscr{B}_\Lambda \stackrel{\text{def}}{=} \{B \in \mathscr{B} : B \cap \Lambda \neq \varnothing\}$ and $\sigma_B(X) = (-1)^{|B \cap X|}$ as before. The partition function of the system is given by

$$Z_{\Lambda,\beta}^Y \stackrel{\text{def}}{=} \sum_{X \subset \Lambda} \exp\left(-\beta \mathscr{H}_\Lambda^Y(X)\right) = \sum_{X \subset \Lambda} \exp\left(\sum_{B \in \mathscr{B}_\Lambda} K_B \sigma_B(X \cdot (Y \cap \Lambda^c))\right), \tag{2.4}$$

with $K_B \stackrel{\text{def}}{=} \beta J_B$. To avoid problems with convergence, we will always assume that the couplings J_B are regular. Moreover, we will always assume that the couplings J_B are *ferromagnetic*, in the sense that $J_B \geq 0$ for all B. This implies, in particular, that Griffiths inequalities always hold. More precisely, under this hypothesis, for any collection of spins σ_A and σ_B one has $\langle \sigma_A \sigma_B \rangle \geq \langle \sigma_A \rangle \langle \sigma_B \rangle$ for empty boundary conditions.

It is worthy to note that for any countable set $\mathscr{L}$ one has a pairing $\langle \cdot, \cdot \rangle : \mathscr{P}_f(\mathscr{L}) \times \mathscr{P}(\mathscr{L}) \to \{-1, 1\}$ given by $\langle X, Y \rangle \stackrel{\text{def}}{=} \sigma_X(Y) = (-1)^{|X \cap Y|}$, which is well-defined since X is finite. We can use this pairing to "take adjoints" of homomorphisms $f : \mathscr{P}(\mathscr{B}) \to \mathscr{P}(\mathbb{Z}^d)$ to yield another homomorphism $g : \mathscr{P}(\mathbb{Z}^d) \to \mathscr{P}(\mathscr{B})$. The next lemma formalizes this construction.

Lemma 2.5. *Let $\langle\cdot,\cdot\rangle_{\mathscr{P}(\mathscr{B})}$ and $\langle\cdot,\cdot\rangle_{\mathscr{P}(\mathbb{Z}^d)}$ be the pairings defined above in their respective spaces. Then, for any homomorphism $f : \mathscr{P}(\mathscr{B}) \to \mathscr{P}(\mathbb{Z}^d)$ satisfying $f(\mathscr{P}_f(\mathscr{B})) \subset \mathscr{P}_f(\mathbb{Z}^d)$, there is a unique homomorphism $g : \mathscr{P}(\mathbb{Z}^d) \to \mathscr{P}(\mathscr{B})$ such that*

$$\langle X, f(\underline{B})\rangle_{\mathscr{P}(\mathbb{Z}^d)} = \langle g(X), \underline{B}\rangle_{\mathscr{P}(\mathscr{B})}$$

For all $X \in \mathscr{P}(\mathbb{Z}^d)$ and $\underline{B} \in \mathscr{P}_f(\mathscr{B})$.

Proof. We will split the proof in four parts.

1. (Classification of homomorphism of $\mathscr{P}_f(\mathscr{L})$ to $\{-1,1\}$): First, we will show that for any homomorphism $T : \mathscr{P}_f(\mathscr{L}) \to \{-1,1\}$ with $\mathscr{L}$ countable there is a set $A \in \mathscr{P}(\mathscr{L})$ such that $T(X) = \sigma_A(X)$. In fact, pick $A \stackrel{\text{def}}{=} \{x \in \mathscr{L} : T(\{x\}) = -1\}$, so that we can split any $X \in \mathscr{P}_f(\mathscr{L})$ as $X = (X \cap A) \cup (X \cap A^c) = (X \cap A) \cdot (X \cap A^c)$. Therefore

$$\begin{aligned} T(X) = T(X \cap A)T(X \cap A^c) &= \left(\prod_{x \in X \cap A} T(x)\right)\left(\prod_{x \in X \cap A^c} T(x)\right) \\ &= \prod_{x \in X \cap A} (-1) = (-1)^{|X \cap A|} = \sigma_A(X). \end{aligned}$$

2. (Non-degeneracy of $\langle\cdot,\cdot\rangle$): Suppose that $\langle \underline{B}, \underline{B_1}\rangle_{\mathscr{P}(\mathscr{B})} = \langle \underline{B}, \underline{B_2}\rangle_{\mathscr{P}(\mathscr{B})}$ for all finite $\underline{B}$. We will prove that $\underline{B_1} = \underline{B_2}$.

   In fact, given any $B \in \underline{B_1}$, then $(-1)^{\{B\} \cap \underline{B_2}} = \sigma_{\{B\}}(\underline{B_2}) = \langle \{B\}, \underline{B_2}\rangle_{\mathscr{P}(\mathscr{B})} = \langle \{B\}, \underline{B_1}\rangle_{\mathscr{P}(\mathscr{B})} = \sigma_{\{B\}}(\underline{B_1}) = (-1)^{\{B\} \cap \underline{B_1}} = -1$. If we had $B \notin \underline{B_2}$, the equality above would not hold. Hence, we must have $\underline{B_1} \subset \underline{B_2}$ and reversing the roles of $\underline{B_1}$ and $\underline{B_2}$ we get the opposite inclusion.

3. (Existence): Now, fix any $X \in \mathscr{P}(\mathbb{Z}^d)$ and consider the map $T_X : \mathscr{P}_f(\mathscr{B}) \to \{-1,1\}$ defined by $T(\underline{B}) = \langle X, f(\underline{B})\rangle_{\mathscr{P}(\mathbb{Z}^d)}$. This map is well-defined since $f(\mathscr{P}_f(\mathscr{B})) \subset \mathscr{P}_f(\mathbb{Z}^d)$ and it is clear that any pairing on a countable set $\mathscr{L}$ satisfies $\langle X, Y \cdot Z\rangle_{\mathscr{P}(\mathscr{L})} = \sigma_X(Y \cdot Z) = \sigma_X(Y)\sigma_X(Z) = \langle X, Y\rangle_{\mathscr{P}(\mathscr{L})}\langle X, Z\rangle_{\mathscr{P}(\mathscr{L})}$ for any two finite Y, Z. In special, T_X is a homomorphism. There is hence a set of bonds $g(X) \in \mathscr{P}(\mathscr{B})$ such that $T_X(\underline{B}) = \sigma_{g(X)}(\underline{B})$, that is

$$\langle X, f(\underline{B})\rangle_{\mathscr{P}(\mathbb{Z}^d)} = \langle g(X), \underline{B}\rangle_{\mathscr{P}(\mathscr{B})}$$

   This defines a map $\mathscr{P}(\mathbb{Z}^d) \ni X \to g(X) \in \mathscr{P}(\mathscr{B})$. It is clearly a homomorphism, since for all finite $\underline{B}$ we have

$$\begin{aligned} \langle g(X \cdot Y), \underline{B}\rangle_{\mathscr{P}(\mathscr{B})} &= \langle X \cdot Y, f(\underline{B})\rangle_{\mathscr{P}(\mathbb{Z}^d)} = \langle X, f(\underline{B}))\rangle_{\mathscr{P}(\mathbb{Z}^d)}\langle Y, f(\underline{B})\rangle_{\mathscr{P}(\mathbb{Z}^d)} \\ &= \langle g(X), \underline{B}\rangle_{\mathscr{P}(\mathscr{B})}\langle g(Y), \underline{B}\rangle_{\mathscr{P}(\mathscr{B})} = \langle g(X) \cdot g(Y), \underline{B}\rangle_{\mathscr{P}(\mathscr{B})} \\ &\implies g(X \cdot Y) = g(X) \cdot g(Y) \end{aligned}$$

4. (Uniqueness): If we had two homomorphisms g_1, g_2 such that $\langle X, f(\underline{B})\rangle_{\mathscr{P}(\mathbb{Z}^d)} = \langle g_1(X), \underline{B}\rangle_{\mathscr{P}(\mathscr{B})} = \langle g_2(X), \underline{B}\rangle_{\mathscr{P}(\mathscr{B})}$ for all $\underline{B} \in \mathscr{P}_f(\mathscr{B})$, then non-degeneracy forces $g_1 = g_2$.

$\square$

Note that by the first part of the proof, it follows that for any set $\mathscr{L}$ the homomorphisms $T : \mathscr{P}_f(\mathscr{L}) \to \{-1,1\}$ are precisely of the form $Y \mapsto \sigma_A(Y)$, for some $A \in \mathscr{P}(\mathscr{L})$. If $\mathscr{L}$ is finite, we recover the result that the set $\{\sigma_A : A \in \mathscr{P}(\mathscr{L})\}$ is precisely the set of characters of $\mathscr{P}(\mathscr{L})$.

In special, by the orthogonality relations of characters,

$$\sum_{B\in\mathscr{P}(\mathscr{L})}\sigma_B(X)\sigma_B(Y)=|\mathscr{P}(\mathscr{L})|\delta_X^Y \tag{2.5}$$

Now, consider the map:

$$\pi:\mathscr{P}(\mathscr{B})\to\mathscr{P}(\mathbb{Z}^d)$$
$$\underline{B}\mapsto\prod_{B\in\underline{B}}B$$

Clearly, for any finite set of bonds the resulting image is in $\mathscr{P}_f(\mathbb{Z}^d)$. The next lemma also shows that π is a homomorphism.

Lemma 2.6. *The map π defined above is a homomorphism.*

Proof. Given $\underline{B_1},\underline{B_2}\in\mathscr{P}(\mathscr{B})$, we have

$$\pi(\underline{B_1}\cdot\underline{B_2})=\prod_{B\in\underline{B_1}\cdot\underline{B_2}}B=\prod_{B\in\underline{B_1}\setminus\underline{B_2}\cup\in\underline{B_2}\setminus\underline{B_1}}B=\left(\prod_{B_1\in\underline{B_1}\setminus\underline{B_2}}B_1\right)\cdot\left(\prod_{B_2\in\underline{B_2}\setminus\underline{B_1}}B_2\right)$$

Since $B^2=\varnothing$ for all group elements, we can write:

$$\begin{aligned}\pi(\underline{B_1}\cdot\underline{B_2})&=\left(\prod_{B_1\in\underline{B_1}\setminus\underline{B_2}}B_1\right)\cdot\left(\prod_{B\in\underline{B_1}\cap\underline{B_2}}B\right)\cdot\left(\prod_{B\in\underline{B_1}\cap\underline{B_2}}B\right)\left(\prod_{B_2\in\underline{B_2}\setminus\underline{B_1}}B_2\right)\\&=\left(\prod_{B_1\in\underline{B_1}}B_1\right)\cdot\left(\prod_{B_2\in\underline{B_2}}B_2\right)=\pi(\underline{B_1})\cdot\pi(\underline{B_2})\end{aligned}$$

□

Therefore, lemma 2.5 can be applied to π to yield its adjoint, which we call $\gamma:\mathscr{P}(\mathbb{Z}^d)\to\mathscr{P}(\mathscr{B})$. We note that the explicit form of γ can be found by the pairing identity applied to each set of the form $\{B\}$,

$$\sigma_{\gamma(X)}(\{B\})=\langle\{B\},\gamma(X)\rangle_{\mathscr{P}(\mathscr{B})}=\langle\pi(\{B\}),X\rangle_{\mathscr{P}(\mathscr{B})}=\langle B,X\rangle_{\mathscr{P}(\mathscr{B})}=\sigma_B(X)$$

So that $B\in\gamma(X)$ if, and only if $\sigma_B(X)=-1$. This implies that:

$$\gamma(X)=\{B\in\mathscr{P}(\mathscr{B}):\sigma_B(X)=-1\}$$

If $X\subset\Lambda$, then for $\sigma_B(X)$ to yield -1 it is necessary for B to intersect Λ. Therefore

$$\gamma(X)=\{B\in\mathscr{P}(\mathscr{B}_\Lambda):\sigma_B(X)=-1\},\ \text{if } X\subset\Lambda. \tag{2.6}$$

We could, have defined γ this way, but proving the homomorphism property of γ would have been a very convoluted proof, and moreover this construction we have done is far more elegant.

Some important subgroups of $\mathscr{P}(\mathscr{B})$ and $\mathscr{P}(\mathbb{Z}^d)$ are the following:

1. The interaction group $\overline{\mathscr{B}}$ is the subgroup of $\mathscr{P}_f(\mathscr{B})$ generated by $\mathscr{B}$. Moreover, we define $\overline{\mathscr{B}_\Lambda}$ to be the subgroup generated by $\mathscr{B}_\Lambda=\{B\in\mathscr{B}:B\cap\Lambda\neq\varnothing\}$.
2. The internal symmetry groups $\mathscr{S}$ and $\mathscr{S}_\Lambda$ are subgroups of $\mathscr{P}(\mathbb{Z}^d)$ (the configuration space) and $\mathscr{P}(\Lambda)$ defined by:

$$\mathscr{S}\stackrel{\text{def}}{=}\{S\in\mathscr{P}(\mathbb{Z}^d):\sigma_B(S)=1,\ \text{for all } B\in\mathscr{B}\} \tag{2.7}$$

$$\mathscr{S}_\Lambda \stackrel{\text{def}}{=} \{S \in \mathscr{P}(\Lambda) : \sigma_B(S) = 1, \text{ for all } B \in \mathscr{B}_\Lambda\} \tag{2.8}$$

Clearly, any $S \in \mathscr{S}$ satisfies $\mathscr{H}_\Lambda^Y(X \cdot S) = \mathscr{H}_\Lambda^Y(X)$ for every $X \in \mathscr{P}(\mathbb{Z}^d)$ and $\mathscr{S}_\Lambda$ is a finite group.

