UNIVERSIDADE DE SÃO PAULO

Instituto de Ciências Matemáticas e de Computação

**On The Enclosure of Solutions of Nonlinear Equations in
Banach Space with Application to Boundary Value Problems**

Eduardo Ramos

Tese de Doutorado do Programa de Pós-Graduação em
Matemática (PPG-Mat)

SERVIÇO DE PÓS-GRADUAÇÃO DO ICMC-USP

Data de Depósito:

Assinatura: _____

Eduardo Ramos

On The Enclosure of Solutions of Nonlinear Equations in Banach Space with Application to Boundary Value Problems

Doctoral dissertation submitted to the Institute of Mathematics and Computer Sciences – ICMC-USP, in partial fulfillment of the requirements for the degree of the Doctorate Program in Mathematics.
EXAMINATION BOARD PRESENTATION COPY

Concentration Area: Mathematics

Advisor: Prof. Dr. Márcio Fuzeto Gameiro

USP – São Carlos
October 2019

Ficha catalográfica elaborada pela Biblioteca Prof. Achille Bassi
e Seção Técnica de Informática, ICMC/USP,
com os dados inseridos pelo(a) autor(a)

R175o Ramos, Eduardo
 On The Enclosure of Solutions of Nonlinear
Equations in Banach Space with Application to
Boundary Value Problems / Eduardo Ramos; orientador
Márcio Gameiro. -- São Carlos, 2019.
 82 p.

 Tese (Doutorado - Programa de Pós-Graduação em
Matemática) -- Instituto de Ciências Matemáticas e
de Computação, Universidade de São Paulo, 2019.

 1. Nonlinear Equation Solutions, Rigorous
Numerics. 2. Inexact Newton Method. 3. Two Point
Boundary Value Problem. 4. Neumann Boundary
Condition. I. Gameiro, Márcio, orient. II. Título.

Eduardo Ramos

**Isolamento de Raízes para Equações Não lineares em
Espaço de Banach com Aplicação em Problemas de Valor
de Contorno**

Tese apresentada ao Instituto de Ciências
Matemáticas e de Computação – ICMC-USP,
como parte dos requisitos para obtenção do título
de Doutor em Ciências – Matemática. *EXEMPLAR
DE DEFESA*

Área de Concentração: Matemática

Orientador: Prof. Dr. Márcio Fuzeto Gameiro

USP – São Carlos
Outubro de 2019

*Este trabalho é dedicado aos que acreditam na matemática
não apenas como um meio, mas como um fim,
pois sem estes a matemática, como a conhecemos, não existiria.*

ACKNOWLEDGEMENTS

Agradeço primeiramente a meu irmão Pedro Luiz Ramos por todo o suporte dado na organização e planejamento da tese aqui apresentada. Em segundo lugar agradeço a meus pais, em especial a minha mãe, por todo o suporte emocional e financeiro dados. Agradeço também à comunidade matemática, dentre os quais muitos amigos meus de longa data, que ajudaram em parte na construção da tese através de sugestões ou idéias.

Por fim mas não menos importante, agradeço a meu orientador Márcio Fuzeto Gameiro, e à USP de São Carlos, pela oportunidade que me deram de trabalhar com o que mais amo: a Matemática.

*“The true spirit of delight, the exaltation,
the sense of being more than man,
which is the touchstone of highest excellence,
is to be found in mathematics as surely as in poetry”
(Bertrand Russel)*

RESUMO

RAMOS, E. **Isolamento de Raízes para Equações Não lineares em Espaço de Banach com Aplicação em Problemas de Valor de Contorno**. 2019. 82 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2019.

Este trabalho é dedicado ao problema de se isolar raízes de equações em bolas fechadas em espaços de Banach. Desta forma, será apresentado neste trabalho um algoritmo com tal finalidade, com garantias de término, sob certas condições de computabilidade e regularidade da função que representa a equação. Também apresentaremos um teorema novo para factibilidade e convergência do Método de Newton Inexato, tal que o erro cometido não necessariamente deva tender a zero para que o método seja factível, condição esta imposta em outros trabalhos. Tal resultado é, até onde vai o conhecimento do autor, o primeiro resultado deste tipo na literatura. Finalmente, mostraremos como utilizar tal algoritmo de isolamento de raízes e o Método de Newton Inexato para isolar e aproximar raízes equações diferenciais com valor de contorno do tipo Neumann.

Palavras-chave: Soluções de Equações Não Lineares, Matemática Verificativa, Método de Newton Inexato, Equações Diferenciais, Condição de Contorno de Neumann.

ABSTRACT

RAMOS, E. **On The Enclosure of Solutions of Nonlinear Equations in Banach Space with Application to Boundary Value Problems.** 2019. 82 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2019.

This paper is dedicated to the problem of isolating zeros of non-linear equations in closed balls of Banach spaces. Thus, in this thesis we shall present an algorithm with such purpose, with terminating guarantees, under certain compatibility and regularity conditions of the function that represents the equation. We shall also present a new theorem for the feasibility and convergence of the Inexact Newton Method, such that the error committed does not necessarily must go to zero for the method to be feasible, where this conditions is usually imposed other works. This results is, to the best of the author knowledge, the first of this kind in the literature. Finally, we shall show how to apply the proposed zero isolation algorithm and the Inexact Newton Method to enclosure and approximate zeros of two point boundary value problem of Neumann type.

Keywords: Nonlinear Equation Solutions, Rigorous Numerics, Inexact Newton Method, Two Point Boundary Value Problem, Neumann Boundary Condition.

LIST OF FIGURES

Figure 1 – Non-existenc of zeros radius for $h(x,y)$ at $(x_0,y_0) = (0,0)$ and $(x_0,y_0) = (0.2,0.2)$	40
Figure 2 – Existence and unicity of zeros radius for $h(x,y)$ at $(x_0,y_0) = (0.54,0.54)$ and $(x_0,y_0) = (0.52,0.52)$	41
Figure 3 – Inexact Newton convergence radius for $(x_0,y_0) = (1/1.95, 1/1.95)$	42
Figure 4 – Subdivision of the circle and first step the algorithm applied to $h(x)$	47
Figure 5 – Step 3 and 4 of Algorithm 1 applied to $h(x,y)$	47

CONTENTS

1	INTRODUCTION	19
2	ENCLOSURE OF SOLUTIONS IN BANNACH SPACE EQUATIONS	21
2.1	Preliminaries	21
2.1.1	<i>Functional Analysis</i>	21
2.1.2	<i>Differential Calculus in Bannach Spaces</i>	22
2.1.3	<i>Differentiation in $\mathbb{R}^n$</i>	24
2.2	The bijectivity modulus, and norm computations.	26
2.2.1	<i>The bijectivity modulus</i>	26
2.2.2	<i>Computation of norms of Df and D^2f</i>	29
2.2.3	<i>Computation of $j(A)_2$, $\ A\ _2$ and $(A^{-1})v^T$</i>	31
2.3	Kantorovich Theorem and Inexact Newton Method	33
2.3.1	<i>Newton Method and Kantorovich Theorem.</i>	33
2.3.2	<i>Theorems for Isolation of Zeros</i>	38
2.3.3	<i>Interval Zeros Isolation Theorem</i>	42
2.3.4	<i>Isolation of Zeros Algorithm</i>	45
2.3.5	<i>Convergence of the Algorithm</i>	47
3	RIGOROUS COMPUTATION OF TWO-POINT BOUNDARY VALUE PROBLEM OF NEUMANN TYPE	53
3.1	Preliminaries	53
3.1.1	<i>Absolutely Continuous Functions</i>	53
3.1.2	<i>Fourier Analysis for cosine and sine series</i>	55
3.1.3	<i>The Sobolev Space $H^q(I)$.</i>	56
3.2	Approximation Theorems for Sobolev Space	58
3.2.1	<i>Truncation estimates for sine and cosine series</i>	58
3.2.2	<i>The Space $H_N^q(I)$.</i>	60
3.2.3	<i>Three Lemmas for $H^q(I)$.</i>	62
3.2.4	<i>Projection estimates on $H_N^2(I)$</i>	69
4	CONCLUSION	79
	BIBLIOGRAPHY	81

INTRODUCTION

Rigorous verification procedures for existence of solutions in non-linear partial differential equations have a great importance due to the fact that usually one can not obtain analytical solutions for such problems and moreover such problems can have multiple solutions. Different authors have proposed verification methods of solutions of partial differential equations such as the analytical method based on a Kantorovich-like theorem in (PLUM, 1995), the semi-analytical method in (NAKAO, 1988) and (NAKAO; YAMAMOTO, 1995), and the analytical method based on radii polynomials in (GAMEIRO; LESSARD, 2010) and (GAMEIRO; LESSARD; PUGLIESE, 2016).

On the other hand, it should be noted that, to date, no algorithm for isolation of all zeros of boundary values problems has been proposed with termination guarantee. Indeed, to the authors knowledge, the only current rigorous algorithm with guarantee of isolating all zeros of a general differentiable non-linear function in a given region is that of (YAP, 2009), which in turn works only for unidimensional functions.

With this in mind, we shall provide in the first chapter such an algorithm aimed to the general setting of Bannach Space non-linear equations, which we consider to be of great importance to the area, since this is, to the authors knowledge, the first rigorous method with such guarantees for the infinite dimensional setting.

Additionally, after a zero is proved to exist, one may expect to approximate it using for example the classical Newton Method. Regarding semi-local guarantees of convergence for the Newton Method we have the celebrated Kantorovich Theorem, which gives a sufficient condition for the convergence of the Newton Method $x_{k+1} = x_k - Df(x_k)f(x_k)$ to a zero of f . On the other hand, it should be noted that the exact Newton Method is usually impracticable for the purpose of rigorous computation since we cannot expect in general to perform computations of the kind $x_k - Df(x_k)f(x_k)$ without committing errors. Thus we can consider the Inexact Newton Method $x_{k+1}x_k - Df(x_k)f(x_k) + r_k$ in which r_k represents the error carried during each step of

the computation.

In this regard, an important question is to give semi-local conditions analogous to the Kantorovich Theorem, for the Inexact Newton method to be feasible and converge under conditions on the decaying rate of r_k . By checking on recent literature such as (ARGYROS IOANNIS K.; GEORGE, 2019), (ARGYROS IOANNIS K.; HILOUT, 2014), (SVAITER, 2012), (WU, 2008) and (LI, 2009), we notice the Newton Method has been proven to be feasible and converge only under certain conditions in which $\|r_k\|$ must converge to 0 with certain given rates.

On the other hand, we shall show in the second chapter a condition for the feasibility under the minor condition that the error r_k be bounded by a small constant to be computed, and we shall prove almost convergence to the roots under the minor condition that $r_k \rightarrow 0$ as $k \rightarrow \infty$. To the authors knowledge, this is the first time that such kind of conditions has been given to the Inexact Newton Method.

Finally, to show how these algorithms can be applied in practice, we shall show how to apply it to the general two point boundary value problem of Neumann type

$$u'' = f(x, u, u'), u'(0) = u'(1) = 0.$$

in which f is a twice-differentiable function.

The remainder of this work is organized as follows: In chapter two, we introduce the Kantorovich Theorem and show a Kantorovich-like theorem aimed for the Inexact Newton Method. We then show a theorem aimed to apply such theorem to isolate all zeros of Bannach Space non-linear equations $f(x) = 0$, with guaranteed termination proved under certain computability and regularity assumptions on f . In chapter three we show how to apply it to the two point Boundary Value Problem of Neumann Type presented above. Finally, in chapter 4 we present some general comments and future work interests.

ENCLOSURE OF SOLUTIONS IN BANNACH SPACE EQUATIONS

2.1 Preliminaries

This section contains the preliminary results which will be used during this thesis.

2.1.1 Functional Analysis

In this thesis, $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ will always denote Bannach spaces. Moreover we shall let $(\mathcal{L}(X, Y), \|\cdot\|_{\mathcal{L}(X, Y)})$ be the Bannach space of bounded linear operators from X to Y under the strong norm $\|\cdot\|_{\mathcal{L}(X, Y)}$ given by $\|G\|_{\mathcal{L}(X, Y)} = \sup_{\|x\|_X \leq 1} \|G(x)\|_Y$ for all $G \in \mathcal{L}(X, Y)$. Moreover, $B(x_0, r)_X$ and $\bar{B}(x_0, r)_X$ will always denote, respectively, the open ball and closed ball with radius r centered at x_0 under the $\|\cdot\|_X$ norm.

The following well known result tells us that the open ball of radius one centered at the identity in $\mathcal{L}(X, Y)$ is composed only of invertible linear operators.

Proposition 1. Given $F \in \mathcal{L}(X, Y)$, suppose that $\|I - F\| < 1$. Then F is invertible with $F^{-1} \in \mathcal{L}(Y, X)$ and moreover:

$$\|F^{-1}\| \leq \frac{1}{1 - \|I - F\|}.$$

For a proof check ([TAYLOR, 1958](#)), Theorem 1.4, page 192.

The following is a direct consequence of the proposition above

Corollary 1. Let $F \in \mathcal{L}(X, Y)$ and $G \in \mathcal{L}(X, Y)$ such that F is invertible with $F \in \mathcal{L}(Y, X)$ and suppose that $\|1 - FG\| < 1$. Then G is invertible with $G^{-1} \in \mathcal{L}(X, Y)$ and moreover

$$\|G^{-1}F^{-1}\| \leq \frac{1}{1 - \|I - FG\|}$$

Proof. Due to Proposition 1 we know that $H = FG$ is invertible with $H^{-1} \in \mathcal{L}(X, Y)$ and

$$\|H^{-1}\| \leq \frac{1}{1 - \|I - FG\|}. \quad (2.1)$$

On the other hand, from the associativity property of compositions of linear operators we have that

$$(H^{-1}F)G = H^{-1}(FG) = H^{-1}H = I \text{ and} \\ G(H^{-1}F) = ((F^{-1}F)G)(H^{-1}F) = (F^{-1}H)(H^{-1}F) = I$$

Thus it follows that G is invertible with inverse $G^{-1} = H^{-1}F$ and thus $G^{-1}F^{-1} = H^{-1}$. Therefore the follows from (2.1). $\square$

The following classical consequence of the Hahn-Bannach Theorem (see (TAYLOR, 1958), Theorem 3.2, page 134) will also be useful:

Proposition 2. Given a null non-null element $x \in X$, there exists $\phi \in \mathcal{L}(X, \mathbb{R})$ such that $\|\phi\|_{\mathcal{L}(X, \mathbb{R})} = 1$ and $\phi(x) = \|x\|_X$.

Finally, we point out the following classical consequence of the open map theorem:

Proposition 3. If $F \in \mathcal{L}(X, Y)$ is invertible, then $F^{-1} \in \mathcal{L}(Y, X)$.

2.1.2 Differential Calculus in Bannach Spaces

The following defines the Fréchet differentiability of a function in a Bannach space.

Definition 1. Given $f : U \subset X \rightarrow Y$, where U is an open set we shall say f is Fréchet differentiable in $x \in U$ if there exists a bounded linear operator $Df(x) \in \mathcal{L}(X, Y)$ such that

$$\lim_{h \rightarrow 0} \frac{\|f(x+h) - f(x) - (Df(x))(h)\|_Y}{\|h\|_X} = 0$$

Moreover if f is Fréchet differentiable in all $x \in U$, we define $Df : U \rightarrow \mathcal{L}(X, Y)$ to be the operator which given $x \in U$ returns the bounded linear operator $Df(x) \in \mathcal{L}(X, Y)$.

In general we shall say only that f is differentiable at x instead of Fréchet differentiable at x . Regarding this definition, if f is Fréchet differentiable in each $x \in U$ we say that f is *Fréchet differentiable* or simply *differentiable*, and if additionally $Df : U \rightarrow \mathcal{L}(X, Y)$ is continuous, we say f is a C^1 function. Since $\mathcal{L}(X, Y)$ is itself Bannach space, we can apply the definition above to define the differentiability of $Df : U \rightarrow \mathcal{L}(X, Y)$ giving rise to the following:

Definition 2. Given $f : U \subset X \rightarrow Y$, where U is an open set we shall say f is two times differentiable in $x \in U$ if f is differentiable in U such that $Df : U \rightarrow \mathcal{L}(X, Y)$ is also differentiable in x , where with its derivative in x will be denoted by $D^2f(x)$.

Notice that, since $Df(x) \in \mathcal{L}(X, Y)$ for all $x \in U$, by definition it follows that $D^2f(x)$ is an element of $\mathcal{L}(X, \mathcal{L}(X, Y))$. For economy, $\mathcal{L}(X, \mathcal{L}(X, Y))$ shall be denoted $\mathcal{L}^2(X, Y)$ for the remaining of this thesis.

Regarding the properties of the derivative Df we have the Chain Rule, Mean Value Theorem as follows and Inverse Function Theorem as follows. For the following we let X, Y and Z be Bannach spaces.

Proposition 4 (Chain Rule). Let $f : U \subset X \rightarrow Y$ and $g : V \subset Y \rightarrow Z$ be functions with U and V open, such that $f(U) \subset V$. Then if f is differentiable in $x \in U$ and g is differentiable at the point $y = f(x)$, it will follow that $h = g \circ f : U \rightarrow Z$ is differentiable in x with

$$Dh = Dg(y) \circ Df(x).$$

Theorem 1 (Mean Value Theorem). Let $f : U \subset X \rightarrow Y$ differentiable in the open set U , let $A \subset U$ be convex and suppose that $\|Df(x)\|_{\mathcal{L}(X, Y)} \leq M \in \mathbb{R}_+$ for all $x \in A$. Then, for all x and y in A it follows that

$$\|f(y) - f(x)\|_Y \leq M \|y - x\|_X.$$

Theorem 2 (Inverse Function Theorem). Let $f : U \subset X \rightarrow Y$ be a C^1 function in the open set U and given $x \in X$, suppose that $Df(x)$ is invertible. Then there exists an open set $V \subset U$ containing x and an open $V \subset Y$ containing $y = f(x)$ such that f is invertible in V , $W = f(V)$ and $f^{-1} : W \subset Y \rightarrow X$ is a C^1 function.

For proof of the previous three results see (CARTAN, 1971), Prop. 2.2.1 page 27, Theo. 3.3.2 page 41, and Theo. 4.2.1 page 51, respectively.

Finally, the following theorem can be seen as a generalization of the Taylor Theorem for twice differentiable functions, suitable for functions f with Lipschitz continuous derivative, thus, we shall call it simply by Taylor Theorem. This next theorem is usually not proved in such generality, since it usually asks for f to be twice differentiable, and thus we shall need to prove it.

Theorem 3 (Taylor Theorem). Let $f : U \subset X \rightarrow Y$ be differentiable in the open set U , let $A \subset U$ be convex, $x_0 \in A$, and suppose that

$$\|Df(x) - Df(x_0)\|_{\mathcal{L}(X, Y)} \leq K \|x - x_0\|_X \text{ for all } x \in A.$$

Then it follows that

$$\|f(x) - f(x_0) - Df(x_0)(x - x_0)\|_Y \leq \frac{K}{2} \|x - x_0\|_X^2$$

for all $x \in A$.

Proof. If $x = x_0$, the theorem is trivially true. Otherwise let

$$r = \frac{1}{\|x - x_0\|_X^2} (f(x) - f(x_0) - Df(x_0)(x - x_0))$$

The theorem is thus equivalent to proving that $\|r\|_Y \leq \frac{M}{2}$.

If $r = 0$, then once again there is nothing to prove. Otherwise, by Proposition 2, there exists $\phi \in \mathcal{L}(Y, \mathbb{R})$ such that $\|\phi\|_{\mathcal{L}(Y, \mathbb{R})} = 1$ and $\phi(r) = r$.

Since A is convex, it follows that $x_0 + t(x - x_0) \in A$ for all $t \in [0, 1]$ and since U is open, there must exist an open set I such that $[0, 1] \subset I$ and $x_0 + t(x - x_0) \in U$ for all $t \in I$.

Thus, if we denote $h = x - x_0$, by the chain rule it follows that the mapping $g : I \rightarrow \mathbb{R}$ defined by $g(t) = \phi(f(x_0 + th))$ is differentiable with $g'(t) = \phi(Df(x_0 + th)(h))$ for all $t \in [0, 1]$. Moreover, since $\|\phi\|_{\mathcal{L}(Y)} = 1$ and since ϕ is linear, it follows by hypothesis that

$$\begin{aligned} |g'(t) - g'(0)| &= |\phi((Df(x_0 + th) - Df(x_0))(h))| \leq \\ \|\phi\|_{\mathcal{L}(Y, \mathbb{R})} \|Df(x_0 + th) - Df(x_0)\|_{\mathcal{L}(X, Y)} \|h\|_X &\leq K \|th\|_X \|h\|_X = Kt \|h\|_X^2 \end{aligned}$$

for all $t \in [0, 1]$. In special it follows that $g'(t) - g'(0)$ is integrable, and thus from the Fundamental Theorem of Calculus we have that

$$\begin{aligned} |g(1) - g(0) - g'(0)| &= \left| \int_0^1 (g'(t) - g'(0)) dt \right| \leq \\ \int_0^1 |g'(t) - g'(0)| dt &\leq \int_0^1 Kt \|h\|_X^2 dt = \frac{K \|h\|_X^2}{2} \end{aligned}$$

Finally, since $|\phi(r)| = \phi(r) = \|r\|_X$, $g(0) = \phi(f(x_0))$, $g(1) = \phi(f(x))$, $g'(0) = \phi(Df(x_0)(h))$, and due to the linearity of ϕ we have that

$$\begin{aligned} \|r\|_X = |\phi(r)| &= \left| \frac{1}{\|h\|_X^2} (\phi(f(x)) - \phi(f(x_0)) - \phi(Df(x_0)(h))) \right| = \\ &= \frac{1}{\|h\|_X^2} |g(1) - g(0) - g'(0)| \leq \frac{K}{2} \end{aligned}$$

which proves the proposition. $\square$

2.1.3 Differentiation in $\mathbb{R}^n$

During this thesis, $\mathbb{R}^n$ will always denote the Bannach space $\mathbb{R}^n$ under the norm $\|\cdot\|_2$ given by

$$\|(a_1, \dots, a_m)\|_2 = \sqrt{\sum_{i=1}^m a_i^2}$$

Thus, $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ will denote the Bannach space of linear operators from the Bannach space $\mathbb{R}^n$ to the Bannach space $\mathbb{R}^m$ under the strong norm, and analogously for $\mathcal{L}^2(\mathbb{R}^n, \mathbb{R}^m) = \mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m))$.

The first observation that we should make is that, since all norms in $\mathbb{R}^n$ are equivalent, it follows that the Fréchet differentiability of f in one norm $\|\cdot\|$ in $\mathbb{R}^n$, imply in its Fréchet differentiability for all other norms in $\mathbb{R}^n$.

Definition 3. Given $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$, U open, in case f has a partial derivative in relation to the i -th coordinate, in the classical sense, we shall denote this partial derivative either by $\frac{\partial f}{\partial x_j} : U \rightarrow \mathbb{R}$ or by $f_{x_j} : U \rightarrow \mathbb{R}$. Moreover, if $\frac{\partial f}{\partial x_i} : U \rightarrow \mathbb{R}$ is differentiable in all $x \in U$ in relation to the j -th coordinate in $\mathbb{R}^n$, then we shall denote this partial derivative either by $\frac{\partial^2 f}{\partial x_j \partial x_i} : U \rightarrow \mathbb{R}$ or by $f_{x_j x_i} : U \rightarrow \mathbb{R}$.

The next proposition gives a relation between the Fréchet differentiability of f and the partial derivatives of f .

Proposition 5. Given $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$, with $f = (f_1, \dots, f_m)$, and U open we have:

- i) If f is Fréchet differentiable, then all partial derivatives $\frac{\partial f_i}{\partial x_j} : U \rightarrow \mathbb{R}^m$ exist and $Df(x)(v) = (y_1, \dots, y_m)$ for all $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ where

$$y_i = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} v_j \text{ for all } 1 \leq i \leq m.$$

- ii) If all partial derivatives $\frac{\partial f_i}{\partial x_j} : U \rightarrow \mathbb{R}^m$ exist and are continuous in U for $1 \leq i \leq m$ and $1 \leq j \leq n$, then f is Fréchet differentiable.

The next proposition on the other hand gives us a condition for f to be twice differentiable in terms of its second order partial derivatives:

Proposition 6. Given $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$, $f = (f_1, \dots, f_m)$, U open, we have

- i) If f is two times Fréchet differentiable, then all second order partial derivatives $\frac{\partial^2 f_i}{\partial x_j \partial x_k} : U \rightarrow \mathbb{R}^m$ exist and $(D^2 f(x)v)w = (y_1, \dots, y_m)$ for all $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ and $w = (w_1, \dots, w_n) \in \mathbb{R}^n$, where

$$y_i = \sum_{j=1}^n \sum_{k=1}^n \frac{\partial^2 f_i}{\partial x_j \partial x_k} v_j v_k, \text{ for all } 1 \leq i \leq m.$$

- ii) If all second order partial derivatives $\frac{\partial^2 f_i}{\partial x_j \partial x_k} : U \rightarrow \mathbb{R}^m$ exist and are continuous in U for $1 \leq i \leq m$, $1 \leq j \leq n$ and $1 \leq k \leq n$, then f is two times Fréchet differentiable.

For proofs of the last two results check either (CARTAN, 1971) or (GARLING, 2014).

Definition 4. Given $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$, U open, such that all partial derivatives $\frac{\partial f_i}{\partial x_j} : U \rightarrow \mathbb{R}^m$ exist in U , we let

$$J_f(x) = (a_{ij}(x)) \in M_{m,n}(\mathbb{R})$$

for all $x \in U$, where $a_{ij}(x) = \frac{\partial f_i}{\partial x_j}(x)$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Moreover, $J_f : U \rightarrow M_{m,n}(\mathbb{R})$ will be called the Jacobian of f .

