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Instituto de Ciências Matemáticas e de Computação

# Poisson quasi-Nijenhuis manifolds and Dirac structures: A geometrical approach to deformation and involutive theorems


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Tese de Doutorado do Programa de Pós-Graduação em Matemática (PPG-Mat)

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**Murilo do Nascimento Luiz**

# Variedades de Poisson quasi-Nijenhuis e estruturas de Dirac: uma abordagem geométrica para os teoremas de deformação e involução

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências – Matemática. *VERSÃO REVISADA*

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*Este trabalho é dedicado à minha família, que sempre me apoiou e foi um refúgio nos momentos mais difíceis.*

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*"If you're always frozen in fear and taking too long to think about what to do, you'll miss your opportunity."*

*(Jason Mendoza – The Good Place)*

# RESUMO

LUIZ, M. N. **Variedades de Poisson quasi-Nijenhuis e estruturas de Dirac: uma abordagem geométrica para os teoremas de deformação e involução**. 2024. 101 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2024.

Neste trabalho, analisamos a conexão entre estruturas de Poisson quase-Nijenhuis, quase-Lie bialgebróides e algebróides de Courant. Demonstramos como deformar uma variedade de Poisson quase-Nijenhuis usando uma 2-forma fechada dentro do contexto dos algebróides de Courant e estruturas de Dirac. Depois, interpretamos este procedimento no contexto de super variedades, como uma instância específica do chamado twisting de um proto-bialgebróide. Por fim, investigamos as aplicações de variedades de Poisson quasi-Nijenhuis dentro da teoria de sistemas integráveis. Os principais resultados desta tese estão relatados em (LUIZ; MENCATTINI; PEDRONI, 2024).

**Palavras-chave:** Variedade de Poisson quasi-Nijenhuis. Quasi-Lie bialgebroides. Estuturas de Dirac. Twisting de um quase-Lie bialgebroid.

# ABSTRACT

LUIZ, M. N. **Poisson quasi-Nijenhuis manifolds and Dirac structures: A geometrical approach to deformation and involutive theorems**. 2024. 101 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2024.

In this work, we analyze the connection between Poisson quasi-Nijenhuis structures, quasi-Lie bialgebroids, and Courant algebroids. We demonstrate how to deform a Poisson quasi-Nijenhuis manifold using a closed 2-form within the context of Courant algebroids and Dirac structures. Then, we interpret this procedure in the context of supermanifolds, as a specific instance of the so-called twisting of a proto-bialgebroid. Finally, we investigate the applications of Poisson quasi-Nijenhuis manifolds in the theory of integrable systems. The main results of this thesis are reported in (LUIZ; MENCATTINI; PEDRONI, 2024).

**Keywords:** Poisson quasi-Nijenhuis manifolds. Quasi-Lie bialgebroids. Courant algebroids. Dirac structures. Twists of quasi-Lie bialgebroids .

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CHAPTER 1

# INTRODUCTION

The concept of integrable systems originated in physics, and the interest in it dates back to Newton's explicit solution of Kepler's two-body problem in a gravitational field. Since Newton's work, significant progress has been made in this field.

The first formal definition of integrability emerged from Liouville's work, linking the existence of a sufficient number of appropriate conserved quantities with a method for solving associated differential equations, known as quadrature methods.

In Poisson geometry, the term "system" is equivalent to "Hamiltonian systems". These are dynamic systems defined by a Poisson bivector and a scalar field, the latter being the system's Hamiltonian. One of the most important features of such systems is that the Hamiltonian is conserved over time.

Although Liouville's definition is somewhat restrictive, the identification of appropriate conserved quantities yields invaluable insights into the system's behavior. To achieve this, numerous geometric and algebraic tools have been developed to capture the intrinsic symmetries of these problems. The works of Lax, Noether, and Magri are notable contributions to this field.

In its simplest form (i.e., within a symplectic framework), the concept can be articulated as it follows. Consider a $2n$-dimensional symplectic manifold $(M,\omega)$ and a smooth function $H : M \to \mathbb{R}$. The Hamiltonian dynamical system defined by $H$ is completely integrable in the Liouville sense if one can identify functions $f_1,\dots,f_n \in C^\infty(M)$ that:

1. Are generally independent on $M$, that is, $df_1 \wedge \cdots \wedge df_n = 0$ in a dense subset;

2. For any pair of $i, j$,
$$\{f_1, f_j\} = 0;$$

3. Include the function $H$, satisfying $\{H, f_i\}_\omega = 0$ for all $i = 1,\dots,n$.

Under these conditions, the fiber $M_b = F^{-1}(b)$ of the momentum map $F = (f_1, \ldots, f_n)$ : $M \to \mathbb{R}^n$ is an embedded, typically non-connected, Lagrangian submanifold of $(M, \omega)$. Assuming the completeness of the flows generated by the $f_i$, each connected component of $M_b$ is diffeomorphic to a cylinder $\mathbb{T}^k \times \mathbb{R}^{n-k}$. Additionally, in the neighborhood of any such fiber, it's possible to introduce suitable coordinates in which the Hamiltonian flow of any Hamiltonian vector field of $f_i$ becomes linear. This observation underscores the integrability of the Hamiltonian flow generated by $H$. See, e.g., (ARNOLD, 1997).

For a $(1,1)$ tensor field $N\colon TM \to TM$, one can define its Nijenhuis torsion as

$$T_N(X,Y) = [NX,NY] - N([NX,Y] + [X,NY] - N[X,Y]).$$

We say that $N$ is a Nijenhuis tensor if the Nijenhuis torsion $T_N$ vanishes. Franco Magri and Carlo Morosi define the notion of compatibility of a Nijenhuis tensor and a Poisson structure as follows.

For a Poisson manifold $(M, \pi)$, we say that $N\colon TM \to TM$ and $\pi\colon T^*M \to TM$ are compatible if

$$N\pi = \pi N^*,$$
$$\mathscr{L}_{\pi^\sharp(\alpha)}(N)X - \pi^\sharp \mathscr{L}_X(N^*\alpha) + \pi^\sharp \mathscr{L}_{NX}(\alpha) = 0,$$

for all 1-form $\alpha$ and vector fields $X$. For all $X, Y \in \Gamma(TM)$. Therefore, we have reached the idea of the Poisson Nijenhuis manifold.

**Definition 1.0.1.** A Poisson-Nijenhuis manifold is a triple $(M, \pi, N)$, where $M$ is a differentiable manifold, $\pi$ is a Poisson tensor, and $N$ is a Nijenhuis operator, such that $\pi$ and $N$ are compatible.

Originating in the theory of soliton equations, the notion of Lenard chain is an important tool for constructing families of functions in involution. Let $\pi_1$ and $\pi_2$ be two Poisson bivectors. A sequence of functions $\{f_j\}_{j\in\mathbb{Z}}$ is said to satisfy the Lenard recursion relations, and is called a Lenard chain, if

$$\pi_1(df_j) = \pi_2(df_{j+1}) \qquad \text{for all } j \in \mathbb{Z}.$$

A noteworthy property of the Lenard chains is that the functions $f_j$ are pairwise in involution with respect to both Poisson brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$. See, e. g. (MAGRI; MOROSI, 1984).

The relevance of Poisson-Nijenhuis manifolds in the theory of integrable systems is summarized in the following result:

**Theorem 1.0.2.** On a PN manifold, the functions $f_k = \dfrac{1}{k}\operatorname{tr} N^k$ satisfy the Lenard recursion relation.

The concept of Poisson quasi-Nijenhuis manifolds, defined in (STIÉNON; XU, 2007), generalizes the notion of Poisson-Nijenhuis manifold, and it has important applications in the study of generalized complex structures. Roughly speaking, it is a Poisson manifold with a compatible (1,1)-tensor in which the vanishing of the Nijenhuis torsion is weakened in a suitable sense. More precisely

**Definition 1.0.3.** A Poisson quasi-Nijenhuis manifold is a quadruple $(M,\pi,N,\phi)$ such that:

- the Poisson bivector $\pi$ and the $(1,1)$ tensor field $N$ are compatible;
- the 3-forms $\phi$ and $i_N\phi$ are closed;
- $T_N(X,Y)=\pi^\sharp(i_{X\wedge Y}\phi)$ for all vector fields $X$ and $Y$, where $i_{X\wedge Y}\phi$ is the 1-form defined as $\langle i_{X\wedge Y}\phi, Z\rangle = \phi(X,Y,Z)$.

The relation between Poisson quasi-Nijenhuis manifolds and integrable systems was studied in (FALQUI *et al.*, 2020). Differently from the Poisson-Nijenhuis, the functions defined as the traces of the powers of the (1,1)-tensor do not necessarily satisfy the Lenard-Magri relations.

Presented by (FALQUI *et al.*, 2020), the deformation theorem of a PN manifold into a PqN manifold is a tool that enables the construction of a new PqN manifold $(M,\pi,\hat{N},\phi)$ from a PN manifold $(M,\pi,N)$ and a closed 2-form $\Omega$. Here,

$$\hat{N} = N + \pi^\sharp\Omega^\flat,$$
$$\phi = d_N + [\Omega,\Omega]_N.$$

This thesis provides a geometric interpretation of the results presented in (FALQUI *et al.*, 2020) and (FALQUI; MENCATTINI; PEDRONI, 2023) using the connections between Dirac structures, supermanifolds, Lie algebroids, and Poisson quasi-Nijenhuis manifolds. As a result, two generalizations of the deformation theorem are presented. The thesis is structured as follows:

Chapter 2 – Courant algebroid and Dirac structures: We introduce the concept of Courant algebroid and discuss the equivalence between the three main definitions of Courant algebroid. We also analyze the theory of Dirac structures in a Courant algebroid and their relations with Lie bialgebroids.

Chapter 3 – Deformation theorem for a Poisson quasi-Nijenhuis Manifold: We introduce the so-called Lie quasi-bialgebroids and explore their connections with Courant algebroids and with Poisson quasi-Nijenhuis manifolds. Using the theory of Dirac structures, we demonstrate a generalization of the deformation theorem, but now the initial manifold can be Poisson quasi-Nijenhuis.

Chapter 4 – Dirac-Nijenhuis structures: In this chapter, we introduce the theory of Dirac-Nijenhuis structures, a generalization of Poisson-Nijenhuis structures, and demonstrate that, when $N = Id$, the deformation defines a Dirac-Nijenhuis structure.

Chapter 5 – The big bracket formalism for the deformation theorem: In this chapter, we discuss the theory of Courant algebroid in the context of supermanifolds and present the definition of Poisson quasi-Nijenhuis structures in an arbitrary Lie algebroid. As a final result, we present a deformation theorem for Poisson quasi-Nijenhuis structures in Lie algebroids.

Chapter 6 – Bi-differential calculi from a Dirac perspective and an involutivity theorem: In this chapter, we present the connection between Poisson quasi-Nijenhuis manifolds and integrable systems and present a Dirac approach to the involutivity theorem of (FALQUI *et al.*, 2020).

Appendix 1 – Differential calculus on Lie algebroids: We summarize the main results concerning the theory of differential calculus on Lie algebroids. These results will be used throughout the text.

The results concerning the deformation theorem are reported in (LUIZ; MENCATTINI; PEDRONI, 2024).

CHAPTER

2

# COURANT ALGEBROID AND DIRAC STRUCTURES

This chapter provides a comprehensive analysis of Courant algebroids and their relations with Lie bialgebroids and Dirac structures. The definitions and results compiled in this chapter will be used throughout the rest of this thesis.

We start by presenting the three main definitions of a Courant algebroid and illustrating the connections between these definitions. The first one is the original definition given by (LIU; WEINSTEIN; XU, 1997). The second is the simplification of the original definition given by (UCHINO, 2002). The final one is the definition presented by (KOSMANN-SCHWARZBACH, 2005), using the non-skew-symmetric bracket studied by (ROYTENBERG, 1999).

Subsequently, we discuss the main result of (LIU; WEINSTEIN; XU, 1997), which introduces Lie bialgebroids as an important example of Courant algebroids. Finally, we present the definition of Dirac structures. All these structures will be used to demonstrate the Deformation Theorem 3.2.9 in Chapter. 3.

## 2.1 Courant algebroids

In 1990, T. Courant introduced the first example of a Courant algebroid in his paper (COURANT, 1990). These algebroids serve as a natural ambient space for Dirac structures, which are geometric objects that extend the idea of pre-symplectic and Poisson structures. Roughly speaking, Courant algebroids are vector bundles equipped with a non-degenerate symmetric bilinear form and a bracket operation on sections that satisfy a set of axioms.

Let $E \to M$ be a vector bundle equipped with a non-degenerate symmetric bilinear form $\langle \cdot \mid \cdot \rangle$, a skew-symmetric bracket $[\![\cdot,\cdot]\!]$ on $\Gamma(E)$ and a bundle map $\rho\colon E \to TM$. Given

$e_1,e_2,e_3 \in \Gamma(E)$, we define $T(e_1,e_2,e_3)$ as the function on $C^\infty(M)$ given by:

$$T(e_1,e_2,e_3) = \frac{1}{3}\left(\langle [\![e_1,e_2]\!] \mid e_3\rangle + \langle [\![e_2,e_3]\!] \mid e_1\rangle + \langle [\![e_3,e_1]\!] \mid e_2\rangle\right),$$

and $D : C^\infty(M) \longrightarrow \Gamma(E)$ as the map defined by $D = \frac{1}{2}\beta^{-1}\rho^* d$, where $\beta$ is the isomorphism between $E$ and $E^*$ given by the bilinear form. In other words,

$$\langle Df \mid e\rangle = \frac{1}{2}\rho(e)(f). \tag{2.1}$$

For every $e_1,e_2,e_3 \in \Gamma(E)$, we define the Jacobiator $J\colon \Gamma(E)\times\Gamma(E)\times\Gamma(E) \to \Gamma(E)$ by

$$J(e_1,e_2,e_3) = [\![[\![e_1,e_2]\!],e_3]\!] + [\![[\![e_2,e_3]\!],e_1]\!] + [\![[\![e_3,e_1]\!],e_2]\!]. \tag{2.2}$$

**Remark 2.1.1.** We denote the non-degenerate symmetric bilinear form on sections of $E$ by $\langle \cdot \mid \cdot\rangle$ to avoid confusion with the usual pairing between vector fields and 1-forms. There is a similar notation in (KOSMANN-SCHWARZBACH, 2005).

### 2.1.1 First definition

The first definition of a Courant algebroid in an abstract context was presented in (LIU; WEINSTEIN; XU, 1997). This definition generalizes the structure studied by Courant and provides a framework for defining Dirac structures in a more general context.

**Definition 2.1.2** ((LIU; WEINSTEIN; XU, 1997), Definition 2.1)**.** A Courant algebroid is a vector bundle $E \to M$ equipped with a non-degenerate symmetric bilinear form $\langle \cdot \mid \cdot\rangle$ on the bundle, a skew-symmetric bracket $[\![\cdot,\cdot]\!]$ on $\Gamma(E)$ and a bundle map $\rho\colon E \to TM$ satisfying the following properties:

(1) for any $e_1,e_2,e_3 \in \Gamma(E)$,

$$J(e_1,e_2,e_3) = DT(e_1,e_2,e_3);$$

(2) for any $e_1,e_2 \in \Gamma(E)$,

$$\rho[\![e_1,e_2]\!] = [\rho(e_1),\rho(e_2)];$$

(3) for any $e_1,e_2 \in \Gamma(E)$ and $f \in C^\infty(M)$,

$$[\![e_1,fe_2]\!] = f[\![e_1,e_2]\!] + (\rho(e_1)f)e_2 - \langle e_1 \mid e_2\rangle Df;$$

(4) for any $f,g \in C^\infty(M)$, $\rho\circ D = 0$, that is,

$$\langle Df \mid Dg\rangle = 0;$$

(5) for any $e,e_1,e_2 \in \Gamma(E)$,

$$\rho(e)\langle e_1 \mid e_2\rangle = \langle [\![e,e_1]\!] + D\langle e \mid e_1\rangle \mid e_2\rangle + \langle e_1 \mid [\![e,e_2]\!] + D\langle e \mid e_2\rangle\rangle.$$

Courant presented the following example in (COURANT, 1990).

**Example 2.1.3** (Standard Courant algebroid)**.** If $E = TM \oplus T^*M$ is equipped with the skew-symmetric bracket

$$[\![X \oplus \alpha, Y \oplus \beta]\!] = [X,Y] \oplus \left( \mathscr{L}_X(\beta) - \mathscr{L}_Y(\alpha) + \frac{1}{2}\,\mathrm{d}(\alpha(Y) - \beta(X)) \right),$$

the pairing $\langle X \oplus \alpha \mid Y \oplus \beta \rangle = \alpha(Y) + \beta(X)$, and the anchor $\rho(X \oplus \alpha) = X$, then $E$ is a Courant algebroid. In this case, we have that, for all f $\in C^\infty(M)$,

$$Df = \frac{1}{2} df.$$

The following lemma, as demonstrated in (ROYTENBERG, 1999), establishes that the set of sections of $E$ which are $Df$ for some function forms an ideal of the algebra $[\![\cdot,\cdot]\!]$.

**Lemma 2.1.4.** For any $e \in \Gamma(E)$, $f \in C^\infty(M)$ one has

$$[\![e, Df]\!] = D\langle e \mid Df \rangle. \tag{2.3}$$

*Proof.* By Property 5, we have that, for $Df, e_1, e_2 \in \Gamma(E)$,

$$\begin{aligned}
\rho(Df)\langle e_1, e_2 \rangle &= \langle [\![Df, e_1]\!] + D\langle Df \mid e_1 \rangle \mid e_2 \rangle + \langle e_1 \mid [\![Df, e_2]\!] + D\langle Df \mid e_2 \rangle \rangle \\
\rho(e_1)\langle e_2, Df \rangle &= \langle [\![e_1, e_2]\!] + D\langle e_1 \mid e_2 \rangle \mid Df \rangle + \langle e_2 \mid [\![e_1, Df]\!] + D\langle e_1 \mid Df \rangle \rangle \\
\rho(e_2)\langle Df, e_1 \rangle &= \langle [\![e_2, Df]\!] + D\langle e_2 \mid Df \rangle \mid e_1 \rangle + \langle Df \mid [\![e_2, e_1]\!] + D\langle e_2 \mid e_1 \rangle \rangle.
\end{aligned}$$

Add the first two equations and subtract the third. After manipulating the expression, we have that

$$\rho\left([\![e_1, e_2]\!]\right) f = \langle Df \mid 4[\![e_1, e_2]\!] \rangle + \langle e_1 \mid 4[\![Df, e_2]\!] \rangle + \langle e_2 \mid 4D\langle Df \mid e_1 \rangle \rangle.$$

Using the definition of $D$, we have that

$$\begin{aligned}
0 &= \rho\left([\![e_1, e_2]\!]\right) f + \langle e_1 \mid 4[\![Df, e_2]\!] \rangle + 2\rho(e_2)\langle e_1 \mid Df \rangle \\
&= \rho\left([\![e_1, e_2]\!]\right) f + \langle e_1 \mid 4[\![Df, e_2]\!] \rangle + \rho(e_2)(\rho(e_1) f) \\
&= \rho(e_1)(\rho(e_2) f) + \langle e_1 \mid 4[\![Df, e_2]\!] \rangle \\
&= \langle e_1 \mid 2D(\rho(e_2) f) + 4[\![Df, e_2]\!] \rangle \\
&= \langle e_1 \mid 4D\langle e_2 \mid Df \rangle + 4[\![Df, e_2]\!] \rangle
\end{aligned}$$

Since $\langle \cdot \mid \cdot \rangle$ is non-degenerate, the statement holds. □

### 2.1.2 Second definition

Later, the definition of Courant algebroid was studied and simplified by (ROYTENBERG, 1999) and (UCHINO, 2002). In the latter, the author shows that the number of axioms in the definition of Courant algebroid can be reduced. Now, we will present those results.

**Proposition 2.1.5** ((UCHINO, 2002), Proposition 1)**.** Let $E \to M$ be a vector bundle equipped with:

- a non-degenerate symmetric bilinear form $\langle \cdot \mid \cdot \rangle$;
- a skew-symmetric bracket $[\![\cdot, \cdot]\!]$;
- a bundle map $\rho\colon E \to TM$;
- a map $D\colon C^\infty(M) \to \Gamma(E)$.

If

(i) for any $e_1, e_2 \in \Gamma(E)$,
$$\rho[\![e_1, e_2]\!] = [\rho(e_1), \rho(e_2)];$$

(ii) for all $f, g \in C^\infty(M)$,
$$D(fg) = fD(g) + gD(f);$$

(iii) for any $e, e_1, e_2 \in \Gamma(E)$,
$$\rho(e)\langle e_1 \mid e_2 \rangle = \langle [\![e, e_1]\!] + D\langle e \mid e_1 \rangle \mid e_2 \rangle + \langle e_1 \mid [\![e, e_2]\!] + D\langle e \mid e_2 \rangle \rangle;$$

then

(iv) for any $e_1, e_2 \in \Gamma(E)$ and $f \in C^\infty(M)$,
$$[\![e_1, fe_2]\!] = f[\![e_1, e_2]\!] + (\rho(e_1)f)e_2 - \langle e_1 \mid e_2 \rangle Df;$$

(v) for any $f, g \in C^\infty(M)$,
$$\langle Df \mid Dg \rangle = 0.$$

*Proof.* First, we will show $(iv)$. Through $(iii)$
$$\rho(e_1)\langle fe_2 \mid e \rangle = \langle [\![e_1, fe_2]\!] + D\langle e_1 \mid fe_2 \rangle \mid e \rangle + \langle fe_2 \mid [\![e_1, e]\!] + D\langle e_1 \mid e \rangle \rangle.$$

On the other hand, applying the Leibniz rule for the vector field, we have that
$$\rho(e_1)\langle fe_2 \mid e \rangle = f\rho(e_1)\langle e_2 \mid e \rangle + \langle e_2 \mid e \rangle \rho(e_1)(f).$$

Applying $(iii)$ on $\rho(e_1)\langle e_2 \mid e \rangle$, we have that
$$\rho(e_1)\langle fe_2 \mid e \rangle = \langle f[\![e_1, e_2]\!] + fD\langle e_1 \mid e_2 \rangle \mid e \rangle + \langle fe_2 \mid [\![e_1, e]\!] + D\langle e_1 \mid e \rangle \rangle + \langle e_2 \mid e \rangle \rho(e_1)(f).$$

Thus,
$$\langle f[\![e_1, e_2]\!] + fD\langle e_1 \mid e_2 \rangle \mid e \rangle + \langle e_2 \mid e \rangle \rho(e_1)(f) = \langle [\![e_1, fe_2]\!] + D\langle e_1 \mid fe_2 \rangle \mid e \rangle.$$

Since the bilinear form $\langle \cdot \mid \cdot \rangle$ is non-degenerate, we have that

$$f[\![e_1,e_2]\!] + fD\langle e_1 \mid e_2\rangle + \rho(e_1)(f)e_2 = [\![e_1, fe_2]\!] + D\langle e_1 \mid fe_2\rangle.$$

Applying $(ii)$, we have that

$$D\langle e_1 \mid fe_2\rangle = fD\langle e_1 \mid e_2\rangle + \langle e_1 \mid e_2\rangle Df.$$

And, finally, we obtained $(iv)$

$$[\![e_1, fe_2]\!] = f[\![e_1,e_2]\!] + \rho(e_1)(f)e_2 - \langle e_1 \mid e_2\rangle Df.$$

Now, we show $(v)$. By $(i)$, we have that

$$\rho[\![e_1, fe_2]\!] = [\rho(e_1), f\rho(e_2)] = f[\rho(e_1),\rho(e_2)] + \rho(e_1)(f)\rho(e_2).$$

Applying $\rho$ in $(iv)$, we have that

$$\langle e_1 \mid e_2\rangle \rho(Df) = 0$$

for all $e_1, e_2 \in \Gamma(E)$ and $f \in C^\infty(f)$. Then, we concluded that

$$\rho \circ D = 0.$$

□

The following proposition gives an expression to the map $D$.

**Proposition 2.1.6** ((UCHINO, 2002), Proposition 2 )**.** Under the conditions of Proposition 2.1.5, we have that

$$\langle Df \mid e\rangle = \frac{1}{2}\rho(e)(f) \tag{2.4}$$

for all $e \in \Gamma(E)$ and $f \in C^\infty(M)$.

*Proof.* For $e = 0$, the identity holds. Let us prove it for $e \neq 0$. By $(iii)$, we have that

$$\rho(fe_1)\langle e \mid e\rangle = \langle [\![fe_1,e]\!] + D\langle fe_1 \mid e\rangle \mid e\rangle + \langle [\![fe_1,e]\!] + D\langle fe_1 \mid e\rangle \mid e\rangle.$$

Applying $(iv)$ and $(ii)$, we have

$$\begin{aligned}\rho(fe_1)\langle e \mid e\rangle =& 2\langle f[\![e_1,e]\!] - \rho(e)(f)e_1 + \langle e_1 \mid e\rangle Df + fD\langle e_1 \mid e\rangle + \langle e_1 \mid e\rangle Df \mid e\rangle \\ =& 2f\langle [\![e_1,e]\!] + D\langle e_1 \mid e\rangle \mid e\rangle - 2\langle \rho(e)(f)e_1 - 2\langle e_1 \mid e\rangle Df \mid e\rangle.\end{aligned}$$

And, applying $(iii)$ again, we got that

$$\rho(fe_1)\langle e \mid e\rangle = f\rho(e_1)\langle e \mid e\rangle - 2\langle \rho(e)(f)e_1 - 2\langle e_1 \mid e\rangle Df \mid e\rangle.$$

Thus

$$0 = \langle -\rho(e)(f)e_1 + 2\langle e_1 \mid e\rangle Df \mid e\rangle = -\rho(e)(f)\langle e_1 \mid e\rangle + 2\langle e_1 \mid e\rangle\langle Df \mid e\rangle.$$

Since $\langle\cdot\mid\cdot\rangle$ is non-degenerate, we have that

$$(-\rho(e)(f)+2\langle Df\mid e\rangle)\langle e_1\mid e\rangle=0 \implies \langle Df\mid e\rangle=\frac{1}{2}\rho(e)(f)$$

□

Note that if we define $D$ by (2.1), then $D$ satisfies $(ii)$. Thus, through the work of (UCHINO, 2002), Definition 2.1.2 is equivalent to the following.

**Definition 2.1.7.** A Courant algebroid is a vector bundle $E\to M$ endowed with a non-degenerate symmetric bilinear form $\langle\cdot\mid\cdot\rangle$, a skew-symmetric bracket $[\![\cdot,\cdot]\!]$ and a bundle map $\rho: E\to TM$ satisfying the following properties:

(C1) For any $e_1,e_2,e_3\in\Gamma(E)$,

$$J(e_1,e_2,e_3)=DT(e_1,e_2,e_3);$$

(C2) for any $e_1,e_2\in\Gamma(E)$,

$$\rho[\![e_1,e_2]\!]=[\rho(e_1),\rho(e_2)];$$

(C3) for any $e,e_1,e_2\in\Gamma(E)$,

$$\rho(e)\langle e_1\mid e_2\rangle=\langle[\![e,e_1]\!]+D\langle e\mid e_1\rangle\mid e_2\rangle+\langle e_1\mid[\![e,e_2]\!]+D\langle e\mid e_2\rangle\rangle,$$

where $J$ is given by Equation (2.2) and $D$ is given by Equation (2.1).

### 2.1.3 Third definition

One notable fact about the definition of Courant algebroid is that the bracket $[\![\cdot,\cdot]\!]$ do not satisfy the Jacobi identity. In (LIU; WEINSTEIN; XU, 1997), the authors propose a new bracket defined by

$$[\![e_1,e_2]\!]^J=[\![e_1,e_2]\!]+D\langle e_1\mid e_2\rangle. \tag{2.5}$$

This new bracket satisfies the Jacoby identity, but the price to pay is that it is not skew-symmetric. In (ROYTENBERG, 1999), another equivalent definition of Courant algebroid is presented in terms of the bracket $[\![\cdot,\cdot]\!]^J$. Now, we will use the ideas of (UCHINO, 2002), presented in Section 2.1.2, and (ROYTENBERG, 1999) to show the equivalence between Definition 2.1.2 and the one presented in (KOSMANN-SCHWARZBACH, 2005).

**Definition 2.1.8** ((KOSMANN-SCHWARZBACH, 2005), Definition 2.1)**.** A Courant algebroid is a vector bundle $E\to M$ together with a non-degenerate symmetric bilinear form $\langle\cdot\mid\cdot\rangle$, a bilinear operation $[\![\cdot,\cdot]\!]^J$, and a bundle map $\rho: E\to TM$ satisfying the following properties:

(J1) For any $e_1, e_2, e_3 \in \Gamma(E)$,

$$[\![e_1, [\![e_2, e_3]\!]^J]\!]^J = [\![[\![e_1, e_2]\!]^J, e_3]\!]^J + [\![e_2, [\![e_1, e_3]\!]^J]\!]^J;$$

(J2) for any $e, e_1, e_2 \in \Gamma(E)$,

$$\rho(e)\langle e_1 \mid e_2 \rangle = \langle e \mid [\![e_1, e_2]\!]^J + [\![e_2, e_1]\!]^J \rangle;$$

(J3) for any $e, e_1, e_2 \in \Gamma(E)$,

$$\rho(e)\langle e_1 \mid e_2 \rangle = \langle [\![e, e_1]\!]^J \mid e_2 \rangle + \langle e_1 \mid [\![e, e_2]\!]^J \rangle.$$

**Remark 2.1.9.** Note that Property (J2) is equivalent to

$$2D\langle e_1 \mid e_2 \rangle = [\![e_1, e_2]\!]^J + [\![e_2, e_1]\!]^J. \tag{2.6}$$

Indeed, taking the bilinear form $\langle \cdot \mid \cdot \rangle$ between the above equation and an arbitrary $e \in \Gamma(E)$, we can recover (J2).

The following lemma gives us the relation between the brackets introduced in Definition 2.1.8 and 2.1.2.

**Lemma 2.1.10.** Let $[\![\cdot, \cdot]\!]^J$ satisfy

(J2) $\rho(e)\langle e_1 \mid e_2 \rangle = \langle e \mid [\![e_1, e_2]\!]^J + [\![e_2, e_1]\!]^J \rangle$;

Then,

$$[\![e_1, e_2]\!]^J = [\![e_1, e_2]\!] + D\langle e_1 \mid e_2 \rangle, \tag{2.7}$$

where

$$[\![e_1, e_2]\!] = \frac{1}{2}\left([\![e_1, e_2]\!]^J - [\![e_2, e_1]\!]^J\right). \tag{2.8}$$

In other words, the symmetric part of $[\![e_1, e_2]\!]^J$ is $D\langle e_1 \mid e_2 \rangle$.

*Proof.* The idea is the same as polarization identity. Equation (2.6) implies

$$2[\![e_1, e_2]\!]^J = 2D\langle e_1 \mid e_2 \rangle + \left([\![e_1, e_2]\!]^J - [\![e_2, e_1]\!]^J\right).$$

Thus

$$[\![e_1, e_2]\!]^J = D\langle e_1 \mid e_2 \rangle + \frac{1}{2}\left([\![e_1, e_2]\!]^J - [\![e_2, e_1]\!]^J\right).$$

□

Similar to what was done in the work of (UCHINO, 2002) for the first definition, (KOSMANN-SCHWARZBACH, 2005) simplifies the definition of (ROYTENBERG, 1999).

