Distribuição exata não assintótica de tempos de entrada

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Resumo

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O tempo decorrido até a primeira ocorrência de um observável em uma realização de um processo estocástico é um objeto de estudo clássico. É conhecido que a distribuição do tempo de entrada, quando reescalada adequadamente, converge para uma lei exponencial.

Neste trabalho, apresentamos a forma exata da distribuição do tempo de entrada de uma sequência finita fixa em um processo independente e identicamente distribuído, e definido sobre um alfabeto finito ou enumerável. Isto é, obtemos o resultado que não é apenas assintótico. Mostramos que a distribuição exata do tempo de entrada é uma soma de exponenciais. Provamos que esta soma possui um termo dominante e que os demais convergem para zero.

Palavras-chave: Tempo de entrada. Relação de recorrência.

Abstract

Julia Faria Codas. **Non-asymptotic exact distribution for hitting times:** . Thesis (Masters). Institute of Mathematics and Statistics, University of São Paulo, São Paulo, 2020.

The time elapsed until the first occurrence of an observable in a realization of a stochastic process is a classic object of study. It is a known result that the distribution of the hitting time, when properly rescaled, converges to an exponential law.

In this work, we present the exact form of the distribution of the hitting time of a fixed finite sequence in an independent and identically distributed process, which is defined over a finite or countable alphabet. That is, we get the result that is not just asymptotic. We show that the exact distribution of the hitting time is a sum of exponentials. We prove that this sum has a dominant term and that the others converge to zero.

Keywords: Hitting time. Recurrence relation.

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Introduction

The Law of Large Numbers is one of the main results in Probability Theory. In its stationary version, it says that, for independent and identically distributed random variables, it is possible to make statistics in the sense that the sample averages tell the truth: the sample averages almost certainly approximate to the spatial average.

It is a macroscopic result in the sense that we must see the whole sample to calculate the proportion of observations made of the target object. If we want to understand how the sequence of repetitions of the observable behave, we must ask further questions such as: how long does it take until the first observation, how long does it take until the second observation, and generally how much time elapses between any two observations, among others. In this paper, we focus on the first one.

The asymptotic study of this problem is already a classic object. Amidst the most outstanding results, we can cite Aldous and Brown (1993)[4], Galves and Schmidt (1997)[8], Hirata, Saussol and Vaienti (1999)[10], Abadi (2004)[2]. Among the works that summarize the state-of-the-art, we can mention Coelho (1997)[6], Abadi and Galves (2001)[1], Haydn (2013)[9]. An exponential law is obtained at the limit not only in independent systems, but some correlation decay already appears as sufficient to obtain this result.

In this work, we introduce two aspects little considered in the literature. On the one hand, we are looking for exact results for fixed observables, not only asymptotic ones. On the other hand, the technique developed uses recurrence relations (as opposed to the classic "cut" the sample into "quasi" independent blocks). Although we focus on systems that are sequences of independent random variables, the observables of interest are sets defined by more than one random variable. Typically, the target set is defined as a fixed finite sequence of values (cylinder). In this case, it implies that the sequence of occurrences of this target set is not independent, despite that the original process is.

The technique, as said before, is based on recurrence relations. It goes through various classical problems of mathematics, such as finding roots of polynomials and solutions of systems of linear equations. In this work, we also construct the spectral gap and the spectral radius of an operator associated with the recurrence. The problem of local recurrence (also called in the literature the first possible return, shortest return, or periodicity) appears explicitly. We show cases in which we can explicitly solve this problem. It corresponds to the cases that the recurrence relations are linear and homogeneous.

Chapter 1

Hitting times via recurrence relations

1.1 General setting

Let $\mathcal{A}$ be a non-empty finite or countable set and define $\Omega = \mathcal{A}^{\mathbb{N}}$. For each non-negative integer n, $X_n : \Omega \longrightarrow \mathbb{R}$ is the n-coordinate projection. We define a cylinder of size n as the set of the form

$$A = \{\omega \in \Omega : (X_0(\omega), \cdots, X_{n-1}(\omega)) = (a_0, \cdots, a_{n-1})\},$$

for some $a_i \in \mathcal{A}$, $i = 1, \cdots, n$. In this case, we say that A is the cylinder defined by a_0^{n-1}, where a_0^{n-1} is a shorthand notation for the sequence $(a_0, \cdots, a_{n-1})$. We also fix the notation A_k for the set of realizations ω in Ω for which the observation of a_0^{n-1} start at time k, that is,

$$A_k = \{\omega \in \Omega : (X_k(\omega), \cdots, X_{k+n-1}(\omega)) = (a_0, \cdots, a_{n-1})\}.$$

Consider $\mathcal{F}$ as the σ-algebra generated by all cylinders of all sizes and let $\mathbb{P}$ be a probability measure defined over $\mathcal{F}$. To avoid uninteresting cases, we suppose that $0 < \mathbb{P}(X_0 = a) < 1$, for all $a \in \mathcal{A}$.

Given a cylinder A defined by the sequence a_0^{n-1}, the hitting time of a realization ω of Ω to the cylinder A, denoted by $\tau_A(\omega)$, is given by the function $\tau_A : \Omega \longrightarrow \mathbb{N} \cup \{\infty\}$ defined as

$$\tau_A(\omega) = \inf\{k \geq 1 : (X_k(\omega), \cdots, X_{k+n-1}(\omega)) = (a_0, \cdots, a_{n-1})\},$$

or infinity otherwise. Note that the following equality holds for all positive integers t

$$\{\omega \in \Omega : \tau_A(\omega) > t\} = \bigcap_{k=1}^{t} A_k^c,$$

where B^c stands for the complementary set of B. We also denote the conditional probability with respect to the event B

$$\mathbb{P}(U|B) = \frac{\mathbb{P}(U \cap B)}{\mathbb{P}(B)},$$

for any U in $\mathcal{F}$.

Our goal in this work is to study the exact form of the probability distribution $\mathbb{P}(\tau_A > t)$. Our strategy to tackle this question is different from the previous ones found in the literature, and it consists of building homogeneous linear recurrence relations for $\mathbb{P}(\tau_A > t)$.