3. The high/low temperatures subgroups of $\mathscr{P}(\mathscr{B}_\Lambda)$, defined by $\mathscr{K}_\Lambda \stackrel{\text{def}}{=} \ker(\pi|_{\mathscr{P}(\mathscr{B}_\Lambda)})$ and $\Gamma_\Lambda \stackrel{\text{def}}{=} \operatorname{im}(\gamma|_{\mathscr{P}(\Lambda)})$ respectively. The name of these subgroups will be justified by their appearance in the high-low temperature expansion for the partition function, in the next section.

   Note that

$$\begin{aligned}\ker(\gamma|_{\mathscr{P}(\Lambda)}) &= \{X \in \mathscr{P}(\Lambda) : \{B \in \mathscr{P}(\mathscr{B}_\Lambda) : \sigma_B(X) = -1\} = \varnothing\} \\ &= \{X \in \mathscr{P}(\Lambda) : \sigma_B(S) = 1, \text{ for all } B \in \mathscr{B}_\Lambda\} = \mathscr{S}_\Lambda\end{aligned}$$

We will use the following lemma in the next section.

Lemma 2.7. *Let G be any finite group, H any group and $\phi : G \to H$ a homomorphism. Suppose that $T : G \to \mathbb{R}$ is any function such that $T(g) = T(h)$ whenever $\phi(g) = \phi(h)$. Then*

$$|ker(\phi)| \sum_{h \in im(\phi)} T(g_h) = \sum_{g \in G} T(g),$$

where g_h is any element of G such that $\phi(g_h) = h$ (by hypothesis, T is independent of the choice of g_h).

Proof. Note that, for any $h \in \operatorname{im}(\phi)$, one has $|\phi^{-1}(\{h\})| = |\ker(\phi)|$, since $\phi^{-1}(\{h\}) = g\ker(\phi)$ for any $g \in G$ with $\phi(g) = h$ and $|g\ker(\phi)| = |\ker(\phi)|$. Hence

$$\begin{aligned}|\ker(\phi)| \sum_{h \in \operatorname{im}(\phi)} T(g_h) &= \sum_{h \in \operatorname{im}(\phi)} |\phi^{-1}(\{h\})| T(g_h) \\ &= \sum_{h \in \operatorname{im}(\phi)} \sum_{\substack{g \in G \\ \phi(g) = h}} T(g) = \sum_{g \in G} T(g)\end{aligned}$$

□

2.1.1 High-Low Temperature Expansions for the Partiton Function

In what follows, we will derive the high and low temperature expansions of the partition function. In this section, we always assume as boundary conditions $Y = \varnothing$, so the spins outside the underlying finite box Λ are all $+1$. It is still worthy to note that the same calculations hold exactly the same when replacing $\mathscr{B}_\Lambda$ with the set of bonds strictly inside Λ, for which we will use the notation $Z_{\Lambda,\beta}^Y$ to remember this observation.

$$Z_{\Lambda,\beta}^Y = \sum_{X \subset \Lambda} \exp\left(\sum_{B \in \mathscr{B}_\Lambda} K_B \sigma_B(X)\right).$$

First, assume that $\mathscr{B}_\Lambda$ is finite and we note that for any $x \in \mathbb{R}$ one has $\exp(\pm x) = \cosh(x)(1 \pm \tanh(x))$. Now, for any configuration $X \subset \Lambda$ and $B \in \mathscr{B}_\Lambda$ one has either $\sigma_B(X) = 1$ or $\sigma_B(X) = -1$. In either case, by our last observation, we have $\exp(K(B)\sigma_B(X)) = \cosh(K(B))(1+\sigma_B(X)\tanh(K(B)))$. Hence, the partition function becomes

$$Z_{\Lambda,\beta}^Y = \sum_{X \in \mathscr{P}(\Lambda)} \prod_{B \in \mathscr{B}_\Lambda} \cosh(K(B))(1 + \sigma_B(X)\tanh(K(B)))$$

$$= \sum_{X\in\mathscr{P}(\Lambda)} \prod_{B\in\mathscr{B}_\Lambda} \cosh(K(B)) \prod_{B\in\mathscr{B}_\Lambda} (1+\sigma_B(X)\tanh(K(B))).$$

However, the last product may we written as $\prod_{B\in\mathscr{B}_\Lambda}(1+\sigma_B(X)\tanh(K(B))) = \sum_{(B_1,...,B_n)\subset\mathscr{B}_\Lambda} \prod_{i=1}^n \sigma_{B_i}(X)\tanh(K(B_i))$, with the summation taking all values of n, up to $|\mathscr{P}(\mathscr{B}_\Lambda)|$. After splitting the last sum over the $(B_1,...,B_n)$ such that $\prod_{i=1}^n B_i = \varnothing$ and $\prod_{i=1}^n B_i \neq \varnothing$, the partition function becomes

$$Z^Y_{\Lambda,\beta} = \sum_{X\in\mathscr{P}(\Lambda)} \prod_{B\in\mathscr{B}_\Lambda} \cosh(K(B)) \sum_{\substack{(B_1,...,B_n)\subset\mathscr{B}_\Lambda \\ \prod_{i=1}^n B_i=\varnothing}} \sigma_{(\prod_{i=1}^n B_i)}(X)\prod_{i=1}^n \tanh(K(B_i))$$

$$+ \prod_{B\in\mathscr{B}_\Lambda} \cosh(K(B)) \sum_{\substack{(B_1,...,B_n)\subset\mathscr{B}_\Lambda \\ \prod_{i=1}^n B_i\neq\varnothing}} \sum_{X\in\mathscr{P}(\Lambda)} \sigma_{(\prod_{i=1}^n B_i)}(X)\prod_{i=1}^n \tanh(K(B_i))$$

Now, given any non-empty $B\in\mathscr{P}(\mathscr{B}_\Lambda)$, we can write $B=\tilde{B}\cdot B'$ with $\tilde{B}\subset\Lambda$ and $B'\subset\Lambda^c$. Since $B'\cap X=\varnothing$ for all $X\subset\Lambda$, then $\sigma_X(B')=1$ for all such X and hence

$$\sum_{X\subset\Lambda}\sigma_X(B) = \sum_{X\subset\Lambda}\sigma_X(\tilde{B})\sigma_X(B') = \sum_{X\subset\Lambda}\sigma_X(\tilde{B}) = 0.$$

Where we used the fact that for $Y\neq\varnothing$ element of $\mathscr{P}(\Lambda)$ we have $\sum_{X\in\mathscr{P}(\Lambda)}\sigma_Y(X)=0$ by equality 2.5 with $\mathscr{L}=\Lambda$.

This means that the second term in the expansion of the partition function is zero, and we are left with

$$Z^Y_{\Lambda,\beta} = \sum_{X\in\mathscr{P}(\Lambda)} \prod_{B\in\mathscr{B}_\Lambda} \cosh(K(B)) \sum_{\substack{(B_1,...,B_n)\subset\mathscr{B}_\Lambda \\ \prod_{i=1}^n B_i=\varnothing}} \sigma_\varnothing(X)\prod_{i=1}^n \tanh(K(B_i))$$

Of course, for all configurations X we have $\sigma_\varnothing(X)=1$. Hence, all the depence of X in the partition function is gone, and the summation over $X\in\mathscr{P}(\Lambda)$ yields a multiplicative term of $2^{|\Lambda|}$. With the definition of the high-temperature subgroup $\mathscr{K}_\Lambda$, this yields

$$Z^Y_{\Lambda,\beta} = 2^{|\Lambda|}\prod_{B\in\mathscr{B}_\Lambda}\cosh(K(B))\sum_{\underline{B}\in\mathscr{K}_\Lambda}\prod_{B\in\underline{B}}\tanh(K(B)), \tag{2.9}$$

The expression in 2.9 is known as the high temperature expansion of the partition function. There is also a low temperature expansion, which is obtained by the following steps:

$$Z^Y_{\Lambda,\beta} = \sum_{X\subset\Lambda}\exp\left(\sum_{B\in\mathscr{B}_\Lambda}K(B)\sigma_B(X)\right) = \sum_{X\subset\Lambda}\prod_{B\in\mathscr{B}_\Lambda}\exp\left(K(B)[\sigma_B(X)+1-1]\right)$$

$$= \sum_{X\subset\Lambda}\prod_{B\in\mathscr{B}_\Lambda}\exp\left(K(B)\right)\prod_{B\in\mathscr{B}_\Lambda}\exp(K(B)[\sigma_B(X)-1])$$

$$= \prod_{B\in\mathscr{B}_\Lambda}\exp\left(K(B)\right)\sum_{X\subset\Lambda}\prod_{B\in\mathscr{B}_\Lambda}\exp\left(-2K(B)\frac{1-\sigma_B(X)}{2}\right)$$

$$= \prod_{B\in\mathscr{B}_\Lambda}\exp\left(K(B)\right)\sum_{X\subset\Lambda}\prod_{B\in\gamma|_{\mathscr{P}(\Lambda)}(X)}\exp\left(-2K(B)\right).$$

Now, the mappings $\mathscr{P}(\Lambda)\ni X\mapsto\gamma|_{\mathscr{P}_\Lambda}(X)$ and $T:\mathscr{P}(\Lambda)\to\mathbb{R}$ given by

$$T(X) = \prod_{B \in \gamma|_{\mathscr{P}_\Lambda}(X)} \exp(-2K(B))$$

obviously satisfy $T(X) = T(Y)$ whenever $\gamma|_{\mathscr{P}_\Lambda}(X) = \gamma|_{\mathscr{P}_\Lambda}(Y)$. Hence, we may apply lemma 2.7 to get

$$Z^Y_{\Lambda,\beta} = 2^{N_S} \prod_{B \in \mathscr{B}_\Lambda} \exp\left(K(B)\right) \sum_{\underline{B} \in \Gamma_\Lambda} \prod_{B \in \underline{B}} \exp(-2K(B)) \tag{2.10}$$

Which is the aforementioned low temperature expansion. Note that the sum on the right-hand side has finitely many terms even in the case where $\mathscr{B}_\Lambda$ is not finite.

Remember that we deduced these equations in the special case where $\mathscr{B}_\Lambda$ is finite. If the set is infinite, we pick a sequence $\mathscr{B}_n$ increasing to $\mathscr{B}_\Lambda$ where each $\mathscr{B}_n$ is finite. Starting from the standard expression for the partition function, for the high temperature expansion case we have

$$\begin{aligned} Z^Y_{\Lambda,\beta} &= \lim_{n\to\infty} \sum_{X \subset \Lambda} \left(\sum_{B \in \mathscr{B}_n} K_B \sigma_B(X) \right) \\ &= 2^{|\Lambda|} \lim_{n\to\infty} \prod_{B \in \mathscr{B}_n} \cosh(K(B)) \sum_{\underline{B} \in \mathscr{K}_n} \prod_{B \in \underline{B}} \tanh(K(B)). \end{aligned}$$

To pass the limit inside the terms, we need to prove that each individual term converges. The next lemmas ensure the convergence of each term, including the ones in the low temperature expansion.

Lemma 2.8. *Let $(a_n)_{n\geq 1}$ be a sequence of non-negative real numbers. Then $\prod_{n=1}^\infty (1+a_n)$ converges if, and only if $\sum_{n=1}^\infty a_n$ converges. Moreover, if $\sum_{n=1}^\infty a_n$ converges to a non-zero real number, then so does $\prod_{n=1}^\infty a_n$.*

Proof. Let N be any natural number. By noticing that the expansion of $(1+a_1)(1+a_2)...(1+a_N)$ equals $\sum_{S \subset [N]} \prod_{s \in S} a_s$, where $[N] \overset{\text{def}}{=} \{1, 2, ..., N\}$, then we restrict the sum over all $S \subset [N]$ by only those S with $|S| = 1$. Since all a_n are non-negative, this restriction can only lower the initial sum, from where we have

$$\sum_{n=1}^N a_n \leq \prod_{n=1}^N (1 + a_n).$$

Using $e^x \geq 1 + x$ we get

$$\prod_{n=1}^N (1 + a_n) \leq \prod_{n=1}^N e^{a_n} = e^{\sum_{n=1}^N a_n},$$

from where

$$\sum_{n=1}^N a_n \leq \prod_{n=1}^N (1 + a_n) \leq e^{\sum_{n=1}^N a_n}.$$

□

Lemma 2.9. *Let z_n be a sequence of complex numbers such that $\sum_{n\geq 1} |z_n| < \infty$. Then:*

$$\prod_{n=1}^\infty \cosh(z_n) < \infty;$$

$$\sum_{S \Subset \mathbb{N}} \prod_{s \in S} \tanh(z_s) < \infty;$$

$$\prod_{n=1}^{\infty} \exp(z_n) < \infty,$$

and every term above is a non-zero complex number.

