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Some properties of the gluing of schemes

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Algumas propriedades da colagem de esquemas

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutora em Ciências – Matemática. *VERSÃO REVISADA*

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RESUMO

CALIXTO, R. A. **Algumas propriedades da colagem de esquemas**. 2024. 78 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2024.

O estudo das propriedades das construções de colagem abrange diversas categorias, sendo particularmente relevante na categoria de esquemas, onde permite a formação de novos esquemas. Este trabalho investiga as propriedades da colagem de dois esquemas ao longo de um terceiro, especificamente ao longo de um subesquema fechado. A partir de propriedades do produto fibra de anéis existentes na literatura, a análise revela que essa colagem resulta em um esquema localmente Noetheriano. Além disso, são estabelecidas condições para que o esquema resultante da colagem seja Cohen-Macaulay, e de maneira parcial Gorenstein e singular. O estudo também aborda a multiplicidade dessa colagem de esquemas, apresentando fórmulas correspondentes. Como aplicação, são obtidas fórmulas para a colagem de esquemas relacionados a variedades algébricas afins. No contexto da categoria de esquemas formais, demonstra-se que, sob condições específicas, a colagem de esquemas formais constitui um esquema formal. Este trabalho proporciona uma compreensão aprofundada das propriedades e características resultantes da colagem de esquemas, contribuindo para o avanço do conhecimento nesse domínio.

Palavras-chave: Colagem de esquemas, Anéis de produtos de fibra, Invariantes.

ABSTRACT

CALIXTO, R. A. **Some properties of the gluing of schemes**. 2024. 78 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2024.

The study of the properties of collage constructions spans various categories, being particularly relevant in the schemes category, where it allows the formation of new schemes. This work investigates the properties of gluing two schemes along a third, specifically along a closed sub-scheme. Based on properties of the fiber product of rings found in the literature, the analysis reveals that this gluing results in a locally Noetherian scheme. Additionally, conditions are established for the resulting scheme from the gluing to be Cohen-Macaulay, and partially Gorenstein and singular. The study also addresses the multiplicity of this scheme gluing, presenting corresponding formulas. As an application, formulas are derived for the gluing of schemes related to affine algebraic varieties. In the context of the formal schemes category, it is demonstrated that, under specific conditions, the gluing of formal schemes constitutes a formal scheme. This work provides a comprehensive understanding of the properties and characteristics resulting from scheme gluing, contributing to the advancement of knowledge in this domain.

Keywords: Gluing of schemes, Fiber product rings, Invariants.

CONTENTS

CHAPTER 1

INTRODUCTION

The classical algebraic geometry is an important area of mathematics that studies the geometric properties of solutions to polynomial equations and their relationships with underlying algebraic structures. Its relevance is due to the fact that it merges concepts and techniques from algebra with geometry to analyze algebraic sets and their geometric properties. More specifically, consider $f_i \in k[x_1, \dots, x_n]$ $(1 \leq i \leq r)$ as polynomials in n variables with coefficients in an algebraically closed field k. The set of common zeros of all f_i is a subset of k^n, denoted by $V(f_1, \dots, f_r)$, which are the closed sets of a topology defined in k^n, called the Zariski topology. If $V := V(f_1, \dots, f_r)$ cannot be expressed as the union of two proper closed subsets, we say that V is *irreducible*. An *affine algebraic variety* is an irreducible closed subset of k^n (with the induced topology) and corresponds to a finitely generated integral domain over a field (see (HARTSHORNE, 1977, Corollary 3.8)), called the *coordinate ring*. This powerful correspondence between geometry and algebra allows us to understand algebraic techniques to study geometric problems. However, working on an algebraically closed field and with finitely generated k-algebras it greatly limits the universe of investigation. Some of the main topics covered by classical algebraic geometry include:

- Algebraic Varieties: The study of sets of solutions to systems of polynomial equations. These sets can be understood as geometric spaces.
- Rings and Ideals: The use of algebraic structures, such as rings and ideals, to understand algebraic properties of geometric objects associated with solutions to polynomial equations.
- Rational Functions and Morphisms: The investigation of functions defined on algebraic varieties and morphisms (transformations) between these varieties.

The theory of schemes in modern algebraic geometry plays a crucial role in generalizing and extending the foundational principles of classical algebraic geometry. Spearheaded by Alexander Grothendieck and his collaborators, this remarkable advancement broadens the scope of the traditional approach to algebraic varieties, providing a more flexible and abstract framework for studying algebraic geometric objects.

By generalizing classical algebraic geometry, schemes allow for a more comprehensive and global analysis of varieties, including those with singularities and other complexities. Grothendieck emphasizes the importance of transcending the limitations of the classical approach in his works, such as "Éléments de géométrie algébrique", where he highlights the significance of this broader perspective for advancing theory.

In the specific context of the theory of abelian groups associated with schemes, this generalization provides a broader and more powerful theoretical foundation. The use of techniques such as group cohomology in a scheme-theoretic context allows for a deeper understanding of the properties of abelian groups, as discussed in Grothendieck's contributions, such as "Le groupe de Brauer III: Exemples et compléments".

Thus, the theory of schemes not only expands but also enriches algebraic geometry, offering a more expansive structure for understanding algebraic varieties and associated groups. This more abstract and global approach represents a relevant advancement in the evolution of algebraic geometry and its applications across various mathematical domains.

In these two subareas of the algebraic geometry, significant ramifications emerge. For instance, if we consider the completion of $k[x_1, \dots, x_n]$ with respect to the $(x_1, \dots, x_n)$-adic topology, we arrive at the ring of power series $k[[x_1, \dots, x_n]]$. Further exploration involves considering $f_i \in k[[x_1, \dots, x_n]]$ $(1 \leq i \leq r)$ and scrutinizing the solutions or zeros of these elements, leading to the encounter with sets known as formal spaces.

Within $k[[x_1, \dots, x_n]]$, a focus on convergent power series or analytic functions leads to another ring denoted by $k\{x_1, \dots, x_n\}$. Similarly, if $f_i \in k\{x_1, \dots, x_n\}$ $(1 \leq i \leq r)$, the zeros of these elements yield sets referred to as analytic spaces. Notably, when $k = \mathbb{C}$, this latter space is extensively studied in the theory of complex singularity.

It is crucial to realize that when considering the aforementioned sets, a significant distinction arises between affine algebraic varieties and affine formal (or analytic) spaces. The former is endowed with the Zariski topology, while the latter adopts the Hausdorff topology. This difference becomes particularly pronounced in the study of singularities, as the Zariski topology is coarser, while the alternative topology is more refined and aligns with the characteristics of the Hausdorff topology.

Similarly to the formal and analytic sub-areas, the modern algebraic geometry encompasses various branches, particularly within the theory of schemes. As is widely recognized, algebraic geometry is a vast and rich field, teeming with numerous conjectures and open problems awaiting exploration. Precisely within the theory of schemes, many open problems arise from investigations within classical algebraic geometry. One of the most current research problems comes from the gluing constructions.

For instance, consider the gluing of two topological spaces X and Y along Z, denoted by $X \sqcup_Z Y$. This operation involves constructing the disjoint union of X and Z, as well as the disjoint union of Y and Z, followed by the identification of certain subsets of Z in both spaces using homeomorphisms. The result of this operation is a new topological space that encapsulates the relationship between X and Y with respect to Z, all while preserving the coherence and continuity of the topological structures involved. Within this context, several overarching questions arise:

Question 1.1. Given the known algebraic or geometric structures of X, Y, and Z, does $X \sqcup_Z Y$ retain these same structures?

In general terms, giving positive answers to the previous question brings with it an alternative path of investigation for more complex spaces, which are spaces that arise from the gluing of other spaces. More specifically, we pose the following inquiries:

Question 1.2.

1. If X, Y, and Z are analytic (formal) spaces, is $X \sqcup_Z Y$ also an analytic (formal) space?
2. If X, Y, and Z are schemes, is $X \sqcup_Z Y$ a scheme?
3. Under what conditions is $X \sqcup_Z Y$ a locally Noetherian scheme?
4. When is $X \sqcup_Z Y$ a Cohen-Macaulay (Gorenstein) scheme?
5. What can be inferred about the singularities of $X \sqcup_Z Y$?
6. Is there any relation between of the multiplicities of X, Y and Z with the multiplicity of $X \sqcup_Z Y$?
7. How are deformations and equisingularity observed in $X \sqcup_Z Y$?

To address these questions, particularly those in Question 1.2, focusing on items 1 and 2, previous research by (FREITAS; PÉREZ; MIRANDA, 2021) and (FREITAS; PÉREZ; MIRANDA, 2022) has explored the analytic and formal contexts. Additionally, investigations in the realm of modern algebraic geometry, specifically within the category of schemes, have been conducted by (OLARTE; RIZZO, 2023), (FERRAND, 2003), and (SCHWEDE, 2005). However, despite these efforts, there remains a dearth of exploration in the analytic and formal contexts, and even less within the category of schemes. Consequently, this thesis aims to investigate and provide insights into the questions posed in Question 1.2, with a primary focus on the theory of schemes.

It is important to realize that in some of these works, it is clear that the class of fiber product rings is a key ingredient in obtaining properties of the spaces studied, thus making research into this class of rings important. In this direction, several authors have been investigated the structure of the fiber product $R \times_T S$ of Noetherian rings, for instance, (OGOMA, 1985; ANANTHNARAYAN; AVRAMOV; MOORE, 2012; ENDO; GOTO; ISOBE, 2021; CHRISTENSEN; STRIULI; VELICHE, 2010; NASSEH *et al.*, 2019; NASSEH; SATHER-WAGSTAFF, 2017; GELLER, 2022; MOORE, 2009; NASSEH; TAKAHASHI, 2020). Further, (FACCHINI, 1982) proves an important result for our work, that describes the spectrum of a fiber product ring that arising form surjective homomorphisms. Precisely, his paper shows that spectrum $\mathrm{Spec}(R \times_T S)$ is obtained by pasting together $\mathrm{Spec}(R)$ and $\mathrm{Spec}(S)$ along two closed sets homeomorphic to $\mathrm{Spec}(T)$. In summary, the fiber product ring theory will play a crucial role in the results obtained here.

The thesis is organized as follows. Since the theory of schemes and commutative algebra are closely intertwined, Chapter 2 revisits fundamental concepts and results necessary for understanding the subsequent chapters, such as ring dimension, and the theory of fiber product rings.

In Chapter 3, we lay the groundwork for our work by introducing the essential concepts in scheme theory, including sheaves, stalks, ringed spaces, and morphisms. Moreover, this chapter introduces schemes and closed immersion of schemes, central concepts for later discussions.

Chapter 4 furnishes answers for the Questions 1.2 2-6. Actually, this chapter gives properties concerning the gluing of schemes, providing insights into the preservation of certain properties in the process. This includes investigations into when the resulting scheme is Noetherian (Theorem 4.2), their dimension (Proposition 4.17) and Zariski tangent space (Theorem 4.19). Also, we show some cases when the gluing of schemes is Cohen-Macaulay, Gorenstein and singular (see Theorems 4.22 and 4.26). In addition, we investigate the gluing of schemes over a field k. More precisely, knowing that k-schemes (locally) of finite type are (locally) Noetherian, Proposition 4.11 shows that that the gluing of k-schemes (finite type or locally of finite type) is a k-scheme (finite type or locally of finite type). Also, the last part of this chapter investigates the relationship between the multiplicities of X, Y, and Z, and the multiplicity of $X \sqcup_Z Y$, giving a formula for the multiplicity of the gluing $X \sqcup_Z Y$ (Theorem 4.29).

Chapter 5 focuses on formal schemes, first elucidating their properties and exploring the gluing process in this framework. Furthermore, the chapter investigates the Noetherianness of formal schemes and its implications, thus establishing a new structure to be investigated.

CHAPTER 2

SOME FACTS OF COMMUTATIVE ALGEBRA

In the current chapter, we will recall some concepts of commutative algebra that will be needed throughout this work. For the proofs of the results, we refer (BRUNS; HERZOG, 1998) and (MATSUMURA, 1987). Also, the references used for the fiber product ring theory were (ANANTHNARAYAN; AVRAMOV; MOORE, 2012) and (ENDO; GOTO; ISOBE, 2021). By convention, all rings in this text are commutative, Noetherian with identity 1, unless explicitly stated otherwise, and all ring homomorphisms send 1 to 1.

2.1 Cohen-Macaulay, regular, complete intersection and Gorenstein rings

We begin this section by recalling the definition of the Krull dimension of a ring. Let R be a ring and let $\mathfrak{p}$ be a prime ideal of R. Then,

$$\dim(R) := \sup\{n \in \mathbb{N} \,|\, \exists\, \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ chain of prime ideals of } R\}$$

is called the *Krull dimension* (or simply the *dimension*) of the ring R. Also,

$$\operatorname{ht}(\mathfrak{p}) := \dim(R_{\mathfrak{p}})$$

is the *height of* $\mathfrak{p}$. It is the supremum of the lengths of the strictly ascending chains of prime ideals contained in $\mathfrak{p}$.

Now, we provide a significant definition for the most fundamental numerical invariant of a Noetherian local ring R.

Definition 2.1. Let $(R, \mathfrak{m})$ be a Noetherian local ring.

1. We say that a sequence of elements $a_1, \dots, a_n \in \mathfrak{m}$ of R is *regular* in $\mathfrak{m}$ if:

    - a_1 is not a zero-divisor on R;

- a_i not a divisor of zero on $R/(a_1,\ldots,a_{i-1})$ for every $2 \le i \le n$;

2. An regular sequence $a_1,\ldots,a_n$ in $\mathfrak{m}$ is said to be *maximal* (in $\mathfrak{m}$), if $a_1,\ldots,a_{n+1}$ is not regular for any $a_{n+1} \in \mathfrak{m}$. Since R is Noetherian, all maximal regular sequence in $\mathfrak{m}$ have the same length, called the *depth of* R, and write $\operatorname{depth}(R)$.

In general, we have the following elementary property of depth. Before stating it, recall that if $(R,\mathfrak{m})$ is an n-dimensional Noetherian local ring, a *system of parameters* of R is a sequence of elements $a_1,\ldots,a_n \in \mathfrak{m}$ such that $\sqrt{(a_1,\ldots,a_n)} = \mathfrak{m}$.

Proposition 2.2. (BRUNS; HERZOG, 1998, Proposition 1.2.12) Let $(R,\mathfrak{m})$ be a Noetherian local ring. Then every regular sequence is part of a system of parameters of R. In particular $\operatorname{depth}(R) \le \dim(R)$.

When the equality occurs, we obtain an important class of rings.

Definition 2.3. A Noetherian local ring $(R,\mathfrak{m})$ is said to be *Cohen–Macaulay* (CM) provided

$$\operatorname{depth}(R) = \dim(R).$$

In the following, we define some important subclasses of Cohen-Macaulay rings.

Definition 2.4. Let $k = R/\mathfrak{m}$ be the residue field of $(R,\mathfrak{m})$. We say that R is *regular* if

$$\mu(\mathfrak{m}) := \dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim(R),$$

where $\mu(\mathfrak{m})$ denotes the minimal number of generators of maximal ideal $\mathfrak{m}$. Also, $\mu(\mathfrak{m})$ is called the *embedding dimension* of R.

Definition 2.5. A Noetherian local ring $(R,\mathfrak{m})$ is said to be a *complete intersection* provided the completion $\widehat{R}$ (where completion $\widehat{R}$ is defined as $\varprojlim R/\mathfrak{m}^n$), with respect the the maximal ideal is isomorphic to $S/(a_1,\ldots,a_t)$, where S is a regular ring and $a_1,\ldots,a_t$ is a regular sequence in S. If $\widehat{R} \cong S/\mathfrak{a}$, where $\mathfrak{a}$ is a principal ideal, then R is said to be a *hypersurface*.

We now introduce Gorenstein rings, which also belong to the Cohen-Macaulay ring class. There are many equivalent definitions for a Gorenstein ring, but here we give the following one.

Definition 2.6. A Noetherian local ring is *Gorenstein* provided the following conditions are met:

(i) R is Cohen-Macaulay,

(ii) There exists a system of parameters, say $a_1, \ldots, a_d$, such that the generated ideal $(a_1, \ldots, a_d)$ is irreducible, i.e., for any ideals $\mathfrak{b}$ and $\mathfrak{b}'$ of R

$$(a_1, \ldots, a_d) \neq \mathfrak{b} \cap \mathfrak{b}' \text{ if } (a_1, \ldots, a_d) \neq \mathfrak{b} \text{ and } (a_1, \ldots, a_d) \neq \mathfrak{b}'.$$

As mentioned above, there is a relationship between these classes of rings, which is established in the following result.

Proposition 2.7. (BRUNS; HERZOG, 1998, Proposition 3.1.20) We have the following chain of implications for Noetherian local rings:

$$\text{regular} \Rightarrow \text{complete intersection} \Rightarrow \text{Gorenstein} \Rightarrow \text{Cohen-Macaulay}.$$

.

Remark 2.8. In general the arrows are not reversible. Let $k[x, y, z]$ be a ring of polynomials and let $\mathfrak{a} = (x^2, y^2, xy, yz, xy - z^2)$ be an ideal of $k[x, y, z]$. Consider $R = k[x, y, z]/\mathfrak{a}$. Then, R is a local Gorenstein ring that is not a complete intersection (see (EISENBUD, 1995, Example 21.7)).

Example 2.9. The following are well-known examples in the literature that are frequently used to illustrate when rings are regular, Cohen-Macaulay, Gorenstein, and complete intersection rings. For further details, refer to (MATSUMURA, 2012, Theorem 19.5 and 21.2) and (BRUNS; HERZOG, 1998, Proposition 2.2.2).

(1) Let $k[x_1, \ldots, x_n]$ be a ring polynomials in n variables with coefficients in a field k. The localization of $k[x_1, \ldots, x_n]$ with the maximal ideal $(x_1, \ldots, x_n)$, denoted by $k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$, is a local regular ring.

(2) The formal power series ring $k[[x_1, \ldots, x_n]]$ in n indeterminates over a field k is a local ring with maximal ideal $(x_1, \ldots, x_n)$, is a regular ring.

(3) (JONG; PFISTER, 2000, Example 4.3.4) The ring $\mathbb{C}\{x_1, \ldots, x_n\}$ of convergent power series over $\mathbb{C}$ is a local regular ring.

(4) Let $f_1, \ldots, f_s$ be a regular sequence of the rings $k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$, $k[[x_1, \ldots, x_n]]$ or $\mathbb{C}\{x_1, \ldots, x_n\}$, respectively. Then,

$$\frac{k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}}{(f_1, \ldots, f_s)}, \ \frac{k[[x_1, \ldots, x_n]]}{(f_1, \ldots, f_s)} \text{ or } \frac{\mathbb{C}\{x_1, \ldots, x_n\}}{(f_1, \ldots, f_s)}$$

are complete intersection. In addition, Proposition 2.7 gives that these rings are Gorenstein and Cohen-Macaulay.

(5) If R is 1-dimensional local domain ring, then R is Cohen-Macaulay.

2.2 Fiber product rings

In this section, we will revisit the concept of a key element crucial for the remainder of this work: the fiber product rings. We will commence with the definition of this category of rings and present some pertinent results for the upcoming chapters.

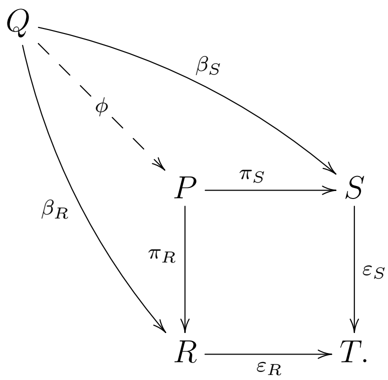
Definition 2.10. Let R, S and T be commutative rings, and let $\varepsilon_R : R \to T, \varepsilon_S : S \to T$ be homomorphisms of rings. The *fiber product* of R and S over T, is the set

$$R \times_T S = \{(r, s) \in R \times S \mid \varepsilon_R(r) = \varepsilon_S(s)\}.$$

The fiber product ring $P := R \times_T S$ is a subring of the usual direct product $R \times S$ and it is the pulback of ε_R and ε_S, i.e., we have a commutative diagram of rings

$$\begin{CD} R \times_T S @>{\pi_S}>> S \\ @V{\pi_R}VV @VV{\varepsilon_S}V \\ R @>{\varepsilon_R}>> T, \end{CD} \tag{2.1}$$

where the maps $\pi_R : R \times_T S \longrightarrow R$, $(r, s) \longmapsto r$ and $\pi_S : R \times_T S \longrightarrow S$, $(r, s) \longmapsto s$ stand for the natural projections. Moreover, for any other ring Q and any two morphism of rings $\beta_R : Q \longrightarrow R$ and $\beta_S : Q \longrightarrow S$ such that $\varepsilon_R \circ \beta_R = \varepsilon_S \circ \beta_S$, there exists a unique morphism $\phi : Q \longrightarrow P$ such that $\beta_R = \pi_R \circ \phi$ and $\beta_S = \pi_S \circ \phi$.



The following setup and notation are in force for the rest of this section: from now on, we assume that the fiber product $P := R \times_T S$ is non-trivial, i.e., $R \neq T \neq S$. In addition, suppose that $\varepsilon_R : R \to T$, $\varepsilon_S : S \to T$ are surjective morphisms. Also, all the rings are local with the same residue field, i.e., let $(R, \mathfrak{m}_R, k), (S, \mathfrak{m}_S, k), (T, \mathfrak{m}_T, k)$ be Noetherian local rings.

Remark 2.11. (ANANTHNARAYAN; AVRAMOV; MOORE, 2012, (1.1.2)) Let η denote the inclusion of rings $R \times_T S \to R \times S$. The rings above are related through exact sequences of P-modules

$$0 \longrightarrow R \times_T S \xrightarrow{\eta} R \oplus S \xrightarrow{(\varepsilon_R, -\varepsilon_S)} T \longrightarrow 0.$$

Proposition 2.12. The ring $R \times_T S$ is local, with maximal ideal $\mathfrak{m} = \mathfrak{m}_R \times_{\mathfrak{m}_T} \mathfrak{m}_S$.

Proof. See (ANANTHNARAYAN; AVRAMOV; MOORE, 2012, Lemma 1.2). □

Proposition 2.13. The following (in)equalities hold:

(a) $\operatorname{edim}(R \times_T S) \geq \operatorname{edim}(R) + \operatorname{edim}(S) - \operatorname{edim}(T)$;

(b) $\dim(R \times_T S) = \max\{\dim(R), \dim(S)\} \geq \min\{\dim(R), \dim(S)\} \geq \dim(T)$;

(c) $\operatorname{depth}(R \times_T S) \geq \min\{\operatorname{depth}(R), \operatorname{depth}(S), \operatorname{depth}(T) + 1\}$;

(d) $\operatorname{depth}(T) \geq \min\{\operatorname{depth}(R), \operatorname{depth}(S), \operatorname{depth}(R \times_T S) - 1\}$.

Proof. See (ANANTHNARAYAN; AVRAMOV; MOORE, 2012, Lemma 1.5). □

When $T = k$, Lescot (1981) has shown the following result.