**Proposition 2.1.11.** The following conditions

(J1) For any $e_1, e_2, e_3 \in \Gamma(E)$,

$$[\![e_1, [\![e_2, e_3]\!]^J]\!]^J = [\![[\![e_1, e_2]\!]^J, e_3]\!]^J + [\![e_2, [\![e_1, e_3]\!]^J]\!]^J;$$

(J3) for all $e, e_1, e_2 \in \Gamma(E)$,

$$\rho(e)\langle e_1 \mid e_2\rangle = \langle [\![e, e_1]\!]^J \mid e_2\rangle + \langle e_1 \mid [\![e, e_2]\!]^J\rangle$$

imply

(J4) for any $e_1, e_2 \in \Gamma(E)$,

$$\rho[\![e_1, e_2]\!]^J = [\rho(e_1), \rho(e_2)];$$

(J5) for any $e_1, e_2 \in \Gamma(E)$ and $f \in C^\infty(M)$,

$$[\![e_1, fe_2]\!]^J = f[\![e_1, e_2]\!]^J + (\rho(e_1)f)e_2.$$

*Proof.* First, we will prove the property (J5). By the Leibniz rule for vector fields, we have that

$$\rho(e_1)\langle fe_2 \mid e\rangle = (\rho(e_1)f)\langle e_2 \mid e\rangle + f(\rho(e_1)\langle e_2 \mid e\rangle).$$

Using the property (J3), we have that

$$\langle [\![e_1, fe_2]\!]^J \mid e\rangle + \langle fe_2 \mid [\![e_1, e]\!]^J\rangle = (\rho(e_1)f)\langle e_2 \mid e\rangle + f\langle [\![e_1, e_2]\!]^J \mid e\rangle + f\langle e_2 \mid [\![e_1, e]\!]^J\rangle.$$

Thus,

$$\langle [\![e_1, fe_2]\!]^J \mid e\rangle = (\rho(e_1)f)\langle e_2 \mid e\rangle + f\langle [\![e_1, e_2]\!]^J \mid e\rangle.$$

Since $\langle \cdot \mid \cdot \rangle$ is non-degenerate, we have the property (J5).

For the property (J4), by (J1), we have that

$$[\![e_1, [\![e_2, fe]\!]^J]\!]^J = [\![[\![e_1, e_2]\!]^J, fe]\!]^J + [\![e_2, [\![e_1, fe]\!]^J]\!]^J.$$

Using the property (J5), we have that

$$[\![e_1, [\![e_2, fe]\!]^J]\!]^J = f[\![e_1, [\![e_2, e]\!]^J]\!]^J + (\rho(e_1)f)[\![e_2, e]\!]^J + (\rho(e_2)f)[\![e_1, e]\!]^J + (\rho(e_1)(\rho(e_2)f))e$$
$$[\![[\![e_1, e_2]\!]^J, fe]\!]^J = f[\![[\![e_1, e_2]\!]^J, e]\!]^J + (\rho([\![e_1, e_2]\!]^J)f)e$$
$$[\![e_2, [\![e_1, fe]\!]^J]\!]^J = f[\![e_2, [\![e_1, e]\!]^J]\!]^J + (\rho(e_2)f)[\![e_1, e]\!]^J + (\rho(e_1)f)[\![e_2, e]\!]^J + (\rho(e_2)(\rho(e_1)f))e$$

Using again (J1), we have that

$$(\rho([\![e_1, e_2]\!]^J)f)e = (\rho(e_1)(\rho(e_2)f))e - (\rho(e_2)(\rho(e_1)f))e.$$

Thus, we have the property (J4). □

Similar to Lemma 2.1.4, (ROYTENBERG, 1999) shows that the set of sections of E which are $Df$ for some function forms an ideal of the algebra $[\![\cdot,\cdot]\!]^J$.

**Lemma 2.1.12.** If $(E, \langle\cdot\mid\cdot\rangle, [\![\cdot,\cdot]\!]^J, \rho)$ satisfies

(J3) for all $e, e_1, e_2 \in \Gamma(E)$,

$$\rho(e)\langle e_1 \mid e_2\rangle = \langle [\![e, e_1]\!]^J \mid e_2\rangle + \langle e_1 \mid [\![e, e_2]\!]^J\rangle;$$

(J4) for any $e_1, e_2 \in \Gamma(E)$,

$$\rho[\![e_1, e_2]\!] = [\rho(e_1), \rho(e_2)],$$

then, for all $e \in \Gamma(E)$ and $f \in C^\infty(M)$, we have that

$$[\![e, Df]\!]^J = 2D\langle e \mid Df\rangle$$
$$[\![Df, e]\!]^J = 0.$$

*Proof.* Given $e_1, e_2 \in \Gamma(E)$, by the definition of $D$ and (J3), we have that

$$\begin{aligned}\rho(e_1)(\rho(e_2)f) = 2\rho(e_1)\langle Df \mid e_2\rangle &= 2\langle [\![e_1, Df]\!]^J \mid e_2\rangle + 2\langle Df \mid [\![e_1, e_2]\!]^J\rangle \\ &= 2\langle [\![e_1, Df]\!]^J \mid e_2\rangle + \rho\left([\![e_1, e_2]\!]^J\right) f,\end{aligned}$$

by (J4), we have that

$$\rho\left([\![e_1, e_2]\!]^J\right) f = [\rho(e_1), \rho(e_2)](f) = \rho(e_1)(\rho(e_2)f) - \rho(e_2)(\rho(e_1)f).$$

Thus,

$$\langle [\![e_1, Df]\!]^J \mid e_2\rangle = \frac{1}{2}\rho(e_2)(\rho(e_1)f) = 2\langle e_2 \mid D\langle e_1 \mid Df\rangle\rangle.$$

Since $\langle\cdot\mid\cdot\rangle$ is non-degenerate, we have

$$[\![e, Df]\!]^J = 2D\langle e \mid Df\rangle.$$

On the other hand,

$$\begin{aligned}[\![Df, e]\!]^J &= [\![Df, e]\!]^J + [\![e, Df]\!]^J - [\![e, Df]\!]^J \\ &= [\![Df, e]\!] + D\langle Df \mid e\rangle + [\![e, Df]\!] + D\langle e \mid Df\rangle - [\![e, Df]\!]^J.\end{aligned}$$

Since $[\![e, Df]\!]^J = 2D\langle e \mid Df\rangle$, we have that

$$[\![Df, e]\!]^J = 2D\langle e \mid Df\rangle - 2D\langle e \mid Df\rangle = 0.$$

□

The following lemma will be used to show the equivalence between Property (1) for Definition 2.1.2 and Property (J1) of Definition 2.1.8.

**Lemma 2.1.13.** If $(E, \langle \cdot \mid \cdot \rangle, [\![\cdot,\cdot]\!]^J, \rho)$ satisfies

(J2) for any $e, e_1, e_2 \in \Gamma(E)$,

$$\rho(e)\langle e_1 \mid e_2 \rangle = \langle e \mid [\![e_1, e_2]\!]^J + [\![e_2, e_1]\!]^J \rangle;$$

(J3 ) for any $e, e_1, e_2 \in \Gamma(E)$,

$$\rho(e)\langle e_1 \mid e_2 \rangle = \langle [\![e, e_1]\!]^J \mid e_2 \rangle + \langle e_1 \mid [\![e, e_2]\!]^J \rangle,$$

then,

$$K(e_1, e_2, e_3) = [\![[\![e_1, e_2]\!]^J, e_3]\!]^J + [\![e_2, [\![e_1, e_3]\!]^J]\!]^J - [\![e_1, [\![e_2, e_3]\!]^J]\!]^J$$

is completely skew-symmetric.

*Proof.* We will show that if two entries coincide, then $K$ vanishes. Indeed, first note that Equation (2.5) implies that

$$[\![e, e]\!]^J = D\langle e \mid e \rangle,$$

thus

$$K(e_1, e_1, e_3) = [\![[\![e_1, e_1]\!]^J, e_3]\!]^J + [\![e_1, [\![e_1, e_3]\!]^J]\!]^J - [\![e_1, [\![e_1, e_3]\!]^J]\!]^J = [\![D\langle e_1 \mid e_1 \rangle, e_3]\!]^J.$$

Through Lemma 2.1.12 we have

$$K(e_1, e_1, e_3) = 0.$$

On the other hand, by (J2)

$$\begin{aligned} K(e_1, e_2, e_2) &= [\![[\![e_1, e_2]\!]^J, e_2]\!]^J + [\![e_2, [\![e_1, e_2]\!]^J]\!]^J - [\![e_1, [\![e_2, e_2]\!]^J]\!]^J \\ &= 2D\left(\langle [\![e_1, e_2]\!]^J \mid e_2 \rangle\right) - [\![e_1, [\![e_2, e_2]\!]^J]\!]^J. \end{aligned}$$

Through Lemma 2.1.12 we have

$$[\![e_1, [\![e_2, e_2]\!]^J]\!]^J = [\![e_1, D\langle e_2 \mid e_2 \rangle]\!]^J = 2D\langle e_1 \mid D\langle e_2 \mid e_2 \rangle\rangle = \rho(e_1)(\langle e_2 \mid e_2 \rangle),$$

and by $(J2)$

$$\rho(e_1)(\langle e_2 \mid e_2 \rangle) = 2D\left(\langle [\![e_1, e_2]\!]^J \mid e_2 \rangle\right).$$

Thus

$$K(e_1, e_2, e_2) = 2D\left(\langle [\![e_1, e_2]\!]^J \mid e_2 \rangle\right) - 2D\left(\langle [\![e_1, e_2]\!]^J \mid e_2 \rangle\right) = 0.$$

And finally, using again $(J2)$, $(J3)$ and Lemma 2.1.12, we have that

$$\begin{aligned}
K(e_1,e_2,e_1) &= [\![[\![e_1,e_2]\!]^J,e_1]\!]^J + [\![e_2,[\![e_1,e_1]\!]^J]\!]^J - [\![e_1,[\![e_2,e_1]\!]^J]\!]^J \\
&= [\![[\![e_1,e_2]\!]^J,e_1]\!]^J + [\![e_2,[\![e_1,e_1]\!]^J]\!]^J - [\![e_1,[\![e_2,e_1]\!]^J]\!]^J \\
&\quad + [\![[\![e_2,e_1]\!]^J,e_1]\!]^J - [\![[\![e_2,e_1]\!]^J,e_1]\!]^J \\
&= 2[\![D\langle e_1 \mid e_2\rangle,e_1]\!]^J - 2D\langle [\![e_2,e_1]\!]^J \mid e_1\rangle + 2D\langle e_2 \mid [\![e_1,e_1]\!]^J\rangle \\
&= -2D\left(\langle [\![e_2,e_1]\!]^J \mid e_1\rangle - \langle e_2 \mid [\![e_1,e_1]\!]^J\rangle\right) \\
&= -2D\left(\langle [\![e_2,e_1]\!]^J \mid e_1\rangle + \langle [\![e_1,e_2]\!]^J \mid e_1\rangle - 2\langle D\langle e_2 \mid e_1\rangle \mid e_1\rangle\right) \\
&= -2D\left(2\langle D\langle e_2 \mid e_1\rangle \mid e_1\rangle - 2\langle D\langle e_2 \mid e_1\rangle \mid e_1\rangle\right) \\
&= 0
\end{aligned}$$

□

We can state the equivalence between Property (1) and Property (J1) in the following way:

**Proposition 2.1.14.** Let $E \to M$ be a vector bundle endowed with a non-degenerate symmetric bilinear form $\langle \cdot \mid \cdot \rangle$ and a bundle map $\rho : E \to TM$. Let $[\![\cdot,\cdot]\!]$, satisfying Properties 2-5, and $[\![\cdot,\cdot]\!]^J$, satisfying Properties (J2) and $(J3)$, be two brackets on the sections of $E$ related by Lemma 2.1.3. Then $[\![\cdot,\cdot]\!]$ satisfies Property (1) if and only if $[\![\cdot,\cdot]\!]^J$ satisfies Property (J1).

*Proof.* By Equation (2.7), we have that

$$H(e_1,e_2,e_3) = J(e_1,e_2,e_3) + R(e_1,e_2,e_3),$$

where

$$\begin{aligned}
R(e_1,e_2,e_3) =& \frac{1}{2}\left([\![D\langle e_1 \mid e_2\rangle,e_3]\!] + [\![e_2,D\langle e_1 \mid e_3\rangle]\!] - [\![e_1,D\langle e_2 \mid e_3\rangle]\!]\right) \\
&+ \frac{1}{2}D\left(\langle [\![e_1,e_2]\!]^J \mid e_3\rangle + \langle e_2 \mid [\![e_1,e_3]\!]^J\rangle - \langle e_1 \mid [\![e_2,e_3]\!]^J\rangle\right).
\end{aligned}$$

But, applying Lemma 2.1.4, we have

$$\begin{aligned}
&\frac{1}{2}\left([\![D\langle e_1 \mid e_2\rangle,e_3]\!] + [\![e_2,D\langle e_1 \mid e_3\rangle]\!] - [\![e_1,D\langle e_2 \mid e_3\rangle]\!]\right) \\
&\qquad = -\frac{1}{2}D\left(\langle D\langle e_1 \mid e_2\rangle \mid e_3\rangle - \langle e_2 \mid D\langle e_1 \mid e_3\rangle\rangle + \langle e_1 \mid D\langle e_2 \mid e_3\rangle\rangle\right),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}D\left(\langle [\![e_1,e_2]\!]^J \mid e_3\rangle + \langle e_2 \mid [\![e_1,e_3]\!]^J\rangle - \langle e_1 \mid [\![e_2,e_3]\!]^J\rangle\right) \\
&\qquad = \frac{1}{2}D\left(\langle [\![e_1,e_2]\!] \mid e_3\rangle + \langle e_2 \mid [\![e_1,e_3]\!]\rangle - \langle e_1 \mid [\![e_2,e_3]\!]\rangle\right) \\
&\qquad\quad + \frac{1}{2}D\left(\langle D\langle e_1 \mid e_2\rangle \mid e_3\rangle + \langle e_2 \mid D\langle e_1 \mid e_3\rangle\rangle - \langle e_1 \mid D\langle e_2 \mid e_3\rangle\rangle\right),
\end{aligned}$$

then

$$R(e_1,e_2,e_3) = \frac{1}{2}D\left(\langle [\![e_1,e_2]\!] \mid e_3\rangle - \langle [\![e_3,e_1]\!] \mid e_2\rangle - \langle [\![e_2,e_3]\!] \mid e_1\rangle\right) \\ + D\left(\langle e_2 \mid D\langle e_1 \mid e_3\rangle\rangle - \langle e_1 \mid D\langle e_2 \mid e_3\rangle\rangle\right).$$

Since $J$ and $H$ are completely skew-symmetric, so is $R$. The result follows since the skew-symmetrization of $R$ is equal to $-DT(e_1,e_2,e_3)$. □

**Proposition 2.1.15.** Let $(E,\langle\cdot\mid\cdot\rangle,[\![\cdot,\cdot]\!],\rho)$ be a Courant algebroid in the sense of Definition 2.1.2. Then $(E,\langle\cdot\mid\cdot\rangle,[\![\cdot,\cdot]\!]^J,\rho)$ is a Courant algebroid in the sense of Definition 2.1.8, where $[\![\cdot,\cdot]\!]^J$ is given by Equation (2.5).

Conversely, if $(E,\langle\cdot\mid\cdot\rangle,[\![\cdot,\cdot]\!]^J,\rho)$ is a Courant algebroid in the sense of Definition 2.1.8, then $(E,\langle\cdot\mid\cdot\rangle,[\![\cdot,\cdot]\!],\rho)$ is a Courant algebroid in the sense of Definition 2.1.2, where $[\![\cdot,\cdot]\!]$ is given by Equation (2.8).

*Proof.* Suppose that $(E,\langle\cdot\mid\cdot\rangle,[\![\cdot,\cdot]\!]^J,\rho)$ is a Courant algebroid in the sense of Definition 2.1.8. The condition (5) of Definition 2.1.2 is clearly equivalent to $(J3)$, and, by Proposition 2.1.14, we have the equivalence between $(J1)$ and (1). Now, for Property (2), we have that

$$\begin{aligned}\rho[\![e_1,e_2]\!] &= \frac{1}{2}\rho\left([\![e_1,e_2]\!]^J - [\![e_2,e_1]\!]^J\right)\\ &= \frac{1}{2}\left(\rho[\![e_1,e_2]\!]^J - \rho[\![e_2,e_1]\!]^J\right)\\ &= \frac{1}{2}\left([\rho(e_1),\rho(e_2)] - [\rho(e_2),\rho(e_1)]\right)\\ &= [\rho(e_1),\rho(e_2)].\end{aligned}$$

Thus, Property $(J4)$ implies Property (2), and the results follows from Proposition 2.1.5.

On the other hand, suppose that $(E,\langle\cdot\mid\cdot\rangle,[\![\cdot,\cdot]\!],\rho)$ is a Courant algebroid in the sense of Definition 2.1.2. Again, we have that Properties $(J1)$ and $(J3)$ hold. By Equation (2.7), we have that

$$\begin{aligned}[\![e_1,e_2]\!]^J &= [\![e_1,e_2]\!] + D\langle e_1 \mid e_2\rangle\\ [\![e_2,e_1]\!]^J &= [\![e_2,e_1]\!] + D\langle e_1 \mid e_2\rangle,\end{aligned}$$

thus, we have that

$$2D\langle e_1 \mid e_2\rangle = [\![e_1,e_2]\!]^J + [\![e_2,e_1]\!]^J.$$

□

**Example 2.1.16.** For the Example 2.1.3, we have that the new bracket is given by

$$[\![X\oplus\alpha, Y\oplus\beta]\!]^J = [X,Y]\oplus\left(\mathscr{L}_X(\beta) - i_Y d\alpha\right).$$

## 2.2 Lie bialgebroids

Another important instance of Courant algebroid, which has numerous applications in the study of Poisson-Nijenhuis manifolds and Dirac structures, is the one defined by the double of a Lie bialgebroid. A Lie bialgebroid is a framework introduced by (MACKENZIE; XU, 1994) that extends the concept of Lie bialgebra. It consists of two Lie algebroids, denoted by $A$ and $A^*$, over a manifold $M$, satisfying some suitable compatibility conditions.

**Remark 2.2.1.** In this section, we will use results from the theory of differential calculus on Lie algebroids. Details can be found in the Appendix A.

**Definition 2.2.2.** Let $(A,[\cdot,\cdot]_A,\rho_A)$ be a Lie algebroid and let $d_{A^*}$ be a degree 1 derivation of $(\Gamma(\wedge^\bullet A),\wedge)$. We say that $(A,d_{A^*})$ is a Lie bialgebroid if the following equations hold

$$d_{A^*}[P,Q]_A = [d_{A^*}P,Q]_A + (-1)^{p-1}[P,d_{A^*}Q]_A \tag{2.9}$$

$$d_{A^*}^2 = 0 \tag{2.10}$$

for all $P \in \wedge^p A$ and $Q \in \wedge^\bullet A$.

**Remark 2.2.3.** Algebraically, a Lie bialgebroid corresponds to a differential Gerstenhaber algebra. See Appendix A.

**Remark 2.2.4.** Exploring the relation between Lie algebroid structures and differential operators $d_{A^*}$ satisfying $d_{A^*}^2 = 0$, a Lie bialgebroid can be alternatively defined requiring two Lie algebroids $(A,[\cdot,\cdot]_A,\rho_A)$ and $(A^*,[\cdot,\cdot]_{A^*},\rho_{A^*})$ that are compatible in the sense of Equation (2.9).

A simple example of Lie bialgebroid is provided by the pair $(TM,T^*M)$, where the tangent bundle carries its canonical structure of Lie algebroid, i.e., $(TM,Id,[\cdot,\cdot])$, while $T^*M$ carries the trivial Lie algebroid structure, i.e., the one with zero anchor and trivial bracket. A less trivial example of Lie bialgebroid is provided again by the pair $(TM,T^*M)$, where $M$ is a PN manifold, in the sense of (MAGRI; MOROSI; RAGNISCO, 1985; KOSMANN-SCHWARZBACH; MAGRI, 1990).

**Definition 2.2.5.** Let $N\colon TM \to TM$ be a (1,1)-tensor on a manifold $M$. We define the Nijehuis torsion of $N$ by

$$T_N(X,Y) = [NX,NY] - N\left([NX,Y] + [X,NY] - N[X,Y]\right).$$

We say that $N$ is a Nijenhuis operator, or a Nijenhuis tensor, if $T_N = 0$.

**Definition 2.2.6.** Let $\pi$ be a Poisson bivector and $N$ be a Nijenhuis operator. We call the triple $(M,\pi,N)$ a Poisson-Nijenhuis manifold, PN manifold from now on, if $\pi$ and $N$ are compatible, that is, for all $\alpha,\beta \in \Gamma(T^*M)$

$$N\pi^\sharp = \pi^\sharp N^*, \qquad [\alpha,\beta]_{N\pi} = [N^*\alpha,\beta]_\pi + [\alpha,N^*\beta]_\pi - N[\alpha,\beta]_\pi. \tag{2.11}$$

**Lemma 2.2.7** ((KOSMANN-SCHWARZBACH, 1996), Proposition 3.2)**.** Assume that $\pi \in \wedge^2 TM$ is a Poisson tensor and $N\colon TM \to TM$ a (1,1)-tensor on $M$. The differential $d_N$ is a derivation of the graded Lie algebra $(\wedge^\bullet T^*M, [\cdot,\cdot]_\pi)$ if, and only if, $\pi$ and $N$ are compatible.

*Proof.* We must to show that, given $\sigma_1 \in \wedge^p T^*M$ and $\sigma_2 \in \wedge^\bullet T^*M$,

$$A(\sigma_1,\sigma_2) = d_N[\sigma_1,\sigma_2]_\pi - [d_N\sigma_1,\sigma_2]_\pi - (-1)^{p+1}[\sigma_1, d_N\sigma_2]_\pi = 0.$$

For $f,g \in C^\infty(M)$,

$$\begin{aligned} A(f,g) =& - [N^*df,g]_\pi + [f,N^*dg]_\pi = -\langle df, \pi^\sharp N^*df\rangle - \langle df, \pi^\sharp N^*dg\rangle \\ =& \langle df, (N\pi^\sharp - \pi^\sharp N^*)dg\rangle. \end{aligned}$$

Thus $A(f,g) = 0$ is equivalent to $N\pi^\sharp dg = \pi^\sharp N^* dg$ . Now, we compute $A(df,g)$. To this aim, we use the fact that $d$ is a derivation of $[\cdot,\cdot]_\pi$.

$$\begin{aligned} A(df,g) =& d_N[df,g]_\pi - [d_N df,g]_\pi + [df,d_N g]_\pi \\ =& N^*[df,dg]_\pi + [d(N^*df),g]_\pi - [df,N^*dg]_\pi \\ =& N^*[df,dg]_\pi - [N^*df,dg]_\pi - [df,N^*dg]_\pi + d[N^*df,g]_\pi \\ =& C(\pi,N)(df,dg) - [df,dg]_{NP} + d\langle dg, \pi^\sharp N^*df\rangle \\ =& C(\pi,N)(df,dg) - d\langle dg, (N\pi^\sharp - \pi^\sharp N^*)df\rangle \\ =& C(\pi,N)(df,dg) + d(A(f,g)), \end{aligned}$$

where $C(\pi,N)(df,dg) = [\alpha,\beta]_\pi - ([N^*\alpha,\beta]_\pi + [\alpha,N^*\beta]_\pi - N[\alpha,\beta]_\pi)$. Thus $A(df,g) = 0$ is equivalent to 2.11 evaluated on exact 1-forms. Finally, we compute

$$\begin{aligned} A(df,dg) &= d_N[df,dg]_\pi - [d_N df,dg]_\pi - [df,d_N dg]_\pi \\ &= d(N^*[df,dg]_\pi - [d(N^*df),dg]_\pi - [df,dN^*dg]_\pi. \\ &= -d(N^*[df,dg]_\pi) + d[N^*df,dg]_\pi + d[df,N^*dg]_\pi \\ &= -d(C(\pi,N)(df,dg)) - d[df,dg]_{N\pi} \\ &= -d(C(\pi,N)(df,dg)). \end{aligned}$$

□

Then, we have the following relation between PN manifolds and Lie biagebroids.

**Theorem 2.2.8.** Let $N$ be a Nijenhuis operator and $\pi$ be a Poisson bivector field such that $N\pi^\sharp = \pi^\sharp N^*$. Then, $(TM,N,[\cdot,\cdot]_N)$ and $(T^*M,\pi,[\cdot,\cdot]_\pi)$ form a Lie bialgebroid if and only if $(M,\pi,N)$ is a PN manifold.

Let $(A,A^*)$ be a Lie bialgebroid It is possible to define a Courant algebroid structure on $E = A \oplus A^*$. We start defining two non-degenerate bilinear forms on $E$ by

$$\langle X_1 \oplus \xi_1 \mid X_2 \oplus \xi_2\rangle_+ = (\langle \xi_1, X_2\rangle + \langle \xi_2, X_1\rangle),$$

$$\langle X_1 \oplus \xi_1 \mid X_2 \oplus \xi_2 \rangle_- = (\langle \xi_1, X_2 \rangle - \langle \xi_2, X_1 \rangle),$$

and a bracket on $\Gamma(E)$ as follows:

$$\begin{aligned}\llbracket X_1 \oplus \xi_1, X_2 \oplus \xi_2 \rrbracket = &\left( [X_1, X_2]_A + \mathscr{L}_{\xi_1}(X_2) - \mathscr{L}_{\xi_2}(X_1) - \frac{1}{2} d_{A*}(\langle X_1 \oplus \xi_1 \mid X_2 \oplus \xi_2 \rangle_- \right) \\ &\oplus \left( [\xi_1, \xi_2]_{A^*} + \mathscr{L}_{X_1}(\xi_2) - \mathscr{L}_{X_2}(\xi_1) + \frac{1}{2} d_A(\langle X_1 \oplus \xi_1 \mid X_2 \oplus \xi_2 \rangle_- \right).\end{aligned} \tag{2.12}$$

Finally, let $\rho \colon E \to TM$ be the bundle map defined by

$$\rho(X \oplus \xi) = \rho_A(X) + \rho_{A^*}(\xi)$$

and $D \colon C^\infty(M) \to E$ defined by

$$D = d_{A^*} \oplus d_A.$$

Without assuming that the pair $(A, A^*)$ is a Lie bialgebroid, the author computes the following expression for the Jacobiator associated with the bracket defined in 2.12, see Theorem 3 of (LIU; WEINSTEIN; XU, 1997).

**Proposition 2.2.9.** Let $(A, [\cdot,\cdot]_A, \rho_A)$ and $(A^*, [\cdot,\cdot]_{A^*}, \rho_{A^*})$ be both Lie algebroids. Then, for every $e_i = X_i \oplus \alpha_i \in \Gamma(A \oplus A^*)$, $i = 1,2,3$, we have that

$$J(e_1, e_2, e_3) = DT(e_1, e_2, e_3) - (J_1 + J_2 + \text{c.p}),$$

where

$$\begin{aligned} J_1 = &i_{\alpha_3}\left( d_{A^*}[X_1, X_2]_A - \mathscr{L}^A_{X_1}(d_{A^*}X_2) + \mathscr{L}^A_{X_2}(d_{A^*}X_1) \right) \\ &\oplus i_{X_3}\left( d_A[\alpha_1, \alpha_2]_{A^*} - \mathscr{L}^{A^*}_{\alpha_1}(d_A\alpha_2) + \mathscr{L}^{A^*}_{\alpha_2}(d_A\alpha_1) \right), \end{aligned}$$

$$\begin{aligned} J_2 = &\mathscr{L}^A_{d_A\langle e_1 \mid e_2 \rangle_-}(X_3) + [d_{A^*}\langle e_1 \mid e_2 \rangle_-, X_3]_A \\ &\oplus \mathscr{L}^A_{d_{A^*}\langle e_1 \mid e_2 \rangle_-}(\alpha_3) + [d_A\langle e_1 \mid e_2 \rangle_-, \alpha_3]_{A^*} \end{aligned}$$

and c.p means the cyclic permutations.

In (LIU; WEINSTEIN; XU, 1997), it was proved that the double of a Lie bialgebroid, together with the above structure, is a Courant algebroid.

**Theorem 2.2.10.** If $(A, A^*)$ is a Lie bialgebroid, then $E = A \oplus A^*$ together with $(\llbracket \cdot,\cdot \rrbracket, \rho, \langle \cdot \mid \cdot \rangle_+)$ is a Courant algebroid.

This statement can be proven by ensuring the five properties described in Definition 2.1.2 hold true. To make this process more organized, we have decided to draft the demonstration of each of these five properties as a separate proposition. Our approach will follow the demonstrations presented in (LIU; WEINSTEIN; XU, 1997).

Regarding Property (1), it is proved after straightforward computations that if $(A,A^*)$ is a Lie bialgebroid, then $J_1+J_2+\text{c.p.}=0$. Thus, the results follow from Proposition 2.2.9. The next proposition demonstrates that Property (2) is valid.

**Proposition 2.2.11.** For any $f\in C^\infty(M)$ and $e_1,e_2\in\Gamma(A\oplus A^*)$, we have that

$$[\![e_1,fe_2]\!]=f[\![e_1,e_2]\!]+(\rho(e_1)f)e_2-\langle e_1\mid e_2\rangle Df$$

*Proof.* Since $e_1=X_1\oplus\alpha_1$ and $e_2=X_2\oplus\alpha_2$ for $X_1,X_2\in\Gamma(A)$ and $\alpha_1,\alpha_2\in\Gamma(A^*)$, we have that

$$\begin{aligned}
[\![X_1,fX_2]\!]&=f[X_1,X_2]_A+(\rho_A(X_1)f)X_2;\\
[\![X_1,f\alpha_2]\!]&=f[\![X_1,\alpha_2]\!]+(\rho_A(X_1)f)\alpha_2-\frac{1}{2}\langle X_1,\alpha_2\rangle Df;\\
[\![\alpha_1,fX_2]\!]&=f[\![\alpha_1,X_2]\!]+(\rho_{A^*}(\alpha_1)f)X_2-\frac{1}{2}\langle X_2,\alpha_1\rangle Df;\\
[\![\alpha_1,f\alpha_2]\!]&=f[\alpha_1,\alpha_2]_{A^*}+(\rho_{A^*}(\alpha_1)f)\alpha_2.
\end{aligned}$$

The result follows using the bilinearity of $[\![\cdot,\cdot]\!]$. □

The following proposition shows that Property (3) holds.