1.2 The general recurrence relation

Without any other assumption for $\mathbb{P}$, the probability distribution $\mathbb{P}(\tau_A > t)$ can be written as

$$\mathbb{P}(\tau_A > t) = \mathbb{P}(\tau_A > t-1) - \mathbb{P}(\tau_A = t),$$

for all positive integer t. Since $\{\tau_A = t\}$ is equal to the intersection $\{\tau_A > t-1\} \cap A_t$, then

$$\mathbb{P}(\tau_A > t) = \mathbb{P}(\tau_A > t-1) - \mathbb{P}(\tau_A > t-1|A_t)\mathbb{P}(A_t).$$

Setting $y_t = \mathbb{P}(\tau_A > t)$, $z_{t-1} = \mathbb{P}(\tau_A > t-1|A_t)$ and $\alpha_{n,t} = \mathbb{P}(A_t)$, we obtain

$$y_t = y_{t-1} - \alpha_{n,t}\, z_{t-1}. \tag{1.1}$$

We can already see in (1.1) a linear recurrence relation between y_t and y_{t-1}. But it is not clear yet what role the conditional probability z_{t-1} plays in the recurrence. In order to turn (1.1) in a homogeneous linear recurrence relation, we need to write z_{t-1} as a sum of terms y_{t-k}, for some positive integers k taken in a fixed set of indexes, where the elements of this set do not depend on the value of t.

From now on, we assume that $\{X_n : n \in \mathbb{N}\}$ are independent and identically distributed random variables. Note that this means that the probability measure $\mathbb{P}$ is stationary. Thus, henceforth we can omit the index t in the definition of $\alpha_{n,t}$ in (1.1), that is,

$$y_t = y_{t-1} - \alpha_n\, z_{t-1}. \tag{1.2}$$

We can also use a structural quantity called the first possible return of a_0^{n-1}, denoted $T(a_0^{n-1})$ and defined as

$$T(a_0^{n-1}) = \inf\{\tau_A(\omega) : \omega \in A\}.$$

The first possible return $T(a_0^{n-1})$ can be seen as the minimum number of shifts necessary to occur the first overlap between the sequence a_0^{n-1} and a translated copy of itself. For example, consider $a_0^5 = (1,0,1,0,1,0)$

$$
\begin{array}{rcccccccc}
 & 1 & 0 & \vdots\, 1 & 0 & 1 & 0\, \vdots & & \\
1^{st} \text{ shift} & & 1 & \vdots\, 0 & 1 & 0 & 1\, \vdots & 0 & \\
2^{nd} \text{ shift} & & & \vdots\, 1 & 0 & 1 & 0\, \vdots & 1 & 0
\end{array}
$$

Thus we have $T(a_0^5) = 2$.

Note that $1 \leqslant T(a_0^{n-1}) \leqslant n$. Indeed, since we take the infimum over all realizations ω in Ω that start with a_0^{n-1}, if the first overlap does not happen in less than n shifts, then there exists a realization ω such that

$$\omega = (a_0, \cdots, a_{n-1}, a_0, \cdots, a_{n-1}, \cdots).$$

In the following sections, we explore the differences that can be found in the term z_{t-1} when we vary the value of $T(a_0^{n-1})$.

1.3 The case $T(a_0^{n-1}) = n$

We first consider a cylinder A defined by a_0^{n-1} such that $T(a_0^{n-1}) = n$, that is, the sequence a_0^{n-1} does not have an overlap. Two examples of sequences of size 6 that have this property are $(1,0,0,0,0,0)$ and $(0,1,0,0,1,1)$.

In the next proposition, we show that in this case we can write (1.2) as a homogeneous linear recurrence relation of order n.

Proposition 1. *Let A be the cylinder defined by the sequence a_0^{n-1} such that $T(a_0^{n-1}) = n$, then*

$$y_t = y_{t-1} - \alpha_n y_{t-n}.$$

Proof. We need to show that $z_{t-1} = y_{t-n}$. Note that if a realization ω is an element of A_t, then it is not possible to observe a_0^{n-1} in ω from time $t-n+1$ until $t-1$ and also from time $t+1$ until $t+n-1$. Hence, if ω is in A_t, then

$$\omega \in \bigcap_{j=t-n+1}^{t-1} A_j^c.$$

Therefore

$$\begin{aligned} z_{t-1} &= \mathbb{P}(\tau_A > t-1|A_t) \\ &= \mathbb{P}\left(\bigcap_{j=1}^{t-1} A_j^c \middle| A_t\right) \\ &= \mathbb{P}\left(\bigcap_{j=1}^{t-n} A_j^c \middle| A_t\right) \\ &= \mathbb{P}(\tau_A > t-n|A_t). \end{aligned}$$

Since $\{\tau_A > t-n\}$ is defined through $X_1, \cdots, X_{t-1}$ and A_t is defined through $X_t, \cdots, X_{t+n-1}$, then the events are independent and the result follows. □

1.4 The case $T(a_0^{n-1}) = 1$

Now, we consider a cylinder A defined by a_0^{n-1} with $T(a_0^{n-1}) = 1$, this means that it is only necessary one shift to see an overlap bewteen a_0^{n-1} and a translated copy. Note that the sequences that are such that $T(a_0^{n-1}) = 1$ are precisely the sequences $(a, a, \cdots, a)$ of size n, where a is some element of $\mathcal{A}$.

Unlike the previous case, if we observe at time t the sequence a_0^{n-1} in some realization ω in Ω, then it is possible to observe again a_0^{n-1} in ω from time $t-n+1$ until $t-1$ and also from time $t+1$ until $t+n-1$. Nevertheless, we can still prove a similar result to Proposition 1, but in this case we can write (1.2) as a homogeneous linear recurrence relation of order $n+1$.

Proposition 2. *Let A be the cylinder defined by the sequence $a_0^{n-1} = (a, \cdots, a)$, for some a in $\mathcal{A}$, then*

$$y_t = y_{t-1} - \alpha_n \theta\, y_{t-n-1},$$

where $\theta = 1 - \mathbb{P}(X_0 = a)$.