Thus, item i) of Proposition 5 tells us that if f is Fréchet differentiable, then the Jacobian $J_f(x)$ exist and $Df(x)(v) = (J_f(x))v^T$ for all $v \in \mathbb{R}^n$.

2.2 The bijectivity modulus, and norm computations.

2.2.1 The bijectivity modulus

In the following, if $F \in \mathcal{L}(X, Y)$ then we let $F(B(0, 1)_X)$ denote the range of F over the open ball $B(0, 1)_X$. For brevity, during this chapter we shall denote the Bannach space $\mathcal{L}(X, Y)$ simply by Z . The following operator $\lambda_Z : \mathcal{L}(X, Y) \rightarrow \mathbb{R}$ will be important through this thesis:

Definition 5. If $F \in \mathcal{L}(X, Y)$ then we define

- $j(F)_Z = \inf_{\|x\|_X=1} \|Fx\|_Y$, called the injectivity modulus of F .
- $k(F)_Z = \sup\{r \geq 0 \mid B(0, r)_Y \subset F(B(0, 1)_X)\}$, called the surjectivity modulus of F .
- $\lambda(F)_Z = \min\{j(F)_Z, k(F)_Z\}$, called the *minimum modulus of F* or *bijectivity modulus of F* .

For brevity, in the remaining of this subsection and other sections we shall hide the index $Z = \mathcal{L}(X, Y)$ in $\lambda(F)_Z$, $j(F)_Z$ and $k(F)_Z$, but in subsequent sections this index might be shown in order to make the presentation clear.

While the definitions of j and k are usual in the spectral theory of linear operators (see (MÜLLER, 2007), Definition 3, page 86, for instance), we proposed the definition of the bijectivity modulus λ above in order to formulate our theory of isolation of zeros in the next chapters.

We have the following basic properties of the operators $j(F)$ and $k(F)$. For the following we call $F(X) = \{f(x) \mid x \in X\}$ the *range of f* .

Proposition 7. If $F \in \mathcal{L}(X, Y)$ then

- $j(F) > 0$ if and only if F is injective with range closed.
- $k(F) > 0$ if and only if F is surjective.

- iii) If F is invertible then $j(F) = k(F) = \|F^{-1}\|^{-1}$.
- iv) $|j(F) - j(G)| \leq \|F - G\|$ for all $G \in \mathcal{L}(X, Y)$.
- v) $|k(F) - k(G)| \leq \|F - G\|$ for all $G \in \mathcal{L}(X, Y)$.

For a proof check (MÜLLER, 2007), Theorem 4 page 86, Theorem 7 page 87 and Proposition 9 page 88.

Although the definition of injectivity and surjectivity modulus are well know, to the author knowledge the definitions of bijectivity modulus is new.

We propose the following properties for λ :

Proposition 8. Given $F \in \mathcal{L}(X, Y)$ we have that:

- i) F is invertible if and only if $\lambda(F) > 0$.
- ii) If F is invertible then $\lambda(F) = \|F^{-1}\|^{-1}$.
- iii) $|\lambda(F) - \lambda(G)| \leq \|F - G\|$ for all $G \in \mathcal{L}(X, Y)$.
- iv) If F being injective implies that F is invertible as well, then we have $\lambda(F) = j(F)$.

Proof. Item i):

If F is invertible then it is both injective with range $F(X) = Y$ closed and surjective, and thus item i) and ii) of Proposition 7 implies that $j(F) > 0$ and $k(F) > 0$. Thus it follows that

$$\lambda(F) = \min\{j(F), k(F)\} > 0$$

under this hypothesis.

On the other hand, if $\lambda(F) > 0$ then, since $\lambda(F) = \min\{j(F), k(F)\}$, it follows that $j(F) > 0$ and $k(F) > 0$, and thus, due to item i) and ii) of Proposition 7, it follows that F is invertible.

Item ii)

If F is invertible then due to item iii) of Proposition 7 it follows that $j(F) = k(F) = \|F^{-1}\|^{-1}$ and thus $\lambda(F) = \min\{j(F), k(F)\} = \|F^{-1}\|^{-1}$.

Item iii)

If F is not injective then $j(F) = 0$, which implies that $\lambda(F) = \min\{j(F), k(F)\} = 0$ and thus, from *iv*) of Proposition 7, we have that

$$\begin{aligned} |\lambda(F) - \lambda(G)| &= |\lambda(G)| = \lambda(G) \leq j(G) = |j(G)| \\ &= |j(G) - j(F)| \leq \|F - G\|. \end{aligned}$$

On the other hand, if F is not onto then $k(F) = 0$ which implies $\lambda(F) = \min\{j(F), k(F)\} = 0$ as well, and thus, from *v*) of Proposition 7, we have that

$$\begin{aligned} |\lambda(F) - \lambda(G)| &= |\lambda(G)| = \lambda(G) \leq k(G) = |k(G)| \\ &= |k(G) - k(F)| \leq \|F - G\|. \end{aligned}$$

Analogously it follows that if either G is not injective or if G is not onto, then the proposition is true.

Finally, if F and G are both invertible, then due to item *iii*) of Proposition 7 and due to item *ii*) we have that $\lambda(F) = \|F^{-1}\|^{-1} = j(F)$ and $\lambda(G) = \|G^{-1}\|^{-1} = j(G)$ and thus by item *iv*) of Proposition 7 we have that

$$|\lambda(F) - \lambda(G)| = |j(F) - j(G)| \leq \|F - G\|.$$

Thus the item *iii*) is proved.

Item *iv*)

If F is not injective then due to item *i*) of Proposition 7 it follows that $j(F) = 0$ and thus $\lambda(F) = 0 = j(F)$. On the other hand, if F is injective, then by hypothesis F is invertible as well and thus by item *ii*) we would have that $\lambda(F) = \|F^{-1}\|^{-1}$ and thus by item *v*) of Proposition 7 it would follow that

$$j(F) = \|F^{-1}\|^{-1} = \lambda(F).$$

Therefore the item *iv*) is proved. □

In particular, from items *i*) and *ii*) of Proposition 8 we have the following alternate characterization for $\lambda(F)$:

$$\lambda(F) = \begin{cases} \|F^{-1}\|^{-1} & \text{if } F \text{ is invertible,} \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 2. Suppose $F \in \mathcal{L}(X, Y)$ is invertible and suppose that $G \in \mathcal{L}(X, Y)$ satisfies $\|G - F\| < \lambda(F)$. Then G is invertible as well with $G^{-1} \in \mathcal{L}(Y, X)$ satisfying

$$\|G^{-1} - F^{-1}\| \leq \frac{\|G - F\|}{\lambda(G)\lambda(F)}$$

Proof. Due to item iii) of Proposition 8 we have that

$$\lambda(F) - \lambda(G) \leq \|F - G\| \Rightarrow \lambda(G) \geq \lambda(F) - \|F - G\|$$

and since by hypothesis $\lambda(F) - \|F - G\| > 0$ we conclude that $\lambda(G) > 0$. Thus, by item i) of Proposition 8 it follows that G . Moreover we have that

$$\begin{aligned} \|G^{-1} - F^{-1}\| &= \|G^{-1}(I - GF^{-1})\| \leq \|G^{-1}\| \|I - GF^{-1}\| \\ &= \|G^{-1}\| \|(F - G)F^{-1}\| \leq \|G^{-1}\| \|F - G\| \|F^{-1}\|. \end{aligned}$$

and since F and G are both invertible, item ii) of Proposition 8 implies that $\|G^{-1}\| = \lambda(G)^{-1}$ and $\|F^{-1}\| = \lambda(F)^{-1}$, which, together with the above inequality, proves the corollary. $\square$

2.2.2 Computation of norms of Df and D^2f

Definition 6. Given $A \in M_{m,n}(\mathbb{R}^n)$ we denote by $\lambda(A)_2$, $j(A)_2$ and $\|A\|_2$, the modules and norms $\lambda(A)_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)}$, $j(A)_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)}$ and $\|A\|_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)}$, respectively, where here $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is interpreted as a linear operator in $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ by letting $A(v) = Av^T$ for all $v \in \mathbb{R}^n$. Moreover $\|A\|_2$ is called the *spectral norm* of $A \in M_{m,n}(\mathbb{R})$.

As a direct corollary of Propositions 5 and Definition 4 we have:

Proposition 9. Let $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ be differentiable with $f = (f_1, \dots, f_m)$, then for all $x \in U$ we have that

$$\begin{aligned} \lambda(Df(x))_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)} &= \lambda(J_f(x))_2 \text{ and} \\ \|Df(x)\|_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)} &= \|J_f(x)\|_2 \end{aligned}$$

where $J_f(x) \in M_{m,n}(\mathbb{R})$ denotes the Jacobian of f at x .

Thus the above proposition gives us a mean to compute $\lambda(Df)_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)}$ and $\|Df\|_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)}$ via these computations over matrices. The following on the other hand tells us how to easily compute a bound for $\|D^2f(x)\|_{\mathcal{L}^2(\mathbb{R}^n, \mathbb{R}^m)}$.

Proposition 10. Let $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ be twice differentiable with $f = (f_1, \dots, f_m)$, then for all $x \in U$ we have that

$$\|D^2f(x)\|_{\mathcal{L}^2(\mathbb{R}^n, \mathbb{R}^m)} \leq \sqrt{\sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^n \left(\frac{\partial^2 f_i(x)}{\partial x_j \partial x_k} \right)^2}$$

Proof. Notice that

$$\begin{aligned} \|D^2f(x)\|_{\mathcal{L}^2(\mathbb{R}^n, \mathbb{R}^m)} &= \sup_{\|a\|_2=1} \|D^2f(x)(a)\|_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)} \\ &= \sup_{\|a\|_2=1, \|b\|_2=1} \|(D^2f(x)(a))(b)\|_2 \end{aligned}$$

On the other hand, if $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ with $\|a\|_2 = \|b\|_2 = 1$ it follows due to Proposition 6 that $(D^2 f(x)(a))(b) = (y_1, \dots, y_m)$ where

$$y_i = \sum_{j=1}^n \left(\sum_{k=1}^n \frac{\partial^2 f_i(x)}{\partial x_j \partial x_k} a_k \right) b_j.$$

But since $\|a\|_2 = 1$, we have from the Cauchy-Schwarz inequality that

$$\sum_{k=1}^n \frac{\partial^2 f_i(x)}{\partial x_j \partial x_k} a_k \leq \sqrt{\sum_{k=1}^n \left(\frac{\partial^2 f_i(x)}{\partial x_j \partial x_k} \right)^2} \quad (2.2)$$

and thus, using again the Cauchy-Schwarz inequality and from $\|b\|_2 = 1$ it follows that

$$y_i \leq \sum_{j=1}^n \left(\sqrt{\sum_{k=1}^n \left(\frac{\partial^2 f_i(x)}{\partial x_j \partial x_k} \right)^2} \right) b_j \leq \sqrt{\sum_{j=1}^n \sum_{k=1}^n \left(\frac{\partial^2 f_i(x)}{\partial x_j \partial x_k} \right)^2}$$

Finally, since $\|(D^2 f(x)(a))(b)\|_2 = \sqrt{\sum_{i=1}^m y_i^2}$ it follows that

$$\|(D^2 f(x)(a))(b)\|_2 \leq \sqrt{\sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^n \left(\frac{\partial^2 f_i(x)}{\partial x_j \partial x_k} \right)^2},$$

which proves the proposition. $\square$

Finally, the next definition and proposition shows us how to cover a closed ball $\bar{B}(x, r)_{\mathbb{R}^n}$ with smaller closed balls $\bar{B}(x_i, r_i)$. For the following let $\lfloor z \rfloor$ denote the nearest integer below $z \in \mathbb{Z}$

Definition 7. Given $r \in \mathbb{R}$ with $r \geq 1$ and $k \in \mathbb{N}$, and letting

$$I_k(r) = \left\{ -\lfloor r \rfloor, \dots, \frac{1}{k}, 0, \frac{1}{k}, \dots, \lfloor r \rfloor \right\}$$

and $I_k^n(r) = \underbrace{I_k(r) \times \dots \times I_k(r)}_{n \text{ times}}$ we define $J_k^n(r) = I_k^n(r) \cap \bar{B}(0, r)_{\mathbb{R}^n}$.

For the following we let $\|x - y\|_\infty = \sup_{1 \leq j \leq n} |x(j) - y(j)| \leq k$ and let $\bar{R}(x, r)_{\mathbb{R}^n} = \{y \in \mathbb{R}^n \mid |x - y|_\infty \leq r\}$ be the rectangle of center x and radius r in $\mathbb{R}^n$.

Proposition 11. Given $k > 0$, $n > 0$ and $A = \bar{B}(0, r)_{\mathbb{R}^n}$ we have that

$$A \subset \bigcup_{x \in J_k^n(r)} \bar{R}\left(x, \frac{1}{k}\right)_{\mathbb{R}^n}$$

Proof. Given $y = (a_1, \dots, a_n) \in A$, let $k_i = \text{sign}(a_i) \lfloor b_i \rfloor$ where $b_i = |a_i|$ for $1 \leq i \leq n$. Then clearly $0 \leq k_i \leq k$, $\|k\|_2 \leq \|a\|_2 \leq r$ and moreover $|ka_i - k_i| \leq 1$ by definition. Thus, letting $x = \left(\frac{k_1}{k}, \dots, \frac{k_n}{k}\right) \in I_k^n$ it follows that

$$\|x - y\|_\infty = \sup_{1 \leq i \leq n} \left(\frac{ka_i - k_i}{k} \right) \leq \sqrt{\sum_{i=1}^n \frac{1}{(2k)^2}} = \frac{\sqrt{n}}{k}$$

and thus $y \in \bar{B}\left(x, \frac{\sqrt{n}}{2k}\right)$, proving the result. $\square$

2.2.3 Computation of $j(A)_2$, $\|A\|_2$ and $(A^{-1})v^T$

In this section we are concerned with the computation of $j(A)_2$, $\|A\|_2$ and $A^{-1}v^T$ where $A \in M_n(\mathbb{R})$ and $v \in \mathbb{R}^n$.

We notice right away that $\lambda(A)_2 = j(A)_2$ due to item iv) of Proposition 8 since by the Rank-Nullity Theorem if $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is injective then it is invertible as well. Thus the computation of $j(A)_2$ implies in the computation of $\lambda(A)_2$ for $A \in M_n(\mathbb{R})$.

The next well know proposition relates the computation of $j(A)_2$ and $\|A\|_2$ to the computation of its singular values.

Proposition 12. Given $A \in M_n(\mathbb{R})$ we have that

- i) $j(A)_2 =$ smallest eigenvalue σ_1 of A .
- ii) $\|A\|_2$ largest eigenvalue σ_n of A .

Proof. If A^T is the transpose of A we have that

$$\|Av\|_2 = \sqrt{\langle Av, Av \rangle} = \sqrt{\langle (A^T A)v, v \rangle}$$

and since from the min max theorem we have that

$$\min \langle (A^T A)v, v \rangle = \sigma_1$$

where σ_1 is the smallest eigenvalue of $A^T A$, the property for $j(A)_2$ given in the proposition follows. The proof of the property for $\|A\|_2$ is completely analogous. $\square$

As a corollary we have the following verificative principle to compute $j(A)_2$ and $\|A\|_2$.

Proposition 13. Given $A \in M_n(\mathbb{R})$, let $D \in M_n(\mathbb{R})$ be a diagonal matrix and $U \in M_n(\mathbb{R})$ is a orthogonal matrix satisfying

$$\|A^T A - UDU^T\|_2 \leq \delta$$

and let d_1 be the smallest diagonal entry in absolute value of D and d_n be the largest diagonal entry in absolute value of D . Then

$$\left| j(A)_2 - \sqrt{|d_1|} \right| \leq \sqrt{\delta} \text{ and } \left| \|A\|_2 - \sqrt{|d_n|} \right| \leq \sqrt{\delta}$$

Proof. Since AA^T is symmetric and semi-definite positive, its greatest singular value coincides with its largest eigenvalue σ_n and similarly its smallest singular value coincides with its smallest eigenvalue σ_1 . Thus, due to Proposition 12 it follows that

$$j(AA^T)_2 = \sigma_1 = j(A)_2^2 \text{ and } \|AA^T\|_2 = \sigma_n = \|A\|_2^2;$$

and similarly we have that

$$j(UDU^T)_2 = |d_1| \text{ and } \|UDU^T\|_2 = |d_n|.$$

On the other hand item *iv*) of Proposition 7 tells us that

$$|j(AA^T)_2 - j(UDU^T)_2| \leq \|AA^T - UDU^T\| \leq \delta$$

and we conclude that

$$|j(A)_2^2 - |d_1|| \leq \delta \text{ and } \left| \|A\|_2^2 - |d_n| \right| \leq \delta$$

thus it follows that

$$j(A) \geq \sqrt{||d_1| - \delta|} \geq \sqrt{|d_1|} - \sqrt{\delta}$$

and

$$j(A) \leq \sqrt{|d_1| + \delta} \leq \sqrt{|d_1|} + \sqrt{\delta}$$

which implies $|j(A) - \sqrt{|d_1|}| \leq \sqrt{\delta_1}$ and the same follows analogously for $\|A\|_2$. $\square$

Finally, just as in the proof of Proposition 10, it is easy to verify through the Cauchy-Schwarz inequality the following proposition.

Proposition 14. Letting $\|A\|_F$ be the Frobenius norm of $A \in M_n(\mathbb{R})$, defined by $\|A\|_F = \sqrt{\sum_{i=1}^n \sum_{j=1}^n a_{i,j}^2}$ we have that $\|A\|_2 \leq \|A\|_F$ for all $A \in M_n(\mathbb{R})$.

Thus, using Proposition 13, we can compute through Intervalar Arithmetic (see (ALEFELD; HERZBERGER, 1983), (MOORE, 1987) (NEUMAIER, 1990)), rigorous enclosures to $\lambda(A)_2$ and $\|A\|_2$ by initially using floating point arithmetic to obtain a candidate $[U, D, U^T]$ for the spectral decomposition of AA^* and then computing a rigorous bound to $\|AA^T - UDU^T\|_2$ by means of the Frobenius norm of $AA^T - UDU^T$. which implies in rigorous enclosures for $\lambda(A)_2$ and $\|A\|_2$ due to . Moreover, since $\|B\|_F$ goes to 0 as $\|B\|_2$ goes to 0, it follows this rigorous enclosure can be as good as one wishes, as long as the spectral decomposition $[U, D, U^T]$ of AA^T is close enough to it in the $\|\cdot\|_2$ norm.

The next proposition uses this same principle to compute rigorous bounds to $A^{-1}v$ via non-rigorous candidates:

Proposition 15. If $A \in M_n(\mathbb{R})$ is an invertible matrix such that $\|v - Ab\|_2 \leq \delta$, then we have that

$$\|A^{-1}v - b\|_2 \leq \frac{\delta}{j(A)}$$

Proof. Since A is invertible we have from item iii) of Proposition 7 that $\|A^{-1}\|_2 = j(A)_2^{-1}$, and thus

$$\|A^{-1}v - b\|_2 = \|A^{-1}(v - Ab)\|_2 \leq \|A\|_2 \|v - Ab\|_2 \leq \frac{\delta}{\lambda(A)}$$

proving the proposition. $\square$

Thus we shall always compute $A^{-1}v$ by first finding non-rigorous solutions for the equation $Ab = v$, and then computing rigorously the error committed via the above proposition. In practice this method is much faster than computing rigorously the matrix A^{-1} and then multiplying by v , since computing inverses is known to be computationally expensive. Moreover, we once again point-out that methods to obtain non-rigorous candidates b for equations $Ab = v$ are readily available and optimized in most mathematical softwares.

2.3 Kantorovich Theorem and Inexact Newton Method

The consecrated Newton method, defined by the recurrence $x_{k+1} = x_k - Df(x_k)f(x_k)$ remains one of the main computational methods to approximate solutions of many kinds of equations.

2.3.1 Newton Method and Kantorovich Theorem.

We begin by defining the Newton Operator and Newton Method.

Definition 8 (Newton Operator). Let X and Y be Banach spaces, $U \subset X$ open, $f : U \subset X \rightarrow Y$ differentiable in U . Then we shall define the Newton Operator $T_f : U \rightarrow X$ by:

$$T_f(x) = \begin{cases} x - Df(x)^{-1}f(x) & \text{if } Df(x) \text{ is invertible.} \\ x & \text{otherwise.} \end{cases}$$

Definition 9 (Feasibility for Newton Method). Let X and Y be Banach spaces, $U \subset X$ open, $f : U \subset X \rightarrow Y$ differentiable in U , and let $(x_k)_{k \in \mathbb{N}}$ be a sequence in X . We shall say $(x_k)_{k \in \mathbb{N}}$ is feasible for the Newton Method over f if $x_k \in U$ and $Df(x_k)$ is invertible for all $k \in \mathbb{N}$.

Thus, if a sequence $(x_k)_{k \in \mathbb{N}}$ is feasible for the Newton method, it follows that $T(x_k) = x_k - Df(x_k)^{-1}f(x_k)$ for all $k \in \mathbb{N}$, and thus, if $(x_k)_{k \in \mathbb{N}}$ additionally satisfies the recurrence $x_{k+1} = T_f(x_k)$ for all $k \in \mathbb{N}$, then it follows that

$$x_{k+1} = x_k - Df(x_k)^{-1}f(x_k), \text{ for all } k \in \mathbb{N},$$

which corresponds to the classical Newton Method.

Regarding the feasibility and convergence of the classical Newton Method, we have the celebrated Kantorovich Theorem as follows.

Theorem 4 (Kantorovich). Let X and Y be Bannach spaces, $U \subset X$ an open convex set, $f : U \subset X \rightarrow Y$ differentiable function in U , $A \subset U$ convex, $x_0 \in A$, $R > 0$ satisfying $\bar{B}(x_0, R) \subset A$, and suppose

- i) $Df(x_0)$ is invertible with $Df(x_0)^{-1} \in \mathcal{L}(X, Y)$
- ii) $\|Df(x_0)^{-1}f(x_0)\| \leq \eta$, and $\|Df(x_0)^{-1}(Df(x) - Df(y))\| \leq K\|x - y\| \in \mathbb{R}_+^*$ for all $x, y \in A$.
- iii) $g_1(t) = \eta - t + \frac{Kt^2}{2}$ has zeros $t^* < t^{**}$ with $t^* \in [0, R]$.

Then

- I) f has a zero $x^* \in \bar{B}(x_0, t^*)$ and no other zeros in $B(x_0, t^{**}) \cap A$
- II) The sequence $x_{k+1} = T_f(x_k)$ is feasible for the Newton Method over f .
- III) $\|x^* - x_k\| \leq \frac{1}{K} \frac{(1 - \sqrt{1 - 2h})^{2^n}}{2^n}$ for all $k \in \mathbb{N}$, where $h = K\eta \leq \frac{1}{2}$.

For proofs, see (KANTOROVICH; AKILOV, 1982), (LECERF; SAADÉ, 2016) and (FERNÁNDEZ; VERÓN, 2017)

We propose the following theorem regarding the feasibility of the Newton Method when errors are committed. We should point out this is the first such kind of Theorem presented, since we don't make the assumption that the error should go to zero during the method, and instead prove the feasibility under a bounding condition on the error.

Theorem 5. Let X and Y be Bannach spaces, $U \subset E$ an open convex set, $f : U \subset X \rightarrow Y$ a twice differentiable function in U , $x_0 \in U$, $A \subset U$ convex, $R > 0$ satisfying $\bar{B}(x_0, R) \subset A$, let $(r_k)_{k \in \mathbb{N}}$ be a sequence in X and suppose:

- i) $Df(x_0)$ is invertible with $Df(x_0)^{-1} \in \mathcal{L}(X, Y)$.
- ii) $\|Df(x_0)^{-1}f(x_0)\| \leq \eta$ and $\|Df(x_0)^{-1}(Df(x) - Df(y))\| \leq K\|x - y\|$ for all $x, y \in A$, where $K \in \mathbb{R}_+^*$.
- iii) $g_2(t) = \eta + d - t(1 + Kd) + \frac{3K}{2}t^2$ has zeros $t_d^* < t_d^{**}$, with $t_d^* \leq R$ and $d \leq \frac{1}{K}$.
- iv) $\|r_k\| \leq d$ for all $k \in \mathbb{N}$.

Then $g_1(t) = \eta - t + \frac{K}{2}t^2$ has two distinct zeros $t^* < t^{**}$ and moreover:

- I) f has a zero $x^* \in \bar{B}(x_0, t^*)$, no other zeros in $B(x_0, t^{**}) \cap U$, and $t^* \leq t_d^*$.

II) The sequence $(x_k)_{k \in \mathbb{N}}$ defined by $x_{k+1} = x_k - Df(x_k)^{-1}f(x_k) + r_k$ for all $k \in \mathbb{N}$, is feasible for the Newton Method on U .

III) $\|x^* - x_n\| \leq \frac{1}{K} \frac{(1 - \sqrt{1 - 2h})^{2^n}}{2^n} + 2d$ for all $n \in \mathbb{N}$, where $h = K\eta \leq \frac{1}{6}$.

Proof. Proof of I).