Proposition 2.14. $\operatorname{depth}(R \times_k S) = \min\{\operatorname{depth}(R), \operatorname{depth}(S), 1\}$.

Below we summarize some structural results concerning the fiber product rings.

Proposition 2.15. Assume that T is Cohen-Macaulay, and set $d = \dim(T)$. The ring $R \times_T S$ is Cohen-Macaulay of dimension d if and only if R and S are Cohen-Macaulay.

Proof. See (ANANTHNARAYAN; AVRAMOV; MOORE, 2012, Proposition 1.7). □

We also have the following reformulation of the result above.

Proposition 2.16. (ENDO; GOTO; ISOBE, 2021, Lemma 2.1) We have the following properties for the fiber product ring.

(a) P is a Noetherian ring if and only if R and S are Noetherian rings.

(b) $(P, \mathfrak{m})$ is a local ring if and only if $(R, \mathfrak{m}_R)$ and $(S, \mathfrak{m}_S)$ are local rings. In this case, $\mathfrak{m} = (\mathfrak{m}_R \times \mathfrak{m}_S) \cap P$.

(c) If $(R, \mathfrak{m}_R)$, $(S, \mathfrak{m}_S)$ are Cohen-Macaulay local rings with $\dim(R) = \dim(S) = d > 0$ and $\operatorname{depth}(T) \geq d - 1$, then $(P, \mathfrak{m})$ is a Cohen-Macaulay local ring and $\dim(P) = d$.

When $T = k$, one has the following structural characterization of the fiber product $R \times_k S$. The next result shows that, the class of hypersurfaces and Gorenstein are equal for the rings $R \times_k S$.

Proposition 2.17. (NASSEH *et al.*, 2019, Corollary 2.7) Let $R \times_k S$ be a fiber product ring. The following statements are equivalent:

(i) $R \times_k S$ is Gorenstein.

(ii) $R \times_k S$ is a 1-dimensional hypersurface.

(iii) R and S are discrete valuation rings.

Remark 2.18. Hariharan (2009) gives some explicit ways to produce the fiber product.

(i) (HARIHARAN, 2009, Corollary 4.4) Let R be a Noetherian local ring. Let I and J be two non-zero ideals of R. Note that the map $\psi : R \to R/I \times_{R/I+J} R/J$, given by $\psi(r) = (r + I, r + J)$, is surjective. Therefore $R/I \cap J \cong R/I \times_{R/I+J} R/J$.

(ii) (HARIHARAN, 2009, Theorem 4.19) Let $R = k[x_1, \dots, x_n]$ and $S = k[y_1, \dots, y_m]$ be two polynomial rings, where k is a field. Let I and J be ideals of R and S, respectively. Then

$$\frac{k[x_1, \dots, x_n]}{I} \times_k \frac{k[y_1, \dots, y_m]}{J} \cong \frac{k[x_1, \dots, x_n, y_1, \dots, y_m]}{(I + J + (x_i y_j))}.$$

Similarly, if $R = k[[x_1, \dots, x_n]]$ and $S = k[[y_1, \dots, y_m]]$ be two power series ring, where k is a field, then

$$\frac{k[[x_1, \dots, x_n]]}{I} \times_k \frac{k[[y_1, \dots, y_m]]}{J} \cong \frac{k[[x_1, \dots, x_n, y_1, \dots, y_m]]}{(I + J + (x_i y_j))}.$$

(iii) Let R be a ring. Let $S = R[x_1, \dots, x_m]$ and $T = R[y_1, \dots, y_n]$. Let $I \subseteq (x_1, \dots, x_m)$ be an ideal of S and $J \subseteq (y_1, \dots, y_n)$ be an ideal of T. Then, by item (i), we get

$$\frac{S}{I} \times_R \frac{T}{J} \cong \frac{R[x_1, \dots, x_m, y_1, \dots, y_n]}{(I + J + (x_1, \dots, x_m)(y_1, \dots, y_n))}.$$

Remark 2.19. For I a non-zero proper ideal of R, D'Anna (D'ANNA, 2006) introduced the ring

$$R \bowtie I = \{(r, s) \in R \times R \mid r - s \in I\},$$

called *amalgamated duplication of the ring R along the ideal I*, which is a subring of $R \times R$. If we assume $\varepsilon_R = \varepsilon_S : R \to R/I$ (Diagram 2.1) are the canonical maps, then $R \bowtie I \cong R \times_{R/I} R$.

This ring has been studied by several authors, since this new construction, in the case $I^2 = 0$, coincides with the Nagata's idealization $R \ltimes I$ (also called trivial extension) (cf. (NAGATA, 1962, p. 2)). Further, this notion brought with it interesting geometric applications to curve singularities. Actually, D'Anna (D'ANNA, 2006) showed that if R is an algebroid curve with h branches, then $R \bowtie I$ is also an algebroid curve with $2h$ branches, and more, a explicit form to construct Gorenstein algebroid curves was provided.

CHAPTER 3

SOME CONCEPTS AND RESULTS OF SCHEMES

This chapter gives an introduction to the main concepts of theory of schemes, which is the basic subject of this thesis. The main notions, in particular, definitions of sheaves and ringed spaces are provided at the beginning of this chapter. We will show that the spectrum of a ring has a structure sheaf which transforms it into a ringed space called an affine scheme. Also, general schemes are obtained by a gluing process of affine schemes. We cite (HARTSHORNE, 1977), (LIU, 2002) and (GÖRTZ; WEDHORN, 2010) for details concerning all the topics of study furnished here.

3.1 Sheaves

The theory of sheaves enables us to gather local data, providing a foundation for deducing global information. Sheaves serve as a fundamental tool in modern algebraic geometry, where they have been successfully employed to solve numerous longstanding problems. In this section, we will briefly introduce some of the basic definitions of sheaf theory, accompanied by illustrative examples.

Before defining sheaves, we begin with the notion of a presheaf.

Definition 3.1. Let X be a topological space. A *presheaf* $\mathcal{F}$ of abelian groups on X consists of the following data,

(a) for every open subset $U \subseteq X$, an abelian group $\mathcal{F}(U)$, and

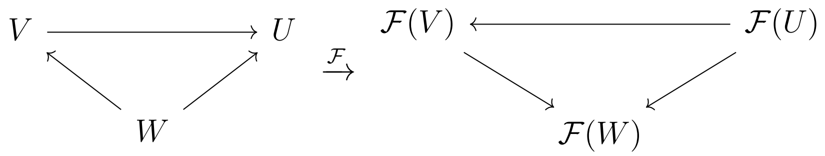
(b) a morphism of rings $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ (called *restriction map*), for every inclusion $V \subseteq U$ of open subsets of X,

which verify the following conditions:

(1) $\mathcal{F}(\emptyset) = 0$, where $\emptyset$ is the empty set,

(2) $\rho_{UU} = Id_{\mathcal{F}(U)}$ for every open set $U \subseteq X$, and

(3) if $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.



An element $s \in \mathcal{F}(U)$ is called a *section of* $\mathcal{F}$ *over* U and an element of $\mathcal{F}(X)$ is called a *global section*. We use the notation $s|_V = \rho_{UV}(s) \in \mathcal{F}(V)$ for $s \in \mathcal{F}(U)$ and $V \subseteq U$, it is read "s restricted to V".

Remark 3.2. We can think of the topological space X as a category, and thus we have an equivalent way of describing presheaves. Let $\mathrm{Opn}(X)$ be the category whose objects are the open sets of X and, for two open sets $U, V \subseteq X$,

$$\mathrm{Hom}(U,V) = \begin{cases} \emptyset & \text{if } U \not\subseteq V, \\ \{\text{the inclusion map } U \hookrightarrow V\} & \text{if } U \subseteq V \end{cases}$$

(composition of morphisms being the composition of the inclusion maps). Then a presheaf is the same as a contravariant functor $\mathcal{F}$ from the category $\mathrm{Opn}(X)$ to the category (Ab) of abelian groups.

The notion of a presheaf is not confined to presheaves of abelian groups. One may speak about presheaves of sets, rings, vector spaces or whatever you want: indeed, for any category **C** one may define presheaves with values in **C**. The definition goes just like for abelian groups, the only difference being that one requires the gadgets $\mathcal{F}(U)$ to be objects from the category **C**, and of course, the restriction maps are all required to be morphisms in the category **C**.

A presheaf satisfying certain extra conditions is called a sheaf.

Definition 3.3. A presheaf $\mathcal{F}$ on a topological space X is called a *sheaf*, provided it satisfies the following supplementary conditions:

(4) (*Uniqueness*) Let U be an open subset of X, $s \in \mathcal{F}(U)$, $\{U_i\}_i$ a covering of U by open subsets U_i. If $s|_{U_i} = 0$ for every i, then $s = 0$.

(5) (*Gluing local sections*) Let us keep the notation of (4). Let $s_i \in \mathcal{F}(U_i), i \in I$, be sections such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, for each i, j. Then there exists a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for each i.

Note that (4) implies that the section s obtained by (5) is unique.

Remark 3.4. (BOSCH, 2013, Lemma 4, p. 246) Let $\mathcal{B}$ be a basis of the topology of X such that $U \cap V \in \mathcal{B}$, for every $U, V \in \mathcal{B}$. Replacing "open subset U of X" by "open set U belonging to $\mathcal{B}$" in the previous definitions, we may define what we call the $\mathcal{B}$-*presheaf* and $\mathcal{B}$-*sheaf*. We have that every $\mathcal{B}$-sheaf $\mathcal{F}_0$ extends to a sheaf $\mathcal{F}$ on X (unique up to unique isomorphism). Precisely, given $U \subseteq X$, one has that $\mathcal{F}$ is defined as

$$\mathcal{F}(U) = \varprojlim_{V \subseteq U, V \in \mathcal{B}} \mathcal{F}_0(V).$$

Remark 3.5. If $\mathcal{F}$ is a sheaf on a topological space X and U is an open subset of X, then $\mathcal{F}|_U$ is a sheaf on U by setting $\mathcal{F}|_U(V) = \mathcal{F}(V)$, for every open subset V of U, i.e., its *restriction* of $\mathcal{F}$ to U. We also have a natural notion of *subsheaf* $\mathcal{F}'$ of $\mathcal{F}$: $\mathcal{F}'(U)$ is a subgroup of $\mathcal{F}(U)$, and the restriction ρ'_{UV} is induced by ρ_{UV}, where $\mathcal{F}'(U)$ is also a sheaf on X.

Example 3.6 (Continuous functions)**.** Let X be a topological space. For any open subset U of X, let $\mathcal{C}(U) = \mathcal{C}^0(U, \mathbb{R})$ be the set of continuous functions from U to $\mathbb{R}$. The restrictions ρ_{UV} are the usual restrictions of functions. Then $\mathcal{C}$ is a sheaf of rings on X. Indeed, any function $f : X \to \mathbb{R}$, which restricts to zero on an open covering of X is the zero function. Also, given continuous functions $f_i : U_i \to \mathbb{R}$ that agree on the overlaps $U_i \cap U_j$, we can form the continuous function $f : U \to \mathbb{R}$ by setting $f(x) = f_i(x)$, for any i such that $x \in U_i$. If we let $\mathcal{F}(U) = \mathbb{R}^U$ be the set of functions on U with values in $\mathbb{R}$, this defines a sheaf $\mathcal{F}$ of which $\mathcal{C}$ is a subsheaf.

Example 3.7 (Holomorphic functions)**.** Let $X \subseteq \mathbb{C}$ be an open set. On X one has the sheaf $\mathcal{A}_X$ of holomorphic functions. That is, for any open $U \subseteq X$, the section $\mathcal{A}_X(U)$ is the ring of complex differentiable functions on U. One checks that $\mathcal{A}_X$ forms a sheaf.

Example 3.8 (Algebraic varieties)**.** Let X be an irredutcible algebraic set in $\mathbb{A}^n_k$ with the Zariski topology. For each open $U \subseteq X$, define the preasheaf

$$\mathcal{O}_X(U) = \{f : U \to k \,|\, f \text{ is regular}\},$$

where f is regular if for each point $x \in U$, there is an affine neighbourhood for which f can be represented as a quotient of polynomials g/h with $h(x) \neq 0$.

This is indeed a sheaf: uniqueness holds, because if $f : U \to k$ restricts to the zero function on an open covering, it is the zero function. If we consider regular functions $f_i : U_i \to k$ on an open overing U_i of U, that agree on the overlaps, they certainly glue to a continuous function $f : U \to k$. Define $f : U \to k$ by $f(x) = f_i(x)$, whenever $x \in U_i$. The function f is also regular, because it restricts to f_i on U_i, and f_i is locally expressible as g/h there.

Given a sheaf $\mathcal{F}$ of abelian groups in a topological space X, we can associate each point $x \in X$ with a group $\mathcal{F}_x$, called the *stalk* of $\mathcal{F}$ in x. In the context of sheaf theory, stalks emerge as a generalization of the concept of rings of germs. Stalks provide a mechanism to analyze the behavior of sections within localized neighborhoods around a point x, disregarding their differences across various open sets of X.

Definition 3.9. Let $\mathcal{F}$ be a presheaf of rings on X, and let $x \in X$. The *stalk* of $\mathcal{F}$ at x is defined as the ring

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U),$$

the direct limit being taken over the open neighbirhoods U of x (ordered by inclusion).

Precisely, $\mathcal{F}_x$ is the set of equivalence classes of pairs (U, s), where U is an open neighborhood of x and $s \in \mathcal{F}(U)$. In fact, two pairs (U_1, s_1) and (U_2, s_2) are equivalent, if there exists an open neighborhood V of x with $V \subseteq U_1 \cap U_2$ such that $s_1|_V = s_2|_V$.

For any open neighbourhood U of $x \in X$, there is a natural ring morphism $\mathcal{F}(U) \to \mathcal{F}_x$ sending a section s to the equivalence class where the pair (s, U) belongs. This class is called the *germ* of s at x, and we denote it by s_x.

A familiar example of stalks are given when we consider $X = \mathbb{C}$ and $\mathcal{A}_X$ to be sheaf of holomorphic functions. In this case, the germ of an analytic function at point x is represented by its Taylor series. Indeed, two holomorphic functions having the same Taylor series are equal. So the stalk $\mathcal{A}_{X,x}$ is naturally identified with the ring of power series converging in a neighbourhood of x.

Definition 3.10. A *morphism of sheaves* on X, $\varphi : \mathcal{F} \to \mathcal{G}$ consists of a morphism of rings $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set U of X, such that whenever $V \subseteq U$ is an inclusion, the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \downarrow{\scriptstyle \rho_{UV}} & & \downarrow{\scriptstyle \rho'_{UV}} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

is commutative, where ρ and ρ' are the restriction maps in $\mathcal{F}$ and $\mathcal{G}$.

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on X. For any $x \in X$, φ canonically induces a ring homomorphism $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ such that $(\varphi(U)(s))_x = \varphi_x(s_x)$, for any open subset U of X, $s \in \mathcal{F}(U)$, and $x \in U$. We say that a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves is *surjective* if φ_x is surjective for all $x \in X$. An *isomorphism* is an invertible morphism φ, i.e., $\varphi(U)$ is an isomorphism for every open subset U of X.

Given a presheaf $\mathcal{F}$, there is a canonical way of defining an sheaf $\mathcal{F}^+$ while preserving the stalks. The following proposition summarizes the properties of $\mathcal{F}^+$.

Proposition 3.11. (GÖRTZ; WEDHORN, 2010, Proposition 2.24) Given a presheaf $\mathcal{F}$ on X, there is a sheaf $\mathcal{F}^+$ (unique up to isomorphism) together with a morphism of presheaves $\theta : \mathcal{F} \longrightarrow \mathcal{F}^+$ satisfying the following:

(i) For every morphism $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$, where $\mathcal{G}$ is a sheaf, there exists a unique morphism $\tilde{\alpha} : \mathcal{F}^+ \longrightarrow \mathcal{G}$ such that $\alpha = \tilde{\alpha} \circ \theta$.

(ii) For all $x \in X$, the map on stalks $\theta_x : \mathcal{F}_x \longrightarrow \mathcal{F}^+_x$ is an isomorphism.

The sheaf $\mathcal{F}^+$ is called the *sheafification* of $\mathcal{F}$.

3.2 Ringed spaces

In this section, we introduce the fundamental concepts of ringed spaces and locally ringed spaces, which are important algebraic-topological structures in the theory of schemes. We will present the notion of morphisms between ringed spaces and locally ringed spaces. Finally, we will provide some important examples of ringed spaces and locally ringed spaces to illustrate these concepts.

Definition 3.12. A *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. We call $\mathcal{O}_X$ as the *structure sheaf* of $(X, \mathcal{O}_X)$. Often we simply write X instead of $(X, \mathcal{O}_X)$.

A *morphism of ringed spaces* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^{\#})$ where $f : X \to Y$ is a continuous map and $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves of rings on Y. Here $f_*\mathcal{O}_X$ stands for the sheaf on Y that is given by $(f_*\mathcal{O}_X)(V) = \mathcal{O}_X(f^{-1}(V))$, for an open subset $V \subseteq Y$ and canonical restriction morphisms.

It should be noted that given a point $x \in X$, the morphism of sheaves $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ induces a homomorphism of rings $f^{\#}_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$.

Definition 3.13. A ringed space $(X, \mathcal{O}_X)$ is said to be a *locally ringed space* if, for each point $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring. In this case, we denote $\mathfrak{m}_{X,x}$ by the maximal ideal of the local ring $\mathcal{O}_{X,x}$, and $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ its residue field.

A *morphism of locally ringed spaces is a morphism* $(f, f^{\#})$ of ringed spaces, such that for each $x \in X$, the induced map of local rings

$$f^{\#}_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

is a local homomorphism of local rings, i.e., ${f^{\#}_x}^{-1}(\mathfrak{m}_{X,x}) = \mathfrak{m}_{Y,f(x)}$ or, equivalently, $f^{\#}_x(\mathfrak{m}_{Y,f(x)}) \subseteq \mathfrak{m}_{X,x}$. An *isomorphism* of locally ringed spaces is a morphism with a two-sided inverse. Then a morphism $(f, f^{\#})$ is an isomorphism if and only if f is a homeomorphism of the underlying topological spaces, and $f^{\#}$ is an isomorphism of sheaves.

3.2.1 Examples

Example 3.14. Analytic Spaces: Let $\mathbb{C}^n$ be a complex space with the Hausdorff topology, and let $W \subset \mathbb{C}^n$ be an open subset. For an open subset $U \subset W$, consider

$$\mathcal{O}_W^{\text{hol}}(U) := \{f \mid f \text{ is an holomorphic function on } U\}.$$

For open subsets $V \subset U$, set $\rho_{UV} : \mathcal{O}_W^{\text{hol}}(U) \to \mathcal{O}_W^{\text{hol}}(V)$ as the restriction $f \to f|_V$ of a map. Then $\mathcal{O}_W^{\text{hol}}$ is a sheaf of rings on W. Note that $\mathcal{O}_W^{\text{hol}}$ is a sheaf of $\mathbb{C}$-algebras and subsheaf of $\mathcal{F}_{W,\mathbb{C}}$.

Let $X \subset W$ be the zero set of a finite number of holomorphic functions $f_1, \ldots, f_r$ on W. The subset X is called an *analytic set.*

For an open subset $U \subset W$, define

$$\mathcal{I}(U) := \{f \mid f \text{ is an holomorphic function on } U \text{ such that } f|_{X\cap U=0}\}.$$

Set ρ_V^U as the restriction of fuctions as before. Then $\mathcal{I}$ is a sheaf of abelian group and it is a subsheaf of $\mathcal{O}_W^{\text{hol}}$ defining X. The presheaf defined by $\mathcal{O}_W^{\text{hol}}(U)/\mathcal{I}(U)$ is a presheaf of commutatives rings. Therefore the sheafification $\mathcal{O}_W^{\text{hol}}/\mathcal{I}$ is a sheaf of commutatives rings on W, moreover, a sheaf of $\mathbb{C}$-algebras. This is a sheaf on W, but it is also considered as a sheaf on X. Indeed a subset V of X is represented as $V = U \cap X$ by using an open subset U of W. Consider

$$\mathcal{O}_X^{\text{hol}}(V) := \mathcal{O}_W^{\text{hol}}/\mathcal{I}(U).$$

Then the right-hand side is independent of a choice of an open subset U, therefore $\mathcal{O}_X^{\text{hol}}$ is a sheaf of commutative rings on X. Here, the pair $(X, \mathcal{O}_X^{\text{hol}})$ is a ringed space and called *reduced analytic local model* and $\mathcal{O}_X^{\text{hol}}$ the structure sheaf.

Definition 3.15. A locally ringed space $(X, \mathcal{O}_X)$ is called an *analytic space* provided X is a Hausdorff space and there exist an open covering $X = \cup_i U_i$ and an analytic local model $(V_i, \mathcal{O}_{V_i}^{\text{hol}})$ which is isomorphic to $(U_i, \mathcal{O}_X|_{U_i})$ for each i, as locally ringed spaces.

Remark 3.16. Suppose that $(X, \mathcal{O}_X)$ is a analytic space. Let $x = (a_1, \ldots, a_n) \in X \subset W \subset \mathbb{C}^n$. Then, the stalk $\mathcal{O}_{X,x} \cong \mathcal{O}_{W,x}^{\text{hol}}$, where

$$\mathcal{O}_{W,x}^{\text{hol}} \cong \mathcal{O}_{\mathbb{C}^n,x}/\mathcal{I}_{X,x} \cong \mathbb{C}\{x_1 - a_1, \ldots, x_n - a_n\}/\mathcal{I}_{X,x},$$

and $\mathcal{I}_{X,x} = \{f_x \in \mathcal{O}_{\mathbb{C}^n,x} \mid \exists\ f \in \mathcal{O}_{\mathbb{C}^n}(U) \text{ representing } f_x \text{ and } f|_{U\cap X} = 0\}$. Since $\mathcal{O}_{\mathbb{C}^n,x}$ is Noetherian, the ideal $\mathcal{I}_{X,x}$ is finitely generated, and so there exists $f_1, \ldots, f_k \in \mathcal{O}_{\mathbb{C}^n,x}$ such that $\mathcal{I}_{X,x} = \langle f_1, \ldots, f_k \rangle$. In this work, $\mathcal{I}_{X,x}$ is an ideal that defines the germ (X, x) of an analytic space. It should be noted that $\mathcal{O}_{X,x}$ is an analytic $\mathbb{C}$-algebra and is a local ring with maximal ideal $\mathfrak{m}_{X,x} = \{f \in \mathcal{O}_{X,x} \mid f(x) = 0\}$.

Example 3.17. Algebraic Varieties: For an algebraically closed field k, an algebraic set in k^n is defined as the zero set of $f_1 \ldots, f_r \in k[x_1, \ldots, x_n]$. Let k^n be an affine space with the Zariski topology. For $U \subset k^n$ an open subset, we define

$$\mathcal{O}_{k^n}(U) := \{f/g \mid f, g \in k[x_1, \ldots, x_n], g \neq 0 \text{ on } U\}.$$

Then one obtains the canonical homomorphism $\rho_{UV} : \mathcal{O}_{k^n}(U) \to \mathcal{O}_{k^n}(V)$ for every pair of open subsets $V \subset U$. By these definitions, $\mathcal{O}_{k^n}$ is a sheaf of rings on k^n.