Proof. We need to show that $z_{t-1} = \theta\, y_{t-n-1}$. First note that

$$\begin{aligned} \mathbb{P}(\tau_A > t-1|A_t) &= \sum_{a_i \in \mathcal{A}} \mathbb{P}(\{\tau_A > t-1\} \cap \{X_{t-1} = a_i\}|A_t) \\ &= \mathbb{P}(\{\tau_A > t-1\} \cap \{X_{t-1} \neq a\}|A_t), \end{aligned}$$

where the last equality follows from the fact that if ω is in the intersection $\{\tau_A > t-1\} \cap A_t$, then $X_{t-1}(\omega) \neq a$. Therefore, it follows that

$$\omega \in \bigcap_{j=t-n}^{t-1} A_j^c.$$

Then

$$\begin{aligned} z_{t-1} &= \mathbb{P}(\tau_A > t-1 | A_t) \\ &= \mathbb{P}(\{\tau_A > t-1\} \cap \{X_{t-1} \neq a\} | A_t) \\ &= \mathbb{P}\left(\bigcap_{j=1}^{t-1} A_j^c, \{X_{t-1} \neq a\} \middle| A_t \right) \\ &= \mathbb{P}\left(\bigcap_{j=1}^{t-n-1} A_j^c, \{X_{t-1} \neq a\} \middle| A_t \right) \\ &= \mathbb{P}(\{\tau_A > t-n-1\} \cap \{X_{t-1} \neq a\} | A_t). \end{aligned}$$

We conclude the proof by noting that since $\{\tau_A > t-n-1\}$ is defined through $X_1, \cdots, X_{t-2}$ and A_t is defined through $X_t, \cdots, X_{t+n-1}$, then the events are independent and both are independent of $\{X_{t-1} \neq a\}$. □

1.5 The case $1 < T(a_0^{n-1}) < n$

We recall that the general recurrence formula is

$$y_t = y_{t-1} - \alpha_n z_{t-1}$$

with $y_t = \mathbb{P}(\tau_A > t)$, $z_{t-1} = \mathbb{P}(\tau_A > t-1 | A_t)$ and $\alpha_n = \mathbb{P}(A_t)$. Our challenge is writing the term z_{t-1} as sum of y_k's.

The next example illustrates some differences that may appear in the case which the cylinder A is defined by a_0^{n-1} with $T(a_0^{n-1}) = T$ such that $2 \leqslant T \leqslant n-1$.

Example 1. *Suppose that $\mathcal{A} = \{0, 1\}$ and denote $p = \mathbb{P}(X_0 = 1)$. Consider the cylinder A defined by the sequence*

$$a_0^{n-1} = (\underbrace{0, \cdots, 0}_{\lfloor \frac{n}{2} \rfloor \text{ times}}, 1, \underbrace{0, \cdots, 0}_{\lfloor \frac{n}{2} \rfloor \text{ times}}),$$

where n is a non-negative odd integer. Thus, the first possible return of the sequence a_0^{n-1} is $T(a_0^{n-1}) = \lceil \frac{n}{2} \rceil$.

We partition the sample space Ω into the events

$$\begin{aligned} \Omega_0 &= \{(X_1, \cdots, X_{t-1}) = (0, \cdots, 0)\}, \\ \Omega_1 &= \bigcup_{j=1}^{t-1} \{X_j = 1, X_k = 0,\ k \in \{1, ..., t-1\} \backslash \{j\}\}, \\ \Omega_2 &= \bigcup_{1 \leqslant i < j \leqslant t-1} \{X_i = 1, X_j = 1, X_k = 0,\ k \in \{1, ..., t-1\} \backslash \{i, j\}\}. \end{aligned}$$

We are interested in analysing the intersection $\{\tau_A > t-1\} \cap \Omega_i$, conditioned to the event A_t, for $i = 0, 1, 2$.

Note that Ω_0 *is a subset of* $\{\tau_A > t-1\}$ *and that* Ω_0 *and* A_t *are independent. Hence,*

$$\mathbb{P}(\{\tau_A > t-1\} \cap \Omega_0 | A_t) = (1-p)^{t-1}. \tag{1.3}$$

Now, note that if ω *is an element of* $\Omega_1 \cap A_t$ *such that* $X_j = 1$, *with* j *in* $\{\lceil \frac{n}{2} \rceil, \cdots, t-1\}$, *then* ω *is not an element of* $\{\tau_A > t-1\}$. *This means that if* ω *is in the intersection* $\{\tau_A > t-1\} \cap \Omega_1 \cap A_t$, *then* $X_j(\omega) = 1$, *for some* j *in* $\{1, \cdots, \lfloor \frac{n}{2} \rfloor\}$. *Hence*

$$\mathbb{P}(\{\tau_A > t-1\} \cap \Omega_1 | A_t) = \left\lfloor \frac{n}{2} \right\rfloor p(1-p)^{t-2}. \tag{1.4}$$

Equations (1.3) and (1.4) show that, in this example, the recurrence for y_t *have some non-homogeneous terms. To illustrate what happens to the intersection* $\{\tau_A > t-1\} \cap \Omega_2$, *conditioned to the event* A_t, *we use the set*

$$\widetilde{\Omega}_2 = \bigcup_{i=1}^{t-3} \{X_i = 1, X_{i+1} = 1, X_k = 0,\ k \in \{i+2, ..., t-1\}\},$$

which is a subset of Ω_2. *Then*

$$\mathbb{P}(\{\tau_A > t-1\} \cap \widetilde{\Omega}_2 | A_t) = \sum_{i=1}^{t-3} p^2(1-p)^{t-i-2} \mathbb{P}(\tau_A > i-n). \tag{1.5}$$

It has appeared in (1.5) a sum of terms of the form y_j, *for* $j = 1, \cdots, t-n-3$. *This implies that the degree of the recurrence relation in this case also depends on t, and hence the problem cannot be solved using the usual method for finite linear recurrences.*

Even if we could find an exact form for the recurrence y_t for all values of $T(a_0^{n-1})$, the preceding example suggests that if the first possible return is a value between 2 and $t-1$, then the term z_{t-1} is a sum of terms of the form y_k, where the number of indexes k may depend on the value of t. Also, as in Example 1, the recurrence may have a sum of non-homogeneous terms. A way to work around these problems is to find recurrence inequalities for y_t.

Proposition 3. *Let* A *be the cylinder defined by the sequence* a_0^{n-1} *and suppose* $T(a_0^{n-1}) = T$ *with* $2 \leqslant T \leqslant n-1$. *Then*

1. $y_t \geqslant y_{t-1} - \alpha_n \theta\, y_{t-n-T+1}$,
2. $y_t \leqslant y_{t-1} - \alpha_n \theta\, y_{t-n-T+1} + (n-1)\alpha_n^2\, y_{t-2n-T+1}$.

where $\theta = \mathbb{P}(A_{t-T}^c | A_t)$.