Proof. We will use the following result: a product $\prod_{n\geq 1}(1+z_n)$ converges to a non-zero complex number if, and only if $\prod_{n\geq 1}(1+|z_n|)$ is finite. Using the expression $\cosh(z) = 1 + 2\sinh^2\left(\frac{z}{2}\right)$, the first expression will converge if, and only if

$$\sum_{n=1}^{\infty} \left|\sinh\left(\frac{z_n}{2}\right)\right|^2 < \infty.$$

Since $|\sinh(z)|^2 = \sinh^2(x)\cos^2(y) + \cosh^2(x)\sin^2(y)$ for $z = x + iy$, we only need to prove that the series:

1. $\sum_{n\geq 1} \sinh^2(x_n)\cos^2(y_n)$
2. $\sum_{n\geq 1} \cosh^2(x_n)\sin^2(y_n)$

are convergent, where $z_n = x_n + iy_n$. For the first series, we bound $\cos^2(y_n) \leq 1$ and note that $\lim_{n\to\infty} \frac{1}{2}\frac{\sinh^2(x_n)}{x_n^2} = \frac{1}{2} \leq 1$ (since $x_n \to 0$ as its series converges). Therefore, we have $\sinh^2(x_n) \leq 2x_n^2$ for all big enough $n \in \mathbb{N}$. Moreover, since $\sum_{n\geq 1}|x_n|$ converges, then also converges the series $\sum_{n\geq 1} x_n^2$, from where series (1) is convergent.

The second series also converges, since $\cosh^2(x_n) \to 1$ and hence $\cosh^2(x_n) \leq 2$ for all big enough $n \in \mathbb{N}$ and since $\sin^2(y_n) \leq y_n^2$. This proves that the first product in convergent and non-zero.

For the second term, it is convergent to an explicit term:

$$\prod_{n=1}^{\infty}(1+\tanh(z_n)) = \prod_{n=1}^{\infty} \frac{e^{z_n}}{\cosh(z_n)} = \frac{e^{\sum_{n=1}^{\infty} z_n}}{\prod_{n=1}^{\infty}\cosh(z_n)},$$

where we used the identity $1 + \tanh(z) = \frac{e^z}{\cosh(z)}$. Note that the final expression converges to a non-zero complex number, since $e^z \neq 0$ for all $z \in \mathbb{C}$.

The product $\prod_{n=1}^{\infty} \exp(z_n) = \exp\left(\sum_{n=1}^{\infty} z_n\right)$ also converges since $\sum_{n=1}^{\infty} z_n < \infty$.

□

Corollary 2.10. *The partition function is non-zero.*

Corollary 2.11. *The partition function in any finite box Λ can be written in the following forms:*

$$Z_{\Lambda,\beta}^{Y} = 2^{|\Lambda|} \prod_{B\in\mathscr{B}_\Lambda} \cosh(K(B)) \sum_{\underline{B}\in\mathscr{K}_\Lambda} \prod_{B\in\underline{B}} \tanh(K(B))$$

$$Z_{\Lambda,\beta}^{Y} = 2^{N_S} \prod_{B\in\mathscr{B}_\Lambda} \exp\left(K(B)\right) \sum_{\underline{B}\in\Gamma_\Lambda} \prod_{B\in\underline{B}} \exp(-2K(B))$$

2.2 Duality Relation

By looking at the partition function expressions of corollary 2.11, a relation between the high-temperature and low-temperature expressions can be established if a map between Γ_Λ and $\mathscr{K}_\Lambda$ is found. To be explicit, We will consider two lattice systems $(\Lambda, J, \mathscr{B})$ and $(\Lambda^*, J^*, \mathscr{B}^*)$. The set of bonds of (Λ^*, J^*) being denoted by $\mathscr{B}^*$, suppose we are given a transformation between the bonds by a map

$$d : \mathscr{B} \to \mathscr{B}^*$$
$$B \mapsto B^*.$$

This yields a map between sets of bonds,

$$D : \mathscr{P}(\mathscr{B}) \to \mathscr{P}(\mathscr{B}^*)$$
$$\underline{B} = (B_1, ..., B_n) \mapsto \underline{B}^* = (B_1^*, ..., B_n^*).$$

We can then ask that D maps bijectively $\mathscr{K}_\Lambda$ to Γ_Λ. Our condition of choice to ensure $D|_{\mathscr{K}}$ is an injective homomorphism is the following lemma.

Lemma 2.12. *Let $\mathscr{G}$ be any subgroup of $\mathscr{P}(\mathscr{B})$. If there exists a generating set $\{\underline{B}_1, ..., \underline{B}_n\}$ of $\mathscr{G}$ such that*

$$\underline{B}_j = \{B \in \mathscr{B} : B^* \in \underline{B}_j^*\}$$

then $D|_{\mathscr{G}}$ is an injective group homomorphism.

Proof. For any product $\underline{B} = \prod_{i=1}^n \underline{B}_i$ of generators, one may write

$$\underline{B} = \prod_{i=1}^n \{B : B^* \in \underline{B}_i^*\} = \{B : B^* \in \underline{B}_1^*\} \cdots \{B : B^* \in \underline{B}_n^*\}$$

In general, the n-fold symmetric difference is given by the collection of all points belonging to an odd number of sets in the product. Hence, the only surviving terms in the product above are those B intersecting an odd number of sets in the product above, i.e, the B's such that $B^* \in \underline{B}_{n_1}^* \cap ... \cap \underline{B}_{n_B}^*$ with n_B and odd number. Therefore:

$$D(\underline{B}) = D(\{B : B^* \in \underline{B}_{n_1}^* \cap ... \cap \underline{B}_{n_B}^*, (-1)^{n_B} = -1\})$$
$$= \{B^* : B^* \in \underline{B}_{n_1}^* \cap ... \cap \underline{B}_{n_B}^*, (-1)^{n_B} = -1\} = \prod_{i=1}^n \underline{B}_i^* = \prod_{i=1}^n D(\underline{B}_i)$$
$$\implies D\left(\prod_{i=1}^n \underline{B}_i\right) = \prod_{i=1}^n D(\underline{B}_i)$$

The property holds for an arbitrary product of general elements of $\mathscr{G}$ as well, since these are given by products of the generators. This proves that $D|_{\mathscr{G}}$ is an homomorphism. Note that for any $\underline{B} \in \mathscr{G}$ such that $D(\underline{B}) = \varnothing$ then $\underline{B}$ must be empty, since otherwise we may write $\underline{B} = \{B_1, ..., B_n\}$ so that

$$\varnothing = \{B_1^*, ..., B_n^*\}$$

A contradiction. Hence the kernel of $D|_{\mathscr{G}}$ is trivial and it is injective. □

Note that if d is surjective, then so is D: given $(B_1^*, ..., B_n^*) \in \mathscr{P}(\mathscr{B}^*)$, then pick $A_1, ..., A_n \in \mathscr{B}$ such that $A_1^* = B_1^*$, ..., $A_n^* = B_n^*$ and therefore $D(\{A_1, ..., A_n\}) = \{B_1^*, ..., B_n^*\}$. Moreover, if d is injective, then the condition of the lemma above (for $\mathscr{G} = \mathscr{P}(\mathscr{B})$) is satisfied: taking as generators the whole $\mathscr{P}(\mathscr{B})$ and given arbitrary $\underline{B} = \{C_1, ..., C_n\} \in \mathscr{P}(\mathscr{B})$ and $B \in \mathscr{B}$ such that $B^* \in \underline{B}^*$, then $d(B) \in \underline{B}^*$, i.e, $d(B) = d(C_j)$, for some $C_j \in \underline{B}$. By injectivity, $B = C_j \in \underline{B}$ and hence

$$\underline{B} \supset \{B : B^* \in \underline{B}^*\}$$

Of course, for every $C_i \in \underline{B}$ one has $C_i^* \in \underline{B}^*$ so the other inclusion holds and equality is achieved. Therefore

Corollary 2.13. *If $d : \mathscr{B} \to \mathscr{B}^*$ is bijective, then the map $D : \mathscr{P}(\mathscr{B}) \to \mathscr{P}(\mathscr{B}^*)$ is a group isomorphism.*

Note, however, that asking for d to be bijective is not necessary to ensure that $D|_{\mathscr{K}}$ to be a group isomorphism. We can ask instead the weaker conditions of surjectivity for d and that D maps $\mathscr{K}_\Lambda$ to Γ^*_Λ while satisfying the conditions of the above lemma. This ensures that $D|_{\mathscr{K}_\Lambda} \to \Gamma^*_\Lambda$ is an isomorphism, as we wanted. Therefore, we arrive at the following definition of duality:

Definition 2.14. *Let (Λ, K) and (Λ^*, K^*) be finite lattice systems and consider the maps $d : \mathscr{B} \to \mathscr{B}^*$ and $D : \mathscr{P}(\mathscr{B}) \to \mathscr{P}(\mathscr{B}^*)$ as before. If the following conditions are satisfied:*

1. *The map d is surjective and the conditions of lemma 2.12 are satisfied for $\mathscr{K}$;*
2. *$D(\mathscr{K}_\Lambda) = \Gamma^*_\Lambda$;*
3. *The interactions are related by:*

$$\exp(-2K^*(B^*)) = \prod_{B \in d^{-1}(B^*)} \tanh(K(B))$$

Then (Λ^*, K^*) is called a dual lattice system for (Λ, K).

Note that the dual set of bonds $\mathscr{B}^*$ is not specified. There is, however, a general way of constructing dual lattice systems for our ferromagnetic case, and in this construction the dual bonds will be specified naturally. The type of duality specified in the definition above is called "HT-LT" duality, which is short for high temperature - low temperature duality. There are other types, like LT-HT, HT-HT and LT-LT duality. The difference between these definitions is the relation of $\mathscr{K}_\Lambda$ and Γ_Λ and the map D, so for example for LT-HT duality we have $D(\Gamma_\Lambda) = \mathscr{K}^*_\Lambda$ and for HT-HT duality we have $D(\mathscr{K}_\Lambda) = \mathscr{K}^*_\Lambda$.

Let us suppose that d is bijective and extract the first consequence of the duality relations above. Starting from the low temperature partition function, we have

$$\begin{aligned}
Z^+_{\Lambda,\beta} &= 2^{N^*_S} \prod_{B^* \in d(\mathscr{B}_\Lambda)} \exp\left(K^*(B^*)\right) \sum_{\underline{B}^* \in \Gamma^*_\Lambda} \prod_{B^* \in \underline{B}^*} \exp(-2K^*(B^*)) \\
&= 2^{N^*_S} \prod_{B^* \in d(\mathscr{B}_\Lambda)} \exp\left(-2K^*(B^*)\right)^{-\frac{1}{2}} \sum_{\underline{B}^* \in \Gamma^*_\Lambda} \prod_{B^* \in \underline{B}^*} \tanh K(d^{-1}(B^*)) \\
&= 2^{N^*_S} \prod_{B^* \in d(\mathscr{B}_\Lambda)} \sqrt{\frac{\cosh K(d^{-1}(B^*))}{\sinh K(d^{-1}(B^*))}} \sum_{\underline{B} \in \mathscr{K}_\Lambda} \prod_{B \in \underline{B}} \tanh K(B) \\
&= 2^{N^*_S} \prod_{B \in \mathscr{B}_\Lambda} \sqrt{\frac{\cosh K(B)}{\sinh K(B)}} \sum_{\underline{B} \in \mathscr{K}_\Lambda} \prod_{B \in \underline{B}} \tanh K(B).
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{Z_{\Lambda,\beta}}{Z_{\Lambda^*,\beta}} &= 2^{|\Lambda| - N^*_S} \prod_{B \in \mathscr{B}_\Lambda} \sqrt{\frac{\sinh K(B)}{\cosh K(B)}} \cosh K(B) \\
&= 2^{|\Lambda| - N^*_S - |\mathscr{B}_\Lambda|} \prod_{B \in \mathscr{B}_\Lambda} \sqrt{2 \sinh K(B) \cosh K(B)} \\
&= 2^{|\Lambda| - N^*_S - |\mathscr{B}_\Lambda|} \prod_{B \in \mathscr{B}_\Lambda} \sqrt{\sinh 2K(B)}, \qquad (2.11)
\end{aligned}$$

which is the duality relation for the partition function.

Lemma 2.15. (Λ^*, K^*) *is a dual lattice for* (Λ, K) *if the following conditions hold:*

1. *The map* $d : \mathscr{B} \to \mathscr{B}^*$ *is surjective, the conditions of lemma 2.12 are satisfied and the image of a set of generators for* $\mathscr{K}_\Lambda$ *by* D *is a set of generators of* Γ^*_Λ*;*

2. *The interactions are related by:*

$$\exp(-2K^*(B^*)) = \prod_{B \in d^{-1}(B^*)} \tanh(K(B))$$

Proof. We only need to prove $D(\mathscr{K}_\Lambda) = \Gamma^*_\Lambda$. Since the conditions of lemma 2.12 are satisfied for $\mathscr{K}_\Lambda$, then we already know that D is a homomorphism. Fix a set of generators $\{\underline{B}_1, ..., \underline{B}_n\}$ of $\mathscr{K}_\Lambda$ such that $\{\underline{B}^*_1, ..., \underline{B}^*_n\}$ generates Γ^*_Λ. Thus, any $\underline{B} \in \mathscr{K}_\Lambda$ can be written as a product:

$$\underline{B} = \prod_{i=1}^{k} \underline{B}_{n_i}$$

and we have $D(\underline{B}) = D\left(\prod_{i=1}^k \underline{B}_{n_i}\right) = \prod_{i=1}^k \underline{B}^*_{n_i} \in \Gamma^*_\Lambda$, since Γ^*_Λ is a subgroup. Moreover, given any $\underline{B}^* \in \Gamma^*_\Lambda$ we have $\underline{B}^* = \prod_{i=1}^k \underline{B}^*_{n_i} = \prod_{i=1}^k D(\underline{B}_{n_i}) = D\left(\prod_{i=1}^k \underline{B}_{n_i}\right) \in D(\mathscr{K}_\Lambda)$, since $\mathscr{K}_\Lambda$ is a subgroup. This finishes the proof.