Notice that, defining $h(t) = t^2 - 4t + (1 - 6K\eta)$, since $g_2(t) = \eta + d - (1 + Kd)t + \frac{3K}{2}t^2$ has a real zero, by the quadratic formula it follows that

$$\begin{aligned} (1 + Kd)^2 - 6K(\eta + d) &\geq 0 \Rightarrow \\ (Kd)^2 - 4Kd + (1 - 6K\eta) &\geq 0 \Rightarrow h(Kd) \geq 0 \end{aligned}$$

On the other hand, by the quadratic formula we notice that $h(t)$ has two non-negative zeros t_1 and t_2 given by

$$t_1 = 2 - \sqrt{3 + 6K\eta} \text{ and } t_2 = 2 + \sqrt{3 + 6K\eta},$$

and moreover $h(t) < 0$ for all $t \in (t_1, t_2)$. Thus, since due to item *iii*) we have that $Kd \leq 1 \leq t_2$, and since $h(Kd) \geq 0$, it follows that $Kd \leq t_1 = 2 - \sqrt{3 + 6K\eta}$, and since $Kd \geq 0$ it follows that

$$2 - \sqrt{3 + 6K\eta} \geq 0 \Rightarrow 6K\eta \leq 1 \Rightarrow h \leq \frac{1}{6} \quad (2.3)$$

where $h = K\eta$. In special it follows that $1 - 2K\eta \geq 0$ and thus the quadratic formula tells us that $g_1(t) = \eta - \nu t + \frac{K}{2}t^2$ have two non-negative zeros $t^* < t^{**}$ and moreover $g_1(t) \leq 0$ if and only $t^* \leq t \leq t^{**}$.

On the other hand, the quadratic formula tells us that the smallest zero t_d^* of $g_2(t)$ is given by

$$t_d^* = \frac{1 + Kd - \sqrt{(1 + Kd)^2 - 6K(\eta + d)}}{3K} \leq \frac{1 + Kd}{3K}$$

and since we proved above that $Kd \leq 2 - \sqrt{3 + 6K\eta}$, it follows that

$$Kt_d^* \leq \frac{1 + Kd}{3} \leq 1 - \sqrt{\frac{1}{3} + \frac{2}{3}K\eta} \quad (2.4)$$

In special we have that $Kt_d^* < 1 \Rightarrow 1 - Kt_d^* > 0$. Therefore, from $g_2(t_d^*) = 0$ we have that:

$$\begin{aligned} \eta + d - (1 + Kd)t_d^* + \frac{3K}{2}(t_d^*)^2 &= 0 \Rightarrow \\ \eta - (1 - Kt_d^*)(t_d^* - d) + \frac{K}{2}(t_d^*)^2 &= 0 \Rightarrow \\ \frac{\eta}{1 - Kt_d^*} - t_d^* + \frac{K}{2(1 - Kt_d^*)}(t_d^*)^2 &= -d \leq 0. \end{aligned} \quad (2.5)$$

Therefore, since $1 \leq \frac{1}{1 - Kt_d^*}$ we have that

$$\eta - t_d^* + \frac{K}{2}(t_d^*)^2 \leq \frac{\eta}{1 - Kt_d^*} - t_d^* + \frac{K}{2(1 - Kt_d^*)}(t_d^*)^2 \leq 0.$$

and thus $g(t_d^*) \leq 0$. Hence it follows that $t^* \leq t_d^*$, and since $t_d^* \leq R$, it follows that $t^* \leq R$ as well. Thus we can apply the Kantorovich Theorem and it follows that f has a zero x^* in $\bar{B}(x_0, t^*)$ and no other zeros in $B(x_0, t^{**})$, which proves *I*).

Proof of *II*):

Fixing $z \in B(x_0, t_d^*)$, it follows due to the Mean Value Theorem (Theorem 1) and (2.4) that

$$\begin{aligned} \|1 - Df(x_0)^{-1}Df(z)\| &= \|Df(x_0)^{-1}(Df(z) - Df(x_0))\| \leq \\ &K\|z - x_0\| \leq Kt_d^* < 1, \end{aligned}$$

Therefore, since $Df(x_0)^{-1}$ is invertible with inverse $Df(x_0) \in \mathcal{L}(X, Y)$, due to Corollary 1 we have that $Df(z)$ is invertible with $Df(z) \in \mathcal{L}(X, Y)$ such that $H = Df(z)^{-1}Df(x_0)$ satisfies

$$\|H\| \leq \frac{1}{1 - Kt_d^*} \quad (2.6)$$

Now, defining $L(x) = x - Df(z)^{-1}f(x)$ for $x \in A$, it follows from the chain rule that L is twice differentiable with $DL(y) = I - Df(z)^{-1}Df(y)$ and thus $DL(y) - DL(z) = Df(z)^{-1}(Df(y) - Df(z))$ for all $y \in A$ and therefore from (2.6) it follows that

$$\|DL(y) - DL(z)\| \leq \|H(Df(x_0)^{-1}(Df(y) - Df(z)))\| \leq \frac{K}{1 - Kt_d^*}$$

for all $y \in A$. Therefore, since $DL(z) = I - Df(z)^{-1}Df(z) = 0$, from the Taylor Theorem (Theorem 3) we have

$$\|L(x) - L(z)\| = \|L(x) - L(z) - DL(z)(x - z)\| \leq \frac{K\|x - z\|^2}{2(1 - Kt_d^*)}. \quad (2.7)$$

for all $x \in A$.

Moreover, since $L(x_0) - x_0 = Df(z)^{-1}f(x_0)$ by definition, it follows from (2.6) that

$$\|L(x_0) - x_0\| = \|H(Df(x_0)^{-1}f(x_0))\| \leq \frac{\eta}{1 - Kt_d^*} \quad (2.8)$$

Additionally, since $g_2(t^*) = 0$, we have due to (2.5) that

$$\begin{aligned} \frac{\eta}{1 - Kt_d^*} - t_d^* + \frac{K}{2(1 - Kt_d^*)}(t_d^*)^2 &= -d \Rightarrow \\ \frac{1}{1 - Kt_d^*} \left(\eta + \frac{K}{2}(t_d^*)^2 \right) &= t_d^* - d \end{aligned} \quad (2.9)$$

Thus from (2.7), (2.8) and (2.9) it follows that

$$\begin{aligned} \|x_0 - L(z)\| &\leq \|x_0 - L(x_0)\| + \|L(x_0) - L(z)\| \\ &\leq \frac{1}{1 - Kt_d^*} \left(\eta + \frac{K(t_d^*)^2}{2} \right) = t_d^* - d \end{aligned}$$

Finally, since $L(z) - x_0 = z - Df(z)^{-1}f(z) - x_0$ it follows that if $r \in X$ with $\|r\| \leq d$ then

$$\|z - Df(z)^{-1}f(z) + r - x_0\| \leq \|x_0 - L(z)\| + d \leq t_d^*.$$

Combining the above results we can conclude that, given $z \in \bar{B}(x_0, t_d^*)$, it follows that $Df(z)$ is invertible and $z - Df(z)^{-1}f(z) + r \in \bar{B}(x_0, t_d^*)$ for all and $r \in X$ with $\|r\| \leq d$. Thus, by induction, the sequence $x_{m+1} = x_m - Df(x_m)^{-1}f(x_m) + r_m$ is well defined for all $m \in \mathbb{N}$ and $x_m \in \bar{B}(x_0, t_d^*)$, which proves *II*).

Proof of *III*)

Since we already proved in *I*) that the zero x^* of f satisfies $x^* \in \bar{B}(x_0, t^*)$ it follows that $\|x^* - x_0\| \leq t^* = \frac{1}{K}(1 - \sqrt{1 - 2h})$, and therefore *III*) is true for $k = 0$.

On the other hand, denoting $L_0(x) = x - Df(x_0)^{-1}f(x)$ for $x \in B(x_0, t')$, since $x^* = x^* - Df(x_0)^{-1}f(x^*) = L_0(x^*)$ and since $L_0(x)$ is differentiable with $L_0(x_0) = 0$ and $DL_0(x) - DL_0(y) = Df(x_0)^{-1}(Df(x) - Df(y))$, by the Taylor Theorem (Theorem 3) we have that

$$\begin{aligned} \|x^* - x_1\| &= \|L_0(x^*) - L_0(x_0) - DL_0(x_0)(x^* - x_0) - r_0\| \leq \\ &\frac{K}{2}\|x^* - x_0\|^2 + d = \frac{K}{2}(t^*)^2 + d = \frac{1}{K} \frac{(1 - \sqrt{1 - 2h})^2}{2} + d \end{aligned}$$

and thus *III*) is also true for $k = 1$.

Now suppose by induction that *III*) is true for $k = m \geq 1$ and we shall prove it for $k = m + 1$. Fixing $z = x_m$, let L be defined by $L(x) = x - Df(z)^{-1}Df(x)$ as in the proof of item *II*). Then it follows that $x^* = x^* - Df(z)^{-1}f(x^*)$, and thus from inequality (2.7) we have that

$$\|x^* - x_{m+1}\| = \|L(x^*) - L(x_m) - r_m\| \leq \frac{K\|x^* - x_m\|^2}{2(1 - Kt_d^*)} + d, \quad (2.10)$$

On the other hand, due to (2.4) we have that

$$Kt_d^* \leq 1 - \frac{1}{\sqrt{3}} \Rightarrow \frac{1}{2(1 - Kt_d^*)} \leq \frac{\sqrt{3}}{2}. \quad (2.11)$$

Therefore, denoting $h' = 1 - \sqrt{1 - 2h}$, due to (2.10) and (2.11) and from the induction hypothesis we have that

$$\begin{aligned} \|x^* - x_{m+1}\| &\leq \frac{\sqrt{3}}{2}K\|x^* - x_m\|^2 + d \\ &\leq \frac{\sqrt{3}}{2}K\|x^* - x_m\| \left(\frac{1}{K} \frac{(h')^{2m}}{2^m} + 2d \right) + d. \end{aligned} \quad (2.12)$$

On the other hand, from the induction hypothesis and $m \geq 1$ we have that, since $h \leq \frac{1}{3}$

$$\frac{\sqrt{3}}{2}(h')^2 \leq \frac{\sqrt{3}}{2} \left(1 - \frac{1}{\sqrt{3}} \right)^2 = \frac{2\sqrt{3}}{3} - 1$$

$$\begin{aligned}
\frac{\sqrt{3}}{2}K \|x^* - x_m\| \left(\frac{1}{K} \frac{(h')^{2m}}{2^m} \right) &\leq \frac{\sqrt{3}}{2} \left(\frac{(h')^{2m}}{2^m} + 2d \right) \frac{(h')^2}{2} \\
&\leq \frac{(h')^{2m}}{2^m} \frac{(h')^2}{2} + \frac{\sqrt{3}d}{2} (h')^2 \leq \frac{(h')^{2m+1}}{2^{m+1}} + \left(\frac{2\sqrt{3}}{3} - 1 \right) d
\end{aligned} \tag{2.13}$$

and since $Kt_d^* \leq 1 - \frac{1}{\sqrt{3}}$ we have that

$$\frac{\sqrt{3}}{2}K \|x^* - x_m\| 2d \leq \sqrt{3}Kt_d^* \leq \sqrt{3} - 1. \tag{2.14}$$

and thus we conclude from (2.12), (2.14) and (2.13) that

$$\begin{aligned}
\|x^* - x_{m+1}\| &\leq \frac{(h')^{2m+1}}{2^{m+1}} + (\sqrt{3} - 1)d + \left(\frac{2\sqrt{3}}{3} - 1 \right) d + d \\
&= \frac{(h')^{2m+1}}{2^{m+1}} + \left(\frac{5\sqrt{3}}{3} - 1 \right) d.
\end{aligned}$$

But since $\sqrt{3} \leq \frac{7}{4}$, it follows that $\frac{5\sqrt{3}}{3} - 1 \leq \frac{23}{12} < 2$ and therefore *III*) is also valid for $k = m + 1$. Thus item *III*) is true by induction. □

2.3.2 Theorems for Isolation of Zeros

In this section we propose theorems for isolation of zeros of differentiable functions f , based on the Kantorovich Theorem and on the definition λ given in section 2.

The following theorem gives us a radius of nonexistence of zeros for a twice differentiable function $f : U \subset X \rightarrow Y$.

Theorem 6. Let X and Y be Bannach spaces, $U \subset X$ open, $f : U \subset X \rightarrow Y$ be twice differentiable in U , $A \subset U$ a convex set with $x_0 \in A$ and suppose:

- i) $\|f(x_0)\| \geq \eta$, $\|Df(x_0)\| \leq \mu$ and $\|Df(x) - Df(x_0)\| \leq K \|x - x_0\|$ for all $x \in A$, where $K \in \mathbb{R}_+^*$;
- ii) $g_0(t) = \eta - \mu t - \frac{K}{2}t^2$ has a non-negative real zero t' ;

Then f has no zeros in $B(x_0, t') \cap A$.

Proof. Since $g_0(t') = 0$ by hypothesis it follows that

$$\eta - \mu t' - \frac{K}{2}(t')^2 = 0 \Rightarrow \mu t' + \frac{K}{2}(t')^2 = \eta.$$

On the other hand, if $x_1 \in B(x_0, t') \cap A$, by the Triangle Inequality and Taylor Theorem it follows that

$$\begin{aligned} \|f(x_1) - f(x_0)\| &\leq \|Df(x_0)(x_1 - x_0)\| + \|f(x_1) - f(x_0) - Df(x_0)(x_1 - x_0)\| \\ &\leq \|Df(x_0)\| \|x_1 - x_0\| + \frac{K}{2} \|x_1 - x_0\|^2 \\ &< \mu t' + \frac{K}{2} (t')^2 = \eta \leq \|f(x_0)\|, \end{aligned}$$

where the strict inequality $<$ above occurs due to the fact that $K > 0$ and $\|x_1 - x_0\| < t'$. Therefore, if it happened that $f(x_1) = 0$ then it would follow that $\|f(x_0)\| < \|f(x_0)\|$, a contradiction. Thus f can have no zeros in $B(x_0, t') \cap A$. $\square$

Due to the above theorem, if η , ν and K satisfies the property i) above we shall call $g_0(t) = \eta - \mu t + \frac{Kt^2}{2}$ an *lower unidimensional taylor projection* of f .

The computations in the next example and the examples tha follow were carried rigorously using the package INTLAB (INTerval LABoratory) on the software MATLAB 2016.

Example 1. To exemplify an application of Theorem 6 we consider the function $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $h = (h_1, h_2)$ where $h_1(x, y) = 2x^2 + 2y^2 - 1$ and $h_2(x) = x - y$. We notice due to Proposition 10 that $K = 3$ serves as an upper bound for the second derivative of h in the $\mathcal{L}^2(\mathbb{R}^2, \mathbb{R}^n)$ norm, while the computation of $\|h(x_0, y_0)\|_2$ and $\|Dh(x_0, y_0)\|_{\mathcal{L}^2(\mathbb{R}^2, \mathbb{R}^2)}$ is done as described in section 2.2.3. Moreover Notice h has exactly two zeros, given by $x = y = \frac{1}{2}$ and $x = y = -\frac{1}{2}$.

For $z_0 = (x_0, y_0) = (0, 0)$ and $z_1 = (x_1, y_1) = (0.45, 0.45)$ we computed

$$\begin{aligned} \|h(z_0)\| &\leq 1 \text{ and} \\ \|Dh(z_0)\| &\geq 0.387907304066808 \\ \|h(z_1)\| &\leq 0.19 \text{ and} \\ \|Dh(z_1)\| &\geq 2.545584412271571, \end{aligned}$$

which generated, via the polynomial $g_0(t)$ as in Theorem 6, a nonexistence radius t' for h at z_0 and z_1 of

$$\begin{aligned} t'(z_0) &\geq 0.387907304066808 \text{ and} \\ t'(z_1) &\geq 0.069024217519696 \end{aligned}$$

respectively, where the nonexistence radius t' for h are represented in red in Figure 1.

Theorem 7. Let X and Y be Bannach spaces, $U \subset X$ an open convex set, $f : U \subset X \rightarrow Y$ differentiable function in U , $A \subset U$ convex, $x_0 \in A$, $R > 0$ satisfying $\bar{B}(x_0, R) \subset A$, and suppose that

$$\begin{aligned} i) \quad &\|f(x_0)\| \leq \eta, \lambda(Df(x)) \geq \nu \text{ and } \|Df(x) - Df(y)\| \leq K\|x - y\| \in \mathbb{R}_+^* \text{ for all } x, y \in A, \\ &\text{where } K \in \mathbb{R}_+^*. \end{aligned}$$

Figure 1 – Non-existence of zeros radius for $h(x, y)$ at $(x_0, y_0) = (0, 0)$ and $(x_0, y_0) = (0.2, 0.2)$

ii) $g_1(t) = \eta - vt + \frac{Kt^2}{2}$ has zeros $t^* < t^{**}$ with $t^* \in [0, R]$.

Then f has a zero $x^* \in \bar{B}(x_0, t^*)$ and no other zeros in $B(x_0, t^{**}) \cap A$

Proof. Notice that if $v = 0$, then g cannot possibly have distinct zeros. Thus $v > 0$ and $\lambda(Df(x_0)) \geq v > 0$, therefore Proposition 7 implies that $\lambda(Df(x_0))$ is invertible with $\|Df(x_0)^{-1}\| = \lambda(Df(x_0))^{-1} \leq v^{-1}$. Now, since

$$\|Df(x_0)^{-1}f(x_0)\| \leq \|Df(x_0)^{-1}\| \|f(x_0)\|$$

and

$$\|Df(x_0)^{-1}(Df(x) - Df(y))\| \leq \|Df(x_0)^{-1}\| \|(Df(x) - Df(y))\|$$

the theorem follows from the Kantorovich Theorem. $\square$

Due to the above theorem, if η , v and K satisfies the property i) above we shall call $g_1(t) = \eta - vt + \frac{Kt^2}{2}$ an *upper unidimensional taylor projection* of f .

Example 2. To exemplify an application of the Theorem 7 we applied to $h(x, y) = (2x^2 + 2y^2 - 1, x - y)$ restricted to $\bar{B}(0, 1)_{\mathbb{R}^2}$ for some points near the zeros of $h(x, y)$. Once again we let $K = 3$ be a bound for the second derivative of h , and the computation of $\|h(x_0, y_0)\|_2$ and $\gamma(Dh(x_0, y_0))_{\mathcal{L}^2(\mathbb{R}^2, \mathbb{R}^2)}$ are done as described in section 2.2.3.

For $z_0 = (x_0, y_0) = (0.54, 0.54)$ and $z_1 = (x_1, y_1) = (0.52, 0.52)$ we computed

$$\|h(z_0)\| \geq 0.1664,$$

$$\gamma(Dh(z_0)) \leq 1.414213562373095,$$

$$\|h(z_1)\| \geq 0.0816, \text{ and}$$

$$\gamma(Dh(z_1)) \leq 1.414213562373095$$

Figure 2 – Existence and unicity of zeros radius for $h(x,y)$ at $(x_0, y_0) = (0.54, 0.54)$ and $(x_0, y_0) = (0.52, 0.52)$

which generated, via the polynomial $g_1(t)$ as in Theorem 7, a existence radius and unicity radius for h at z_0 of

$$\begin{aligned} t^*(z_0) &\geq 0.226274169979694, \\ t^{**}(z_0) &\leq 0.245130350811329, \\ t^*(z_1) &\geq 0.067311181191315, \text{ and} \\ t^{**}(z_1) &\leq 0.404093339599717 \end{aligned}$$

respectively, where the existence radius t^* for h are represented in green in Figure 2 and the unicity radius t^{**} for h are represented in blue in Figure 2.

Theorem 8. Let X and Y be Bannach spaces, $U \subset E$ an open convex set, $f : U \subset X \rightarrow Y$ a twice differentiable function in U , $x_0 \in U$, $A \subset U$ convex, $R > 0$ satisfying $\bar{B}(x_0, R) \subset A$, let $(r_k)_{k \in \mathbb{N}}$ be a sequence in X and suppose:

- i) $\|f(x_0)\| \leq \eta$, $\lambda(f(x_0)) \geq \nu$, and $\|Df(x) - Df(y)\| \leq K \|x - y\|$ for all $x, y \in A$, where $K \in \mathbb{R}_+^*$.
- ii) $g_2(t) = \eta_d - \nu_d t + \frac{3K}{2} t^2$ has zeros $t_d^* < t_d^{**}$ such that $t_d^* \leq R$, where $\eta_d = \eta + \nu d$, $\nu_d = \nu + Kd$, and $d \leq \frac{\nu}{K}$.

Then $g_1(t) = \eta - \nu t + \frac{K}{2} t^2$ has two distinct zeros $t^* < t^{**}$ with $t^* \leq t_d^*$ and moreover:

- I) f has a zero $x^* \in \bar{B}(x_0, t^*)$ and no other zeros in $B(x_0, t^{**}) \cap U$.
- II) If $x_{k+1} \in \bar{B}(T_f(x_k), c)$ for all $k \in \mathbb{N}$ then $(x_k)_{k \in \mathbb{N}}$ is feasible for the Newton Method on U and item III) holds,

$$\text{III) } \|x^* - x_n\| \leq \frac{\nu}{K} \frac{(1 - \sqrt{1 - 2h})^{2^n}}{2^n} + 2d \text{ for all } n \in \mathbb{N}, \text{ where } h = \frac{K\eta}{\nu^2} \leq \frac{1}{6}.$$

Proof. Proceeding analogously as in Theorem 8, we can see the theorem is simply a consequence of Proposition 7 and Theorem 5. $\square$

Example 3. We exemplify an application of Theorem 8 for $h(x, y) = (2x^2 + 2y^2 - 1, x - y)$ restricted to $\bar{B}(0, 1)_{\mathbb{R}^2}$. Once again we let $K = 3$ be a bound for the second derivative of h , and the computation of $\|h(x_0, y_0)\|_2$ and $\gamma(Dh(x_0, y_0))_{\mathcal{L}^2(\mathbb{R}^2, \mathbb{R}^2)}$ are done as described in section 2.2.3.

For $z_0 = (x_0, y_0) = (0.51, 0.51)$ we considered the error in the Inexact Newton Method to be $d = 10^{-6}$ and computed

$$\begin{aligned}\|h(z_0)\| &\geq 0.0404, \\ \gamma(Dh(z_0)) &\leq 1.414213562373095,\end{aligned}$$

which generated, via the polynomial $g_2(t)$ as in Theorem 7, a existence for h at z_0 of

$$t_d^*(z_0) \geq 0.037531452770462,$$

where the feasibility radius t_d^* for the Inexact Newton method for h is represented in green in Figure 2.

Figure 3 – Inexact Newton convergence radius for $(x_0, y_0) = (1/1.95, 1/1.95)$

2.3.3 Interval Zeros Isolation Theorem

We now define the main theorem for zeros isolation in Bannach Spaces. We shall see soon that this theorem provides directly a method to isolate all zeros, and shall prove this method converges.

Definition 10. We define $\mathbb{IR}$ to be the set of closed intervals $\eta = [a, b] \subset \mathbb{R}$, where $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$, and we define $\mathbb{IR}_+$ to be the set of such intervals with $a \geq 0$. Moreover, given $\eta \in \mathbb{IR}$ with $\eta = [a, b]$ we define $\underline{\eta} = a$ and $\overline{\eta} = b$.

The following theorem is the interval isolation theorem of zeros of $f : U \subset X \rightarrow Y$. For the following if $A \subset X$, we let A^c denote the complement of A in X .

Definition 11. We define the functions $\rho_0 : \mathbb{R}_+^3 \rightarrow \mathbb{R}$, $\rho_1 : \mathbb{R}_+^3 \rightarrow \mathbb{R}$ and $\rho_2 : \mathbb{R}_+^4 \rightarrow \mathbb{R}$ by

$$\begin{aligned}\rho_0(a, b, c) &= \frac{-b + \sqrt{b^2 + 4ac}}{2a} \\ \rho_1(a, b, c, d) &= \frac{\sqrt{\max\{b_d^2 - 4ac_d, 0\}}}{2a}, \\ \rho_2(a, b, c, d) &= \frac{b_d - \sqrt{\max\{b_d^2 - 4ac_d, 0\}}}{2a}\end{aligned}$$

where $b_d = b + ad$ and $c_d = c + bd$.

Theorem 9. Let X and Y be Bannach spaces, $U \subset X$ open, $f : U \subset X \rightarrow Y$ a differentiable function in U , $A \subset U$ a convex set, and suppose $\omega = (K, d, \eta, \mu, \nu, \alpha, \beta, \gamma) \in (\mathbb{R}_+^*)^2 \times \mathbb{R}_+^6$ satisfy

- i) $\|Df(x) - Df(y)\| \leq K \|x - y\|$ for all $x, y \in A$;
- ii) $\|f(x_0)\| \in \eta$, $\|Df(x_0)\| \in \mu$ and $\lambda(f(x_0)) \in \nu$;
- iii) $\rho_0\left(\frac{K}{2}, \bar{\mu}, \underline{\eta}\right) \in \alpha$, $\rho_1\left(\frac{K}{2}, \underline{\nu}, \bar{\eta}, 0\right) \in \beta$;
- iv) $\min\left\{\underline{\beta} - \rho_2\left(\frac{K}{2}, \underline{\nu}, \bar{\eta}, 0\right), \text{dist}(x_0, A^c) - \rho_2\left(\frac{3K}{2}, \underline{\nu}, \bar{\eta}, d\right), \rho_1\left(\frac{3K}{2}, \underline{\nu}, \bar{\eta}, d\right), \frac{\underline{\nu}}{K} - d\right\} \in \gamma$.
- v) $r_0(\omega) = \underline{\alpha}$, $r_1(\omega) = \underline{\beta}$ if $\underline{\gamma} > 0$ and $r_1(\omega) = 0$ otherwise

Then, denoting $h = \frac{K\bar{\eta}}{\underline{\nu}^2}$, we have that

I) f has no zeros in $B(x_0, r_0(\omega))$;

II) If $r_1(\omega) \neq 0$, then f has a unique zero x^* in $B(x_0, r_1(\omega)) \cap A$, any sequence $(x_k)_{k \in \mathbb{N}}$ satisfying $T_f(x_k) \in \bar{B}(x_{k+1}, d)$ for all $k \in \mathbb{N}$ is feasible for the Newton Method over f ,

$$\|x^* - x_k\| \leq \frac{\underline{\nu}}{K} \frac{(1 - \sqrt{1 - 2h})^{2^k}}{2^k} + 2d \text{ for all } k \in \mathbb{N}, \text{ and } h \leq \frac{1}{6}.$$

Proof. Item I)

If we let $g_0(t) = \underline{\eta} - \bar{\mu}t - \frac{K}{2}t^2$, then the quadratic formula tells us that $g_0(t)$ has a unique non-negative zero t' where

$$t' = \frac{-\bar{\mu} + \sqrt{(\bar{\mu})^2 + 2K\underline{\eta}}}{K} = \rho_0\left(\frac{K}{2}, \bar{\mu}, \underline{\eta}\right) \in \alpha.$$

On the other hand, by Theorem 6 f has no zeros in $B(x_0, t') \cap A$, and since $r_0(\omega) = \underline{\alpha} \leq t'$, it follows that f has no zeros in $B(x_0, r_0(\omega)) \cap A$ as well.