Let $X \subset k^n$ be an algebraic set with the induced topology from k^n. For $U \subset k^n$ we define

$$\mathcal{I}(U) := \{\varphi \in \mathcal{O}_{k^n}(U) | \ \varphi|_{U \cap X} = 0\},$$

then $\mathcal{I}$ is a subsheaf of ideals of $\mathcal{O}_{k^n}$ defining X. In the same way as in the case analytic the quotient sheaf $\mathcal{O}_X := \mathcal{O}_{k^n}/\mathcal{I}$ is a sheaf of commutative rings on X. A pair $(X, \mathcal{O}_X)$ is called a *reduced affine variety* and $\mathcal{O}_X$ the structure sheaf.

In short: If X is an affine variety, $k[X]$ its coordinate ring, and $k(X)$ its function field, then

$$\mathcal{O}_X(U) := \{f/g \mid f, g \in k[X], g(p) \neq 0 \text{ for all } p \in U\},$$

i.e., $f/g \in k(X)$ such that $f/g \in k[X]_g$, where $p \in D(g)$.

Definition 3.18. A locally ringed space $(X, \mathcal{O}_X)$ is called an *algebraic variety* if there exist an open covering $X = \cup_i U_i$ and an affine variety $(V_i, \mathcal{O}_{V_i})$ which is isomorphic to $(U_i, \mathcal{O}_X|_{U_i})$ for each i.

Remark 3.19. Suppose that $(X, \mathcal{O}_X)$ is a algebraic variety. Let $x = (a_1, \ldots, a_n) \in X \subset k^n$ Then, the stalk $\mathcal{O}_{X,x} \cong k[X]_{\mathfrak{m}_x}$, where $\mathfrak{m}_x$ is a maximal ideal of $k[X]$ of regular functions of X vanishing at the point x.

3.3 The spectrum of a ring as a locally ringed space

In this section we will show that the spectrum of a commutative ring can be endowed with a topological space structure and we will construct a structure sheaf in this space. This makes the spectrum a so-called affine scheme. This type of scheme provides a local part for the construction of general schemes.

Definition 3.20. Let R be a commutative ring with unit. The set

$$\mathrm{Spec}(R) := \{\mathfrak{p} \subset R \mid \mathfrak{p} \text{ prime ideal in } R\}$$

is called the *spectrum* of R. By convention, the unit ideal is not a prime ideal.

Now, we will put in Spec(R) a topology that generalizes the Zariski topology on an algebraic set (also called the *Zariski topology*). For this purpose, we define the the subsets, for each ideal $\mathfrak{a} \subseteq R$:

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{a} \subseteq \mathfrak{p}\}.$$

If $f \in R$, set

$$\begin{aligned} V(f) &= \{\mathfrak{p} \in \operatorname{Spec}(R) \mid f \in \mathfrak{p}\} \text{ and} \\ D(f) &= \operatorname{Spec}(R) - V(f) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid f \notin \mathfrak{p}\}. \end{aligned}$$

The following result shows that sets of the form $V(\mathfrak{a})$ satisfy the properties of closed sets of a topology.

Proposition 3.21. (UENO, 1999, Proposition 2.1) Let R be a ring and assume that $\{\mathfrak{a}_i\}_{i\in I}$ is a family of ideals in R. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals in R. Then the following statements hold true:

(a) $V(R) = \emptyset$ and $V(0) = \operatorname{Spec}(R)$.

(b) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$.

(c) $\bigcap_i V(\mathfrak{a}_i) = V(\sum_i \mathfrak{a}_i)$.

In particular, there exists a unique topology on Spec(R), so-called *Zariski topology*, whose closed subsets are the sets of the form $V(\mathfrak{a})$, for some ideal $\mathfrak{a}$ of R. Moreover, the family of subsets $\{D(f)\}_{f\in R}$ of Spec(R) forms a basis for this topology.

Lemma 3.22. (UENO, 1999, Lemma 2.10) Let R be a ring and $f, g \in R$. We have:

(a) $D(f) \cap D(g) = D(fg)$.

(b) $D(g) \subseteq D(f) \iff g \in \sqrt{f}$. In particular, one has $D(f) = D(f^n)$ for all n.

(c) The family $\{D(f_i)\}_{i\in I}$ forms an open covering of Spec(R) if and only if f_i generate the unit ideal, i.e., there is a relation

$$1 = a_1 f_{i_1} + \cdots + a_n f_{i_n},$$

where $i_1, \ldots, i_n \in I$ and $a_1, \ldots, a_n \in R$. In particular, we have that Spec(R) is quasi-compact[1].

[1] A topological space X is called quasi-compact if every open covering of X has a finite subcovering (see (GÖRTZ; WEDHORN, 2010, Definition 1.22)).

Proposition 3.23. (LIU, 2002, Lemma 1.7, pg. 28) Let $\phi : R \to S$ be a ring homomorphism. Then ϕ induces a map

$$
\begin{aligned}
{}^a\phi := \operatorname{Spec}(\phi) : \operatorname{Spec}(S) &\to \operatorname{Spec}(R) \\
\mathfrak{p} &\mapsto \phi^{-1}(\mathfrak{p})
\end{aligned}
$$

which has the following properties:

(a) ${}^a\phi$ is continuous with respect to Zariski topology on $\operatorname{Spec}(S)$ and $\operatorname{Spec}(R)$.

(b) If ϕ is surjective, then ${}^a\phi$ induces a homeomorphism from $\operatorname{Spec}(S)$ onto the closed subset $V(\operatorname{Ker}\,\phi)$ of $\operatorname{Spec}(R)$. In particular, if $\phi : R \longrightarrow R/\mathfrak{a}$ is the canonical projection, then $\operatorname{Spec}(R/\mathfrak{a})$ is homeomorphic to $V(\mathfrak{a})$, where $V(\mathfrak{a})$ is equipped with the subspace topology obtained from the Zariski topology of $\operatorname{Spec}(R)$.

(c) If ϕ is a localization morphism $R \longrightarrow S^{-1}R$, then ${}^a\phi$ is a homeomorphism from $\operatorname{Spec}(S^{-1}R)$ onto the subspace $\{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{p} \cap S = \emptyset\}$ of $\operatorname{Spec}(R)$. In particular $\operatorname{Spec}(R_f)$ is homeomorphic to R_f.

Now that we have a topology on $X = \operatorname{Spec}(R)$, our next step is to construct a sheaf of rings $\mathcal{O}_X$ on X. We know that the family $\mathcal{B} = \{D(f)\}_{f \in R}$ forms a basis for the Zariski topology on X, therefore we start by defining a $\mathcal{B}$-presheaf and then prove that the sheaf axioms are satisfied with respect to $\mathcal{B}$, i.e., it is a $\mathcal{B}$-sheaf. In this way, the sheaf $\mathcal{O}_X$ on X will be the only sheaf which is an extension of this $\mathcal{B}$-sheaf (Remark 3.4).

Let us consider X and $\mathcal{B}$ as above. Set

$$\mathcal{O}_X(D(f)) := R_f.$$

We need to define the restriction maps for $D(g) \subseteq D(f)$. For this recall that $D(g) \subseteq D(f)$ if and only if $g \in \sqrt{(f)}$ (Lemma 3.22 (b)). Hence, there exist $m \geq 1$ and $b \in R$ such that $g^m = fb$. Thus f is invertible in R_g. In this case, by the universal property of the localization map $\rho_g : R \longrightarrow R_g$, there is a unique homomorphism

$$
\begin{aligned}
\rho_{fg} : R_f &\longrightarrow R_g \\
\frac{a}{f^n} &\longmapsto \frac{ab^n}{g^{nm}}
\end{aligned}
$$

such that $\rho_{fg} \circ \rho_f = \rho_g$. Observe that $\rho_{ff} : R_f \longrightarrow R_f$ for any $f \in R$ is the identity map. Furthermore, whenever $D(f) \subseteq D(g) \subseteq D(h)$, then $\rho_{hf} = \rho_{gf} \circ \rho_{hg}$ (UENO, 1999, Lemma 2.11). In particular, if $D(f) = D(g)$, we easily verify that $\rho_{fg} : R_f \to R_g$ is an isomorphism. Therefore, we can define the restriction maps for $D(g) \subseteq D(f)$ to be ρ_{fg} and so deduce a $\mathcal{B}$-presheaf of rings that does not depend on the choice of f. Next we will see that the $\mathcal{B}$-presheaf $\mathcal{O}_X$ as defined above is in fact a sheaf.

Proposition 3.24. With the notation above, we have to $\mathcal{O}_X$ is a $\mathcal{B}$-sheaf of rings.

Proof. Let $D(f)$ be a basic open set and an open covering $D(f) = \bigcup_{i\in I} D(f_i)$. We have to show the following two properties.

1. (*Uniqueness*) Let $s \in \mathcal{O}_X(D(f))$ be such that $s|_{D(f_i)} = 0$ for all $i \in I$. Then $s = 0$.
2. (*Gluing local sections*) For $i \in I$ let $s_i \in \mathcal{O}_X(D(f_i))$ be such that $s_i|_{D(f_i)\cap D(f_j)} = s_j|_{D(f_i)\cap D(f_j)}$ for all $i, j \in I$. Then there exists $s \in \mathcal{O}_X(D(f))$ such that $s|_{D(f_i)} = s_i$ for all $i \in I$.

We may assume without loss of generality $f = 1$ and, hence, $D(f) = X = \mathrm{Spec}(R)$. Since X is quasi-compact, we can assume that I is finite, say of type $X = \bigcup_{i=j}^{n} D(f_i)$ for functions $f_1, \ldots, f_n \in R$.

Uniqueness: Let $s = a \in R$ be such that $a/1 = 0$ in each localization R_{f_i}. Then for each i, $1 \leq i \leq n$, there is an expoent $n_i \in \mathbb{N}$ such that $f_i^{n_i} a = 0 \in R$. Since there are only finitely many n_i, we may choose an m that works for all f_i, that is, $f_i^m a = 0$ in R. Furthermore, $X = \bigcup_{i=1}^{n} D(f_i)$ is equivalent to

$$\emptyset = \bigcap_{i=1}^{n} V(f_i) = V(f_1, ..., f_n),$$

and hence $R = (f_1, \ldots, f_n)$ or even to $R = (f_1^m, \ldots, f_n^m)$ (Proposition 3.21), since $D(f_i) = D(f_i^m)$ (Lema 3.22). Hence, there are elements $a_1, \ldots, a_n \in R$ such that

$$a = \sum_{i=1}^{n} a_j f_{i_j}^{n} a = \sum_{j=1}^{n} a_j = 0.$$

This establishes the first sheaf property.

Gluing local sections: Let $a_i \in R_{f_i}$ such that a_i and a_j are mapped to the same element in $R_{f_i f_j}$ for every pair i, j of indices. Each a_i can be written as $a_i = b_i / f_i^{n_i}$, where $b_i \in R$, and since the indices are finite, one may replace n_i with $n = \max_i n_i$. That a_i and a_j induce the same element in the localization $R_{f_i f_j}$, means that we have the equations

$$f_i^N f_j^N \left(b_i f_j^n - b_j f_i^n\right) = 0, \tag{3.1}$$

where N a priori depends on i and j, but again due to there being only finitely many indices, it can be chosen to work for all. Equation (3.1) gives

$$b_i f_i^N f_j^m - b_j f_j^N f_i^m = 0 \tag{3.2}$$

where $m = N + n$. Putting $b_i' = b_i f_i^N$ we see that a_i equals b_i'/f_i^m in R_{f_i} and Equation (3.2) takes the form

$$b_i' f_j^m - b_j' f_i^m = 0.$$

As above, write $1 = \sum_i c_i f_i^m$. Defining $a = \sum_i c_i b_i'$, one has

$$a f_j^m = \sum_i c_i b_i' f_j^m = \sum_i c_i b_j' f_i^m = b_j' \sum_i c_i f_i^m = b_j'.$$

This means that the image of a in R_{f_j} is $a_j = b_j'/f_j^m$. □

Therefore $\mathcal{O}_X$ extends to a sheaf of rings on X, which we will also denote by $\mathcal{O}_X$.

Proposition 3.25. The sheaf $\mathcal{O}_X$ on $X = \operatorname{Spec}(R)$ as defined above is a sheaf of rings satisfying the following properties:

(a) $\mathcal{O}_X(\operatorname{Spec}(R)) = R$

(b) For any $\mathfrak{p} \in X$, the stalk $\mathcal{O}_{X,\mathfrak{p}}$ is canonically isomorphic to $R_\mathfrak{p}$. In particular $(X, \mathcal{O}_X)$ is a locally ringed space.

Proof. (a) The proof follows directly from the definition of $\mathcal{O}_X$ by taking $f = 1$.

(b) The open set $D(f)$ contains $\mathfrak{p}$ if and only if $f \notin \mathfrak{p}$. So, it suffices to show that the canonical homomorphism

$$\varphi : \varinjlim_{f \notin \mathfrak{p}} R_f \to R_\mathfrak{p}$$

is an isomorphism. Every element α of $R_\mathfrak{p}$ can be written as $\alpha = af^{-1}$ for some $f \notin \mathfrak{p}$. It follows that α is in the image of R_f. Hence φ is surjective. On the other hand, if $af^{-n} \in R_f$ $(f \notin \mathfrak{p})$ is mapped to 0 in $R_\mathfrak{p}$, there exists a $g \notin \mathfrak{p}$ such that $ga = 0$. It follows that $af^{-n} = 0$ in R_{gf}. Hence φ is injective, as desired. □

Now, we recall a key definition for the rest of this work.

Definition 3.26. An *affine scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic to the $(\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)})$, for some ring R.

Example 3.27. Let k be a field and set $X := \mathbb{A}_k^1 = \operatorname{Spec}(k[x])$. Every non-empty subset U of X is the form $U = D(P(x))$, where $P(x) \in k[x] \setminus \{0\}$. We have

$$\mathcal{O}_X(U) = k[x]_{P(x)} = k[x, 1/P(x)],$$

i.e., the set of rational fractions whose denominator is only divisible by the irreducible factors of $P(x)$. If k is algebraically closed, we can consider a rational fraction as a function $k \to k \cup \{+\infty\}$. Then $\mathcal{O}_X(U)$ consists exactly of those without a pole in U. Note also that in the case of k being algebraically closed, the closed points of X are in one-to-one correspondence with elements of k. The scheme X differs from the variety only in that the scheme contains one more point, called the generic point of X, corresponding to the ideal (0). The closure of the point (0) is all of X.

Example 3.28. Let R be an integral domain and K its field of fractions. Let $X = \mathrm{Spec}(R)$. By the definition of the structure sheaf, one obtains $\mathcal{O}_X(D(f)) = R_f$, and so for all open subset $U \subseteq X$, we have

$$\mathcal{O}_X(U) = \bigcap_{D(f)\subset U} R_f.$$

As $R_\mathfrak{p} = \mathcal{O}_{X,\mathfrak{p}}$, we see that

$$\mathcal{O}_X(U) = \bigcap_{\mathfrak{p}\in U} \mathcal{O}_{X,\mathfrak{p}}.$$

Remark 3.29. From the previous example, note that the construction of the sheave structure $\mathcal{O}_{\mathrm{Spec}(R)}$ generalizes the construction of the ring of regular functions on an affine variety (see (HARTSHORNE, 1977, Ch. I, Section 3) or Example 3.17).

Definition 3.30. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ such that X admits an open covering $(U_i)_i$ where $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme, for every i. We will often denote $(X, \mathcal{O}_X)$ simply by X. A *morphism of schemes* is a morphism of locally ringed spaces. An *isomorphism of schemes* is an isomorphism of locally ringed spaces.

Example 3.31. Let X_1 and X_2 be schemes, let $U_1 \subseteq X_1$ and $U_2 \subseteq X_2$ be open subsets, and let $\varphi : (U_1, \mathcal{O}_{X_1}|_{U_1}) \to (U_2, \mathcal{O}_{X_2}|_{U_2})$ be an isomorphism locally ringed spaces. Then we can define a scheme X, obtained by the *gluing* of X_1 and X_2 along U_1 and U_2 via the isomorphism φ. The topological space of X is the quotient of the disjoint union $X_1 \cup X_2$ by the equivalence relation $x_1 \sim \varphi(x_1)$ for each $x_1 \in U_1$, with the quotient topology. Thus, there are maps $i_1 : X_1 \to X$ and $i_2 : X_2 \to X$, and a subset $V \subseteq X$ is open if and only if $i_1^{-1}(V)$ is open in X_1 and $i_2^{-1}(V)$ is open in X_2. The structure sheaf $\mathcal{O}_X$ is defined as follows: for any open set $V \subseteq X$,

$$\mathcal{O}_X(V) = \left\{(s_1, s_2) \in \mathcal{O}_{X_1}(i_1^{-1}(V)) \times \mathcal{O}_{X_2}(i_2^{-1}(V)) \middle|\ \varphi(s_1|_{i_1^{-1}(V)\cap U_1}) = s_2|_{i_2^{-1}(V)\cap U_2}\right\}.$$

We have that $\mathcal{O}_X$ is a sheaf, and that $(X, \mathcal{O}_X)$ is a locally ringed. Further, since X_1 and X_2 are schemes, one has that every point of X has a neighborhood which is affine, hence X is a scheme (see more details in (GÖRTZ; WEDHORN, 2010, Proposition 3.10)).
To illustrate this fact, consider the "affine line with the a double point": Let k be a field and consider $X_1 = X_2 = \mathbb{A}_k^1 = \mathrm{Spec}(k[x])$. Fix a closed point $p \in \mathbb{A}_k^1$ and let $U_1 = U_2 = X_1 - \{p\}$. We define a gluing isomorphism $\varphi : X_1 \to X_2$ as the identity morphism. Let X be the gluing of X_1 and X_2 along U_1 and U_2 via φ. The scheme X should be thought of as an affine line with the point p doubled.

Now, we will introduce the notions of open subscheme and closed subscheme of a scheme X. For any open subset $U \subset X$, the pair $(U, \mathcal{O}_X|_U)$ constitutes a scheme once again (refer to (LIU, 2002, Proposition 3.9, Ch. 2)). This is commonly referred to as the *open subscheme of* X. However, the definition of closed subschemes requires a more careful approach. Consider the following example.

Example 3.32. Let R be a ring, and let $\mathfrak{a}$ be an ideal of R. Set $Y = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(R/\mathfrak{a})$. The ring homomorphism $R \to R/\mathfrak{a}$ induces a morphism of schemes $f : X \to Y$. The map f is a homeomorphism of X onto the closed subset $V(\mathfrak{a}) \subseteq Y$ (Proposition 3.23). Equipping $V(\mathfrak{a})$ with the structure sheaf induced from $R/\mathfrak{a}$, we get an affine scheme that we may call a *closed subscheme* of $\mathrm{Spec}(R)$.

The previous example gives us an intuitive idea of how to generalize the notion of closed subscheme. Using the notation $f : X \longrightarrow Y$ for a morphism of scheme instead of $(f, f^{\#})$, we call f a *closed immersion* provided it yields a homeomorphism of X onto a closed subset of Y and $f^{\#} : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ is surjective. A *closed subscheme* of a scheme X is a closed subset Z of X endowed with the structure $(Z, \mathcal{O}_Z)$ of a scheme and with a closed immersion $(j, j^{\#}) : (Z, \mathcal{O}_Z) \longrightarrow (X, \mathcal{O}_X)$, where $j : Z \longrightarrow X$ is the canonical injection.

Proposition 3.33. (LIU, 2002, Proposition 3.20, pg. 47) Let $X = \mathrm{Spec}(R)$ be an affine scheme. Let $j : Z \longrightarrow X$ be a closed immersion of schemes. Then Z is affine and there exists a unique ideal J of R such that j induces an isomorphism from Z onto $\mathrm{Spec}(R/J)$.

In particular, the previous result gives an important fact that every closed subscheme of an affine scheme is affine.

Remark 3.34. The ideal J in the above proposition is given by $\mathcal{I}(X)$, where $\mathcal{I} = \mathrm{Ker}\, j^{\#}$ is the subsheaf of $\mathcal{O}_X$ defined by

$$(\mathrm{Ker}\, j^{\#})(U) = \mathrm{Ker}(j^{\#}(U)).$$

In the situation in the Example 3.32, the sheaf $\mathcal{I} = \mathrm{Ker}\, f^{\#}$ on $\mathrm{Spec}(R)$ on basic open subsets $D(f) \subseteq \mathrm{Spec}(R)$ where $f \in R$, is given by

$$\mathcal{I}(D(f)) = \mathfrak{a} \otimes R_f = \mathfrak{a}_f.$$

We call $\mathcal{I}$ the *sheaf associated* to the ideal $\mathfrak{a}$.

In the following, we will present some characterizations of closed immersions that will be relevant throughout the work.

Remark 3.35. (BOSCH, 2013, Proposition 9, p. 308) A morphism $f : X \longrightarrow Y$ of schemes is a closed immersion if f satisfies one of the following equivalent conditions:

(i) For every affine open subscheme $U \subseteq Y$, $f^{-1}(U)$ is affine and

$$f^{\#}(U) : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}(U))$$

is surjective.

(ii) There exists an affine open covering $(U_i)_{i\in I}$ of Y, such that $f^{-1}(U_i)$ is affine and

$$f^{\#}(U) : \mathcal{O}_Y(U_i) \longrightarrow \mathcal{O}_X(f^{-1}(U_i))$$

is surjective.

Below we give another concept that will be necessary in the next chapter.

Definition 3.36. A scheme X is said to be *locally Noetherian*, if X admits an affine open cover $X = \bigcup_{i\in I} U_i$, such that $\mathcal{O}_X(U_i)$ is a Noetherian ring for every i. If in addition X is quasi-compact, X is called *Noetherian*.

Remark 3.37. Note that any open or closed subschemes of locally Noetherian schemes are locally Noetherian and all the local rings $\mathcal{O}_{X,x}$ of a locally Noetherian scheme X are Noetherian. But the Noetherianess of $\mathcal{O}_{X,x}$ for all $x \in X$, does not imply that X is Noetherian (see (GÖRTZ; WEDHORN, 2010, Exercise 3.21)).

Example 3.38. Let $\mathcal{P}$ be the set of all prime numbers. Denote by $X_p = \mathrm{Spec}(\mathbb{Z}_{(p)})$ for $p \in \mathcal{P}$. We can build the scheme $X_{\mathcal{P}}$ by gluing the different X_p's together along the generic points. The glued scheme $X_{\mathcal{P}}$ it is neither affine nor Noetherian, but it is locally Noetherian (For more details see (ELLINGSRUD; OTTEM, 2023, Example 7.11, p. 114)).

CHAPTER

4

ON THE GLUING OF SCHEMES

As a refinement to the investigation given by Schwede (2005), in this chapter we explore the structure of the gluing of schemes. Actually, in Section 4.1, we begin by presenting the formal definition of gluing and we show that if both X and Y are affine schemes, then $X \sqcup_Z Y$ inherits the affine scheme structure. Furthermore, we establish that the gluing of two schemes is itself a scheme. These two assertions was showed in (SCHWEDE, 2005, Theorem 3.4 and Corollary 3.9). Here, we simply present an alternative way of viewing the topological space $X \sqcup_Z Y$ using some algebraic aspects. The approach used makes use of the theory of fiber product rings (showed in Chapther 2) and with this gives us an alternative and easier way to obtain deeper results on the structure of the gluing of schemes. Section 4.2 deals of the gluing of k-schemes, focusing on schemes over an arbitrary field k that are locally of finite type or of finite type.