□

To end the section, we shall prove the following result:

Lemma 2.16. *Let* $\{\Lambda, \mathscr{B}_\Lambda, K\}$ *be any finite lattice system. Then, for any HT-LT dual* $\{\Lambda^*, \mathscr{B}^*{}_{\Lambda^*}, K^*\}$ *we have*

$$\langle \sigma_{\underline{B}} \rangle_{\Lambda, \mathscr{B}_\Lambda, K} = \langle \mu_{\underline{B}^*} \rangle_{\Lambda^*, \mathscr{B}^*{}_{\Lambda^*}, K^*},$$

where $\mu_{\underline{B}} = \prod_{B \in \underline{B}} \exp(-2K(B)\sigma_B)$.

Proof. First, write $\sigma_B = -ie^{i\frac{\pi}{2}\sigma_B}$ and rewrite the expected value as

$$\langle \sigma_{\underline{B}} \rangle_{\{\Lambda, K\}} = Z^{-1}_{\{\Lambda, K\}} \sum_{X \subset \Lambda} \sigma_{\underline{B}}(X) e^{\sum_{B \in \mathscr{B}_\Lambda} K(B)\sigma_B(X)}$$

$$= Z^{-1}_{\{\Lambda, K\}} \sum_{X \subset \Lambda} \left(\prod_{B \in \underline{B}} (-i) e^{i\frac{\pi}{2}\sigma_B(X)} \right) e^{\sum_{B \in \mathscr{B}_\Lambda} K(B)\sigma_B(X)}$$

$$= (-i)^{|\underline{B}|} Z^{-1}_{\{\Lambda, K\}} \sum_{X \subset \Lambda} e^{\sum_{B \in \underline{B}} i\frac{\pi}{2}\sigma_B(X)} e^{\sum_{B \in \mathscr{B}_\Lambda} K(B)\sigma_B(X)}$$

$$= (-i)^{|\underline{B}|} Z^{-1}_{\{\Lambda, K\}} \sum_{X \subset \Lambda} e^{\sum_{B \in \mathscr{B}_\Lambda} \tilde{K}(B)\sigma_B(X)},$$

where we define the new interaction $\tilde{K}$ by

$$\tilde{K}(B) = \begin{cases} K(B), \text{ if } B \notin \underline{B}; \\ K(B) + i\frac{\pi}{2}, \text{ if } B \in \underline{B}. \end{cases}$$

Therefore, with respect to the finite lattice system $(\Lambda, \tilde{K})$, we have

$$\langle \sigma_{\underline{B}} \rangle_{\{\Lambda, K\}} = (-i)^{|\underline{B}|} \frac{Z_{\{\Lambda, \tilde{K}\}}}{Z_{\{\Lambda, K\}}} = (-i)^{|\underline{B}|} \frac{Z_{\{\Lambda, \tilde{K}\}}}{Z_{\{\Lambda^*, \tilde{K}^*\}}} \frac{Z_{\{\Lambda^*, \tilde{K}^*\}}}{Z_{\{\Lambda^*, K^*\}}} \frac{Z_{\{\Lambda^*, K^*\}}}{Z_{\{\Lambda, K\}}} \tag{2.12}$$

The first and last terms can be computed by the duality relations of the partiton function (see equation 2.11), and they yield

$$\frac{Z_{\{\Lambda,K\}}}{Z_{\{\Lambda^*,K^*\}}} = C \prod_{B\in\mathscr{B}_\Lambda} \sqrt{\sinh 2K(B)}$$

$$\frac{Z_{\{\Lambda,\tilde{K}\}}}{Z_{\{\Lambda^*,\tilde{K}^*\}}} = C \prod_{B\in\mathscr{B}_\Lambda} \sqrt{\sinh 2\tilde{K}(B)},$$

where $C = 2^{|\Lambda|-N_S^*-|\mathscr{B}_\Lambda|}$ is a constant depending only on $|\Lambda|, N_S^*$ and $\mathscr{B}_\Lambda$ and hence is indeed equal for both terms. Therefore, we have

$$\frac{Z_{\{\Lambda,\tilde{K}\}}}{Z_{\{\Lambda^*,\tilde{K}^*\}}}\frac{Z_{\{\Lambda^*,K^*\}}}{Z_{\{\Lambda,K\}}} = \prod_{B\in\mathscr{B}} \sqrt{\frac{\sinh 2\tilde{K}(B)}{\sinh 2K(B)}} = i^{|\underline{B}|}, \tag{2.13}$$

and we noted that for every $B \in \mathscr{B}_\Lambda$ such that $B \notin \underline{B}$, we have $\tilde{K}(B) = K(B)$ and hence the corresponding term in (2.13) equals 1. If otherwise, then $\sinh(2\tilde{K}(B)) = \sinh(2K(B) + i\pi) = -\sinh 2K(B)$ and hence the corresponding term is i, so overall the value of (2.13) is in fact $i^{\underline{B}}$. As for the middle term in (2.12), we use the duality relations between the interactions

$$e^{-2K^*(B^*)} = \tanh K(B)$$

$$e^{-2\tilde{K}^*(B^*)} = \tanh(\tilde{K}(B))$$

$$\implies \tilde{K}^*(B^*) - K^*(B^*) = -\frac{1}{2}\log\tanh\tilde{K}(B) + \frac{1}{2}\log\tanh K(B) = \frac{1}{2}\log\frac{\tanh K(B)}{\tanh\tilde{K}(B)}.$$

By using the relation $\tanh(x - i\frac{\pi}{2}) = \coth(x)$ and defining $\varphi(B) \stackrel{\text{def}}{=} \log\tanh K(B)$, we get[3] $\tilde{K}^*(B^*) - K^*(B^*) = \varphi(B)\delta_{B\in\underline{B}}$, so that $e^{\tilde{K}^*(B^*)\sigma_{B^*}(\cdot)} = e^{K^*(B^*)\sigma_{B^*}(\cdot)}e^{\varphi(B)\sigma_{B^*}(\cdot)\delta_{B\in\underline{B}}}$ and the middle term in (2.12) is

$$\frac{Z_{\{\Lambda^*,\tilde{K}^*\}}}{Z_{\{\Lambda^*,K^*\}}} = Z^{-1}_{\{\Lambda^*,K^*\}} \sum_{X^*\subset\Lambda^*} e^{\sum_{B^*\in\mathscr{B}^*_{\Lambda^*}} \tilde{K}^*(B^*)\sigma_{B^*}(X^*)}$$

$$= Z^{-1}_{\{\Lambda^*,K^*\}} \sum_{X^*\subset\Lambda^*} \prod_{B^*\in\mathscr{B}^*_{\Lambda^*}} e^{\tilde{K}^*(B^*)\sigma_{B^*}(X^*)} = Z^{-1}_{\{\Lambda^*,K^*\}} \sum_{X^*\subset\Lambda^*} \prod_{B^*\in\mathscr{B}^*_{\Lambda^*}} e^{K^*(B^*)\sigma_{B^*}(X^*)} e^{\varphi(B)\delta_{B\in\underline{B}}\sigma_{B^*}(X^*)}$$

$$= Z^{-1}_{\{\Lambda^*,K^*\}} \sum_{X^*\subset\Lambda^*} \left(\prod_{B^*\in\mathscr{B}^*_{\Lambda^*}} e^{K^*(B^*)\sigma_{B^*}(X^*)}\right)\left(\prod_{B^*\in\mathscr{B}^*_{\Lambda^*}} e^{\varphi(B)\sigma_{B^*}(X^*)\delta_{B\in\underline{B}}}\right)$$

$$= Z^{-1}_{\{\Lambda^*,K^*\}} \sum_{X^*\subset\Lambda^*} \left(\prod_{B\in\underline{B}} e^{\varphi(B)\sigma_{B^*}(X^*)}\right)\left(\prod_{B^*\in\mathscr{B}^*_{\Lambda^*}} e^{K^*(B^*)\sigma_{B^*}(X^*)}\right)$$

After using again the duality relations between the interactions, we note that $\varphi(B) = -2K^*(B^*)$. Substituting this in the above, we get:

$$\frac{Z_{\{\Lambda^*,\tilde{K}^*\}}}{Z_{\{\Lambda^*,K^*\}}} = Z^{-1}_{\{\Lambda^*,K^*\}} \sum_{X^*\subset\Lambda^*} \left(\prod_{B^*\in\underline{B}^*} e^{-2K^*(B^*)\sigma_{B^*}(X^*)}\right)\left(\prod_{B^*\in\mathscr{B}^*} e^{-K^*(B^*)\sigma_{B^*}(X^*)}\right)$$

[3] If $B \in \underline{B}$, then $\tilde{K}(B) = K(B) - i\frac{\pi}{2}$, so that $\tanh(\tilde{K}(B)) = \tanh(K(B) - i\frac{\pi}{2}) = \frac{1}{\tanh(K(B))}$ and hence $\frac{1}{2}\log\frac{\tanh K(B)}{\tanh\tilde{K}(B)} = \log\tanh(K(B)) = \varphi(B)$. If $B \notin \underline{B}$, then $\tilde{K}(B) = K(B)$ and the logarithm term yields zero.

$$= \left\langle \prod_{B^*\in \underline{B}^*} e^{-2K^*(B^*)\sigma_{B^*}(\cdot)} \right\rangle_{\{\Lambda^*,K^*\}} \tag{2.14}$$

Plugging equations (2.13) and (2.14) into (2.12), we have, finally,

$$\langle \sigma_{\underline{B}} \rangle_{\{\Lambda,K\}} = \left\langle \prod_{B^*\in \underline{B}^*} e^{-2K^*(B^*)\sigma_{B^*}(\cdot)} \right\rangle_{\{\Lambda^*,K^*\}}$$

□

It is not hard to see that the same result holds if we replace $\mathscr{B}_\Lambda$ with $\mathscr{B}^f_\Lambda$, and for LT-HT duality we get $\langle \mu_{\underline{B}} \rangle_{\Lambda,\mathscr{B}_\Lambda,K} = \langle \sigma_{\underline{B}^*} \rangle_{\Lambda^*,\mathscr{B}^*{}_{\Lambda^*},K^*}$.

2.2.1 Construction of Dual Lattices

For the rest of this section, $\mathscr{K}_\Lambda \subset \mathscr{P}(\mathscr{B})$ will denote the subgroup of all $\underline{B} = \{B_1, ..., B_n\}$ contained in Λ with product being the identity. For the rest of the thesis, $\mathscr{B}^f_\Lambda$ will denote the collection of all bonds contained in Λ.

Suppose that we found a generator $\mathscr{K}_0 \subset \mathscr{K}_\Lambda$ of $\mathscr{K}_\Lambda$ and denote its elements by $\underline{B}_1, ..., \underline{B}_n$. To each $\underline{B}_i$ we define a point $r^*_{\underline{B}_i}$ (usually the barycenter of $\bigcup_{B\in\underline{B}_i} B$) and we define Λ^* as the collection of all points of this form.

For any given bond $B \in \mathscr{B}$, we define the dual bond B^* as

$$B^* \stackrel{\text{def}}{=} \{r^*_{\underline{B}_i} : \underline{B}_i \ni B\},$$

and we define $\mathscr{B}^*$ as the collection of all subsets of this form. This defines a map $d : \mathscr{B} \to \mathscr{B}^*$. With these definitions, we have

Proposition 2.17. *The lattice system* $(\Lambda^*, \mathscr{B}^*)$ *constructed above is a dual lattice system for* $(\Lambda, \mathscr{B})$.

Proof. First we will show that $\underline{B}_i = D^{-1}(D(\underline{B}_i))$, and this will imply that $D|_{\mathscr{K}}$ is an injective group homomorphism. To fix notations, set $\underline{B}_i = (B_1, ..., B_n)$ and we shall prove that $B^* \in \underline{B}^*_i$ if, and only if $r^*_{\underline{B}_i} \in B^*$.

First, suppose that $B^* \in \underline{B}^*_i$, so that there is some k satisfying $B^* = B^*_k$. Since $B_k \in \underline{B}_i$, then:

$$r^*_{\underline{B}_i} \in \{r^*_{\underline{B}_j} : B_k \in \underline{B}_j\} = B^*_k = B^*.$$

Now, suppose that $r^*_{\underline{B}_i} \in B^*$. Then, since $\{r^*_{\underline{B}_j} : B \in \underline{B}_j\} = B^*$, this implies $B \in \underline{B}_i$ and therefore $B^* \in \underline{B}^*_i$, as we wanted. This show the equivalence we wanted. Note that in particular this implies the following useful equality:

$$\underline{B}^*_i = \{B^* : B^* \in \underline{B}^*_i\} = \{B^* : r^*_{\underline{B}_i} \in B^*\} \tag{2.15}$$

By gluing together our argument thus far, we now have a sequence of implications:

$$B^* \in \underline{B}^*_i \implies r^*_{\underline{B}_i} \in B^* \implies B \in \underline{B}_i.$$

Since the other side of the implications is trivially true, $B^* \in \underline{B}^*_i$ if, and only if $B \in \underline{B}_i$, which is the same as saying $\underline{B}^*_i = \underline{B}^*$ implies $\underline{B}_i = \underline{B}$ or $D^{-1}(\underline{B}^*_i) = \underline{B}_i$, i.e, $D^{-1}(D(\underline{B}_i)) = \underline{B}_i$.