Item II)

If $\underline{\gamma} \leq 0$ then by definition $r_1(\omega) = 0$, in which case the item II) is trivially true. Thus we shall suppose that $\underline{\gamma} > 0$.

Now, since by hypothesis $\underline{\gamma} \leq \frac{\underline{v}}{K} - d$, from $\underline{\gamma} > 0$ we conclude that $\underline{v} > Kd \geq 0$ and thus $\lambda(Df(x_0)) \geq \underline{v} > 0$. Therefore by items i) and ii) of Proposition 8 it follows that $Df(x_0)$ is invertible with $\lambda(Df(x_0)) = \|Df(x_0)^{-1}\|^{-1}$. Thus

$$\|Df(x_0)^{-1}\| = \lambda(Df(x_0))^{-1} \leq \underline{v}^{-1}$$

Now, denoting $\eta_d = \underline{\eta} + Kd$ and $\eta_d = \bar{\eta} + \bar{\eta}d$, by definition we have that $\rho_1\left(\frac{3K}{2}, \underline{v}, \bar{\eta}, d\right) = \underline{v}_d^2 - 6K\eta_d$. Thus, since $\underline{v}_d^2 - 6K\eta_d = \rho_1\left(\frac{3K}{2}, \underline{v}, \bar{\eta}, d\right) \geq \underline{\gamma} > 0$, it follows by the quadratic formula that $g_2(t) = \eta_d - \underline{v}_d t + \frac{3K}{2}t^2$ has distinct zeros $t_d^* < t_d^{**}$, where

$$t_d^* = \frac{\underline{v}_d - \sqrt{\underline{v}_d^2 - 6K\eta_d}}{3K} = \rho_2\left(\frac{3K}{2}, \underline{v}, \bar{\eta}\right) \text{ and } t_d^{**} = \frac{\underline{v}_d + \sqrt{\underline{v}_d^2 - 6K\eta_d}}{3K}$$

and since $\text{dist}(x_0, A^c) - \rho_2\left(\frac{3K}{2}, \underline{v}, \bar{\eta}\right) \geq \underline{\gamma} > 0$, it follows that $t_d^* = \rho_2\left(\frac{3K}{2}, \underline{v}, \bar{\eta}\right) < \text{dist}(x_0, A^c)$ and thus it follows that $\bar{B}(x_0, t_d^*) \subset A$. Therefore, due to Theorem 9 we have that f has a zero x^* in $\bar{B}(x_0, t_d^*)$ and that $g_1(t) = \bar{\eta} - \underline{v}t + \frac{K}{2}t^2$ has two distinct zeros, which are given by

$$t^* = \frac{\underline{v} - \sqrt{\underline{v}^2 - 2K\bar{\eta}}}{K} = \rho_2\left(\frac{K}{2}, \underline{v}, \bar{\eta}, 0\right) \text{ and } t^{**} = \frac{\underline{v} + \sqrt{\underline{v}^2 - 2K\bar{\eta}}}{K}.$$

Thus, applying Theorems 7 and 8, the theorem follows. $\square$

Observation: During the proof of Theorem 9 it was proved in special that if $r_1(\omega) \neq 0$ then f has a zero x^* in $\bar{B}(x_0, t^*)$ and no other zeros in $B(x_0, t^{**}) \cap A$ where

$$t^* = \frac{\underline{v} - \sqrt{\underline{v}^2 - 2K\bar{\eta}}}{K} \text{ and } t^{**} = \frac{\underline{v} + \sqrt{\underline{v}^2 - 2K\bar{\eta}}}{K}$$

This result will be useful to prove Proposition 16 in the next subsection.

Definition 12. Given $f : U \subset X \rightarrow Y$ differentiable, U open, $A \subset U$ convex, and $x \in A$, we say $\omega = (K, d, \eta, \mu, v, \alpha, \beta, \gamma) \in \mathbb{R}_+^* \times \mathbb{IR}_+^6$ is *feasible* for (f, x) in Theorem 9 if its coordinates satisfies conditions i) to iv) of Theorem 9 for $x_0 = x$. Moreover we define

$$\varepsilon(\omega) = \max \{d, \text{diam } \eta, \text{diam } \mu, \text{diam } v, \text{diam } \alpha, \text{diam } \beta, \text{diam } \gamma\}$$

to be the *computation error* of ω , and we define $r_0(\omega)$ and $r_1(\omega)$ in the same way as in Theorem 9.

2.3.4 Isolation of Zeros Algorithm

The next proposition tells us that if the ball $B(x_0, r_1(\omega))$ given in Theorem 9 is computed in distinct points and if such balls intersect, then they contain the same zero x^* of f .

Proposition 16 (Intersection Property). Given $f : A \subset U \subset X \rightarrow Y$ as in Theorem 9, suppose that:

- i) $\omega_1 \in (\mathbb{R}_+^*)^2 \times \mathbb{IR}_+^6$ is feasible for (f, x_1) in Theorem 9 for $x_1 \in A$;
- ii) $\omega_2 \in (\mathbb{R}_+^*)^2 \times \mathbb{IR}_+^6$ is feasible for (f, x_2) in Theorem 9 for $x_2 \in A$;
- iii) $B(x_1, r_1(\omega_1)) \cap B(x_2, r_1(\omega_2)) \neq \emptyset$.

Then there exists a zero x^* of f with $x^* \in B(x_1, r_1(\omega_1)) \cap B(x_2, r_1(\omega_2)) \cap A$.

Proof. Let

$$\omega_1 = (K_1, d_1, \eta_1, \mu_1, v_1, \alpha_1, \beta_1, \gamma_1) \in (\mathbb{R}_+^*)^2 \times \mathbb{IR}_+^6$$

and

$$\omega_2 = (K_2, d_2, \eta_2, \mu_2, v_2, \alpha_2, \beta_2, \gamma_2) \in (\mathbb{R}_+^*)^2 \times \mathbb{IR}_+^6,$$

satisfying the hypothesis i) to iii) above.

Let x_1^* be a zero of f in $B(x_1, r_1(\omega_1))$ and let x_2^* be a zero of f in $B(x_2, r_1(\omega_2))$ and let us suppose by contradiction that $x_1^* \neq x_2^*$.

Since by hypothesis there exists at least an element $z \in B(x_1, r_1(\omega_1)) \cap B(x_2, r_1(\omega_2))$, it follows that

$$\|x_2 - x_1\| \leq \|x_2 - z\| + \|z - x_1\| \leq r_1(\omega_1) + r_2(\omega_2) \quad (2.15)$$

On the other hand, since $B(x_1, r_1(\omega_1)) \cap B(x_2, r_1(\omega_2)) \neq \emptyset$, it follows that $r_1(\omega_1) \neq 0$ and $r_1(\omega_2) \neq 0$ and thus it follows that $\underline{v}_1^2 - 2K_1\overline{\eta}_1 > 0$, $\underline{v}_2^2 - 2K_2\overline{\eta}_2 > 0$ and that

$$\begin{aligned} r_1(\omega_1) &= \underline{\beta}_1 \leq \rho_1 \left(\frac{K}{2}, \underline{v}_1, \overline{\eta}_1, 0 \right) = \frac{\sqrt{\underline{v}_1^2 - 2K_1\overline{\eta}_1}}{K_1} \text{ and} \\ r_1(\omega_2) &= \underline{\beta}_2 \leq \rho_1 \left(\frac{K}{2}, \underline{v}_2, \overline{\eta}_2, 0 \right) = \frac{\sqrt{\underline{v}_2^2 - 2K_2\overline{\eta}_2}}{K_2}. \end{aligned}$$

Now, due to the observation of Theorem 9, letting

$$u_1 = \frac{\underline{v}_1 - \sqrt{\underline{v}_1^2 - 2K_1\overline{\eta}_1}}{K_1} \text{ and } u_2 = \frac{\underline{v}_2 - \sqrt{\underline{v}_2^2 - 2K_2\overline{\eta}_2}}{K_2}.$$

it follows that $x_1^* \in \bar{B}(x_1, u_1) \cap A$ and $x_2^* \in \bar{B}(x_2, u_2) \cap A$, and also due to the observation in Theorem 9, letting

$$w_1 = \frac{\underline{v}_1 + \sqrt{\underline{v}_1^2 - 2K_1\overline{\eta}_1}}{K_1} \text{ and } w_2 = \frac{\underline{v}_2 + \sqrt{\underline{v}_2^2 - 2K_2\overline{\eta}_2}}{K_2},$$

it follows that x_1^* is the unique zero of f in $B(x_1, w_1) \cap A$ and x_2^* is the unique zero of f in $B(x_2, w_2) \cap A$. In particular, since $x_1^* \neq x_2^*$ it follows that $x_1^* \notin B(x_2, w_2)$ and $x_2^* \notin B(x_1, w_1)$.

Now notice that, if it happened by absurd that $\|x_2 - x_1\| \leq w_2 - u_1$, then it would follow that $\|x_2 - x_1^*\| \leq \|x_2 - x_1\| + \|x_1 - x_1^*\| \leq w_2$ and thus $x_1^* \in B(x_2, w_2)$, a contradiction, and thus we conclude that $\|x_2 - x_1\| > w_2 - u_1$. Analogously it follows that $\|x_1 - x_2\| > w_1 - u_2$. Therefore we have that

$$2\|x_2 - x_1\| > w_2 - u_1 + w_1 - u_2 \Rightarrow \|x_2 - x_1\| > \frac{w_2 - u_2}{2} + \frac{w_1 - u_1}{2} = \frac{\sqrt{v_1^2 - 2K_1\eta_1}}{K_1} + \frac{\sqrt{v_2^2 - 2K_2\eta_2}}{K_2} \geq r_1(\omega_1) + r_2(\omega_2),$$

a contradiction with (2.15).

Thus it follows that $x_1^* = x_2^*$, and since $x_1^* \in B(x_1, r_1(\omega_1)) \cap A$ and $x_2^* \in B(x_2, r_1(\omega_2)) \cap A$ the proposition is proved. $\square$

The zero isolation algorithm follows therefore from the following steps:

Given a closed ball $\bar{B}(x_0, r) \subset U \subset X$, $f : U \rightarrow Y$ differentiable and given $\varepsilon_0 > 0$ and $\delta_0 > 0$ we suppose that:

- We can always cover $\bar{B}(x_0, r)$ by a collection $\mathcal{B}$ of non-empty sets $S_1, \dots, S_m$ such that $\text{diam}(S_j) \leq \delta_0$ for all $1 \leq j \leq m$.
- We suppose for each S_j an element $x_j \in S_j$ can be selected and that we can always find a $\omega_j \in (\mathbb{R}_+^*)^2 \times \mathbb{R}^6$ feasible for (f, x_j) such that $\varepsilon(\omega_j) \leq \varepsilon_0$

Under these suppositions the algorithm proceeds as follows:

- step 1 Given a set $\mathcal{A}$ initially empty, for each $1 \leq j \leq m$, we compute $r_0(\omega_j)$ and check if $S_j \subset r_0(\omega_j)$. If this is the case then due to Theorem 9 there are no zeros of f in S_j and thus we can eliminate S from $\mathcal{B}$.
- step 2 For each S_j non eliminated we test if $S_j \subset r_1(\omega_j)$ and if this is the case, due to Theorem 9 there exists a unique zero of f in $r_1(\omega_j)$ and thus shall eliminate S_j from $\mathcal{B}$ and we add the closed ball $\bar{B}(x_j, r_1(\omega_j))$ to $\mathcal{A}$
- step 3 If $\mathcal{B}$ is empty, we partition $\mathcal{A}$ into a set $\mathcal{A}_1, \dots, \mathcal{A}_n$ such all balls of $\mathcal{A}_j$ intersect between themselves but intersect no balls from $\mathcal{A}_i$ for $i \neq j$. This is always possible due to the Intersection Property proved above for Theorem 9.
- step 4 For each $1 \leq i \leq m$ we select a single ball $\bar{B}(x_m, r) \in \mathcal{A}_i$ and use the Inexact Newton algorithm $x_{m,k+1} = T_f(x_{m,k})$, $x_{m,0} = x_m$, k times, estimating the distance $\varepsilon_m = \|x_{m,k} - x_m^*\|$ via the error formula in Theorem 9

Due to the Intersection Property for Theorem 9, n will be the number of zeros.

Example 4. We exemplify it to isolate all zeros of $h(x, y) = (2x^2 + 2y^2 - 1, x - y)$ restricted to $\bar{B}(0, 1)_{\mathbb{R}^2}$.

Figure 4 – Subdivision of the circle and first step the algorithm applied to $h(x)$. Figure 5 – Step 3 and 4 of Algorithm 1 applied to $h(x, y)$.

First we have a covering $\bar{B}(0, 1)_{\mathbb{R}^2}$ via smaller closed rectangles as in Proposition 11, where these rectangles covering $\bar{B}(0, 1)_{\mathbb{R}^2}$ are represented in yellow in the left figure of Figure 4.

On the other hand, in the right figure of Figure 4 we illustrate the elimination process carried step 1 of Algorithm 1, corresponding to the elimination of the rectangles that do not contain zeros of h .

Moreover, in the left figure of Figure 5 we have the steps 2 and 3 of Algorithm 1 applied, where the existence and unicity radius generated are represented in green and the points that generate them are represented in black.

Finally, in the right figure of Figure 5 we have the Inexact Newton Method applied to approximate the zeros of h , where a tight enclosure for solutions are represented in red.

2.3.5 Convergence of the Algorithm

Finally, the following theorem gives us regularity conditions (conditions i) and ii) below) for the Algorithm to terminate.

Theorem 10. Let $f : U \subset X \rightarrow Y$ be differentiable, U open, $A \subset U$ bounded and convex, $\Omega \subset A$ compact and let $L > 0$. Then there exists $\varepsilon > 0$ and $\delta > 0$ such that if the following hold:

- i) All zeroes z of f in A lie in Ω and satisfy $z \notin \partial A$;
- ii) $Df(y)$ is invertible for all zeros y of f in A ;
- iii) $\omega = (K, d, \eta, \mu, \nu, \alpha, \beta, \gamma)$ is feasible for (f, x) in Theorem 9 and
- iv) $K \leq L$ and $\varepsilon(\omega) \leq \varepsilon$

then $\bar{B}(x, \delta) \subset B(x, r_0(\omega))$ or $\bar{B}(x, \delta) \subset B(x, r_1(\omega))$.

Proof. Let $S = \{z \in A \mid f(z) = 0\} \subset \Omega$. Since by ii), $Df(z)$ is invertible for all $z \in S$, it follows due to the Inverse Function Theorem that all elements of S are isolated and thus the compactity of Ω implies that S is finite. Thus $S = \{y_1, \dots, y_m\}$ for some $m > 0$ and $y_i \in \Omega, i = 1, \dots, m$.

For the remaining of the proof we denote

$$\Omega_a = \{x \in \Omega \mid \exists y \in S, \|x - y\| < a\} \text{ and}$$

$$\Omega_a^* = \Omega - \Omega_a = \{x \in \Omega \mid \|x - y\| \geq a \forall y \in S\}.$$

for all $a \in \mathbb{R}$.

We shall now prove the following two items:

- I) There exist $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that, under the hypothesis i) to iv) of the theorem, if $\varepsilon(\omega) \leq \varepsilon_1$ and if $x \in \Omega_{\delta_1}$, then $\bar{B}(x, \delta_1) \subset B(x, r_1(\omega))$
- II) We shall prove there exists $\varepsilon_2 > 0$ and $\delta_2 > 0$, such that, given δ_1 as in item I), and under the hypothesis i) to iv) of the theorem, if $\varepsilon(\omega) \leq \varepsilon_2$ and if $x \in \Omega_{\delta_1}^*$, then $\bar{B}(x, \delta_2) \subset B(x, r_1(\omega))$

Item I):

Before defining ε_1 and δ_1 we must prove some bounds.

We first notice that, to the continuity of f and the fact that S is finite, there must exist $N_0 > 0$ and $\delta_0 > 0$ such that if $\|x - y\| < \delta_0$ for some $y \in S$, then $\|f(x)\| \leq N_0$. Thus we conclude that

$$\|f(x)\| \leq N_0 \text{ for all } x \in \Omega_{\delta_0}. \quad (2.16)$$

On the other hand, since by hypothesis $Df(y)$ is invertible for all $y \in S$ and since S is finite, it follows that there exist $N_1 > 0$ such that $\|Df(y)^{-1}\| \leq (3N_1)^{-1}$ for all $y \in S$ and thus by ii) of Proposition 8 we have that $\lambda(Df(y)) = \|Df(y)^{-1}\|^{-1} \geq 2N_1$ for all $y \in S$. Therefore, it

follows that if x satisfies $\|x - y\| \leq \frac{N_1}{K}$, then due to item *iii*) of Proposition 8 and the hypothesis we have that

$$\begin{aligned} \|\lambda(Df(x)) - \lambda(Df(y))\| &\leq \|Df(x) - Df(y)\| \leq M \|x - y\| \leq N_1 \Rightarrow \\ \lambda(Df(x)) &\geq \lambda(Df(y)) - N_1 \geq 2N_1 - N_1 = N_1. \end{aligned}$$

Thus, letting $\delta_0^* = \frac{N_1}{K}$ we conclude that

$$\lambda(Df(x)) \geq N_1 \text{ for all } x \in \Omega_{\delta_0^*} \quad (2.17)$$

Now let $N_2 = \min_{1 \leq i \leq m} \text{dist}(y_i, A^c)$. Notice that $N_2 < \infty$ since A is bounded in X . Moreover, we have that $N_2 \neq 0$, since the equality $\text{dist}(y_i, A^c) = 0$ would imply $y_i \in \overline{A^c}$, which combined with $y_i \in A$ would imply in $y_i \in \partial A$, contradicting the hypothesis. Therefore, letting $\delta_0^{**} = \frac{N_2}{2}$ it follows that if $\|x - y\| \leq \delta_1$ then due to the triangle inequality we have that

$$\begin{aligned} \text{dist}(y, A^c) &\leq \text{dist}(y, x) + \text{dist}(x, A^c) \Rightarrow \\ \text{dist}(x, A^c) &\geq \text{dist}(y, A^c) - \|x - y\| \geq N_2 - \delta_1 = \frac{N_2}{2} \end{aligned}$$

and thus we conclude that

$$\text{dist}(x, A^c) \geq \frac{N_2}{2} \text{ for all } x \in \Omega_{\delta_0^{**}}. \quad (2.18)$$

Now, given $N = \min \left\{ \sqrt{12KN_0}, \frac{1}{3}N_1, \frac{1}{2}KN_2 \right\}$, we let

$$\begin{aligned} \varepsilon_1 &= \min \left\{ \frac{N^2}{12K}, \frac{N}{12K}, N \right\}, \text{ and} \\ \delta_1 &= \min \left\{ \delta_0, \delta_0^*, \delta_0^{**}, \frac{N}{K} \right\} \end{aligned}$$

and suppose that $\varepsilon(\omega) \leq \varepsilon_1$ and $x \in \Omega_{\delta_1}$. The first thing we need to prove now is that $\underline{\gamma} > 0$, which we do as follows.

Notice that $\Omega_{\delta_1} \subset \Omega_{\delta_0^*}$, and thus, since by hypothesis $\text{diam } \nu \leq \varepsilon(\omega) \leq \varepsilon_1$, and since $\lambda(Df(x)) \in \nu$, it follows from 2.17 that

$$\underline{\nu} \geq \bar{\nu} - \varepsilon_1 \geq \lambda(Df(x)) - \varepsilon_1 \geq N_1 - N \geq 2N. \quad (2.19)$$

Moreover, $\Omega_{\delta_1} \subset \Omega_{\delta_0}$, and thus, since by hypothesis $\text{diam } \eta \leq \varepsilon(\omega) \leq \varepsilon_1 \leq \frac{N^2}{12K}$, and since $\|f(x)\| \in \eta$, it follows from (2.16) that

$$\bar{\eta} \leq \underline{\eta} + \varepsilon_1 \leq \|f(x)\| + \varepsilon_1 \leq N_0 + \frac{N^2}{12K} \leq \frac{N^2}{6K},$$

and since due to (2.19) we have that $N \leq \frac{\underline{\nu}}{2}$, it follows that

$$\bar{\eta} \leq \frac{N^2}{6K} \leq \frac{\underline{\nu}N}{12K} \text{ and} \quad (2.20)$$

$$\bar{\eta} \leq \frac{\underline{\nu}N}{12} \leq \frac{\underline{\nu}^2}{24K} \quad (2.21)$$

On the other hand, since by hypothesis $d \leq \varepsilon(\omega)$, it follows that

$$d \leq \varepsilon(\omega) \leq \varepsilon_1 \leq \frac{N}{6K} \leq \frac{\underline{v}}{12K}. \quad (2.22)$$

Thus it follows from (2.19) and (2.20) that

$$\underline{v}^2 - 2K\bar{\eta} \geq \underline{v}^2 - 2K\left(\frac{\underline{v}^2}{24K}\right) = \frac{11}{12}\underline{v}^2 \geq \left(\frac{\underline{v}}{2}\right)^2 \quad (2.23)$$

and since due (2.19) we have that $2\bar{\eta} \leq \frac{\underline{v}N}{6K}$ it follows from (2.23) that

$$\begin{aligned} \rho_2\left(\frac{K}{2}, \underline{v}, \bar{\eta}, 0\right) &= \frac{\underline{v} - \sqrt{\max\{\underline{v}^2 - 2K\bar{\eta}, 0\}}}{K} = \frac{\underline{v} - \sqrt{\underline{v}^2 - 2K\bar{\eta}}}{K} \\ &= \frac{2\bar{\eta}}{\underline{v} + \sqrt{\underline{v}^2 - 2K\bar{\eta}}} \leq \frac{\underline{v}N}{6K\left(\underline{v} + \frac{\underline{v}}{2}\right)} = \frac{2N}{9K}. \end{aligned} \quad (2.24)$$

Moreover, from (2.23) and (2.19) we have that

$$\rho_1\left(\frac{K}{2}, \underline{v}, \bar{\eta}, 0\right) = \frac{\sqrt{\max\{\underline{v}^2 - 2K\bar{\eta}, 0\}}}{K} \geq \frac{3\underline{v}}{4K} \geq \frac{3N}{2K}.$$

and since by hypothesis $\text{diam } \beta \leq \varepsilon(\omega) \leq \varepsilon_1 \leq \frac{N}{6K}$ and $\rho_1\left(\frac{K}{2}, \underline{v}, \bar{\eta}, 0\right) \in \beta$, it follows that

$$\underline{\beta} \geq \bar{\beta} - \frac{N}{6K} \geq \rho_1\left(\frac{K}{2}, \underline{v}, \bar{\eta}, 0\right) - \frac{N}{6K} \geq \frac{3N}{2K} - \frac{N}{6K} = \frac{4N}{3K} \quad (2.25)$$

which combined with (2.24) implies that

$$\underline{\beta} - \rho_2\left(\frac{K}{2}, \underline{v}, \bar{\eta}, 0\right) \geq \frac{4N}{3K} - \frac{2N}{9K} = \frac{10N}{9K} \quad (2.26)$$

Now, denoting $\eta_d = \bar{\eta} + \underline{v}d$ and $v_d = \underline{v} + Kd$, since $v_d \geq \underline{v}$, it follows from the inequalities (2.20), (2.22) and (2.19) that

$$\begin{aligned} v_d^2 - 6K\eta_d &\geq \underline{v}^2 - 6K\bar{\eta} - 6K\underline{v}d \\ &\geq \underline{v}^2 - 6K\left(\frac{\underline{v}^2}{24K}\right) - 6K\underline{v}\left(\frac{\underline{v}}{12K}\right) = \underline{v}^2 - \frac{\underline{v}^2}{4} - \frac{\underline{v}^2}{2} = \frac{\underline{v}^2}{4} \geq N^2 \end{aligned}$$

and thus

$$\rho_1\left(\frac{3K}{2}, \underline{v}, \bar{\eta}, d\right) = \frac{\sqrt{\max\{v_d^2 - 6K\eta_d, 0\}}}{3K} \geq \frac{N}{3K} \quad (2.27)$$

Moreover, from inequalities (2.20) and (2.22) we have that

$$\eta_d = \bar{\eta} + \underline{v}d \leq \frac{\underline{v}N}{6K} + \frac{\underline{v}N}{6K} = \frac{\underline{v}N}{3K}$$

and thus it follows that

$$\begin{aligned} \rho_2\left(\frac{3K}{2}, \underline{v}, \bar{\eta}, d\right) &= \frac{v_d - \sqrt{\max\{v_d^2 - 6K\eta_d, 0\}}}{3K} = \frac{v_d - \sqrt{v_d^2 - 6K\eta_d}}{3K} \\ &= \frac{2\eta_d}{v_d + \sqrt{v_d^2 - 6K\eta_d}} \leq \frac{2\underline{v}N}{3K\left(\underline{v} + \sqrt{\frac{\underline{v}}{4}}\right)} = \frac{8N}{9K}. \end{aligned}$$

and thus, since $\Omega_{\delta_1} \subset \Omega_{\delta_0^{**}}$, from (2.18) it follows that

$$\text{dist}(x, A^c) - \rho_2 \left(\frac{3K}{2}, \underline{v}, \bar{\eta}, d \right) \geq \frac{N_2}{2} - \frac{8N}{9K} \geq \frac{N}{K} - \frac{8N}{9K} = \frac{N}{9K} \quad (2.28)$$

On the other hand, since from (2.22) we have that $d \leq \frac{v}{12K}$, it follows from (2.19) that

$$\frac{v}{K} - d \geq \frac{11v}{12K} \geq \frac{11N}{6K} \quad (2.29)$$

Thus, combining inequalities (2.26), (2.27), (2.28) and (2.29) it follows that, letting

$$\tau = \min \left\{ \underline{\beta} - \rho_2 \left(\frac{K}{2}, \underline{v}, \bar{\eta}, 0 \right), \text{dist}(x_0, A^c) - \rho_2 \left(\frac{3K}{2}, \underline{v}, \bar{\eta}, d \right), \rho_1 \left(\frac{3K}{2}, \underline{v}, \bar{\eta}, d \right), \frac{v}{K} - d \right\}$$

then

$$\tau \geq \min \left\{ \frac{10N}{9K}, \frac{N}{3K}, \frac{N}{9K}, \frac{11N}{6K} \right\} \geq \frac{N}{9K}.$$

Therefore, since by hypothesis $\text{diam } \gamma \leq \varepsilon(\omega) \leq \varepsilon_1$ and $\tau \in \gamma$, it follows that

$$\underline{\gamma} \geq \bar{\gamma} - \varepsilon_1 \geq \tau - \varepsilon_1 \geq \frac{N}{9K} - \frac{N}{12K} \geq \frac{N}{36K} > 0$$

that is $\underline{\gamma} > 0$.