**Proposition 2.2.12.** For any $e_1,e_2\in\Gamma(A\oplus A^*)$, we have

$$\rho[\![e_1,e_2]\!]=[\![\rho(e_1),\rho(e_2)]\!].$$

*Proof.* Let $e_1=X_1\oplus\alpha_1$ and $e_2=X_2\oplus\alpha_2$ for $X_1,X_2\in\Gamma(A)$ and $\alpha_1,\alpha_2\in\Gamma(A^*)$.

$$\begin{aligned}
\rho[\![e_1,e_2]\!]=\rho_A&\left([X_1,X_2]_A+\mathscr{L}^{A^*}_{\alpha_1}(X_2)-\mathscr{L}^{A^*}_{\alpha_2}(X_1)-d_{A^*}\langle e_1\mid e_2\rangle_-\right)\\
&+\rho_{A^*}\left([\alpha_1,\alpha_2]_{A^*}+\mathscr{L}^{A}_{X_1}(\alpha_2)-\mathscr{L}^{A}_{X_2}(\alpha_1)+d_A\langle e_1\mid e_2\rangle_-\right).
\end{aligned}$$

Since $(A,[\cdot,\cdot]_A,\rho_A)$ and $(A^*,[\cdot,\cdot]_{A^*},\rho_{A^*})$ are Lie algebroids, we must check that, for all $X\in\Gamma(A)$ and $\alpha\in\Gamma(A^*)$,

$$[\rho_A(X),\rho_{A^*}(\alpha)]=\rho_{A^*}(\mathscr{L}^A_X(\alpha))-\rho_A(\mathscr{L}^{A^*}_\alpha(X))+\rho_A\rho^*_{A^*}d_A\langle\alpha,X\rangle.$$

Indeed, for any $f\in C^\infty(M)$,

$$\begin{aligned}
(\rho_A\rho^*_{A^*}d_A\langle\alpha,X\rangle)(f)&=\langle d_{A^*}\langle\alpha,X\rangle,d_Af\rangle\\
&=\mathscr{L}^{A^*}_{d_Af}(\langle\alpha,X\rangle)\\
&=\langle\mathscr{L}^{A}_{d_{A^*}f}(\alpha),X\rangle+\langle\alpha,\mathscr{L}^{A^*}_{d_Af}(X)\rangle.
\end{aligned}$$

Since $\mathscr{L}^{A^*}_{d_Af}(X)=[X,d_{A^*}f]_A$, see Proposition 3.4 of (MACKENZIE; XU, 1994), we have that

$$\begin{aligned}
(\rho_A\rho_{A^*}d_A\langle\alpha,X\rangle)(f)&=-\langle\mathscr{L}^{A^*}_\alpha(d_Af),X\rangle+\langle\alpha,[X,d_{A^*}f]_A\rangle\\
&=[\rho_A(X),\rho_{A^*}(\alpha)](f)-\rho_{A^*}(\mathscr{L}^A_X(\alpha))f+\rho_A(\mathscr{L}^{A^*}_\alpha(X))f.
\end{aligned}$$

□

For property $(iv)$, we have that, for all $f,g \in C^\infty(M)$,

$$\langle Df, Dg\rangle = \langle d_A f, d_{A^*} g\rangle + \langle d_{A^*} f, d_A g\rangle = \langle \rho_{A^*} \circ \rho_A^*(df), dg\rangle + \langle df, \rho_{A^*} \circ \rho_A^*(dg)\rangle.$$

Thus, property $(iv)$ is equivalent to the operator $\rho_{A^*} \circ \rho_A^*\colon T^*M \to TM$ being skew symmetric. In (MACKENZIE; XU, 1994), the authors proved the following proposition.

**Proposition 2.2.13.** Suppose that $(A, A^*)$ is a Lie bialgebroid. Then $\pi_M^* = \rho_A \circ \rho_{A^*}^* : T^*M \to TM$ defines a Poisson structure on $M$, and so does $\bar{\pi}_M^* = \rho_{A^*} \circ \rho_A^* : T^*M \to TM$.

In particular, Property $(iv)$ holds. Concerning Property $(v)$, after straightforward computations, we have the following.

**Proposition 2.2.14.** For any $e, h_1, h_2 \in \Gamma(E)$, we have

$$\rho(e)\langle h_1 \mid h_2\rangle = \langle [\![e, h_1]\!] + D\langle e \mid h_1\rangle \mid h_2\rangle + \langle h_1 \mid [\![e, h_2]\!] + D\langle e \mid h_2\rangle\rangle.$$

**Remark 2.2.15.** In the case where $A = (TM, [\cdot,\cdot], Id)$ and $A^* = (T^*M, 0, 0)$, we recover the standard Courant algebroid, see Example 2.1.3.

**Example 2.2.16.** For a PN manifold, we have that:

- $\langle X \oplus \alpha \mid Y \oplus \beta\rangle = \alpha(Y) + \beta(X)$;
- $\rho(X \oplus \alpha) = NX + \pi^\sharp(\alpha)$;
- $D = d_\pi + d_N$;
- $$[\![X \oplus \alpha, Y \oplus \beta]\!] = \left([X,Y]_N + \mathscr{L}_\alpha^\pi(Y) - \mathscr{L}_\beta^\pi(X) - \tfrac{1}{2} d_\pi(\alpha(Y) - \beta(X))\right) \oplus \left([\alpha,\beta]_\pi + \mathscr{L}_X^N(\beta) - \mathscr{L}_Y^N(\alpha) + \tfrac{1}{2} d_N(\alpha(Y) - \beta(X))\right);$$
- $[\![X \oplus \alpha, Y \oplus \beta]\!]^J = \left([X,Y]_N + \mathscr{L}_\alpha^\pi(Y) - i_\beta(d_\pi X)\right) \oplus \left([\alpha,\beta]_\pi + \mathscr{L}_X^N(\beta) - i_Y(d_N\alpha)\right).$

## 2.3 Dirac structures

In general, the bracket $[\![\cdot,\cdot]\!]$ in the definition of a Courant algebroid does not satisfy the Jacobi identity. However, there are specific subbundles $L \subset E$ in which the restricted bracket $[\![\cdot,\cdot]\!]\big|_L$ satisfies the Jacobi identity. Dirac structures were initially defined for the standard Courant algebroid by Courant in his paper (COURANT, 1990) and extended to arbitrary Courant algebroids in (LIU; WEINSTEIN; XU, 1997).

To arrive at the notion of a Dirac structure, first, we need the concepts enclosed in the following.

**Definition 2.3.1.** Let $E$ be a Courant algebroid. A subbundle $L \subset E$ is called isotropic if, for all $\sigma_1, \sigma_2 \in L$,

$$\langle \sigma_1, \sigma_2 \rangle = 0.$$

If the isotropic subbundle is also maximal, it is called Lagrangian. A subbundle is called integrable if $\Gamma(L)$ is closed under the bracket $[\![\cdot,\cdot]\!]$.

Now, we are ready to define a Dirac subbundle.

**Definition 2.3.2.** A Dirac structure, or Dirac subbundle, in a Courant algebroid $(E, [\![\cdot,\cdot]\!], \rho, \langle \cdot \mid \cdot \rangle)$, is a subbundle $L \subset A \oplus A^*$ which is maximal, isotropic, and integrable.

The next proposition shows that the definition of Dirac structure is independent of the choice of the bracket $[\![\cdot,\cdot]\!]$ or $[\![\cdot,\cdot]\!]^J$.

**Proposition 2.3.3.** Let $L \in E$ be a Lagrangian subbundle. Then, L is closed with respect to $[\![\cdot,\cdot]\!]$ if and only if it is closed with respect to $[\![\cdot,\cdot]\!]^J$.

*Proof.* Since being Lagrangian depends only on the $\langle \cdot \mid \cdot \rangle$, we just have to check the equivalence of being closed in both brackets. Through Equation (2.7), we see that the restriction of both brackets coincides in the Lagrangian subspace, that is, $[\![\cdot,\cdot]\!]\big|_L = [\![\cdot,\cdot]\!]^J\big|_L$. □

**Example 2.3.4.** Let $\pi$ be a Poisson tensor. Then $\mathrm{Graph}(\pi)$ is a Dirac structure on the canonical Courant algebroid.

**Example 2.3.5.** Let $\Omega$ be a 2-form such that $d\Omega = 0$. Then $\mathrm{Graph}(\Omega)$ is a Dirac structure on the canonical Courant algebroid.

A Dirac structure can also be seen as a Lie algebroid, that is, the bracket of the Dirac structure satisfies the Jacobi identity. Precisely (LIU; WEINSTEIN; XU, 1997),

**Proposition 2.3.6.** Suppose that $L$ is a Dirac subbundle of $(E, [\![\cdot,\cdot]\!], \rho, \langle \cdot \mid \cdot \rangle)$. Then $\left(L, \rho|_L, [\![\cdot,\cdot]\!]\big|_L\right)$ is a Lie algebroid.

*Proof.* Once the $[\![\cdot,\cdot]\!]$ is closed, the restriction $[\![\cdot,\cdot]\!]\big|_L$ is meaningful. Since the bracket in the Courant algebroid is skew-symmetric, we only need to prove that the restriction satisfies the Jacobi identity and the Leibniz rule.

Since $L$ is isotropic, $T(e_1, e_2, e_3) = 0$ for all $e_1, e_2, e_3 \in L$, which implies that the condition $(1)$ of the definition of Courant algebroid becomes the Jacobi identity. Moreover, the isotropy of $L$ reduces the condition $(3)$ to the Leibniz rule.

□

Now, we have the notion of transversal Lagrangian subbundles.

**Definition 2.3.7.** Let $L_1$ and $L_2$ be two Lagrangian subbundles in a Courant algebroid $(E, [\![\cdot,\cdot]\!], \rho, \langle\cdot\mid\cdot\rangle)$. We say that $L_1$ and $L_2$ are transversal to each other if

$$E = L_1 \oplus L_2.$$

Theorem 2.6 in (LIU; WEINSTEIN; XU, 1997), is the converse of our Theorem 2.2.10. More precisely,

**Theorem 2.3.8.** Let $(E, [\![\cdot,\cdot]\!], \langle\cdot\mid\cdot\rangle, \rho)$ be a Courant algebroid and let $A, L$ be two transversal Dirac subbundles. Then $(A, L)$ is a Lie bialgebroid, where $L$ is identified with the dual bundle of $A$ under the pairing $\langle\cdot\mid\cdot\rangle$.

*Proof.* Let $(E, [\![\cdot,\cdot]\!], \rho, \langle\cdot\mid\cdot\rangle)$ be a Courant algebroid and let $A, L \subset E$ be two transverse Dirac structures. Through Proposition 2.3.6, the ambient space induces a Lie algebroid structure on $A$ and $L$. Since $\langle\cdot\mid\cdot\rangle$ is non-degenerate and $A$ is transverse to $L$, we can define the following identification

$$\begin{aligned} &L \to A^* \\ &\alpha \mapsto \hat{\alpha}(X) = 2\langle\alpha, X\rangle \end{aligned}$$

Note that, $\mathscr{L}_{d_{A^*}f}(\alpha) = -[\![df, \alpha]\!]$. Indeed, since $\rho_A \circ d_{A^*} = -\rho_{A^*} \circ d_A$, we have that, for all $\alpha \in A$,

$$\begin{aligned} [\rho_{A^*}\alpha, \rho_A X] = [\rho\alpha, \rho X] = \rho[\![\alpha, X]\!] \\ &= \rho\left(\mathscr{L}_\alpha(X) - \frac{1}{2}d_{A*}(\langle\alpha, X\rangle) \oplus -\mathscr{L}_X(\alpha) + \frac{1}{2}d_A(\langle\alpha, X\rangle)\right) \\ &= \rho_A\left(\mathscr{L}_\alpha(X) - \frac{1}{2}d_{A*}(\langle\alpha, X\rangle)\right) + \rho_{A^*}\left(-\mathscr{L}_X(\alpha) + \frac{1}{2}d_A(\langle\alpha, X\rangle)\right) \\ &= \rho_A(\mathscr{L}_\alpha(X)) - \rho_{A^*}(-\mathscr{L}_X(\alpha)) + (\rho_{A^*} \circ d_A)(\langle\alpha, X\rangle). \end{aligned}$$

On the other hand,

$$\begin{aligned} (\rho_{A^*} \circ d_A)(\langle\alpha, X\rangle(f)) = (\rho_A(d_{A^*}f))\langle\alpha, X\rangle \\ &= \left\langle \mathscr{L}_{d_{A^*}f}(\alpha), X\right\rangle + \langle\alpha, [d_{A^*}f, X]_A\rangle \\ &= \langle[\alpha, d_A f]_A, X\rangle - \langle\alpha, \mathscr{L}_X(d_{A^*}f)\rangle + \left\langle \mathscr{L}_{d_{A^*}f}(\alpha) + [d_A f, \alpha]_{A^*}, X\right\rangle \\ &= \rho_{A^*}(\alpha)\rho_A(X)f - \langle d_A f, \mathscr{L}_\alpha(X)\rangle - \rho_A(X)\rho_{A^*}(\alpha)f \\ &\quad + \langle\mathscr{L}_X(\alpha), d_{A*}f\rangle + \left\langle \mathscr{L}_{d_{A^*}f}(\alpha) + [d_A f, \alpha]_A, X\right\rangle \\ &= [\rho_{A^*}(\alpha), \rho_A(X)](f) - \rho_A(\mathscr{L}_\alpha(X))f + \rho_{A^*}(\mathscr{L}_X(\alpha))f \\ &\quad + \left\langle \mathscr{L}_{d_{A^*}f}(\alpha) + [d_A f, \alpha]_{A^*}, X\right\rangle. \end{aligned}$$

Thus, we have that $\langle\mathscr{L}_{d_{A^*}f}(\alpha) + [\![df, \alpha]\!], X\rangle = 0$. In a similar way, we can prove that $\mathscr{L}_{d_A f}(X) = -[\![d_{A^*}f, X]\!]$.

From Proposition 2.2.9, we have that, for any $e_1, e_2, e_3 \in \Gamma(A \oplus A^*)$, $J_1 + J_2 + \text{c.p} = 0$. Using the equations $\mathscr{L}_{d_{A^*}f}(\alpha) = -[\![df, \alpha]\!]$ and $\mathscr{L}_{d_A f}(X) = -[\![d_{A^*}f, X]\!]$, we have that $J_1 + \text{c.p} = 0$. In particular, if $e_1 = X_1$, $e_2 = X_2$ and $e_3 = \alpha$, we have that

$$i_\alpha \left(d_{A^*}[X_1, X_2]_A - \mathscr{L}_{X_1}(d_{A^*}X_2) + \mathscr{L}_{X_2}(d_{A^*}X_1)\right) = 0,$$

which implies that $(A, A^*)$ is a Lie bialgebroid. □

Following (LIU, 2000), we will characterize Dirac structures in terms of their characteristic pairs. First, we will define a characteristic pair of a Dirac structure.

Let $D \subset A$ be a distribution and $H \in \Gamma(\wedge^2 A)$ be a 2-section. It is easy to check that the subbundle

$$L = \left\{ \left(X + H^{\#}(\alpha)\right) \oplus \alpha \mid X \in D, \alpha \in D^{\perp} \right\}$$

is maximal and isotropic.

**Definition 2.3.9.** We call $(D, H)$ the characteristic pair of $L$.

**Remark 2.3.10.** Note that, if $(D, H_1)$ and $(D, H_2)$ correspond to the same maximal isotropic subbundle, then $(H_1 - H_2)|_D = 0$.

The following theorem gives a necessary and sufficient condition for a maximal isotropic subbundle, with a characteristic pair $(D, H)$, to be a Dirac structure.

**Theorem 2.3.11** ((LIU, 2000), Theorem 3.1)**.** Let $L$ be a maximal isotropic subbundle of a Lie bialgebroid $(A, A^*)$ with a characteristic pair $(D, H)$. Then $L$ is integrable if and only if the following three conditions hold:

(1) For all $X, Y \in \Gamma(D)$,

$$[X, Y]_A \in \Gamma(D);$$

(2) For every $\alpha, \beta, \gamma \in D^{\perp}$,

$$\left(d_A H + \frac{1}{2}[H, H]_{A^*}\right)(\alpha, \beta, \gamma) = 0;$$

(3) $\Gamma\left(D^{\perp}\right)$ is closed under the bracket $[\cdot,\cdot]_{A^*} + [\cdot,\cdot]_H$, where $[\cdot,\cdot]_H$ is the bracket induced by $H$, that is,

$$[\alpha, \beta]_H = \mathscr{L}_{H^\sharp(\alpha)}(\beta) - \mathscr{L}_{H^\sharp(\beta)}(\alpha) - d_{A^*}(H(\alpha, \beta)),$$

for all $\alpha, \beta \in \Gamma(A^*)$.

*Proof.* Suppose that the $(1),(2)$ and $(3)$ hold true:

Since $L$ is a maximal isotropic subbundle, we only have to show that, for all $X,Y \in D$, and $\alpha,\beta \in D^{\perp}$,

$$[\![(X+H^{\sharp}(\alpha))\oplus\alpha,(Y+H^{\sharp}(\beta))\oplus\beta]\!]\in\Gamma(L).$$

First, note that for every $X,Y \in D$,

$$[\![X\oplus 0,Y\oplus 0]\!]=[X,Y]_A\,.$$

By condition (1), $[\![X\oplus 0,Y\oplus 0]\!]\in\Gamma(L)$.

Using that

$$\begin{aligned}[H^{\sharp}(\alpha),H^{\sharp}(\alpha)]_{A^*}&=H^{\sharp}[\alpha,\beta]+\frac{1}{2}[H,H]_A(\alpha,\beta);\\ (d_{A^*}H)(\alpha,\beta)&=H^{\sharp}[\alpha,\beta]+\mathscr{L}_{\alpha}(H^{\sharp}(\beta))-\mathscr{L}_{\beta}(H^{\sharp}(\alpha))-d_{A^*}H(\alpha,\beta),\end{aligned}$$

we have that, for all $\alpha,\beta\in\Gamma\left(D^{\perp}\right)$,

$$\begin{aligned}[\![H^{\#}\alpha\oplus\alpha,H^{\#}\beta\oplus\beta]\!]=&\left(d_*H+\frac{1}{2}[H,H]\right)^{\#}(\alpha,\beta)+H^{\#}\left([\alpha,\beta]_{A^*}+[\alpha,\beta]_H\right)\\ &\oplus\left([\alpha,\beta]_{A^*}+[\alpha,\beta]_H\right).\end{aligned}$$

By conditions (2) and (3), we have that

$$\left(d_*H+\frac{1}{2}[H,H]\right)^{\#}(\alpha,\beta)\in\Gamma(D),\qquad\text{and}\qquad[\alpha,\beta]_{A^*}+[\alpha,\beta]_H\in\Gamma\left(D^{\perp}\right),$$

thus $\left[\Omega^{\#}\alpha\oplus\alpha,\Omega^{\#}\beta\oplus\beta\right]\in\Gamma(L)$.

Moreover, for all $X\in\Gamma(D)$ and $\alpha\in\Gamma\left(D^{\perp}\right)$, we have

$$[\![X,H^{\#}\alpha\oplus\alpha]\!]=\left(\left[X,H^{\#}\alpha\right]_A-\mathscr{L}_{\alpha}(X)+\frac{1}{2}d_A(\alpha(X))\right)\oplus\left(\mathscr{L}_X(\alpha)-\frac{1}{2}d_{A*}(\alpha(X))\right).$$

Since $\alpha(X)=0$, summing and subtracting $H^{\sharp}\left(\mathscr{L}_X(\alpha)\right)$, we have that

$$[\![X,H^{\sharp}\alpha\oplus\alpha]\!]=\left(\left[X,H^{\sharp}\alpha\right]_A-\mathscr{L}_{\alpha}(X)-H^{\sharp}(\mathscr{L}_X(\alpha))+H^{\sharp}(\mathscr{L}_X(\alpha))\right)\oplus\mathscr{L}_X(\alpha).$$

Condition (1) implies that $\mathscr{L}_X(\alpha)\in\Gamma\left(D^{\perp}\right)$. Thus, $[\![X,H^{\sharp}\alpha\oplus\alpha]\!]\in\Gamma(L)$ if and only if

$$\left[X,H^{\sharp}\alpha\right]-\mathscr{L}_{\alpha}(X)-H^{\sharp}\left(\mathscr{L}_X(\alpha)\right)\in\Gamma(D).\tag{2.13}$$

Now, we take any $\beta \in \Gamma\left(D^{\perp}\right)$, thus

$$\begin{aligned}\langle [\alpha,\beta]_{A^*} + [\alpha,\beta]_H \mid X\rangle =& \langle [\alpha,\beta]_{A^*} \mid X\rangle + \langle \mathscr{L}_{H^\sharp\alpha}(\beta) \mid X\rangle \\ &- \langle \mathscr{L}_{H^\# \beta}(\alpha) \mid X\rangle - \langle H^\sharp \alpha \mid \beta\rangle \\ =& -\langle \beta \mid \mathscr{L}_\alpha(X)\rangle + \langle \beta \mid \left[X, H^\# \alpha\right]\rangle - \langle \beta \mid H^\# \left(\mathscr{L}_X(\alpha)\right)\rangle \\ =& \langle \beta \mid \left[X, H^\# \alpha\right] - \mathscr{L}_\alpha(X) - H^\# \left(\mathscr{L}_X(\alpha)\right)\rangle .\end{aligned}$$

So, Equation 2.13 is equivalent to condition (3).

On the other hand, suppose that $L$ is integrable. For every $X, Y \in D$, $[\![X \oplus 0, Y \oplus 0]\!] = [X,Y]_A \in \Gamma(L)$, thus condition (1) holds true.

For all $\alpha, \beta \in \Gamma\left(D^{\perp}\right)$,

$$\begin{aligned}[\![H^\# \alpha \oplus \alpha, H^\# \beta \oplus \beta]\!] = & \left(d_* H + \frac{1}{2}[H,H]\right)^\# (\alpha,\beta) + H^\# \left([\alpha,\beta]_{A^*} + [\alpha,\beta]_H\right) \\ & \oplus \left([\alpha,\beta]_{A^*} + [\alpha,\beta]_H\right).\end{aligned}$$

Since $[\![H^\# \alpha \oplus \alpha, H^\# \beta \oplus \beta]\!] \in \Gamma(L)$, conditions (2) and (3) hold true.

□

**Remark 2.3.12.** If we choose $D = \{0\}$, we recover the following theorem stated in (LIU; WEINSTEIN; XU, 1997).

**Theorem 2.3.13.** Let $H \in \Gamma(\bigwedge^2 A)$. $L_H = \{H^\sharp(\alpha) \oplus \alpha \mid \alpha \in A^*\}$ is a Dirac subbbundle of $(A, A^*)$ if and only if $H$ is skew-symmetric and satisfies the following Maurer-Cartan type equation:

$$d_{A^*} H + \frac{1}{2}[H,H]_A = 0.$$

**Example 2.3.14.** In the case of the standard Courant algebroid, as illustrated in Example 2.1.3, the Maurer-Cartan equation takes the following form:

1. $d\Omega = 0$, for 2-forms;
2. $[\pi,\pi] = 0$, for bivector fields.

Therefore, $L = graph(\Omega) = \{X \oplus \Omega^\flat(X) \mid X \in \Gamma(TM)\}$ is a Dirac structure if and only if $\Omega$ is closed. Similarly, $L = graph(\pi) = \{\pi^\sharp \oplus \alpha \mid \alpha \in \Gamma(T^*M)\}$ is a Dirac structure if and only if $\pi$ is a Poisson tensor.

In (FALQUI; MENCATTINI; PEDRONI, 2023), a new result is presented that allows for the deformation of a Poisson-Nijenhuis manifold into a Poisson quasi-Nijenhuis manifold. We will discuss this result in Chapter 3. The authors provide some examples that can be understood in terms of the theory of Dirac structures.

**Example 2.3.15** (Deformations of the canonical PN structure)**.** Let $(\mathbb{R}^{2n}, \pi, N)$ be the canonical PN structure on $\mathbb{R}^{2n}$, that is, in the canonical coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$

$$\pi = \sum_{i=1}^{n} \partial_{p_i} \wedge \partial_{q_i}, \qquad N = \sum_{i=1}^{n} p_i(\partial_{q_i} \otimes dq_i + \partial_{p_i} \otimes dp_i).$$

Note that $\pi$ is the Canonical symplectic structure in Darboux's coordinate for $\mathbb{R}^{2n}$. Consider the following 2-form

$$\Omega = \sum_{i<j} (v_i(q_i - q_{i+1}) dq_{i+1} \wedge dq_i + dp_{i+1} \wedge dp_i),$$

where $V_{i,i+1} \in C^\infty(\mathbb{R})$ on the variable $q_i - q_{i+1}$. Given a subbundle $D \subset TM$, we can use Theorem 2.3.11 to decide when the Lagrangian subbundle

$$L = \{(\alpha + \Omega^\flat(X)) \oplus X \mid X \in D,\, \alpha \in D^\perp\}$$

is a Dirac structure. First, after some computations, we have that

$$d_N\Omega = \sum_{i=1}^{n} V_{i,i+1} dq_i \wedge dq_{i+1} \wedge (dp_i + dp_j),$$
$$[\Omega, \Omega]_\pi = -2 \sum_i V'_{i,i+1} dq_i \wedge dq_{i+1} \wedge (dp_i + dp_{i+1}).$$

If $D = TM$, then the Theorem 2.3.11 becomes the Theorem 2.3.13 and $L$ is a Dirac structure if and only if

$$d_N\Omega + \frac{1}{2}[\Omega, \Omega]_\pi = 0.$$

It will hold if and only if, for every $i$,

$$V'_{i,i+1} = -V_{i,i+1}.$$

CHAPTER

3

# DEFORMATION THEOREM FOR A POISSON QUASI-NIJENHUIS MANIFOLD

The results presented in this chapter are reported in (LUIZ; MENCATTINI; PEDRONI, 2024). This work generalizes the deformation theorem initially introduced in (FALQUI; MENCATTINI; PEDRONI, 2023) to the case of Poisson quasi-Nijenhuis manifolds (hereafter referred to as PqN manifolds). Our approach includes two equivalent proofs: the first employs the formalism of Courant algebroids and Dirac structures, while the second utilizes the concept of twisting a quasi-Lie bialgebroid. We expect that this new formulation may elucidate the relationship between PqN manifolds and integrable systems, as discussed in (FALQUI *et al.*, 2020).

In this chapter, we will explore the relationships between Poisson-Nijenhuis structures, Lie bialgebroids and Dirac structures to state and provide the first proof Theorem of 3.2.9.

## 3.1 Poisson quasi-Nijenhuis manifolds and quasi-Lie bialgebroid

In (STIÉNON; XU, 2007) a Poisson quasi-Nijenhuis manifold was defined as follows:

**Definition 3.1.1.** A Poisson quasi-Nijenhuis manifold is a quadruple $(M,\pi,N,\phi)$ such that:

- the Poisson bivector $\pi$ and the $(1,1)$ tensor field $N$ are compatible in the sense of Definition 2.2.6;
- the 3-forms $\phi$ and $i_N\phi$ are closed;
- $T_N(X,Y) = \pi^\sharp(i_{X\wedge Y}\phi)$ for all vector fields $X$ and $Y$, where $i_{X\wedge Y}\phi$ is the 1-form defined as $\langle i_{X\wedge Y}\phi, Z\rangle = \phi(X,Y,Z)$, and

$$T_N(X,Y) = [NX,NY] - N[X,Y]_N \tag{3.1}$$

is the Nijenhuis torsion of $N$.

Similar to the case of PN manifolds, associated with a PqN manifold we have a quasi-Lie bialgebroid. A quasi-Lie bialgebroid extends the definition of a Lie bialgebroid, allowing the differential operator $d_{A^*}$ to deviate from being a square zero. This deviation means that $d_{A^*}$ does not define a Lie algebroid in the conventional sense. To be precise, we have the following definition:

**Definition 3.1.2.** A quasi-Lie bialgebroid is a triple $(A, d_{A^*}, \phi)$ consisting of a Lie algebroid A, a degree 1 derivation $d_{A^*}$ of both the algebras $(\Gamma(\wedge^\bullet A), \wedge)$ and $(\Gamma(\wedge^\bullet A), [\cdot,\cdot]_A)$, and an element $\phi \in \Gamma(\wedge^3 A)$ such that $d_{A^*}^2 = [\phi, \cdot]_A$ and $d_{A^*}\phi = 0$.

Let $(M, \pi, N, \phi)$ be a PqN manifold. The Lie algebroid associated with the Poisson tensor, $\left(T^*M, [\cdot,\cdot]_\pi, \pi^\sharp\right)$ will be denoted as $(T^*M)_\pi$. According to Proposition 3.5 in (STIÉNON; XU, 2007), $\mathbb{T}M' = \left((T^*M)_\pi, d_N, \phi\right)$ is a quasi-Lie bialgebroid.

**Proposition 3.1.3** ((STIÉNON; XU, 2007), Proposition 3.5 )**.** The quadruple $(M, \pi, N, \phi)$ is a Poisson quasi-Nijenhuis manifold if and only if $((T^*M)_\pi, d_N, \phi)$ is a quasi-Lie algebroid and $\phi$ is a closed 3-form.

**Remark 3.1.4.** In the case $\phi = 0$, the Kosmann-Schwarzbach correspondence between Lie bialgebroids and PN manifolds is obtained, as stated in Theorem 2.2.8.

Now we recall how quasi-Lie bialgebroids are related to Courant algebroids, see e.g. (ROYTENBERG, 2002; STIÉNON; XU, 2007). Let $(A, d_{A^*}, \phi)$ be a quasi-Lie bialgebroid, thus it defines a differential pre-Lie algebra over $A$, see Appendix A. We can define a morphism $\rho_{A^*}\colon A^* \to TM$ by

$$\rho_{A^*}(\alpha)(f) = \alpha(d_{A^*}f), \forall \alpha \in \Gamma(A^*), \forall f \in C^\infty(M),$$

and a bracket in $\Gamma(A^*)$ by

$$[\alpha_1, \alpha_2]_{A^*}(X) = \rho_{A^*}(\alpha_1)(X(\alpha_2)) - \rho_{A^*}(\alpha_2)(X(\alpha_1)) - d_{A^*}X(\alpha_1, \alpha_2),$$

for all $X \in \Gamma(A)$. Then $E = A \oplus A^*$ together with the bundle map $\rho\colon E \to TM$

$$\rho(X \oplus \alpha) = \rho_A(X) + \rho_{A^*}(\alpha),$$

the non-degenerated symmetric pairing on E

$$\langle X_1 \oplus \alpha_1 \mid X_2 \oplus \alpha_2 \rangle = \frac{1}{2}\left(X_1(\alpha_2) + X_2(\alpha_1)\right),$$

and the bracket in $\Gamma(E)$,

$$[\![X_1, X_2]\!] = [X_1, X_2]_A, \forall X_1, X_2 \in A;$$

$$[\![\alpha_1, \alpha_2]\!] = \phi(\alpha_1, \alpha_2, \cdot) \oplus [\alpha_1, \alpha_2]_{A^*}, \forall \alpha_1, \alpha_2 \in A^*;$$

$$[\![X, \alpha]\!] = \left(-i_\alpha(d_{A^*}X) - \frac{1}{2}d_{A^*}(X(\alpha))\right) \oplus \left(i_X(d_A\alpha) + \frac{1}{2}d_A(X(\alpha))\right), \forall X \in A, \alpha \in A^*,$$

is a Courant algebroid.

On the other hand, let $(E, [\![\cdot,\cdot]\!], \rho, \langle \cdot \mid \cdot \rangle)$ be a Courant algebroid and $A$ be a Dirac structure transverse to a Lagrangian subbundle $L$. Since $\langle \cdot \mid \cdot \rangle$ is non-degenerated and $E = A \oplus L$, we can identify $L^* \simeq A$ through

$$\begin{aligned} &A \to L^* \\ &\alpha \mapsto \hat{\alpha}(X) = 2\langle \alpha, X \rangle \end{aligned}$$

Let $\phi \in \Gamma(\wedge^3 A)$ be defined by

$$\phi(\alpha_1, \alpha_2, \alpha_3) = 2\langle [\![\alpha_1, \alpha_2]\!] \mid \alpha_3 \rangle,$$

for all $\alpha_1, \alpha_2, \alpha_3 \in \Gamma(L)$. Since $E = A \oplus L$, for all $\alpha_1, \alpha_2 \in L$, we have the following decomposition

$$[\![\alpha_1, \alpha_2]\!] = [\alpha_1, \alpha_2]_L \oplus [\alpha_1, \alpha_2]_A,$$

where $[\alpha_1, \alpha_2]_L \in L$ and $[\alpha_1, \alpha_2]_A \in A$.