Now, note that the family of all $\{x\}$ with $x \in \Lambda$ generates $\mathscr{P}(\Lambda)$. Since γ is a homomorphism, then the image of this family by γ generates Γ_Λ, so the sets $\gamma(\{x\})$ generate $\Gamma_\Lambda = \gamma(\mathscr{P}(\Lambda))$. However, we note the following:

$$\gamma(\{x\}) = \{B : \sigma_B(\{x\}) = -1\} = \{B : B \ni x\}.$$

Applying this result to the dual lattice we built, we get that:

$$D(\underline{B}_i) = \{B^* : r^*_{\underline{B}_i} \in B^*\} = \gamma^*(r^*_{\underline{B}_i}),$$

where we have used equation 2.15. Hence, D maps a generator of $\mathscr{K}_\Lambda$ into a generator of $\Gamma^*_{\Lambda^*}$. This proves the proposition.

□

This proves dual systems always exist. We will now provide explicit examples for a few models. The main strategy is the following: by the first isomorphism theorem for groups, we have[4]

$$\mathscr{P}(\mathscr{B}^f_\Lambda)/\mathscr{K}_\Lambda = \pi(\mathscr{P}(\mathscr{B}^f_\Lambda)) = \overline{\mathscr{B}^f_\Lambda},$$

which implies $|\mathscr{K}_\Lambda| = 2^{|\mathscr{B}^f_\Lambda| - N_i}$. This yields a way of finding generating subsets of $\mathscr{K}_\Lambda$: we only need to find independent (in the sense of $\mathbb{Z}_2$-vector spaces) subsets of $\mathscr{K}_\Lambda$ with $|\mathscr{B}^f_\Lambda| - N_i$ elements. In general, these subsets are not too hard to find.

First, we will provide examples in which there are external fields present. This means that every singleton $\{x\}$ with $x \in \Lambda$ is a bond. In special, this implies that $\{x : x \in \Lambda\} \subset \mathscr{B}^f_\Lambda$, and hence $\mathscr{P}(\Lambda) = \overline{\{x : x \in \Lambda\}} \subset \overline{\mathscr{B}^f_\Lambda}$. Since the other inclusion holds trivially, we have $\overline{\mathscr{B}^f_\Lambda} = \mathscr{P}(\Lambda)$, which implies $N_i = |\Lambda|$, so that $\mathscr{K}_\Lambda$ is generated by $|\mathscr{B}^f_\Lambda| - |\Lambda|$ elements.

Considering the family of all $\underline{B} = \{B, x_1, ..., x_n\}$, where $B \in \mathscr{B}^f_\Lambda$ and $\{x_1, ..., x_n\} = B$ (this does not include the singleton sets of the form $\{x\}$ with $x \in \Lambda$), there are a total of $|\mathscr{B}^f_\Lambda|$ sets minus the singletons, which account for $|\Lambda|$, as wanted. Now, in general, an n-fold symmetric difference $\Delta^n_{i=1} A_i$ is empty if every $x \in \bigcup_{i=1}^{n} A_i$ is contained in an even number of the A_i. This is the case for the sets $\{B, x_1, ..., x_n\}$ above, since every $x \in B$ is contained only in a singleton set and in B itself, implying that $\underline{B} \in \mathscr{K}_\Lambda$. Moreover, the collection is also independent: by labeling $\underline{B} = \{B, x_1, ..., x_n\}$ with $B = \{x_1, ..., x_n\}$, then if

$$\prod_{\substack{B \in \mathscr{B}^f_\Lambda \\ |B| \geq 2}} \underline{B} = \varnothing$$

with every term in the product being distinct from the others implies that, for any $B' \in \mathscr{B}^f_\Lambda$ with $|B'| \geq 2$, we have

$$\underline{B'} = \prod_{\substack{B \in \mathscr{B}^f_\Lambda \\ |B| \geq 2 \\ B \neq B'}} \underline{B}.$$

In special, if $\underline{B'}$ is not empty then it must contain B' and hence B' must be in some $\underline{B}$ with $B \neq B'$. This is impossible, since $B' \in \underline{B}$ implies $B' = B$. Therefore, this proves that

$$\{\underline{B} = \{B, x_1, ..., x_n\} : B = \{x_1, ..., x_n\} \text{ and } |B| \geq 2\}$$

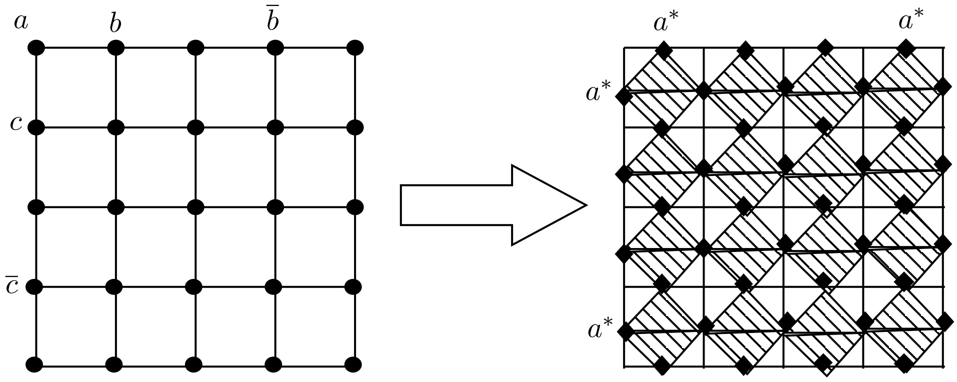
is independent and hence a generating set for $\mathscr{K}_\Lambda$. The map $d : \mathscr{B} \to \mathscr{B}^*$ then gives the following description: for any $B \in \mathscr{B}^f_\Lambda$, we have $B^* = \{r^*_{\underline{B}} : B \in \underline{B}\} = r^*_{\{B, x_1, ..., x_n\}}$, since the only $\underline{B}$ in the generating set containing B must the $\{B, x_1, ..., x_n\}$. In this way, we see that the dual model maps bonds B with $|B| \geq 2$ to an external field in the barycenter of B. In special, in this specific

[4] $\mathscr{K}_\Lambda$ is the kernel of π, and $\pi(\mathscr{P}(\mathscr{B}^f_\Lambda))$ is the group consisting of all products of elements of $\mathscr{P}(\mathscr{B}^f_\Lambda)$ by the definition of the map π, which coincides with the subgroup generated by $\mathscr{P}(\mathscr{B}^f_\Lambda)$.

construction we have done, dual models to models with external field always have external field themselves.

Moreovever, any singleton bond $\{x\}$ corresponding to an external field is mapped to $B^* = \{r^*_{\underline{B}} : \{x\} \in \underline{B}\}$. This is more easily seen in the following way: given some $x \in \Lambda$, the corresponding dual interaction is built by finding all $B \in \mathscr{B}$ containing this point. Then, the set of all the barycenters of these B define the dual interaction. We now turn to the examples.

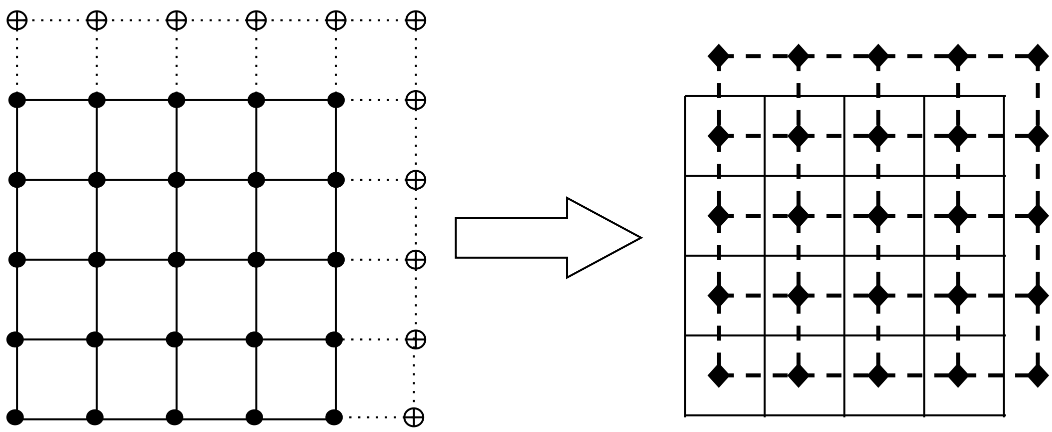
- Nearest neighbor Ising model with periodic boundary conditions and external field:



Here, we begin with the box on the left, where the black dots represent the vertices of the lattice. Since the interactions respect a periodic boundary condition, the vertex a interacts with $b, \overline{b}$, c and $\overline{c}$.

Passing through the arrow, the dual vertexes are the barycenters of the nearest neighbor interactions, which are represented in the lattice on the right by the black losangles. The interactions are given by the procedure described before and in this case are four-body interactions. The interactions strictly inside the dual lattice happen only on the boundary of the non-shaded losangles (those with barycenter consisting of a "normal" vertex), and the vertexes on the boundary give rise to the four-body interactions described by the periodicity of the boundary condition. For example, the dual four-body interaction corresponding to the vertex a is the collection of all dual vertexes labeled a^* in the dual lattice. Note that the dual systems inherits the periodic boundary conditions.

- Nearest neighbors 4-body interaction Ising model with periodic boundary conditions and external field:



The dotted lines in the lattice represent layers of the periodic boundary conditions. The barycenters of each square representing an interaction define the dual lattice and the procedure described above also yiels a four-body interaction for the dual model. Since both the system and its dual have the same lattices and interaction type (up to a translation), we say that the system is self-dual.

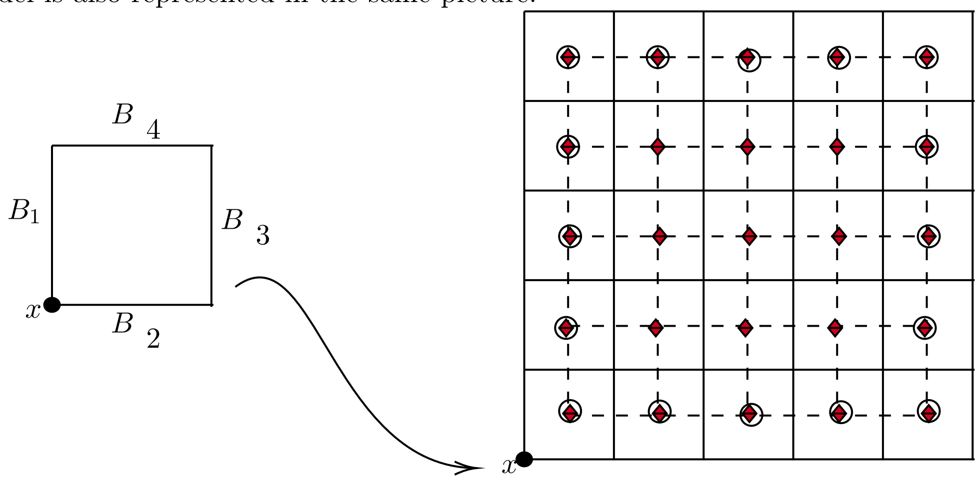
- Nearest neighbor Ising model with free boundary conditions and without external field:

The main difficulty in this example is the fact that $|\overline{\mathscr{B}}|$ is not equal to $2^{|\Lambda|}$ anymore, since there is no external field. Instead, we have to compute by more direct means the minimal number of generators of $\mathscr{K}_\Lambda$.

The new strategy is to use lemma 2.3 to get $N_S + N_i = |\Lambda|$. Since in the non-external field case there are exactly two elements in $\mathscr{S}_\Lambda$, being the $+$ and $-$ configurations everywhere, then $N_S = 1$ and hence $N_i = N^2 - 1 = (N-1)(N+1)$, where we put Λ as a square of side N. Moreover, it is not hard to see that there are $2N(N-1)$ bonds. Then, the number of minimal generators of $\mathscr{K}_\Lambda$ is

$$|\mathscr{B}| - N_i = 2N(N-1) - (N-1)(N+1) = (N-1)(2N - N - 1) = (N-1)^2.$$

It is very easy to see that every loop of bonds in this model is an element of $\mathscr{K}_\Lambda$. Thus, for each $x \in [N-1, N-1]^2 \cap \mathbb{Z}^2$ we can associate the set $\underline{x} = \{B_1, B_2, B_3, B_4\}$ defined below. The dual model is also represented in the same picture:



It is also not hard to see that the collection of all $\underline{x}$ is independent: if $\prod_{x\in[N-1]^2\cap\mathbb{Z}^2} \underline{x} = \varnothing$ then all the boundary $\underline{x}$ (those with x at the boundary) terms must be empty, otherwise we could take a bond $B \in \underline{x}$ which intersects only $\underline{x}$ and hence would be in the symmetric difference, yielding a contradiction. We now repeat the process as many times as needed, erasing the new boundary terms each time to get that all $\underline{x}$ must be empty.

As we can see from the general construction, the corresponding dual model is a nearest neighbor Ising model but with external fields only in the boundary, since it is at the boundary bonds that only one generator contains the given bond. In the picture, the losangles represent the dual vertices, and the losangles within the circles represent the dual vertices having an external field. The dual interactions are also of two-body type.