Thus by definition it follows that $r_1(\omega) = \underline{\beta}$. On the other hand, due to (2.25) we have that $\underline{\beta} \geq \frac{4N}{3K} \geq \frac{N}{K} \geq \delta_1$ and thus $\bar{B}(x, \delta_1) \subset \bar{B}(x, r_1(\omega))$ which proves *I*).

Item *II*).

Letting ε_1 and δ_1 be as in the proof of item *I*), since S is finite it follows that

$$\Omega_{\delta_1} = \{x \in X \mid \exists y \in S, \|f(x) - y\| < \delta_1\} = \cup_{y \in S} f^{-1}(B(y, \delta_1))$$

is open in X . Therefore, since Ω is closed in X , it follows that $\Omega_{\delta_1}^* = \Omega - \Omega_{\delta_1}$ is closed as well, and since $\Omega_{\delta_1}^* \subset \Omega$, and Ω is compact, it follows that $\Omega_{\delta_1}^*$ is compact as well.

Thus, due to the continuity of f and compactness of $\Omega_{\delta_1}^*$, it follows that $\inf_{y \in \Omega_{\delta_1}^*} \|f(y)\| = \|f(z)\|$ for some $z \in \Omega_{\delta_1}^*$, and since by definition $f(z) \neq 0$ for all $z \in \Omega_{\delta_1}^*$, it follows that $\|f(z)\| > 0$. Thus, letting $M_0 = \frac{\|f(z)\|}{2}$ we have that

$$\|f(y)\| \geq 2M_0 \text{ for all } y \in \Omega_{\delta_1}^*.$$

.

Analogously, from the compactness of Ω_{δ_1} and continuity of Df , there must exist a $M_1 > 0$ such that

$$\|Df(y)\| \leq \frac{M_1}{2} \text{ for all } y \in \Omega_{\delta_1}^*$$

Thus we let

$$\delta_2 = \frac{\sqrt{M_0}}{2M_1^2 \sqrt{M_0^{-1}} + \sqrt{2K}} \text{ and}$$

$$\varepsilon_2 = \min\{M_0, M_1, \delta_2\}$$

Notice that, since by definition $\text{diam } \eta \varepsilon(\omega) \leq \varepsilon_2$ and since $\|f(x)\| \in \eta$, it follows that

$$\underline{\eta} \geq \bar{\eta} - \varepsilon_2 \geq \|f(x)\| - \varepsilon_2 \geq 2M_0 - M_0 \geq M_0 \quad (2.30)$$

Analogously, from $\text{diam } \mu \leq \varepsilon(\omega) \leq \varepsilon_2$ we have that

$$\bar{\mu} \leq \bar{\mu} + \varepsilon_2 \leq \|Df(x)\| + \varepsilon_2 \leq \frac{M_1}{2} + \frac{M_1}{2} \leq M_1. \quad (2.31)$$

Therefore, from inequalities (2.30) and (2.31) and the elementary fact that $\sqrt{e+f} \leq \sqrt{e} + \sqrt{f}$ for all e and f in $\mathbb{R}_+$ we have that

$$\begin{aligned} & \bar{\mu} + \sqrt{(\bar{\mu})^2 + 2K\underline{\eta}} \leq 2(\bar{\mu})^2 + \sqrt{2K\underline{\eta}} \\ & = \sqrt{\underline{\eta}} \left(2(\bar{\mu})^2 (\sqrt{\underline{\eta}})^{-1} + \sqrt{2K} \right) \leq \sqrt{\underline{\eta}} \left(2M_1^2 (\sqrt{M_0})^{-1} + \sqrt{2K} \right) \end{aligned}$$

and thus

$$\begin{aligned} \rho_0 \left(\frac{K}{2}, \bar{\mu}, \underline{\eta} \right) &= \frac{-\bar{\mu} + \sqrt{(\bar{\mu})^2 + 2K\underline{\eta}}}{K} = \frac{2\underline{\eta}}{\bar{\mu} + \sqrt{(\bar{\mu})^2 + 2K\underline{\eta}}} \\ &\geq \frac{2\sqrt{\underline{\eta}}}{2M_1^2 (\sqrt{M_0})^{-1} + \sqrt{2K}} \geq \frac{2\sqrt{M_0}}{2M_1^2 (\sqrt{M_0})^{-1} + \sqrt{2K}} = 2\delta_2 \end{aligned}$$

but since $\text{diam } \alpha \leq \varepsilon_2$, and since $r_0(\omega) = \underline{\alpha}$ by definition, it follows that

$$r_0(\omega) = \underline{\alpha} \geq \bar{\alpha} - \varepsilon_2 \geq 2\delta_2 - \delta_2 = \delta_2$$

which implies $\bar{B}(x, \delta_2) \subset B(x, r_0(\omega))$, proving item II).

Finally, due to item I) and item II), letting $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and $\delta = \min\{\delta_1, \delta_2\}$, if $x \in \Omega$ then it follows that either $x \in \Omega_{\delta_1}$, in which case the theorem is true due to item I), or $x \in \Omega_{\delta_1}^*$, in which case the theorem is true due to item II). Thus, the theorem is proved. $\square$

From this Theorem the convergence of the algorithm should be obvious, since it tells us that every element S_j of $\mathcal{B}$ will eventually be eliminated, either for not containing zeros, via the condition $S_j \subset r_0(\omega_j)$ or for containing a unique zero, via the condition $S_j \subset r_1(\omega_j)$.

RIGOROUS COMPUTATION OF TWO-POINT BOUNDARY VALUE PROBLEM OF NEUMANN TYPE

In this chapter we shall show how to apply the zeros isolation algorithm provided in the first part of this thesis, to semi-linear two point-boundary value problems of the form

$$u''(x) - f(x, u(x), u'(x)) = 0, \text{ for all } x \in [0, 1], \quad (3.1)$$

under the Neumann boundary conditions

$$u'(0) = 0 \text{ and } u'(1) = 0,$$

The algorithm will provide a method to find all solutions $u \in C^2(I)$ in closed balls $B(0, r)_{H^2(I)}$ of the H^2 norm, where $I = (0, 1)$, under certain computability and regularity conditions on f .

3.1 Preliminaries

3.1.1 Absolutely Continuous Functions

In the following we let $L^1([a, b])$ denote the Lebesgue integrable functions such that if u and v are Lebesgue integrable and differ on a set of measure 0 in $[a, b]$, then u and v are identified as the same function in $L^1([a, b])$. It is well known that $L^1([a, b])$ is a Banach space under the norm $\int_a^b |u(x)| dx$.

Definition 13. Given $u : [a, b] \rightarrow \mathbb{R}$, we say that u is absolutely continuous in $[a, b]$ if there exists $v \in L^1([a, b])$ such that

$$u(x) = u(a) + \int_a^x v(s) ds \quad \forall x \in [a, b]. \quad (3.2)$$

Theorem 11 (Lebesgue Differentiation Theorem). Let $u : [a, b] \rightarrow \mathbb{R}$ and $v : [a, b] \rightarrow \mathbb{R}^n$ be Lebesgue integrable with

$$u(x) = u(a) + \int_a^x v(s) ds \quad \forall x \in [a, b].$$

Then $u'(x)$ exists for a.e. $x \in [a, b]$ with $u'(x) = v(x)$ in such points.

For a proof see (FOLLAND, 1999) (Folland, Theo. 3.21, pg. 98).

Thus, due to the Lebesgue Differentiation Theorem, we know that if u is absolutely continuous, then such a v satisfying equation (3.2) is unique up to a null set in $[a, b]$, and therefore v is unique as an element of $L^1([a, b])$. Therefore we shall call v the *derivative of u as an absolutely continuous function*.

The following is the chain rule for absolutely continuous functions.

Proposition 17. Let $f_1, \dots, f_n$ be absolutely continuous functions on $[a, b]$, let $f = (f_1, \dots, f_n) : [a, b] \rightarrow \mathbb{R}^n$, let $U \subset \mathbb{R}^n$ be open, let $A \subset U$ be convex, let $\phi : U \rightarrow \mathbb{R}$ be differentiable, suppose that $D\phi : U \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R})$ is bounded (in the strong norm) in A and suppose that $f(x) \in A$ for all $x \in [a, b]$. Then $g = \phi \circ f$ is absolutely continuous in $[a, b]$, with

$$g'(x) = (D\phi(f(x)))(f'(x)) \text{ a.e. in } [a, b]$$

Proof. Letting $K = \sup_{y \in A} \|D\phi(y)\|_{\mathcal{L}(\mathbb{R}^n, \mathbb{R})}$, since $K < \infty$ by hypothesis, due to the Mean Value Theorem it follows that

$$|\phi(y_1) - \phi(y_2)| \leq K \|y_1 - y_2\|_{\mathbb{R}^n}$$

for all y_1, y_2 in A , and thus it follows that ϕ is Lipschitz continuous in A . Therefore, due to (BOGACHEV, 2007) (Bogachev, Lemma 5.3.2., pg. 352) we have that $g = \phi \circ f$ is absolutely continuous in $[a, b]$.

On the other hand since the intersection of sets with full measure still have full measure, it follows due to the Lebesgue Differentiation Theorem (Theorem 11) that there exist $J \subset (a, b)$ with full measure on (a, b) such that $f_1, \dots, f_n$ are differentiable in J . Therefore $f = (f_1, \dots, f_n)$ is differentiable in all $y \in J \subset (a, b)$ and by the classical chain rule for differentiable functions we have that $g = \phi \circ f$ is differentiable in J with

$$g'(x) = (D\phi(f(x)))(f'(x)) \text{ for all } x \in J$$

and since J has full measure in (a, b) , the proposition follows. $\square$

The following is also a classical theorem regarding absolutely continuous functions (see (MCSHANE, 1947)).

Proposition 18. If u and v are absolutely continuous in $[a, b]$, then uv is absolutely continuous in $[a, b]$ as well, and $u'v + uv'$ is the derivative of uv as an absolutely continuous function in $[a, b]$.

3.1.2 Fourier Analysis for cosine and sine series

In the following we let $I = (0, 1)$, and $L^2(I)$ will denote the square integrable functions such that if u and v are square integrable with $u(x) = v(x)$ up to a null set in I then u and v are identified as the same function in $L^2(I)$. It is well known that $L^2(I)$ is a Hilbert space under the inner product $\langle u, v \rangle_{L^2(I)} = \int_0^1 u(x)v(x)dx$.

Definition 14. Given $u \in L^2(I)$, we define the cosine and sine series expansion of u by

$$\begin{aligned} u(x) &\sim \widehat{u}_{\cos}(0) + \sum_{k=1}^{\infty} \widehat{u}_{\cos}(k) \sqrt{2} \cos(k\pi x) \text{ and} \\ u(x) &\sim \widehat{u}_{\sin}(0) + \sum_{k=1}^{\infty} \widehat{u}_{\sin}(k) \sqrt{2} \sin(k\pi x) \end{aligned} \quad (3.3)$$

respectively, where

$$\begin{aligned} \widehat{u}_{\cos}(0) &= \int_0^1 u(x)dx, \widehat{u}_{\sin}(0) = 0, \\ \widehat{u}_{\cos}(k) &= \int_0^1 u(x) \sqrt{2} \cos(k\pi x)dx \text{ for } k \geq 1, \\ \widehat{u}_{\sin}(k) &= \int_0^1 u(x) \sqrt{2} \sin(k\pi x)dx \text{ for } k \geq 1, \end{aligned}$$

and moreover we denote $\widehat{u}_{\cos} = (\widehat{u}_{\cos}(0), \widehat{u}_{\cos}(1), \dots)$ and $\widehat{u}_{\sin} = (\widehat{u}_{\sin}(0), \widehat{u}_{\sin}(1), \dots)$.

The following is the equivalent to the Parseval Theorem for the cosine and sine series.

Proposition 19. Given $u \in L^2(I)$ then we have that

$$\begin{aligned} \widehat{u}_{\cos} &\in l^2(\mathbb{N}), \widehat{u}_{\sin} \in l^2(\mathbb{N}) \text{ and} \\ \|u\|_{L^2(I)} &= \|\widehat{u}_{\cos}\|_{l^2(\mathbb{N})} = \|\widehat{u}_{\sin}\|_{l^2(\mathbb{N})} \end{aligned}$$

Proof. Let $u(x) \in L^2(I)$, and define

$$u^*(x) = \begin{cases} u(x) & \text{if } x \geq 0 \\ u(-x) & \text{if } x < 0 \end{cases}$$

Then clearly $u^*(x) \in L^2([-1, 1])$ and its Fourier series expansion is given by

$$u^*(x) \sim \sum_{i=-\infty}^{\infty} \widehat{u}^*(k) e^{ik\pi x}$$

where $\widehat{u}^*(0) = \int_{-1}^1 u^*(x) e^{-ik\pi x} dx = 2 \int_0^1 u(x) dx = 2\widehat{u}_{\cos}(0)$ and

$$\begin{aligned} \widehat{u}^*(k) &= \frac{1}{2} \int_{-1}^1 u^*(x) e^{-ik\pi x} dx = \frac{1}{2} \int_{-1}^1 u^*(x) \cos(k\pi x) dx = \\ &= \int_0^1 u(x) \cos(k\pi x) dx = \frac{1}{\sqrt{2}} \widehat{u}_{\cos}(|k|) \end{aligned}$$

for $k \geq 1$, where $|k|$ stands for the absolute value of k . Now, by Parseval Theorem (see (STEIN, 2003))

$$\begin{aligned} \|u^*\|_{L_p([-1,1])}^2 &= 2 \left(\sum_{i=-\infty}^{\infty} \widehat{u^*}(n)^2 \right) = 2 \left(\widehat{u}_{\cos}(0)^2 + 2 \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{2}} \widehat{u}_{\cos}(k) \right)^2 \right) = \\ &= 2 \sum_{k=0}^{\infty} \widehat{u}_{\cos}(k)^2 \Rightarrow \sum_{k=0}^{\infty} \widehat{u}_{\cos}(k)^2 = \|u\|_{L^2(I)}^2 \end{aligned}$$

therefore $u \in l^2(\mathbb{N})$ with $\|\widehat{u}_{\cos}\|_{l^2(\mathbb{N})} = \|u\|_{L^2(I)}$.

Now let $u(x) \in L^2(I)$, and define

$$u^{**}(x) = \begin{cases} u(x) & \text{if } x \geq 0 \\ -u(-x) & \text{if } x < 0 \end{cases}$$

Then clearly $u^{**}(x) \in L^2([-1,1])$ and its fourier series expansion is given by

$$u^{**}(x) \sim \sum_{i=-\infty}^{\infty} \widehat{u^{**}}(k) e^{ik\pi x}$$

where $\widehat{u}(0) = \int_{-1}^1 u^{**}(x) dx = 0$ and

$$\begin{aligned} \widehat{u^{**}}(k) &= \frac{1}{2} \int_{-1}^1 u^{**}(x) e^{-ik\pi x} dx = \frac{1}{2} \int_{-1}^1 u^{**}(x) \sin(-k\pi x) = \\ &= - \int_0^1 u(x) \sin(k\pi x) dx = - \frac{(\text{sign } k)}{\sqrt{2}} \widehat{u}_{\sin}(|k|) \end{aligned}$$

where $\text{sign } k = -1$ if $k < 0$ and $\text{sign}(k) = 1$ if $k \geq 0$. Now, by Parseval Theorem we have

$$\begin{aligned} \|u^{**}\|_{L_p([-1,1])}^2 &= 2 \left(\sum_{i=-\infty}^{\infty} \widehat{u^{**}}(n)^2 \right) = 2 \left(\widehat{u}_{\sin}(0)^2 + 2 \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{2}} \widehat{u}_{\sin}(k) \right)^2 \right) = \\ &= 2 \sum_{k=0}^{\infty} \widehat{u}_{\sin}(k)^2 \Rightarrow \sum_{k=0}^{\infty} \widehat{u}_{\sin}(k)^2 = \|u\|_{L^2(I)}^2 \end{aligned}$$

and therefore $\widehat{u}_{\sin} \in l^2(\mathbb{N})$ with $\|\widehat{u}_{\sin}\|_{l^2(\mathbb{N})} = \|u\|_{L^2(I)}$ □

3.1.3 The Sobolev Space $H^q(I)$.

In this section we let $I = (0, 1)$, $I' = [0, 1]$ and we shall define the Sobolev space $H^q(I)$.

Definition 15. If $u \in L^1(I)$ and $v \in L^1(I)$, we say that v is the m^{th} -weak derivative of u if

$$\int_0^1 u(x) \psi^{(m)}(x) dx = (-1)^m \int_0^1 v(x) \psi(x) dx$$

for every $\psi(x) \in C_C^\infty(I')$.

Following (EVANS, 1998) (Evans, pg. 243), the m^{th} -weak derivative of u in I is unique as an element of $L^2(I)$ and we shall denote it by $u^{(m)}$ for all $m > 0$.

Definition 16. For $q \geq 1$ we let $H^q(I)$ be the set of functions $u \in L^2(I)$ having a m^{th} -weak derivative for every $1 \leq m \leq q$, with $u^{(m)} \in L^2(I)$, where two elements u and v of $H^q(I)$ are identified as the same element if they are equivalent in $L^2(I)$. Moreover we set $H^0(I) = L^2(I)$.

Proposition 20. For $q \geq 0$, $H^q(I)$ is a Hilbert space under the inner product

$$\langle u, v \rangle_{H^q(I)} = \sum_{k=0}^q \langle u^{(k)}, v^{(k)} \rangle_{L^2(I)}$$

For a proof of this proposition check (EVANS, 1998) (Evans, Theo. 2, pg. 249).

Notice the norm $\|\cdot\|_{H^q(I)}$ induced by $\langle \cdot, \cdot \rangle_{H^q(I)}$ is given by

$$\|u\|_{H^q(I)} = \sqrt{\sum_{k=0}^q \|u^{(k)}\|_{L^2(I)}^2}.$$

The following relates the set of absolutely continuous functions with the set $H^q(I)$. For a proof, check (BREZIS, 2011) (Brezis, Theo. 8.2, pg. 204).

Proposition 21. If $q \geq 1$ and $u \in H^q(I)$, there exist $v \in H^q(I)$, such that $u(x) = v(x)$ a.e. in I , $v \in C^{(q-1)}(I')$ and such that $v^{(q-1)}$ is absolutely continuous in I' , and moreover, the derivative of $v^{(q-1)}$ as an absolutely continuous function coincides in $L^2(I)$ with $v^{(q)}$.

Due to this proposition, if $q \geq 1$, it follows that $u \in H^q(I)$ can be identified uniquely as an element of $C^{(q-1)}(I')$, and thus if $u \in H^q(I)$ we shall always suppose $u \in C^{(q-1)}(I')$ and in special the values $u^{(k)}(0)$ and $u^{(k)}(1)$ are well defined for all $1 \leq k \leq q-1$.

Definition 17. We define $H_0^1(I)$ to be set of functions u in $H^1(I)$ such that $u(0) = u(1) = 0$.

Proposition 22. If $u \in H^1(I)$ then:

- i) $\widehat{u}'_{\sin}(k) = -k\pi\widehat{u}_{\cos}(k)$ for all $k \in \mathbb{N}$.
- ii) $\widehat{u}'_{\cos}(k) = k\pi\widehat{u}_{\sin}(k)$ for all $k \in \mathbb{N}$ if $u \in H_0^1(I)$.

Proof. Item i):

If $u \in H^1(I)$ then, due to Proposition 21, u is absolutely continuous and its derivative as an absolutely continuous function coincides in $L^2(I)$ with its weak derivative u' . Therefore, due to the integration by parts rule for absolutely continuous functions we have:

$$\int_0^1 u'(x) \sin(k\pi x) dx = -k\pi \int_0^1 u(x) \cos(k\pi x) dx$$

And thus $\widehat{u}'_{\sin}(k) = -k\pi\widehat{u}_{\cos}(k)$ for all $k \in \mathbb{N}$.

Item ii):

If $u \in H_0^1(I)$ then $u(0) = u(1) = 0$, and thus, once again due Proposition 21 and the integration by parts rule for absolutely continuous functions we have that

$$\int_0^1 u'(x) \sqrt{2} \cos(k\pi x) dx = k\pi \int_0^1 u(x) \sqrt{2} \sin(k\pi x) dx$$

for all $k \in \mathbb{N}$, which proves the item. $\square$

3.2 Approximation Theorems for Sobolev Space

3.2.1 Truncation estimates for sine and cosine series

Definition 18. Given $m \geq 1$, we define the truncation operators $\rho_{\cos,m} : L^2(I) \rightarrow C^\infty(I')$ and $\rho_{\sin,m} : L^2(I) \rightarrow C^\infty(I')$ by

$$\begin{aligned} \rho_{\cos,m}(u) &= \widehat{u}_{\cos}(0) + \sum_{i=1}^{m-1} \widehat{u}_{\cos}(k) \sqrt{2} \cos(k\pi x) \text{ and} \\ \rho_{\sin,m}(u) &= \widehat{u}_{\sin}(0) + \sum_{i=1}^{m-1} \widehat{u}_{\sin}(k) \sqrt{2} \sin(k\pi x), \end{aligned}$$

respectively

Proposition 23. Given $u \in H^1(I)$ we have that:

- i) $\|u\|_{C(I')} \leq \frac{2\sqrt{3}}{3} \|u\|_{H^1(I)}$;
- ii) $\|\rho_{\cos,m}(u)\|_{C(I')} \leq \frac{2\sqrt{3}}{3} \|u\|_{H^1(I)}$;
- iii) $\|u\|_{C(I')} \leq \frac{\sqrt{3}}{3} \|u\|_{H^1(I)}$ in case $u \in H_0^1(I)$;
- iv) $\|\rho_{\sin,m}(u)\|_{C(I')} \leq \frac{\sqrt{3}}{3} \|u\|_{H^1(I)}$ in case $u \in H_0^1(I)$.

Proof. Item i):

Since $u' \in L^2(I)$, due to Proposition 19 we have that $\widehat{u'}_{\sin} \in \ell^2(\mathbb{N})$ with $\|\widehat{u'}_{\sin}\|_{\ell^2(\mathbb{N})} = \|u'\|_{L^2(I)}$. Moreover, from Proposition 22 we have that $\widehat{u}_{\cos}(k) = -\frac{1}{k\pi} \widehat{u'}_{\sin}(k)$ for all $k > 0$. Now, let $v_0(x) = \widehat{u}_{\cos}(k)$ and $v_k(x) = \sqrt{2} \widehat{u}_{\cos}(k) \cos(k\pi x)$ for all $k > 0$. Since $|v_k(x)| = |\widehat{u}_{\cos}(k) \sqrt{2} \cos(k\pi x)| \leq \frac{\sqrt{2}}{k\pi} |\widehat{u}_{\sin}(k)|$ for all $k > 0$ by the Cauchy Inequality and the classical

identity $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ we have that

$$\begin{aligned}
 \sum_{k=0}^m |v_k(x)| &= |\widehat{u}(0)| + \sum_{k=1}^m \frac{\sqrt{2}}{\pi k} |\widehat{u}'_{\sin}(k)| \\
 &\leq \sqrt{1 + \frac{2}{\pi^2} \sum_{k=1}^m \frac{1}{k^2}} \sqrt{|\widehat{u}_{\cos}(0)|^2 + \sum_{k=1}^m \widehat{u}'_{\sin}(k)^2} \\
 &\leq \sqrt{1 + \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2}} \sqrt{|\widehat{u}_{\cos}(0)|^2 + \|\widehat{u}'_{\sin}\|_{l^2(\mathbb{N})}^2} \\
 &= \frac{2\sqrt{3}}{3} \sqrt{\left| \int_0^1 u(x) dx \right|^2 + \|u'\|_{L^2(I)}^2} \\
 &\leq \frac{2\sqrt{3}}{3} \sqrt{\|u\|_{L^2(I)}^2 + \|u'\|_{L^2(I)}^2} = \frac{2\sqrt{3}}{3} \|u\|_{H^1(I)} < \infty
 \end{aligned}$$

Thus, the Weierstrass M -test combined with the Uniform Limit Theorem imply that $u_m = \sum_{k=0}^m v_k$ converges uniformly to a function $u^* \in C(I')$ with

$$\|u^*\|_{C(I')} \leq \frac{2\sqrt{3}}{3} \|u\|_{H^1(I)}$$

In particular it follows that $u_m = \sum_{k=0}^m v_k$ converges to u^* in the $L^2(I)$ norm. On the other hand, since by the Parseval Theorem (Proposition 19) u_m converges to u in the $L^2(I)$ norm, and since $L^2(I)$ is Hausdorff, it follows that u and u^* correspond to the same function in $L^2(I)$, that is, $u(x) = u^*(x)$ a.e. in I , and since u and u^* are continuous in I' , it follows that $u = u^*$ in $C(I')$ and the item is proved.