In Section 4.3 we explore some properties of the gluing of schemes. In Section 4.4 we display the concept of multiplicity of schemes and give formulas for the multiplicity of the gluing of schemes. In the last section we present some applications.

4.1 The gluing of schemes

In order to give more details concerning the structure of the gluing of schemes, first we recall some basic notions. For the next definition, $X \amalg Y$ denotes the co-product or disjoint union of sets X and Y.

Definition 4.1. Let $\alpha : Z \to X$ and $\beta : Z \to Y$ be morphisms of ringed spaces. Set

$$X \sqcup_Z Y = X \amalg Y / \sim,$$

where the relation $\sim$ is generated by relations of the form $x \sim y$ ($x \in X$, $y \in Y$) if there exists $z \in Z$ such that $\alpha(z) = x$ and $\beta(z) = y$.

Namely, it is the smallest equivalence relation on $X \amalg Y$ such that after passing to the quotient $X \amalg Y/\sim$ the following square becomes commutative

$$\begin{array}{ccc} Z & \xrightarrow{\alpha} & X \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & X \sqcup_Z Y, \end{array}$$

where f and g are the continuous natural maps.

Since $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ and $(Z, \mathcal{O}_Z)$ are ringed spaces, (SCHWEDE, 2005, Proposition 2.2) provides that $(X \sqcup_Z Y, \mathcal{O}_{X \sqcup_Z Y})$ is also a ringed space. Therefore, f and g becomes morphisms of ringed spaces. Note that this definition also satisfies the universal property (SCHWEDE, 2005, Theorem 2.3).

As previously mentioned, in the next result, the statements (i)-(ii) are given in (SCHWEDE, 2005, Theorem 3.4 and Corollary 3.9). What we are doing here is simply giving an alternative way to looking at the topological spaces, using the notion of fiber product rings. As a consequence of this approach, we derive (iii) and some other consequences in this section.

Theorem 4.2. Let X, Y and Z be schemes such that $\alpha : Z \to X$ and $\beta : Z \to Y$ are closed immersions of schemes.

(i) If X and Y are affine schemes, then $X \sqcup_Z Y$ is also an affine scheme.

(ii) The gluing $X \sqcup_Z Y$ is a scheme.

(iii) If X and Y are locally Noetherian, then $X \sqcup_Z Y$ is also a locally Noetherian scheme.

Proof. (i) Since X and Y are affine schemes and Z is a closed subscheme of X and Y, then $X = \text{Spec}(R)$, $Y = \text{Spec}(S)$ and $Z = \text{Spec}(T)$ for some commutative rings R, S, T, respectively. By (FACCHINI, 1982, Proposition 2.1)

$$X \sqcup_Z Y = \text{Spec}(R) \sqcup_{\text{Spec}(T)} \text{Spec}(S) \cong \text{Spec}(R \times_T S),$$

and this provides the desired statement.

(ii) It suffices to show that the points of Z in $X \sqcup_Z Y$ have an affine neighborhood, since outside of Z we have a scheme. So let $z \in Z$ and $w \in X \sqcup_Z Y$ such that $w = \alpha(z) \sqcup_z \beta(z)$. Then there exist affine subsets U_i and V_j of the X and Y, respectively, where $z \in W_{i,j} := \alpha^{-1}(U_i) = \beta^{-1}(V_j)$, because the maps $\alpha : Z \to X$ and $\beta : Z \to Y$ are closed immersions of schemes. Note that $W_{i,j}$ is an open affine and by (SCHWEDE, 2005, Lemma 2.4) $U_i \sqcup_{W_{i,j}} V_j$ is an open neighborhood of w. Thus, since $U_i = \mathrm{Spec}(R_i)$, $V_j = \mathrm{Spec}(S_j)$ and $W_{i,j} = \mathrm{Spec}(T_{i,j})$, one has

$$\begin{aligned} U_i \sqcup_{W_{i,j}} V_j &\cong \mathrm{Spec}(R_i) \sqcup_{\mathrm{Spec}(T_{i,j})} \mathrm{Spec}(S_j) \\ &\cong \mathrm{Spec}(R_i \times_{T_{i,j}} S_j), \quad \text{by (FACCHINI, 1982, Proposition 2.1).} \end{aligned}$$

This proves that Z is covered by affine open sets, as desired.

(iii) By item (ii), since the gluing of schemes is also a scheme, for each i, j, it sufficient to show that $\mathcal{O}_{X \sqcup_Z Y}(U_i \sqcup_{W_{i,j}} V_j)$ is a Noetherian ring. Since, $\alpha : Z \to X$ and $\beta : Z \to Y$ are closed immersions of schemes, there exists ideals $I_i \subset R_i$ and $J_i \subset S_i$ such that $W_{i,j} = \alpha^{-1}(\mathrm{Spec}(R_i)) = \mathrm{Spec}(R_i/I_i)$ and $W_{i,j} = \beta^{-1}(\mathrm{Spec}(S_j)) = \mathrm{Spec}(S_j/J_j)$, so $W_{i,j} = \mathrm{Spec}(T_{i,j})$ with $T_{i,j}$ Noetherian, since R_i and S_j are Noetherian. By Proposition 2.16, one obtains that $R_i \times_{T_{i,j}} S_j$ is a Noetherian ring. Again, since $U_i \sqcup_{W_{i,j}} V_j \cong \mathrm{Spec}(R_i \times_{T_{i,j}} S_j)$ (FACCHINI, 1982, Proposition 2.1), the desired conclusion follows. □

The next example justifies the hypotheses imposed in (iii) in the previous result.

Example 4.3. (SCHWEDE, 2005, Example 3.7) If k is a field, consider the Noetherian schemes $X = \mathrm{Spec}(k[x, y])$, $Y = \mathrm{Spec}(k)$ and $Z = \mathrm{Spec}(k[x, y]/(x))$. Let $\mathcal{O}_X(X) \twoheadrightarrow \mathcal{O}_Z(Z)$ and $\mathcal{O}_Y(Y) \hookrightarrow \mathcal{O}_Z(Z)$ be two morphism of rings. Note that the gluing $X \sqcup_Z Y$ is not a Noetherian scheme, because $X \sqcup_Z Y = \mathrm{Spec}(k[x, xy, xy^2, xy^3, \dots])$.

Corollary 4.4. Let $X = \mathrm{Spec}(R)$ be an affine Noetherian scheme. Let $i : Y \to X$ and $j : Z \to X$ be closed immersions of schemes. Then $Y \sqcup_{Y \cap Z} Z$ is a Noetherian affine scheme equal to $Y \cup Z$.

Proof. Since $i : Y \to X$ and $j : Z \to X$ are closed immersions, one obtains that Y and Z are also affine Noetherian schemes. Hence $Y \sqcup_{Y \cap Z} Z$ is an affine Noetherian scheme by Theorem 4.2 (*ii*). In addition, there exists unique ideals I and J of R, such that $Y \cong \mathrm{Spec}(R/I)$ and $Z \cong \mathrm{Spec}(R/J)$. Further, $Y \cap Z \cong \mathrm{Spec}(\frac{R}{I+J})$. Since $R/I \cap J \cong R/I \times_{R/I+J} R/J$ (Remark 2.18 (i)) one has

$$\begin{aligned} Y \sqcup_{Y \cap Z} Z &\cong \mathrm{Spec}(R/I) \sqcup_{\mathrm{Spec}(\frac{R}{I+J})} \mathrm{Spec}(R/J) \\ &\cong \mathrm{Spec}(R/I \times_{R/I+J} R/J) \qquad \text{(by (FACCHINI, 1982, Proposition 2.1))} \\ &\cong \mathrm{Spec}\left(\frac{R}{I \cap J}\right). \end{aligned}$$

□

The proof of the next result is similar to Corollary 4.4.

Corollary 4.5. Let X be a locally Noetherian scheme. Let $i : Y \to X$ be a closed immersion. Then, $X \sqcup_Y X$ is locally Noetherian scheme. In addition, if $X = \mathrm{Spec}(R)$ is an affine scheme, then exists unique ideal I of R such that

$$X \sqcup_Y X \cong \mathrm{Spec}(R \times_{R/I} R).$$

Remark 4.6. In certain sense, the previous result gives an interesting and nice geometric interpretation. Let $R = k[[x_1, \ldots, x_n]]/(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_h)$, where k is an algebraically closed field and $\mathfrak{p}_1, \cdots, \mathfrak{p}_h$ are prime ideals of height $n-1$ in $k[[x_1, \ldots, x_n]]$. Let $X = \mathrm{Spec}(R)$ be an algebroid curve with h branches and let $Y = \mathrm{Spec}(R/\mathfrak{a})$, where I is a regular proper ideal of R (i.e. $\mathfrak{a}$ contains a nonzero divisor of R). By Corollary 4.5,

$$X \sqcup_Y X \cong \mathrm{Spec}(R \times_{R/\mathfrak{a}} R).$$

As in Remark 2.19, $R \times_{R/\mathfrak{a}} R \cong R \bowtie \mathfrak{a}$ is the *amalgamated duplication of the ring R along the ideal $\mathfrak{a}$* (D'ANNA, 2006). Hence $X \sqcup_Y X \cong \mathrm{Spec}(R \bowtie \mathfrak{a})$, and therefore (D'ANNA, 2006, Theorem 14) furnishes that $X \sqcup_Y X$ is an algebroid curve with $2h$ branches. Motivated by this fact, the scheme $X \sqcup_Y X$ will be called *amalgamated duplication of the scheme X along Y.*

With the help of Remark 2.18 and Theorem 4.2, we can easily construct some examples that show that gluing of (affine) schemes are also related (affine) schemes.

Example 4.7. (i) Consider the Whitney umbrella $X = \mathrm{Spec}(\mathbb{C}[[x, y, z]]/(x^2 - y^2 z))$ and the curve alpha given by $Y = \mathrm{Spec}(\mathbb{C}[[t, s]]/(s^2 - t^2(t+1)))$. The gluing of these related schemes (that is, a hypersurface and a curve) by Remark 2.18 (ii), is given by

$$X \sqcup_{\mathrm{Spec}(\mathbb{C})} Y = \mathrm{Spec}(\mathbb{C}[[x, y, z, t, s]]/((x^2 - y^2 z, s^2 - t^2(t+1)) + (x, y, z)(t, s))).$$

(ii) Consider the Whitney umbrella $X = \mathrm{Spec}(\mathbb{C}[w][x, y, z]/(x^2 - y^2 z))$ and the curve alpha given by $Y = \mathrm{Spec}(\mathbb{C}[w][t, s]/(s^2 - t^2(t+1)))$ with coefficients in $\mathbb{C}[w]$. The gluing of these related schemes, by Remark 2.18 (iii), is

$$X \sqcup_{\mathrm{Spec}(\mathbb{C}[w])} Y = \mathrm{Spec}(\mathbb{C}[w][x, y, z, t, s]/((x^2 - y^2 z, s^2 - t^2(t+1)) + (x, y, z)(t, s))).$$

(iii) Consider the hypersurface $X = \mathrm{Spec}(\mathbb{Z}[w]/(w^2 - 3)[x, y, z]/(x^2 - y^2 z))$ and the curve given by $Y = \mathrm{Spec}(\mathbb{Z}[w]/(w^2 - 3)[t, s]/(s^2 - t^3))$ with coefficients in $\mathbb{Z}[w]/(w^2 - 3)$. By Remark 2.18 (iii), the gluing of these related schemes is

$$X \sqcup_{\mathrm{Spec}(\mathbb{Z}[w]/(w^2-3))} Y = \mathrm{Spec}\left(\frac{\mathbb{Z}[w]/(w^2 - 3)[x, y, z, t, s]}{((x^2 - y^2 z, s^2 - t^3) + (x, y, z)(t, s))}\right).$$

In other words, this last scheme is the gluing of two schemes over Gaussian integers, since $\mathbb{Z}[w]/(w^2 - 3) \cong \mathbb{Z}[\sqrt{3}]$, i.e.,

$$X \sqcup_{\mathrm{Spec}(\mathbb{Z}[w]/(w^2-3))} Y = \mathrm{Spec}\left(\frac{\mathbb{Z}[\sqrt{3}][x, y, z, t, s]}{((x^2 - y^2 z, s^2 - t^3) + (x, y, z)(t, s))}\right).$$

4.2 The gluing of k-schemes

Our goal in this section is to investigate the behavior of the gluing of k-schemes. We start with the basic definition below.

Definition 4.8. Let k be an arbitrary field, and let $X \to \operatorname{Spec}(k)$ be a k-scheme. We call X a k-scheme *locally of finite type* or say that X is *locally of finite type over* k, if there is an affine open cover $X = \bigcup_{i\in I} U_i$ such that for all i, $U_i = \operatorname{Spec}(R_i)$ is the spectrum of a finitely generated k-algebra R_i. We say that X is of *finite type over* k if X is locally of finite type and quasi-compact.

Remark 4.9. As an immediate consequence of the definition is that every k-scheme (locally) of finite type is (locally) Noetherian because every finitely generated k-algebra is Noetherian.

Motivated by the previous investigation concerning the gluing of schemes given in this chapter, it is natural to pose the following question:

Question 4.10. Is the gluing of k-schemes (finite type or locally of finite type) a k-scheme (finite type or locally of finite type)?

The next result gives a partial answer for the previous question.

Proposition 4.11. Let X, Y and Z be k-schemes locally of finite type such that $Z \to X$ and $Z \to Y$ are closed immersions of schemes.

(i) If $Z = \{\text{point}\}$, then $X \sqcup_{\{\text{point}\}} Y$ is a k-scheme locally of finite type.

(ii) Let W be an affine k-scheme and let $i : X \to W$ and $j : Y \to W$ be closed immersions of schemes. Then, $X \sqcup_{X\cap Y} Y$ is a affine k-scheme.

Proof. (i) Since X and Y are k-scheme locally of finite type, there exists an affine open covers of $X = \cup_{i\in I} U_i$ and $Y = \cup_{j\in I} V_j$ such that for all i, j, $U_i = \operatorname{Spec}(R_i)$ and $V_j = \operatorname{Spec}(S_j)$ are the spectrum of finitely generated k-algebras R_i and S_j, respectively. Since, $\cup_{i,j\in I} U_i \sqcup_{\operatorname{Spec}(k)} V_j$ cover $X \sqcup_{\operatorname{Spec}(k)} Y$, it is sufficient to show that, for all i, j, the affine open $U_i \sqcup_{\operatorname{Spec}(k)} V_j$ is a spectrum of a finitely generated k-algebra.

In fact, note that $U_i = \operatorname{Spec}(R_i) \subset X$ and $V_j = \operatorname{Spec}(S_j) \subset Y$ where $R_i \cong \frac{k[x_1,\ldots,x_n]}{I}$ and $S_j \cong \frac{k[y_1,\ldots,y_m]}{J}$, respectively. Therefore,

$$\begin{aligned} U_i \sqcup_{\operatorname{Spec}(k)} V_j &\cong \operatorname{Spec}(R_i) \sqcup_{\operatorname{Spec}(k)} \operatorname{Spec}(S_j) \\ &\cong \operatorname{Spec}(R_i \times_k S_j) \quad \text{(by (FACCHINI, 1982, Proposition 2.1))} \\ &\cong \operatorname{Spec}\left(\frac{k[x_1,\ldots,x_n,y_1,\ldots,y_m]}{I+J+(x_i y_j)}\right), \end{aligned}$$

where the last isomorphism follows by Remark 2.18 (ii).

(ii) The result is a consequence of Corollary 4.4 and Remark 2.18 (i). □

Consider the following types of k-algebras:

1. Let $k[[x_1, \ldots, x_n]]/I$ be a formal local k-algebra, where $k[[x_1, \ldots, x_n]]$ denotes power series ring and I is an ideal in $k[[x_1, \ldots, x_n]]$.

2. Consider $\mathbb{C}\{x_1, \ldots, x_n\}/I$ the analytic local $\mathbb{C}$-algebra, where the ring $\mathbb{C}\{x_1, \ldots, x_n\}$ denotes the convergent power series ring and I is an ideal in $\mathbb{C}\{x_1, \ldots, x_n\}$.

A ringed space X is called a *k-scheme locally formal* provided all its open sets $U_i = \operatorname{Spec}(R_i)$, where $R_i \cong k[[x_1, \ldots, x_n]]/I$ (refer to Definition 4.8). Similarly, a ringed space X is termed a $\mathbb{C}$-scheme locally analytic if all its open sets are of the form $U_i = \operatorname{Spec}(R_i)$, where $R_i \cong \mathbb{C}\{x_1, \ldots, x_n\}/I$.

It's important to realize that there are few answers available in the literature for Question 4.10 (for instance (FREITAS; PÉREZ; MIRANDA, 2021; FREITAS; PÉREZ; MIRANDA, 2022)). As a consequence these papers cited, we derive the following result.

Proposition 4.12. (i) If X, Y and Z are k-schemes locally formal such that $\alpha : Z \to X$ and $\beta : Z \to Y$ are closed immersions of schemes, then $X \sqcup_Z Y$ is k-scheme locally formal.

(ii) If X, Y and Z are $\mathbb{C}$-schemes locally analytic such that $\alpha : Z \to X$ and $\beta : Z \to Y$ are closed immersions of schemes, then $X \sqcup_Z Y$ is a $\mathbb{C}$-scheme locally analytic.

Proof. (i) Let $\{U_i\}$ be an open covering of X, where $U_i = \operatorname{Spec}(R_i)$ and $R_i \cong k[[x_1, \ldots, x_{n_i}]]/I_i$. Also, let $\{V_j\}$ be an open covering of Y such that $V_j = \operatorname{Spec}(S_j)$ and $R_i \cong k[[y_1, \ldots, y_{m_j}]]/J_j$. Since Z is a k-scheme locally formal and the maps α and β are closed immersions, one has an open covering $\{W_{i,j}\}$ of Z, where $W_{i,j} := \alpha^{-1}(U_i) = \beta^{-1}(V_j)$ and $W_{i,j} = R_i \cong k[[z_1, \ldots, z_{s_{i,j}}]]/K_{i,j}$. Then, one obtains an open covering $\{U_i \sqcup_{W_{i,j}} V_j\}$ of $X \sqcup_Z Y$, such that

$$\begin{aligned} U_i \sqcup_{W_{i,j}} V_j &\cong \operatorname{Spec}\left(\frac{k[[x_1, \ldots, x_{n_i}]]}{I_i}\right) \sqcup_{\operatorname{Spec}\left(\frac{k[[z_1,\ldots,z_{s_{i,j}}]]}{K_{i,j}}\right)} \operatorname{Spec}\left(\frac{k[[y_1, \ldots, y_{m_j}]]}{J_j}\right) \\ &\cong \operatorname{Spec}\left(\frac{k[[x_1, \ldots, x_{n_i}]]}{I_i} \times_{\frac{k[[z_1,\ldots,z_{s_{i,j}}]]}{K_{i,j}}} \frac{k[[y_1, \ldots, y_{m_j}]]}{J_j}\right) \\ &\cong \operatorname{Spec}\left(\widehat{\frac{k[[x_1,\ldots,x_{n_i}]]}{I_i} \times_{\frac{k[[z_1,\ldots,z_{s_{i,j}}]]}{K_{i,j}}} \frac{k[[y_1,\ldots,y_{m_j}]]}{J_j}}\right) \end{aligned}$$

The second and last isomorphisms follow from (FACCHINI, 1982, Proposition 2.1) and the completeness of the rings, respectively. Now, since that the ring $\frac{k[[x_1,\ldots,x_{n_i}]]}{I_i} \times_{\frac{k[[z_1,\ldots,z_{s_{i,j}}]]}{K_{i,j}}}$

$\frac{k[[y_1,\ldots,y_{m_j}]]}{J_j}$ is a k-algebra, the Cohen-Structure Theorem yields

$$\overbrace{\frac{k[[x_1,\ldots,x_{n_i}]]}{I_i} \times_{\frac{k[[z_1,\ldots,z_{s_{i,j}}]]}{K_{i,j}}} \frac{k[[y_1,\ldots,y_{m_j}]]}{J_j}} \cong k[[w_1,\ldots,w_r]]/K.$$

Therefore $X \sqcup_Z Y$ is covered by formal local k-algebras, as desired.

(ii) Let $\{U_i\}$ be an open covering of X, where $U_i = \mathrm{Spec}(R_i)$ and $R_i \cong \mathbb{C}\{x_1,\ldots,x_{n_i}\}/I_i$. Also, let $\{V_j\}$ be an open covering of Y such that $V_j = \mathrm{Spec}(S_j)$ and $R_i \cong \mathbb{C}\{y_1,\ldots,y_{m_j}\}/J_j$. Since Z is a $\mathbb{C}$-scheme locally analytic and the maps α and β are closed immersions, one has an open covering $\{W_{i,j}\}$ of Z, where $W_{i,j} := \alpha^{-1}(U_i) = \beta^{-1}(V_j)$ and $W_{i,j} = R_i \cong \mathbb{C}\{z_1,\ldots,z_{s_{i,j}}\}/K_{i,j}$. Then, one obtains an open covering $\{U_i \sqcup_{W_{i,j}} V_j\}$ of $X \sqcup_Z Y$, such that

$$\begin{aligned} U_i \sqcup_{W_{i,j}} V_j &\cong \mathrm{Spec}\left(\frac{\mathbb{C}\{x_1,\ldots,x_{n_i}\}}{I_i}\right) \sqcup_{\mathrm{Spec}\left(\frac{\mathbb{C}\{z_1,\ldots,z_{s_{i,j}}\}}{K_{i,j}}\right)} \mathrm{Spec}\left(\frac{\mathbb{C}\{y_1,\ldots,y_{m_j}\}}{J_j}\right) \\ &\cong \mathrm{Spec}\left(\frac{\mathbb{C}\{x_1,\ldots,x_{n_i}\}}{I_i} \times_{\frac{\mathbb{C}\{z_1,\ldots,z_{s_{i,j}}\}}{K_{i,j}}} \frac{\mathbb{C}\{y_1,\ldots,y_{m_j}\}}{J_j}\right) \\ &\cong \mathrm{Spec}\left(\mathbb{C}\{w_1,\ldots,w_r\}/K\right) \end{aligned}$$

The second and last isomorphisms follow from (FACCHINI, 1982, Proposition 2.1) and (FREITAS; PÉREZ; MIRANDA, 2021, Lemma 3.1), respectively. Therefore $X \sqcup_Z Y$ is covered by analytic local $\mathbb{C}$-algebras, as desired. □

4.3 Some properties

The stalks $\mathcal{O}_{X,x}$ of a scheme X play a crucial role in the study of scheme theory. Several concepts in this theory can be defined in terms of their stalks, for instance, we can mention the notions of Cohen-Macaulay schemes and Gorenstein schemes. Further, we also have geometric notions of dimension, Zariski cotangent space, and Zariski tangent space.