2.2.2 Duality in Infinite Systems

Instead of working with finite lattices, in this section we will define dual lattice systems for infinite systems.

To start, the main difference is that in infinite systems we do not specify the finite box, so it is just a pair $(\mathscr{B}, K)$ of a set of bonds and the interaction between them.

Remember that the main point of the duality in the finite case was a map $d : \mathscr{B} \to \mathscr{B}^*$ mapping the bonds onto the dual bonds such that $d(\mathscr{K}_\Lambda) = \Gamma^*_\Lambda$. To generalize, we will consider a map $d : \mathscr{B} \to \mathscr{B}^*$ such that $d(\mathscr{K}_f) = \Gamma^{(f)^*}$, where we define

$$\mathscr{K}_f \stackrel{\text{def}}{=} \ker(\pi|_{\mathscr{P}_f(\mathscr{B})});$$

$$\Gamma \stackrel{\text{def}}{=} \operatorname{im}(\gamma) \text{ and } \Gamma^{(f)} \stackrel{\text{def}}{=} \operatorname{im}(\gamma|_{\mathscr{P}_f(\mathbb{Z}^d)}).$$

These are the infinite volume counterparts of the already defined sets $\mathscr{K}_\Lambda$ and Γ_Λ. Consider the two conditions bellow:

1. $\Gamma^{(f)} = \Gamma \cap \mathscr{P}_f(\mathscr{B})$;
2. There exists a sequence of finite boxes $\Lambda_i \to \mathbb{Z}^d$ such that for all i and $|X| < \infty$ we have that $\sigma_B(X) = +1$ for all $B \in \mathscr{B} \cap \mathscr{P}(\Lambda_i^c)$ implies a decomposition $X = YS_f$ with $Y \subset \Lambda_i$ and $S_f \in \mathscr{S}$.

Note that condition (b) says that if a configuration X satisfies the definition of a ground state only for bonds outside the fixed box Λ_i, then the configuration can be decomposed as a product of a configuration inside the box and a ground state. As for condition (a), it says that $|\gamma(X)| < \infty$ is equivalent to the existence of some finite X' (not necessarily equal to X) such that $\gamma(X) = \gamma(X')$.

It is not hard to see that the construction made for the finite systems still works for infinite systems, if the condition $\Gamma^{(f)} = \Gamma \cap \mathscr{P}_f(\mathscr{B})$ is respected. One only needs to find a generating subset $\mathscr{K}_0 \subset \mathscr{K}_f$ and repeat the procedure. The main difference is that the methods we used to prove that the subsets $\mathscr{K}_0$ generate $\mathscr{K}_\Lambda$ in the finite case only work in finite lattices, so in the infinite case we need to show this fact directly. Usually, one needs to know what the elements of $\mathscr{K}_f$ look like to then find a suitable generator.

Now, even though dual systems satisfy $d(\mathscr{K}_f) = \Gamma^* \cap \mathscr{P}_f(\mathscr{B}^*) = \Gamma^{(f)^*}$ by definition, condition (a) also implies the converse (see [GHMMS77], chapter 4), that is, $d(\Gamma^{(f)}) = \mathscr{K}_f^*$.

Using both conditions (a) and (b) above, one can transfer the duality in an infinite system to duality of a growing sequence of finite systems. In this construction, we build the dual systems by collecting all the sites covered by the $\mathscr{B}_{\Lambda_i}$ into a new lattice Λ_i^*, i.e, we set $\Lambda_i^* := \bigcup_{B \in \mathscr{B}_{\Lambda_i}} B$. Note that Λ_i^* will only be finite in the case where the interactions are short-range. This effectivily transfers the " + " boundary condition into free boundary conditions in the dual system, as the next lemma shows:

Lemma 2.18. *Let $\{\mathbb{Z}^d, \mathscr{B}, K\}$ be a ferromagnetic system satisfying the conditions (a) and (b) above. Then for any dual system $\{\mathbb{Z}^d, \mathscr{B}^*, K^*\}$ we have:*

- $(\langle\cdot\rangle_+)^* = \langle\cdot\rangle_f^*$;
- *If the dual system satisfies the same hypothesis, then* $(\langle\cdot\rangle_f)^* = \langle\cdot\rangle_+^*$

where, for any Gibbs measure $\langle\cdot\rangle$ in $\{\mathbb{Z}^d, \mathscr{B}, K\}$, we define $(\langle\sigma_{\underline{B}^*}\rangle)^* = \langle\mu_{\underline{B}}\rangle$ and $\langle\cdot\rangle_Y^*$ means the usual Gibbs measure with boundary condition Y on the dual system. Moreover, for any Gibbs measure $\langle\cdot\rangle$, if $(\langle\cdot\rangle)^*$ is a Gibbs measure for the dual system then we have the inequalities

- $\langle\sigma_{\underline{B}}\rangle^+ \geq \langle\sigma_{\underline{B}}\rangle^Y \geq \langle\sigma_{\underline{B}}\rangle_f \geq \prod_{B\in\underline{B}} \tanh K(B)$;
- $1 \geq \langle\mu_{\underline{B}}\rangle_f \geq \langle\mu_{\underline{B}}\rangle^Y \geq \langle\mu_{\underline{B}}\rangle^+$.

for any boundary condition Y for which the thermodynamical limit exists.

Proof. Applying condition (b), for each i we have $\Gamma_{\Lambda_i} = \Gamma^{(f)} \cap \mathscr{P}(\mathscr{B}_{\Lambda_i})$. Indeed, the inclusion $\subset$ is obvious and, as for the other inclusion, for every X with $|X| < \infty$ and such that $\gamma(X) \subset \mathscr{B}_{\Lambda_i}$ we have by definition of γ that every bond B such that $\sigma_B(X) = -1$ is in $\mathscr{B}_{\Lambda_i}$ and hence all bonds B not intersecting Λ_i must satisfy $\sigma_B(X) = +1$. By condition (b), this implies a decomposition $X = YS_f$ with $Y \subset \Lambda_i$ and $S_f \in \mathscr{S}$. Since γ is a homomorphism and $\gamma(S) = \varnothing$ for all $S \in \mathscr{S}$ (just compare the definitions of γ and $\mathscr{S}$), then $\gamma(X) = \gamma(Y)$. Since $Y \subset \Lambda_i$, this proves the inclusion $\supset$.

Since we have $d(\Gamma^{(f)}) = \mathscr{K}_f^*$, then

$$d(\Gamma_{\Lambda_i}) = d(\Gamma^{(f)}) \cap \mathscr{P}(d\mathscr{B}_{\Lambda_i}) = \mathscr{K}_f^* \cap \mathscr{P}(d\mathscr{B}_{\Lambda_i})$$

The last equation says that $(\Lambda_i^*, d(\mathscr{B}_{\Lambda_i}))$ is a $LT-HT$ dual of $(\Lambda_i, \mathscr{B}_{\Lambda_i})$, where $\Lambda_i^* \stackrel{\text{def}}{=} \bigcup_{B\in\mathscr{B}_{\Lambda_i}} B^*$. Now, let $\underline{B}^* \in d(\mathscr{B}_{\Lambda_i})$ be any and take a sequence $\widetilde{\Lambda_i^*} \subset \Lambda_i^*$ of maximal volumes satisfying $\mathscr{B}^f_{\widetilde{\Lambda_i^*}} \subset d(\mathscr{B}_{\Lambda_i})$. Then, since by Griffith's inequalities the correlation functions are non-decreasing functions of the interactions, we start in the system $(\Lambda_i^*, d(\mathscr{B}_{\Lambda_i}))$ and tune down all interactions K_B with $B \in d(\mathscr{B}_{\Lambda_i}) \setminus \mathscr{B}^f_{\widetilde{\Lambda_i^*}}$ to zero. After doing this, all interactions in the region $\Lambda_i^* \setminus \widetilde{\Lambda_i^*}$ are killed, so the left-over system is just $\widetilde{\Lambda_i^*}$ with free boundary conditions. Thus, we achieve the inequality

$$\langle \sigma_{\underline{B}^*}\rangle_{\{\widetilde{\Lambda_i^*}, \mathscr{B}^f_{\widetilde{\Lambda_i^*}}\}} \leq \langle \sigma_{\underline{B}^*}\rangle_{\{\Lambda_i^*, d(\mathscr{B}_{\Lambda_i})\}}.$$

By the same argument of Griffith's inequalities, since $d(\mathscr{B}_{\Lambda_i}) \subset \mathscr{B}^f_{\Lambda_i^*}$ by the definition of Λ_i^*, similarly we have $\langle \sigma_{\underline{B}^*}\rangle_{\{\Lambda_i^*, d(\mathscr{B}_{\Lambda_i})\}} \leq \langle \sigma_{\underline{B}^*}\rangle_{\{\Lambda_i^*, \mathscr{B}^f_{\Lambda_i^*}\}}$ and in total we obtain

$$\langle \sigma_{\underline{B}^*}\rangle_{\{\widetilde{\Lambda_i^*}, \mathscr{B}^f_{\widetilde{\Lambda_i^*}}\}} \leq \langle \sigma_{\underline{B}^*}\rangle_{\{\Lambda_i^*, d(\mathscr{B}_{\Lambda_i})\}} \leq \langle \sigma_{\underline{B}^*}\rangle_{\{\Lambda_i^*, \mathscr{B}^f_{\Lambda_i^*}\}}.$$

Then, as i goes to infinity, the volumes $\widetilde{\Lambda_i^*}$ and Λ_i^* cover $\mathbb{Z}^d$ and for hence the thermodynamical limit yields

$$\lim_{\Lambda_i\to\mathbb{Z}^d} \langle \sigma_{\underline{B}^*}\rangle_{\{\Lambda_i^*, d(\mathscr{B}_{\Lambda_i})\}} = \langle \sigma_{\underline{B}^*}\rangle_f^*.$$

Finally, using lemma 2.16, we have

$$(\langle \sigma_{\underline{B}^*}\rangle_+)^* = \langle \mu_{\underline{B}}\rangle_+ = \lim_{\Lambda_i\to\mathbb{Z}^d} \langle \mu_{\underline{B}}\rangle_{\{\Lambda_i, \mathscr{B}_{\Lambda_i}\}} = \lim_{\Lambda_i^*\to\mathbb{Z}^d} \langle \sigma_{\underline{B}^*}\rangle_{\{\Lambda_i^*, d(\mathscr{B}_{\Lambda_i})\}} = \langle \sigma_{\underline{B}^*}\rangle_f^*. \tag{2.16}$$

We now proceed to prove the set of inequalities of the proposition. Pick any boundary condition Y and transform $\langle \sigma_{\underline{B}^*}\rangle^*_{\{\Lambda_i, Y\}}$ into a correlation function in terms of free boundary conditions in the following way: we first rewrite $\exp\left(-\beta\mathscr{H}_\Lambda^Y(\cdot)\right)$ as

$$\begin{aligned}
e^{-\beta\mathscr{H}^Y_{\Lambda_i}(X)} &= \prod_{B\in\mathscr{B}_{\Lambda_i}} e^{\sigma_B(X)\sigma_B(Y\cap\Lambda_i^c)K_B} = \prod_{\substack{B\in\mathscr{B}:\\ B\cap\Lambda_i\neq\varnothing}} e^{\sigma_B(X)\sigma_B(Y\cap\Lambda_i^c)K_B} \\
&= \left(\prod_{\substack{B\in\mathscr{B}:\\ B\subset\Lambda_i}} e^{\sigma_B(X)\sigma_B(Y\cap\Lambda_i^c)K_B}\right)\left(\prod_{\substack{B\in\mathscr{B}:\\ B\cap\Lambda_i\neq\{\varnothing,B\}}} e^{\sigma_B(X)\sigma_B(Y\cap\Lambda_i^c)K_B}\right) \\
&= \left(\prod_{\substack{B\in\mathscr{B}:\\ B\subset\Lambda_i}} e^{\sigma_B(X)\sigma_B(Y\cap\Lambda_i^c)K_B}\right)\left(\prod_{\substack{B\in\mathscr{B}:\\ B\cap\Lambda_i\neq\{\varnothing,B\}\\ \sigma_B(Y\cap\Lambda_i^c)=1}} e^{\sigma_B(X)K_B}\right)\left(\prod_{\substack{B\in\mathscr{B}:\\ B\cap\Lambda_i\neq\{\varnothing,B\}\\ \sigma_B(Y\cap\Lambda_i^c)=-1}} e^{-\sigma_B(X)K_B}\right).
\end{aligned} \tag{2.17}$$