We define $d_L$, given $P \in \Gamma(\wedge^k A)$, for all $\alpha_1, \dots, \alpha_{k+1} \in \Gamma(L)$,

$$\begin{aligned} (d_L P)(\alpha_1, \dots, \alpha_{k+1}) = &\sum_{i<j} (-1)^{i+j} P([\alpha_i, \alpha_j]_L, \alpha_1 \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \alpha_K) \\ &+ \sum_{i=1}^{k+1} (-1)^{i+1} \rho(\alpha_i) \left( P(\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_{k+1}) \right), \end{aligned}$$

then $(A, d_L, \phi)$ is a quasi-Lie bialgebroid.

The following theorem is an extension of Theorem 2.3.8 to quasi-Lie bialgebroids, see e.g. (STIÉNON; XU, 2007).

**Theorem 3.1.5.** There is a 1-1 correspondence between quasi-Lie bialgebroids and Dirac structures with transversal isotropic complements in $A \oplus A^*$ with the Courant algebroid structure defined by the quasi-Lie bialgebroid.

## 3.2 Deformation theorem

In the previous section, we discussed how the quasi-Lie bialgebroid $\mathbb{T}M' = ((T^*M)_\pi, d_N, \phi)$ endows a Courant algebroid structure upon $T^*M \oplus TM$.

Given a differential form $\Omega \in \Gamma\left(\bigwedge^2 T^*M\right)$ with the property $d\Omega = 0$, we proceed to define a new endomorphism $\hat{N} \colon \Gamma(TM) \to \Gamma(TM)$ as $\hat{N} = N + \pi^\sharp \Omega^\flat$. Our goal is to demonstrate the existence of a Dirac structure $D \in \mathbb{T}M'$ that possesses transversal isotropic complements $L$ such that $(\Gamma(\wedge^\bullet L), d_L) \simeq (\Gamma(\wedge^\bullet TM), d_{\hat{N}})$, and conclude that the quasi-Lie algebroid $(D, d_L, \phi)$ is the one associated with some PqN structure $(M, \pi, \hat{N}, \hat{\phi})$ .

Indeed,

$$L = \text{graph}(\Omega) = \left\{\Omega^\flat(X) \oplus X \mid X \in TM\right\}$$

is a Lagrangian subbundle of $T^*M \oplus TM$. Note that $T^*M \oplus \{0\} \subset \mathrm{T}M'$ is a Dirac structure and it is transverse to $L$. By Theorem 3.1.5, we have that $((T^*M)_\pi, d_L, \phi')$ is a quasi-Lie bialgebroid.

Now, let us compute $\phi'$. For this end, we need to compute $[\![X \oplus \Omega^\flat(X), Y \oplus \Omega^\flat(Y)]\!]$ for every $X, Y \in \Gamma(TM)$. First, note that

**Proposition 3.2.1.** Let $\pi$ be a bivector field and $\Omega$ be a 2-form. Then, for every $X, Y \in \Gamma(TM)$,

$$\mathscr{L}^\pi_{\Omega^\flat(X)} Y = \mathscr{L}_{\pi^\sharp \Omega^\flat X}(Y) + \pi^\sharp(i_Y(d(\Omega^\flat(X)))).$$

*Proof.* For every $\beta \in \Gamma(T^*M)$, by definition,

$$\begin{aligned}
\mathscr{L}^\pi_{\Omega^\flat(X)} Y(\beta) &= \pi^\sharp(\Omega^\flat(X))\langle\beta, Y\rangle - \langle[\Omega^\flat(X), \beta]_\pi, Y\rangle \\
&= \langle\mathscr{L}_{\pi^\sharp\Omega^\flat(X)}(\beta), Y\rangle + \langle\beta, \mathscr{L}_{\pi^\sharp\Omega^\flat(X)}(Y)\rangle - \langle[\Omega^\flat(X), \beta]_\pi, Y\rangle \\
&= \langle\mathscr{L}_{\pi^\sharp\Omega^\flat(X)}(\beta), Y\rangle + \langle\beta, \mathscr{L}_{\pi^\sharp\Omega^\flat(X)}(Y)\rangle - \langle\mathscr{L}_{\pi^\sharp\Omega^\flat(X)}(\beta), Y\rangle \\
&\quad + \langle\mathscr{L}_{\pi^\sharp(\beta)}(\Omega^\flat(X)), Y\rangle + \langle d(\pi(\Omega^\flat(X), \beta)), Y\rangle.
\end{aligned}$$

Note that,

$$\begin{aligned}
\langle\mathscr{L}_{\pi^\sharp(\beta)}(\Omega^\flat(X)), Y\rangle &= \langle i_{\pi^\sharp(\beta)} d(\Omega^\flat(X)) + d(\Omega(X, \pi^\sharp(\beta))), Y\rangle \\
\langle d(\pi(\Omega^\flat(X), \beta)), Y\rangle &= -\langle d(\Omega(X, \pi^\sharp(\beta))), Y\rangle.
\end{aligned}$$

Thus,

$$\mathscr{L}^\pi_{\Omega^\flat(X)} Y(\beta) = \langle\beta, \mathscr{L}_{\pi^\sharp\Omega^\flat X}(Y) + \pi^\sharp(i_Y(d(\Omega^\flat(X))))\rangle.$$

□

**Proposition 3.2.2.** Let $\pi$ be a Poisson tensor and $\Omega$ be a closed 2-form, then

$$[X_i, X_j]^\pi_\Omega = [X_i, X_j]_{\pi^\sharp\Omega^\flat},$$

where

$$[X, Y]^\pi_\Omega = \mathscr{L}^\pi_{\Omega^\flat(X)} Y - \mathscr{L}^\pi_{\Omega^\flat(Y)} X - d_\pi(\Omega(X, Y)).$$

*Proof.* By Proposition 3.2.1, we have that

$$\begin{aligned}
[X, Y]^\pi_\Omega &= \mathscr{L}_{\pi^\sharp\Omega^\flat(X)}(Y) + \pi^\sharp(i_Y(d(\Omega^\flat(X)))) - \mathscr{L}_{\pi^\sharp\Omega^\flat(Y)}(X) - \pi^\sharp(i_X(d(\Omega^\flat(Y)))) - d_\pi(\Omega(X, Y)) \\
&= [\pi^\sharp\Omega^\flat(X), Y] + [X, \pi^\sharp\Omega^\flat(Y)] + \pi^\sharp(i_Y(d(\Omega^\flat(X)))) - \pi^\sharp(i_X(d(\Omega^\flat(Y)))) - d_\pi(\Omega(X, Y))
\end{aligned}$$

Note that,

$$\begin{aligned}
\pi^\sharp\Omega^\flat[X, Y] &= \pi^\sharp(\Omega^\flat[X, Y]) = \pi^\sharp(i_{\mathscr{L}_X(Y)}\Omega) = \pi^\sharp(\mathscr{L}_X(i_Y\Omega) - i_Y(\mathscr{L}_X(\Omega))) \\
&= \pi^\sharp(i_X d(i_Y\Omega) + d(i_X i_Y\Omega) - i_Y d(i_X\Omega) - i_Y i_X d\Omega).
\end{aligned}$$

If $d\Omega = 0$, we have that

$$[X,Y]^{\pi}_{\Omega} = [\pi^{\sharp}\Omega^{\flat}(X),Y] + [X,\pi^{\sharp}\Omega^{\flat}(Y)] - \pi^{\sharp}\Omega^{\flat}[X,Y] = [X,Y]_{\pi^{\sharp}\Omega^{\flat}}.$$

□

As shown in Appendix A, see Example A.1.11, we have

**Proposition 3.2.3.** Let $d^{\pi}_{\Omega}$ be the differential operator given by Equation (A.3) corresponding to the anchor $\pi^{\sharp}$ and bracket $[\cdot,\cdot]^{\pi}_{\Omega}$, then

$$d^{\pi}_{\Omega} = [\Omega,\cdot]_{\pi}.$$

**Proposition 3.2.4.** Let $(M,\pi)$ be a Poisson manifold and $\Omega \in \Gamma(\wedge^2 T^*M)$. Then

$$[\Omega^{\flat}(X),\Omega^{\flat}(Y)]_{\pi} = \Omega^{\flat}[X,Y]^{\pi}_{\Omega} + \frac{1}{2} i_Y \left( i_X \left( [\Omega,\Omega]_{\pi} \right) \right).$$

*Proof.* By Proposition (A.1.10),

$$[\Omega,\Omega]_{\pi}(X,Y,Z) = -2\left( \langle \mathscr{L}^{\pi}_{\Omega^{\flat}(X)}(Z),\Omega^{\flat}(Y)\rangle + \langle \mathscr{L}^{\pi}_{\Omega^{\flat}(Y)}(X),\Omega^{\flat}(Z)\rangle + \langle \mathscr{L}^{\pi}_{\Omega^{\flat}(Z)}(Y),\Omega^{\flat}(X)\rangle \right)$$

and, for every $\alpha,\beta \in \Gamma(T^*M)$, and $X \in \Gamma(TM)$,

$$i_{\beta}\mathscr{L}^{\pi}_{\alpha}(X) = i_{\pi^{\sharp}(\alpha)}(d(X(\beta))) - X([\alpha,\beta]_{\pi})$$

Thus, we have that,

$$\begin{aligned}\frac{1}{2}[\Omega,\Omega]_{\pi}(X,Y,Z) =& - i_{\pi^{\sharp}(\Omega^{\flat}(X))}(d\Omega(Y,Z)) + i_Z([\Omega^{\flat}(X),\Omega^{\flat}(Y)]_{\pi}) \\ & - i_{\Omega^{\flat}(Z)}\mathscr{L}^{\pi}_{\Omega^{\flat}(Y)}(X) \\ & - i_{\pi^{\sharp}(\Omega^{\flat}(Z))}(d\Omega(X,Y)) + i_Y([\Omega^{\flat}(Z),\Omega^{\flat}(X)]_{\pi}).\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}i_Z(\Omega^{\flat}([X,Y]^{\pi}_{\Omega})) =& - i_{\Omega^{\flat}(Z)}(\mathscr{L}^{\pi}_{\Omega^{\flat}(X)}(Y)) + i_{\Omega^{\flat}(Z)}(\mathscr{L}^{\pi}_{\Omega^{\flat}(Y)}(X)) + i_{\Omega^{\flat}(Z)}(d_{\pi}(\Omega(X,Y))) \\ =& - i_{\pi^{\sharp}(\Omega^{\flat}(X))}(d\Omega(Z,Y)) + i_Y([\Omega^{\flat}(X),\Omega^{\flat}(Z)]_{\pi}) + i_{\Omega^{\flat}(Z)}(\mathscr{L}^{\pi}_{\Omega^{\flat}(Y)}(X)) \\ & + i_{\pi^{\sharp}(\Omega^{\flat}(Z))}(d\Omega(X,Y)).\end{aligned}$$

Using the skew-symmetry of $d\Omega$ and $[\cdot,\cdot]_{\pi}$, we have that

$$\frac{1}{2}[\Omega,\Omega]_{\pi}(X,Y,Z) + i_Z(\Omega^{\flat}([X,Y]^{\pi}_{\Omega})) = +i_Z([\Omega^{\flat}(X),\Omega^{\flat}(Y)]_{\pi}).$$

Thus,

$$[\Omega^{\flat}(X),\Omega^{\flat}(Y)]_{\pi} = \Omega^{\flat}[X,Y]^{\pi}_{\Omega} + \frac{1}{2} i_Y \left( i_X \left( [\Omega,\Omega]_{\pi} \right) \right).$$

□

Then, we have that

**Proposition 3.2.5.** For every $X,Y \in \Gamma(TM)$,

$$\llbracket X \oplus \Omega^\flat(X), Y \oplus \Omega^\flat(Y) \rrbracket = i_Y i_X \left( d_N\Omega + \frac{1}{2}[\Omega,\Omega]_\pi + \phi \right) + \Omega^\flat\left([X,Y]_N + [X,Y]^\pi_\Omega\right) \oplus \left([X,Y]_N + [X,Y]^\pi_\Omega\right).$$

*Proof.* Since

$$\llbracket X, \Omega^\flat(Y) \rrbracket = \left( -i_{\Omega^\flat(Y)}(d_\pi X) - \frac{1}{2} d_\pi(\Omega(Y,X)) \right) \oplus \left( i_X(d_N \Omega^\flat(Y)) + \frac{1}{2} d_N(\Omega(Y,X)) \right)$$

$$\llbracket \Omega^\flat(X), Y \rrbracket = \left( i_{\Omega^\flat(X)}(d_\pi Y) + \frac{1}{2} d_\pi(\Omega(X,Y)) \right) \oplus \left( -i_Y(d_N \Omega^\flat(X)) - \frac{1}{2} d_N(\Omega(X,Y)) \right)$$

and, by definition,

$$\begin{aligned}[X,Y]^\pi_\Omega &= \mathscr{L}^\pi_{\Omega^\flat(X)}(Y) - \mathscr{L}^\pi_{\Omega^\flat(Y)}(X) + d_\pi\langle \Omega^\flat(Y), X\rangle \\ &= i_{\Omega^\flat(X)}(d_\pi Y) - i_{\Omega^\flat(Y)}(d_\pi X) + d_\pi(\Omega(X,Y)),\end{aligned}$$

thus,

$$\llbracket \Omega^\flat(X), Y \rrbracket + \llbracket X, \Omega^\flat(Y) \rrbracket = [X,Y]^\pi_\Omega \oplus (i_X(d_N\Omega^\flat(Y))) - i_Y(d_N\Omega^\flat(X)) - d_N(\Omega(X,Y)).$$

By Proposition (3.2.4),

$$[\Omega^\flat(X), \Omega^\flat(Y)]_\pi = \Omega^\flat[X,Y]^\pi_\Omega + \frac{1}{2} i_Y\left(i_X\left([\Omega,\Omega]_\pi\right)\right).$$

Thus,

$$\begin{aligned}\llbracket X \oplus \Omega^\flat(X), Y \oplus \Omega^\flat(Y) \rrbracket =& \llbracket \Omega^\flat(X), \Omega^\flat(Y) \rrbracket + \llbracket X, \Omega^\flat(Y) \rrbracket + \llbracket \Omega^\flat(X), Y \rrbracket + \llbracket X, Y \rrbracket \\ =& \Omega^\flat[X,Y]^\pi_\Omega + \frac{1}{2} i_y(i_x([\Omega,\Omega]_\pi)) + i_Y i_X \phi + i_X(d_N\Omega^\flat(Y)) \\ & - i_Y(d_N\Omega^\flat(X)) - d_N(\Omega(X,Y)) \oplus ([X,Y]_N + [X,Y]^\pi_\Omega).\end{aligned}$$

Now, we will show that

$$i_X(d_N\Omega^\flat(Y)) - i_Y(d_N\Omega^\flat(X)) - d_N(\Omega(X,Y)) = i_Y(i_X(d_N\Omega)) + \Omega^\flat([X,Y]_N).$$

As presented in Section 3.1, we have that

$$\begin{aligned}d_N\Omega(X,Y,Z) =& i_X(d_N\Omega(Y,Z)) - i_Y(d_N\Omega(X,Z)) + i_Z(d_N\Omega(X,Y)) \\ & - \Omega([X,Y]_N, Z) + \Omega([X,Z]_N, Y) - \Omega([Y,Z]_N, X).\end{aligned}$$

By definition:

$$\Omega([X,Z]_N, Y) = -i_{\Omega^\flat(Y)}[X,Z]_N = -i_X(d_N(\Omega(Y,Z))) + i_Z(d_N(\Omega(Y,X))) + i_Z(i_X(d_N\Omega^\flat(Y)))$$

and

$$-\Omega([Y,Z]_N,X) = i_{\Omega^\flat(X)}[Y,Z]_N = i_Y(d_N(\Omega(X,Z))) - i_Z(d_N(\Omega(X,Y))) - i_Z(i_Y(d_N\Omega^\flat(X))).$$

Thus,

$$d_N\Omega(X,Y,Z) = -i_Z(\Omega^\flat[X,Y]_N) + i_Z(i_X(d_N\Omega^\flat(Y))) - i_Z(i_Y(d_N\Omega^\flat(X))) - i_Z(d_N(\Omega(X,Y)))$$

and

$$i_X(d_N\Omega^\flat(Y)) - i_Y(d_N\Omega^\flat(X)) - d_N(\Omega(X,Y)) = i_Y(i_X(d_N\Omega)) + \Omega^\flat[X,Y]_N.$$

Finally,

$$\begin{aligned}[\![X\oplus\Omega^\flat(X), Y\oplus\Omega^\flat(Y)]\!] =& i_Y i_X\left(d_N\Omega + \frac{1}{2}[\Omega,\Omega]_\pi + \phi\right) + \Omega^\flat\left([X,Y]_N + [X,Y]^\pi_\Omega\right) \\ &\oplus\left([X,Y]_N + [X,Y]^\pi_\Omega\right)\end{aligned}$$

□

And finally, we have that

$$\begin{aligned}\phi'(X,Y,Z) &= 2\langle[\![X\oplus\Omega^\flat(X), Y\oplus\Omega^\flat(Y)]\!], Z\oplus\Omega^\flat(Z)\rangle \\ &= \frac{2}{2}\left(d_N\Omega + \frac{1}{2}[\Omega,\Omega]_\pi + \phi\right)(X,Y,Z) + \Omega\left([X,Y]_N + [X,Y]^\pi_\Omega, Z\right) \\ &\quad + \Omega\left(Z, [X,Y]_N + [X,Y]_\Omega\right) \\ &= \left(d_N\Omega + \frac{1}{2}[\Omega,\Omega]_\pi + \phi\right)(X,Y,Z).\end{aligned}$$

Now, we will demonstrate that when we identify $L$ with $TM$, it follows that $(\Gamma(\wedge^\bullet L), d_L) \simeq (\Gamma(\wedge^\bullet TM), d_{\hat{N}}),$.

**Proposition 3.2.6.** Let $(M,\pi,N)$ be a PN manifold and $\Omega$ be a closed 2-form. Suppose that $L = \text{graph}(\Omega^\flat)$ is a Lagrangian subbundle in $\mathbb{T}M'$, than, when we identify $\bigwedge^\bullet L^*$ with $\bigwedge^\bullet T^*M$ through

$$\sigma(\Omega^\flat(X_1)\oplus X_1, \dots, \Omega^\flat(X_k)\oplus X_k) = \sigma(X_1,\dots,X_k),$$

we have that

$$(\Gamma(\wedge^\bullet L), d_L) \simeq (\Gamma(\wedge^\bullet TM), d_{\hat{N}}),$$

for every $\sigma \in \bigwedge^\bullet T^*M$. In particular, if $L$ is a Dirac structure, we have that $(TM, [\cdot,\cdot]_{\hat{N}}, N)$ is isomorphic as a Lie algebroid to $(L, [\cdot,\cdot]_L, \rho_L)$.

*Proof.* When $L \subset \mathbb{T}M'$ we have the following Lie algebroid structure: the anchor is given by

$$\rho_L\left(\Omega^\flat(X)\oplus X\right) = \pi^\sharp\Omega^\flat(X) + NX,$$

and the Lie bracket is given by

$$[\Omega^\flat(X)\oplus X,\Omega^\flat(Y)\oplus Y]_L=\Omega^\flat\left([X,Y]_N+[X,Y]^\pi_\Omega\right)\oplus\left([X,Y]_N+[X,Y]^\pi_\Omega\right),$$

where

$$[X,Y]^\pi_\Omega=\mathscr{L}^\pi_{\Omega^\flat(X)}Y-\mathscr{L}^\pi_{\Omega^\flat(Y)}X-d_\pi(\Omega(X,Y)).$$

Associated with this Lie algebroid we have the following differential operator, denoted by $\hat{d}_L$. Using the same identification $\wedge^k L^*\approx\wedge^k T^*M$, for $\sigma\in\wedge^{k-1}T^*M$

$$\begin{aligned}
d_L\sigma&\left(\Omega^\flat\,(X_1)\oplus X_1,\dots,\Omega^\flat(X_k)\oplus X_k\right)=\\
&\sum_{i<j}(-1)^{i+j}\sigma\left(\Omega^\flat\,(X_i)\oplus X_i,\Omega^\flat\,(X_j)\oplus X_j]_L,\Omega^\flat\,(X_1)\oplus X_1,\dots,\Omega^\flat\,(X_k)\oplus X_k\right)\\
&+\sum_{i=1}^k(-1)^{i+1}\mathscr{L}_{\rho_L\left(\Omega^\flat(X_i)\oplus X_i\right)}\left(\sigma\left(\Omega^\flat\,(X_1)\oplus X_1,\dots,\Omega^\flat\,(X_k)\oplus X_k\right)\right)\\
=&\sum_{i<j}(-1)^{i+j}\sigma\left(\left[X_i,X_j\right]_N,X_1,\dots,X_k\right)+\sum_{i<j}(-1)^{i+j}\sigma\left(\left[X_i,X_j\right]^\pi_\Omega,X_1,\dots,X_k\right)\\
&+\sum_{i=1}^k(-1)^{i+1}\mathscr{L}_{NX_i}\left(\sigma\left(X_1,\dots,X_k\right)\right)+\sum_{i=1}^k(-1)^{i+1}\mathscr{L}_{\pi^\sharp\Omega^\flat(X_i)}\left(\sigma\left(X_1,\dots,X_k\right)\right)\\
=&d_N\sigma\left(X_1,\dots,X_k\right)+\sum_{i<j}(-1)^{i+j}\sigma\left(\left[X_i,X_j\right]^\pi_\Omega,X_1,\dots,X_k\right)\\
&+\sum_{i=1}^k(-1)^{i+1}\mathscr{L}_{\pi^\sharp\Omega^\flat(X_i)}\left(\sigma\left(X_1,\dots,X_k\right)\right).
\end{aligned}$$

Since $d\Omega=0$, we can apply Proposition 3.2.2 and conclude that $\left[X_i,X_j\right]^\pi_\Omega=\left[X_i,X_j\right]_{\pi^\sharp\Omega^\flat}$, thus,

$$(\Gamma(\wedge^\bullet L),d_L)\simeq(\Gamma(\wedge^\bullet TM),d_{\hat{N}}),$$

□

**Remark 3.2.7.** Note that it is not necessarily true that $d_L^2=0$. It will happen if and only if $L$ is a Dirac structure in $\mathbb{T}M'$.

The following proposition is proven in (STIÉNON; XU, 2007):

**Proposition 3.2.8** ((STIÉNON; XU, 2007), Proposition 3.5)**.** The quadruple $(M,\pi,N,\phi)$ is a Poisson quasi-Nijenhuis manifold if and only if $((T^*M)_\pi,d_N,\phi)$ is a quasi-Lie bialgebroid and $\phi$ is a closed 3-form.

In this way, we have proved the main theorem of this chapter.

**Theorem 3.2.9.** Let $(M,\pi,N,\phi)$ be a PqN manifold and let $\Omega$ be a closed 2-form. If $\hat{N} = N+\pi^\sharp\Omega^\flat$ and

$$\phi' = \phi + d_N\Omega + \frac{1}{2}[\Omega,\Omega]_\pi,$$

then $(M,\pi,\hat{N},\phi')$ is a PqN manifold. In particular, if

$$\phi + d_N\Omega + \frac{1}{2}[\Omega,\Omega]_\pi = 0,$$

then $(M,\pi,\hat{N})$ is a PN manifold.

The proof can be summarized in the diagram shown in Figure 1. In Chapter 5 we presented another proof for this theorem using the twists of Proto-bialgebroids.

**Example 3.2.10** (The closed Toda lattice)**.** In (FALQUI *et al.*, 2020), it is shown how to deform the PN structure of the open Toda lattice to obtain PqN structure for the closed one. More than that, they show that this PqN structure is involutive.

The PN structure for the open Toda lattice is given by the manifold $\mathbb{R}^{2n}$ endowed with the Poisson tensor which is given in the canonical coordinates by

$$\pi = \sum_{i=1}^{n} \partial_{p_i} \wedge \partial_{q_i}$$

and the (1,1)-tensor

$$\begin{aligned} N = &\sum_{i=1}^{n} p_i \left(\partial_{q_i} \otimes \mathrm{d}q_i + \partial_{p_i} \otimes \mathrm{d}p_i\right) + \sum_{i<j} \left(\partial_{q_i} \otimes \mathrm{d}p_j - \partial_{q_j} \otimes \mathrm{d}p_i\right) \\ &+ \sum_{i=1}^{n} e^{q_i - q_{i+1}} \left(\partial_{p_j} \otimes \mathrm{d}q_i - \partial_{p_i} \otimes \mathrm{d}q_j\right). \end{aligned}$$

We have that

$$\frac{1}{2}I_1 = \frac{1}{2}\mathrm{Tr}(N) = \sum_{i=1}^{n} p_i, \qquad \frac{1}{2}I_2 = \frac{1}{4}\mathrm{Tr}(N^2) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} \mathrm{e}^{q_i - q_{i+1}} \tag{3.2}$$

are respectively the total momentum and the energy of the n-particle open Toda lattice.

If we apply the Theorem 3.2.9 to the above PN structure using the closed 2-form $\Omega = \mathrm{e}^{q_n - q_1} d_{q_n} \wedge dq_1$ we get the deformated tensor

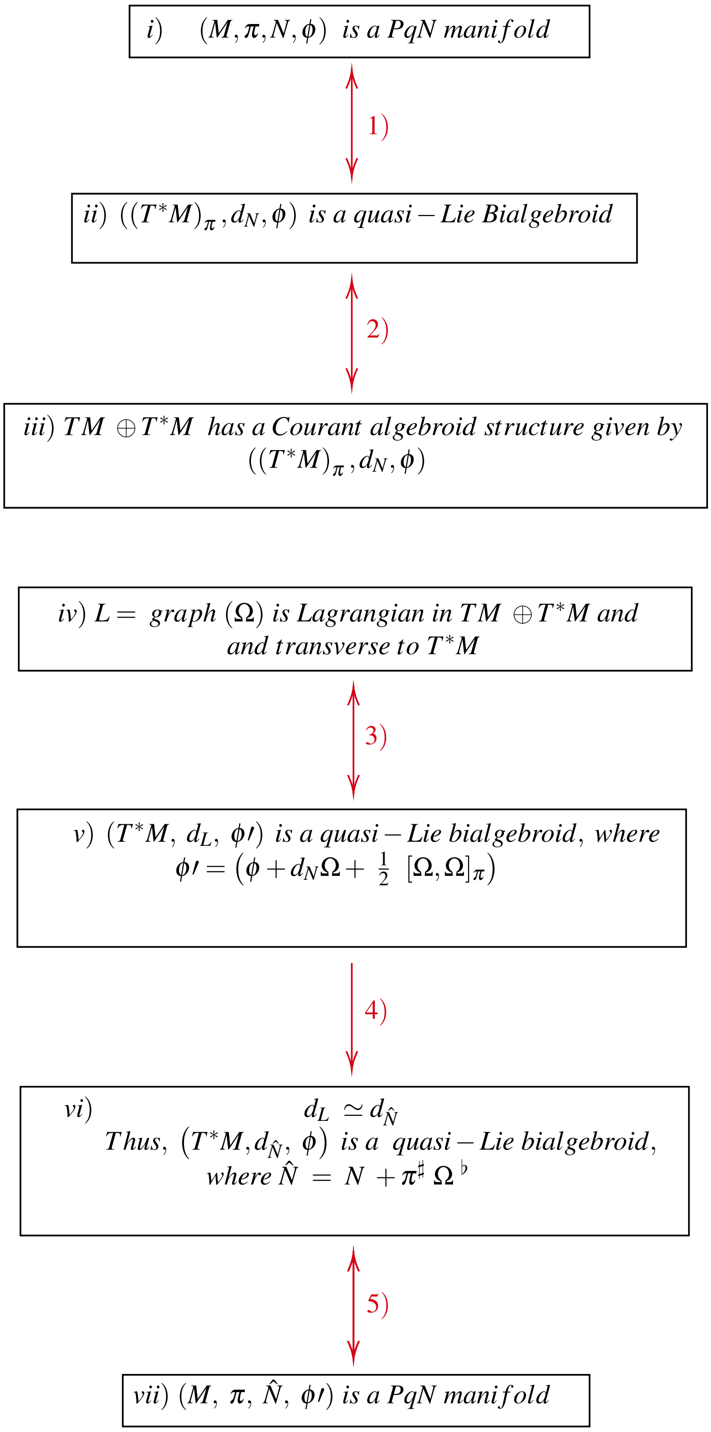
$$\begin{aligned} \hat{N} = &\sum_{i=1}^{n} p_i \left(\partial_{q_i} \otimes dq_i + \partial_{p_i} \otimes dp_i\right) + \sum_{i<j} \left(\partial_{q_i} \otimes dp_j - \partial_{q_j} \otimes dp_i\right) \\ &+ \sum_{i=1}^{n-1} \mathrm{e}^{q_i - q_{i+1}} \left(\partial_{p_{i+1}} \otimes dq_i - \partial_{p_i} \otimes dq_{i+1}\right) \\ &- \mathrm{e}^{q_n - q_1} \left(\partial_{p_1} \otimes dq_n - \partial_{p_n} \otimes dq_1\right), \end{aligned} \tag{3.3}$$

while $\phi' = d_N\Omega = dI_1 \wedge \Omega$. We recover the Hamiltonian of the n-particle close toda by

$$\frac{1}{4}\mathrm{Tr}\left(\hat{N}^2\right) = \sum_{i=1}^{n} \left(\frac{1}{2}p_i^2 + \mathrm{e}^{q_i - q_{i+1}}\right), \tag{3.4}$$

where $q_{n+1} = q_1$. The other $I_k$'s are the well know constants of the motion of the closed Toda chain.

Figure 1 – Proof of Theorem 3.2.9




1) *Proposition* 3.5 *of "Poisson Quasi – Nijenhuis Manifolds" by Stinon & Xu*

2) *Theorem* 2.6 *of "Poisson Quasi – Nijenhuis Manifolds" by Stinon & Xu*

3) *Theorem* 2.6 *of "Poisson Quasi – Nijenhuis Manifolds" by Stinon & Xu*

4) *Proposition* (3.2.6)

5) *Theorem* 3.5 *of "Poisson Quasi – Nijenhuis Manifolds" by Stinon & Xu*

**Remark 3.2.11.** If $\mathfrak{G}_{\Omega_c^2}$ is the (additive) group of closed 2-forms on $M$, the Theorem 3.2.9 implies that the set of the PqN-structures on $M$ carries the following $\mathfrak{G}_{\Omega_c^2}$-action:

$$\Omega \cdot (M, \pi, N, \phi) = (M, \pi, N + \pi^\sharp \Omega^\flat, \phi + d_N \Omega + \frac{1}{2}[\Omega, \Omega]_\pi).$$

CHAPTER

4

# DIRAC-NIJENHUIS STRUCTURES

The concept of Dirac-Nijenhuis structures is not uniformly defined in the literature, as evidenced by various definitions found in references (CLEMENTE-GALLARDO; COSTA, 2004), (HE; LIU, 2006) and (BURSZTYN; DRUMMOND; NETTO, 2023). Despite the varying definitions, they all build upon the concept of Poisson-Nijenhuis structures. In simple terms, a Dirac-Nijenhuis structure is a pair $(L,D)$, where $L \subset E$ is a Dirac structure on the Courant algebroid $(E, \langle\cdot \mid \cdot\rangle, [\![\cdot,\cdot]\!]^J, \rho)$ and $D\colon E \to E$ is a (1,1)-tensor that is compatible with $L$ in some suitable sense.