In Section 4.1 we saw that under certain conditions, the gluing of schemes $X \sqcup_Z Y$ forms a scheme. Therefore, the main focus of this section is to investigate some properties of the gluing $X \sqcup_Z Y$ of schemes.

For this purpose, we begin by introducing the foundational Lemma 4.14, which establishes a crucial correspondence between the local rings of $X \sqcup_Z Y$ and the local rings of its individual components, namely X, Y, and Z. This important result enables us to establish conditions under which $X \sqcup_Z Y$ is Cohen-Macaulay, for instance (Theorem 4.22). In addition, we provide a relation between the dimensions of tangent spaces of $X \sqcup_Z Y$ and the tangent spaces of X, Y, and Z (Theorem 4.19).

For the definitions and fundamental facts of the theory presented here, readers can refer to (EISENBUD; HARRIS, 2000) and (GÖRTZ; WEDHORN, 2010).

Remark 4.13. Recall that if $X = \mathrm{Spec}(R)$ is an affine scheme and $x = [\mathfrak{p}] \in X$, then

$$\mathcal{O}_{X,x} = \varinjlim_{f \notin \mathfrak{p}} R_f = R_{\mathfrak{p}} \quad \text{(Proposition 3.25)}$$

and $\mathfrak{m}_{X,x} := \varinjlim_{f \notin \mathfrak{p}} \mathfrak{p} R_f = \mathfrak{p} R_{\mathfrak{p}}$ is the maximal ideal of $\mathcal{O}_{X,x}$.

The next lemma is an important result for the rest of this work.

Lemma 4.14. Let X, Y and Z be schemes, such that $\alpha : Z \to X$ and $\beta : Z \to Y$ are closed immersions of schemes. Let $z \in Z$ and $w \in X \sqcup_Z Y$ such that $w = \alpha(z) \sqcup_z \beta(z)$. Then,

$$\mathcal{O}_{X \sqcup_Z Y, w} \cong \mathcal{O}_{X,\alpha(z)} \times_{\mathcal{O}_{Z,z}} \mathcal{O}_{Y,\beta(z)}.$$

Proof. Since the stalk is a local fact, it is sufficient to show the statement in the affine case. Let $X = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(S)$ and $Z = \mathrm{Spec}(T)$, where R, S and T are commutative rings. By the diagram

$$\begin{array}{ccc} R \times_T S & \xrightarrow{\pi_R} & R \\ {\scriptstyle \pi_S}\downarrow & & \downarrow{\scriptstyle \varepsilon_R} \\ S & \xrightarrow{\varepsilon_S} & T, \end{array}$$

with surjective arrows, one has that each element $h \in R \times_T S$ corresponds to $f \in R$ and $g \in S$ such that $\varepsilon_R(f) = \varepsilon_S(g) =: t$. In addition, $S_h = S_g$, $R_h = R_f$ and $T_{\varepsilon_R(f)} = T_{\varepsilon_S(g)} = T_h$ taking the localization as $R \times_T S$-modules. Consider $z = [\mathfrak{p}_T] \in Z$, $\alpha(z) = [\mathfrak{p}_R] \in X$ and $\beta(z) = [\mathfrak{p}_S] \in Y$ such that $w = \alpha(z) \sqcup_z \beta(z)$. Then,

$$\varinjlim_{h \notin \mathfrak{p}_R \sqcup_{\mathfrak{p}_T} \mathfrak{p}_S} R_h = \varinjlim_{f \notin \mathfrak{p}_R} R_f, \qquad \varinjlim_{h \notin \mathfrak{p}_R \sqcup_{\mathfrak{p}_T} \mathfrak{p}_S} S_h = \varinjlim_{g \notin \mathfrak{p}_S} S_g$$

and

$$\varinjlim_{h \notin \mathfrak{p}_R \sqcup_{\mathfrak{p}_T} \mathfrak{p}_S} T_h = \varinjlim_{t \notin \mathfrak{p}_T} T_t.$$

By Remark 4.13, the exactness of localization and direct limit, the following exact sequence follows

$$0 \longrightarrow \mathcal{O}_{X \sqcup_Z Y, w} \longrightarrow \mathcal{O}_{X,\alpha(z)} \oplus \mathcal{O}_{Y,\alpha(z)} \longrightarrow \mathcal{O}_{Z,z} \longrightarrow 0.$$

This gives the desired statement. □

Now, the focus is to introduce the concept of dimension of a scheme. By Lemma 4.14, we can present a result that establishes a correlation between the dimension of the scheme $X \sqcup_Z Y$ and the dimensions of the schemes X and Y. Before we fix the following notation.

Notation 4.15. Let X, Y be locally Noetherian schemes and let Z be a closed subscheme of both X and Y. So there exists $\alpha : Z \to X$ and $\beta : Z \to Y$ closed immersions of schemes. Let $z \in Z$ and $w \in X \sqcup_Z Y$ such that $w = \alpha(z) \sqcup_z \beta(z)$ (see diagram in Definition 4.1). For the rest of the chapter, $\alpha(z) = x$ and $\beta(z) = y$.

In addition, we define the *germ of a scheme* X at x as being $\mathrm{Spec}(\mathcal{O}_{X,x})$, which we denote by (X,x). Now, similarly for the morphisms of germs $\alpha_z : (Z,z) \to (X,\alpha(z))$ and $\beta_z : (Z,z) \to (Y,\beta(z))$, the definition above can be made for germs of schemes (X,x), (Y,y) and (Z,z), and denoted by $(X,\beta(z)) \sqcup_{(Z,z)} (Y,\alpha(z))$. For the rest of this thesis, $(X,\beta(z)) \sqcup_{(Z,z)} (Y,\alpha(z))$ will be denoted by $(X,x) \sqcup_{(Z,z)} (Y,y)$, where $\alpha(z) = x$ and $\beta(z) = y$. When a germ (Z,z) is a reduced point, i.e, $(Z,z) = (z,z)$, we will denote (Z,z) by $\{z\}$.

Definition 4.16. The *dimension of a scheme X at a point $x \in X$*, namely $\dim(X,x)$, is the (Krull) dimension of the local ring $\mathcal{O}_{X,x}$ (see Chapter 2). The *dimension* of X, is defined by

$$\dim X := \sup\{\dim \mathcal{O}_{X,x} \mid x \in X\}.$$

An *irreducible component* of a non-empty scheme X is a maximal closed irreducible subset of X. We say that a scheme X is *equidimensional* or *pure* if all of its irreducible components have the same dimension.

The next result shows that the dimension of the gluing of schemes is given in terms of the dimensions of the respective schemes involved.

Proposition 4.17. Let X, Y be locally Noetherian schemes and let Z be a closed subscheme of both X and Y. Then

$$\dim X \sqcup_Z Y = \max\{\dim X, \dim Y\}.$$

Proof. By hypothesis, there exists $\alpha : Z \to X$ and $\beta : Z \to Y$ closed immersions of schemes. In addition, since X, Y and Z are locally Noetherian schemes, Remark 3.37 gives that $\mathcal{O}_{X,\alpha(z)}$, $\mathcal{O}_{Y,\beta(z)}$ and $\mathcal{O}_{Z,z}$ are Noetherian local rings. Therefore

$$\begin{aligned}
\dim X \sqcup_Z Y &= \sup\{\dim \mathcal{O}_{X\sqcup_Z Y,w} \mid w = \alpha(z) \sqcup_z \beta(z) \in X \sqcup_Z Y \text{ and } z \in Z\} \\
&= \sup\{\dim(\mathcal{O}_{X,\alpha(z)} \times_{\mathcal{O}_{Z,z}} \mathcal{O}_{Y,\beta(z)}) \mid z \in Z\} \text{ (by Lemma 4.14)} \\
&= \sup\{\max\{\dim \mathcal{O}_{X,\alpha(z)}, \dim \mathcal{O}_{Y,\beta(z)}\} \mid z \in Z\} \text{ (by Proposition 2.13(b))} \\
&= \max\{\dim X, \dim Y\}.
\end{aligned}$$

□

With the previous result, the main focus of the rest of the section is to investigate when the gluing of schemes is Cohen-Macaulay and Gorenstein. For this purpose, a key ingredient is the notion of Zariski tangent space, given below.

Definition 4.18. Let X be a scheme, and let $x \in X$. Then $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ is a vector space over $k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$, called *Zariski cotangent* space to X at x. The Zariski *tangent* *space* of X in x is, by definition, the dual vector space

$$T_x X := (\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2)^\vee.$$

The following result establishes a relationship between the dimension of the tangent space of $X \sqcup_Z Y$ at a point $x \sqcup_z y$ and the dimensions of the tangent spaces of its components X, Y, and Z at points x, y and z, respectively.

Theorem 4.19. If X, Y are Noetherian schemes and Z is a closed subscheme of both X and Y, then

$$\dim T_{x \sqcup_z y} X \sqcup_Z Y \geq \dim T_x X + \dim T_y Y - \dim T_z Z.$$

Proof. By Lemma 4.14, one has $\mathcal{O}_{X\sqcup_Z Y, x\sqcup_z y} \cong \mathcal{O}_{X,x} \times_{\mathcal{O}_{Z,z}} \mathcal{O}_{Y,y}$. Since the maximal ideal of $\mathcal{O}_{X,x} \times_{\mathcal{O}_{Z,z}} \mathcal{O}_{Y,y}$ is the form $\mathfrak{m} = \mathfrak{m}_{X,x} \times_{\mathfrak{m}_{Z,z}} \mathfrak{m}_{Y,y}$ (Proposition 2.12), where $\mathfrak{m}_{X,x}$, $\mathfrak{m}_{Y,y}$ and $\mathfrak{m}_{Z,z}$ are the maximal ideals of $\mathcal{O}_{X,x}$, $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{Z,z}$, respectively, one obtains the following exact sequence

$$0 \longrightarrow \mathfrak{m} \longrightarrow \mathfrak{m}_{X,x} \oplus \mathfrak{m}_{Y,y} \longrightarrow \mathfrak{m}_{Z,z} \longrightarrow 0.$$

Tensoring this sequence by $\mathcal{O}_{X\sqcup_Z Y, x\sqcup_z y}/\mathfrak{m}$ over $\mathcal{O}_{X\sqcup_Z Y, x\sqcup_z y}$, we deduce an exact sequence of $k(x)$-vector spaces given by

$$0 \longrightarrow K \longrightarrow \frac{\mathfrak{m}}{\mathfrak{m}^2} \longrightarrow \frac{\mathfrak{m}_{X,x} \oplus \mathfrak{m}_{Y,y}}{\mathfrak{m}(\mathfrak{m}_{X,x} \oplus \mathfrak{m}_{Y,y})} \longrightarrow \frac{\mathfrak{m}_{Z,z}}{\mathfrak{m}\mathfrak{m}_{Z,z}} \longrightarrow 0, \tag{4.1}$$

for some $\mathcal{O}_{X\sqcup_Z Y, x\sqcup_z y}$-module K. Now, since

$$\mathfrak{m}\mathfrak{m}_{X,x} = \mathfrak{m}_{X,x}^2, \mathfrak{m}\mathfrak{m}_{Y,y} = \mathfrak{m}_{Y,y}^2 \text{ and } \mathfrak{m}\mathfrak{m}_{Z,z} = \mathfrak{m}_{Z,z}^2,$$

we get

$$0 \longrightarrow K \longrightarrow \frac{\mathfrak{m}}{\mathfrak{m}^2} \longrightarrow \frac{\mathfrak{m}_{X,x}}{\mathfrak{m}_{X,x}^2} \oplus \frac{\mathfrak{m}_{Y,y}}{\mathfrak{m}_{Y,y}^2} \longrightarrow \frac{\mathfrak{m}_{Z,z}}{\mathfrak{m}_{Z,z}^2} \longrightarrow 0. \tag{4.2}$$

The field $k(x)$ is injective (as a module over itself is injective), and so the functor $(-)^\vee := \mathrm{Hom}_{k(x)}(-, k(x))$ is exact. Applying this functor to (4.2), the following exact sequence is provided

$$0 \longrightarrow \left(\frac{\mathfrak{m}_{Z,z}}{\mathfrak{m}_{Z,z}^2}\right)^\vee \longrightarrow \left(\frac{\mathfrak{m}_{X,x}}{\mathfrak{m}_{X,x}^2}\right)^\vee \oplus \left(\frac{\mathfrak{m}_{Y,y}}{\mathfrak{m}_{Y,y}^2}\right)^\vee \longrightarrow \left(\frac{\mathfrak{m}}{\mathfrak{m}^2}\right)^\vee \longrightarrow K^\vee \longrightarrow 0.$$

The desired statement follows. □

Proposition 4.20. If X, Y are Noetherian schemes and $Z = \{\text{point}\}$ is a closed subscheme of both X and Y, then

$$T_{x\sqcup_{\{\text{point}\}} y} X \sqcup_{\{\text{point}\}} Y \cong T_x X \oplus T_y Y.$$

In particular, $\dim T_{x\sqcup_{\{\text{point}\}} y} X \sqcup_{\{\text{point}\}} Y = \dim T_x X + \dim T_y Y$.

Proof. First, recall that the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{X,x} \times_k \mathcal{O}_{Y,y}$ is $\mathfrak{m} \cong \mathfrak{m}_{X,x} \oplus \mathfrak{m}_{Y,y}$. Similarly to the proof of Theorem 4.19, one has

$$\frac{\mathfrak{m}}{\mathfrak{m}^2} \cong \frac{\mathfrak{m}_{X,x}}{\mathfrak{m}_{X,x}^2} \oplus \frac{\mathfrak{m}_{Y,y}}{\mathfrak{m}_{Y,y}^2},$$

with is sufficient for the proof of the desired result. □

For the next results, we will recall the concepts of Cohen-Macaulay and Gorenstein schemes.

Definition 4.21. We say that a locally Noetherian scheme X is *Cohen-Macaulay* if $\mathcal{O}_{X,x}$ is Cohen-Macaulay for every $x \in X$. Similarly, X is *Gorenstein* if X is locally Noetherian and $\mathcal{O}_{X,x}$ is Gorenstein for all $x \in X$.

In the Chapter 2 we saw the notion of depth for a local ring. The general notion of depth can be reduced to the local case by result in (BĂNICĂ; STĂNĂŞILĂ, 1976, Ch. II, Corollary 1.22). Thus, for a sheafified notion of depth, let X be a scheme, $Z \subset X$ a closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$, and $\mathcal{F}$ a coherent sheaf on X. We define

$$\operatorname{depth}_Z \mathcal{F} = \inf\{\operatorname{depth}_{\mathcal{I}_x} \mathcal{F}_x | x \in Z\}.$$

If $\mathcal{F} = \mathcal{O}_Z$, $\operatorname{depth}_Z \mathcal{F}$ is denoted by $\operatorname{depth} Z$.

We are able to prove one of the main structural results.

Theorem 4.22. Let X, Y be locally Noetherian equidimensional schemes and let Z be a closed subscheme of both X and Y.

(i) Assume that Z is Cohen-Macaulay of dimension d. Then $X \sqcup_Z Y$ is Cohen-Macaulay of dimension d if and only if X and Y are.

(ii) If X and Y are Cohen-Macaulay with $\dim X = \dim Y = d > 0$ and $\operatorname{depth} Z \geq d - 1$, then $X \sqcup_Z Y$ is Cohen-Macaulay and $\dim X \sqcup_Z Y = d$.

Proof. Without loss of generality, we may assume that the schemes are affine Noetherian schemes. The result follows applying Lemma 4.14, Definition 4.21 and the results that have been established in Proposition 2.15 and Proposition 2.16 (c). □

Example 4.23. Next, we will show that the gluing of Cohen-Macaulay schemes is not always Cohen-Macaulay. Consider $X = \operatorname{Spec}(\mathbb{C}[x, y])$ and $Y = \operatorname{Spec}(\mathbb{C}[z])$ as two affine Cohen-Macaulay schemes. Then $X \sqcup_{\operatorname{Spec}(\mathbb{C})} Y$ is not Cohen-Macaulay. In fact, from Remark 2.18 and Proposition 4.11, we have

$$X \sqcup_{\operatorname{Spec}(\mathbb{C})} Y = \operatorname{Spec}(\mathbb{C}[x, y, z]/(xz, yz)).$$

Now, since $\mathcal{O}_{X\sqcup_{\mathrm{Spec}(\mathbb{C})}Y,(0\sqcup_0 0)}$ is not Cohen-Macaulay, we conclude that $X \sqcup_{\mathrm{Spec}(\mathbb{C})} Y$ is not Cohen-Macaulay at $(0 \sqcup_0 0)$. Hence, $X \sqcup_{\mathrm{Spec}(\mathbb{C})} Y$ is not Cohen-Macaulay. This example also illustrates the necessity of imposing the condition $\dim X = \dim Y$.

Example 4.24. Let k be a field and $a, b, c, d \geq 2$ integers. Consider the affine schemes $X = \mathrm{Spec}(k[[x, y]]/(x^a - y^b))$ and $Y = \mathrm{Spec}(k[[z, w]]/(z^c - w^d))$, both of which are Cohen-Macaulay. Then, $X \sqcup_{\mathrm{Spec}(k)} Y = \mathrm{Spec}(k[[x, y, z, w]]/((x, y)(z, w) + (x^a - y^b, z^c - w^d)))$ is also Cohen-Macaulay. To prove this, consider the canonical projections $f : R \to k$ and $g : S \to k$, where $R = k[[x, y]]/(x^a - y^b)$ and $S = k[[z, w]]/(z^c - w^d)$. We have

$$R \times_k S \cong k[[x, y, z, w]]/((x, y)(z, w) + (x^a - y^b, z^c - w^d))$$

However, since $\mathrm{depth}(R \times_k S) = 1$ and $\dim(R \times_k S) = 1$, we conclude that $R \times_k S$ is Cohen-Macaulay. Since $X \sqcup_{\mathrm{Spec}(k)} Y = \mathrm{Spec}(R \times_k S)$, then $X \sqcup_{\mathrm{Spec}(k)} Y$ is Cohen-Macaulay.

Recall that a locally Noetherian scheme X is said to be *nonsingular* (or *regular*) at $x \in X$ if the Zariski tangent space to X at x has dimension equal to $\dim(X, x)$, that is, $\mathcal{O}_{X,x}$ is regular (see Definition 2.4); otherwise, we say that X is *singular at* x. We say that X is *regular* if it is regular at all of its points. We say that the germ (X, x) is Cohen-Macaulay if and only if $\mathcal{O}_{X,x}$ is Cohen-Macaulay.

Proposition 4.25. Let (X, x) and (Y, y) be two germs of schemes of dimension ≥ 1, and let $Z = \{z\}$ be a closed subscheme of both (X, x) and (Y, y). The gluing $(X, x) \sqcup_{\{z\}} (Y, y)$ is Cohen-Macaulay if and only if $\dim(X, x) \sqcup_z (Y, y) = 1$ and (X, x) and (Y, y) are Cohen-Macaulay with $\dim(X, x) \sqcup_z (Y, y) = \dim(X, x) = \dim(Y, y)$.

Proof. In order to deduce the result, consider the previous definition of Cohen-Macaulay for germs of schemes and apply (VANDEBOGERT, 2017, Theorem 3.2.1). □

Theorem 4.26. Let (X, x) and (Y, y) be two germs of curve schemes, and let $Z = \{\text{point}\}$ be a closed subscheme of both (X, x) and (Y, y).

(i) Then $(X, x) \sqcup_{\{z\}} (Y, y)$ is a 1-dimensional Gorenstein if and only if (X, x) and (Y, y) are smooth.

(ii) $(X, x) \sqcup_{\{z\}} (Y, y)$ is singular.

Proof. Item (i) follows by (CHRISTENSEN; STRIULI; VELICHE, 2010, Observation 3.2) or Proposition 2.17, by the previous comment and definitions. For (ii), suppose that $(X, x) \sqcup_{\{z\}} (Y, y)$ is regular. Since $(X, x) \sqcup_{\{z\}} (Y, y)$ is Cohen-Macaulay, Theorem 4.22 (iii), Definition 4.21 and Proposition 4.17 give that $\dim(X, x) \sqcup_{\{z\}} (Y, y) = 1$. By assumption,

$$1 = \dim(X, x) \sqcup_{\{z\}} (Y, y) = \dim T_{x\sqcup_{\{z\}}y} X \sqcup_{\{z\}} Y.$$

Now, Corollary 4.20 provides a contradiction, because $\dim T_x X \geq 1$ and $\dim T_y Y \geq 1$. □

Proposition 4.27. Let X be a Cohen-Macaulay scheme and let $i : Y \to X$ be a closed immersion of schemes. Then, $X \sqcup_Y X$ is Gorenstein if and only if X has canonical module ω_X over Y and $\omega_X \cong \mathcal{I}$, where $\mathcal{I}$ is the sheaf of ideals defining Y.

Proof. Without loss of generality, we may assume that $X = \operatorname{Spec}(R)$ is an affine Noetherian scheme. Since $i : Y \to X$ is a closed immersion, then there exists an ideal $I \subset R$ such that $Y = \operatorname{Spec}(R/I)$. By definition, $X \sqcup_Y X$ is Gorenstein if and only if $\mathcal{O}_{X \sqcup_Y X, x \sqcup_y x}$ is Gorenstein for all $x \sqcup_y x \in X \sqcup_Y X$. But, Lemma 4.14, Remark 4.13 and Remark 2.19 yield

$$\mathcal{O}_{X \sqcup_Y X, x \sqcup_y x} \cong \mathcal{O}_{X,x} \times_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \cong R_{\mathfrak{p}} \times_{R_{\mathfrak{p}}/I_{\mathfrak{p}}} R_{\mathfrak{p}} \cong R_{\mathfrak{p}} \bowtie I_{\mathfrak{p}} \cong \mathcal{O}_{X,x} \bowtie \mathcal{I}_x$$

for all $x = [\mathfrak{p}] \in Y$. By (D'ANNA, 2006, Theorem 11) one obtains that $\mathcal{O}_{X,x} \bowtie \mathcal{I}_x$ is Gorenstein if and only if $\mathcal{O}_{X,x}$ has canonical module $\omega_{\mathcal{O}_{X,x}} =: \omega_{X,x}$ and $\omega_{X,x} \cong \mathcal{I}_x$. Therefore, the desired statement follows. □

4.4 The multiplicity of the gluing of schemes

Consider X an arbitrary locally Noetherian scheme, $x \in X$ a point, $\mathcal{O}_{X,x}$ the local ring of X at x and $\mathfrak{m}_{X,x}$ the maximal ideal of $\mathcal{O}_{X,x}$. Similarly to algebraic varieties, below we define the multiplicity of a scheme X at point $x \in X$. For details see (EISENBUD; HARRIS, 2000).

First, let's begin by recalling that the *(Hilbert-Samuel) multiplicity* of a local Noetherian ring $(R, \mathfrak{m})$ is given by

$$e(R) = \lim_{n \to \infty} \frac{d!}{n^d} \ell(R/\mathfrak{m}^n),$$

where $\ell(-)$ denotes the length of modules. In other words, $e(R)$ is $d!$ times the leading coefficient of of the Hilbert polynomial representing the Hilbert function $\ell(R/\mathfrak{m}^n)$ for $n \gg 0$ (HUNEKE; SWANSON, 2006, Theorem 11.1.3).