To proceed, we will insert the term

$$\left(\prod_{\substack{B\in\mathscr{B}:\\ B\cap\Lambda_i\neq\{\varnothing,B\}\\ \sigma_B(Y\cap\Lambda_i^c)=-1}} e^{\sigma_B(X)K_B}\right)\left(\prod_{\substack{B\in\mathscr{B}:\\ B\cap\Lambda_i\neq\{\varnothing,B\}\\ \sigma_B(Y\cap\Lambda_i^c)=-1}} e^{-\sigma_B(X)K_B}\right) = 1$$

between the last two products and then simplify similar terms. Moreover, we note that if $B \subset \Lambda_i$, then $\sigma_B(Y \cap \Lambda_i^c) = 1$. In this way, equation (2.17) becomes

$$\left(\prod_{\substack{B\in\mathscr{B}:\\ B\subset\Lambda_i}} e^{\sigma_B(X)K_B}\right)\left(\prod_{\substack{B\in\mathscr{B}:\\ B\cap\Lambda_i\neq\{\varnothing,B\}}} e^{\sigma_B(X)K_B}\right)\left(\prod_{\substack{B\in\mathscr{B}:\\ B\cap\Lambda_i\neq\{\varnothing,B\}\\ \sigma_B(Y\cap\Lambda_i^c)=-1}} e^{-2\sigma_B(X)K_B}\right)$$

$$= \mu_b(X)\left(\prod_{B\in\mathscr{B}_{\Lambda_i}} e^{\sigma_B(X)K_B}\right) = \mu_b(X)\exp(-\mathscr{H}^+_{\Lambda_i}(X)),$$

where $b := \{B \in \mathscr{B} : B \cap \Lambda_i \neq \{\varnothing, B\}$ and $\sigma_B(Y \cap \Lambda_i^c) = -1\}$. In terms of correlation functions, for any $\underline{B} \subset \mathscr{B}_{\Lambda_i}$ this yields

$$\langle\mu_{\underline{B}}\rangle^Y_{\Lambda_i} = \frac{\sum_{X\subset\Lambda_i}\mu_{\underline{B}}(X)\exp(-\beta\mathscr{H}^Y_{\Lambda_i}(X))}{\sum_{X\subset\Lambda_i}\exp(-\beta\mathscr{H}^Y_{\Lambda_i}(X))} = \frac{\sum_{X\subset\Lambda_i}\mu_{\underline{B}}(X)\mu_b(X)\exp(-\beta\mathscr{H}^+_{\Lambda_i}(X))}{\sum_{X\subset\Lambda_i}\mu_b(X)\exp(-\beta\mathscr{H}^+_{\Lambda_i}(X))}$$

$$= \frac{\sum_{X\subset\Lambda_i}\mu_{\underline{B}}(X)\mu_b(X)\exp(-\beta\mathscr{H}^+_{\Lambda_i}(X))}{\sum_{X\subset\Lambda_i}\exp(-\beta\mathscr{H}^+_{\Lambda_i}(X))}\frac{\sum_{X\subset\Lambda_i}\exp(-\beta\mathscr{H}^+_{\Lambda_i}(X))}{\sum_{X\subset\Lambda_i}\mu_b(X)\exp(-\beta\mathscr{H}^+_{\Lambda_i}(X))}$$

$$= \frac{\langle\mu_{\underline{B}\cup b}\rangle_{\{\Lambda_i,\mathscr{B}_{\Lambda_i}\}}}{\langle\mu_b\rangle_{\{\Lambda_i,\mathscr{B}_{\Lambda_i}\}}}.$$

Since $\{\Lambda_i^*, d(\mathscr{B}_{\Lambda_i})\}$ is a $LT - HT$ dual for $\{\Lambda_i, \mathscr{B}_{\Lambda_i}\}$, by duality and by second Griffith's inequality, we have

$$\langle\mu_{\underline{B}}\rangle^Y_{\{\Lambda_i\}} = \frac{\langle\sigma_{\underline{B}^*}\sigma_{b^*}\rangle_{\{\Lambda_i^*,d\mathscr{B}_{\Lambda_i}\}}}{\langle\sigma_{b^*}\rangle_{\{\Lambda_i^*,d\mathscr{B}_{\Lambda_i}\}}} \geq \langle\sigma_{\underline{B}^*}\rangle^*_{\{\Lambda_i^*,f\}},$$

where we remember again that $(\Lambda_i^*, d\mathscr{B}_{\Lambda_i})$ is a finite system with free boundary conditions. Taking the thermodynamical limit and letting $\langle\cdot\rangle$ denote the limit of the LHS, we get

$$\langle\mu_{\underline{B}}\rangle^Y \geq \langle\sigma_{\underline{B}^*}\rangle^*_f = \langle\mu_{\underline{B}}\rangle_+.$$

Moreover, if the dual system satisfies the same conditions of the theorem, then we also get

$$\langle\mu_{\underline{B}^*}\rangle^*_Y \geq \langle\mu_{\underline{B}^*}\rangle^*_+,$$

which translates to $\langle\sigma_{\underline{B}}\rangle^Y \geq \langle\sigma_{\underline{B}}\rangle_f$. To end the proof, all there is left to prove is the last lower bound. To achieve it, simply write

$$\langle\sigma_{\underline{B}}\rangle_{\Lambda_i,f} = \langle\mu_{\underline{B}^*}\rangle_{\Lambda_i^*,f} = \frac{\sum_{X\subset\Lambda_i^*}\prod_{B^*\in\underline{B}^*}\exp(-2K^*(B^*)\sigma_{B^*}(X))\exp(-H_{\Lambda_i^*,f}(X))}{\sum_{X\subset\Lambda_i^*}(X)\exp(-\beta\mathscr{H}_{\Lambda_i^*}(X))}$$

$$\geq \prod_{B^*\in\underline{B}^*}\exp(-2K^*(B^*)) = \prod_{B^*\in\underline{B}^*}\tanh(K(B)),$$

where we bounded $\exp(-2K^*(B^*)\sigma_{B^*}(X)) \geq \exp(-2K^*(B^*))$ and used the duality relation between the interactions.

□

2.3 Surface Tension: Duality Approach

In this section we will prove the convergence of the surface tension for Ising-like models. The first step is to split $\mathbb{Z}^d = \mathbb{Z}^d_u \cup \mathbb{Z}^d_l$ (upper and lower parts, respectively), where $\mathbb{Z}^d_u = \{x \in \mathbb{Z}^d : x_d > 0\}$

and $\mathbb{Z}^d_l = \{x \in \mathbb{Z}^d : x_d \leq 0\}$ and consider the boxes Λ of side lengths $L_1, L_2..., L_{d-1}, 2M$ with $L_1, ..., L_{d-1}, M > 0$.

We let $S \in \mathscr{S}$ be any and consider the boundary condition $(S, +)$ such that the spin in the region $\mathbb{Z}^d_u$ equals 1 and the spin in the region $\mathbb{Z}^d_l$ is the same as S, that is, $(S, +) = S \cap \mathbb{Z}^d_l$.

The Ising-like Hamiltonians to be worked with are of the form

$$\mathscr{H}^Y_\Lambda(X) \stackrel{\text{def}}{=} -\sum_{B \in \mathscr{B}_\Lambda} J_B \sigma_B(X \cdot Y_{\Lambda^c}), \tag{2.18}$$

where $Y_{\Lambda^c} = Y \cap \Lambda^c$. Here it is crucial, again, that we work in a model of short-range interactions, meaning that all $B \in \mathscr{B}$ have their diameter uniformly bounded by some finite constant. This guarantees, for example, that $\mathscr{B}_\Lambda$ has finitely many elements. The limit of interest here is the one in equation 1.39 and, up to the negative sign and the inverse temperature, is equal to

$$\lim_{L_1,...,L_{d-1}\to\infty} \lim_{M\to\infty} \frac{1}{L_1 L_2 ... L_{d-1}} \log\left(\frac{Z^{(S,+)}_{\Lambda,\beta}}{Z^+_{\Lambda,\beta}}\right).$$

We will prove that it indeed exists in the ferromagnetic case and it is uniformly bounded above. Putting $Y = S \cap \mathbb{Z}^d_l$, the Hamiltonian with respect to this boundary condition can be written as

$$\mathscr{H}^{(S,+)}_\Lambda(X) = -\sum_{B \in \mathscr{B}_\Lambda} J_B \sigma_B(X \cdot Y_{\Lambda^c}) = -\sum_{B \in \mathscr{B}_\Lambda} J_B \sigma_B(X) \sigma_B(S \cap \mathbb{Z}^d_l \cap \Lambda^c).$$

After adding and subtracting $\mathscr{H}^+_\Lambda(X)$ from both sides of the equality, one gets

$$\mathscr{H}^{(S,+)}_\Lambda(X) - \mathscr{H}^+_\Lambda(X) = -\sum_{B \in \mathscr{B}_\Lambda} J_B \sigma_B(X) \{\sigma_B(S \cap \Lambda^c \cap \mathbb{Z}^d_l) - 1\}.$$

Now, for all $B \in \gamma(Y_{\Lambda^c})$ by definition $\sigma_B(S \cap \mathbb{Z}^d_l \cap \Lambda^c) = -1$ and hence there is an overall factor of 2 in the right hand side and for all $B \notin \gamma(Y_{\Lambda^c})$ the factor in parenthesis is zero. Hence

$$\mathscr{H}^{(S,+)}_\Lambda(X) - \mathscr{H}^+_\Lambda(X) = 2 \sum_{B \in \gamma(Y_{\Lambda^c}) \cap \mathscr{B}_\Lambda} J_B \sigma_B(X). \tag{2.19}$$

In terms of the partition function, we have

$$\frac{Z^{(S,+)}_\Lambda}{Z^+_\Lambda} = \frac{\sum_{X \subset \Lambda} \exp(-\beta \mathscr{H}^{(S,+)}_\Lambda(X))}{\sum_{X \subset \Lambda} \exp(-\beta \mathscr{H}^+_\Lambda(X))} = \frac{\sum_{X \subset \Lambda} \exp(-\beta \mathscr{H}^+_\Lambda(X)) \exp(-2 \sum\limits_{B \in \gamma(Y_{\Lambda^c}) \cap \mathscr{B}_\Lambda} J_B \sigma_B(X))}{\sum_{X \subset \Lambda} \exp(-\beta \mathscr{H}^+_\Lambda(X))}$$

$$= \sum_{X \subset \Lambda} \prod_{B \in \gamma(Y^c_\Lambda \cap \mathscr{B}_\Lambda)} \mu_B(X) \frac{\exp(-\beta \mathscr{H}^+_\Lambda(X))}{\sum\limits_{X \subset \Lambda} \exp(-\beta \mathscr{H}^+_\Lambda(X))} = \left\langle \prod_{B \in \gamma(Y_{\Lambda^c}) \cap \mathscr{B}_\Lambda} \mu_B \right\rangle^+_\Lambda$$

We now note that, defining $\Lambda_l \stackrel{\text{def}}{=} \Lambda \cap \mathbb{Z}^d_l$ and $\Lambda_u \stackrel{\text{def}}{=} \Lambda \cap \mathbb{Z}^d_u$,

$$(S \cap \Lambda^c_l) \cdot Y_{\Lambda^c} = (S \cap \Lambda^c_l) \cdot (S \cap \mathbb{Z}^d_l \cap \Lambda^c) = (S \cap \Lambda^c_l) \setminus (S \cap \mathbb{Z}^d_l \cap \Lambda^c) = S \cap \mathbb{Z}^d_u,$$

and therefore $Y_{\Lambda^c} = (S \cap \Lambda^c_l) \cdot (S \cap \mathbb{Z}^d_u)$. Thus,

$$\begin{aligned} \gamma(Y_{\Lambda^c}) \cap \mathscr{B}_\Lambda &= \{\gamma(S \cap \Lambda^c_l) \cdot \gamma(S \cap \mathbb{Z}^d_u)\} \cap \mathscr{B}_\Lambda \\ &= \{\gamma(S \cap \Lambda^c_l) \cap \mathscr{B}_\Lambda\} \cdot \{\gamma(S \cap \mathbb{Z}^d_u) \cap \mathscr{B}_\Lambda\} =: \beta^1_\Lambda \cdot \underline{B}_\Lambda. \end{aligned}$$

Note that for all $B \in \gamma(S \cap \mathbb{Z}^d_u)\} \cap \mathscr{B}_\Lambda = \underline{B}_\Lambda$ the bond intersects the box Λ and $\mathbb{Z}^d_u$[5] and

[5]since if $B \cap \mathbb{Z}^d_u = \varnothing$ then we would have $\sigma_B(S \cap B \cap \mathbb{Z}^d_u) = 1$, contradicting the definition of $\gamma(S \cap \mathbb{Z}^d_u)$

hence intersects the upper part Λ_u of the box. If, however, we had $B \subset \mathbb{Z}_u^d$ then we would get $-1 = \sigma_B(S \cap \mathbb{Z}_u^d) = (-1)^{|S\cap B\cap \mathbb{Z}_u^d|} = (-1)^{|S\cap B|} = \sigma_B(S) = 1$, a contradiction. Therefore, every $B \in \gamma(S \cap \mathbb{Z}_u^d)$ intersects $\mathbb{Z}_l^d$ and also $\mathbb{Z}_u^d$. Since we are working with short range interactions, this implies that for M large enough the number of bonds in $\gamma(S \cap \mathbb{Z}_u^d)\} \cap \mathscr{B}_\Lambda = \underline{B}_\Lambda$ is constant.