The focus of our discussion is on the definition proposed in the paper (BURSZTYN; DRUMMOND; NETTO, 2023). We aim to show that by using the deformation process on a PN manifold with the Nijenhuis tensor as the identity, we can establish a Dira-Nijenhuis structure.

In the abovementioned paper, the authors define Dirac-Nijenhuis structures using the concept of one-derivations, a specific type of generalized derivations formally introduced and explored by (BURSZTYN; DRUMMOND, 2019) and (DRUMMOND, 2022).

**Remark 4.0.1.** In this chapter, we will use the third definition of the Courant algebroid, that is, we will use the one involving the non-skew-symmetric bracket $[\![\cdot,\cdot]\!]^J$. See Definition 2.1.8.

**Definition 4.0.2.** Let $E \to M$ be a real vector bundle. A one-derivation on $E$ is a triple $\mathscr{D} = (D, l, N)$, where $N\colon TM \to TM$ and $l\colon E \to E$ are vector bundles maps covering the identity, and $D\colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ is a $\mathbb{R}$-linear map satisfying the following Leibniz-type condition:

$$D_X(fe) = fD_X(e) + (\mathscr{L}_x(f))l(e) - (\mathscr{L}_{NX}(f))e,$$

where $X \in \Gamma(TM)$, $e \in \Gamma(E)$, $f \in C^\infty(M)$ and $D_X : \Gamma(E) \to \Gamma(E)$ is defined by

$$D_X(e) = i_X(D(e)).$$

Associated to a one-derivation $\mathscr{D} = (D, l, N)$ on $E$ one can define a one-derivation on $E^*$ by $\mathscr{D}^* = (D^*, l^*, N)$, where $l^* : E^* \to E^*$ is dual to $l$, and $D^*$ is defined by the equation

$$\langle D^*_X(\xi), e\rangle = \mathscr{L}_X\langle \xi, l(e)\rangle - \mathscr{L}_{N(X)}\langle \xi, e\rangle - \langle \xi, D_X(e)\rangle$$

for $e \in \Gamma(E), \xi \in \Gamma(E^*)$, and $X \in \Gamma(TM)$.

An important property of a one-derivation is its relationship with linear (1,1)-vector fields. Given a vector bundle $q : E \to M$, its tangent prolongation is the vector bundle $T(q) : TE \to TM$, where $T(q)$ is the differential of the projection $q$. Consider $D \in \Gamma(T^*E \oplus TE)$. We say that $D$ is linear if there exists a morphism $N : TM \to TM$ that turns $D$ into a vector bundle morphism.

$$\begin{array}{ccc} TE & \xrightarrow{\ D\ } & TE \\ {\scriptstyle T(q)}\downarrow & & \downarrow{\scriptstyle T(q)} \\ TM & \xrightarrow{\ N\ } & TM \end{array}$$

In (BURSZTYN; DRUMMOND, 2019), the authors demonstrated the existence of a 1–1 correspondence between one-derivations and linear (1,1)-tensors on $E$. The following definition gives the compatibility between a one- derivation on $E$ and the Courant algebroid structure.

**Definition 4.0.3.** Let $(E, \langle\cdot \mid \cdot\rangle, [\![\cdot,\cdot]\!]^J, \rho)$ be a Courant algebroid. A Courant one-derivation is a one-derivation $\mathscr{D} = (D, l, N)$ on the vector bundle $E$ such that $\mathscr{D} = \mathscr{D}^*$ and satisfying the following compatibility equations:

(CN1) $\rho \circ l = N \circ \rho$;

(CN2) $\rho\left(D_X(e)\right) = D^N_X(\rho(e))$;

(CN3) $l\left([\![e_1, e_2]\!]^J\right) = [\![e_1, l(e_2)]\!]^J - D_{\rho(e_2)}(e_1) - \rho^*(C(e_1, e_2))$;

(CN4) $D_X\left([\![e_1, e_2]\!]^J\right) = [\![e_1, D_X(e_2)]\!]^J - [\![e_2, D_X(e_1)]\!]^J + D_{[\rho(e_2), X]}(e_1) - D_{[\rho(e_1), X]}(e_2)$
$- \rho^*(i_X dC(e_1, e_2))$,

for all $e_1, e_2 \in \Gamma(E)$ and $X \in \Gamma(TM)$, where

$$C(e_1, e_2) = \langle D_{(\cdot)}(e_1) \mid e_2\rangle.$$

**Definition 4.0.4.** We call the one-derivation $\mathscr{D} = (D, l, N)$ a Nijenhuis one-derivation if the following equations hold

$$T_N = 0, \quad (4.1)$$

$$D_X(l(e)) - l(D_X(e)) = 0, \quad (4.2)$$

$$l(D_{[X,Y]}(e)) - [D_X, D_Y](e) - D_{[X,Y]_N}(e) = 0. \quad (4.3)$$

**Definition 4.0.5.** A Courant-Nijenhuis one-derivation is a Courant one-derivation that is also a Nijenhuis one-derivation. A Courant algebroid equipped with a Courant-Nijenhuis one-derivation is called Courant-Nijenhuis algebroid.

**Definition 4.0.6.** Let $(E, \langle \cdot \mid \cdot \rangle, [\![\cdot,\cdot]\!]^J, \rho)$ be a Courant algebroid and $\mathscr{D} = (D, l, N)$ be a Courant one-derivation. A Lagrangian subbundle $L$ is called $\mathscr{D}$-invariant if

- $l(L) \subset L$;
- $D_X(\gamma(L)) \subset \Gamma(L)$,

for all $X \in \Gamma(TM)$.

**Proposition 4.0.7.** Let $\mathscr{D} = (D, l, N)$ be a Courant one-derivation. A Lagrangian subbundle $L$ is $\mathscr{D}$-invariant if and only if

$$S\big|_L = C\big|_L = 0,$$

where $S$ is the symmetric 2-form defined by

$$S(e_1, e_2) = \langle l(e_1) \mid e_2 \rangle,$$

and $C$ is defined by

$$C(e_1, e_2) = \langle D_{(\cdot)}(e_1) \mid e_2 \rangle.$$

**Definition 4.0.8.** A Dirac structure $L$ is called a Dirac-Nijenhuis structure if it is $\mathscr{D}$-invariant.

## 4.1 How Dirac-Nijenhuis structures generalize Poisson-Nijenhuis Manifolds

Given a $(1,1)$-tensor $N\colon TM \to TM$, we can define $D^N : \Gamma(TM) \to \Gamma(T^*M \otimes TM)$ by

$$D^N_X(Y) = i_X D^N(Y) = [Y, NX] - N[Y, X], \text{ for all } X, Y \in \Gamma(TM), \tag{4.4}$$

and $D^{N*} : \Gamma(T^*M) \to \Gamma(T^*M \otimes T^*M)$ by

$$D^{N*}_X(\alpha) = i_X D^{N*}(\alpha) = \mathscr{L}_X(N^*\alpha) - \mathscr{L}_{NX}(\alpha), \text{ for all } X \in \Gamma(TM) \text{ and } \alpha \in \Gamma(T^*M). \tag{4.5}$$

On (BURSZTYN; DRUMMOND; NETTO, 2022), it is proven that $\mathrm{D}^N = ((D^N, D^{N*}), (N, N^*), N)$ is a Courant one-derivation of $\mathbb{T}M = ((TM, [\cdot,\cdot], Id), (T^*M, 0, 0))$. Additionally, $N$ is a Nijenhuis tensor if and only if $\mathrm{D}^N$ is a Nijenhuis one-derivation.

**Proposition 4.1.1.** Let $\pi \in \Gamma(\wedge^2 TM)$ and $L_\pi = \mathrm{graph}(\pi^\sharp)$ and $N\colon TM \to TM$ be a Nijenhuis tensor. The pair $(L_\pi, N)$ is a Dirac-Nijenhus structure if and only if $(\pi, N)$ is a Poisson Nijenhuis structure.

**Definition 4.1.2.** Let $\Omega \in \Gamma(\wedge^2 A^*)$ and $N\colon \Gamma(A) \to \Gamma(A)$. We say that $(\Omega, N)$ is a $\Omega N$-structure if $\Omega^\flat \circ N = N^* \circ \Omega^\flat$, $N$ is a Nijenhuis tensor, and both $\Omega$ and $\Omega_N$ are closed, where $\Omega_N(X,Y) = \Omega(NX,Y)$.

**Proposition 4.1.3.** Let $\Omega$ be a closed 2-form, $L_\Omega = \mathrm{graph}(\Omega^\flat)$ and $N\colon TM \to TM$ be a Nijenhuis tensor. The pair $(L_\Omega, N)$ is a Dirac-Nijenhus structure if and only if $(\Omega, N)$ is a $\Omega N$-structure.

## 4.2 Deformation of the Identity

In this last subsection we will prove the following result:

**Theorem 4.2.1.** If $(M,\pi)$ is a Poisson manifold and $\Omega$ satisfies the conditions

$$[\Omega,\Omega]_\pi = 0 \tag{4.6}$$

$$d\Omega = 0, \tag{4.7}$$

then $(L_\Omega, \hat{N})$ is a Dirac-Nijenhuis structure, where $\hat{N} = Id + \pi^\sharp\Omega^\flat$.

Through Theorem 3.2.9, we have that $(M,\pi,\hat{N})$ is a PN manifold. Associated with every deformation of this type, we have a $\Omega N$-structure. Precisely,

**Proposition 4.2.2.** Let $(M,\pi,Id)$ and $\Omega$ as above. Then $(\Omega, \pi^\sharp\Omega^\flat)$ is a $\Omega N$-structure.

*Proof.* First, note that

$$Id \circ \Omega^\flat + (\pi^\sharp \circ \Omega^\flat)^* \circ \Omega^\flat = Id \circ \Omega^\flat + (\Omega^\flat \circ \pi^\sharp) \circ \Omega^\flat = Id \circ \Omega^\flat + \Omega^\flat \circ (\pi^\sharp \circ \Omega^\flat).$$

By Theorem 3.2.9, we have that $\hat{N}$ is a Nijenhuis tensor. Now, we must show that $\Omega_{\hat{N}}$ is closed, but seeing that

$$(d + d_{\pi^\sharp\Omega^\flat})\Omega = i_{\pi^\sharp\Omega^\flat} d\Omega - d(i_{\pi^\sharp\Omega^\flat}\Omega),$$

and $\frac{1}{2} i_{\pi^\sharp\Omega^\flat}\Omega = \Omega_{\pi^\sharp\Omega^\flat}$, we have that $\Omega_{\pi^\sharp\Omega^\flat}$ is closed. □

Thus, by Proposition 4.1.3, we have that $(L_\Omega, \hat{N})$ is a Dirac-Nijenhuis structure, and Theorem 4.2.1 is proved.

**Remark 4.2.3.** Although the exigence of $N = Id$ is somewhat restrictive, as shown in (FALQUI; MENCATTINI; PEDRONI, 2023), some important examples of PqN structures can be constructed as a deformation of the identity. We can cite Example 2.3.15, the closed Toda lattice, and the rational Calogero system.

CHAPTER

5

# THE BIG BRACKET FORMALISM FOR THE DEFORMATION THEOREM

pIn this chapter, we will use the theory of supermanifolds and the works of (ANTUNES, 2008), (KOSMANN-SCHWARZBACH; RUBTSOV, 2010) and (ROYTENBERG, 2002) to present an alternative proof of Theorem 3.2.9 in the setting of twists of Proto-bialgebroids.

## 5.1 Big bracket

The theory of graded manifolds is a generalization of the theory of smooth manifolds in which one can work with functions on $\mathbb{Z}$-graded variables. One of the applications of this formalism is that we can see the graded algebra of sections $\wedge^\bullet A$ of some vector bundle $A$ as functions on a supermanifold. See (ROYTENBERG, 1999; VORONOV, 2002).

Let $A \to M$ be a vector bundle. Let $\{x^1,\dots,x^n\}$ be a local coordinate system for $U \subset M$, and let $\{e^1,\dots,e^m\}$ be a local frame in $U$. Then we have the induced local coordinate system $\{x^1,\dots,x^n,\xi^1,\dots,\xi^m\}$ for the supermanifold $\Pi A$, where $x^i$ are the commutative coordinates on the base and $\xi^j$ are the anti-commutative coordinates for the fibers. We have the isomorphism $C^\infty(\Pi A^*) \simeq \Gamma(T^*M)$.

The cotangent bundle $T^*\Pi A^*$ has a canonical symplectic structure with Darboux coordinates $\{x^i,\xi^j,p_i,\theta_j\}$, where $\{\theta_j\}$ are the generators of the algebra $\Gamma(\wedge^\bullet A\big|_U)$ corresponding to $\{\xi^j\}$. We also can consider $T^*\Pi A$ with the canonical symplectic structure. Then, there is a symplectomorphism $L\colon T^*\Pi A^* \to T^*\Pi A$ given in local coordinates by

$$(x^i,\xi^j,p_i,\theta_j) \to (x^i,\theta_j,p_i,\xi^j).$$

See e.g. (ROYTENBERG, 1999; ROYTENBERG, 2002; VORONOV, 2002). Thus, we can see both $\Gamma(\wedge^\bullet A^*)$ and $\Gamma(\wedge^\bullet A)$ inside $C^\infty(T^*\Pi A^*)$.

$$\begin{array}{ccc} T^*\Pi A^* & \xrightarrow{\quad L \quad} & T^*\Pi A \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle pr_2} \\ \Pi A^* & & \Pi A \end{array}$$

To the coordinates of $T^*\Pi A^*$ two degrees are assigned as follows: a bidegree $(\varepsilon, \delta)$ given by

| $x^i$ | $\xi^j$ | $p_i$ | $\theta_j$ |
|---|---|---|---|
| (0,0) | (0,1) | (1,1) | (1,0) |

resulting in a total degree of $\varepsilon + \delta$,

| $\lvert x^i \rvert$ | $\lvert \xi^j \rvert$ | $\lvert p_i \rvert$ | $\lvert \theta_j \rvert$ |
|---|---|---|---|
| 0 | 1 | 2 | 1 |

The Poisson bracket $\{\cdot,\cdot\}$ on $T^*\Pi A^*$ with bidegree (-1,-1) is known as the “big bracket”, see e.g. (KOSMANN-SCHWARZBACH, 2005; KOSMANN-SCHWARZBACH; RUBTSOV, 2010). It satisfies the following properties:

(i) $\{P_1, P_2\} = 0$, if $P_1, P_2 \in \Gamma(\wedge^\bullet A)$;

(ii) $\{\sigma_1, \sigma_2\} = 0$, if $\sigma_1, \sigma_2 \in \Gamma(\wedge^\bullet A^*)$;

(iii) $\{X, \alpha\} = \alpha(X)$, if $X \in \Gamma(A)$ and $\alpha \in \Gamma(A^*)$;

(iv) $\{\Theta_1, \Theta_2\} = -(-1)^{|\Theta_1||\Theta_2|}\{\Theta_2, \Theta_1\}$;

(v) $\{\Theta_1, \Theta_2 \wedge \Theta_3\} = \{\Theta_1, \Theta_2\} \wedge \Theta_3 + (-1)^{\theta_1\theta_2}\Theta_2 \wedge \{\Theta_1, \Theta_3\}$, for all $\Theta_1 \in \Gamma(\wedge^{\theta_1}(A \oplus A^*))$ and $\Theta_2 \in \Gamma(\wedge^{\theta_2}(A \oplus A^*))$.

Additionally, it satisfies the Jacobi identity:

$$\{\Theta_1, \{\Theta_2, \Theta_3\}\} = \{\{\Theta_1, \Theta_2\}, \Theta_3\} + (-1)^{|\Theta_1||\Theta_2|}\{\Theta_2, \{\Theta_1, \Theta_3\}\}.$$

Or, equivalently,

$$\{\{\Theta_1, \Theta_2\}, \Theta_3\} = \{\Theta_1, \{\Theta_2, \Theta_3\}\} + (-1)^{|\Theta_2||\Theta_3|}\{\{\Theta_1, \Theta_3\}, \Theta_2\}.$$

In local coordinates:

$$\{x^i, p_j\} = \delta_i^j,$$
$$\{\xi^i, \theta_j\} = \delta_i^j,$$

while it vanishes in other combinations.

**Proposition 5.1.1.** Given elements $e_i = X_i \oplus \alpha_i \in A \oplus A^*$, where $i \in \{1,2,3\}$, then

$$\{e_1\{e_2,e_3\}\} = 0.$$

*Proof.* Firstly, observe that $\{e_2,e_3\} = \langle e_2 \mid e_3 \rangle \in C^\infty(M)$ possesses bidegree (0,0). Additionally, $e_1 = X_1 + \alpha_1$, where $X_1$ holds a bidegree of (1,0) and $\alpha_1$ a bidegree of (0,1). The result follows from the fact that the big bracket has a bidegree of (-1,-1). □

## 5.2 Lie algebroid

Given a vector bundle $A$, a Lie algebroid structure on $A$ can be given by a square zero differential operator $d_A$. This operator acts on $C^\infty(\Pi A)$ in the following way

$$\begin{aligned} &d_A(f)(X) = \rho_A(X)f, \\ &d_A(\alpha)(X_1,X_2) = \rho_A(X_1)(\alpha(X_2)) - \rho_A(X_2)(\alpha(X_1)) - \alpha([X_1,X_2]_A), \end{aligned} \tag{5.1}$$

where $f \in C^\infty(M)$, $\alpha \in \Gamma(A^*)$ and $X_1, X_2 \in \Gamma(A)$. Then, one can see $d_A$ as an element inside $T^*\Pi A$, which is given in local coordinates by

$$d_A = \xi_j A_i^j(X)\frac{\partial}{\partial x_i} - \frac{1}{2}C_k^{ij}(x)\xi_i\xi_j\frac{\partial}{\partial \xi_k} = \xi_j A_i^j(X)p^i - \frac{1}{2}C_k^{ij}(x)\xi_i\xi_j\theta^k, \tag{5.2}$$

where $\rho_A(\xi_i) = A^i_j\frac{\partial}{\partial x_j}$ and $[\xi_i,\xi_j]_A = C_k^{ij}(x)\xi_k$.

**Proposition 5.2.1.** The condition $d_A^2 = 0$ is equivalent to

$$\{d_A, d_A\} = 0.$$

*Proof.* First, note that under the inclusion of $\Gamma(\wedge^\bullet A^*) \subset C^\infty(T^*\Pi A)$, we have that, given $\sigma \in \Gamma(\wedge^k A^*)$, $\{d_A,\sigma\} \in \Gamma(\wedge^{k+1}A^*)$. From the properties of the Poisson bracket, we have that $\{d_A,\cdot\}$ is a degree 1 derivation of $(\Gamma(\wedge^\bullet A^*),\wedge)$.

By means of Equation (5.2), $\{d_A,\cdot\}$ acts in $(\Gamma(\wedge^\bullet A^*),\wedge)$ in the same way $d_A$ does. Applying the Jacobi identity, we have

$$d_A^2(\sigma) = \{d_A,\{d_A,\sigma\}\} = \{\{d_A,d_A\},\sigma\} - \{d_A,\{d_A,\sigma\}\}.$$

Thus,

$$d_A^2(\sigma) = \{d_A,\{d_A,\sigma\}\} = \frac{1}{2}\{\{d_A,d_A\},\sigma\}.$$

Hence, $d_A^2 = 0$ if and only if $\{d_A,d_A\} = 0$. □

Thus, we can understand a Lie algebroid in terms of the big bracket as follows.

**Definition 5.2.2.** A Lie algebroid structure on a vector bundle $A \to M$ is the supermanifold $T^*\Pi A$ together with a function $\mu$ with bidegree (1,2) such that

$$\{\mu,\mu\} = 0.$$

The Lie algebroid structure defines a Schouten bracket on the section of $\wedge^\bullet A$ by

$$[P,Q]_\mu = -\{\{P,\mu\},Q\},$$

for all $P,Q \in \Gamma(\wedge^\bullet A)$. It also defines an anchor map $\rho_\mu : A \to TM$, by

$$\rho_\mu(X)f = \{\{X,\mu\},f\},$$

for all $X \in \Gamma(A)$ and $f \in C^\infty(M)$.

The differential operator of the Lie algebroid is given by

$$d_\mu = \{\mu,\cdot\},$$

and, given $X \in \Gamma(A)$, the Lie derivative is defined in terms of the differential and contraction operators by

$$\mathscr{L}_X^\mu = [i_X, d_\mu].$$

**Remark 5.2.3.** Through the identification $L\colon T^*\Pi A^* \to T^*\Pi A$, a Lie algebroid $(A^*, [\cdot,\cdot]_{A^*}, \rho_{A^*})$ can be described by a function $\gamma$ with bidegree (2,1) on $C^\infty(T^*\Pi A^*)$ such that $\{\gamma,\gamma\} = 0$.

## 5.3 Bivectors, 2-forms and (1,1)-tensor

To characterize Poisson quasi-Nijenhuis structures within a Lie algebroid, it's crucial to grasp how (1,1)-tensors and 2-sections of both $A$ and $A^*$ are incorporated into this framework

**Lemma 5.3.1.** Let $\pi \in \Gamma(\wedge^2 A)$, $\Omega \in \Gamma(\wedge^2 A^*)$ and $N\colon A \to A$ be an endomorphism induced by a (1,1)-tensor. Then,

$$\begin{aligned}\pi^\sharp(\alpha) &= \{\alpha,\pi\}, \text{ for all } \alpha \in \Gamma(\wedge A^*),\\ \Omega^\flat(X) &= \{\Omega,X\}, \text{ for all } X \in \Gamma(\wedge A),\\ NX &= \{X,N\}, \text{ for all } X \in \Gamma(\wedge A).\end{aligned}$$

*Proof.* The results follow directly comparing the local expression of both sides of the equations. □

We can also consider the (1,1)-tensor defined by $\pi^\sharp\Omega^\flat$ .

**Lemma 5.3.2.** The map $N = \pi^\sharp \circ \Omega^\flat$ is given by

$$N = \{\pi,\Omega\}.$$

*Proof.* Since, for every $X \in \Gamma(A)$, $X = \sum_i X_i\theta^i$ and $\pi = \sum_{i,j}\frac{1}{2}\pi_{ij}\theta^i\theta^j$, we have that

$$\{X,\pi\} = 0.$$

Applying the Jacobi identity for $\{\cdot,\cdot\}$, we have that

$$\pi^\sharp(\Omega^\flat X) = \{\Omega^\flat X, \pi\} = \{\{\Omega, X\}, \pi\} = \{\{\Omega, \pi\}, X\} = \{X, \{\pi, \Omega\}\} = \{X, N\}.$$

□

**Lemma 5.3.3.** Let $\pi \in \Gamma(\wedge^2 A)$ and $\phi \in \Gamma(\wedge^3 A^*)$, then, for all $X, Y \in \Gamma(\wedge A)$,

$$\pi^\sharp(i_{X\wedge Y}\phi) = \{\{X, \{\phi, \pi\}\}, Y\}.$$

*Proof.* By the Jacobi identity, we have

$$\begin{aligned}\{\{X, \{\phi, \pi\}\}, Y\} &= \{\{\{X, \phi\}, \pi\}, Y\} = \{\{\{X, \phi\}, Y\}, \pi\} = \{\{i_X\phi, Y\}, \pi\} = \{i_y(i_x\phi), \pi\} \\ &= \pi^\sharp(i_{X\wedge Y}\phi).\end{aligned}$$

□

If $\pi$ is a 2-section, $\gamma_\pi = \{\pi, \mu\}$ has degree (2,1). Consequently, if $\{\gamma_\pi, \gamma_\pi\} = 0$, it defines a Lie algebroid structure on the dual bundle $A^*$. The following lemma provides the necessary and sufficient conditions for the 2-section $\pi$ to define a Lie algebroid structure in $A^*$.

**Lemma 5.3.4.** $\gamma_\pi$ defines a Lie algebroid structure on $A^*$ if and only if

$$\{\mu, [\pi, \pi]_\mu\} = 0.$$

*Proof.* By definition,

$$\{\gamma_\pi, \gamma_\pi\} = \{\{\pi, \mu\}, \{\pi, \mu\}\}.$$

Using the Jacobi identity,

$$\{\gamma_\pi, \gamma_\pi\} = \{\{\{\pi, \mu\}, \pi\}, \mu\} + \{\pi, \{\{\pi, \mu\}, \mu\}\} = \{[\pi, \pi]_\mu, \mu\}.$$

□

The notion of the Poisson bi-vector field can be extended to an arbitrary Lie algebroid.

**Definition 5.3.5.** Let $(A, \mu)$ be a Lie algebroid and $\pi \in \Gamma(\wedge^2 A)$. We say that $\pi$ is a Poisson structure on $(A, \mu)$ if

$$[\pi, \pi]_\mu = \{\{\pi, \mu\}, \pi\} = 0.$$

Note that, if $\pi$ is a Poisson structure, Lemma 5.3.4 ensures that $\gamma_\pi$ defines a Lie algebroid structure on $A^*$, but it does not work both ways. That is, if we start with $\pi \in \Gamma(\wedge^2 A)$ such that $\{\gamma_\pi, \gamma_\pi\} = 0$, we have that

$$\{\mu, [\pi, \pi]_\mu\} = 0.$$

The above equation means that $[\pi,\pi]_\mu$ vanishes on the $d_\mu$-exact 1-sections of $A$. Indeed, through the Jacobi identity,

$$\{[\pi,\pi]_\mu, d_\mu f\} = \{[\pi,\pi]_\mu, \{\mu, f\}\} = \{\{[\pi,\pi]_\mu, \mu\}, f\} + \{\mu, \{[\pi,\pi]_\mu, f\}\}.$$

Since $\{\mu,[\pi,\pi]_\mu\} = 0$ and $[\pi,\pi]_\mu$ has not components on $p^k$, we have that

$$\{[\pi,\pi]_\mu, d_\mu f\} = 0, \text{ for all } f \in C^\infty(M).$$

When the $d_\mu$-exact 1-forms generate $\Gamma(A^*)$ locally, we have that

$$\{\mu,[\pi,\pi]_\mu\} = 0 \Longleftrightarrow [\pi,\pi]_\mu = 0.$$

**Remark 5.3.6.** We will define the Schouten bracket associated with $\gamma_\pi$ using a different convention than the one defined for $\mu$. Specifically, we denote it as:

$$[\alpha,\beta]_\gamma = \{\{\gamma_\pi,\alpha\},\beta\}.$$

This choice is made to maintain consistency with the sign defined in Chapter 3 and align with the results presented in (FALQUI; MENCATTINI; PEDRONI, 2023). The bracket defined by $\gamma_\pi$ on $\Gamma(\wedge^\bullet A^*)$ is the Koszul bracket of forms. For $C^\infty(M)$ and $\Gamma(A^*)$,

$$\{\{\{\pi,\mu\},\alpha\},f\} = ((\rho_\mu \circ \pi^\sharp)\alpha)f,$$
$$\{\{\{\pi,\mu\},\alpha\},\beta\} = \mathscr{L}^\mu_{\pi^\sharp\alpha}(\beta) - \mathscr{L}^\mu_{\pi^\sharp\beta}(\alpha) - d_\mu(\pi(\alpha,\beta)).$$

Similarly to how the endomorphism of the tangent bundle alters the Lie algebroid structure of $(A,[\cdot,\cdot],Id)$, a (1,1)-tensor $N \in \Gamma(A^*\otimes A)$ deforms the structure associated with the function $\mu$ of bidegree (0,2) in the following manner:

$$\mu_N = \{N,\mu\}.$$

The function $\mu_N$ is $d_\mu$-closed.

**Lemma 5.3.7.** Given a (1,1)-tensor $N \in \Gamma(A^*\otimes A)$ on the Lie algebroid $(A,\mu)$. Then

$$\{\mu,\mu_N\} = 0.$$

*Proof.* Applying the Jacobi identity,

$$\{\mu,\{N,\mu\}\} = \{\{\mu,N\},\mu\} + \{N,\{\mu,\mu\}\}.$$

But, $\{\mu,N\} = -\{N,\mu\}$ and $\{\mu,\mu\} = 0$, thus

$$2\{\mu,\{N,\mu\}\} = 0.$$

□

This new function $\mu_N$ has also bidegree (0,2) and defines an anchor $\rho_\mu^N = \rho_\mu \circ N$ and a skew-symmetric bracket on $A$ as

$$[X,Y]_N^\mu = \{X,\{N,\mu\}\},Y\}, \text{ for all } X,Y \in \Gamma(A).$$

It is called the deformed bracket.

**Lemma 5.3.8.** For all $X,Y \in \Gamma(A)$,

$$[X,Y]_N^\mu = [NX,Y]_\mu + [X,NY]_\mu - N[X,Y]_\mu.$$

*Proof.* Using the Jacobi identity to $\{\cdot,\cdot\}$ and the definitions of $[\cdot,\cdot]_\mu$ and $[\cdot,\cdot]_N^\mu$, we have

$$\begin{aligned}[X,Y]_N^\mu &= \{\{X,\{N,\mu\}\},Y\} = \{\{\{X,N\},\mu\},Y\} + \{\{N,\{X,\mu\}\},Y\} \\ &= [NX,Y]_\mu + \{N,\{\{X,\mu\},Y\}\} + \{\{N,Y\},\{X,\mu\}\} \\ &= [NX,Y]_\mu + [X,NY]_\mu - N[X,Y]_\mu.\end{aligned}$$

□

The Nijenhuis torsion of $N$ is defined in the usual way, for all $X,Y \in \Gamma(A)$,

$$T_\mu N(X,Y) = [Nx,NY]_\mu - N([X,Y]_\mu^N) = [Nx,NY]_\mu - N([NX,Y]_\mu + [X,NY]_\mu - N[X,Y]_\mu). \tag{5.3}$$

The following Lemma gives the description of the Nijenhuis torsion in terms of the Big bracket.

**Lemma 5.3.9.** The Nijenhuis torsion satisfies the following equations:

$$T_\mu N = \frac{1}{2}(\{N,\{N,\mu\}\} - \{N^2,\mu\}), \tag{5.4}$$

$$\frac{1}{2}\{\{N,\mu\},\{N,\mu\}\} = \{\mu,T_\mu N\}. \tag{5.5}$$

*Proof.* After applying many times the Jacobi identity, we can prove the Equation (5.4). See e.g. (GRABOWSKI, 2006; KOSMANN-SCHWARZBACH; RUBTSOV, 2010). Now, let us prove Formula (5.5). Note that, thanks to Lemma 5.3.7, $\{\mu,\{N^2,\mu\}\} = 0$, and consequently,

$$\{\mu,T_\mu N\} = \frac{1}{2}\{\mu\{N,\{N,\mu\}\}\}.$$

Using the Jacobi identity, we have

$$\{\mu\{N,\{N,\mu\}\}\} = \{\{\mu,N\},\{N,\mu\}\} + \{N,\{\mu,\{N,\mu\}\}\}.$$

Since $\{\mu,\mu\} = 0$, $\{\mu,\{N,\mu\}\} = 0$. Thus,

$$\{\mu,T_\mu N\} = \frac{1}{2}\{\{N,\mu\},\{N,\mu\}\}.$$

□

We define a Nijenhuis tensor as follows.