Proof. Since $T(a_0^{n-1}) = T$, if ω is in A_t, then

$$(X_j(\omega), \cdots, X_{j+n-1}(\omega)) \neq (a_0, \cdots, a_{n-1}),$$

for $j = t - T + 1, \cdots, t - 1$, that is,

$$\omega \in \bigcap_{j=t-T+1}^{t-1} A_j^c.$$

Therefore,

$$\begin{aligned}
z_{t-1} &= \mathbb{P}(\tau_A > t-1 | A_t) \\
&= \mathbb{P}\left(\bigcap_{j=1}^{t-1} A_j^c \middle| A_t \right) \\
&= \mathbb{P}\left(\bigcap_{j=1}^{t-T} A_j^c \middle| A_t \right) \\
&\leqslant \mathbb{P}\left(\bigcap_{j=1}^{t-n-T} A_j^c, A_{t-T}^c | A_t \right) \\
&= \mathbb{P}\left(\bigcap_{j=1}^{t-n-T} A_j^c \right) \mathbb{P}(A_{t-T}^c | A_t) \\
&= \theta\, \mathbb{P}(\tau_A > t - n - T) \\
&= \theta\, y_{t-n-T+1}
\end{aligned}$$

and this proves inequality 1.

For inequality 2, we begin in the same way, but now we will subtract the probability of what we added in the inequality 1.

$$\begin{aligned}
z_{t-1} &= \mathbb{P}(\tau_A > t-1 | A_t) \\
&= \mathbb{P}\left(\bigcap_{j=1}^{t-1} A_j^c \middle| A_t \right) \\
&= \mathbb{P}\left(\bigcap_{j=1}^{t-T} A_j^c \middle| A_t \right) \\
&= \mathbb{P}\left(\bigcap_{j=1}^{t-n-T} A_j^c, A_{t-T}^c \middle| A_t \right) - \mathbb{P}\left(\bigcap_{j=1}^{t-n-T} A_j^c, \bigcup_{j=t-n-T+1}^{t-T-1} A_j, A_{t-T}^c \middle| A_t \right) \\
&\geqslant \theta\, \mathbb{P}(\tau_A > t - n - T) - \mathbb{P}\left(\bigcap_{j=1}^{t-2n-T} A_j^c \right) \mathbb{P}\left(\bigcup_{j=t-n-T+1}^{t-T-1} A_j \right) \\
&\geqslant \theta\, \mathbb{P}(\tau_A > t - n - T) - (n-1)\alpha_n \mathbb{P}(\tau_A > t - 2n - T) \\
&= \alpha_n \theta\, y_{t-n-T+1} + (n-1)\alpha_n^2\, y_{t-2n-T+1},
\end{aligned}$$

where in the last inequality we used the following fact

$$\mathbb{P}\left(\bigcup_{j=t-n-T+1}^{t-T-1} A_j\right) \leqslant \sum_{j=t-n-T+1}^{t-T-1} \mathbb{P}(A_j) = (n-1)\alpha_n.$$

□

1.6 The exact distribution of the hitting time

We found in Propositions 1 and 2 that when the first possible return $T(a_0^{n-1})$ is equal to 1 or to n, then the general recurrence (1.2) can be written as a homogeneous linear recurrence relation with constant coefficients. To solve it, we use classical results.[7]

We begin with the case where the first possible return is $T(a_0^{n-1}) = n$. Then the characteristic polynomial associated to the recurrence found in Proposition 1 is

$$f(x) = x^n - x^{n-1} + \alpha_n. \tag{1.6}$$

It follows from the theory of homogeneous linear recurrence relations that we can write the explicit formula for the solution of the recurrence in Proposition 1 in terms of the roots of f. Also, the multiplicities of these roots have to be considered so we can decide the number of terms and the form of the coefficients in the general solution.

Hence, in the next theorem, we give the exact form of the distribution of the hitting time τ_A by finding the multiplicities of the roots of f.

Theorem 1. *Suppose A is a cylinder defined by the sequence a_0^{n-1} such that $T(a_0^{n-1}) = n$. Let f be the polynomial defined as in (1.6). If $\alpha_n \neq \frac{(n-1)^{n-1}}{n^n}$, then the distribution of the hitting time τ_A is given by*

$$\mathbb{P}(\tau_A > t) = \sum_{j=1}^{n} C_j r_j^t,$$

for some constants $C_1, \cdots, C_n$, and $r_1, \cdots, r_n$ are the distinct roots of f. And if $\alpha_n = \frac{(n-1)^{n-1}}{n^n}$, then the distribution of the hitting time τ_A is given by

$$\mathbb{P}(\tau_A > t) = \sum_{j=1}^{n-2} D_j s_j^t + (D_{n-1} + tD_n)s_{n-1}^t,$$

for some constants $D_1, \cdots, D_n$, where $s_1, \cdots, s_{n-2}$ are the roots of f with multiplicity 1, and s_{n-1} is the root of f with multiplicity 2.

Proof. It is sufficient to show that if $\alpha_n \neq \frac{(n-1)^{n-1}}{n^n}$, then f has n distinct roots $r_1, \cdots, r_n$. And if $\alpha_n = \frac{(n-1)^{n-1}}{n^n}$, then f has $n-2$ roots $s_1, \cdots, s_{n-2}$ with multiplicity 1 and one root s_{n-1} with multiplicity 2.

Note that the derivative of f is

$$f'(x) = x^{n-2}(nx - n + 1),$$

so the roots of f' are 0, with multiplicity $n-2$, and $\frac{n-1}{n}$ with multiplicity 1. Since $\alpha_n > 0$, then 0 cannot be a root of f. On the other hand, note that

$$f(\tfrac{n-1}{n}) = (\tfrac{n-1}{n})^{n-1}(\tfrac{n-1}{n} - 1) + \alpha_n = -\tfrac{(n-1)^{n-1}}{n^n} + \alpha_n.$$

Thus, $\frac{n-1}{n}$ is a root of f with multiplicity 2 if and only if $\alpha_n = \frac{(n-1)^{n-1}}{n^n}$. □

In order to find the constants mentioned in the above theorem, we can use the set $\{\mathbb{P}(\tau_A > j), j = 0, \cdots, n-1\}$ as initial conditions, where

$$\mathbb{P}(\tau_A > j) = 1 - j\alpha_n, \tag{1.7}$$

$j = 0, \cdots, n-1$. To prove (1.7), note that $\mathbb{P}(\tau_A > 0) = 1$ and

$$\begin{aligned}\mathbb{P}(\tau_A = j) &= \mathbb{P}(A_j)\mathbb{P}\left(\bigcap_{i=1}^{j-1} A_i^c \middle| A_j\right)\\ &= \mathbb{P}(A_j)\\ &= \alpha_n,\end{aligned}$$

for $j = 1, \cdots, n-1$. Therefore,

$$\begin{aligned}\mathbb{P}(\tau_A > j) &= 1 - \mathbb{P}(\tau_A \leqslant j)\\ &= 1 - \sum_{i=1}^{j} \mathbb{P}(\tau_A = i)\\ &= 1 - j\alpha_n,\end{aligned}$$

for $j = 1, \cdots, n-1$.