Item ii):

Following similar steps and computations as done in item i) we have that

$$\begin{aligned}
 |\rho_{m,\cos}(u)(x)| &\leq |\widehat{u}(0)| + \sum_{k=1}^m \sqrt{2} |\widehat{u}_{\cos}(k)| = |\widehat{u}(0)| + \sum_{k=1}^m \frac{\sqrt{2}}{\pi k} |\widehat{u}'_{\sin}(k)| \\
 &\leq \frac{2\sqrt{3}}{3} \sqrt{\|u\|_{L^2(I)}^2 + \|u'\|_{L^2(I)}^2} = \frac{2\sqrt{3}}{3} \|u\|_{H^1(I)}
 \end{aligned}$$

for all $x \in I'$, which proves item ii).

Item iii):

Since $u' \in L^2(I)$, due to Proposition 19 we have that $\widehat{u}'_{\cos} \in l^2(\mathbb{N})$ with $\|\widehat{u}'_{\cos}\|_{l^2(\mathbb{N})} = \|u'\|_{L^2(I)}$. On the other hand, from Proposition 22 we have that $\widehat{u}_{\cos}(k) = \frac{1}{k\pi} \widehat{u}'_{\sin}(k)$ for all $k > 0$ and thus, letting $v_k(x) = \sqrt{2} \widehat{u}_{\sin}(k) \sin(k\pi x)$ for all $k > 0$ we have that

$$\begin{aligned}
\sum_{k=1}^m |v_k(x)| &\leq \sum_{k=1}^m \sqrt{2} |\widehat{u}_{\sin}(k)| \leq \sum_{k=1}^m \frac{\sqrt{2}}{\pi k} |\widehat{u}'_{\sin}(k)| \\
&\leq \sqrt{\frac{2}{\pi^2} \sum_{k=1}^m \frac{1}{k^2}} \sqrt{\sum_{k=1}^m \widehat{u}'_{\sin}(k)^2} \leq \sqrt{\frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2}} \sqrt{\|\widehat{u}'_{\sin}\|_{l^2(\mathbb{N})}^2} \\
&= \frac{\sqrt{3}}{3} \sqrt{\|u'\|_{L^2(I)}^2} \leq \frac{\sqrt{3}}{3} \|u\|_{H^1(I)} < \infty
\end{aligned}$$

And analogously to item *i*) we can conclude that v_k converges uniformly to u , proving the result.

Item *iv*):

Following similar steps and computations as done in item *iii*) we have that

$$\begin{aligned}
|\rho_{m,\sin}(u)(x)| &\leq \sum_{k=1}^m \sqrt{2} |\widehat{u}_{\sin}(k)| = \sum_{k=1}^m \frac{\sqrt{2}}{\pi k} |\widehat{u}'_{\cos}(k)| \\
&\leq \frac{\sqrt{3}}{3} \sqrt{\|u'\|_{L^2(I)}^2} = \frac{\sqrt{3}}{3} \|u\|_{H^1(I)}
\end{aligned}$$

for all $x \in I'$, which proves item *iv*). □

3.2.2 The Space $H_N^q(I)$.

Again we let $I = (0, 1)$ and $I' = [0, 1]$. The set H_N^q defined bellow will be useful for the analysis of the boundary value problem of Neumann type.

Definition 19. Given $q \geq 2$ we define by $H_N^q(I)$ the set of functions $u \in H^q(I)$ such that $u'(0) = u'(1) = 0$.

Here $H_N^q(I)$ stands for q -weakly differentiable functions of Neumann type.

Proposition 24. $H_N^q(I)$ is Banach under the norm $\|\cdot\|_{H^q(I)}$ for all $q > 1$.

Proof. We just need to prove that $H_N^q(I)$ is closed in $H^q(I)$ for $q > 1$.

Thus, let $(u_m)_{m \in \mathbb{N}}$ be a sequence in $H_N^q(I)$, satisfying $\lim_{m \rightarrow \infty} u_m = u \in H^q(I)$ on the $H^q(I)$ norm. Then, due to Proposition 23, it follows that

$$\lim_{m \rightarrow \infty} \|u'_m - u'\|_{C(I')} \leq \lim_{m \rightarrow \infty} \frac{2\sqrt{3}}{3} \|u'_m - u'\|_{H^1(I)} = 0$$

and therefore $u'(0) = \lim_{m \rightarrow \infty} u'_m(0) = 0$ and $u'(1) = \lim_{m \rightarrow \infty} u'_m(1) = 0$. Thus by definition we have that $u \in H_N^q(I)$ and therefore $H_N^q(I)$ is closed in $H^q(I)$. □

For the following, let $\omega(k) = \sqrt{1 + k^2 + k^4}$ for all $k \in \mathbb{N}$.

Proposition 25. $H_N^2(I)$ has the set

$$B_N = \{1, \omega(1)^{-1}\sqrt{2}\cos(\pi x), \omega(2)^{-1}\sqrt{2}\cos(2\pi x), \dots\}$$

as a Hilbert basis and $h_{\cos} : H_N^2(I) \rightarrow l^2(\mathbb{N})$ defined by

$$(h_{\cos}(u))(k) = \omega(k)\widehat{u}_{\cos}(k) \text{ for all } k \in \mathbb{N}$$

is an isometry between $H_N^2(I)$ and $l^2(\mathbb{N})$.

Proof. It is direct to see that B_N forms an ortonormal set in $H_N^2(I)$. To see it is ortonormal, letting e_k^* denote the $(k+1)^{\text{th}}$ element of B_N for $k \in \mathbb{N}$, we have directly that $\|e_0^*\|_{H_N^2(I)} = 1$ and for $k \geq 1$ we have that

$$\begin{aligned} \|e_k\|_{H_N^2(I)}^2 &= \int_0^1 (\omega(k)^{-1}\sqrt{2}\cos(k\pi x))^2 dx + \int_0^1 (k\omega(k)^{-1}\sqrt{2}\sin(k\pi x))^2 dx \\ &\quad + \int_0^1 (k^2\omega(k)^{-1}\sqrt{2}\cos(k\pi x))^2 dx \\ &= \omega(k)^{-2}(1+k^2+k^4) = 1 \end{aligned}$$

Now, to prove that the linear span of B_N is dense in $H_N^2(I)$, given $v \in H_N^2(I)$, consider $(u_m)_{m \geq 2}$ defined by

$$u_m = \widehat{v}_{\cos, m} + \sum_{k=1}^{m-1} (\omega(k)\widehat{v}_{\cos}(k)) \omega(k)^{-1}\sqrt{2}\cos(k\pi x)$$

It is clear u_m is in the linear span of B_N . Now, let $w_n = v - u_m$ for all $m \in \mathbb{N}$. Since

$$(\widehat{w_m})_{\cos}(k) = 0$$

for all $0 \leq k \leq m-1$ and $(\widehat{w_m})_{\cos}(k) = \widehat{v}_{\cos, m}$ for $k \geq m$. Thus by Propositions 19 and 22 we have that

$$\|v - u_m\|_{H^2(I)} = \sqrt{\sum_{k=m}^{\infty} \widehat{v}_{\cos}(k)^2 + \sum_{k=m}^{\infty} \widehat{v}'_{\sin}(k)^2 + \sum_{k=m}^{\infty} \widehat{v}''_{\cos}(k)^2}.$$

Moreover, since $v \in H^2(I)$, it follows due to Proposition 19 that $\widehat{v}_{\cos} \in l^2(\mathbb{N})$, $\widehat{v}'_{\sin} \in l^2(\mathbb{N})$ and $\widehat{v}''_{\sin} \in l^2(\mathbb{N})$ and thus by the above equation we have that $\lim_{m \rightarrow \infty} \|v - u_m\|_{H^2(I)} = 0$, proving thus the density of the span of B_N over $H_N^2(I)$.

Finally, notice that $h_{\cos} : H_N^2(I) \rightarrow l^2(\mathbb{N})$, as defined above, sends the m^{th} element of B_N to the m^{th} element of the canonical basis $B_{l^2} = \{e_0, e_1, \dots\}$ of $l^2(\mathbb{N})$. Thus it follows that $h_{\cos}$ is an isometry between $H_N^2(I)$ and $l^2(\mathbb{N})$. $\square$

The operator $h_{\cos}$ and its inverse $h_{\cos}^{-1}$ will play an important role the next chapters. From the definition of $h_{\cos}$, it is direct to see that $h_{\cos}^{-1} : l^2(\mathbb{N}) \rightarrow H_N^2(I)$ is defined by

$$(h_{\cos}(a))(k) = \omega(k)^{-1}a(k)$$

for all $k \in \mathbb{N}$.

3.2.3 Three Lemmas for $H^q(I)$.

The next lemma tells us how much some functions $u \in H^q(I)$ are perturbed normwise when truncating its sine and cosine series.

Lemma 1. Given $u \in H^1(I)$ and $m \geq 1$ it follows that

- i) $\|u - \rho_{\cos,m}(u)\|_{L^2(I)} \leq \frac{1}{\pi m} \|u'\|_{L^2(I)}$;
- ii) $\|u - \rho_{\sin,m}(u)\|_{L^2(I)} \leq \frac{1}{\pi m} \|u'\|_{L^2(I)}$ if $u \in H_0^1(I)$.
- iii) $\|u - \rho_{\cos,m}(u)\|_{H^2(I)} \leq \frac{1}{\pi m} \|u'\|_{H^2(I)}$ if $u \in H_N^3(I)$

Proof. Item i)

Since $u' \in L^2(I)$, due to Proposition 19 we have that $\widehat{u'}_{\sin} \in l^2(\mathbb{N})$ with $\|\widehat{u'}_{\sin}\|_{l^2(\mathbb{N})} = \|u'\|_{L^2(I)}$ and from Proposition 22 we have that $\widehat{u}_{\cos}(k) = -\frac{1}{k\pi} \widehat{u'}_{\sin}(k)$ for all $k \in \mathbb{N}$. Thus, from Proposition 19 it follows for $m \geq 1$ that

$$\begin{aligned} \|u - \rho_{\cos,m}(u)\|_{L^2(I)} &= \sqrt{\sum_{k=m}^{\infty} \widehat{u}_{\cos}(k)^2} = \sqrt{\sum_{k=m}^{\infty} \left(\frac{1}{\pi k} \widehat{u'}_{\sin}(k)\right)^2} \\ &\leq \frac{1}{\pi m} \|\widehat{u'}_{\sin}\|_{l^2(\mathbb{N})} = \frac{1}{\pi m} \|u'\|_{L^2(I)} \end{aligned}$$

which proves item i).

Item ii)

Since $u' \in L^2(I)$, due to Proposition 19 we have that $\widehat{u'}_{\cos} \in l^2(\mathbb{N})$ with $\|\widehat{u'}_{\cos}\|_{l^2(\mathbb{N})} = \|u'\|_{L^2(I)}$ and from Proposition 22 and $u(0) = u(1) = 0$ we have that $\widehat{u}_{\sin}(k) = \frac{1}{k\pi} \widehat{u'}_{\cos}(k)$ for all $k \in \mathbb{N}$. Thus, from Proposition 19 it follows for $m \geq 1$ that

$$\begin{aligned} \|u - \rho_{\sin,m}(u)\|_{L^2(I)} &= \sqrt{\sum_{k=m}^{\infty} \widehat{u}_{\sin}(k)^2} = \sqrt{\sum_{k=m}^{\infty} \left(\frac{1}{\pi k} \widehat{u'}_{\cos}(k)\right)^2} \\ &\leq \frac{1}{\pi m} \|\widehat{u'}_{\cos}\|_{l^2(\mathbb{N})} = \frac{1}{\pi m} \|u'\|_{L^2(I)} \end{aligned}$$

which proves item ii).

Item iii)

Since $u' \in L^2(I)$, $u'' \in L^2(I)$ and $u^{(3)} \in L^2(I)$ by hypothesis due to Proposition 19 we have that $\widehat{u'}_{\sin} \in l^2(\mathbb{N})$, $\widehat{u''}_{\cos} \in l^2(\mathbb{N})$ and $\widehat{u^{(3)}}_{\cos} \in l^2(\mathbb{N})$ and since $u' \in H_0^1(I)$, from Proposition

22 we have that

$$\begin{aligned}\widehat{u}_{\cos}(k) &= -\frac{1}{k\pi}\widehat{u}'_{\sin}(k), \\ \widehat{u}'_{\sin}(k) &= \frac{1}{k\pi}\widehat{u}''_{\cos}(k), \text{ and} \\ \widehat{u}''_{\cos}(k) &= -\frac{1}{k\pi}\widehat{u^{(3)}_{\sin}}(k)\end{aligned}$$

for all $k \in \mathbb{N}$. Thus, from Proposition 19 it follows for $m \geq 1$ that

$$\begin{aligned}\|u - \rho_{\cos,m}(u)\|_{H^2(I)} &= \sqrt{\sum_{k=m}^{\infty} \widehat{u}_{\cos}(k)^2 + \sum_{k=m}^{\infty} \widehat{u}'_{\sin}(k)^2 + \sum_{k=m}^{\infty} \widehat{u}''_{\cos}(k)^2} \\ &= \sqrt{\sum_{k=m}^{\infty} \left(\frac{1}{\pi k} \widehat{u}'_{\sin}(k)\right)^2 + \sum_{k=m}^{\infty} \left(\frac{1}{\pi k} \widehat{u}''_{\cos}(k)\right)^2 + \sum_{k=m}^{\infty} \left(\frac{1}{\pi k} \widehat{u^{(3)}_{\sin}}(k)\right)^2} \\ &\leq \frac{1}{\pi m} \sqrt{\|\widehat{u}'_{\sin}\|_{l^2(\mathbb{N})}^2 + \|\widehat{u}''_{\cos}\|_{l^2(\mathbb{N})^2}^2 + \|\widehat{u^{(3)}_{\sin}}\|_{l^2(\mathbb{N})^2}^2} = \frac{1}{\pi m} \|u'\|_{H^2(I)}\end{aligned}$$

which proves item iii).

□

Lemma 2. Suppose $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $g : \mathbb{R}^3 \rightarrow \mathbb{R}$ are differentiable with

$$|f(x, u, v)| \leq M_1 \text{ and } \|Df(x, u, v)\|_{\mathcal{L}(\mathbb{R}^3, \mathbb{R})} \leq M_2$$

for all $(x, u, v) \in I' \times \bar{B}\left(0, \frac{2\sqrt{6}}{3}r\right)_{\mathbb{R}^2}$ and suppose that $w \in \bar{B}(0, r)_{H^2(I)}$. Then, given $M = M_2 \sqrt{1 + \|w\|_{H^2(I)}^2}$ and $K = \sqrt{M_1^2 + M_2^2} \left(1 + \|w\|_{H^2(I)}^2\right)$ we have that:

- i) $f \in H^2(I)$ with $\|(f(x, w, w'))'\|_{L^2(I)} \leq M$
- ii) $\|f(x, w, w') - \rho_{m,\cos}(f(x, w, w'))\|_{L^2(I)} \leq \frac{M}{\pi m}$;
- iii) $\|f(x, w, w')v - \rho_{\cos,m}(f(x, w, w'))\rho_{\cos,m}(v)\|_{L^2(I)} \leq \frac{2\sqrt{3}K\|v\|_{H^2(I)}}{\pi m}$ for all $v \in H^1(I)$;
- iv) $\|f(x, w, w')v - \rho_{\cos,m}(f(x, w, w'))\rho_{\sin,m}(v)\|_{L^2(I)} \leq \frac{\sqrt{3}K\|v\|_{H^2(I)}}{\pi m}$ for all $v \in H_0^1(I)$;

Proof. Item i):

First notice that, since $w \in \bar{B}(0, r)_{H^2(I)}$, and since due to Proposition 23 we have that $|w(x)| \leq \|w\|_{H^2(I)}$ and $\|w'(x)\| \leq \|w\|_{H^2(I)}$ for all $x \in I'$, it follows that

$$\begin{aligned}\|(w(x), w'(x))\|_{\mathbb{R}^2} &\leq \sqrt{|w(x)|^2 + |w'(x)|^2} \\ &\leq \frac{2\sqrt{3}}{3} \sqrt{2\|w\|_{H^2(I)}^2} = \frac{2\sqrt{6}}{3} \|w\|_{H^2(I)} \leq \frac{2\sqrt{6}}{3} r\end{aligned}$$

Therefore $(x, w(x), w'(x)) \in I' \times \bar{B}\left(0, \frac{2\sqrt{6}}{3}r\right)_{\mathbb{R}^2}$ for all $x \in I'$. Thus, by hypothesis, it follows that

$$|f(x, w(x), w'(x))| \leq M_1$$

for all $x \in I'$ and thus

$$\|f(x, w, w')\|_{L^2(I)} \leq M_2. \quad (3.4)$$

On the other hand, letting $A = I' \times \bar{B}\left(0, \frac{2\sqrt{6}}{3}r\right)_{\mathbb{R}^2}$, since we proved that $(x, w(x), w'(x)) \in A$ for all $x \in I'$ and since A is convex, it follows due to Proposition 17 that $y = f(x, w, w')$ is absolutely continuous in I' with

$$y'(x) = (Df(x, w(x), w'(x)))(1, w'(x), w''(x)) \text{ a.e. in } I'$$

and thus

$$|y'(x)| \leq |(Df(x, w(x), w'(x)))(1, w'(x), w''(x))| \leq M_2 \sqrt{1 + |w'(x)|^2 + |w''(x)|^2}$$

a.e. in I' . Therefore it follows that

$$\|y'\|_{L^2(I)}^2 \leq M_2^2 \int_0^1 (1 + w'(x)^2 + w''(x)^2) dx = M_2^2 \left(1 + \|w\|_{H^1(I)}^2\right) = M, \quad (3.5)$$

and thus

$$\|(f(x, w, w'))'\|_{L^2(I)} = \|y'\|_{L^2(I)} \leq M,$$

and in particular $f \in H^2(I)$, proving item *i*).

Item *ii*):

Direct from item *i*) and from item *i*) of Lemma 1.

Since $|f(x, u, v)| \leq M_1$ for all $(x, u, v) \in I' \times \bar{B}\left(0, \frac{2\sqrt{6}}{3}r\right)_{\mathbb{R}^2}$, just as in the proof of item *i*).

Item *iii*):

Let $y = f(x, w, w')$, $y_0 = yv$ and $y_1 = y\rho_{\cos, m}(v)$.

In item *i*) it was proved that $y = f(x, w, w') \in H^1(I)$ with $\|y\|_{H^1(I)} \leq M$ and therefore item *i*) of Proposition 23 implies that

$$\|y\|_{C(I')} \leq \frac{2\sqrt{3}}{3}M. \quad (3.6)$$

In special it follows that $y \in C(I')$ and since $v \in H^1(I)$ it follows that $y_0 = yv \in L^2(I)$ and $y_1 = y\rho_{\cos, m}(v) \in L^2(I)$. Moreover, since $v \in H^1(I)$, we have due to item *i*) of Lemma 1 that

$$\|v - \rho_{\cos, m}(v)\|_{L^2(I)} \leq \frac{\|v'\|_{L^2(I)}}{\pi m} \leq \frac{\|v\|_{H^2(I)}}{\pi m},$$

and thus, combining the above inequalities, we have that

$$\begin{aligned} \|y_1 - y_0\|_{L^2(I)} &= \|y(v - \rho_{\cos,m}(v))\|_{L^2(I)} \leq \|y\|_{C(I')} \|v - \rho_{\cos,m}(v)\|_{L^2(I)} \\ &\leq \frac{2\sqrt{3}M \|v\|_{H^2(I)}}{3\pi m} \end{aligned} \quad (3.7)$$

We now estimate $\|\rho_{\cos,m}(y_1) - y_1\|_{L^2(I)}$.

Since $v \in H_N^2(I)$ it follows that $v' \in H_0^1(I)$ and due to Proposition 18 we know that $y_1 = y\rho_{\sin,m}(v)$ is absolutely continuous with

$$y_1' = y' \rho_{\cos,m}(v) + y \rho_{\sin,m}(v').$$

Now, since we proved in item *i*) that $y' \in L^2(I)$ and since $\rho_{\cos,m}(v) \in C(I')$, it follows that $y' \rho_{\cos,m}(v) \in L^2(I)$ and item *ii*) of Proposition 23 implies that

$$\|y' \rho_{\cos,m}(v)\|_{L^2(I)} \leq \|y'\|_{L^2(I)} \|\rho_{\cos,m}(v)\|_{C(I')} \leq \frac{2\sqrt{3}}{3} \|y'\|_{L^2(I)} \|v\|_{H^1(I)}.$$

On the other hand, since $y \in C(I')$ and $\rho_{\sin,m}(v') \in L^2(I)$, it follows that $y \rho_{\sin,m}(v') \in L^2(I)$, and inequality (3.6) implies that

$$\|y \rho_{\sin,m}(v')\|_{L^2(I)} \leq \|y\|_{C(I')} \|\rho_{\sin,m}(v')\|_{L^2(I)} \leq \frac{2\sqrt{3}}{3} \|y\|_{H^1(I)} \|v'\|_{L^2(I)}.$$

Thus it follows that $y_1' \in L^2(I)$ with

$$\begin{aligned} \|y_1'\|_{L^2(I)} &\leq \frac{2\sqrt{3}}{3} \|y'\|_{L^2(I)} \|v\|_{H^1(I)} + \frac{2\sqrt{3}}{3} \|y\|_{H^1(I)} \|v'\|_{L^2(I)} \\ &\leq \frac{4\sqrt{3}}{3} \|y\|_{H^1(I)} \|v\|_{H^1(I)} \end{aligned}$$

Hence, by item *i*) of Lemma 1 and by (3.5) it follows that

$$\|\rho_{\cos,m}(y_1) - y_1\| \leq \frac{4\sqrt{3}M \|v\|_{H^1(I)}}{3\pi m} \quad (3.8)$$

Finally, combining (3.7) and (3.8) it follows that

$$\|\rho_{\cos,m}(y_1) - y_0\|_{L^2(I)} \leq \|\rho_{\cos,m}(y_1) - y_1\| + \|y_1 - y_0\| \leq \frac{2\sqrt{3}M \|v\|_{H^1(I)}}{\pi m},$$

which proves item *iii*).

Item *iv*):

Let $y = f(x, w, w')$, $y_0 = v$, and $y_2 = y\rho_{\sin,m}(v)$. Following the same exact steps that led to (3.7) in item *ii*) we can see that y_0 and y_2 are square integrable with

$$\|y_2 - y_0\|_{L^2(I)} \leq \|y\|_{C(I')} \|v - \rho_{\sin,m}(v)\|_{L^2(I)} \leq \frac{2\sqrt{3}M}{3\pi m} \|v\|_{H^1(I)} \quad (3.9)$$

Now, due to Proposition 18 we have that y_2 is absolutely continuous with $y_2' = y' \rho_{\sin, m}(v) + y \rho_{\cos, m}(v')$. Thus, using items *ii* and *iv*) of Proposition 23 and proceeding analogously as in the proof of item *ii*), since $y \in H_0^1(I)$, it follows that

$$\begin{aligned} \|y' \rho_{\sin, m}(v)\|_{L^2(I)} &\leq \|y'\|_{L^2(I)} \|\rho_{\sin, m}(v)\|_{C(I')} \leq \frac{\sqrt{3}}{3} \|y'\|_{L^2(I)} \|v\|_{H^1(I)} \cdot \\ \|y \rho_{\cos, m}(v')\|_{L^2(I)} &\leq \|y\|_{C(I')} \|\rho_{\cos, m}(v')\|_{L^2(I)} \leq \frac{2\sqrt{3}}{3} \|y\|_{H^1(I)} \|v'\|_{L^2(I)} \cdot \end{aligned}$$

and thus $y_2' \in L^2(I)$ with

$$\|y_2'\|_{L^2(I)} \leq \sqrt{3} \|y'\|_{L^2(I)} \|v\|_{H^1(I)}.$$

Hence, by item *i*) of Lemma 1 and by (3.5) it follows that

$$\|\rho_{\cos, m}(y_2) - y_2\| \leq \frac{\sqrt{3}M}{\pi m} \|v\|_{H^1(I)} \quad (3.10)$$

Finally, from (3.9), (3.10) and the Triangle inequality it follows that

$$\|y_2 - y_0\|_{L^2(I)} \leq \|y_2 - y_1\| + \|y_1 - y_0\| \leq \frac{\sqrt{3}M}{\pi m} \|v\|_{H^1(I)}$$

□

The next lemma regards the differentiability of an operator $\mathcal{F}$ related to equation (3.1). In the proof of the following lemma, $[a, b]_{\mathbb{R}}$ will denote the closed interval $[a, b] \subset \mathbb{R}$ if $a \geq b$, and $[a, b]_{\mathbb{R}}$ will denote the closed interval $[b, a]$ if $b < a$.