**Definition 5.3.10.** Let $N \in \Gamma(A \times A^*)$ be a (1,1)-tensor of the Lie algebroid $(A,\mu)$. We say that $N$ is a Nijenhuis tensor if

$$T_\mu N = 0.$$

Similar to the Poisson structure, $(A,\mu_N)$ forms a Lie algebroid if $N$ is a Nijenhuis tensor, but the opposite direction does not necessarily hold true. In fact, equation (5.3) shows that $T_\mu N$ has no components in $p^i$, thus, if $\{\mu, T_\mu N\} = 0$, for every $f \in C^\infty(M)$,

$$\{T_\mu N, d_\mu f\} = \{\{T_\mu N, \mu\}, f\} + \{\mu, \{T_\mu N, f\}\} = 0.$$

If we assume that locally the $d_\mu$-exact 1-forms generate $\Gamma(A^*)$, then we have that

$$\{\mu, T_\mu N\} = 0 \Longleftrightarrow T_\mu N = 0.$$

## 5.4 Proto-bialgebroid

Proto-bialgebroid generalizes the notion of Manin triples for Lie bialgebroids.

**Definition 5.4.1.** A proto-bialgebroid is the supermanifold $T^*\Pi A$ together with a function $\Theta$ such that $|\Theta| = 3$ and $\{\Theta,\Theta\} = 0$.

A function with total degree 3 can be written as $\Theta = \mu + \gamma + \phi + \psi$ such that $\mu$ has bi-degree $(1,2)$, $\gamma$ has bi-degree $(2,1)$, $\phi$ has bi-degree $(0,3)$, and $\psi$ has bi-degree $(3,0)$. The condition $\{\Theta,\Theta\} = 0$ becomes the following equations

$$\frac{1}{2}\{\mu,\mu\} + \{\gamma,\phi\} = 0 \tag{5.6}$$

$$\frac{1}{2}\{\gamma,\gamma\} + \{\mu,\psi\} = 0 \tag{5.7}$$

$$\{\mu,\gamma\} + \{\phi,\psi\} = 0 \tag{5.8}$$

$$\{\mu,\phi\} = 0 \tag{5.9}$$

$$\{\gamma,\psi\} = 0 \tag{5.10}$$

**Definition 5.4.2.** Let $\Theta = \mu + \gamma + \phi + \psi$ be a proto-bialgebroid structure.

- $(A,A^*)$ is a Lie bialgebroid if $\phi = \psi = 0$. Then both $A$ and $A^*$ are Lie algebroids.
- $(A,A^*)$ is a quasi-Lie bialgebroid if $\psi = 0$. Then $A$ is Lie algebroid.
- $(A,A^*)$ is a Lie quasi-bialgebroid if $\phi = 0$. Then $A^*$ is Lie algebroid.

In (ROYTENBERG, 1999), Roytenberg proves that a proto-bialgebroid induces a Courant algebroid structure on $A \oplus A^*$. More precisely,

**Theorem 5.4.3.** If $\Theta = \mu + \gamma + \phi + \psi$ is a proto-bialgebroid structure on $T^*\Pi A$, then

$$\langle X_1 \oplus \alpha_1, X_2 \oplus \alpha_2 \rangle = \{X_1 \oplus \alpha_1, X_2 \oplus \alpha_2\}$$
$$\rho_\Theta(X \oplus \alpha)(f) = \{\{X, \mu\}, f\} + \{\{\alpha, \gamma\}, f\}$$
$$[\![X_1 \oplus \alpha_1, X_2 \oplus \alpha_2]\!]^J_\Theta = \{\{\Theta, X_1 \oplus \alpha_1\}, X_2 \oplus \alpha_2\}$$

define a structure of Courant algebroid on $A \oplus A^*$.

*Proof.* We will show that the 3 proprieties of Definition 2.1.8 holds. For every $e_1, e_2, e_3 \in A \oplus A^*$,

$$[\![e_1, [\![e_2, e_3]\!]^J_\Theta]\!]^J_\Theta = \{\{\Theta, e_1\}, \{\{\Theta, e_2\}, e_3\}\}.$$

Using the Jacobi identity,

$$\{\{\Theta, e_1\}, \{\{\Theta, e_2\}, e_3\} = \{\{\{\Theta, e_1\}, \{\Theta, e_2\}\}, e_3\} - \{\{\Theta, e_2\}, \{\{\Theta, e_1\}, e_3\}\}.$$

But

$$\{\{\{\Theta, e_1\}, \{\Theta, e_2\}\}, e_3\} = \{\{\Theta, \{e_1, \{\Theta, e_2\}\}\}, e_3\} + \{\{\{\Theta, \{\Theta, e_2\}\}, e_1\}, e_3\}$$

and

$$\{\{\Theta, e_2\}, \{\{\Theta, e_1\}, e_3\}\} = \{\Theta, \{e_2, \{\{\Theta, e_1\}, e_3\}\} + \{\{\Theta, \{\{\Theta, e_1\}, e_3\}\}, e_2\}.$$

Thus, (J1) holds if

$$\{\{\{\Theta, \{\Theta, e_2\}\}, e_1\}, e_3\} - \{\Theta, \{e_2, \{\{\Theta, e_1\}, e_3\}\} = 0.$$

After applying the Jacobi identity many times, the equation above holds.

For (J2), we must show that

$$\{\{e, \Theta\}, \{e_1, e_2\}\} = \{e, \{\{\Theta, e_1\}, e_2\} + \{\{\Theta, e_2\}, e_1\}\}.$$

But applying the Jacobi identity,

$$\{e, \{\{\Theta, e_1\}, e_2\} = \{e, \{\Theta, \{e_1, e_2\}\}\} - \{e, \{\{\Theta, e_2\}, e_1\}\}$$

and

$$\{e, \{\Theta, \{e_1, e_2\}\}\} = \{\{e, \Theta\}, \{e_1, e_2\}\} - \{\theta, \{e, \{e_1, e_2\}\}\}.$$

By Proposition 5.1.1, we have that $\{e, \{e_1, e_2\}\} = 0$. Thus,

$$\{e, \{\{\Theta, e_1\}, e_2\} + \{\{\Theta, e_2\}, e_1\}\} = \{\{e, \Theta\}, \{e_1, e_2\}\},$$

and propriety (J2) holds.

Using the Jacobi identity on $\rho(e)\langle e_1 \mid e_2 \rangle$, we have

$$\{\{e, \Theta\}, \{e_1, e_2\}\} = \{\{\{\Theta, e\}, e_1\}, e_2\} + \{e_1, \{\{\Theta, e\}, e_2\}\}.$$

Thus, (J3) holds. □

Let $(A, [\cdot,\cdot]_A, \rho_A)$ be the Lie algebroid structure induced by $\mu$ and $(A^*, [\cdot,\cdot]_{A^*}, \rho_{A^*})$ be the Lie algebroid structure induced by $\gamma$. Then, the Courant algebroid structure is given by

$$\begin{aligned}\langle X_1 \oplus \alpha_1, X_2 \oplus \alpha_2 \rangle =& \alpha_2(X_1) + \alpha_1(X_2)\\ \rho_\theta(X \oplus \alpha)(f) =& \rho_A(X) + \rho_{A^*}(\alpha)\\ [\![X_1 \oplus \alpha_1, X_2 \oplus \alpha_2]\!]_\theta^J =& \Big([X_1, Y_2]_A + \mathscr{L}_{\alpha_1}^{A^*}(X_2) - i_{\alpha_2} d_{A^*} X_1 - \psi(\alpha_1,\alpha_2)\Big)\\ &\oplus \Big([\alpha_1,\alpha_2]_{A^*} + \mathscr{L}_{X_1}^{A}(\alpha_2) - i_{X_2} d_A \alpha_1 - \phi(X_1, X_2)\Big).\end{aligned}$$

## 5.5 Twisting

Let $\Omega \in C^\infty(T^*\Pi M)$ with bi-degree (0,2). Let the transformation $e^\Omega \colon C^\infty(T^*\Pi M) \to C^\infty(T^*\Pi M)$ defined by the series

$$e^\Omega(a) = a + \{\Omega, a\} + \frac{1}{2!}\{\Omega,\{\Omega,a\}\} + \frac{1}{3!}\{\Omega,\{\Omega,\{\Omega,a\}\}\} + \cdots$$

In the Darboux coordinates, $\Omega = \sum_{ij} \frac{1}{2}\Omega_{ij}\xi_i\xi_j$, then $e^\Omega$ acts on the coordinates as follows

$$\begin{aligned}e^\Omega(x_i) &= x_i,\\ e^\Omega(\xi_i) &= \xi_i,\\ e^\Omega(p^i) &= p^i - \frac{1}{2}\frac{\partial \Omega_{ij}}{\partial X_i}\xi_i\xi_j,\\ e^\Omega(\theta^i) &= \theta^i - \Omega_{ij}\theta^j.\end{aligned}$$

Applying $e^\Omega(\Theta)$, we have the so called twisted structure $\Theta_\Omega = \phi_\Omega + \gamma_\Omega + \mu_\Omega + \psi_\Omega$, where

$$\begin{aligned}\mu_\Omega =& \mu - \{\gamma,\Omega\} + \frac{1}{2}\{\{\psi,\Omega\},\Omega\},\\ \phi_\Omega =& \phi - \{\mu,\Omega\} + \frac{1}{2}\{\{\gamma,\Omega\},\Omega\} - \frac{1}{6}\{\{\{\psi,\Omega\},\Omega\},\Omega\},\\ \gamma_\Omega =& \gamma - \{\psi,\Omega\},\\ \psi_\Omega =& \psi.\end{aligned}$$

Note that if we start with a quasi-Lie bialgebroid, i.e., when $\psi = 0$, then the twisted structure $\Theta_\Omega$ constitutes, once again, a quasi-Lie bialgebroid given by

$$\begin{aligned}\mu_\Omega =& \mu - \{\gamma,\Omega\},\\ \phi_\Omega =& \phi - \{\mu,\Omega\} + \frac{1}{2}\{\{\gamma,\Omega\},\Omega\},\\ \gamma_\Omega =& \gamma,\\ \psi_\Omega =& 0.\end{aligned}$$

## 5.6 Poisson-Nijenhuis structure

In (ANTUNES, 2008) the definition of a Poisson quasi-Nijenhuis manifold is extended to arbitrary proto-Lie algebroid. These extended structures are referred to as Poisson quasi-Nijenhuis structures with background and specialized to yield the following definition for PqN structures applicable to arbitrary Lie algebroids.

Let $\pi \in \Gamma(\wedge^2 A)$ and $N \in \Gamma(A^* \otimes A)$ and consider the structures $\mu_N$ on $A$ and $\gamma_\pi$ on $A^*$. Additionally, we suppose that

$$N \circ \pi^\sharp = \pi^\sharp \circ N^*,$$

thus, $N \circ \pi^\sharp$ is skew-symetric and define a bivector

$$\pi_N = \frac{1}{2}\{\pi, N\}.$$

We ask the following compatibility condition for $\pi$ and $N$:

$$C_\mu(\pi, N) = \{N, \{\pi, \mu\}\} + \{\pi, \{N, \mu\}\} = 0. \tag{5.11}$$

The Equation (5.11) implies that the bracket $[\cdot,\cdot]_N^\mu$ twisted by $\pi$ is equal to the bracket $[\cdot,\cdot]_\pi$ deformed by $N$.

**Lemma 5.6.1.** For any $\pi \in \Gamma(\wedge^2 A)$ and $N \in \Gamma(A^* \oplus A)$ in the Lie algebroid $(A, \mu)$, we have that

$$2\{\mu_N, \gamma_\pi\} = \{\mu, C_\mu(\pi, N)\}.$$

*Proof.* Using the Jacobi identity, we have

$$\begin{aligned} 2\{\{N,\mu\},\{\pi,\mu\}\} &= \{\{N,\mu\},\{\pi,\mu\}\} + \{\{\pi,\mu\},\{N,\mu\}\} \\ &\{\{\{N,\mu\},\pi\},\mu\} + \{\{\{\pi,\mu\},N\},\mu\} = \{\mu, C_\mu(\pi,N)\}. \end{aligned}$$

□

Similar to $T\mu N$ and $[\pi,\pi]_\mu$, if $d_\mu$-exact 1-forms generate $\Gamma(A^*)$ locally, we have that

$$\{\mu, C_\mu(\pi,N)\} = 0 \Longleftrightarrow C_\mu(\pi,N) = 0.$$

We can define a PqN structure on a Lie algebroid as follows.

**Definition 5.6.2.** A Poisson quasi-Nijenhuis structure on a Lie algebroid $(A, \mu)$ is a triple $(\pi, N, \phi)$ where $\pi \in \Gamma\left(\wedge^2 A\right)$, $N \in \Gamma(A \otimes A^*)$, $\phi \in \Gamma\left(\wedge^3 A^*\right)$ are such that $N \circ \pi^\sharp = \pi^\sharp \circ N^*$, $d_\mu \phi = \{\mu, \phi\} = 0$ and the following conditions hold:

$$[\pi,\pi]_\mu = \{\{\pi,\mu\},\pi\} = 0, \tag{5.12}$$

$$C_\mu(\pi,N) = \{N,\{\pi,\mu\}\} + \{\pi,\{N,\mu\}\} = 0, \tag{5.13}$$

$$T_\mu(N) = \{N,\{N,\mu\}\} - \{N^2,\mu\} = 2\{\pi,\phi\}, \tag{5.14}$$

$$d_N^\mu(\phi) = \{\{N,\mu\},\phi\} = 0. \tag{5.15}$$

If $\phi = 0$, we have the notion of PN structure on a Lie algebroid. Thus, we have the following theorem, see (KOSMANN-SCHWARZBACH; RUBTSOV, 2010).

**Theorem 5.6.3.** Let $N$ be a Nijenhuis (1, 1)-tensor and $\pi$ a Poisson structure on $(A,\mu)$.

(i) The vanishing of $\{\mu, C_\mu(\pi,N)\}$ is a necessary and sufficient condition for $(\mu_N,\gamma_\pi)$ to define a Lie bialgebroid structure on $(A,A^*)$. In particular, if $\pi$ and $N$ are compatible, then $(\mu_N,\gamma_\pi)$ is a Lie bialgebroid structure.

(ii) If the $d_\mu$-exact 1-forms generate $\Gamma(A^*)$ locally as a $C^\infty(M)$-module, then a Poisson bivector $\pi$ and a Nijenhuis tensor $N$ define a *PN* structure on $(A,\mu)$ if and only if the pair $(\mu_N,\gamma_\pi)$ defines a Lie bialgebroid structure on $(A,A^*)$.

One can wonder if starting with PqN structure on $(A,\mu)$, then $\mu_N+\gamma_\pi+\phi$ define a quasi-Lie bialgebroid, that is, if the following conditions hold

$$\frac{1}{2}\{\mu_N,\mu_N\}+\{\gamma_\pi,\phi\}=0, \tag{5.16}$$

$$\{\gamma_\pi,\gamma_\pi\}=0, \tag{5.17}$$

$$\{\mu_N,\gamma_\pi\}=0, \tag{5.18}$$

$$\{\mu_N,\phi\}=0. \tag{5.19}$$

To answer this question, first note that

$$\{\gamma_\pi,\phi\}=\{\{\pi,\mu\},\phi\}=\{\pi,\{\mu,\phi\}\}-\{\{\pi,\phi\}\mu\}.$$

Since $d_N^\mu(\phi)=0$, using Equation (5.5), we can rewrite Equation (5.16) as

$$\{\mu,T_\mu N\}-\{\mu,\{\pi,\phi\}\}=0.$$

Given that $T_\mu N=2\{\pi,\phi\}$, the Equation (5.16) holds. Equations (5.17) and (5.19) are direct consequences of Equations (5.12) and (5.15). Equation (5.18) is a direct consequence of Equation (5.13) and Lemma 5.6.1.

If we assume that the $d_\mu$-exact 1-forms generate $\Gamma(A^*)$ locally as a $C^\infty(M)$-module, we can state the following relation between PqN structures and quasi-Lie bialgebroids.

**Theorem 5.6.4.** If the $d_\mu$-exact 1-forms generate $\Gamma(A^*)$ locally as a $C^\infty(M)$-module, then a Poisson bivector $\pi$ and a Nijenhuis tensor $N$ define a *PqN*-structure on $(A,\mu)$ if and only if the pair $\mu_N+\gamma_\pi+\phi$ defines a Lie bialgebroid structure on $(A,A^*)$ and $d_\mu\phi=0$.

## 5.7 Deformation theorem

We now possess the necessary tools to extend Theorem 3.2.9 to any arbitrary Lie algebroid. To achieve this objective, we will employ the twisting of a proto-bialgebroid and we use the relationship between quasi-Lie algebroids and PqN structures, as discussed above in the preceding sections.

**Theorem 5.7.1.** Let $(\pi, N, \phi)$ be a PqN structure on $(A, \mu)$ and let $\Omega \in \Gamma(\wedge^2 A^*)$ such that $d_\mu \Omega = 0$. Defining

$$\hat{N} = N + \{\pi, \Omega\} = N + \pi^\sharp \circ \Omega^\flat,$$
$$\hat{\phi} = \phi + \{\mu_N, \Omega\} + \frac{1}{2}\{\{\Omega, \gamma_\pi\}, \Omega\} = \phi - d_N^\mu(\Omega) - \frac{1}{2}[\Omega, \Omega]_\pi.$$

If the $d_\mu$-exact 1-forms generate $\Gamma(A^*)$ locally, then $(\pi, \hat{N}, \hat{\phi})$ is a PqN structure on $(A, \mu)$.

*Proof.* Clearly $\hat{N} \circ \pi^\sharp = \pi^\sharp \circ \hat{N}^*$. Indeed,

$$(N + \pi^\sharp \circ \Omega^\flat) \circ \pi^\sharp = N \circ \pi^\sharp + \pi^\sharp \circ \Omega^\flat \circ \pi^\sharp = \pi^\sharp \circ (N + \Omega^\flat \circ \pi^\sharp) = \pi^\sharp \circ \hat{N}.$$

Let $(\gamma_\pi + \mu_N + \phi)$ be the quasi-Lie algebroid structure associated to $(\Pi, N, \phi)$. Let $\Omega \in \Gamma(\wedge^2 A^*)$, then the twisted quasi Lie bialgebroid $e^{-\Omega}(\gamma_\pi + \mu_N + \phi)$ is given by

$$\mu_\Omega = \mu_N + \{\gamma_\pi, \Omega\},$$
$$\gamma_\Omega = \gamma_\pi,$$
$$\phi_\Omega = \phi + \{\mu_N, \Omega\} + \frac{1}{2}\{\{\gamma_\pi, \Omega\}, \Omega\}.$$

By the Jacobi identity,

$$\{\gamma_\pi, \Omega\} = \{\{\pi, \mu\}, \Omega\} = \{\pi, \{\mu, \Omega\}\} + \{\{\pi, \Omega\}, \mu\},$$

Since $d_\mu(\Omega) = \{\pi, \Omega\} = 0$, through Lemma 5.3.2, we have

$$\{\gamma_\pi, \Omega\} = \{\{\pi, \Omega\}, \mu\} = \mu_{\pi^\sharp \Omega^\flat},$$

and

$$\mu_\Omega = \mu_{\hat{N}}.$$

Since $\Theta_\Omega$ is a proto-bialgebroid, we have that

$$\frac{1}{2}\{\mu_{\hat{N}}, \mu_{\hat{N}}\} + \{\gamma_\pi, \hat{\phi}\} = 0, \tag{5.20}$$
$$\{\gamma_\pi, \gamma_\pi\} = 0, \tag{5.21}$$
$$\{\mu_{\hat{N}}, \gamma_\pi\} = 0, \tag{5.22}$$
$$\{\mu_{\hat{N}}, \hat{\phi}\} = 0. \tag{5.23}$$

Now, we will show that $d_\mu \hat{\phi} = \{\mu, \hat{\phi}\} = 0$.

$$\{\mu, \hat{\phi}\} = \{\mu, \phi\} + \{\mu, \{\mu_N, \Omega\}\} + \frac{1}{2}\{\mu, \{\{\gamma_\pi, \Omega\}, \Omega\}\}$$

The first term on the right side vanishes since $\phi$ is $d_\mu$-closed. For the second one, applying the Jacobi identity, we have

$$\{\mu,\{\{\mu,N\},\Omega\}\} = \{\{\mu,\{\mu,N\}\},\Omega\} - \{\{\mu,N\},\{\mu,\Omega\}\}.$$

But, by hypothesis, $\{\mu,\Omega\} = 0$ and Lemma 5.3.7 says that $\{\mu,\{\mu,N\}\}$. For the third one, we have that

$$\{\mu,\{\{\gamma_\pi,\Omega\},\Omega\}\} = \{\{\mu,\{\gamma_\pi,\Omega\}\},\Omega\} + \{\{\gamma_\pi,\Omega\},\{\mu,\Omega\}\}.$$

But, $d_\mu\Omega = 0$ and

$$\{\mu,\{\{\pi,\mu\},\Omega\}\} = \{\{\mu,\{\pi,\mu\}\},\Omega\} + \{\{\pi,\mu\},\{\mu,\Omega\}\} = 0.$$

Thus $d_\mu(\hat{\phi}) = 0$.

Through Equation (5.5), Equation(5.20) implies that

$$\frac{1}{2}\{\mu,T_\mu N\} + \{\{\pi,\mu\},\hat{\phi}\} = 0.$$

But, through the Jacobi identity,

$$\{\{\pi,\mu\},\hat{\phi}\} = \{\pi,\{\mu,\hat{\phi}\}\} - \{\{\pi,\hat{\phi}\},\mu\}.$$

Since $d_\mu\hat{\phi} = 0$, we have that

$$\frac{1}{2}\{\mu,T_\mu N\} = \{\{\pi,\hat{\phi}\},\mu\}.$$

Thus, if the $d_\mu$-exact 1-forms generate $\Gamma(A^*)$ locally,

$$T_\mu N = 2\{\pi,\hat{\phi}\},$$

and Condition (5.14) holds. The Equation (5.21) can be rewritten as

$$\{\gamma_\pi,\gamma_\pi\} = \{[\pi,\pi]_\mu,\mu\}.$$

Thus, again, if the $d_\mu$-exact 1-forms generate $\Gamma(A^*)$ locally, the Condition (5.12) holds.

Using Lemma 5.6.1 and Formula (5.22), we have

$$\{\mu,C_\mu(\pi,\hat{N})\} = 0,$$

and then Condition (5.13) holds. The Equation (5.23) is exactly the Condition (5.15). □

CHAPTER

6

# BI-DIFFERENTIAL CALCULI FROM A DIRAC PERSPECTIVE AND AN INVOLUTIVITY THEOREM

In this chapter, our aim is to explore results towards creating a more general involutivity theorem than the one presented in (FALQUI *et al.*, 2020). To this end, we will investigate two methods, both of which revolve around the Lenard chains. The first approach is based on the results presented in (DORFMAN, 1993), while the second employs what is known as bi-calculi (CRAMPIN; SARLET; THOMPSON, 2000).

Let $M$ be a manifold and let $\pi_1$ and $\pi_2$ be two Poisson bivector fields defined on $M$. Then a sequence of functions $\{f_j\}_{j\in\mathbb{Z}}$ is said to satisfy the Lenard recursion relations, and is called a Lenard chain, if

$$\pi_1(df_j) = \pi_2(df_{j+1}) \qquad \text{for all } j \in \mathbb{Z}.$$

Under these hypotheses we can state the following important result:

**Theorem 6.0.1.** If two Poisson bivector fields $\pi_1, \pi_2$ are defined on the manifold $M$ and there exists a sequence of (smooth) functions $\{f_j\}_{j\in\mathbb{Z}}$ satisfying the Lenard recursion relations written above, then the functions $f_j$ are pairwise in involution with respect to both Poisson brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$.

If $(M,\pi,N)$ is a PN manifold, we have that $\pi_N^\sharp = N\circ\pi^\sharp$ defines a Poisson tensor and the functions $I_k = \frac{1}{k}\operatorname{Tr}(N^k)$, for $k = 1,2,\ldots$, satisfy the following Lenard recursion relations

$$\pi^\sharp(dI_{k+1}) = \pi_N^\sharp(dI_k). \tag{6.1}$$

**Remark 6.0.2.** The Equations (6.1) are also called Lenard-Magri relations.

To prove the Lenard-Magri relation, we start defining the set of Hamiltonian forms (MAGRI; MOROSI, 1984).

**Definition 6.0.3.** We define the set of Hamiltonian forms by

$$\Omega^1_{Ham}(N) = \{\alpha \in \Gamma(T^*M) \mid d\alpha = d_N\alpha = 0\}.$$

**Proposition 6.0.4.** The set of Hamiltonian forms $\Omega^1_{Ham}(N)$ is $N$-invariant.

*Proof.* If $\alpha \in \Omega^1_{Ham}(N)$, then

$$d(N^*\alpha) = d \circ i_N(\alpha) = (i_N \circ d - d_N)\alpha = 0.$$

Since the anchor $N$ is a Lie algebroid morphism, we have that $N^* \circ d = d_N \circ N^*$, thus

$$d_N(N^*\alpha) = N^*(d\alpha) = 0.$$

□

**Proposition 6.0.5.** Let $(M,\pi,N)$ be a PN manifold. The Hamiltonian forms $\Omega^1_{Ham}(N)$ define a subalgebra with respect to the bracket $[\cdot,\cdot]_\pi$ and the following property holds

$$N^*[\alpha,\beta]_\pi = [N^*\alpha,\beta]_\pi = [\alpha,N^*\beta], \tag{6.2}$$

for all $\alpha,\beta \in \Omega^1_{Ham}(N)$

*Proof.* First, note that $[\alpha,\beta]_\pi$ is a exact form. Indeed,

$$\mathscr{L}_{\pi^\sharp(\alpha)}(\beta) = (d \circ i_{\pi^\sharp(\alpha)} + i_{\pi^\sharp(\alpha)} \circ d)\beta = d(\pi(\alpha,\beta)),$$
$$\mathscr{L}_{\pi^\sharp(\beta)}(\alpha) = (d \circ i_{\pi^\sharp(\beta)} + i_{\pi^\sharp(\beta)} \circ d)\alpha = d(\pi(\beta,\alpha)).$$

Thus,

$$[\alpha,\beta]_\pi = -d\pi(\alpha,\beta).$$

Since $((TM,[\cdot,\cdot]_\pi,\pi^\sharp),d_N)$ is a Lie bialgebroid, we have that

$$d_N[\alpha,\beta]_\pi = [d_N\alpha,\beta]_\pi + [\alpha,d_N\beta]_\pi = [d(N\alpha),\beta]_\pi + [\alpha,d(N\beta)]_\pi = 0.$$

Now, note that

$$[\alpha,\beta]_{\pi_N} = -d\pi_N(\alpha,\beta) = -d\langle\pi^\sharp(\alpha),N^*\beta\rangle = [\alpha,N^*\beta]_\pi$$

but, we also have that

$$[\alpha,\beta]_{\pi_N} = -d\langle\pi^\sharp(N^*\alpha),\beta\rangle = [N^*\alpha,\beta]_\pi.$$

Since $\pi$ and $N$ are compatible, we have

$$[\alpha,\beta]_{\pi_N} = [N^*\alpha,\beta]_\pi + [\alpha,N^*\beta]_\pi - N^*[\alpha,\beta]_\pi,$$

and the results follow. □

**Theorem 6.0.6.** Let $\alpha \in \Omega^1_{Ham}(N)$ and let $\alpha_{j+1} = (N^*)^j\alpha$. For each $l,k = 1,2\cdots$, we have that

$$[\alpha_l, \alpha_k]_\pi = 0. \tag{6.3}$$

Moreover, if $I_k = \frac{1}{k}\operatorname{Tr}(N^k)$, then

$$N^* dI_k = dI_{k+1}. \tag{6.4}$$

*Proof.* Applaying Equation (6.2), we get that, for all $l,k$

$$[\alpha_K, \alpha_l]_\pi = (N^*)^{k+l-2}[\alpha,\alpha]_\pi = 0.$$

The vanishing of the Nijenhuis Torsion is equivalent to that, for every $X \in \Gamma(TM)$,

$$\mathscr{L}_{NX}(N) = N\mathscr{L}_X(N).$$

Thus,

$$\langle X, dI_{k+1}\rangle = \mathrm{Tr}(\mathrm{N}^{\mathrm{k}}\mathscr{L}_{\mathrm{X}}(\mathrm{N})) = \mathrm{Tr}(\mathrm{N}^{\mathrm{k\text{-}1}}\mathscr{L}_{\mathrm{NX}}(\mathrm{N})) = \langle NX, dI_k\rangle = \langle X, N^* dI_k\rangle.$$

□

**Remark 6.0.7.** The Nijenhuis tensor generates a hierarchy of Poisson tensors given by $\pi^k_N = N^k\pi$. In fact, all the previous results hold for all $\pi^k_N$, See (MAGRI; MOROSI, 1984).

If we assume that the Nijenhuis torsion does not vanish, then the Lenard-Magri relations may not hold in general. Indeed, in (FALQUI *et al.*, 2020), it is shown that if $(M,\pi,N,\phi)$ is a PqN manifold, then the so-called generalize Lenard-Magri relations

$$dI_{k+1} = N^* dI_k - \phi_{k-1} \tag{6.5}$$

holds, where

$$\langle \phi_l, X\rangle = \operatorname{Tr}\left((i_X T_N)N^l\right) = \operatorname{Tr}\left(N^l(i_X T_N)\right), \qquad l \geq 0.$$

The additional term $\phi_l$ modifies the usual formula $\{I_k, I_j\}_\pi = \{I_{k-1}, I_{j+1}\}_\pi$. Indeed,

$$\begin{aligned}
\{I_k, I_j\}_\pi =&\langle dI_k, \pi^\sharp(dI_j)\rangle = \langle N^* dI_{k-1}, \pi^\sharp(dI_j)\rangle - \langle \phi_{k-2}, \pi^\sharp(dI_j)\rangle\\
=&\langle dI_{k-1}, N\pi^\sharp(dI_j)\rangle - \langle \phi_{k-2}, \pi^\sharp(dI_j)\rangle\\
=&\langle dI_{k-1}, \pi^\sharp(N^* dI_j)\rangle - \langle \phi_{k-2}, \pi^\sharp(dI_j)\rangle = \langle dI_{k-1}, \pi^\sharp(dI_{j+1})\rangle\\
&+ \langle dI_{k-1}, \pi^\sharp(\phi_{j-1})\rangle - \langle \phi_{k-2}, \pi(dI_j)\rangle\\
=&\{I_{k-1}, I_{j+1}\}_\pi - \left(\langle \phi_{j-1}, \pi^\sharp(dI_{k-1})\rangle + \langle \phi_{k-2}, \pi^\sharp(dI_j)\rangle\right).
\end{aligned}$$

In the case where torsion does not vanish, we have

$$\{I_k, I_j\} - \{I_{k-1}, I_{j+1}\} = -\langle \phi_{j-1}, \pi^\sharp(dI_{k-1})\rangle + \langle \phi_{k-2}, \pi^\sharp(dI_j)\rangle. \tag{6.6}$$

(FALQUI *et al.*, 2020) also shows that there are some PqN manifolds in which the functions $I_k$ are not in involution with respect to $\{\cdot,\cdot\}_\pi$. Consider the $\mathbb{R}^6$ with canonical variables $(q_1,q_2,q_3,p_1,p_2,p_3)$ with the canonical Poisson tensor

$$\pi = \sum_{i=1}^{3} \partial_{p_i} \wedge \partial_{q_i}$$

and let

$$N = \sum_{i=1}^{3} p_i \left(\partial_{q_i} \otimes \mathrm{d}q_i + \partial_{p_i} \otimes \mathrm{d}p_i\right) + \sum_{i<j} \left(\partial_{q_i} \otimes \mathrm{d}p_j - \partial_{q_j} \otimes \mathrm{d}p_i\right) + \sum_{i<j} \frac{1}{(q_i - q_j)} \left(\partial_{p_j} \otimes \mathrm{d}q_i - \partial_{p_i} \otimes \mathrm{d}q_j\right).$$

After straightforward computations, we have that $\{I_1,I_2\}_\pi = \{I_1,I_3\}_\pi = 0$, but $\{I_2,I_3\}_\pi \neq 0$.