Thus, we can find the constants $C_1, \cdots, C_n$, if we solve the system of linear equations

$$\begin{cases}\mathbb{P}(\tau_A > 0) = C_1 + \cdots + C_n\\ \mathbb{P}(\tau_A > 1) = C_1 r_1 + \cdots + C_n r_n\\ \vdots\\ \mathbb{P}(\tau_A > n-1) = C_1 r_1^{n-1} + \cdots + C_n r_n^{n-1}.\end{cases}$$

And we can find the constants $D_1, \cdots, D_n$, if we solve the system of linear equations

$$\begin{cases} \mathbb{P}(\tau_A > 0) = D_1 + \cdots + D_{n-1} \\ \mathbb{P}(\tau_A > 1) = D_1 s_1 + \cdots + D_{n-2} r_{n-2} + (D_{n-1} + D_n) s_{n-1} \\ \vdots \\ \mathbb{P}(\tau_A > n-1) = D_1 s_1^{n-1} + \cdots + D_{n-2} r_{n-2}^{n-1} + (D_{n-1} + (n-1) D_n) s_{n-1}^{n-1}. \end{cases}$$

Now, we consider the case that the first possible return is $T(a_0^{n-1}) = 1$. The characteristic polynomial associated to the recurrence found in Proposition 2 is

$$g(x) = x^{n+1} - x^n + \alpha_n \theta. \tag{1.8}$$

We proceed as before in the next theorem, we give the exact distribution of τ_A by defining the multiplicities of the roots of g.

Theorem 2. *Suppose A is a cylinder defined by the sequence a_0^{n-1} such that $T(a_0^{n-1}) = 1$. Let g be the polynomial defined as in (1.8). If $\alpha_n \theta \neq \frac{n^n}{(n+1)^{n+1}}$, then the distribution of the hitting time τ_A is given by*

$$\mathbb{P}(\tau_A > t) = \sum_{j=1}^{n+1} C_j r_j^t,$$

for some constants $C_1, \cdots, C_{n+1}$, where $r_1, \cdots, r_{n+1}$ are the distinct roots of g. And if $\alpha_n \theta = \frac{n^n}{(n+1)^{n+1}}$, then the distribution of the hitting time τ_A is given by

$$\mathbb{P}(\tau_A > t) = \sum_{j=1}^{n-1} D_j s_j^t + (D_n + t D_{n+1}) s_n^t,$$

for some constants $D_1, \cdots, D_{n+1}$, where $s_1, \cdots, s_{n-1}$ are the roots of g with multiplicity 1, and s_n is the root of g with multiplicity 2.

Proof. We only need to show that if $\alpha_n \theta \neq \frac{n^n}{(n+1)^{n+1}}$, then g has $n+1$ distinct roots $r_1, \cdots, r_{n+1}$. And if $\alpha_n \theta = \frac{n^n}{(n+1)^{n+1}}$, then g has $n-1$ roots $s_1, \cdots, s_{n-1}$ with multiplicity 1 and one root s_{n-1} with multiplicity 2.

The derivative of g is such that

$$g'(x) = x^{n-1}((n+1)x - n),$$

then the roots of g' are 0, with multiplicity $n-1$, and $\frac{n}{n+1}$ with multiplicity 1. Since $\alpha_n \theta > 0$, then 0 cannot be a root of g. On the other hand, note that

$$g(\tfrac{n}{n+1}) = (\tfrac{n}{n+1})^n(\tfrac{n}{n+1} - 1) + \alpha_n \theta = -\frac{n^n}{(n+1)^{n+1}} + \alpha_n.$$

Thus, $\frac{n}{n+1}$ is a root of g with multiplicity 2 if and only if $\alpha_n \theta = \frac{n^n}{(n+1)^{n+1}}$. $\square$

Again, to find the constants mentioned in the above theorem, we can use the set of initial conditions $\{\mathbb{P}(\tau_A > j), j = 0, \cdots, n\}$, where $\mathbb{P}(\tau_A > 0) = 1$ and

$$\begin{aligned}\mathbb{P}(\tau_A > j) &= 1 - \mathbb{P}(\tau_A \leqslant j)\\ &= 1 - \sum_{i=1}^{j} \mathbb{P}(\tau_A = i)\\ &= 1 - \alpha_n - (j-1)\alpha_n\theta,\end{aligned}$$

with $j = 1, \cdots, n$.

1.7 Examples

In the next two examples, we suppose $\mathcal{A} = \{0, 1\}$ and denote $p = \mathbb{P}(X_0 = 1)$.

Example 2. *Consider A the cylinder defined by $(1, 0)$ and suppose that $p \neq 0.5$. Using Proposition 1, we have the following recurrence relation*

$$y_t = y_{t-1} - p(1-p)y_{t-2}.$$

So the associated characteristic polynomial is

$$f(x) = x^2 - x + p(1-p),$$

where the roots r_1 and r_2 are

$$\begin{aligned}r_1 &= 1 - p,\\ r_2 &= p.\end{aligned}$$

Thus, the roots are distinct and, by Theorem 1, the general solution is of the form

$$\mathbb{P}(\tau_A > t) = C_1 r_1^t + C_2 r_2^t,$$

where C_1 and C_2 are constants. The initial conditions are

$$\begin{aligned}\mathbb{P}(\tau_A > 0) &= 1,\\ \mathbb{P}(\tau_A > 1) &= 1 - p(1-p).\end{aligned}$$

Then, solving the system of equations

$$\begin{cases}\mathbb{P}(\tau_A > 0) = C_1 + C_2\\ \mathbb{P}(\tau_A > 1) = C_1 r_1 + C_2 r_2\end{cases}$$

we obtain

$$C_1 = \frac{(1-p)^2}{2(1-p)-1},$$
$$C_2 = \frac{p^2}{2p-1}.$$

Example 3. *Consider A the cylinder defined by* $(1,1)$*. By Proposition 2, the recurrence relation for* y_t *is*

$$y_t = y_{t-1} - p^2(1-p)y_{t-3}.$$

So the associated characteristic polynomial is

$$g(x) = x^2(x-1) + p^2(1-p),$$

where the roots of g *are*

$$r_1 = \frac{1-p+\sqrt{(1-p)(1+3p)}}{2},$$
$$r_2 = \frac{1-p-\sqrt{(1-p)(1+3p)}}{2},$$
$$r_3 = p.$$