Now, since X is a locally Noetherian scheme, one obtains that $\mathcal{O}_{X,x}$ is a Noetherian local ring. From what we have seen above, we have the following:

Definition 4.28. Let X be a locally Noetherian scheme. The multiplicity of X at point $x \in X$, denoted by $\operatorname{mult}_x X$, is defined as $\operatorname{mult}_x X := e(\mathcal{O}_{X,x})$, i.e.,

$$\operatorname{mult}_x X := \lim_{n \to \infty} \frac{d!\, \ell(\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}^n)}{n^d},$$

where $d := \dim(X, x)$.

With the previous notion, the following result provides formulas for the multiplicity of the gluing of schemes.

Theorem 4.29. Let X and Y be locally Noetherian schemes, and let Z be a closed subscheme of both X and Y. Then,

$$\mathrm{mult}_{x\sqcup_z y} X \sqcup_Z Y = \begin{cases} \mathrm{mult}_x X + \mathrm{mult}_y Y - \mathrm{mult}_z Z, & \text{if } \dim(X,x) = \dim(Y,y) = \dim(Z,z) \\ \mathrm{mult}_x X + \mathrm{mult}_y Y, & \text{if } \dim(X,x) = \dim(Y,y) > \dim(Z,z) \\ \mathrm{mult}_x X, & \text{if } \dim(X,x) > \dim(Y,y) > \dim(Z,z). \end{cases}$$

Proof. Since the multiplicity is a local property, in order to show the desired statement, it is sufficient to consider the stalks. Set $R := \mathcal{O}_{X\sqcup_Z Y, x\sqcup_z y}$ and note that $R \cong \mathcal{O}_{X,x} \times_{\mathcal{O}_{Z,z}} \mathcal{O}_{Y,y}$, by Lemma 4.14. Consider the maximal ideal $\mathfrak{m} = \mathfrak{m}_{X,x} \times_{\mathfrak{m}_{Z,z}} \mathfrak{m}_{Y,y}$ of R, where $\mathfrak{m}_{X,x}$, $\mathfrak{m}_{Y,y}$ and $\mathfrak{m}_{Z,z}$ are the maximal ideals of $\mathcal{O}_{X,x}$, $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{Z,z}$, respectively. From the exact sequence of R-modules (Proposition 2.12 and Remark 2.11)

$$0 \longrightarrow \mathfrak{m} \longrightarrow \mathfrak{m}_{X,x} \oplus \mathfrak{m}_{Y,y} \longrightarrow \mathfrak{m}_{Z,z} \longrightarrow 0,$$

we derive the exact sequence of vector spaces given by

$$\cdots \longrightarrow \frac{\mathfrak{m}}{\mathfrak{m}\mathfrak{m}^{n-1}} \longrightarrow \frac{\mathfrak{m}_{X,x} \oplus \mathfrak{m}_{Y,y}}{\mathfrak{m}^{n-1}(\mathfrak{m}_{X,x} \oplus \mathfrak{m}_{Y,y})} \longrightarrow \frac{\mathfrak{m}_{Z,z}}{\mathfrak{m}^{n-1}\mathfrak{m}_{Z,z}} \longrightarrow 0, \tag{4.3}$$

tensoring by $R/\mathfrak{m}^{n-1}$ over R. Since $\mathfrak{m}\mathfrak{m}_{X,x} = \mathfrak{m}_{X,x}^2$, $\mathfrak{m}\mathfrak{m}_{Y,y} = \mathfrak{m}_{Y,y}^2$ and $\mathfrak{m}\mathfrak{m}_{Z,z} = \mathfrak{m}_{Z,z}^2$, one has

$$\cdots \longrightarrow \frac{\mathfrak{m}}{\mathfrak{m}^n} \longrightarrow \frac{\mathfrak{m}_{X,x}}{\mathfrak{m}_{X,x}^n} \oplus \frac{\mathfrak{m}_{Y,y}}{\mathfrak{m}_{Y,y}^n} \longrightarrow \frac{\mathfrak{m}_{Z,z}}{\mathfrak{m}_{Z,z}^n} \longrightarrow 0.$$

The previous sequence yields

$$\ell\left(\frac{\mathfrak{m}_{X,x}}{\mathfrak{m}_{X,x}^n}\right) + \ell\left(\frac{\mathfrak{m}_{Y,y}}{\mathfrak{m}_{Y,y}^n}\right) = \ell\left(\frac{\mathfrak{m}_{X,x}}{\mathfrak{m}_{X,x}^n} \oplus \frac{\mathfrak{m}_{Y,y}}{\mathfrak{m}_{Y,y}^n}\right) \le \ell\left(\frac{\mathfrak{m}}{\mathfrak{m}^n}\right) + \ell\left(\frac{\mathfrak{m}_{Z,z}}{\mathfrak{m}_{Z,z}^n}\right),$$

then

$$\left[\ell\left(\frac{\mathcal{O}_{X,x}}{\mathfrak{m}_{X,x}^n}\right) - 1\right] + \left[\ell\left(\frac{\mathcal{O}_{Y,y}}{\mathfrak{m}_{Y,y}^n}\right) - 1\right] \le \left[\ell\left(\frac{R}{\mathfrak{m}^n}\right) - 1\right] + \left[\ell\left(\frac{\mathcal{O}_{Z,z}}{\mathfrak{m}_{Z,z}^n}\right) - 1\right],$$

and hence

$$\ell\left(\frac{\mathcal{O}_{X,x}}{\mathfrak{m}_{X,x}^n}\right) + \ell\left(\frac{\mathcal{O}_{Y,y}}{\mathfrak{m}_{Y,y}^n}\right) \le \ell\left(\frac{R}{\mathfrak{m}^n}\right) + \ell\left(\frac{\mathcal{O}_{Z,z}}{\mathfrak{m}_{Z,z}^n}\right). \tag{4.4}$$

The long exact sequence (4.3) gives the short exact sequence

$$0 \longrightarrow \frac{\mathfrak{m}}{\mathfrak{m}^{n-1}(\mathfrak{m}_{X,x} \oplus \mathfrak{m}_{Y,y}) \cap \mathfrak{m}} \longrightarrow \frac{\mathfrak{m}_{X,x}}{\mathfrak{m}_{X,x}^n} \oplus \frac{\mathfrak{m}_{Y,y}}{\mathfrak{m}_{Y,y}^n} \longrightarrow \frac{\mathfrak{m}_{Z,z}}{\mathfrak{m}_{Z,z}^n} \longrightarrow 0. \tag{4.5}$$

By the Artin-Rees Lemma, there exist $c \in \mathbb{N}$ such that for all $n \ge c+1$

$$\mathfrak{m}^{n-1}(\mathfrak{m}_{X,x} \oplus \mathfrak{m}_{Y,y}) \cap \mathfrak{m} \subseteq \mathfrak{m}^{n-1-c}\mathfrak{m} = \mathfrak{m}^{n-c},$$

and hence

$$\ell\left(\frac{\mathfrak{m}}{\mathfrak{m}^{n-1}(\mathfrak{m}_{X,x} \oplus \mathfrak{m}_{Y,y}) \cap \mathfrak{m}}\right) \ge \ell\left(\frac{\mathfrak{m}}{\mathfrak{m}^{n-c}}\right).$$

By (4.4), (4.5) and doing the same procedure to deduce (4.4), one obtains

$$\ell\left(\frac{R}{\mathfrak{m}^{n-c}}\right)+\ell\left(\frac{\mathcal{O}_{Z,z}}{\mathfrak{m}^n_{Z,z}}\right)\leq \ell\left(\frac{\mathcal{O}_{X,x}}{\mathfrak{m}^n_{X,x}}\right)+\ell\left(\frac{\mathcal{O}_{Y,y}}{\mathfrak{m}^n_{Y,y}}\right)\leq \ell\left(\frac{R}{\mathfrak{m}^{n}}\right)+\ell\left(\frac{\mathcal{O}_{Z,z}}{\mathfrak{m}^n_{Z,z}}\right). \tag{4.6}$$

Since

$$\lim_{n\to\infty}\frac{\dim(R)!}{n^{\dim(R)}}\ell\left(\frac{R}{\mathfrak{m}^{n-c}}\right)=\lim_{n\to\infty}\frac{\dim(R)!}{n^{\dim(R)}}\ell\left(\frac{R}{\mathfrak{m}^{n}}\right)$$

and by Proposition 2.13

$$\dim(R)=\dim((X,x)\sqcup_{(Z,z)}(Y,y))=\max\{\dim(X,x),\dim(Y,y)\}\geq\dim(Z,y),$$

the proof follows multiplying 4.6 by $\frac{\dim(R)!}{n^{\dim(R)}}$, taking limits and Definition 4.28. □

Remark 4.30. The previous result generalizes (FREITAS; PÉREZ; MIRANDA, 2021, Theorem 6.1) in the analytic complex case. It should be noted that the proof is similiar, but here one obtains more generality.

4.5 Applications: A especial case of schemes

Let $k[Y]=k[x_1,\dots,x_n]/I_Y$ be a coordinate ring of an affine algebraic variety Y over the field k (algebraically closed) and let $I_Z\supseteq I_Y$ be a prime ideal in $k[Y]$ defining an irreducible scheme Z and a subvariety of Y. The local ring of Y along Z is defined as the localization of $k[Y]$ at the prime ideal I_Z, that is $\mathcal{O}_{Z,Y}:=k[Y]_{I_Z}$. This ring is called the *local ring of a scheme Y at a closed irreducible subscheme Z.*

Let $\mathfrak{M}_{Z,Y}$ denote the maximal ideal of $\mathcal{O}_{Z,Y}$. For $n\gg 0$ and $\operatorname{codim}(Z,Y):=d=\dim(\mathcal{O}_{Z,Y})$, the Hilbert-Samuel polynomial is given by

$$P(n):=\text{length}\left(\mathcal{O}_{Z,Y}/\mathfrak{M}^{n+1}_{Z,Y}\right)=e_ZY\frac{n^d}{d!}+\text{ lower terms.}$$

The coefficient e_ZY of the leading term of the Hilbert-Samuel polynomial is known as the *algebraic multiplicity* of Y along Z. The coefficient e_ZY is also the multiplicity of the ideal $I_Z\mathcal{O}_{Z,Y}$ in the local ring $\mathcal{O}_{Z,Y}$, denoted $e(I_Z,\mathcal{O}_{Z,Y})$ (see (FULTON, 1998, Example 4.3.1)). This definition is the usual Hilbert-Samuel multiplicity and it is also given by the integer coefficient of $[Z]$ in the Segre class in $s(Z,Y)$ (see (FULTON, 1998, Example 4.3.4)). Note that, if $I_Z=\mathfrak{m}$ is the maximal ideal of $k[Y]$, then $e_ZY=\text{mult}_xY$, i.e., it is the multiplicity of Y at the point $x=V(\mathfrak{m})\in Y$.

Lemma 4.31. Let $X=\operatorname{Spec}(k[Y])$ be a scheme associated to the affine variety Y. Then, for $x=[\mathfrak{p}]\in X$,

$$\text{mult}_xX=e_ZY,$$

where $Z=V(\mathfrak{p})$ is an irreducible subvariety of Y.

Proof. Since $X = \operatorname{Spec}(k[Y])$, by Remark 4.13 one has $\mathcal{O}_{X,x} \cong k[Y]_{\mathfrak{p}}$. Now, since $[\mathfrak{p}] \in X$, then $I_Y \subseteq \mathfrak{p}$. The result follows by Definition 4.28 and the previous comment. □

For the next result, recall that the *codimension* of an irreducible closed subset Z of a scheme X, denoted by $\operatorname{codim}(Z, X)$, is the supremum of integers n such that there exists a chain

$$Z = Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n$$

of distinct closed irreducible subsets of X, beginning with Z. If Y is any closed subset of X, we define

$$\operatorname{codim}(Y, X) = \inf_{Z \subseteq Y} \operatorname{codim}(Z, X),$$

where the infimum is taken over all closed irreducible subsets of Y.

Proposition 4.32. Let X, Y be affine algebraic varieties over k and let Z be a subvariety of both X and Y. Suppose that $W = \operatorname{Spec}(k[X] \times_{k[Z]} k[Y])$ and that $w = x \sqcup_z y \in W$. Then

$$\operatorname{mult}_w W = \begin{cases} e_{V(\mathfrak{p})}X + e_{V(\mathfrak{q})}Y - e_{V(\mathfrak{t})}Z, & \text{if} \quad \operatorname{codim}(V(\mathfrak{p}), X) = \operatorname{codim}(V(\mathfrak{q}), Y) = \operatorname{codim}(V(\mathfrak{t}), Z) \\ e_{V(\mathfrak{p})}X + e_{V(\mathfrak{q})}Y, & \text{if} \quad \operatorname{codim}(V(\mathfrak{p}), X) = \operatorname{codim}(V(\mathfrak{q}), Y) > \operatorname{codim}(V(\mathfrak{t}), Z) \\ e_{V(\mathfrak{p})}X, & \text{if} \quad \operatorname{codim}(V(\mathfrak{p}), X) > \operatorname{codim}(V(\mathfrak{q}), Y) > \operatorname{codim}(V(\mathfrak{t}), Z). \end{cases}$$

where $x = [\mathfrak{p}] \in \operatorname{Spec}(k[X])$, $y = [\mathfrak{q}] \in \operatorname{Spec}(k[Y])$ and $z = [\mathfrak{t}] \in \operatorname{Spec}(k[Z])$.

Proof. By Lemma 4.14 and Remark 4.13, one has

$$\begin{aligned} \mathcal{O}_{W,w} &\cong \mathcal{O}_{\operatorname{Spec}(k[X]),x} \times_{\mathcal{O}_{\operatorname{Spec}(k[Z]),z}} \mathcal{O}_{\operatorname{Spec}(k[Y]),y} \\ &\cong k[X]_{\mathfrak{p}} \times_{k[Z]_{\mathfrak{t}}} k[Y]_{\mathfrak{q}}, \end{aligned}$$

where $x = [\mathfrak{p}] \in \operatorname{Spec}(k[X])$, $y = [\mathfrak{q}] \in \operatorname{Spec}(k[Y])$ and $z = [\mathfrak{t}] \in \operatorname{Spec}(k[Z])$. The desired conclusion now follows by Theorem 4.29 and Lemma 4.31. □

Next, the main focus is to give an explicit and practical formula to calculate the Hilbert-Samuel multiplicity of the gluing of schemes. For this purpose, we first recall the notion of degree of an affine projective variety.

Let k be an algebraically closed field. A projective variety X is called *non-degenerate* if it is not contained in a hyperplane. Let I_X be the ideal of X. Note that I_X is a homogeneous ideal of the polynomial ring $S = k[x_0, x_1, \ldots, x_t]$ and the homogeneous coordinate ring of X is defined as $k[X] := S/I_X$. Then $k[X]$ is a graded ring and we write it as $k[X] = \oplus_{n \geq 0} k[X]_n$, where $k[X]_n = S_n/(I_X \cap S_n)$.

We can see $k[X]$ as a finitely generated graded S-module and that each graded part $k[X]_n$ is a finite-dimensional vector space over k. The Hilbert function of $k[X]$ is $H_{k[X]}(n) = \dim_k(k[X]_n)$. Then, for all $n \gg 0$, the Hilbert polynomial of $k[X]$ is denoted by $P_{k[X]}(n) \in \mathbb{Q}(n)$ and it can be written in the form

$$P_{k[X]}(n) = e_0(k[X]) \frac{n^{\dim(X)-1}}{(\dim(X)-1)!} + \text{ lower order terms},$$

where the integer $e_0(k[X])$ is called the *degree* of X. An another characterization of this number is also denoted and calculated as

$$\deg(X) := \lim_{n\to\infty} \frac{(\dim(X)-1)! P_{k[X]}}{n^{\dim(X)-1}} = e_0(k[X]).$$

Question 4.33. Is there any relation between the Hilbert-Samuel multiplicity $e_Z Y$ and the degree?

A recent progress concerning the previous question has been made by Harris, Helmer and Nanda in (HARRIS; HELMER, 2020, Theorem 5.2) and (HELMER; NANDA, 2023, Proposition 3.3). Before answering this question for gluing of schemes, first let's consider some notations.

Let $Z \subset Y \subset \mathbb{P}^n_k$ be projective varieties. Let I_Z be the ideal that defines Z and let $f_0, \dots, f_r$ be homogeneous polynomials that generate the ideal I_Z of $k[x_0, x_1, \dots, x_n]$. Set

$$V = \{x \in \mathbb{P}^n_k : F_1(x) = \cdots = F_{\dim Y - \dim Z}(x) = 0\},$$

where F_j are homogeneous polynomials that have the form $F_j = \sum_i \lambda_i f_i$, $\lambda_i \in k$. Now, we define

$$g_{\dim Z}(Z, Y) := \deg((Y \cap L \cap V) - Z),$$

where L is a generic $(n - \dim Z)$-dimensional linear subspace of $\mathbb{P}^n_k$. In the following results, we will use projective closures of an affine algebraic set X, denoted by PX. Also, let $\hat{Z}$ denote the flat completion of the a scheme Z.

Lemma 4.34. Let $R = \mathcal{O}_{Z,Y}$ be the local ring of an equidimensional scheme Y at a closed irreducible subscheme Z. Let d be the maximum degree of the generators of I_Z and I_Y. Then,

$$e_{\hat{Z}}\hat{Y} = e_Z Y = e_{PZ} PY = \frac{\deg(PY) d^{\dim PY - \dim PZ} - g_{\dim Z}(PZ, PY)}{\deg(PZ)}.$$

Proof. Since that the map $(R, \mathfrak{M}_{Z,Y}) \to (\hat{R}, \widehat{\mathfrak{M}_{Z,Y}})$ is a local flat homomorphism of local rings and $\widehat{\mathfrak{M}_{Z,Y}} = \mathfrak{M}_{\hat{Z},\hat{Y}}$, for all positive integer n, one obtains

$$\begin{aligned}\text{length}(\mathcal{O}_{Z,Y}/\mathfrak{M}^{n+1}_{Z,Y}) &= \text{length}\left(\widehat{\mathcal{O}_{Z,Y}/\mathfrak{M}^{n+1}_{Z,Y}}\right) \\ &= \text{length}(\mathcal{O}_{\hat{Z},\hat{Y}}/\mathfrak{M}^{n+1}_{\hat{Z},\hat{Y}}).\end{aligned}$$

Further, since $\dim R = \dim \hat{R}$,

$$e_Z Y = \lim_{n\to\infty} \frac{d!\text{length}(\mathcal{O}_{Z,Y}/\mathfrak{M}^{n+1}_{Z,Y})}{n^{\dim R}} = \lim_{n\to\infty} \frac{d!\text{length}(\mathcal{O}_{\hat{Z},\hat{Y}}/\mathfrak{M}^{n+1}_{\hat{Z},\hat{Y}})}{n^{\dim \hat{R}}} = e_{\hat{Z}}\hat{Y}.$$

On the other hand, the equality $e_Z Y = e_{PZ} PY$ was presented within the proof of (HELMER; NANDA, 2023, Proposition 3.2) (in the case $k = \mathbb{C}$). But for an arbitrary field

k, this equality is an immediate consequence of (MATSUMURA, 1987, Theorem 13.8). Now, the last equality follows by (HARRIS; HELMER, 2020, Theorem 5.2) or (HELMER; NANDA, 2023, Proposition 3.2). □

The next result follows easily from Proposition 4.32 and Lemma 4.34.

Corollary 4.35. Let X, Y be affine algebraic varieties over k and Z a subvariety of both X and Y. Suppose that $W = \operatorname{Spec}(k[X] \times_{k[Z]} k[Y])$ and that $w = x \sqcup_z y \in W$, where $x = [\mathfrak{p}] \in \operatorname{Spec}(k[X])$, $y = [\mathfrak{q}] \in \operatorname{Spec}(k[Y])$ and $z = [\mathfrak{t}] \in \operatorname{Spec}(k[Z])$.

(i) If $\operatorname{codim}(V(\mathfrak{p}), X) = \operatorname{codim}(V(\mathfrak{q}), Y) > \operatorname{codim}(V(\mathfrak{t}), Z)$, then

$$e_w(W) = \frac{\deg(PX)d_1^{\dim PX - \dim PV(\mathfrak{p})} - g_{\dim V(\mathfrak{p})}(PV(\mathfrak{p}), PX)}{\deg(PV(\mathfrak{p}))} + \frac{\deg(PY)d_2^{\dim PY - \dim PV(\mathfrak{q})} - g_{\dim V(\mathfrak{q})}(PV(\mathfrak{q}), PY)}{\deg(PV(\mathfrak{q}))},$$

where d_1 and d_2 are the maximum degree of the generators of $\mathfrak{p}$ and I_X; and $\mathfrak{q}$ and I_Y, respectively.

(ii) If $\operatorname{codim}(V(\mathfrak{p}), X) > \operatorname{codim}(V(\mathfrak{q}), Y) > \operatorname{codim}(V(\mathfrak{t}), Z)$, then

$$e_w(W) = \frac{\deg(PX)d^{\dim PX - \dim PV(\mathfrak{p})} - g_{\dim V(\mathfrak{p})}(PV(\mathfrak{p}), PX)}{\deg(PV(\mathfrak{p}))},$$

where d is the maximum degree of the generators of $\mathfrak{p}$ and I_X.

In (HARRIS; HELMER, 2020, Theorem 5.2) and (HELMER; NANDA, 2023, Proposition 3.2), the authors give formulas for the multiplicity for the equidimensional case. It should be noted that in the previous result, the scheme W, which is the gluing of X, Y and Z, in general is not equidimensional (see Proposition 4.17 and Theorem 4.22). Nevertheless, it is possible to obtain formulas for the multiplicity of W.

In next result we employ a result of Samuel (SAMUEL, 1967) (see also (FULTON, 1998, Ex. 12.4.5(b))) which relates the containment of a subvariety in the singular locus of a variety and the algebraic multiplicity.

Proposition 4.36. (SAMUEL, 1967, II, §6.2b) Let Z be a subvariety of a smooth projective toric variety T_Σ and suppose that X is a subvariety of Z. Then $e_X Z = 1$ if and only if X is not contained in the singular locus of Z.

Theorem 4.37. Let X, Y be affine algebraic varieties over k and Z a subvariety of both X and Y. Suppose that $W = \operatorname{Spec}(k[X] \times_{k[Z]} k[Y])$ is equidimensional and that $w = x \sqcup_z y \in W$, where $x = [\mathfrak{p}] \in \operatorname{Spec}(k[X])$, $y = [\mathfrak{q}] \in \operatorname{Spec}(k[Y])$ and $z = [\mathfrak{t}] \in \operatorname{Spec}(k[Z])$.

(i) Suppose that $\operatorname{codim}(V(\mathfrak{p}), X) = \operatorname{codim}(V(\mathfrak{q}), Y) = \operatorname{codim}(V(\mathfrak{t}), Z)$. If $V(\mathfrak{p})$, $V(\mathfrak{t})$ and $V(\mathfrak{q})$ is not contained in the singular locus of X, Z and Y respectively, then W is nonsingular at $w \in W$.

(ii) Suppose that $\operatorname{codim}(V(\mathfrak{p}), X) = \operatorname{codim}(V(\mathfrak{q}), Y) > \operatorname{codim}(V(\mathfrak{t}), Z)$. If $V(\mathfrak{p})$ and $V(\mathfrak{q})$ is not contained in the singular locus of X and Y respectively, then W is singular at $w \in W$.