The theorem below is the first application of PqN manifolds in the theory of integrable systems.

**Theorem 6.0.8** ((FALQUI *et al.*, 2020), Theorem 6)**.** Let $(M,\pi,N)$ be a PN manifold, $\Omega$ a closed 2-form on $M$ such that $[\Omega,\Omega]_\pi = 0$, $\hat{N} = N - \pi^\sharp\Omega^\flat$, and $I_k = \frac{1}{k}\mathrm{Tr}(\hat{N}^k)$. Suppose that

1. $d_N\Omega = dI_1 \wedge \Omega$;
2. $i_{Y_k}\Omega = 0$, where $Y_k = (\hat{N})^{k-1} - X_k$ and $X_k = \pi^\sharp(dI_k)$;
3. $\{I_1,I_k\} = 0$ for all $k \geq 2$.

Then

i) $(M,\pi,\hat{N},d_N\Omega)$ is a PqN manifold;

ii) $\{I_j,I_k\} = 0$ for all $j,k \geq 1$.

Since Example 3.2.10 satisfies the hypotheses of the Theorem above, see Theorem 7 of (FALQUI *et al.*, 2020), the well-known closed Toda chain's constants of motion can be obtained using the PqN structure.

## 6.1 Lenard scheme for Dirac structures

The works of Dorfman, see e.g. (DORFMAN, 1993), provide a way to interpret Nijenhuis torsionas as a relation in a vector space. She also shows that the intersection of two Dirac structures in a subtle way enables the construction of a Lenard-Magri scheme. We begin this section by presenting the results that we need from (DORFMAN, 1993). Later, we connect these results to Theorem 6 of (FALQUI *et al.*, 2020) to understand it from a Dirac perspective.

**Definition 6.1.1.** Let $A$ be a vector space. A relation in $A$ is a linear subspace $\mathscr{R} \subset A \oplus A$. We define the dual of the relation $\mathscr{R}$ as

$$\mathscr{R}^* = \{\alpha \oplus \beta \in A^* \oplus A^* \mid \beta(X) = \alpha(Y), \forall\, X \oplus Y \in \mathscr{R}\}.$$

**Definition 6.1.2.** Let $(A, [\cdot,\cdot]_A)$ be a Lie algebra. A Nijenhuis relation is a relation $\mathscr{N} \in A \oplus A$ such that, for $X_1, X_2, Y_1, Y_2 \in A$ and $\alpha_1, \alpha_2, \alpha_3 \in A^*$ satisfying

$$X_1 \oplus X_2 \in \mathscr{N},\ Y_1 \oplus Y_2 \in \mathscr{N},\ ,\alpha_1 \oplus \alpha_2 \in \mathscr{N}^*,\ \alpha_2 \oplus \alpha_3 \in \mathscr{N}^*,$$

the real-valued function

$$\mathrm{T}(\mathscr{N}) = \langle \alpha_1, [X_2, Y_2]_A \rangle - \langle \alpha_2, [X_2, Y_1]_A + [X_1, Y_2]_A \rangle + \langle \alpha_3, [X_1, Y_1]_A \rangle \tag{6.7}$$

vanishes.

**Remark 6.1.3.** The above definition also holds when the Lie algebra is replaced by a Loday algebra, see (KOSMANN-SCHWARZBACH, 2012).

**Proposition 6.1.4.** The Equation (6.7) for the Nijenhuis torsion is equivalent to

$$\mathrm{T}(\mathscr{N}) = -d_A\alpha_1(X_2, Y_2) + d_A\alpha_2(X_2, Y_1) + d_A\alpha_2(X_1, Y_2) - d_A\alpha_3(X_1, Y_1). \tag{6.8}$$

*Proof.* By definition,

$$\begin{aligned}
d_A\alpha_1(X_2, Y_2) &= X_2(\alpha_1(Y_2)) - Y_2(\alpha_1(X_2)) - \alpha_1([X_2, Y_2]_A),\\
d_A\alpha_2(X_2, Y_1) &= X_2(\alpha_2(Y_1)) - Y_1(\alpha_2(X_2)) - \alpha_2([X_2, Y_1]_A),\\
d_A\alpha_2(X_1, Y_2) &= X_1(\alpha_2(Y_2)) - Y_2(\alpha_2(X_1)) - \alpha_2([X_1, Y_2]_A),\\
d_A\alpha_3(X_1, Y_1) &= X_1(\alpha_3(Y_1)) - Y_1(\alpha_3(X_1)) - \alpha_3([X_1, Y_1]_A).
\end{aligned}$$

Since $X_1 \oplus X_2 \in \mathscr{N}$, $Y_1 \oplus Y_2 \in \mathscr{N}$, and $\alpha_1 \oplus \alpha_2 \in \mathscr{N}^*$, $\alpha_2 \oplus \alpha_3 \in \mathscr{N}^*$,

$$\begin{aligned}
\alpha_1(X_2) &= \alpha_2(X_1), & \alpha_2(X_2) &= \alpha_3(X_1),\\
\alpha_1(Y_2) &= \alpha_2(Y_1), & \alpha_2(Y_2) &= \alpha_3(Y_1).
\end{aligned}$$

Thus,

$$\begin{aligned}
d_A\alpha_1(X_2, Y_2) &= X_2(\alpha_1(Y_2)) - Y_2(\alpha_1(X_2)) - \alpha_1([X_2, Y_2]_A),\\
d_A\alpha_2(X_2, Y_1) &= X_2(\alpha_1(Y_2)) - Y_1(\alpha_3(X_1)) - \alpha_2([X_2, Y_1]_A),\\
d_A\alpha_2(X_1, Y_2) &= X_1(\alpha_3(Y_1)) - Y_2(\alpha_1(X_2)) - \alpha_2([X_1, Y_2]_A),\\
d_A\alpha_3(X_1, Y_1) &= X_1(\alpha_3(Y_1)) - Y_1(\alpha_3(X_1)) - \alpha_3([X_1, Y_1]_A).
\end{aligned}$$

Finally, we have

$$-d_A\alpha_1(X_2, Y_2) + d_A\alpha_2(X_2, Y_1) + d_A\alpha_2(X_1, Y_2) - d_A\alpha_3(X_1, Y_1) = \mathrm{T}(\mathscr{N}).$$

□

**Example 6.1.5.** Let $N\colon A\to A$ be a (1,1)-tensor. We can interpret the $\mathscr{R}_N=\mathrm{graph}(N)=\{X\oplus NX\mid X\in\Gamma(A)\}$ as a relation in $\Gamma(A)\oplus\Gamma(A)$. Then, we have that the dual relation is given by $\mathscr{R}_N^*=\{\alpha\oplus N^*\alpha\mid\alpha\in\Gamma(A^*)\}\subset\Gamma(A^*)\oplus\Gamma(A^*)$. Thus, we have that the Nijenhuis torsion of $\mathscr{R}_N$ is given by

$$\langle\alpha,[NX,NY]_A\rangle-\langle N^*\alpha,[NX,Y]_A+[X,NY]_A\rangle+\langle N^*\alpha,N[X,Y]_A\rangle. \tag{6.9}$$

In (DORFMAN, 1987), the concept of Nijenhuis relation is presented as a generalization of Nijenhuis Torsion.

**Proposition 6.1.6.** A vector bundle morphism $N\colon A\to A$ is a Nijenhuis tensor if and only if $\mathscr{R}_N=\mathrm{graph(N)}$ defines a Nijenhuis relation.

*Proof.* The result follows directly from Equation (6.9), taking into account the non-degeneracy of the pairing. □

We can define a suitable compatibility condition between two Dirac structures using the notion of Nijenhuis relation.

**Definition 6.1.7.** Two Dirac structures $L_1,L_2\subset A\oplus A^*$ are said to constitute a Dirac pair if the set

$$\mathscr{R}(L_1,L_2)=\{X_1\oplus X_2\mid\exists\alpha\in A^*,X_1\oplus\alpha\in L_2,X_2\oplus\alpha\in L_1\}\subset A\oplus A$$

is a Nijenhuis relation.

The Lenard scheme for Dirac pairs can be summarized in the following result:

**Theorem 6.1.8.** Let $L_1,L_2\in A\oplus A^*$ be a Dirac pair. Let there be given two sequence $X_0,X_1,\ldots\in A$ and $\alpha_{-1},\alpha_0,\ldots\in A^*$ such that

$$X_i\oplus\alpha_{i-1}\in L_1,\ X_i\oplus\alpha_i\in L_2. \tag{6.10}$$

Assume that

$$d_A\alpha_{-1}=d_A\alpha_0=0. \tag{6.11}$$

Suppose that the following condition holds: if for some $\alpha\in A^*$, $d\alpha(X,Y)=0$ for $X$ and $Y$ in the projection of $\mathscr{R}(L_1,L_2)$ on $A$, then $d\alpha=0$. Then,

(a) all $\alpha_i$ are closed;

(b) all $f_i\in C^\infty(M)$ such that $d_Af_i=\alpha_i$ are in involution with respect to the Poisson brackets associated with $L_1$ and $L_2$:

$$\{f_i,f_j\}_{L_1}=\{f_i,f_j\}_{L_2}=0. \tag{6.12}$$

*Proof.* First, note that, by definition, for every $Y_1 \oplus Y_2 \in \mathscr{R}(L_1, L_2)$, there exists some $\xi \in A^*$ such that $Y_1 \oplus \xi \in L_2$ and $Y_2 \oplus \xi \in L_1$. Thus, using the isotropic property of the Dirac structure and condition 6.10, we have that

$$\langle Y_1 \oplus \xi \mid X_i \oplus \alpha_i \rangle = \langle Y_2 \oplus \xi \mid X_i \oplus \alpha_{i-1} \rangle = 0,$$

which implies that

$$\alpha_i(Y_1) + \xi(X_i) - \alpha_{i-1}(Y_2) - \xi(x_i) = \alpha_i(Y_1) - \alpha_{i-1}(Y_2) = 0.$$

Since $Y_1 \oplus Y_2$ is an arbitrary element in $\mathscr{R}(L_1, L_2)$, we conclude that

$$\alpha_{i-1} \oplus \alpha_i \in \mathscr{R}^*(L_1, L_2).$$

Now, suppose by induction that, for every $i \leq n$, $d\alpha_i = 0$. Since $L_1$ and $L_2$ constitute a pair of Dirac structures, for arbitrary $Y_1 \oplus Y_2$, $Y_1' \oplus Y_2' \in \mathscr{R}(L_1, L_2)$, there holds

$$\alpha_{n-i}([Y_2, Y_2']) - \alpha_n([Y_2, Y_1'] + [Y_1, Y_2']) + \alpha_{n+1}([Y_1, Y_1']) = 0,$$

or, equivalently,

$$d\alpha_{n-i}(Y_2, Y_2') - d\alpha_n(Y_2, Y_1') - d\alpha_n(Y_1, Y_2') + d\alpha_{n+1}(Y_1, Y_1') = 0.$$

Using condition 6.11, we can conclude that $d\alpha_{n+1}(Y_1, Y_1') = 0$ for arbitrary $Y_1, Y_1'$ from the projection of $\mathscr{R}(L_1, L_2)$. By hypothesis, this is enough to conclude that $d\alpha_{n+1} = 0$.

Now we prove (b). Given $i, j$ arbitrary. Without loss of generality, we can suppose $i < j$. thus

$$\begin{aligned} X_{i+1} \oplus df_i &\in L_1, & X_i \oplus df_i &\in L_2, \\ X_j \oplus df_{j-1} &\in L_1, & X_j \oplus df_j &\in L_2. \end{aligned}$$

By isotropy of $L_1$, we have that

$$\langle X_{i+1} \oplus df_i, X_j \oplus df_{j-1} \rangle = 0 \implies \langle X_{i+1}, df_{j-1} \rangle = -\langle X_j, df_i \rangle.$$

By isotropy of $L_2$, we have that

$$\langle X_i \oplus df_i, X_j \oplus df_j \rangle = 0 \implies \langle X_i, df_j \rangle = -\langle X_j, df_i \rangle.$$

Therefore

$$\{f_i, f_j\} = \langle X_i, df_j \rangle = \langle X_{i+1}, df_{j-1} \rangle$$

Let $s = j - i$, then repeating it $s$-times, we obtain that

$$\{f_i, f_j\} = \{f_{i+s}, f_{j-s}\} = \{f_j, f_k\}.$$

So, as we wish, $\{f_i, f_j\} = 0$ for arbitrary $i, j$. □

If we consider one of the Dirac structures as the $graph(\pi)$ for some Poisson vector field, we have the following

**Corollary 6.1.9.** Let $(M,\pi)$ be a Poisson manifold and $L \subset TM \oplus T^*M$ an isotropic subbundle. If there is a sequence $\{\alpha_k\}$ of closed 1-form such that, for every $k$,

$$\pi(\alpha_k) \oplus \alpha_{k+1} \in L,$$

then, for all $i, j$,

$$\left[\alpha_i, \alpha_j\right]_\pi = 0.$$

In particular, if $\alpha_i = df_i$, then

$$\{f_i, f_j\} = 0.$$

**Remark 6.1.10.** Note that, with the same argument, we could ask $\pi(df_{k+1}) \oplus df_k \in L$ instead $\pi(df_k) \oplus df_{k+1} \in L$.

Let $(M,\pi,N)$ be a PN manifold and $I_k = \frac{1}{k+1}\operatorname{Tr}(N^k)$. Thanks to the Lenard-Magri relations, we get that

$$N\pi(dI_k) \oplus dI_k = \pi(dI_{k+1}) \oplus dI_k \in \operatorname{graph}(N\pi).$$

Then, the Corollary 6.1.9 applies to the sequence $\{dI_k\}$, choosing $(M,\pi)$ as the Poisson manifold and $L = \operatorname{graph}(N\pi) \subset TM \oplus T^*M$ as the graph of the Poisson structure $N\pi$.

However, if $(M,\pi,N,\phi)$ is a PqN manifold, the Lenard-Magri relation does not hold, thus, if we want to understand the involutivity of the family $\{I_k\}$ of a PqN manifold using the Corollary 6.1.9, we must search for a more appropriate Lagrangian subbundle.

For a PqN manifold $(M,\pi,\hat{N},\phi)$ we define $I_k = \frac{1}{k}\operatorname{Tr}(\hat{N}^k)$ and $X_k = \pi^\sharp(dI_k)$. Set $D$ as the distribution generated by $\{X_k\}$, that is,

$$D_p = Span\{(X_k)_p\}.$$

Now, define a Lagrangian subbundle by

$$L_p := \{X_p \oplus \alpha \mid X_p \in D_p, \alpha \in D_p^\perp\}.$$

**Proposition 6.1.11.** Let $(M,\pi,\hat{N},\phi)$ be a PqN manifold, $I_k = \frac{1}{k}\operatorname{Tr}(\hat{N}^k)$ and $X_k = \pi^\sharp(dI_k)$. If the family $\{I_k\}$ is involutive, then the Poisson manifold $(M,\pi)$ and the Lagrangian subbundle $L = \{X \oplus \alpha) \mid X \in D, \alpha \in D^\perp\}$ satisfy the hypotheses of the Corollary 6.1.9.

*Proof.* By hypothesis,

$$\{I_k, I_{j+1}\} = dI_{k+1}(X_k) = 0 \text{ for all } j,k.$$

Thus, for all $k$, $dI_{k+1} \in D^{\perp}$ and

$$X_k \oplus dI_{k+1} \subset L.$$

□

Now, let $L'$ be the gauge transformation of $L$ by a closed 2-form $\omega$, that is,

$$L' := T_{\omega}(L) = \{X \oplus (\omega^{\flat}(X) + \alpha) \mid X \in D, \alpha \in D^{\perp}\}.$$

**Proposition 6.1.12.** Let $(M, \pi, N, \phi)$ be a PqN manifold. Suppose that, for some closed 2-form, the previous Lagrangian subbundle $L'$ satisfies the Corollary 6.1.9. Then,

$$\omega^{\flat}(X_j)(X_k) = dI_{j+1}(X_k), \text{ for all } j,k. \tag{6.13}$$

**Remark 6.1.13.** Note that, to apply Corollary 6.1.9, it is only necessary $L$ to be a Lagrangian subbundle, that is, it is not necessary $\omega$ to be a closed 2-form.

**Proposition 6.1.14.** Let $(M, \pi, \hat{N}, \phi)$ be a PqN manifold, $\omega$ be a closed 2-form, $I_k = \frac{1}{k}\operatorname{Tr}(\hat{N}^k)$ and $X_k = \pi^{\sharp}(dI_k)$. If the sequence $dI_k$, the Poisson manifold $(M, \pi)$ and the subbundle $L' = T_{\omega}(L) = \{X \oplus \omega^{\flat}(X) + \alpha) \mid X \in D, \alpha \in D^{\perp}\}$ satisfy the hypotheses of Corollary 6.1.9, then, for every 2-form $\omega'$ such that

$$\omega'(X_k) \in D^{\perp}, \text{ for all } k,$$

$T_{\omega'+\omega}$ satisfies the hypotheses of Corollary 6.1.9.

*Proof.* If $\omega(X_k) - dI_{k+1} \in D^{\perp}$, so does $\omega(X_k) + \omega'(X_k) - dI_{k+1} \in D^{\perp}$. □

Finally, we can reinterpret the Theorem 6.0.8 as follows:

**Theorem 6.1.15.** Let $(M, \pi, \hat{N}, \phi)$ be a PqN manifold, $I_k = \frac{1}{k}\operatorname{Tr}(\hat{N}^k)$, and $W = \operatorname{Span}\{\pi^{\sharp}(dI_k)\}$. Suppose that there exists a 2-form $\Omega$ and $\alpha \in D^{\perp}$ such that:

(i) $\phi = \alpha \wedge \Omega$;

(ii) $\Omega(X_j, Y_k) = 0$, where $Y_k = \hat{N}^{k-1}\pi^{\sharp}(\alpha) - X_k$ and $X_k = \pi^{\sharp}(dI_k)$.

Suppose that the Poisson tensor $\pi$ is invertible. Define $\omega = \pi^{-1}\hat{N} - 2\Omega$, then

$$L' = \{X \oplus (\omega^{\flat}(X) + \alpha) \mid X \in D, \alpha \in D^{\perp}\}$$

satisfies the hypothesis of Corollary 6.1.9.

*Proof.* Analogously to the demonstration presented in (FALQUI *et al.*, 2020), we have that

$$\begin{aligned}
T_{\hat{N}}(X,Y) &= \pi^{\sharp}\left(i_Y i_X (\alpha \wedge \Omega)\right) = \pi^{\sharp}\left(i_Y \left(\langle \alpha, X\rangle \Omega - \alpha \wedge i_X \Omega\right)\right) \\
&= \pi^{\sharp}\left(\langle \alpha, X\rangle i_Y \Omega - \langle \alpha, Y\rangle i_X \Omega + (i_Y i_X \Omega)\, \alpha\right) \\
&= \langle \alpha, X\rangle \left(\pi^{\sharp}\Omega^{\flat}\right)(Y) - \langle \alpha, Y\rangle \left(\pi^{\sharp}\Omega^{\flat}\right)(X) + \Omega(X,Y)\, \pi^{\sharp}(\alpha)
\end{aligned}$$

for all vector fields $X,Y$, so that

$$i_X T_{\hat{N}} = \langle \alpha, X \rangle \pi^\sharp \Omega^\flat - \left(\pi^\sharp \Omega^\flat\right)(X) \otimes \alpha + \pi^\sharp(\alpha) \otimes i_X \Omega.$$

Thus,

$$\begin{aligned} \langle \phi_k, X_j \rangle &= \mathrm{Tr}\left(\hat{N}^k \left(i_{X_j} T_{\hat{N}}\right)\right) = \mathrm{Tr}\left(\hat{N}^k \left(\langle \alpha, X_j \rangle \pi^\sharp \Omega^b - \left(\pi^\sharp \Omega^b\right)(X_j) \otimes \alpha + \pi^\sharp(\alpha) \otimes i_{X_j}\Omega\right)\right) \\ &= \langle \alpha, X_j \rangle \mathrm{Tr}\left(\hat{N}^k \pi^\sharp \Omega^b\right) - \mathrm{Tr}\left(\left(\hat{N}^k \pi^\sharp \Omega^b\right)(X_j) \otimes \alpha\right) + \mathrm{Tr}\left(\left(\hat{N}^k \pi^\sharp(\alpha)\right) \otimes i_{X_j}\Omega\right). \end{aligned}$$

Since $\langle \alpha, X_j \rangle = 0$,

$$\mathrm{Tr}\left(\left(\hat{N}^k \pi^\sharp \Omega^b\right)(X_j) \otimes \alpha\right) = \left\langle \alpha, \left(\hat{N}^k \pi^\sharp \Omega^b\right)(X_j)\right\rangle = -\Omega(X_j, \hat{N}^k \pi^\sharp(\alpha))$$

and

$$\mathrm{Tr}\left(\left(\hat{N}^k \pi^\sharp(\alpha)\right) \otimes i_{X_j}\Omega\right) = \Omega(X_j, \hat{N}^k \pi^\sharp(\alpha)),$$

we have that

$$\langle \phi_k, X_j \rangle = 2\Omega(X_j, \hat{N}^k \pi^\sharp(\alpha)).$$

Now, thanks to assumption 2, we can substitute $\hat{N}^k \pi^\sharp(\alpha)$ with $X_{k+1}$. Thus,

$$\langle \phi_k, X_j \rangle = -2\Omega(X_{k+1}, X_j). \tag{6.14}$$

The equation 6.14 implies that, for every $k$, it exists some $\alpha_k \in D^\perp$ such that

$$\phi_k = -\Omega^\flat(X_{k+1}) + \alpha_k.$$

Thanks to the generalized Lenard-Magri equation, we have that

$$dI_{k+1} = \hat{\hat{N}} dI_k + \phi_{k-1} = \pi^{-1} \hat{\hat{N}} X_k - \Omega^\flat(X_k) + \alpha_{k-1} = \omega^\flat(X_k) + \alpha_{k-1}.$$

Therefore,

$$X_k \oplus dI_{k+1} \subset L'.$$

□

**Example 6.1.16.** Let $(M, \pi, N)$ be a PN manifold, $\Omega$ be a closed 2-form on $M$ satisfying the hypotheses of the theorem 6 of (FALQUI *et al.*, 2020). Let $\hat{N} = N + \pi\Omega^\flat$, then $(M, \pi, \hat{N}, dI_1 \wedge \Omega)$ satisfy Theorem 6.1.15.

In particular, we have that the n-particle closed Toda satisfies the theorem 6.1.15. Let us do some computation to the example "$n = 2$" presented in section 4.1 of (FALQUI; MENCATTINI; PEDRONI, 2023). Consider the following PqN manifold

$$\pi = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \hat{N} = N + \pi^\sharp \Omega^\flat = \begin{pmatrix} p_1 & 0 & 0 & 1 \\ 0 & p_2 & -1 & 0 \\ 0 & -V(q_1-q_2) & p_1 & 0 \\ V(q_1-q_2) & 0 & 0 & p_2, \end{pmatrix},$$

where

$$N = \begin{pmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_1 & 0 \\ 0 & 0 & 0 & p_2 \end{pmatrix} \quad \textit{and} \quad \Omega = V(q_1-q_2)dq_1 \wedge dq_2 + dp_1 \wedge dp_2.$$

We define the following Dirac structure

$$L'_p = \left\{ X_p \oplus \omega\left(X_p\right) + \alpha \mid X_p \in D_p, \alpha \in D_p^\perp \right\},$$

where $\omega = \pi^{-1}\hat{N} - 2\Omega^\flat = \Omega^\flat$.

If we want to apply the Theorem 6.1.15 on the structures $L'$ and graph$(\pi)$, we must have that

$$\omega\left(X_j\right) + \alpha = dI_{j+1} \Rightarrow \omega\left(X_j, X_k\right) = dI_j\left(X_k\right) \forall j,k,$$

that is

$$\omega\left(X_1, X_2\right) = dI_2\left(X_2\right) = 0,$$

but, in this case, we have that

$$\begin{cases} dI_1 = dp_1 + dp_2 \\ dI_2 = p_1 dp_1 + p_2 dp_2 + \frac{\partial V}{\partial q_1} dq_1 + \frac{\partial V}{\partial q_2} dq_2 \\ X_1 = \partial q_1 + \partial q_2 \\ X_2 = p_1 \partial q_1 + p_2 \partial q_2 - \frac{\partial V}{\partial q_1} \partial p_1 - \frac{\partial V}{\partial q_2} \partial p_2 \end{cases},$$

$$\omega = V(q_1-q_2)\mathrm{d}q_1 \wedge \mathrm{d}q_2 + p_1 \mathrm{d}q_1 \wedge \mathrm{d}p_1 + p_2 \mathrm{d}q_2 \wedge \mathrm{d}p_2 + \mathrm{d}p_1 \wedge \mathrm{d}p_2.$$

Then,

$$\omega^\flat(X_1) = p_1 dp_1 + p_2 dp_2 - V\left(q_1-q_2\right) dq_1 + V\left(q_1-q_2\right) dq_2.$$

Hence,

$$\omega(X_1, X_1) = 0 = dI_2(X_1),$$
$$\omega(X_1, X_2) = (p_2-p_1)V - (p_2-p_1)\frac{\partial V}{\partial q_2}.$$

Thus, the equation 6.13 is satisfied only if $V = \frac{\partial V}{\partial q_2}$, that is,

$$V = Ae^{q_2 - q_1}, \text{ where } A \in \mathbb{R}.$$

## 6.2 Bi-differential calculi

A bi-differential calculi consists of a pair of two distinct differential operators over the same graded algebra $\Gamma(\bigwedge^\bullet A)$. These calculi are compatible in a certain sense. One of the applications of this framework is that, under certain conditions, we can use the pair of differential operators to reconstruct a Lenard-Magri sequence on a Poisson-Nijenhuis manifold (CRAMPIN; SARLET; THOMPSON, 2000).

First, we define a differential calculus.

**Definition 6.2.1.** A differential over a graduated algebra $\bigwedge^\bullet A$ is a linear map $d\colon \wedge^p A \to \wedge^{p+1} A$ with the proprieties

$$d^2 = 0,$$
$$d(P \wedge Q) = (dP) \wedge Q + (-1)^p P \wedge dQ,$$

where $P \in \wedge^p A$ and $Q \in \bigwedge^\bullet A$. The pair $(A, d)$ is called a differential graded algebra.

Now, we can define a bi-differential calculi.

**Definition 6.2.2.** We call bi-diferential calculi a triple $(\bigwedge^\bullet A, d_1, d_2)$ that comprises a graded algebra $(\bigwedge^\bullet A)$ and two differential, $d_1$ and $d_2$, satisfying the condition

$$d_1 \circ d_2 + d_2 \circ d_1 = 0.$$

Suppose that the first cohomology group $H^1_{d_1}$ is trivial, that is, all $d_1$-closed 1-sections are $d_1$-exact. If there is a non-vanishing function $f_0 \in$ such that

$$d_1(f_0) = 0$$

We can inductively construct a sequence of 1-sections that are $d_1$-closed and $d_2$-closed.

Let $\alpha_1 = d_2(f_0)$, then

$$d_1(\alpha_1) = d_1(d_2(f_0)) = -d_2(d_1(f_0)) = 0.$$

Since $H^1_{d_1}$ is trivial, there is some $f_1 \in \wedge^0 A$ such that

$$\alpha_1 = d_1(f_1).$$

Now, we suppose by induction that, for some $m \in \mathbb{N}$, every $k \leq m$ we have that

$$d_1(\alpha_k) = 0,$$

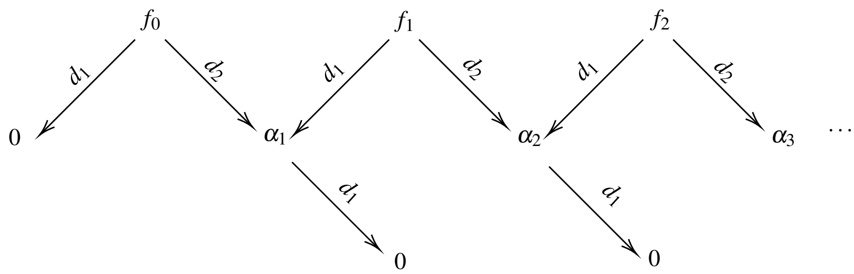
where $\alpha_k = d_2(f_{k-1})$. Since $H^1_{d_1}$ is trivial, there is some $f_m \in \wedge^0 A$ such that

$$\alpha_m = d_1(f_m).$$

Thus, defining $\alpha_{m+1} = d_2(f_m)$, we have that

$$d_1(\alpha_{m+1}) = d_1(d_2(f_m)) = -d_2(d_1(f_m)) = 0.$$

In this way, we can extend the sequence infinitely.



If $L \subset E$ is a Dirac structure, By Proposition 2.3.6, the Courant algebroid structure of $E$ descends to a Lie algebroid structure on $L$. This allows us to obtain a differential operator $d_L$ that acts on $\Gamma(\wedge^\bullet L^*)$. Suppose we have a second Courant algebroid structure on $E$. If $L$ is also Dirac within this second structure, we will have a second differential operator $d_{L'}$ that acts on $\Gamma(\wedge^\bullet L^*)$. A natural question is:

**Question 1.** Let $E$ and $E'$ be two different Courant algebroids structures over the same vector bundle and suppose that $L$ is a Dirac structure in both $E$ and $E'$. Is it true that $d_L$ and $\hat{d}_L$ form a bicalculi?

The following theorem says that the answer to the Question 1 is "not always".

**Theorem 6.2.3.** Let $(M, \pi, \hat{N}, \phi)$ be a PqN manifold. Suppose that, for some $D \subset T^*M$,

$$L = \{\alpha \oplus X \mid \alpha \in D, X \in D^\perp\}$$

is a Dirac structure in both

$$\mathbb{T}M = ((T^*M, [\cdot,\cdot]_\pi, \pi^\sharp), d, 0),$$
$$\mathbb{T}M' = ((T^*M, [\cdot,\cdot]_\pi, \pi^\sharp), d_N, \phi).$$

Then, for all $f \in C^\infty(M)$, $\alpha \oplus X$, $\beta \oplus Y \in \Gamma(L)$

$$(d_L(d_{L'}f) + d_{L'}(d_L f))(\alpha \oplus X, \beta \oplus Y) = \phi(X, Y, \pi^\sharp(df)).$$

*Proof.* First, note that, for every $f \in C^\infty(M)$ and $X,Y \in \Gamma(TM)$, we have that

$$d_N(df)(X,Y) = -df([X,Y]_N) + NX(df(Y)) - NY(df(X)),$$
$$d(d_N f)(X,Y) = -d_N f([X,Y]) + X(d_N f(Y)) - Y(d_N f(X)).$$

Let $[\![\cdot,\cdot]\!]_1$ and $[\![\cdot,\cdot]\!]_2$ be the Courant algebroid bracket of $\mathbb{T}M$ and $\mathbb{T}M'$, respectively. We will denote by $(L,[\cdot,\cdot]_L,\rho_1)$ the Lie algebroid structure in $L$ that descends of $\mathbb{T}M$, and by $d_L$ its differential operator, and we will denote by $(L,[\cdot,\cdot]_{L'},\rho_2)$ and $d_{L'}$ the ones associated to $\mathbb{T}M'$.