Thus, by Theorem 2, the general solution is of the form

$$y_t = C_1 r_1^t + C_2 r_2^t + C_3 r_3^t,$$

where C_j*,* $j = 1,2,3$*, are constants. The initial conditions are*

$$\mathbb{P}(\tau_A > 0) = 1,$$
$$\mathbb{P}(\tau_A > 1) = 1 - p^2,$$
$$\mathbb{P}(\tau_A > 2) = 1 - p^2 - (1-p)p^2,$$

and solving the system of linear equations

$$\begin{cases} \mathbb{P}(\tau_A > 0) = C_1 + C_2 \\ \mathbb{P}(\tau_A > 1) = C_1 r_1 + C_2 r_2 \\ \mathbb{P}(\tau_A > 2) = C_1 r_1^2 + C_2 r_2^2 \end{cases}$$

we find that

$$
\begin{aligned}
C_1 &= \frac{1}{2} + \frac{1 + p - 2p^2}{2\sqrt{(1-p)(1+3p)}}, \\
C_2 &= \frac{1}{2} - \frac{1 + p - 2p^2}{2\sqrt{(1-p)(1+3p)}}, \\
C_3 &= 0.
\end{aligned}
$$

Chapter 2

The parameters of the exact distribution

2.1 Introduction

In this chapter, we focus on the case of a cylinder A defined by a_0^{n-1} such that $T(a_0^{n-1}) = n$. From now on, we consider $\alpha_n < \frac{(n-1)^{n-1}}{n^n}$, since this is the typical case when n is large. Then by Theorem 1, we have that the distribution of τ_A is a sum of exponentials terms, that is,

$$\mathbb{P}(\tau_A > t) = \sum_{j=1}^{n} C_j r_j^t,$$

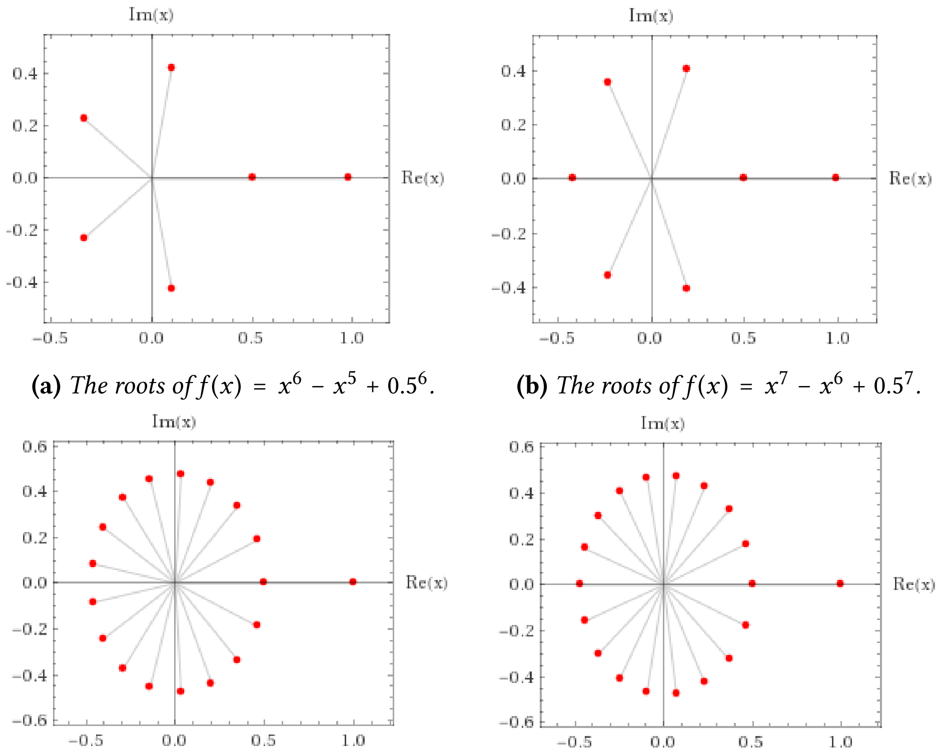
where $r_1, \cdots, r_n$ are the distinct roots of the polynomial $f(x) = x^n - x^{n-1} + \alpha_n$, and $C_1, \cdots, C_n$ are the constants that are uniquely defined after we apply the initial conditions $\mathbb{P}(\tau_A > j) = 1 - j\alpha_n$, for $j = 0, \cdots, n-1$, as shown in (1.7).

In Section 2.2, we study properties of the roots $r_1, \cdots, r_n$. In Section 2.3, we give an explicit formula for the constant C_j in terms of the corresponding root r_j, for $j = 1, \cdots, n$, and we show that the distribution of the hitting time has a dominant term and that the others converge to zero.

2.2 Analysis of the roots

Since we consider $\alpha_n < \frac{(n-1)^{n-1}}{n^n}$, then all the roots $r_1, \cdots, r_n$ of f are distinct. By studying the first derivative of f, we find that f has two positive real roots. The number of negative real roots depends on the parity of the degree of f. If n is odd, then f has one negative real root. If n is even, then f does not have any negative real root. See Figure 2.1.

Suppose that the indexes of the roots are such that $|r_1| > |r_j|$, for $j = 2, \cdots, n$. We are interested in defining upper and lower bounds for the largest root r_1 and we wish to describe a circle with a radius less than $|r_1|$ that contains the remaining roots.



(a) *The roots of* $f(x) = x^6 - x^5 + 0.5^6$. **(b)** *The roots of* $f(x) = x^7 - x^6 + 0.5^7$.

(c) *The roots of* $f(x) = x^{18} - x^{17} + 0.5^{18}$. **(d)** *The roots of* $f(x) = x^{19} - x^{18} + 0.5^{19}$.

Figure 2.1: *Comparison of the graphical representation of the roots of* $f(x) = x^n - x^{n-1} + 0.5^n$ *for different values of n. The polynomial f is associated to the cylinder of size n defined by* $(1, 0, \cdots, 0)$ *in the case that* $\mathcal{A} = \{0, 1\}$ *and* $\mathbb{P}(X_0 = 0) = 0.5$.