By exercise 3.12 of [FV17], for all large enough M we then get that

$$\frac{Z_\Lambda^{(S,+)}}{Z_\Lambda^+} = \left\langle \prod_{B\in\gamma(Y_{\Lambda^c})\cap\mathscr{B}_\Lambda} \mu_B \right\rangle_\Lambda^+ = \langle \sigma_{\underline{B}_\Lambda^*} \rangle_{\{\Lambda^*, d(\mathscr{B}_\Lambda)\}}$$

is a non-decreasing function of M, where $\Lambda^* = \bigcup_{B\in\mathscr{B}_\Lambda} B^*$ (remember that $(\Lambda^*, d(\mathscr{B}_\Lambda))$ is a dual system for $(\Lambda, \mathscr{B}_\Lambda)$, as in lemma 2.18 for a proof). Therefore, the map

$$M \mapsto \log\left(\frac{Z_\Lambda^{(S,+)}}{Z_\Lambda^+}\right)$$

is non-decreasing and bounded above by zero, implying that

$$\lim_{M\to\infty} \log\left(\frac{Z_\Lambda^{(S,+)}}{Z_\Lambda^+}\right)$$

exists. Before proving the existence of the surface tension, we will first prove that it is bounded below if it exists. To do this, we first note that

$$|\underline{B}_\Lambda| \leq \widetilde{C}\prod_{i=1}^{d-1} L_i, \tag{2.20}$$

where R is the maximum range of the interactions, i.e, $R := \max\{|B| : B \in \mathscr{B}\}$ and $\tilde{C} := 2^{|B_{2R}(0)|}$ is the number of subsets in the ball of radius $2R$ in the ℓ_∞ norm. This is true because the number of such bonds is certainly bounded by the number of bonds intersecting the separation plane, and this number is bounded by $\prod_{i=1}^{d-1} L_i$ (i.e, the area of intersection of the box with the separation plane) times the number of bonds intersecting the lower and upper plane and containing a point $x \in \prod_{i=1}^{d-1} L_i$. For each point $x \in \prod_{i=1}^{d-1} L_i$, the number of such bonds is bounded by the number of subsets in the ℓ_∞ norm ball of radius $2R$ with center in x, finishing this argument.

Now, by the end of the proof of lemma 2.18 and the duality relations for the interactions, we have for all sufficiently big values of M

$$\frac{Z_\Lambda^{(S,+)}}{Z_\Lambda^+} = \langle \sigma_{\underline{B^*}_\Lambda^*} \rangle_{\{\Lambda^*, d(\mathscr{B}_\Lambda)\}} \geq \prod_{B\in\underline{B}_\Lambda^*} \tanh(K_{B^*}^*) = \prod_{B\in\underline{B}_\Lambda} e^{-2K(B)}.$$

Applying the logarithm on both sides, bounding $K_B \leq \widetilde{K} := \max\{K_B : B \in \beta_\Lambda\}$ and using the last upper bound, this implies that

$$\log\left(\frac{Z_\Lambda^{(S,+)}}{Z_\Lambda^+}\right) \geq -2\sum_{B\in\underline{B}_\Lambda} K_B \geq -2\widetilde{K}\widetilde{C}\prod_{i=1}^{d-1} L_i. \tag{2.21}$$

Taking the limit first on M and then dividing both sides of the inequality by $\prod_{i=1}^{d-1} L_i$ we get that the surface tension is bounded below by $-2\widetilde{K}\widetilde{C}$, if it exists. To prove why it exists, we first define the function

$$f(L_1, ..., L_{d-1}) := \lim_{M\to\infty} \log\left(\frac{Z_\Lambda^{(S,+)}}{Z_\Lambda^+}\right), \tag{2.22}$$

and we wish to show that f is superadditive in each variable. To see why this holds, pick some $1 \leq i \leq d-1$ and split the box as $\Lambda = \Lambda' \cup \Lambda''$, where Λ', Λ'' are the sub-box with sides $(L_1, ..., L_{i-1}, L_i', L_{i+1}, ..., L_{d-1}, 2M)$ and $(L_1, ..., L_{i-1}, L_i'', L_{i+1}, ..., L_{d-1}, 2M)$ respectively. Note that $\underline{B}_{\Lambda'} \cdot \underline{B}_{\Lambda''}$ consists of bonds $B \in (\gamma(S \cap \mathbb{Z}_u^d) \cap \mathscr{B}_{\Lambda'}) \cdot (\gamma(S \cap \mathbb{Z}_u^d) \cap \mathscr{B}_{\Lambda''})$ and we note that by definition they intersect either Λ' or Λ'', but not both simultaneously. If we add this missing set of bonds into the symmetric difference above, we recover $\underline{B}_\Lambda$.

Hence, if we define $\delta \underline{B}_\Lambda \overset{\text{def}}{=} \underline{B}_\Lambda \setminus (\underline{B}_{\Lambda'} \cdot \underline{B}_{\Lambda''})$ then $\delta \underline{B}_\Lambda$ is disjoint of $\underline{B}_{\Lambda'} \cdot \underline{B}_{\Lambda''}$ implying that $\underline{B}_{\Lambda'} \cdot \underline{B}_{\Lambda''} \cdot \delta \underline{B}_\Lambda = \underline{B}_\Lambda$. One can then repeat the arguments above to get

$$|\delta \underline{B}_\Lambda| \leq \widetilde{C} \prod_{\substack{j=1 \\ j \neq i}}^{d-1} L_j.$$

Before taking the limit in M, note that Griffiths second inequality and exercise 3.12 of [FV17] imply

$$\begin{aligned}
\frac{Z_\Lambda^{(S,+)}}{Z_\Lambda^+} &= \langle \sigma_{\underline{B}_{\Lambda^*}} \rangle_{\{\Lambda^*, d(\mathscr{B}_\Lambda)\}} = \langle \sigma_{\delta \underline{B}_{\Lambda^*}} \sigma_{\underline{B}_{\Lambda^{*\prime}}} \sigma_{\underline{B}_{\Lambda^{*\prime\prime}}} \rangle_{\{\Lambda^*, d(\mathscr{B}_\Lambda)\}} \\
&\geq \langle \sigma_{\underline{B}_{\Lambda^{*\prime}}} \rangle_{\{\Lambda^*, d(\mathscr{B}_\Lambda)\}} \langle \sigma_{\underline{B}_{\Lambda^{*\prime\prime}}} \rangle_{\{\Lambda^*, d(\mathscr{B}_\Lambda)\}} \langle \sigma_{\delta \underline{B}_{\Lambda^*}} \rangle_{\{\Lambda^*, d(\mathscr{B}_\Lambda)\}} \\
&\geq \langle \sigma_{\underline{B}_{\Lambda^{*\prime}}} \rangle_{\{\Lambda^{*\prime}, d(\mathscr{B}_\Lambda)\}} \langle \sigma_{\underline{B}_{\Lambda^{*\prime\prime}}} \rangle_{\{\Lambda^{*\prime\prime}, d(\mathscr{B}_\Lambda)\}} \langle \sigma_{\delta \underline{B}_{\Lambda^*}} \rangle_{\{\Lambda^*, d(\mathscr{B}_\Lambda)\}} \\
&\geq \frac{Z_{\Lambda'}^{(S,+)}}{Z_{\Lambda'}^+} \frac{Z_{\Lambda''}^{(S,+)}}{Z_{\Lambda''}^+} \prod_{B \in \delta \underline{B}_\Lambda} e^{-2K(B)},
\end{aligned}$$

where we have used lemma 2.18 again for the last term. Taking the logarithm on both sides, we have

$$\log\left(\frac{Z_\Lambda^{(S,+)}}{Z_\Lambda^+}\right) \geq \log\left(\frac{Z_{\Lambda'}^{(S,+)}}{Z_{\Lambda'}^+}\right) + \log\left(\frac{Z_{\Lambda''}^{(S,+)}}{Z_{\Lambda''}^+}\right) - 2\widetilde{C}\widetilde{K} \prod_{\substack{j=1 \\ j \neq i}}^{d-1} L_j,$$

and hence

$$f(L_1, ..., L_{i-1}, L_i' + L_i'', L_{i+1}, ..., L_{d-1}) \geq$$
$$f(L_1, ..., L_{i-1}, L_i', L_{i+1}, ..., L_{d-1}) + f(L_1, ..., L_{i-1}, L_i'', L_{i+1}, ..., L_{d-1}) - 2\widetilde{C}\widetilde{K} \prod_{\substack{j=1 \\ j \neq i}}^{d-1} L_i.$$

If we now define the function $g(L_1, ..., L_{d-1}) = f(L_1, ..., L_{d-1}) - 2\widetilde{C}\widetilde{K} L_1 ... L_{d-1} \sum_{j=1}^{d-1} \frac{1}{L_j} = f(L_1, ..., L_{d-1}) - 2\widetilde{C}\widetilde{K}(L_2 L_3 ... L_{d-1} + L_1 L_3 ... L_{d-1} + ... L_1 L_2 ... L_{d-2})$, then g is superadditive in each variable. For example, we have

$$\begin{gathered}
g(L_1' + L_1'', L_2, ..., L_{d-1}) \geq f(L_1', L_2, ..., L_{d-1}) + f(L_1'', L_2, ..., L_{d-1}) \\
-2\widetilde{C}\widetilde{K} L_2 ... L_{d-1} - 2\widetilde{C}\widetilde{K}(L_2 ... L_{d-1} + L_1' L_3 ... L_{d-1} + L_1'' L_3 ... L_{d-1} + ... + L_1' L_2 ... L_{d-2} + L_1'' L_2 ... L_{d-2}) \\
= (f(L_1', L_2, ..., L_{d-1}) - 2\widetilde{C}\widetilde{K}(L_2 ... L_{d-1} + L_1' L_3 ... L_{d-1} + ... + L_1' L_2 ... L_{d-2})) \\
+(f(L_1'', L_2, ..., L_{d-1}) - 2\widetilde{C}\widetilde{K}(L_2 ... L_{d-1} + L_1'' L_3 ... L_{d-1} + ... + L_1'' L_2 ... L_{d-2})) \\
= g(L_1', L_2, ..., L_{d-1}) + g(L_1'', L_2, ..., L_{d-1})
\end{gathered}$$

We now use Fekete's lemma repeatedly: first, we note that the limit on L_1 of $g(L_1, ..., L_{d-1}) \frac{1}{L_1}$

exists, which we denote by $g(L_2, ..., L_{d-1})$. Therefore, the limit on L_1 of $f(L_1, ..., L_{d-1})\frac{1}{L_1}$ also exists and equals $g(L_2, ..., L_{d-1}) + 2\widetilde{C}\widetilde{K}(L_3L_4...L_{d-1} + L_2L_4...L_{d-1} + ... + L_2L_3...L_{d-2})$, and we denote $f(L_2, ..., L_{d-1}) = \lim_{L_1\to\infty} \frac{1}{L_1} f(L_1, L_2, ..., L_{d-1})$. Of course, $g(L_2, ..., L_{d-1})$ is also superadditive, being the limit of superadditive functions. We now iterate the procedure to get the existence of $\lim_{L_{d-1}\to\infty} ... \lim_{L_1\to\infty} \frac{f(L_1,...,L_{d-1})}{L_1...L_{d-1}}$.

Chapter 3

Concluding Remarks

The main theorems of this thesis concern bounds and existence results for the surface tension. These results include interesting applications, such as in the proof that the correlation functions $\langle \sigma_A \rangle^{\pm}$ with $A \subset \mathbb{Z}^3$ and $\tau_\beta(0,1) - 2\beta J$, both depending on βJ, are analytic in $z = e^{-\beta J}$ (see [BLP79]). For this proof, one uses the uniform bound in L found in the last section. Another application, for Potts models with 2^n spins, one can transform the model to a ferromagnetic Ising model ([BLM83]), and hence the surface tension in this case exists by the same technique used here.

One could ask for generalizations of both papers for long range models. One indicative that this is possible in [BKL83] is the existence of long-range extensions of Pirogov-Sinai theory (see for example [Par88a] and its continuation [Par88b]). We note, moreover, that a *cluster expansion* method was employed starting in lemma 1.17. By cluster expansion, one usually means a way to write the free energy of the system as a convergent series in terms of collections of contours, called *polymers* or *clusters*.

In the classical theory of Pirogov and Sinai, the contours are connected and there is no interaction between them. In the long range picture, however, the picture is very different. Here, the contours are usually disconnected and there is no way of making them not interact with each other, since the spin interaction radius is infinite. In the special case of the d-dimensional long range Ising model with interactions $\frac{J}{|x-y|^\alpha}$, a recent work in progress of Lucas Affonso, Rodrigo Bissacot, João Maia, João F. Rodrigues and Kelvyn Welsch proved the convergence of a cluster expansion for all regularity region $\alpha > d \geq 2$ of the form

$$\log \widetilde{Z}_{\Lambda,\beta} = \sum_{X \subset \mathcal{E}^+_\Lambda} \phi^T(X) \prod_{\Gamma \in X} z^+_\beta(\Gamma),$$

where $\mathcal{E}^+_\Lambda$ denotes the collection of external contours with boundary condition $+$ and $z^+_\beta(\Gamma)$ are the activities for the model. Here, the contours are different from the Pirogov-Sinai contours presented in this thesis, as they are partitions of the set of incorrect points satisfying a certain separability condition. A convergent cluster expansion for the one dimensional model was proved for $\alpha \in \left(3 - \frac{\ln 3}{\ln 2}, 2\right]$ in [CMPR14], and these results can be used as a substitute for the cluster expansion used in lemma 1.17.

As for [GHMMS77], problems include that the general construction of dual lattices may not work anymore and some proofs become invalid. In fact, in this case $\mathscr{P}(\mathscr{B}_\Lambda)$ is infinite and therefore $\mathscr{K}_\Lambda$ is infinite, so it may have an infinite set of generators. Following the construction of dual models, this would yield some infinite Λ^* even though Λ may be finite.

Going back to the proof of lemma 2.18, one of the main steps was to construct a finite dual system by setting the lattice as $\Lambda^* \stackrel{\text{def}}{=} \bigcup_{B \in \mathscr{B}_{\Lambda_i}} B^*$. In special, if the duality map d is injective then this Λ^* would be again infinite for long range models.

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