By definition, for every $(X\oplus\alpha),(Y\oplus\beta) \in L$ and $f \in C^\infty(M)$,

$$d_L f(X\oplus\alpha) = \mathscr{L}_{\rho_1(X\oplus\alpha)}(f) = \mathscr{L}_X(f) + \mathscr{L}_{\pi^\sharp(\alpha)}(f) = df(X) - d_\pi f(\alpha),$$
$$d_{L'} f(X\oplus\alpha) = \mathscr{L}_{\rho_2(X\oplus\alpha)}(f) = \mathscr{L}_{NX}(f) + \mathscr{L}_{\pi^\sharp(\alpha)}(f) = d_N f(X) - d_\pi f(\alpha),$$

and

$$[\![X\oplus\alpha, Y\oplus\beta]\!]_1 = \Big([X,Y] + \mathscr{L}^\pi_\alpha(Y) - \mathscr{L}^\pi_\beta(X)\Big) \oplus \left([\alpha,\beta]_\pi + \mathscr{L}_X(\beta) - \mathscr{L}_Y(\alpha)\right),$$
$$[\![X\oplus\alpha, Y\oplus\beta]\!]_2 = \Big([X,Y]_N + \mathscr{L}^\pi_\alpha(Y) - \mathscr{L}^\pi_\beta(X)\Big) \oplus \left([\alpha,\beta]_\pi + \mathscr{L}^N_X(\beta) - \mathscr{L}^N_Y(\alpha) + \phi(X,Y,\cdot)\right).$$

Since $d_L^2 = 0$, we have that

$$\begin{aligned} d_\pi f\left([\alpha,\beta]_\pi + \mathscr{L}_X(\beta) - \mathscr{L}_Y(\alpha)\right) = & df\Big([X,Y] + \mathscr{L}^\pi_\alpha(Y) - \mathscr{L}^\pi_\beta(X))\Big) \\ & - X\left(df(Y) - d_\pi f(\beta)\right) - \pi^\sharp\alpha\left(df(Y) - d_\pi f(\beta)\right) \\ & + Y\left(df(X) - d_\pi f(\alpha)\right) + \pi^\sharp\beta\left(df(X) - d_\pi f(\alpha)\right) \end{aligned}$$

Since $d_{L'}^2 = 0$, we have that

$$\begin{aligned} d_\pi f\left([\alpha,\beta]_\pi + \mathscr{L}^N_X(\beta) - \mathscr{L}^N_Y(\alpha) + \phi(X,Y,\cdot)\right) = & d_N f\Big([X,Y]_N + \mathscr{L}^\pi_\alpha(Y) - \mathscr{L}^\pi_\beta(X))\Big) \\ & - NX\left(d_N f(Y) - d_\pi f(\beta)\right) - \pi^\sharp\alpha\left(d_N f(Y) - d_\pi f(\beta)\right) \\ & + NY\left(d_N f(X) - d_\pi f(\alpha)\right) + \pi^\sharp\beta\left(d_N f(X) - d_\pi f(\alpha)\right) \end{aligned}$$

$$\begin{aligned} d_{L'} d_L f(X\oplus\alpha, Y\oplus\beta) = & -d_L f([\![X\oplus\alpha, Y\oplus\beta]\!]_2) + \mathscr{L}_{\rho_2(X\oplus\alpha)}(d_L f(Y\oplus\beta)) \\ & - \mathscr{L}_{\rho_2(Y\oplus\beta)}(d_L f(X\oplus\alpha)) \\ = & \, d_\pi f([\alpha,\beta]_\pi + \mathscr{L}^N_X(\beta) - \mathscr{L}^N_Y(\alpha) + \phi(X,Y,\cdot)) \\ & - df([X,Y]_N + \mathscr{L}^\pi_\alpha(Y) - \mathscr{L}^\pi_\beta(X)) \\ & + NX(df(Y) - d_\pi f(\beta)) + \pi^\sharp(\alpha)(df(Y) - d_\pi f(\beta)) \\ & - NY(df(X) - d_\pi f(\alpha)) - \pi^\sharp(\beta)(df(X) - d_\pi f(\alpha)) \end{aligned}$$

$$\begin{aligned} d_L d_{L'} f(X\oplus\alpha, Y\oplus\beta) &= -d_{L'} f([\![X\oplus\alpha, Y\oplus\beta]\!]_1) + \mathscr{L}_{\rho_1(X\oplus\alpha)}(d_{L'} f(Y\oplus\beta)) \\ &\quad - \mathscr{L}_{\rho_1(Y\oplus\beta)}(d_{L'} f(X\oplus\alpha)) \\ &= d_\pi f([\alpha,\beta]_\pi + \mathscr{L}_X(\beta) - \mathscr{L}_Y(\alpha))) \\ &\quad - d_N f([X,Y] + \mathscr{L}^\pi_\alpha(Y) - \mathscr{L}^\pi_\beta(X)) \\ &\quad + X(d_N f(Y) - d_\pi f(\beta)) + \pi^\sharp(\alpha)(d_N f(Y) - d_\pi f(\beta)) \\ &\quad - Y(d_N f(X) - d_\pi f(\alpha)) - \pi^\sharp(\beta)(d_N f(X) - d_\pi f(\alpha)) \end{aligned}$$

and, thus,

$$(d_L(d_{L'} f) + d_{L'}(d_L f))(\alpha\oplus X, \beta\oplus Y) = \phi(X, Y, \pi^\sharp(df)).$$

□

**Remark 6.2.4.** On the other hand, if we start from a PN manifold, the Theorem 6.2.3 shows that for any choice of $D\subset T^*M$ such that $L$ is a Dirac structure, $d_L\circ d_{L'} + d_{L'}\circ d_L = 0$ on $C^\infty(M)$.

**Example 6.2.5** (The 4-partle closed Toda case)**.** Consider the PqN structure associated with the closed Toda lattice as described in Example (3.2.10). As proved in (FALQUI *et al.*, 2020), we have that

i) $[\Omega,\Omega]_\pi = 0$;

ii) $d_N\Omega = dI_1\wedge\Omega$, where $I_k = \frac{1}{k}\mathrm{Tr}(\hat{N}^k)$;

iii) $i_{Y_k}\Omega = 0$, where $Y_k = (\hat{N})^{k-1}X_1 - X_k$ and $X_k = \pi^\sharp(dI_k)$;

iv) $\{I_1, I_k\} = 0$ for all $k\geq 2$.

First, note that since $\hat{N} = N + \pi^\sharp\Omega^\flat$, we have $d_{\hat{N}} = d_N + d_{\pi^\sharp\Omega^\flat}$. Since $d_{\pi^\sharp\Omega^\flat} = -[\Omega,\cdot]_\pi$, see e.g., (FALQUI *et al.*, 2020), condition i) implies that $d_{\pi^\sharp\Omega^\flat}(\Omega) = 0$ and $d_{\hat{N}}\Omega = d_N\Omega$.

Let

$$D^\perp = Ker(\Omega^\flat) = \{X\in\Gamma(TM) \mid \Omega^\flat(X) = 0\}.$$

Condition ii) implies that, for all $Z_1, Z_2\in D^\perp$, $\phi(Z_1, Z_2,\cdot) = 0$. We will use the Theorem 2.3.11 to show that the Lagrangian subbundle

$$L = \{X\oplus\alpha \mid \alpha\in D, X\in D^\perp\}$$

is a Dirac structure in both $\mathbb{T}M$ and $\mathbb{T}M'$.

First, note that $D^\perp$ is closed with respect to the usual commutator of vector fields. Indeed, since $d\Omega = 0$ we have that, for all $X, Y\in D^\perp$ and $Z\in\Gamma(TM)$,

$$\begin{aligned} 0 = d\Omega(X,Y,Z) &= \mathscr{L}_X(\Omega(Y,Z)) - \mathscr{L}_Y(\Omega(X,Z)) + \mathscr{L}_Z(\Omega(X,Y)) \\ &\quad - \Omega([X,Y],Z) + \Omega([X,Z],Y) - \Omega([Y,Z],X) \\ &= -\Omega([X,Y],Z). \end{aligned}$$

Since $Z$ is arbitrary, we have that $[X,Y] \in D^{\perp}$.

Now, we show that $D^{\perp}$ is closed with respect to the bracket $[\cdot,\cdot]_{\hat{N}}$. Since $d_N\Omega = \Phi = dI_1 \wedge \Omega$, we have that

$$\begin{aligned} 0 = \phi(X,Y,Z) &= \mathscr{L}_{NX}(\Omega(Y,Z)) - \mathscr{L}_{NY}(\Omega(X,Z)) + \mathscr{L}_{NZ}(\Omega(X,Y)) \\ &\quad - \Omega([X,Y]_N,Z) + \Omega([X,Z]_N,Y) - \Omega([Y,Z]_N,X) \\ &= -\Omega([X,Y]_N,Z), \end{aligned}$$

for all $X,Y \in D^{\perp}$ and $Z \in \Gamma(TM)$. Thus $[X,Y]_N \in D^{\perp}$.

Condition iii) ensures that all $Y_k \in D^{\perp}$. It is easy to check that $D^{\perp}$ is generated by $\{\partial p_1, \partial p_2, \partial p_3, \partial p_4, \partial q_2, \partial q_3\}$, and $D$ by $\{dq_1, dq_4\}$. So, since

$$[dq_i, dq_j]_\pi = d\pi(dq_i, dq_j) = 0,$$

$D$ is closed with respect to $[\cdot,\cdot]_\pi$. Therefore, the hypotheses of the Theorem 2.3.11 are verified for both $\mathbb{T}M$ and $\mathbb{T}M'$. Thus, by Theorem 6.2.3,

$$d_L \circ d_{L'} + d_{L'} \circ d_L = 0$$

on $C^\infty(M)$.

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APPENDIX

A

# DIFFERENTIAL CALCULUS ON LIE ALGEBROIDS

In this section, we will summarize the main definitions and results regarding the theory of differential calculus on Lie algebroids. Given a vector bundle $A$, we can define two graded algebras: $\wedge^\bullet A = \bigoplus_k \wedge^k A$ and $\wedge^\bullet A^* = \bigoplus_k \wedge^k A^*$, both with the product given by the wedge product.

Let $X \in \Gamma(A)$. We can define the interior product $i_X$ on $\wedge^\bullet A^*$ as follows: for every $\eta \in \Gamma(\bigwedge^p A^*)$, the action of $i_X(\eta) \in \Gamma(\wedge^{p-1} A^*)$ on $X_1, \ldots, X_{p-1} \in \Gamma(A)$ is given by:

$$i_X(\eta)(X_1, \ldots, X_{p-1}) = \eta(X, X_1, \ldots, X_{p-1}).$$

The interior product $i_X$ is a derivation of degree $-1$ of the exterior algebra $\bigwedge^\bullet A^*$, that is, for every $\eta \in \Gamma(\bigwedge^p A^*)$ and $\xi \in \Gamma(\bigwedge^\bullet A^*)$

$$i_X(\eta \wedge \xi) = (i_X(\eta)) \wedge \xi + (-1)^p \eta \wedge (i_X(\xi)).$$

The interior product can be extended for any multisection of A: let $P \in \Gamma(\bigwedge^p A)$, $i_P$ is the linear endomorphism of $\bigwedge^\bullet A^*$ of degree $-p$ such that, for every $\eta \in \Gamma(\bigwedge^q A^*)$, the action of $i_P(\eta) \in \Gamma(\bigwedge^{p-1} A^*)$ on $Q \in \Gamma(\bigwedge^{q-p} A)$ is given by

$$(i_P \eta)(Q) = (-1)^{(p-1)\frac{p}{2}} \eta(P \wedge Q).$$

For $p > 1$, $i_P$ is no more a derivation of $P \in \Gamma(\bigwedge^p A)$, but we have that, for every $P \in \Gamma(\bigwedge^p A)$ and $Q \in \Gamma(\bigwedge^q A)$,

$$i_Q \circ i_P = i_{(P \wedge Q)}.$$

(MACKENZIE, 2005) defines a Lie algebroid structure on $A$ as follows:

**Definition A.0.1.** A Lie algebroid is a triple $(A,[\cdot,\cdot]_A,\rho_A)$ consisting of a vector bundle $A$, a Lie bracket $[\cdot,\cdot]_A:\Gamma(A)\times\Gamma(A)\to\Gamma(A)$ and a morphism of vector bundles $\rho_A:\Gamma(A)\to\Gamma(TM)$, called anchor, satisfying the compatibility condition

$$[X_1,fX_2]_A=f[X_1,X_2]_A+(\mathscr{L}_{\rho_A(X_1)}f)X_2, \tag{A.1}$$

$$\rho\left([X_1,X_2]_A\right)=[\rho\left(X_1\right),\rho\left(X_2\right)]_A, \tag{A.2}$$

for all $X_1,X_2\in\Gamma(A)$ and $f\in C^\infty(M)$.

**Example A.0.2.** If $M$ is any manifold and $[\cdot,\cdot]$ is the Lie bracket of vector fields, then $(TM,[\cdot,\cdot],Id)$ is a Lie algebroid.

**Example A.0.3.** Let $\pi$ be a Poisson bivector field. The triple $(T^*M,[\cdot,\cdot]_\pi,\pi^\sharp)$ is a Lie algebroid over $M$, where $\pi^\sharp\colon T^*M\to TM$ is defined by $\pi^\sharp(\alpha)(\beta)=\pi(\alpha,\beta)$ and

$$[\alpha,\beta]_\pi=\mathscr{L}_{\pi^\sharp(\alpha)}(\beta)-\mathscr{L}_{\pi^\sharp(\beta)}(\alpha)-d(\pi(\alpha,\beta)).$$

We refer to (VAISMAN, 1994) for more details.

**Example A.0.4.** Let $N\colon TM\to TM$ be a tensor of type $(1,1)$. If $T_N(X,Y)=0$ for all $X,Y\in\Gamma(TM)$, see Definition 2.2.5, then the triple $(TM,[\cdot,\cdot]_N,N)$ is a Lie algebroid, where

$$[X,Y]_N=[NX,Y]+[X,NY]-N[X,Y].$$

**Remark A.0.5.** Note that if $N=Id$ we recover Example A.0.2.

Let $(A,[\cdot,\cdot]_A,\rho_A)$ be a Lie algebroid, then we can define a Cartan-like calculus on the vector bundle $A$. The following definition generalizes the concept of Cartan's differential.

**Definition A.0.6.** Let $(A,\rho_A,[\cdot,\cdot]_A)$ be a Lie algebroid, we define a differential operator $d_A$ in $\bigwedge^\bullet A^*$ as follows: given $\sigma\in\Gamma(\wedge^kA^*)$, for all $X_1,\ldots,X_{k+1}\in\Gamma(A)$,

$$\begin{aligned}(d_A\sigma)(X_1,\ldots,X_{k+1})=&\sum_{i<j}(-1)^{i+j}\sigma([X_i,X_j]_A,X_1\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_{K+1})\\&+\sum_{i=1}^{k+1}(-1)^{i+1}\mathscr{L}_{\rho_A(X_i)}(\sigma(X_1,\ldots,\hat{X}_i,\ldots,X_{k+1})).\end{aligned} \tag{A.3}$$

It can be checked that $d_A$ is a derivation of degree 1 and that $d_A^2=0$ (MACKENZIE, 2005). We can provide a second graded algebraic structure to $\wedge^\bullet A$, extending the notion of the Schouten bracket and Lie derivative to an arbitrary Lie algebroid.

**Definition A.0.7.** Let $\phi$ and $\psi$ be two endormorphism of graded algebra $A=\bigoplus_{k\in\mathbb{Z}}A^k$. Supouse that $\phi$ and $\psi$ have degrees $p$ and $q$, respectively, then we define their graded commutator by

$$[\phi,\psi]=\phi\circ\psi-(-1)^{pq}\psi\circ\phi.$$

**Proposition A.0.8.** Let $(A, [\cdot,\cdot]_A, \rho_A)$ be a Lie algebroid. For each $P \in \Gamma(\bigwedge^p A)$ and $Q \in \Gamma(\bigwedge^q A)$, there exists a unique element $[P,Q]_A \in \Gamma(\bigwedge^{p+q-1} A)$, called the Schouten bracket of $P$ and $Q$, defined by

$$i_{[P,Q]_A} = [[i_P, d_A], i_Q],$$

such that:

(i) it satisfies $[X, f]_A = \langle \rho_A(X), df \rangle = X(d_A f)$, for all $X \in \Gamma(A)$ and for all $f \in C^\infty(M)$;

(ii) it is antisymmetric in the graded sense, i.e., for every $P \in \wedge^p A$ and $Q \in \wedge^q A$,

$$[P,Q]_A = -(-1)^{(p-1)(q-1)}[Q,P]_A;$$

(iii) for every $P_1 \in \wedge^{p_1} A$, $[P_1, \cdot]_A$ is a derivation of the graded algebra $(\bigwedge^\bullet A, \wedge)$, i. e.,

$$[P_1, P_2 \wedge P_3]_A = [P_1, P_2]_A \wedge P_3 + (-1)^{(p_1-1)p_2} P_2 \wedge [P_1, P_3]_A,$$

for all $P_2 \in \wedge^{p_2} A$ and $P_3 \in \bigwedge^\bullet A$.

**Remark A.0.9.** The Schouten bracket defined on a Lie algebroid generalizes various brackets in graded algebras $(\wedge^\bullet A)$ of a vector bundle $A$. For instance:

- If the Lie algebroid is the standard Lie algebroid in the tangent bundle of a manifold, $(TM, [\cdot,\cdot], \mathrm{Id})$, we recover the standard Schouten bracket for multivector fields. We refer to (MARLE, 1997).
- If $\pi$ is a Poisson tensor on a manifold $M$, then the Schouten bracket associated with the Lie algebroid $(T^*M, [\cdot,\cdot]_\pi, \pi^\sharp)$ yields the Koszul bracket. See (FIORENZA; MANETTI, 2012).

**Proposition A.0.10.** Let $(A, \rho_A, [\cdot,\cdot]_A)$ be a Lie algebroid on a smooth manifold $M$. For every $X \in \Gamma(A)$ there is a unique derivation of degree 0 of the algebra $\bigwedge^\bullet A^*$, denoted by $\mathscr{L}^A_X$ and called the Lie derivative with respect to $X$, that satisfies:

(i) for every $f \in C^\infty(M)$,

$$\mathscr{L}^A_X(f) = i_{\rho_A(X)} df;$$

(ii) for every $\eta \in \bigwedge(A^*)$ of degree $p > 0$,

$$\mathscr{L}^A_X(\eta)(Y_1, \ldots, Y_p) = i_{\rho_A(X)} d(\eta(Y_1, \ldots, Y_p)) + \sum_{k=1}^{p} (-1)^k \eta([X, Y_i]_A, Y_1, \ldots, \hat{Y}_i, \ldots, Y_p),$$

where $Y_1, \ldots Y_p \in \Gamma(A)$ and, as usual, the entries with hat are to be omitted.

For $P \in \bigwedge^p A$, the Lie derivative $\mathscr{L}_X$ will be defined by

$$\mathscr{L}^A_X(P) = [X, P]_A.$$

## A.1 Diferential Lie algebra

In this section, we will study the differential Lie algebras and their generalization: differential pre-Lie algebras. Differential Lie algebras form the pure algebraic framework of Lie algebroids. We will follow the definitions and notations outlined in (KOSMANN-SCHWARZBACH; MAGRI, 1990).

**Definition A.1.1.** Let $K$ be the field of real or complex numbers, and let A be an associative and commutative $K$-algebra whit unit. Let $E$ be a finitely generated projective $A$-module. Let $[\cdot,\cdot]_\mu$ be an antisymmetric, $K$-bilinear map, from $E\times E$ to $E$. We say that $(E,[\cdot,\cdot]_\mu)$ is a differential pre-Lie algebra over $A$ if there exists an $A$-linear map $\mathscr{L}^\mu$ from $E$ to the $K$-vector space of derivations of $A$ such that

$$[X,fY]_\mu = f[X,Y]_\mu + \mathscr{L}^\mu_X(f)Y,$$

for all $X$ and $Y$ in $E$, and for all $f$ in $A$.

**Remark A.1.2.** Here we will assume that the $A$-module $E$ has an element $X\in E$ such that $f\in A$ and $fX=0$ imply $f=0$. Moreover, we will suppose that $E$ is a finitely generated protective $A$-module, thus $(E^*)^*\cong E$.

If we add the requirement that $[\cdot,\cdot]_\mu$ satisfies the Jacobi identity, we get the concept of differential Lie algebra. More precisely,

**Definition A.1.3.** Let $(E,[\cdot,\cdot]_\mu)$ be a differential pre-Lie algebra over $A$. We say that $(E,[\cdot,\cdot]_\mu)$ is a differential Lie algebra over $A$ if $[\cdot,\cdot]_\mu$ is a $K$-Lie algebra structure on $E$ and $\mathscr{L}^\mu$ defines an $E$-module structure on A.

An important consequence of the definition of differential Lie algebra is that $\mathscr{L}^\mu$ is a Lie algebra morphism from $(E,[\cdot,\cdot]_\mu)$ to the algebra of derivations of the ring $A$, that is,

$$\mathscr{L}^\mu_{[X,Y]_\mu} = [\mathscr{L}^\mu_X,\mathscr{L}^\mu_Y],$$

for all $X,Y\in E$.

Since $E$ is finitely generated and projective, we can identify the $A$-module of $q$-linear skewsymmetric maps from $E$ to $A$ with $\wedge^q(E^*)$. We call graded pre-differential algebra a graded commutative algebra, with a derivation of degree 1. Note that this derivation is not necessarily of square 0. Then, we have an equivalence between graded pre-differential algebra on $\wedge^\bullet(E^*)$ and differential pre-Lie algebra on $E$.

**Proposition A.1.4.** Let $(E,[\cdot,\cdot]_\mu)$ be a differential pre-Lie algebra, then there is a correspondent degree one derivation on $\wedge^\bullet(E^*)$ defined as in Equation (A.3)

$$\begin{aligned}(d_\mu\sigma)(X_1,\ldots,X_{k+1}) = &\sum_{i<j}(-1)^{i+j}\sigma([X_i,X_j]_\mu,X_1\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_{K+1})\\ &+\sum_{i=1}^{k+1}(-1)^{i+1}\mathscr{L}^\mu_{X_i}(\sigma(X_1,\ldots,\hat{X}_i,\ldots,X_{k+1}))\end{aligned} \tag{A.4}$$

that make $\wedge^\bullet(E^*)$ a graded pre-differential algebra.

Conversely, if $(\wedge^\bullet(E^*), d_\nu)$ is a differential pre-Lie algebra, then we can define

$$\mathscr{L}_X^\nu(f) = \langle d_\nu f, X\rangle$$

and

$$\langle \alpha, [X,Y]_\nu\rangle = -d_\nu\alpha(X,Y) + \mathscr{L}_X^\nu(\langle\alpha,Y\rangle) - \mathscr{L}_Y^\nu(\langle\alpha,X\rangle),$$

where $X,Y \in E$, $\alpha \in E^*$, and $f \in A$. Moreover, $(E,[\cdot,\cdot]_\nu)$ is a differential pre-Lie algebra.

**Proposition A.1.5.** The differential pre-Lie algebra $(E,[\cdot,\cdot]_\mu)$ over $A$ is a differential Lie algebra if and only if

$$(d_\mu)^2 = 0.$$

**Example A.1.6.** Let $(A, d_{A^*}, \phi)$ be a quasi-Lie algebroid. Then $(A,[\cdot,\cdot]_A)$ is a differential pre-Lie algebra and $(\wedge^\bullet A, d_{A^*})$ is a graded pre-differential algebra.

**Proposition A.1.7.** Let $(E,[\cdot,\cdot]_\mu)$ be a differential pre-Lie algebra over $A$. There exists a unique $K$-bilinear mapping that extending the bracket $[\cdot,\cdot]_\mu$ on the graded algebra $\wedge^\bullet(E)$ satisfying

(i) $[X,f]_\mu = \mathscr{L}_X^\mu(f)$, for all $X \in E$ and $f \in A$;

(ii) it is skewsymmetric in the graded sense, that is,

$$[Q,P]_\mu = -(-1)^{(q-1)(p-1)}[P,Q];$$

(iii) it is a biderivation of the graded algebra $\wedge^\bullet(E)$, that is,

$$[Q,P\wedge W] = [P,Q]\wedge W + (-1)^{(q-1)p} P\wedge[Q,W],$$

for $Q \in \wedge^q(E)$, $P \in \wedge^p(E)$ and $W \in \wedge^\bullet(E)$.

Let $N\colon E \to E$ be an $A$-linear map. We define the deformed bracket by

$$[X,Y]_\mu^N = [NX,Y]_\mu + [X,NY]_\mu - N[X,Y]_\mu,$$

where $X,Y \in E$.

Then $(E,[\cdot,\cdot]_\mu^N)$ is a differential pre-Lie algebra with

$$\mathscr{L}_{NX}^\mu.$$

**Proposition A.1.8.** Let $(E,[\cdot,\cdot]_\mu)$ be a differential pre-Lie algebra and let $N\colon E \to E$ be an $A$-linear map. Then $(E,[\cdot,\cdot]_\mu^N)$ is a differential pre-Lie algebra. Furthermore, the derivation of degree 1 of the graded algebra $\wedge^\bullet(E^*)$ associated with $[\cdot,\cdot]_\mu^N$ and denoted by $d_N^\mu$ satisfies

$$d_N^\mu = [i_N, d_\mu].$$

Let $P \in \wedge^2 E$ be a bivector on $E$, and let $P^\sharp \colon E^* \to E$ be the linear mapping defined by

$$\langle \alpha, P^\sharp(\beta) \rangle = P(\beta, \alpha).$$

We define a bracket in $E^*$ by

$$\begin{aligned}\langle [\alpha,\beta]_P^\mu, X\rangle =& \langle \alpha, [P^\sharp(\beta), X]_\mu\rangle - \langle \beta, [P^\sharp(\alpha), X]_\mu\rangle + \mathscr{L}^\mu_{P^\sharp(\alpha)}(\langle \beta, X\rangle) \\ &- \mathscr{L}^\mu_{P^\sharp(\beta)}(\langle \alpha, X\rangle) - \mathscr{L}^\mu_X(\langle \beta, P^\sharp(\alpha)\rangle)\end{aligned}$$

then, $(E, [\cdot,\cdot]_P^\mu)$ is a differential pre-Lie algebra.

**Proposition A.1.9.** Let $(E, [\cdot,\cdot]_\mu)$ be a differential pre-Lie algebra and let $P$ be a bivector on $E$. the derivation of degree 1 of the graded algebra $\wedge^\bullet E$ associated with the bracket $[\cdot,\cdot]_\mu$ is

$$d_P^\mu = [P, \cdot]_\mu.$$

*Proof.* Let us check that both derivation $d_P^\mu$ and $[P,\cdot]_\mu$ coincide on elements of $A$ and $E$. For all $\alpha \in E^*$ and $f \in A$,

$$d_P^\mu f(\alpha) = \mathscr{L}^\mu_{P^\sharp(\alpha)}(f) = \langle d_\mu f, P^\sharp(\alpha)\rangle = -\langle \alpha, P^\sharp(d_\mu f)\rangle.$$

For $X \in E$,

$$\begin{aligned}d_P^\mu X(\alpha,\beta) &= \mathscr{L}^\mu_{P^\sharp(\alpha)}(\langle \beta, X\rangle) - \mathscr{L}^\mu_{P^\sharp(\beta)}(\langle \alpha, X\rangle) - \langle [\alpha,\beta]_P^\mu, X\rangle \\ &= -\langle \alpha, [P^\sharp(\beta), X]_\mu\rangle + \langle \beta, [P^\sharp(\alpha), X]_\mu\rangle - \mathscr{L}^\mu_X(\langle \alpha, P^\sharp(\beta)\rangle),\end{aligned}$$

on the other hand,

$$\begin{aligned}[P,X]_\mu(\alpha,\beta) &= -\mathscr{L}^\mu_X(P)(\alpha,\beta) = -\mathscr{L}^\mu_X(P(\alpha,\beta)) - \langle \alpha, P^\sharp(\mathscr{L}^\mu_X(\beta))\rangle + \langle \beta, P^\sharp(\mathscr{L}^\mu_X(\alpha))\rangle \\ &= -\langle \alpha, [P^\sharp(\beta), X]_\mu\rangle + \langle \beta, [P^\sharp(\alpha), X]_\mu\rangle - \mathscr{L}^\mu_X(\langle \alpha, P^\sharp(\beta)\rangle).\end{aligned}$$

□

**Proposition A.1.10.** Let $(E, [\cdot,\cdot]_\mu)$ be a differential pre-Lie algebra and let $P$ be a bivector on $E$. Then,

$$[P,P]_\mu(\alpha_1,\alpha_2,\alpha_3) = -2\left(\langle \mathscr{L}^\mu_{P^\sharp(\alpha_1)}(\alpha_3), P^\sharp(\alpha_2)\rangle + \langle \mathscr{L}^\mu_{P^\sharp(\alpha_2)}(\alpha_1), P^\sharp(\alpha_3)\rangle + \langle \mathscr{L}^\mu_{P^\sharp(\alpha_3)}(\alpha_2), P^\sharp(\alpha_1)\rangle\right).$$

*Proof.* First, we use the previous proposition to compute $[P,\cdot]_\mu$ by

$$\begin{aligned}d_P^\mu P(\alpha_1,\alpha_2,\alpha_3) =& \langle [\alpha_1,\alpha_2]_P^\mu, P^\sharp(\alpha_3)\rangle - \langle [\alpha_1,\alpha_3]_P^\mu, P^\sharp(\alpha_2)\rangle + \langle [\alpha_2,\alpha_3]_P^\mu, P^\sharp(\alpha_1)\rangle \\ &+ \mathscr{L}^\mu_{P^\sharp(\alpha_1)}(P(\alpha_2,\alpha_3)) - \mathscr{L}^\mu_{P^\sharp(\alpha_2)}(P(\alpha_1,\alpha_3)) + \mathscr{L}^\mu_{P^\sharp(\alpha_3)}(P(\alpha_1,\alpha_2))\end{aligned}$$

The result follows using the definition $[\cdot,\cdot]_P^\mu$. □

**Example A.1.11.** Let $(T^*M, [\cdot,\cdot]_\pi, \pi^\sharp)$ be the Lie algebroid associated with a Poisson manifold $(M,\pi)$. Given $\Omega \in \Gamma(\wedge^2 T^*M)$, we have that

$$d^\pi_\Omega = [\Omega, \cdot]_\pi$$

and

$$[\Omega,\Omega]_\pi(X_1,X_2,X_3) = -2\left(\langle \mathscr{L}^\pi_{\Omega^\flat(X_1)}(X_3), \Omega^\flat(X_2)\rangle + \langle \mathscr{L}^\pi_{\Omega^\flat(X_2)}(X_1), \Omega^\flat(X_3)\rangle + \langle \mathscr{L}^\pi_{\Omega^\flat(X_3)}(X_2), \Omega^\flat(X_1)\rangle\right).$$

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