Consider then the following polynomial

$$l(x) = x^n - x^{n-1}.$$

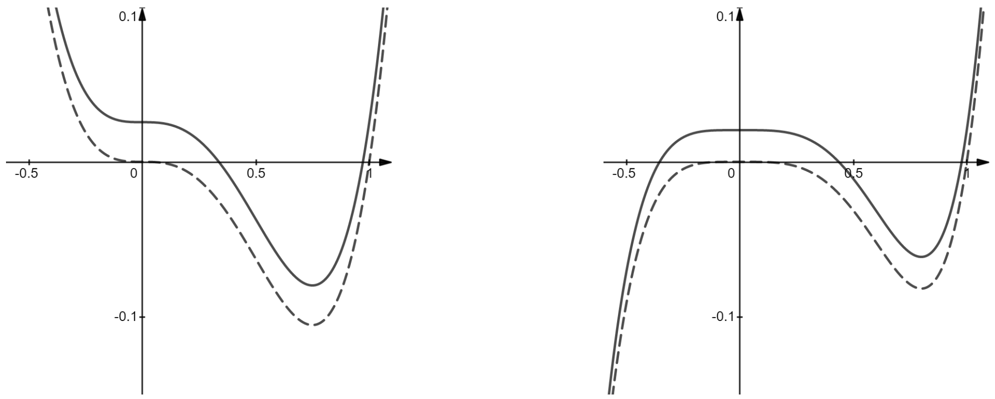
The only difference between the polynomial f and l is the constant coefficient α_n, that is,

$$f(x) - l(x) = \alpha_n,$$

for all complex values x. Also, since the polynomial l can be factored as $l(x) = x^{n-1}(x - 1)$, then the roots of l are 0 and 1, with multiplicities equal to $n - 1$ and 1, respectively. See Figure 2.2.

It is a known result that the roots of a polynomial vary continuously as a function of the coefficients.[12] It means that since f and l differ only by α_n, then $n - 1$ roots of f are inside a neighborhood centered at the origin, and the remaining root of f is inside a neighborhood centered at 1. That is, the roots of f and l are, in a sense, close. We use Rouché's Theorem as a tool to characterize the neighborhood centered at the origin.

Theorem. *(Rouché's Theorem) Suppose that*



(a) *The dashed line represents the polynomial* $l(x) = x^4 - x^3$ *and the solid line represents the polynomial* $f(x) = x^4 - x^3 + 0.8^2 0.2^2$, *which is associated to the cylinder defined by* $(1, 1, 0, 0)$.

(b) *The dashed line represents the polynomial* $l(x) = x^5 - x^4$ *and the solid line represents the polynomial* $f(x) = x^5 - x^4 + 0.8^3 0.2^2$, *which is associated to the cylinder defined by* $(1, 1, 1, 0, 0)$.

Figure 2.2: *Comparison of the graphs of the polynomials f and l for the even n case (a) and for the odd n case (b).*

1. *two functions h_1 and h_2 are analytic inside and on a simple closed contour C;*
2. *$|h_1(x)| > |h_2(x)|$ at each point x in C.*

Then h_1 and $h_1 + h_2$ have the same number of roots, counting multiplicities, inside C.

Note that, if we find ϵ in $(0, 1)$ such that

$$|l(x)| > f(x) - l(x), \tag{2.1}$$

for all complex values x such that $|x| = \epsilon$, then, by Rouché's Theorem, l and f have $n - 1$ roots, counting multiplicities, inside the circle centered at the origin with radius ϵ. Notice that the condition (2.1) can be written as

$$\min\{|l(x)| : x \in \mathbb{C} \text{ with } |x| = \epsilon\} > \alpha_n,$$

which is equivalent to

$$\epsilon^{n-1}(\epsilon - 1) + \alpha_n < 0. \tag{2.2}$$

Hence, by straightforward computations, one can prove the following propositions.

Proposition 4. *Fixed $\delta > 0$, there exists n_0 such that $\epsilon = \alpha_n^{\frac{1}{(n-1)(1+\delta)}}$ satisfies (2.2), for every $n \geqslant n_0$.*

Proposition 5. *Fixed η in $(0, 1)$, there exists n_0 such that*

$$1 - \alpha_n - \alpha_n^{1+\eta} \leqslant r_1 \leqslant 1 - \alpha_n - \alpha_n^2,$$

for every $n \geqslant n_0$.

Proposition 5 says that the largest root r_1 is a positive real number. Since we consider $\alpha_n < \frac{(n-1)^{n-1}}{n^n}$, there is another positive real root, which we denote by r_2. Note that

$$f(\alpha_n^{\frac{1}{n-1}}) = \alpha_n(\alpha_n^{\frac{1}{n-1}} - 1) + \alpha_n > 0. \tag{2.3}$$

Therefore, by Propositions 4 and 5, and by (2.3), if we fix η in $(0,1)$ and fix $\delta > 0$, then for large n, we have

$$1 - \alpha_n - \alpha_n^{1+\eta} \leqslant r_1 \leqslant 1 - \alpha_n - \alpha_n^2, \tag{2.4}$$

$$\alpha_n^{\frac{1}{n-1}} < r_2 < \alpha_n^{\frac{1}{(n-1)(1+\delta)}}, \tag{2.5}$$

and

$$|r_j| < \alpha_n^{\frac{1}{(n-1)(1+\delta)}}, \tag{2.6}$$

for $j = 3, \cdots, n$.

2.3 Analysis of the constants

We want to show that in Theorem 1 the constant C_j can be written in terms of the corresponding root r_j, for $j = 1, \cdots, n$. As said before, to determine $C_1, \cdots, C_n$, we have to solve the system of linear equations

$$\begin{cases} \mathbb{P}(\tau_A > 0) = C_1 + \cdots + C_n \\ \mathbb{P}(\tau_A > 1) = C_1 r_1 + \cdots + C_n r_n \\ \vdots \\ \mathbb{P}(\tau_A > n-1) = C_1 r_1^{n-1} + \cdots + C_n r_n^{n-1} \end{cases} \tag{2.7}$$

where, by (1.7), we have

$$\mathbb{P}(\tau_A > j) = 1 - j\alpha_n,$$

for $j = 0, \cdots, n-1$. Note that we can write (2.7) in the matricial form

$$V \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} - \alpha_n \begin{bmatrix} 0 \\ 1 \\ \vdots \\ n-1 \end{bmatrix},$$

where V is the square matrix of size n, called the Vandermonde matrix, which is defined as

$$V = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^{n-1} \end{bmatrix}.$$

Since all the roots of f are distinct, V is invertible, and therefore we can find the constants by calculating

$$\begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} = V^{-1} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} - \alpha_n V^{-1} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ n-1 \end{bmatrix}. \tag{2.8}$$