Lemma 3. Given $f : J \times \mathbb{R}^2 \rightarrow \mathbb{R}$, $J \subset \mathbb{R}$ open with $I' \subset J$, $r > 0$, and $\mathcal{F} : H^2(I) \rightarrow L^2(I)$ defined by

$$\mathcal{F}(u) = u'' - f(x, u, u').$$

Suppose that:

- i) $f : J \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is differentiable with $f_u : J \times \mathbb{R}^2 \rightarrow \mathbb{R}$ and $f_v : J \times \mathbb{R}^2 \rightarrow \mathbb{R}$ differentiable.
- ii) $\|Df_u(x, u, v)\|_{\mathcal{L}(\mathbb{R}^3, \mathbb{R})} \leq K_0$ and $\|Df_{u'}(x, u, v)\|_{\mathcal{L}(\mathbb{R}^3, \mathbb{R})} \leq K_1$ for all $(x, u, v) \in I' \times \bar{B}\left(0, \frac{2\sqrt{6}}{3}r\right)_{\mathbb{R}^3}$.

Then

- I) $\mathcal{F}$ is differentiable in $H^2(I)$, with derivative $D\mathcal{F}(u) \in \mathcal{L}(H^2(I), L^2(I))$ defined by

$$D\mathcal{F}(u)(z) = z'' - f_v(x, u, u')z' - f_u(x, u, u')z$$

for all $z \in H^2(I)$.

- II) $D\mathcal{F}(u)$ is Lipschitz continuous in $\bar{B}(0, r)_{H^2(I)}$ with Lipschitz constant $K = \frac{2\sqrt{6}}{3} \sqrt{K_0^2 + K_1^2}$.

Proof. Item I):

Given $u \in H^2(I)$, due to Proposition 21 we have that $u \in C(I')$ and $u' \in C(I')$, and since $f_u : J \times \mathbb{R}^2 \rightarrow \mathbb{R}$ and $f_v : J \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous, it follows that $c(x) = f_u(x, u(x), u'(x))$ and $d(x) = f_v(x, u(x), u'(x))$ are continuous as well. Therefore it follows directly that $D\mathcal{F}(u)(z)$ is square integrable with

$$\|D\mathcal{F}(u)(z)\|_{L^2(I)} \leq \left(1 + \sup_{x \in I'} |c(x)| + \sup_{x \in I'} |d(x)|\right) \|z\|_{H^2(I)}$$

for all $z \in H^2(I)$. Thus $D\mathcal{F}(u) \in \mathcal{L}(H^2(I), L^2(I))$.

Now, given $u \in H^2(I)$ and $\varepsilon > 0$, we need to prove that $\mathcal{F}$ is differentiable at u . For this end, let $r' > 0$ be such that $u \in B(0, r')$ and suppose h is small enough in norm such that $u + h \in B(0, r')$. Now notice that, since Df_u and Df_v are continuous, it follows from the Mean Value Theorem that f_u and f_v are Lipschitz continuous in the compact $I' \times \bar{B}(0, r') \times \bar{B}(0, r')$, that is, there exists $L > 0$ such that

$$\begin{aligned} |f_u(x, u_0, u_1) - f_u(x, w_0, w_1)| &\leq L_0 \|(u_0, u_1) - (w_0, w_1)\|_{\mathbb{R}^2} \\ |f_v(x, u_0, u_1) - f_v(x, w_0, w_1)| &\leq L_1 \|(u_0, u_1) - (w_0, w_1)\|_{\mathbb{R}^2} \end{aligned} \quad (3.11)$$

for all $x \in I'$ and $(u_0, u_1), (w_0, w_1) \in \bar{B}(0, r')_{\mathbb{R}^2}$. Now let $D_z f : J \times \mathbb{R}^2 \rightarrow \mathcal{L}(\mathbb{R}^2, \mathbb{R})$ be the partial derivative of f in relation to $z = (u, v)$, that is, $D_z f$ corresponds to the Frechét derivative of $f \circ \pi_{u,v}$ where $\pi_{u,v} : J \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the projection of (t, u, v) to (u, v) . By, the chain rule it follows that $D_z f$ indeed exists and

$$D_z f(x, u_0, w_0)(a, b) = f_u(u_0, w_0)a + f_v(u_0, w_0)b$$

for all $x \in I'$ and $(a, b) \in \mathbb{R}^2$. Thus, denoting $L = \sqrt{L_0^2 + L_1^2}$, it follows from (3.11) and the Cauchy-Schwarz that

$$\begin{aligned} &|(Df_z(x, u_0, u_1) - D_z f(x, w_0, w_1))(a, b)| \\ &\leq \sqrt{L_0^2 + L_1^2} \left(\sqrt{a^2 + b^2}\right) \|(u_0, u_1) - (w_0, w_1)\|_{\mathbb{R}^2} \\ &= L \|(a, b)\|_{\mathbb{R}^2} \|(u_0, u_1) - (w_0, w_1)\|_{\mathbb{R}^2} \end{aligned} \quad (3.12)$$

for all $(u_0, u_1), (w_0, w_1) \in \bar{B}(0, r')_{\mathbb{R}^2}$ and therefore we have that

$$\|D_z f(u_0, u_1) - D_z f(w_0, w_1)\|_{\mathcal{L}(\mathbb{R}^2, \mathbb{R})} \leq L \|(u_0, u_1) - (w_0, w_1)\|_{\mathbb{R}^2} \quad (3.13)$$

for all $(u_0, u_1), (w_0, w_1) \in \bar{B}(0, r')_{\mathbb{R}^2}$. Therefore, the Taylor Theorem (Theorem 3) implies that

$$\begin{aligned} &\|f(x, u_0, u_1) - f(x, w_0, w_1) - D_z f(w_0, w_1)(u_0 - w_0, u_1 - w_1)\|_{\mathbb{R}^2} \\ &\leq \frac{L}{2} \|(u_0, u_1) - (w_0, w_1)\|_{\mathbb{R}^2}^2 \end{aligned} \quad (3.14)$$

for all $(u_0, u_1), (w_0, w_1) \in \bar{B}(0, r')_{\mathbb{R}^2}$. On the other hand, due to Proposition 23 we have that

$$\begin{aligned} \|(u(x), u'(x))\|_{\mathbb{R}^2} &\leq \sqrt{|u(x)|^2 + |u'(x)|^2} \\ &\leq \frac{2\sqrt{3}}{3} \sqrt{\|u\|_{H^1(I)}^2 + \|u'\|_{H^1(I)}^2} \leq \frac{2\sqrt{6}}{3} \|u\|_{H^2(I)} \leq \frac{2\sqrt{6}}{3} r' \end{aligned} \quad (3.15)$$

for all $x \in I'$, and analogously, denoting $w = u + h$, we have that $\|(w(x), w'(x))\|_{\mathbb{R}^2} \leq \frac{2\sqrt{6}}{3} r'$ for all $x \in I'$. Thus it follows that $(u(x), u'(x)), (w(x), w'(x)) \in \bar{B}(0, \frac{2\sqrt{6}}{3} r')_{\mathbb{R}^2}$ for all $x \in I'$ and denoting $y = \mathcal{F}(u + h) - \mathcal{F}(u) - D\mathcal{F}(u)(h)$ we have from (3.14) and Proposition 23 that

$$\begin{aligned} |y(x)| &= |f(x, w(x), w'(x)) - f(x, u(x), u'(x)) - D_z f(u(x), u'(x))(h(x), h'(x))| \\ &\leq \frac{L}{2} \|(h(x), h'(x))\|_{\mathbb{R}^2}^2 = \frac{L}{2} (|h(x)|^2 + |h'(x)|^2) \leq \frac{4L}{3} \|h\|_{H^2(I)}^2 \end{aligned}$$

for all $x \in I'$ and thus we conclude that $\|y\|_{L^2(I)} \leq \frac{4L}{3} \|h\|_{H^2(I)}$ and therefore

$$\lim_{h \rightarrow 0} \frac{\|\mathcal{F}(u + h) - \mathcal{F}(u) - D\mathcal{F}(u)(h)\|_{L^2(\mathbb{R}^2)}}{\|h\|_{H^2(I)}} = \lim_{h \rightarrow 0} \frac{4L}{3} \|h\|_{H^2(I)} = 0.$$

Thus $\mathcal{F}$ is Frechét differentiable with derivative $D\mathcal{F}$, by definition.

Item II):

Let $u, w \in \bar{B}(0, r)_{H^2(I)}$ and $z \in H^2(I)$ with $\|z\|_{H^2(I)} = 1$. Notice that, given $q = D\mathcal{F}(u)(z) - D\mathcal{F}(w)(z) \in L^2(I)$ we have by definition of $\mathcal{D}\mathcal{F}$ that

$$\begin{aligned} q(x) &= \left(\frac{\partial f}{\partial u}(x, u(x), u'(x)) - \frac{\partial f}{\partial u}(x, w(x), w'(x)) \right) z(x) \\ &\quad + \left(\frac{\partial f}{\partial v}(x, u(x), u'(x)) - \frac{\partial f}{\partial v}(x, w(x), w'(x)) \right) z'(x) \\ &= (D_z f(x, u(x), u'(x)) - D_z f(x, w(x), w'(x)))(z(x), z'(x)) \end{aligned}$$

for all $x \in J$. On the other hand, since $h = w - u$ is an element of $H^2(I)$, Proposition 23 implies that

$$\|(h(x), h'(x))\|_{\mathbb{R}^2} \leq \frac{2\sqrt{3}}{3} \sqrt{\|h\|_{H^2(I)}^2 + \|h'\|_{H^2(I)}^2} = \frac{2\sqrt{6}}{3} \|h\|_{H^2(I)} \quad (3.16)$$

for all $x \in I'$. Moreover, denoting $K' = \sqrt{K_0^2 + K_1^2}$, proceeding analogously as in (3.13) we have that

$$\|D_z f(x, u_0, u_1) - D_z f(x, w_0, w_1)\|_{\mathbb{R}^2} \leq K' \|(u_0, u_1) - (w_0, w_1)\|_{\mathbb{R}^2}$$

for all $x \in I'$, and $(u_0, u_1), (w_0, w_1) \in \bar{B}(0, r)_{\mathbb{R}^2}$. On the other hand, since $u, w \in \bar{B}(0, r)_{H^2(I)}$ proceeding analogously as in (3.15), we can prove that $(u(x), u'(x)), (w(x), w'(x)) \in \bar{B}(0, \frac{2\sqrt{6}}{3} r)_{\mathbb{R}^2}$ and therefore by the last inequality we have that

$$\|D_z f(x, u(x), u'(x)) - D_z f(x, w(x), w'(x))\|_{\mathbb{R}^2} \leq K' \|(h(x), h'(x))\|_{\mathbb{R}^2}$$

for all $x \in I'$. Thus by inequality (3.16) it follows that

$$|q(x)| \leq K' \|(h(x), h'(x))\|_{\mathbb{R}^2} \|(z(x), z'(x))\| \leq K \|h\|_{H^2(I)} \|(z(x), z'(x))\|$$

for all $x \in I'$ and since

$$\|(z(x), z'(x))\|^2 = \int_0^1 z(x)^2 + z'(x)^2 dx = \|z\|_{H^1(I)}^2 \leq \|z\|_{H^2(I)}^2 = 1$$

we conclude that

$$\begin{aligned} \|\mathcal{F}(u)(z) - \mathcal{F}(w)(z)\|_{L^2(I)} &\leq \|y\|_{L^2(I)} = \sqrt{\int_0^1 q(x)^2 dx} \\ &\leq K \|h\|_{H^2(I)} \sqrt{\int_0^1 z(x)^2 + z'(x)^2 dx} \leq K \|h\|_{H^2(I)}, \end{aligned}$$

and since z was arbitrarily chosen, it follows that

$$\|D\mathcal{F}(u) - D\mathcal{F}(w)\|_{\mathcal{L}(H^2(I), L^2(I))} \leq K \|h\|_{H^2(I)} = K \|u - w\|_{H^2(I)},$$

which completes the proof. □

3.2.4 Projection estimates on $H_N^2(I)$

In this final subsection we shall prove Theorem 12, which is main result of the second part of this thesis. This theorem shall reduce the computation of the fundamental values defined in the first part of this thesis for linear operators $\mathcal{F} : H_N^2(I) \rightarrow L^2(I)$ of the form

$$\mathcal{F}(u) = u'' - f(x, u, u'),$$

by using a finite dimensional projection of $\mathcal{F}$, which we shall soon define.

During this section we shall always consider $\mathbb{R}^m$ as a subset of $l^2(\mathbb{N})$, through the isometric embedding $\tau(a) = (a(0), \dots, a(m-1), 0, \dots) \in l^2(\mathbb{N})$. Moreover, for the following, once again we let $\omega(k) = \sqrt{1+k^2+k^4}$ for all $k \in \mathbb{N}$.

Definition 20. Given $m \geq 1$, we define the operators $\text{tr}_m : l^2(\mathbb{N}) \rightarrow \mathbb{R}^m$, $\text{tr}_m^* : l^2(\mathbb{N}) \rightarrow l^2(\mathbb{N})$ and $d^{-2} : l^2(\mathbb{N}) \rightarrow l^2(\mathbb{N})$ by

$$\begin{aligned} \text{tr}_m(a) &= (a(0), \dots, a(m-1)), \\ \text{tr}_m^*(a) &= a - \text{tr}_m(a), \\ (d^2(a))(k) &= \frac{k^2}{\omega(k)} a(k), \text{ and} \\ (d^{-2}(a))(k) &= \frac{\omega(k)}{\max\{1, k^2\}} a(k). \end{aligned}$$

for all $k \in \mathbb{N}$ and $a \in l^2(\mathbb{N})$.

Following this definition, it is straightforward to check the following properties

Proposition 26. The following items hold

- i) $(d^{-2} \circ d^2)(b) = (d^2 \circ d^{-2})(b) = b$ for all $b \in l^2(\mathbb{N})$ with $b(0) = 0$
- ii) $d^2 \circ \text{tr}_m = \text{tr}_m \circ d^2$ and $d^{-2} \circ \text{tr}_m = \text{tr}_m \circ d^{-2}$.
- iii) $\rho_{\cos, m} \circ \pi_{\cos} = \pi_{\cos} \circ \text{tr}_m$ and $\rho_{\cos, m} \circ \text{tr}_m = h_{\cos} \circ \text{tr}_m$.
- iv) $b = \text{tr}_m(b) + \text{tr}_m^*(b)$ and $\|b\|_{l^2(\mathbb{N})} = \sqrt{\|\text{tr}_m(b)\|_{\mathbb{R}^m}^2 + \|\text{tr}_m^*(b)\|_{l^2(\mathbb{N})}^2}$

The following defines the cosine projection and the finite dimensional cosine projection of $\mathcal{F}$.

Definition 21. Given $\mathcal{F} : H^2(I) \rightarrow L^2(I)$ we define $\mathcal{F}_{\cos} : l^2(\mathbb{N}) \rightarrow l^2(\mathbb{N})$ and $\mathcal{F}_{\cos, m} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ by

$$\mathcal{F}_{\cos} = \pi_{\cos} \circ \mathcal{F} \circ h_{\cos}^{-1} \text{ and } \mathcal{F}_{\cos, m} = \text{tr}_m \circ \mathcal{F}_{\cos} \circ \text{tr}_m,$$

respectively.

The following is the main result of the second part of this thesis, which is a synthesis of the results obtained so far. For the following we let $Z(\mathbb{R}^m)$ denote the Bannach space $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ and we let $Z(l^2)$ denote the Bannach space $\mathcal{L}(l^2(\mathbb{N}), l^2(\mathbb{N}))$.

Theorem 12. Given $f : J \times \mathbb{R}^2 \rightarrow \mathbb{R}$, $J \subset \mathbb{R}$ open, $I' \subset J$, $\mathcal{F} : H_N^2(I) \rightarrow L^2(I)$ defined by

$$\mathcal{F}(u) = u'' - f(x, u, u'),$$

suppose that $f : J \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is differentiable, suppose that $f_u : J \times \mathbb{R}^2 \rightarrow \mathbb{R}$ and $f_{u'} : J \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are differentiable and suppose that

$$|f(x, u, v)| \leq M_1, \|Df(x, u, v)\|_{\mathcal{L}(\mathbb{R}^3, \mathbb{R})} \leq M_2, \\ \|Df_u(x, u, v)\|_{\mathcal{L}(\mathbb{R}^3, \mathbb{R})} \leq L_1, \|Df_{u'}(x, u, v)\|_{\mathcal{L}(\mathbb{R}^3, \mathbb{R})} \leq L_2,$$

for all $(x, u, v) \in I' \times \bar{B}(0, r)_{H^2(I)}$. Moreover, given $r > 0$, let

$$K = \frac{2\sqrt{6}}{3} \sqrt{L_1^2 + L_2^2}, M = M_2 \sqrt{1 + r^2}, M' = M_2 \sqrt{1 + 2r^2} \text{ and } \\ L = 2\sqrt{3} \sqrt{M_2^2 + L_1^2(1 + r^2)} + \sqrt{3} \sqrt{M_2^2 + L_2^2(1 + r^2)}.$$

Then $\mathcal{F}_{\cos} : l^2(\mathbb{N}) \rightarrow l^2(\mathbb{N})$ and $\mathcal{F}_{\cos, m} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ and given $a \in \bar{B}(0, r)_{l^2(\mathbb{N})}$ we have that:

$$\text{I) } \left| \|\mathcal{F}_{\cos}(a)\|_{l^2(I)} - \|\mathcal{F}_{m, \cos}(a)\|_{\mathbb{R}^m} \right| \leq \frac{M}{\pi m}$$

- II) $\left| \|D\mathcal{F}_{\cos}(a)\|_{Z(l^2)} - \max \left\{ \|D\mathcal{F}_{m,\cos}(a)\|_{Z(\mathbb{R}^m)}, 1 \right\} \right| \leq \frac{L}{\pi m}$
- III) $\left| \lambda(D\mathcal{F}_{\cos}(a))_{Z(l^2)} - \min \left\{ \lambda(D\mathcal{F}_{m,\cos}(a))_{Z(\mathbb{R}^m)}, k^2 \omega(k)^{-1} \right\} \right| \leq \frac{L}{\pi m}$
- IV) $\|D\mathcal{F}_{\cos}(a_1) - D\mathcal{F}_{\cos}(a_2)\|_{Z(l^2)} \leq K \|a_1 - a_2\|_{l^2(\mathbb{N})}$ for all $a_1, a_2 \in \bar{B}(0, r)_{l^2(\mathbb{N})}$.
- V) $\|T_{\mathcal{F}_{\cos}}(a) - T_{\mathcal{F}_{\cos,m}}(a)\| \leq \frac{\delta M + L(r + M_1)}{\delta(\pi m \delta - L)}$ if $\delta = \lambda(D\mathcal{F}_{\cos,m}(a))_{Z(\mathbb{R}^m)} > \frac{L}{\pi m}$.
- VI) Every zero of $\mathcal{F}_{\cos}$ in $\bar{B}(0, r)_{l^2(\mathbb{N})}$ must lie in $\cup_{c \in J_{k,m}} \bar{B}\left(rc, r\frac{\sqrt{m}}{k} + \frac{M'}{\pi m}\right)_{l^2(\mathbb{N})}$.

Proof. Item I):

Letting $w = h_{\cos}^{-1}(a)$, since $h_{\cos}$ is an isometry it follows that $w \in \bar{B}(0, r)_{H^2(I)}$. Thus by the hypothesis and by Lemma 2 we have that

$$\|f(x, w, w') - \rho_{\cos,m}(f(x, w, w'))\|_{L^2(I)} \leq \frac{M_2 \sqrt{1+r^2}}{\pi m} = \frac{M}{\pi m}. \quad (3.17)$$

Now, since $a \in \mathbb{R}^m$, we have that $d^2(a) = d^2(\text{tr}_m(a))$, and since $\text{tr}_m \circ \pi_{\cos} = \rho_{m,\cos} \circ \text{tr}_m$, it follows directly that

$$\mathcal{F}_{\cos}(a) - \mathcal{F}_{\cos,m}(a) = \pi_{\cos}(f(x, w, w') - \rho_{m,\cos}(f(x, w, w'))),$$

Therefore, since $\pi_{\cos} : L^2(I) \rightarrow l^2(\mathbb{N})$ is an isometry, it follows due to (3.17) that

$$\|\mathcal{F}_{\cos}(a) - \mathcal{F}_{\cos,m}(a)\|_{l^2(\mathbb{N})} \leq \frac{M}{\pi m}, \quad (3.18)$$

and thus, due to the Reverse Triangular Inequality we have that

$$\left| \|\mathcal{F}_{\cos}(a)\|_{l^2(\mathbb{N})} - \|\mathcal{F}_{\cos,m}(a)\|_{l^2(\mathbb{N})} \right| \leq \|\mathcal{F}_{\cos}(a) - \mathcal{F}_{\cos,m}(a)\|_{l^2(\mathbb{N})} \leq \frac{M}{\pi m},$$

which completes item I).

Item II):

$$\text{Let } \beta = \|D\mathcal{F}_{\cos,m}(a)\|_{\mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)}.$$

Notice that, since $\text{tr}_m : l^2(\mathbb{N}) \rightarrow \mathbb{R}^m$ is a bounded linear operator and since $\mathcal{F}_{\cos}$ is differentiable, it follows from $\mathcal{F}_{\cos,m} = \text{tr}_m \circ \mathcal{F}_{\cos} \circ \text{tr}_m$ and the chain rule that $\mathcal{F}_{\cos,m}$ is differentiable in $a \in \mathbb{R}^m$ with

$$D\mathcal{F}_{\cos,m}(a) = \text{tr}_m \circ D\mathcal{F}_{\cos}(\text{tr}_m(a)) \circ \text{tr}_m. \quad (3.19)$$

Thus, since $\text{tr}_m \circ \pi_{\cos} = \pi_{\cos} \circ \rho_{\cos,m}$, denoting $v(b) = h_{\cos}^{-1}(\text{tr}_m(b))$ for all $b \in l^2(\mathbb{N})$ we have that

$$\begin{aligned} (D\mathcal{F}_{\cos,m}(a))(b) &= d^2(b) - \pi_{\cos}(\rho_{\cos,m}(f_{u'}(x, w, w')v(b))) \\ &\quad - \pi_{\cos}(\rho_{\cos,m}(f_u(x, w, w')v(b))) \end{aligned}$$

for all $b \in \mathbb{R}^m$.

Now, letting $G \in \mathcal{L}(l^2(\mathbb{N}), l^2(\mathbb{N}))$ be defined by

$$G(b) = d^2(\text{tr}_m^*(b)) + (D\mathcal{F}_{\cos, m}(a))(\text{tr}_m(b)), \quad (3.20)$$

we shall estimate $\|D\mathcal{F}_{\cos} - G\|_{\mathcal{L}(l^2(\mathbb{N}), l^2(\mathbb{N}))}$.

Notice that, since $\pi_{\cos}((h_{\cos}^{-1}(\text{tr}_m(b)))) = d^2(\text{tr}_m(b))$ and $d^2(\text{tr}_m^*(b)) + d^2(\text{tr}_m(b)) = d^2(b)$ for all $b \in l^2(\mathbb{N})$, it follows that

$$\begin{aligned} G(b) &= d^2(b) - \pi_{\cos}(\rho_{\cos, m}(f_{u'}(x, w, w')v(b)')) \\ &\quad - \pi_{\cos}(\rho_{\cos, m}(f_u(x, w, w')v(b))) \end{aligned} \quad (3.21)$$

for all $b \in l^2(\mathbb{N})$ and thus

$$\begin{aligned} D\mathcal{F}_{\cos}(b) - G(b) &= \pi_{\cos}(f_{u'}(x, w, w')v(b)' - \rho_{\cos, m}(f_{u'}(x, w, w')v(b)')) \\ &\quad + \pi_{\cos}(f_u(x, w, w')v(b) - \rho_{\cos, m}(f_u(x, w, w')v(b))). \end{aligned}$$

Therefore, letting $K_1^* = 2\sqrt{3}\sqrt{L_2^2 + M_1^2\|w\|_{H^2(I)}}$ and $K_2^* = \sqrt{3}\sqrt{L_2^2 + M_2^2\|w\|_{H^2(I)}}$, since $\pi_{\cos}$ is an isometry, since by definition $v(b)' \in H_0^1(I)$, due to ites *iii*) and *iv*) of Lemma 2 we have that

$$\begin{aligned} \|D\mathcal{F}_{\cos}(b) - G(b)\|_{l^2(\mathbb{N})} &\leq \|f_{u'}(x, w, w')v(b)' - \rho_{\cos, m}(f_{u'}(x, w, w')v(b)')\|_{L^2(I)} \\ &\quad + \|f_u(x, w, w')v(b) - \rho_{\cos, m}(f_u(x, w, w')v(b))\|_{L^2(I)} \\ &\leq \frac{K_2^* \|v(b)\|_{H^2(I)}}{\pi m} + \frac{K_1^* \|v(b)\|_{H^2(I)}}{\pi m} = \frac{K \|v(b)\|_{H^2(I)}}{\pi m} \end{aligned}$$

for all $b \in l^2(\mathbb{N})$. Moreover, since $h_{\cos}^{-1} : H^2(I) \rightarrow l^2(\mathbb{N})$ is an isometry we have that $\|v(b)\|_{H^2(I)} = \|\text{tr}_m(b)\|_{l^2(\mathbb{N})} \leq \|b\|_{l^2(\mathbb{N})}$ and therefore

$$\|D\mathcal{F}_{\cos} - G\|_{Z(l^2)} \leq \frac{L}{\pi m}. \quad (3.22)$$

which, due to the reverse triangle inequality, implies that

$$\left| \|D\mathcal{F}_{\cos}\|_{Z(l^2)} - \|G\|_{Z(l^2)} \right| \leq \|D\mathcal{F}_{\cos} - G\|_{Z(l^2)} \leq \frac{L}{\pi m}$$

Now, to conclude the proof we just need to prove that $\|G\|_{Z(l^2)} = \max\{\beta, 1\}$.