(iii) If $\operatorname{codim}(V(\mathfrak{p}), X) > \operatorname{codim}(V(\mathfrak{q}), Y) > \operatorname{codim}(V(\mathfrak{t}), Z)$, then W is nonsingular at $w \in W$ if and only if $V(\mathfrak{p})$ is not contained in the singular locus of X.

Proof. (i) Since $V(\mathfrak{p})$, $V(\mathfrak{t})$ and $V(\mathfrak{q})$ are not contained in the singular locus of X, Z and Y respectively, Proposition 4.36 provides

$$e_{V(\mathfrak{p})}X = e_{V(\mathfrak{q})}Y = e_{V(\mathfrak{t})}Z = 1.$$

Hence, by Proposition 4.32 one obtains that $\operatorname{mult}_w W = 1$. Therefore, the result follows by the assumption and again Proposition 4.36. The proof of (ii) and (iii) follows similarly. □

CHAPTER 5

THE GLUING OF FORMAL SCHEMES

In this chapter, we begin by providing a brief overview of the theory of linearly topologized rings, along with an introduction to the concepts of Cauchy sequences and complete rings (5.1.1). Additionally, we explore Hausdorff completions and introduce the notions of admissible and adic rings. As the main consequence, we derive that the category of admissible rings has fiber products (5.1.2). Finally, in Section 5.2, we introduce formal schemes and extend some results obtained in Chapter 3 to this category. We basically refer (ARNAUTOV; GLAVATSKY; MIKHALEV, 1996), (GROTHENDIECK; DIEUDONNÉ, 1971) and (FUJIWARA; KATO, 2018) for the fundamental theory presented here.

5.1 Topological rings

This section is dedicated to recall fundamental concepts related to linearly topologized rings. Actually, our focus lies on admissible rings, which play a crucial role in defining the central subject of this chapter: the formal schemes.

Throughout this section, again all rings are commutative with unity, and a morphism of rings $R \to S$ satisfies $1_R \mapsto 1_S$. We begin with the following observation in the field of general topology. Concerning the theory of topological spaces, let remember the following definitions:

Definition 5.1. Let X be a topological space and $x \in X$.

1. A *neighborhood of* x is any subset of X which contains an open set containing x.
2. A subset U is *open* in X if only if U is a neighborhood of any of its elements.
3. If $x \in X$, then a *fundamental system of neighborhoods* (*basis of neighborhoods*) of x is a nonempty set $\mathcal{S}$ of open neighborhoods of x with the property that if U is open and $x \in U$, then there is $W \in \mathcal{S}$ with $W \subseteq U$.

Definition 5.2. A *topological ring* is a ring R endowed with a topology for which the maps $A \times A \to A$ defined by $(x, y) \mapsto x - y$ and $(x, y) \mapsto xy$ are continuous. We say that a topological ring is *linearly topologized* if there exists a basis of neighborhood of 0 consisting of ideals. In this situation we say that the topology is *linear*.

Example 5.1. Below we will give examples of linearly topologized rings and not linearly topologized rings.

1. The ring of formal power series $R[[x]]$ in a single variable over a ring R, where a system of neighborhoods of 0 is given by powers of the ideal (x). More generally, $R[[x_1, \ldots, x_n]]$ is also an example, with the topology given by powers of $(x_1, \ldots, x_n)$.
2. The ring of integers of a non-archimedean field.
3. The local fields such as $\mathbb{Q}_p$ and $\mathbb{F}_q((t))$ are not linearly topologized rings.

Next we will define a topology in the ring R that transforms R into a topological ring linearly topologized. But first, let us remember the following.

Definition 5.2. A *directed set*, is a non-empty set Λ with the relation $\leq$ satisfying:

(i) $\lambda \leq \lambda$ for all $\lambda \in \Lambda$;

(ii) $\lambda \leq \lambda'$ and $\lambda' \leq \lambda''$ imply $\lambda \leq \lambda''$ for all $\lambda, \lambda', \lambda'' \in \Lambda$;

(iii) $\lambda \leq \lambda'$ and $\lambda' \leq \lambda$ imply $\lambda = \lambda'$;

(iv) for all $\lambda, \lambda' \in \Lambda$, there exists a $\lambda'' \in \Lambda$ with $\lambda \leq \lambda''$ and $\lambda' \leq \lambda''$.

Definition 5.3. Let R be a ring. Let $\mathcal{F} = \{I_\lambda\}_{\lambda \in \Lambda}$ be a family of ideals of R indexed by a directed set Λ, satisfying

$$\lambda_1 \geq \lambda_2 \Rightarrow I_{\lambda_1} \subseteq I_{\lambda_2}$$

for all $\lambda_1, \lambda_2 \in \Lambda$. We call $\mathcal{F}$ as the *descending filtration by ideals*.

It can be shown that then there exists one and only one topology in R such that R is a topological ring with respect to that topology and $\mathcal{F}$ is a basis of neighborhoods of 0 of the topological ring R (see (ARNAUTOV; GLAVATSKY; MIKHALEV, 1996, Theorem 1.2.5) or (ZARISKI; SAMUEL, 2013, pg. 252)). We call this topology as the *topology defined by the filtration* $\mathcal{F}$. Explicitly, a subset $U \subseteq R$ is open in this topology if and only if for any $x \in U$, there exists $\lambda \in \Lambda$ such that $x + I_\lambda \subseteq U$.

A particular case of this topology, widely used in practice, is the so-called *adic topology*, that is, the topology defined by the filtration $\mathcal{F} = \{\mathfrak{a}^n\}_{n \in \mathbb{N}}$ where $\mathfrak{a} \subseteq R$ is an ideal (if we like to spell out the ideal $\mathfrak{a}$, we say it is the $\mathfrak{a}$-*adic topology*).

Remark 5.4. This is the $\mathfrak{a}$-preadic topology according to Grothendieck's terminology (GROTHENDIECK; DIEUDONNÉ, 1971); the latter is referred to be adic if it is separated and complete.

Given a topological ring, a natural question would be whether this ring is Hausdorff. The following result gives us the answer to this question.

Proposition 5.5. (ARNAUTOV; GLAVATSKY; MIKHALEV, 1996, Corollary 1.3.3) For any topological ring R, if $\mathcal{B}_0$ is a basis of neighborhoods of zero of R, then R is Hausdorff if and only if $\bigcap_{V\in\mathcal{B}_0} V = 0$.

We can now define the notion of Cauchy sequences and completion in a topological ring as follows.

Definition 5.6. Let R be a topological ring, and let Λ be a directed set.

1. A *sequence* in R is a function $f : \Lambda \to R$. We represent a sequence $f : \Lambda \to R$ by $(x_\lambda)_{\lambda\in\Lambda}$, where $x_\lambda = f(\lambda)$.
2. An element $x \in R$ is called a *limit* of the sequence $(x_\lambda)_{\lambda\in\Lambda}$, if for any neighborhood V of the point x in R there exists an element $\lambda_0 \in \Lambda$ such that $x_\lambda \in V$ for all $\lambda \geq \lambda_0$. The sequence that has a limit is called *convergent* to the limit.
3. A sequence $(x_\lambda)_{\lambda\in\Lambda}$ is *Cauchy* if, for any neighborhood U of 0, there exists some $\lambda_U \in \Lambda$ such that, for $\lambda, \lambda' \geq \lambda_U$, $x_\lambda - x_{\lambda'} \in U$.
4. An topological ring R is *complete* if every Cauchy sequence in R converges. In addition, if R is Haudorff, we say that R is *complete Hausdorff.*

Two Cauchy sequences (x_λ) and (y_λ) are equivalent if the sequence $(x_\lambda - y_\lambda)$ converges to zero (that is, for every open neighborhood U of 0 there is $N \geq 1$ such that for $\lambda \geq N$, $(x_\lambda - y_\lambda) \in U$). This relationship in fact determines an equivalence relationship. We now let $\widehat{R}$ be the set of all equivalence classes of Cauchy sequences under this relation and we define $[(x_\lambda)] + [(y_\lambda)] := [(x_\lambda + y_\lambda)]$ and $[(x_\lambda)][(y_\lambda)] := [(x_\lambda y_\lambda)]$. It follows that $\widehat{R}$ is a complete ring with unity. Note that this setup generalizes the usual completion seen in analysis, and therefore we refer to $\widehat{R}$ as the *Hausdorff completion* of R.

Given a topological ring, with a basis of neighborhoods of 0 consisting of ideal, there is a nice description of completions, which we will explain below.

Let R be a topological ring. Let also $\{\mathfrak{a}_\lambda\}_{\lambda\in\Lambda}$ be a basis of neighborhoods of 0 in R, where Λ is a directed set, such that for any pair of elements $\alpha, \beta \in \Lambda$, such that $\beta \geq \alpha$, we have $\mathfrak{a}_\beta \subseteq \mathfrak{a}_\alpha$. For any $\alpha, \beta \in \Lambda$, with $\beta \geq \alpha$, consider the natural homomorphisms $f_\alpha^\beta : R/\mathfrak{a}_\beta \to R/\mathfrak{a}_\alpha$, such that

(i) $f_\alpha^\alpha = \text{id}$ for all $\alpha \in \Lambda$;

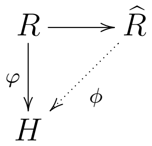
(ii) $f_\alpha^\lambda = f_\alpha^\beta \circ f_\beta^\lambda$ for $\alpha \leq \beta \leq \lambda$.

Then, $\varprojlim_\lambda R/\mathfrak{a}_\lambda$ is a subring of the ring $\prod_{\lambda\in\Lambda} R/\mathfrak{a}_\lambda$. More precisely, $\varprojlim_\lambda R/\mathfrak{a}_\lambda$ is the subset of the product $\prod_{\lambda\in\Lambda} R/\mathfrak{a}_\lambda$ consisting of those points (x_λ) such that $x_\alpha = f_\alpha^\beta(x_\beta)$, whenever $\beta \geq \alpha$.

We have that the Hausdorff completion $\widehat{R}$ is isomorphic to $\varprojlim_\lambda R/\mathfrak{a}_\lambda$. This fact together with the next result justifies the term "Hausdorff" in $\widehat{R}$.

Theorem 5.7. (FUJIWARA; KATO, 2018, Proposition 7.1.8(2)) Let R be an topological ring. Then $\widehat{R}$ is Hausdorff complete with respect to the topology defined by the induced filtration $\widehat{\mathcal{F}} = \{\ker \pi_\lambda\}_{\lambda\in\Lambda}$, where π_λ is the canonical projection map $\pi_\lambda : \widehat{R} \to R/\mathfrak{a}_\lambda$.

Remark 5.8.
1. (FUJIWARA; KATO, 2018, Proposition 7.1.9) In fact, $\widehat{R}$, as defined above, satisfies a universal property: With the previous notation, if $\varphi : R \to H$ is a continuous homomorphism of topological rings, where H is Hausdorff complete with respect to a descending filtration by ideals, then there exists a unique continuous homomorphism $\phi : \widehat{R} \to H$ such that the following diagram commutes:



2. If R is complete, then $\widehat{R}$ is topologically isomorphic to R. In particular, R is separated. This fact follows from (1).

When we construct formal schemes, we will do so by defining the formal spectrum of an admissible ring. In practice, however, the formal schemes that arise are those whose open affines are adic. For this purpose, the following subsection is necessary:

5.1.1 Admissible and adic rings

Let R be a ring endowed with the topology defined by a descending filtration $\mathcal{F} = \{\mathfrak{a}_\lambda\}_{\lambda\in\Lambda}$ of ideals.

Definition 5.9. An ideal $\mathfrak{a} \subseteq R$ is said to be an *ideal of definition* of the topological ring R if the following conditions are satisfied:

(i) $\mathfrak{a}$ is *open*; that is, there exists $\lambda \in \Lambda$ such that $\mathfrak{a}_\lambda \subseteq \mathfrak{a}$;

(ii) $\mathfrak{a}$ is *topologically nilpotent*; that is, for any $\mu \in \Lambda$ there exists $n \geq 0$ such that $\mathfrak{a}^n \subseteq \mathfrak{a}_\mu$.

Remark 5.10. Note that if $\mathfrak{a}$ is an ideal of definition in R, and $\mathfrak{b}$ any open ideal of R, then $\mathfrak{a} \cap \mathfrak{b}$ is also an ideal of definition. Hence, if R admits at least one ideal of definition, it has a fundamental system of open neighborhoods of 0 consisting of ideals of definition, called a *fundamental system of ideals of definition*.

To be or not an ideal of definition $\mathfrak{a} \subseteq R$, depends on the topology being considered in R. For instance, consider the following example.

Example 5.11. Consider the ring $R = \mathbb{Z}_p[x]$. We have that R is a topological ring with the following two topologies:

(1) the topology where a neighborhood basis for 0 is given by $\mathcal{F} = \{(x)^n\}$; and

(2) the topology where a neighborhood basis for 0 is given by $\mathcal{S} = \{(p)^n\}$.

Note that these two topologies are not the same. A easiest way to see this is to take their respective completions: the completion of the former is the ring $\mathbb{Z}_p[[x]]$, whereas the completion of the latter is $\mathbb{Z}_p\{x\}$ (the ring of restricted formal power series).

If we consider R with the topology defined by $\mathcal{F}$, then (x) is a ideal of definition of R. However, if we consider R with the topology defined by $\mathcal{S}$, then (x) is no longer a ideal of definition of R.

To clarify this situation, the following result follows:

Proposition 5.12. (FUJIWARA; KATO, 2018, Proposition 7.2.1) Let R be a ring equipped with the $\mathfrak{a}$-adic topology by an ideal $\mathfrak{a} \subseteq R$. Then $\mathfrak{b} \subseteq R$ is an ideal of definition of R if only if the $\mathfrak{b}$-adic topology coincides with the $\mathfrak{a}$-adic topology.

Definition 5.13. We say that R is an *admissible* ring if R admits an ideal of definition, and is Hausdorff complete. If in addition the topology on R is $\mathfrak{a}$-adic for some ideal $\mathfrak{a} \subseteq R$, R is called *adic ring*. In other words, an adic ring is a ring that is Hausdorff complete with the adic topology.

The next results shows that the classification of adic rings does not depend on the choice of the ideal definition.

Proposition 5.14. (FUJIWARA; KATO, 2018, Lemma 1.1.2) Let R be a ring endowed with the topology defined by a descending filtration $\mathcal{F} = \{\mathfrak{a}_\lambda\}_{\lambda \in \Lambda}$ by ideals, and $\mathfrak{a} \subseteq R$ an ideal. Then the following conditions are equivalent.

(a) The topology on R is the same as the $\mathfrak{a}$-adic topology.

(b) $\mathfrak{a}$ is an ideal of definition, and $\mathfrak{a}^n$ is open for any $n \geq 0$.

(c) $\mathfrak{a}^n$ is an ideal of definition for any $n \geq 1$.

(d) $\{\mathfrak{a}^n\}_{n\geq 0}$ is a fundamental system of open neighborhoods of 0.

Moreover, if these conditions are fulfilled, then for any ideal of definition $\mathfrak{b} \subseteq R$ the topology on R is $\mathfrak{b}$-adic.

The following examples illustrate rings that have an ideal of definition, its topology is $\mathfrak{a}$-adic, but they are not adic rings.

Example 5.15. (1) The ring $R[x]$, where a basis of neighborhoods of 0 is given by powers of the ideal (x). As before, more generally, $R[x_1, \ldots, x_n]$ with a neighborhood basis of 0 given by powers of the ideal $(x_1, \ldots, x_n)$.

(2) $\mathbb{Z}$ equipped with the m-adic topology, for any $m \in \mathbb{Z}$. (Of course, the cases of primary interest are when m is in fact a prime p.)

The following rings are adic:

Example 5.16. (1) Any commutative ring under the discrete topology. With a neighborhood basis is $\{(0)\}$, which is also $\{(0)^n\}$.

(2) $R[[x]]$ (resp. $R[[x_1, \ldots, x_n]]$), with a neighborhood basis $\{(x)^k\}$ (resp. $\{(x_1, \ldots, x_n)^k\}$). This is just the completion of the first ring mentioned above.

(3) The ring of integers of a non-archimedean field.

Remark 5.17. In general, if $\mathfrak{a} \subseteq R$ is a finitely generated ideal, then the $\mathfrak{a}$-adic completion $\widehat{R}$ of a ring R is Haudorff complete with the $\mathfrak{a}$-adic topology, and hence is an adic ring (see (FUJIWARA; KATO, 2018, Proposition 7.2.15)).

The following result will be important in the definition of formal spectrum.

Proposition 5.18. Let $\mathfrak{p}$ be a prime ideal of a ring R with ideal of definition. The following conditions are equivalent:

(i) $\mathfrak{p}$ is open;

(ii) $\mathfrak{p}$ contains every ideal of definition;

(iii) $\mathfrak{p}$ contains an ideal of definition.

Proof. *(i)* $\Rightarrow$ *(ii)* Given any ideal of definition $\mathfrak{a}$, by definition there exists some n (depending on $\mathfrak{a}$) such that $\mathfrak{a}^n \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is prime, $\mathfrak{a} \subseteq \mathfrak{p}$.

(ii) $\Rightarrow$ *(iii)* trivial.

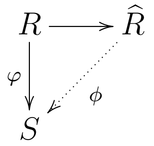
$(iii) \Rightarrow (i)$ Let $\mathfrak{a}$ be an ideal of definition, $\mathfrak{a} \subseteq \mathfrak{p}$. Recall that $\mathfrak{a}$ is open; given any $x \in \mathfrak{p}$, $x + \mathfrak{a} \subseteq \mathfrak{p}$ is an eighborhood of x contained in $\mathfrak{p}$. So $\mathfrak{p}$ is open. □

Let R be a ring that admits an ideal of definition, and $\{\mathfrak{a}_\lambda\}_{\lambda\in\Lambda}$ a fundamental system of ideals of definition. Consider $\widehat{R} = \varprojlim_{\lambda} R/\mathfrak{a}_\lambda$.

Theorem 5.19. If R admits an ideal of definition, then $\widehat{R}$ is admissible.

Proof. By Theorem 5.7, one has that $\widehat{R}$ is separated, complete, and the inverse limit topology naturally makes $\widehat{R}$ a topological ring. To conclude, it is enough to show that $\hat{\mathfrak{a}}_\lambda := \ker \pi_\lambda$ is the ideal of definition of $\widehat{R}$. □

Remark 5.20. The completion $\widehat{R}$ of a ring R satisfies a universal property. In fact, the ring $\widehat{R}$, as defined above, satisfies a universal property: with the previous notation, if $\varphi : R \to S$ is a homomorphism of rings, where S is an admissible ring, then there exists a unique continuous homomorphism $\phi : \widehat{R} \to S$ such that the following diagram commutes:



5.1.2 Completed fiber product

The main purpose of this subsection is to show that the fiber product of tree admissible rings is also admissible. Hence, consider R, S and T tree admissible rings, where $\varepsilon_R : R \to T, \varepsilon_S : S \to T$ are homomorphisms of rings.

Let $\{\mathfrak{a}_\lambda\}_{\lambda\in\Lambda}$, $\{\mathfrak{b}_\beta\}_{\beta\in\Sigma}$ and $\{\mathfrak{c}_\alpha\}_{\alpha\in C}$ be a fundamental system of ideals of definition of R, S and T, respectively. We suppose that for any $\lambda \in \Lambda$, $\beta \in \Sigma$ there exists $\alpha \in C$ such that $\mathfrak{a}_\lambda \subseteq \varepsilon_R^{-1}(\mathfrak{c}_\alpha)$ and $\mathfrak{b}_\beta \subseteq \varepsilon_S^{-1}(\mathfrak{c}_\alpha)$. The *completed fiber product* of R, S and T is given by

$$\widehat{R \times_T S} = \varprojlim_{\lambda,\beta,\alpha} R \times_T S/\mathfrak{a}_\lambda \times_{\mathfrak{c}_\alpha} \mathfrak{b}_\beta,$$

where $\Lambda \times \Sigma \times C$ is the directed set given by

$$\Lambda \times \Sigma \times C = \{(\lambda, \beta, \alpha) \,|\, \varepsilon_R(\mathfrak{a}_\lambda) \subseteq \mathfrak{c}_\alpha, \varepsilon_S(\mathfrak{b}_\beta) \subseteq \mathfrak{c}_\alpha\},$$

considered with the ordering defined by

$$(\lambda, \beta, \alpha) \leq (\lambda', \beta', \alpha') \iff \lambda \leq \lambda', \beta \leq \beta' \text{ and } \alpha \leq \alpha'.$$

The ring $\widehat{R \times_T S}$ is the Hausdorff completion of the fiber product the ring $R \times_T S$ with respect to the topology defined by the filtration

$$H^{\lambda,\beta,\alpha} := \{\mathfrak{a}_\lambda \times_{\mathfrak{c}_\alpha} \mathfrak{b}_\beta\}_{(\lambda,\beta,\alpha)\in\Lambda\times\Sigma\times C}.$$

For each $(\lambda, \beta, \alpha) \in \Lambda \times \Sigma \times C$, let $\widehat{H}^{\lambda,\beta,\alpha}$ be the closure of the image of $H^{\lambda,\beta,\alpha}$ in $\widehat{R \times_T S}$. Then $\widehat{R \times_T S}$ is Hausdorff complete with respect to the topology defined by the filtration $\widehat{H}^{\lambda,\beta,\alpha}$ (see (FUJIWARA; KATO, 2018, Proposition 7.1.8(2), p. 141)).

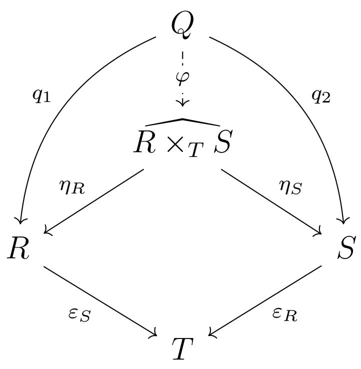
Lemma 5.21. Let R, S and T be rings that admit ideals of definition, and let $\{\mathfrak{a}_\lambda\}_{\lambda\in\Lambda}$, $\{\mathfrak{b}_\beta\}_{\beta\in\Gamma}$ and $\{\mathfrak{c}_\alpha\}_{\alpha\in\Sigma}$ be a fundamental system of ideals of definition of R, S and T, respectively. Then the fiber product ring $P := R \times_T S$ admits an ideal of definition, endowed with the $\{\mathfrak{a}_\lambda \times_{\mathfrak{c}_\alpha} \mathfrak{b}_\beta\}$ topology.