The matrix V^{-1} has the general form (see for instance [11])

$$V^{-1} = \begin{bmatrix} \dfrac{r_1^{n-1} + b_1 r_1^{n-2} + \cdots + b_{n-2} r_1 + b_{n-1}}{\prod_{j=2}^{n} (r_1 - r_j)} & \cdots & \dfrac{r_1^2 + b_1 r_1 + b_2}{\prod_{j=2}^{n} (r_1 - r_j)} & \dfrac{r_1 + b_1}{\prod_{j=2}^{n} (r_1 - r_j)} & \dfrac{1}{\prod_{j=2}^{n} (r_1 - r_j)} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \dfrac{r_n^{n-1} + b_1 r_n^{n-2} + \cdots + b_{n-2} r_n + b_{n-1}}{\prod_{j=1}^{n-1} (r_n - r_j)} & \cdots & \dfrac{r_n^2 + b_1 r_n + b_2}{\prod_{j=1}^{n-1} (r_n - r_j)} & \dfrac{r_n + b_1}{\prod_{j=1}^{n-1} (r_n - r_j)} & \dfrac{1}{\prod_{j=1}^{n-1} (r_n - r_j)} \end{bmatrix},$$

where

$$\begin{aligned} b_1 &= -\sum_{j=1}^{n} r_j, \\ b_2 &= \sum_{1 \leqslant j < k \leqslant n} r_j r_k, \\ &\vdots \\ b_{n-1} &= (-1)^{n-1} \sum_{1 \leqslant j_1 < j_2 < \cdots < j_{n-1} \leqslant n} r_{j_1} r_{j_2} \cdots r_{j_{n-1}}, \\ b_n &= (-1)^n r_1 r_2 \cdots r_n. \end{aligned} \tag{2.9}$$

Since the polynomial f can be factored as

$$f(x) = \prod_{j=1}^{n} (x - r_j),$$

then f can be written in terms of the coefficients b_j, for $j = 1, \cdots, n$. That is,

$$f(x) = x^n + b_1 x^{n-1} + \cdots + b_{n-1} x + b_n.$$

Therefore, we also have

$$\begin{aligned} b_1 &= -1, \\ b_2 &= 0, \\ b_3 &= 0, \\ &\vdots \\ b_{n-1} &= 0, \\ b_n &= \alpha_n. \end{aligned} \tag{2.10}$$

Thus, in our case, V^{-1} has the following simplified form

$$V^{-1} = \begin{bmatrix} \frac{r_1^{n-1} - r_1^{n-2}}{\prod_{j=2}^{n}(r_1 - r_j)} & \cdots & \frac{r_1^2 - r_1}{\prod_{j=2}^{n}(r_1 - r_j)} & \frac{r_1 - 1}{\prod_{j=2}^{n}(r_1 - r_j)} & \frac{1}{\prod_{j=2}^{n}(r_1 - r_j)} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{r_n^{n-1} - r_n^{n-2}}{\prod_{j=1}^{n-1}(r_n - r_j)} & \cdots & \frac{r_n^2 - r_n}{\prod_{j=1}^{n-1}(r_n - r_j)} & \frac{r_n - 1}{\prod_{j=1}^{n-1}(r_n - r_j)} & \frac{1}{\prod_{j=1}^{n-1}(r_n - r_j)} \end{bmatrix}.$$

Then solving (2.8), we get

$$C_j = \frac{r_j^{n-1} - \alpha_n \dfrac{1 - r_j^{n-1}}{1 - r_j}}{\prod_{k \neq j}(r_j - r_k)},$$

for $j = 1, \cdots, n$. Using (2.9) and (2.10), the denominator of C_j can be simplified as

$$\prod_{k \neq j}(r_j - r_k) = n r_j^{n-1} - (n-1) r_j^{n-2},$$

for $j = 1, \cdots, n$. Therefore,

$$C_j = \frac{r_j - \alpha_n \dfrac{r_j^{-(n-2)} - r_j}{1 - r_j}}{n r_j - (n-1)}, \tag{2.11}$$

for $j = 1, \cdots, n$.

Moreover, since r_j is a root of f, then it satisfies

$$r_j^n - r_j^{n-1} + \alpha_n = 0,$$

or equivalently

$$r_j^{n-1} = \frac{\alpha_n}{1 - r_j}, \tag{2.12}$$

for $j = 1, \cdots, n$.

Thus, substituting (2.12) in (2.11), we obtain the final expression of C_j as function of the respective root r_j, that is,

$$C_j = \frac{r_j^n}{nr_j - (n-1)}, \tag{2.13}$$

for $j = 1, \cdots, n$.

Note that the complex roots come in conjugate pairs. Let r be a complex root of f and $\bar{r}$ be its conjugate root. If we denote by C_r and $C_{\bar{r}}$ the associated constants of the roots r and $\bar{r}$, respectively, then using (2.13), we can see that

$$C_{\bar{r}} = \bar{C}_r.$$

Thus, in the general solution, the sum

$$C_r r^t + \bar{C}_r \bar{r}^t$$

is a real number.

Finally, notice the inequality (2.4) shows that, when n is large, the root r_1 approaches the value 1. Hence, by (2.13), the constant C_1 also approaches the value 1. Further, using (2.5), (2.6) and (2.13), we can see that, for large n, the absolute value of the remaining constants are of order $\frac{\alpha_n}{n}$.

Chapter 3

Conclusion

In this work, for the cases that the first possible return $T(a_0^{n-1})$ is equal to 1 or to n, we proved that the exact distribution of the hitting time is a sum of exponentials. We showed that this sum has a dominant term and that the others converge to zero.

In Chapter 2, we focused on the case $T(a_0^{n-1}) = n$ for the analysis of the parameters of the exact distribution. Notice that analogous results proved in Sections 2.2 and 2.3 hold for the case $T(a_0^{n-1}) = 1$, since there are only small differences between the two cases. Namely: the initial conditions used to find the constants, the degree and the constant term of the characteristic polynomials f and g.

For the case $1 < T(a_0^{n-1}) < n$, we found inequalities for y_t in Proposition 3. Our idea is that one can solve the inequalities following the procedure exposed in Chapters 1 and 2 to obtain sharp upper and lower bounds for y_t. Note that the difference between the two inequalities in Proposition 3 is a term of order α_n^2, which is small for large n.

Moreover, we think that the same type of technique used to build the inequalities in Proposition 3 could be used in the case of discrete-time stochastic processes with some type of mixing condition assumed.

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