For that end, notice that, since $\text{tr}_m^*(d^2(b)) = 0$ for all $b \in \mathbb{R}^m$, it follows due to (3.20) that $D\mathcal{F}_{\cos, m}$ is simply the restriction of G to $\mathbb{R}^m$ and thus, in special, it follows that $\|G\|_{Z(l^2)} \geq \|D\mathcal{F}_{\cos, m}\|_{Z(\mathbb{R}^m)} = \beta$. On the other hand, letting e_k denote the k^{th} element of the orthonormal basis of $l^2(\mathbb{N})$, given $k \geq m$ we have that

$$\|G(e_k)\|_{l^2(\mathbb{N})} = \|d^2(e_k)\|_{l^2(\mathbb{N})} = \frac{k^2}{\sqrt{1+k^2+k^4}}$$

and thus

$$\|G\|_{Z(l^2)} \geq \frac{k^2}{\sqrt{1+k^2+k^4}} \text{ for all } k \geq m,$$

but since $\lim_{k \rightarrow \infty} \frac{k^2}{\sqrt{1+k^2+k^4}} = 1$ it follows that $\|G\|_{Z(l^2)} \geq 1$. Combining the inequalities obtained we conclude that $\|G\|_{Z(l^2)} \geq \max\{\beta, 1\}$.

On the other hand, given $b \in l^2(\mathbb{N})$ with $\|b\|_{l^2(\mathbb{N})} = 1$, it follows due to (3.20) that

$$\begin{aligned} \text{tr}_m(G(b)) &= (D\mathcal{F}_{\cos,m}(a))(\text{tr}_m(b)) \text{ and} \\ \text{tr}_m^*(G(b)) &= \text{tr}_m^*(d^2(b)) \end{aligned}$$

and thus we have that

$$\begin{aligned} \|G(b)\| &= \sqrt{\|\text{tr}_m(G(b))\|_{\mathbb{R}^m}^2 + \|\text{tr}_m^*(G(b))\|_{l^2}^2} \\ &= \sqrt{\|(D\mathcal{F}_{\cos,m}(a))(\text{tr}_m(b))\|_{\mathbb{R}^m}^2 + \|d^2(\text{tr}_m^*(b))\|_{l^2(\mathbb{N})}^2} \\ &\leq \sqrt{\left(\beta \|\text{tr}_m(b)\|_{\mathbb{R}^m}^2\right) + \|\text{tr}_m^*(b)\|_{l^2(\mathbb{N})}^2} \\ &\leq \max\{\beta, 1\} \sqrt{\|\text{tr}_m(b)\|_{\mathbb{R}^m}^2 + \|\text{tr}_m^*(b)\|_{l^2(\mathbb{N})}^2} = \max\{\beta, 1\}. \end{aligned}$$

Therefore it follows that $\|G\|_{Z(l^2)} \leq \max\{\beta, 1\}$.

Combining the two obtained inequalities we have that $\|G\|_{Z(l^2)} = \max\{\beta, 1\}$, which concludes item II).

Item III)

To begin the proof of item III), letting G be as in (3.20), we shall prove that if $D\mathcal{F}_{\cos,m}(a) : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is invertible then G is invertible as well.

Thus suppose $D\mathcal{F}_{\cos,m}(a)$ is invertible, and let $G^* : L^2(I) \rightarrow H^2(I)$ be defined by

$$G^*(b) = d^{-2}(\text{tr}^*(b)) + (D\mathcal{F}_{\cos,m}(a))^{-1}(\text{tr}(b)).$$

Since $\text{tr}_m(d^{-2}(\text{tr}^*(b))) = 0$, it follows that

$$\text{tr}_m(G^*(b)) = (D\mathcal{F}_{\cos,m}(a))^{-1}(\text{tr}_m(b)),$$

and since $(D\mathcal{F}_{\cos,m}(a))^{-1}(\text{tr}_m(b)) \in \mathbb{R}^m$, it follows that $\text{tr}_m^*((D\mathcal{F}_{\cos,m}(a))^{-1}(\text{tr}_m(b))) = 0$ and thus

$$\text{tr}_m^*(G^*(b)) = d^{-2}(\text{tr}_m^*(b)).$$

Therefore we have that

$$G(\text{tr}_m(G^*(b))) = (D\mathcal{F}_{\cos,m}(a) \circ D\mathcal{F}_{\cos,m}(a))^{-1}(\text{tr}_m(b)) = \text{tr}_m(b)$$

and

$$G(\mathrm{tr}_m^*(G^*(b))) = d^2(d^{-2}(\mathrm{tr}_m^*(b))) = \mathrm{tr}_m^*(b).$$

Combining these identities it follows that

$$G((G^*(b))) = G(\mathrm{tr}_m(G^*(b)) + \mathrm{tr}_m(G^*(b))) = \mathrm{tr}_m(b) + \mathrm{tr}_m^*(b) = b.$$

Analogously we can prove that $G^*(G(b)) = b$, proving thus that G is invertible with inverse G^* .

Now, if G was injective, since $D\mathcal{F}_{\cos,m}(a)$ is the restriction of G to $\mathbb{R}^m$, it would follow that $D\mathcal{F}_{\cos,m}(a)$ would be injective as well and therefore, by the Rank-nullity Theorem, $D\mathcal{F}_{\cos,m}(a) : \mathbb{R}^m \rightarrow \mathbb{R}^m$ would be invertible, and therefore G would be invertible as well by what we have proved above.

Thus, it follows that if G is injective then G is onto as well and thus due to item iv) of Proposition 8 we that $\lambda(G)_{Z(l^2)} = j(G)_{Z(l^2)}$ where j is injectivity modulus.

We now prove that.

$$\lambda(G)_{Z(l^2)} = \min \{ \lambda(D\mathcal{F}_{\cos,m}(a))_{Z(\mathbb{R}^m)}, m^2 \omega(m)^{-1} \} \quad (3.23)$$

To this end let $\delta = \lambda(D\mathcal{F}_{\cos,m}(a))_{Z(\mathbb{R}^m)} = j(D\mathcal{F}_{\cos,m}(a))_{Z(\mathbb{R}^m)}$ (see Proposition 8). Since $D\mathcal{F}_{\cos,m}$ is simply the restriction of G to $\mathbb{R}^m$ it follows in special that $j(G)_{Z(l^2)} \leq j(D\mathcal{F}_{\cos,m})_{Z(\mathbb{R}^m)} = \delta$. On the other hand, letting e_{m+1} be the $(m+1)^{\text{th}}$ element of the orthonormal basis of $l^2(\mathbb{N})$ we have that

$$\begin{aligned} \|G(e_{m+1})\|_{l^2(\mathbb{N})} &= \|d^2(e_{m+1})\|_{l^2(\mathbb{N})} = m^2 \omega(m)^{-1} \Rightarrow \\ j(G)_{Z(l^2)} &\leq m^2 \omega(m)^{-1} \end{aligned}$$

and therefore we conclude that $j(G)_{Z(l^2)} \leq \min\{\delta, m^2 \omega(m)^{-1}\}$. On the other hand, given $b \in l^2(\mathbb{N})$ with $\|b\|_{l^2(\mathbb{N})} = 1$, it follows that

$$\begin{aligned} \mathrm{tr}_m(G(b)) &= (D\mathcal{F}_{\cos,m}(a))(\mathrm{tr}_m(b)) \text{ and} \\ \mathrm{tr}_m^*(G(b)) &= \mathrm{tr}_m^*(d^2(b)), \end{aligned}$$

which implies that

$$\begin{aligned} \|G(b)\|_{l^2(\mathbb{N})} &= \sqrt{\|(D\mathcal{F}_{\cos,m}(a))(\mathrm{tr}_m(b))\|_{\mathbb{R}^m}^2 + \|d^2(\mathrm{tr}_m^*(b))\|_{l^2(\mathbb{N})}^2} \\ &\geq \sqrt{\left(\delta \|\mathrm{tr}_m(b)\|_{\mathbb{R}^m}^2\right) + (m^2 \omega(m)^{-1}) \|\mathrm{tr}_m^*(b)\|_{l^2(\mathbb{N})}^2} \\ &\geq \min\{\delta, m^2 \omega(m)^{-1}\} \sqrt{\|\mathrm{tr}_m(b)\|_{\mathbb{R}^m}^2 + \|\mathrm{tr}_m^*(b)\|_{l^2(\mathbb{N})}^2} = \min\{\delta, m^2 \omega(m)^{-1}\}, \end{aligned}$$

and thus we conclude that $j(G)_{Z(l^2)} \geq \min\{\delta, m^2 \omega(m)^{-1}\}$. Combining the two inequalities obtained we have that $\lambda(G)_{Z(l^2)} = \min\{\delta, m^2 \omega(m)^{-1}\}$.

Now, to conclude the proof we notice that, due to Proposition 8 we have that

$$\left| \lambda(D\mathcal{F}_{\cos})_{Z(l^2)} - \lambda(G)_{Z(l^2)} \right| \leq \|D\mathcal{F}_{\cos} - G\|_{Z(l^2)}$$

and thus, due to inequality (3.22) it follows that

$$\left| \lambda(D\mathcal{F}_{\cos})_{Z(l^2)} - \lambda(G)_{Z(l^2)} \right| \leq \frac{L}{\pi m}$$

concluding the proof of item III).

Item IV) is just item II) of Lemma 3.

Item V):

Let $\delta = \lambda(D\mathcal{F}_{\cos,m}(a))_{Z(\mathbb{R}^m)}$.

Since by hypothesis $\delta > \frac{L}{\pi m} \geq 0$, it follows from item i) of Proposition 8 that $D\mathcal{F}_{\cos,m}(a)$ is invertible and moreover from item III) it follows that

$$\lambda(D\mathcal{F}_{\cos}(a)) \geq \delta - \frac{L}{\pi m}.$$

and thus $D\mathcal{F}_{\cos}$ is invertible as well. On the other hand, since $D\mathcal{F}_{\cos,m}(a)$ is invertible, it follows due to what was proved in item III) that G is invertible as well with G^{-1} given by

$$G^{-1}(b) = d^{-2}(\text{tr}_m^*(b)) + (D\mathcal{F}_{\cos,m}(a))^{-1}(\text{tr}_m(b))$$

for all $b \in l^2(\mathbb{N})$.

On the other hand, since $\mathcal{F}_{\cos,m}(a) \in \mathbb{R}^m$, it follows that $d^{-2}(\text{tr}_m^*(F)) = 0$ and thus

$$\begin{aligned} a - G^{-1}(\mathcal{F}_{\cos,m}(a)) &= a - (D\mathcal{F}_{\cos,m}(a))^{-1}(\text{tr}_m(\mathcal{F}_{\cos,m}(a))) \\ &= x - (D\mathcal{F}_{\cos,m}(a))^{-1}(\mathcal{F}_{\cos,m}(a)) = T_{\mathcal{F}_{\cos,m}}(a) \end{aligned}$$

which implies

$$\|T_{\mathcal{F}_{\cos}}(a) - T_{\mathcal{F}_{\cos,m}}(a)\| = \|D\mathcal{F}_{\cos}(a)^{-1}(\mathcal{F}_{\cos}(a)) - G^{-1}(\mathcal{F}_{\cos,m}(a))\|_{l^2(\mathbb{N})}$$

On the other hand, since we prove in item i) that

$$\|\mathcal{F}_{\cos}(a) - \mathcal{F}_{\cos,m}(a)\|_{L^2(I)} \leq \frac{M}{\pi m},$$

and since due to item ii) of Proposition 8 we have that $\|D\mathcal{F}_{\cos}(a)^{-1}\|_{Z(l^2)} = \lambda(D\mathcal{F}_{\cos}(a))_{Z(l^2)}^{-1} \leq \frac{1}{\delta - \frac{L}{\pi m}}$, it follows that

$$\|D\mathcal{F}_{\cos}(a)^{-1}(\mathcal{F}_{\cos}(a) - \mathcal{F}_{\cos,m}(a))\|_{l^2(\mathbb{N})} \leq \frac{1}{\delta - \frac{L}{\pi m}} \frac{M}{\pi m} = \frac{M}{\pi m \delta - L}.$$

On the other hand, due to (3.23) we have that $\lambda(G)_{Z(l^2)} = \min\{\delta, m^2\omega(m)\} \leq \delta$ which combined with (3.22) implies in

$$\begin{aligned} \|D\mathcal{F}_{\cos}(a)^{-1} - G^{-1}\|_{Z(l^2)} &= \|D\mathcal{F}_{\cos}(a)^{-1}(G - D\mathcal{F}_{\cos}(a))G\|_{Z(l^2)} \\ &\leq \|D\mathcal{F}_{\cos}^{-1}\|_{l^2(\mathbb{N})} \|G - D\mathcal{F}_{\cos}(a)\|_{Z(l^2)} \|G\|_{l^2(\mathbb{N})} \leq \frac{\|G - D\mathcal{F}_{\cos}(a)\|_{Z(l^2)}}{\lambda(G)_{Z(l^2)}\lambda(D\mathcal{F}_{\cos}(a))_{Z(l^2)}} \\ &\leq \frac{L}{\pi m\delta(\delta - \frac{L}{\pi m})} \leq \frac{L}{\delta(\pi m\delta - L)}. \end{aligned}$$

and since from Lemma 1 $(w(x), w'(x)) \in \bar{B}\left(0, \frac{2\sqrt{6}}{3}\right)$ it follows that

$$\begin{aligned} \|\mathcal{F}_{\cos}(a)\|_{l^2(\mathbb{N})} &= \|d^2(a) - \pi_{\cos}(f(x, w, w'))\|_{l^2(\mathbb{N})} \leq \\ &\|a\|_{l^2(\mathbb{N})} + \|f(x, w, w')\|_{L^2(\mathbb{N})} \leq r + M_1 \end{aligned}$$

and thus

$$\|(D\mathcal{F}_{\cos}(a)^{-1} - G^{-1})\mathcal{F}_{\cos,m}(a)\| \leq \frac{L(r + M_1)}{\delta(\pi m\delta - L)}$$

Thus, combined the obtained inequalities with the triangle inequality it follows that

$$\|D\mathcal{F}_{\cos}(a)^{-1}(\mathcal{F}_{\cos}(a)) - G^{-1}(\mathcal{F}_{\cos,m}(a))\|_{l^2(\mathbb{N})} \leq \frac{M}{\pi m\delta - L} + \frac{L(r + M_1)}{\delta(\pi m\delta - L)}$$

which concludes item V).

Item VI)

Suppose that b is a zero of $\mathcal{F}_{\cos}$ in $\bar{B}\left(0, \frac{\sqrt{6}}{4}r\right)_{\mathbb{R}^m}$ and let $w = h_{\cos}^{-1}(b) \in \bar{B}\left(0, \frac{\sqrt{6}}{4}r\right)_{H^2(I)}$. Thus $\mathcal{F}_{\cos}(w) = 0$ implies in $\mathcal{F}(w) = 0$ and therefore $w'' = f(x, w, w')$ in $L^2(I)$, that is, $w \in H_N^3(I)$. Therefore, due to item i) of Lemma 2 we have that

$$\|w^{(3)}\|_{L^2(I)} = \|(f(x, w, w'))'\|_{L^2(I)} \leq M_2\sqrt{1 + \|w\|_{H^2(I)}^2}$$

and thus

$$\begin{aligned} \|w'\|_{H^2(I)} &= \sqrt{\|w'\|_{L^2(I)}^2 + \|w''\|_{L^2(I)}^2 + \|w^{(3)}\|_{H^1(I)}^2} \\ &\leq \sqrt{\|w\|_{H^2(I)}^2 + M_2^2\left(1 + \|w\|_{H^2(I)}^2\right)} = M_2\sqrt{1 + 2\|w\|_{H^2(I)}^2} = M' \end{aligned}$$

Thus, due to item iii) of Lemma 1 it follows that

$$\|w - \rho_{\cos,m}(w)\|_{H^2(I)} \leq \frac{M'}{\pi m}.$$

and since $w = h_{\cos}^{-1}(b)$ and $\rho_{\cos,m}(w) = \rho_{\cos,m}(h_{\cos}^{-1}(b)) = h_{\cos}^{-1}(\text{tr}_m(b))$ it follows that

$$\|b - \text{tr}_m b\|_{l^2(\mathbb{N})} \leq \frac{M'}{\pi m}.$$

On the other hand, since $\|\mathrm{tr}_m b\|_{\mathbb{R}^m} \leq \|b\|_{L^2(\mathbb{N})} \leq r$, due to what was proved in the preliminaries of the first chapter we have that $\mathrm{tr}_m b \in \bar{B}\left(c, r\frac{\sqrt{m}}{k}\right)_{\mathbb{R}^m}$ for some $c \in J_{k,m}$ and thus

$$\|b - c\|_{L^2(\mathbb{N})} \leq \|b - \mathrm{tr}_m(b)\|_{L^2(\mathbb{N})} + \|\mathrm{tr}_m(b) - c\|_{\mathbb{R}^m} \leq r\frac{\sqrt{m}}{k} + \frac{M'}{\pi m}$$

which proves item VI). □

These items thus provides all tools necessary to isolates zeros of boundary value problems from the algorithm of the first part of the thesis, providing the first theoretically proved algorithm to isolate all zeros of such equations in bounded closed balls of $H^2(I)$.

CONCLUSION

In this work we provided a convergent algorithm for Bannach Space non-linear equations under certain computability and regularity conditions and proved formally its termination. Moreover we also provided a Newton Inexact Method with explicit convergence rate to the zeros. Finally, we showed a theorem for boundary value problems of Neumann type, which trivializes the process to obtain the elements required in the algorithm, via a finite dimension projector of the equation. It should be noted that all of the work we have done to the Two-Point Boundary Value Problem of Neumann type can be reproduced to the Two-Point Boundary Value Problem of Dirichlet type. Besides, we wish to generalize this work to solve the problem of determining parametrized solutions $u_\lambda \in C^2(\Omega')$ for the boundary value problem with Neumann conditions:

$$\begin{aligned} -u''_\lambda(x) + g(x)u'_\lambda(x) + f(x, \lambda, u_\lambda(x)) &= 0 \quad \forall x \in \Omega, \\ u'_\lambda(a) &= u'_\lambda(b) = 0. \end{aligned} \tag{4.1}$$

Indeed, in (GAMEIRO; LESSARD, 2010) and in (GAMEIRO; LESSARD; PUGLIESE, 2016), it was provided a continuation method based on the so called to determine such regions S and parametrized solutions u_λ . On the other hand, our method will be carried by proposing a version of the Kantorovich theorem suitable for implicit surface determination. Indeed, we observed that given a twice differentiable function $\mathcal{F}(\gamma, u)$ satisfying

$$\|\mathcal{F}_{uu}(\gamma, u)\| \leq K, \quad \|\mathcal{F}_{u\gamma}(\gamma, u)\| \leq L \text{ and } \|\mathcal{F}_{\gamma\gamma}(\gamma, u)\| \leq M$$

then we shall see that, just as we showed in the non-parameter case, the non-existence of solutions of $\mathcal{F}(\gamma, u) = 0$ near a point $z_0 = (\gamma_0, u_0)$ is closely related to the non-existence of solutions of $g(h, t) = 0$ near $(h, t) = 0$ where

$$g_0(h, t) = \|\mathcal{F}(z_0)\| - \|\mathcal{F}_u(z_0)\|t - \|\mathcal{F}_\gamma(z_0)\|h - \frac{1}{2}(Kt^2 + 2Lht + Mh^2)$$

while the existence and unicity of parametrized solutions u_γ for $\mathcal{F}(\gamma, u) = 0$ near $z_0 = (\gamma_0, u_0)$ is closely related to the existence of parametrized solutions t_h for $g_1(h, t) = 0$ where

$$g_1(h, t) = \|\mathcal{F}(z_0)\| - \lambda(\mathcal{F}_u(z_0))t + \|\mathcal{F}_\gamma(z_0)\|h + \frac{1}{2}(Kt^2 + 2Lht + Mh^2).$$

Finally, we also wish to generalize the method in this thesis for bidimensional partial differential equations over rectangles.

BIBLIOGRAPHY

ALEFELD, G.; HERZBERGER, J. **Introduction to Interval Computations**. [S.l.]: Academic, 1983. Citation on page [32](#).

ARGYROS IOANNIS K.; GEORGE, S. Extending the applicability of the inexact newton-hss method for solving large systems of nonlinear equations. **Numerical Algorithms**, Springer US, 3 2019. Citation on page [20](#).

ARGYROS IOANNIS K.; HILOUT, S. Robust semi-local convergence analysis for inexact newton method. **Applied Mathematics and Computation**, Elsevier Science, v. 227, 01 2014. Citation on page [20](#).

BOGACHEV, V. I. **Measure Theory Volume 1**. first. New York: Springer, 2007. Citation on page [54](#).

BREZIS, H. **Functional Analysis, Sobolev Spaces and Partial Differential Equations**. [S.l.]: Springer-Verlag New York, 2011. Citation on page [57](#).

CARTAN, H. **Differential Calculus**. [S.l.]: Houghton Mifflin, 1971. Citations on pages [23](#) and [25](#).

EVANS, L. C. **Partial Differential Equations**. [S.l.]: American Mathematical Society, 1998. Citations on pages [56](#) and [57](#).

FERNÁNDEZ, J. A. E.; VERÓN, M. A. H. **Newton's Method: an Updated Approach of Kantorovich's Theory**. [S.l.]: Springer International Publishing, 2017. Citation on page [34](#).

FOLLAND, G. B. **Real Analysis - Modern Techniques and Their Applications**. second. New York: Wiley, 1999. Citation on page [54](#).

GAMEIRO, M.; LESSARD, J. P. Analytic estimates and rigorous continuation for equilibria of higher-dimensional pdes. **Journal of Differential Equations**, Elsevier Science, v. 249, p. 2237–2268, 2010. Citations on pages [19](#) and [79](#).

GAMEIRO, M.; LESSARD, J. P.; PUGLIESE, A. Computation of smooth manifolds via rigorous multi-parameter continuation in infinite dimensions. **Foundations of Computational Mathematics**, Springer-Verlag, v. 16, p. 531–575, 2016. Citations on pages [19](#) and [79](#).

GARLING, D. J. H. **A Course in Mathematical Analysis: Volume 2, Metric and Topological Spaces, Functions of a Vector Variable**. [S.l.]: Cambridge University Press, 2014. (A Course in Mathematical Analysis, Volume II). Citation on page [25](#).

KANTOROVICH, L. V.; AKILOV, G. P. **Functional analysis**. [S.l.]: Pergamon Press, 1982. Citation on page [34](#).

LECERF, G.; SAADÉ, J. **A short survey on Kantorovich-like theorems for Newton's method**. [S.l.: s.n.], 2016. 1–11 p. Citation on page [34](#).

- LI, W. S. C. Kantorovich-type convergence criterion for inexact newton methods. **Applied Numerical Mathematics**, Elsevier Science, v. 59, 2009. Citation on page 20.
- MCSHANE, E. J. **Integration**. 2. ed. Princeton: Princeton University Press, 1947. Citation on page 54.
- MOORE, R. E. **Methods and Applications of Interval Analysis**. [S.l.]: Society for Industrial and Applied Mathematics, 1987. Citation on page 32.
- MÜLLER, V. **Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras**. 2nd. ed. [S.l.]: Birkhäuser, 2007. (Operator Theory: Advances and Applications). Citations on pages 26 and 27.
- NAKAO, M. T. A numerical approach to the proof of existence of solutions for elliptic problems. **Japan J. Appl. Math.**, v. 5, p. 313–332, 1988. Citation on page 19.
- NAKAO, M. T.; YAMAMOTO, N. Numerical verifications for solutions to elliptic equations using residual iterations with higher order finite elements. **J. Comput. Appl. Math.**, v. 60, p. 271–279, 1995. Citation on page 19.
- NEUMAIER, A. **Interval Methods for Systems of Equations**. [S.l.]: Cambridge University Press, 1990. Citation on page 32.
- PLUM, M. Existence and enclosure results for continua of solutions of parameter-dependent nonlinear boundary value problems. **J. Comput. Appl. Math.**, Elsevier Science, v. 60, p. 187–200, 1995. Citation on page 19.
- STEIN, R. S. E. M. **Fourier Analysis: An Introduction**. [S.l.]: Princeton University Press, 2003. Citation on page 56.
- SVAITER, O. F. B. A robust kantorovich’s theorem on the inexact newton method with relative residual error tolerance. **Journal of Complexity**, Elsevier Science, v. 28, 2012. Citation on page 20.
- TAYLOR, A. E. **Introduction to Functional Analysis**. 2nd ed., reprint ed. ed. [S.l.]: R.E. Krieger Pub. Co, 1958. ISBN 0898749514. Citations on pages 21 and 22.
- WU, M. A new semi-local convergence theorem for the inexact newton methods. **Applied Mathematics and Computation**, Elsevier Science, v. 200, 2008. Citation on page 20.
- YAP, C. K. In praise of numerical computation. **Lecture Notes in Computer Science**, Springer, Berlin, v. 5760, 2009. Citation on page 19.