Proof. It is sufficient to show that, for any λ, β, α and $\lambda', \beta', \alpha'$, there exists some n so that $(\mathfrak{a}_\lambda \times_{\mathfrak{c}_\alpha} \mathfrak{b}_\beta)^{2n} \subseteq \mathfrak{a}_{\lambda'} \times_{\mathfrak{c}_{\alpha'}} \mathfrak{b}_{\beta'}$. Then, any element of $\mathfrak{a}_\lambda \times_{\mathfrak{c}_\alpha} \mathfrak{b}_\beta$ can be written as $(r, s) \in \mathfrak{a}_\lambda \times \mathfrak{b}_\beta$ such that $\varepsilon_R(r) = \varepsilon_S(s) = t \in \mathfrak{c}_\alpha$. So, if n is such that $\mathfrak{a}_\lambda^n \subseteq \mathfrak{a}_{\lambda'}$, $\mathfrak{b}_\beta^n \subseteq \mathfrak{b}_{\beta'}$, $\mathfrak{c}_\alpha^n \subseteq \mathfrak{c}_{\alpha'}$; and $r \in \mathfrak{a}_\lambda^n$, $s \in \mathfrak{b}_\beta^n$, $t \in \mathfrak{c}_\alpha^n$, then $r \in \mathfrak{a}_\lambda$, $s \in \mathfrak{b}_\beta$, $t \in \mathfrak{c}_\alpha$. Hence, $(\mathfrak{a}_\lambda \times_{\mathfrak{c}_\alpha} \mathfrak{b}_\beta)^{2n} \subseteq \mathfrak{a}_{\lambda'} \times_{\mathfrak{c}_{\alpha'}} \mathfrak{b}_{\beta'}$. □

Remark 5.22. As a consequence of Lemma 5.21 and Theorem 5.19, we derive that $\widehat{R \times_T S}$ is an admissible ring.

As proposed at the beginning of the section, the next result shows that the category of admissible rings has fiber product.

Theorem 5.23. Let R, S and T be three admissible rings. Then $\widehat{R \times_T S}$ is the pullback (also called the fiber product) in the category of admissible rings. That is, for any other such admissible ring (Q, q_1, q_2), where $q_1 : Q \to R$ and $q_2 : Q \to S$ are morphisms with $\varepsilon_R \circ q_1 = \varepsilon_S \circ q_2$, then there exists a unique continuous map $\varphi : Q \to \widehat{R \times_T S}$ making the following diagram commute:



Proof. We already know that $R \times_T S$ is a pullback (fiber product) in the category of rings; so, given the maps as above, we know there exists a unique continuous map $\phi : Q \to R \times_T S$. Therefore, by Remark 5.20 there exists a unique map of rings $\varphi : Q \to \widehat{R \times_T S}$, satisfying the desired statement. □

Definition 5.24. An admissible ring R is said to be of *finite ideal type* if it has a fundamental system of ideals of definition consisting of finitely generated ideals.

Example 5.25. Let R be a ring, and let $\mathfrak{a} \subseteq R$ be a finitely generated ideal. We consider the $\mathfrak{a}$-adic topology on R. Then the Hausdorff completion $\widehat{R}$ of R with respect to the $\mathfrak{a}$-adic topology is an adic ring. Note that in this situation, the closure $\mathfrak{b}$ of the image of $\mathfrak{a}$ in $\widehat{R}$ coincides with $\mathfrak{a}\widehat{R}$ (see (BOSCH, 2014, Remark 8, p. 156)).

Lemma 5.26. If the rings R, S and T are adic of finite ideal type, then so is $\widehat{R \times_T S}$. Also, if $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ are a finitely generated ideals of definition of R, S and T, respectively, then $H = H^{1,1,1}\widehat{R \times_T S}$ give a finitely generated ideal of $\widehat{R \times_T S}$, where $H^{n,m,k} = \mathfrak{a}^n \times_{\mathfrak{c}^k} \mathfrak{b}^m$ for $n, m, k \geq 0$.

Proof. Clearly, the filtration $\{H^{n,n,n}\}_{n \geq 0}$ gives a fundamental system of neighborhoods of 0 for the ring $R \times_T S$, for the diagonal map $\mathbb{N} \to \mathbb{N}^3$ is cofinal[1]. We calculate easily for $r \geq 0$, that $(H^{1,1,1})^{2r} \subseteq H^{r,r,r}$.

On the other hand, we clearly have $H^{2r,2r,2r} \subseteq (H^{1,1,1})^{2r}$. Hence

$$H^{2r,2r,2r} \subseteq (H^{1,1,1})^{2r} \subseteq H^{r,r,r}$$

holds for any $r \geq 0$, and thus the topology on $R \times_T S$ given by $\{H^{n,m,k}\}_{n,m,k \geq 0}$ is $H^{1,1,1}$-adic. Now, since $H^{1,1,1}$ is a finitely generated ideal of definition, the Hausdorff completion $\widehat{R \times_T S}$ is actually the $H^{1,1,1}$-adic completion (FUJIWARA; KATO, 2018, Proposition 7.2.15, pg. 153), and hence $H = H^{1,1,1}\widehat{R \times_T S}$ is an ideal of definition (FUJIWARA; KATO, 2018, Corollary 7.2.9, p. 150), as desired. □

5.2 Formal schemes

Similarly to the concept of ordinary schemes, formal schemes are constructed from local affine parts, called affine formal schemes. However, unlike the case of ordinary schemes, we say that a formal scheme is a *topologically locally ringed space* isomorphic to an affine formal scheme. By topologically ringed space, we mean the pair $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, where $\mathcal{X}$ is a topological space and $\mathcal{O}_{\mathcal{X}}$ is a sheaf of topological rings on $\mathcal{X}$.

In this section, we will explicitly define the objects mentioned above. We have established that the category of affine formal schemes is the dual category to that of admissible rings. Finally, we will show that the category of formal schemes (which extends the category of schemes) also has gluing.

Throughout our discussion in this subsection, all rings will be admissible, and $\{\mathfrak{a}_\lambda\}_{\lambda \in \Lambda}$ will denote fundamental system of neighborhoods of 0 consisting of all the ideals of definition. Before we make the definition of the formal spectrum, let us make a remark.

1 This means that for each $(a, b, c) \in \mathbb{N}^3$, there is an $n \in \mathbb{N}$ such that $(a, b, c) \leq (n, n, n)$.

Remark 5.27. Let R be any ring.

1. Suppose R is an addmissible ring. Then for any two ideals of definition $\mathfrak{a}$ and $\mathfrak{b}$, we have that $\mathrm{Spec}(R/\mathfrak{a}) = \mathrm{Spec}(R/\mathfrak{b})$ as topological spaces. In fact, each of these subspaces can be identified with the closed subspaces $V(\mathfrak{a})$ and $V(\mathfrak{b})$, respectively. So it suffices to show $V(\mathfrak{a}) = V(\mathfrak{b})$, but this follows by Proposition 5.18. In particular, the topological space $\mathrm{Spec}(R/\mathfrak{a})$ does not depend on the choice of ideal of definition $\mathfrak{a}$. We denote this topological space by $\mathcal{X}$.

2. For each λ, we have a sheaf $\mathcal{I}_\lambda$ on $X = \mathrm{Spec}(R)$ associated to the ideal of definition $\mathfrak{a}_\lambda$ (Remark 3.34). We denote by $\mathcal{O}_\lambda$ the sheaf induced on $\mathcal{X}$ by $\mathcal{O}_X/\mathcal{I}_\lambda$. For each $\mathfrak{a}_\mu \subset \mathfrak{a}_\lambda$ tha canonical moprphism $R/\mathfrak{a}_\mu \to R/\mathfrak{a}_\lambda$ induces a morphism of sheaves of rings $u_{\mu\lambda} : \mathcal{O}_\mu \to \mathcal{O}_\lambda$. Then $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ forms an inverse system of sheaves on the topological space $\mathrm{Spec}(R/\mathfrak{a})$. Follow that $\varprojlim_{\lambda} \mathcal{O}_\lambda$ is a sheaf on $\mathrm{Spec}(R/\mathfrak{a})$ (see (HARTSHORNE, 1977, Proposition 9.2, ch. II)).

3. In particular, if R is $\mathfrak{a}$-adic, the collection $\{\mathfrak{a}^{n+1}\}_{n\geq 0}$ defined by an ideal of definition $\mathfrak{a}$ is cofinal, and thus the above sheaf coincides with the projective $\varprojlim_{n} \mathcal{O}_{\mathrm{Spec}(R/\mathfrak{a}^{n+1})}$.

With the previous comments, we are able to define the formal schemes.

Definition 5.28. Fix some ideal of definition $\mathfrak{a}$. The *formal spectrum* of a ring R (admissible) is the topological space $\mathcal{X} = \mathrm{Spec}(R/\mathfrak{a})$ together with the sheaf of rings $\mathcal{O}_{\mathcal{X}} := \varprojlim_{\lambda} \mathcal{O}_\lambda$, as defined above. We denote by $\mathrm{Spf}(R)$ the ringed space $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.

Next, we define the complete localization, that will be essential for the next results presented in this section.

Let R be an admissible ring, $\{\mathfrak{a}_\lambda\}_{\lambda\in\Lambda}$ a fundamental system of ideals of definition, and $S \subseteq R$ a multiplicative subset. Consider the localization $S^{-1}R$ in the multiplicative set S endowed with the topology defined by $\{S^{-1}\mathfrak{a}_\lambda\}_{\lambda\in\Lambda}$. Let $R_{\{S\}}$ denote the Hausdorff completion of the ring $S^{-1}R$:

$$R_{\{S\}} = \varprojlim_{\lambda\in\Lambda} S^{-1}R/S^{-1}\mathfrak{a}_\lambda.$$

We call $R_{\{S\}}$ as the *complete localization of R with respect to S.*

Proposition 5.29. (GROTHENDIECK; DIEUDONNÉ, 1971, (**I**, 10.1.3) and (**I**, 10.1.4)) Suppose R is an admissible ring. Let $\mathcal{X} = \mathrm{Spf}(R)$ and for every $f \in R$, let $\mathcal{D}(f) = D(f) \cap \mathcal{X}$, then

(i) $\mathcal{O}_{\mathcal{X}}|_{\mathcal{D}(f)}(\mathcal{D}(f)) \cong R_{\{f\}}$

(ii) $\mathcal{O}_{\mathcal{X}}(\mathcal{X}) \cong R$.

Note that the sheaf of ring $\mathcal{O}_{\mathcal{X}}$ of $\mathrm{Spf}(R)$ admits stalk for each $x \in \mathcal{X}$. By the proposition above, $\mathcal{O}_{\mathcal{X},\mathfrak{p}_x}$ can be identified with $\varinjlim_{f \notin \mathfrak{p}} R_{\{f\}}$.

Proposition 5.30. (GROTHENDIECK; DIEUDONNÉ, 1971, (**I**, 10.1.6)) For every open prime ideal $\mathfrak{p} \in \mathcal{X} = \mathrm{Spf}(R)$, the stalk $\mathcal{O}_{\mathcal{X},\mathfrak{p}}$ is a local ring whose residue field is isomorphic to $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.

Notice that a prime ideal $\mathfrak{p} \subseteq R$ is open if and only if it contains at least one (hence all) ideals of definition (see Proposition 5.18). Hence, the formal spectrum $\mathrm{Spf}(R)$ is a locally ringed space (Proposition 5.30) with the underlying set consisting of all open prime ideals of R.

Definition 5.31. An *affine formal scheme* is a topologically ringed space $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ that is isomorphic to $\mathrm{Spf}(R)$ for some admissible ring R. A *formal scheme* is a topologically locally ringed space that is locally isomorphic to an affine formal scheme.

An *open formal subscheme* of a formal scheme $\mathcal{X}$ is a formal scheme of the form $(U, \mathcal{O}_{\mathcal{X}}|_U)$, where U is an open subset of the underlying topological space of $\mathcal{X}$. An open formal subscheme $U \subseteq \mathcal{X}$ is said to be *affine open* if it is an affine formal scheme. Thus any formal scheme $\mathcal{X}$ allows an open covering $\mathcal{X} = \bigcup_\alpha U_\alpha$ consisting of affine open formal subschemes; an open covering of this form is called an *affine (open) covering.*

Just as in the case of ordinary schemes, any open of a formal scheme is a formal scheme (see (GROTHENDIECK; DIEUDONNÉ, 1971, (**I**, 10.4.4))).

Example 5.32. Let X be a scheme and $Y \subset X$ a closed subscheme, defined by a quasicoherent ideal $\mathcal{I} \subset \mathcal{O}_X$. Then consider the sheaf $\mathcal{O}_Y$ obtained by restricting the projective $\varprojlim_n \mathcal{O}_X/\mathcal{I}^n$ to Y. It follows that $(Y, \mathcal{O}_Y)$ is a locally topologically ringed space, the desired *formal completion of X along Y*. Locally, the construction looks as follows: Let $X = \mathrm{Spec}(R)$ and assume that $\mathcal{I}$ is associated to the ideal $\mathfrak{a} \subseteq R$. Then

$$(Y, \mathcal{O}_Y) = \mathrm{Spf}\left(\varprojlim_n R/\mathfrak{a}^n\right) = \mathrm{Spf}(\hat{R})$$

where $\hat{R}$ is the $\mathfrak{a}$-adic completion of R.

Definition 5.33. Given two formal schemes $\mathcal{X}$ and $\mathcal{Y}$, a *morphism (of formal schemes)* from $\mathcal{X}$ to $\mathcal{Y}$ is a morphism $(\Phi, \tilde{\Phi})$ of topologically ringed spaces such that, for all $x \in \mathcal{X}$, $\tilde{\Phi}^{\#}_x$ is a local homomorphism $\mathcal{O}_{\mathcal{Y},\Phi(x)} \longrightarrow \mathcal{O}_{\mathcal{X},x}$.

Remark 5.34. As in the case of ordinary schemes, the functor

$$R \longmapsto \mathrm{Spf}(R)$$

gives rise to a categorical equivalence between the opposite category of the category of admissible rings and the category of affine formal schemes (GROTHENDIECK; DIEUDONNÉ, 1971, (**I**, 10.2)).

With this new notion, it is natural to ask the behaviour of the gluing of formal schemes. This is the main focus of the rest of this section.

Proposition 5.35. Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be affine formal schemes, such that $\tilde{\alpha} : \mathcal{O}_{\mathcal{X}} \longrightarrow \alpha_*\mathcal{O}_{\mathcal{Z}}$ and $\tilde{\beta} : \mathcal{O}_{\mathcal{Y}} \longrightarrow \beta_*\mathcal{O}_{\mathcal{Z}}$ are both surjective homomorphisms. Then, $\mathcal{X} \sqcup_{\mathcal{Z}} \mathcal{Y}$ is an affine formal scheme.

Proof. By Theorem 5.23, the category of admissible rings admits pullback. In addition, since that the category of affine formal schemes is the dual category of admissible rings (Remark 5.34), one has

$$\mathcal{X} \sqcup_{\mathcal{Z}} \mathcal{Y} = \operatorname{Spf}(\widehat{R \times_T S}).$$ □

Theorem 5.36. Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be formal schemes such that $\alpha : \mathcal{Z} \longrightarrow \mathcal{X}$ and $\beta : \mathcal{Z} \longrightarrow \mathcal{Y}$ are homeomorphism onto its image. Now suppose following conditions are satisfied:

(i) for each formal affine open U_i and V_j of $\mathcal{X}$ and $\mathcal{Y}$, respectively, we have $\alpha^{-1}(U_i) \subseteq \mathcal{Z}$ and $\beta^{-1}(V_j) \subseteq \mathcal{Z}$ are formal affine open subsets.

(ii) $\tilde{\alpha} : \mathcal{O}_{\mathcal{X}} \longrightarrow \alpha_*\mathcal{O}_{\mathcal{Z}}$ and $\tilde{\beta} : \mathcal{O}_{\mathcal{Y}} \longrightarrow \beta_*\mathcal{O}_{\mathcal{Z}}$ are both surjective homomorphisms.

Then, $\mathcal{X} \sqcup_{\mathcal{Z}} \mathcal{Y}$ is a formal scheme.

Proof. In order to give the structure sheaf on $\mathcal{W} := \mathcal{X} \sqcup_{\mathcal{Z}} \mathcal{Y}$, one canonical way is the following. First we define a presheaf $\mathcal{F}$ on $\mathcal{W}$ in the following way:

(i) For $p \in \mathcal{W} \setminus \mathcal{Y}$, put $\mathcal{F}_p := \mathcal{O}_{\mathcal{X},p}$.

(ii) For $p \in \mathcal{W} \setminus \mathcal{X}$, put $\mathcal{F}_p := \mathcal{O}_{\mathcal{Y},p}$.

(iii) For $p = \alpha(z) = \beta(z)$, put $\mathcal{F}_p := \mathcal{O}_{\mathcal{X},\alpha(z)} \times_{\mathcal{O}_{\mathcal{Z},z}} \mathcal{O}_{\mathcal{Y},\beta(z)}$.

Now, for an open subset $U \subset \mathcal{W}$ (an open subset in $\mathcal{W}$ is defined as the subset whose pull-back to $\mathcal{X} \sqcup_{\mathcal{Z}} \mathcal{Y}$ is an open subset), we define

$$\mathcal{F}(U) := \prod_{p \in U} \mathcal{F}_p.$$

For open subsets $V \subset U$ of $\mathcal{W}$, define the restriction map $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ in the canonical way. Then $\mathcal{F}$ be comes a presheaf on W. Let $\mathcal{O}_{\mathcal{W}}$ be the sheafication $\mathcal{F}$.

We cover $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ by affine formal schemes respectively. In fact, similarly to Theorem 4.2, we get the open subsets $U_i \sqcup_{W_{i,j}} V_j$ of $\mathcal{X} \sqcup_{\mathcal{Z}} \mathcal{Y}$ that correspond bijectively to open subsets U_i and V_j of the $\mathcal{X}$ and $\mathcal{Y}$, respectively, where $W_{i,j} := \alpha^{-1}(U_i) = \beta^{-1}(V_j)$. Note that $W_{i,j}$ is an open affine, and their union covers the scheme $\mathcal{Z}$, because the maps $\alpha : \mathcal{Z} \to \mathcal{X}$ and $\beta : \mathcal{Z} \to \mathcal{Y}$ are homeomorphism onto its image there. Thus, since $\mathcal{X} = \cup_i U_i$, $\mathcal{Y} = \cup_j V_j$ and $\mathcal{Z} = \cup_{i,j} W_{i,j}$ are such that $U_i = \mathrm{Spf}(R_i)$, $V_j = \mathrm{Spf}(S_j)$ and $W_{i,j} = \mathrm{Spf}(T_{i,j})$, where R_i, S_j, $T_{i,j}$ are admissible rings. So, we have a cover $\mathcal{X} \sqcup_{\mathcal{Z}} \mathcal{Y} = \cup_{i,j}(U_i \sqcup_{W_{i,j}} V_j)$, i.e., it is an union of open subsets such that

$$\begin{aligned} U_i \sqcup_{W_{i,j}} V_j &\cong \mathrm{Spf}(R_i) \sqcup_{\mathrm{Spf}(T_{i,j})} \mathrm{Spf}(S_j) \\ &\cong \mathrm{Spf}(R_i \widehat{\times_{T_{i,j}}} S_j), \quad \text{by Theorem 5.23.} \end{aligned}$$

Therefore, the desired result follows by Proposition 5.35. □

A formal scheme $\mathcal{X}$ is *adic* if there exists a cover of $\mathcal{X}$ by affine open formal schemes $U_i = \mathrm{Spf}(R_i)$ where R_i is adic. A formal scheme $\mathcal{X}$ is *locally Noetherian* if R_i are Noetherian and adic. Also, $\mathcal{X}$ is *Noetherian* if it is locally Noetherian and quasi-compact.

Proposition 5.37. (GROTHENDIECK; DIEUDONNÉ, 1971, (**0**,10.14.1)) Let $\mathcal{X}$ be a locally Noetherian formal scheme, and let $\mathcal{I}$ be a coherent sheaf of ideals of $\mathcal{O}_{\mathcal{X}}$. If we put[2] $\mathcal{Y} := \mathrm{Supp}(\mathcal{O}_{\mathcal{X}}/\mathcal{I})$ then $\mathcal{Y}$ is a closed subset and the topologically ringed space $(\mathcal{Y}, (\mathcal{O}_{\mathcal{X}}/\mathcal{I})|_{\mathcal{Y}})$ is a locally Noetherian formal scheme, which is Noetherian provided $\mathcal{X}$ is.

Definition 5.38. A *closed subscheme* of a locally Noetherian formal scheme $\mathcal{X}$ is any formal scheme $(\mathcal{Y}, (\mathcal{O}_{\mathcal{X}}/\mathcal{I})|_{\mathcal{Y}})$, where $\mathcal{I}$ is a coherent ideal of $\mathcal{O}_{\mathcal{X}}$. This scheme is called the closed subscheme defined by $\mathcal{I}$.

Remark 5.39. Analogously to the case of schemes, given $\mathcal{X}$ a Noetherian formal scheme there exists a bijective correspondence between coherent ideals $\mathcal{I}$ of $\mathcal{O}_{\mathcal{X}}$ and closed subschemes $\mathcal{X}' \hookrightarrow \mathcal{X}$ given by $\mathcal{X}' = \mathrm{Supp}(\mathcal{O}_{\mathcal{X}}/\mathcal{I})$ and $\mathcal{O}_{\mathcal{X}} = (\mathcal{O}_{\mathcal{X}'}/\mathcal{I})|_{\mathcal{X}'}$.

In particular, put $\mathcal{X} = \mathrm{Spf}(R)$ where R is a J-adic Noetherian ring. Given $I \subseteq R$ an ideal the ring R/I is $J(R/I)$-adic and $\mathcal{X}' = \mathrm{Spf}(R/I)$ is a closed subscheme of $\mathcal{X}$.

Furthermore, we have:

Lemma 5.40. (GROTHENDIECK; DIEUDONNÉ, 1971, (**I**, 10.14.4)) Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of locally Noetherian formal schemes, and let (U_i) be a cover of $f(\mathcal{Y})$ by Noetherian formal affine open subsets of $\mathcal{X}$ such that the $f^{-1}(U_i)$ are Noetherian formal

[2] (GROTHENDIECK; DIEUDONNÉ, 1971, (**0**, 3.1.5)) If $\mathcal{F}$ is a sheaf on a topological space X, the support of $\mathcal{F}$ is

$$\mathrm{Supp}(\mathcal{F}) := \{x \in X \text{ such that } \mathcal{F}_x \neq 0\}.$$

affine open subsets of $\mathcal{Y}$. For f to be a closed immersion, it is necessary and sufficient for $f(\mathcal{Y})$ to be a closed subset of $\mathcal{X}$ and, for all i, for the restriction of f to $f^{-1}(U_i)$ to correspond to a surjective homomorphism $\mathcal{O}_{\mathcal{X}}(U_i) \to \mathcal{O}_{\mathcal{Y}}(f^{-1}(U_i))$.

For the next result we need the following Proposition.

Proposition 5.41. (GROTHENDIECK; DIEUDONNÉ, 1971, (**I**, 10.6.5)) Let R be an admissible ring. Then $\mathrm{Spf}(R)$ is Noetherian if and only if R is Noetherian and adic.

With the previous information, we are able to show the following result.

Theorem 5.42. Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be Noetherian formal scheme such that $\alpha : \mathcal{Z} \to \mathcal{X}$ and $\beta : \mathcal{Z} \to \mathcal{Y}$ are closed immersion. Suppose also that for each Noetherian formal affine open U and V that cover $\mathcal{X}$ and $\mathcal{Y}$, respectively, we have $\alpha^{-1}(U) \subseteq \mathcal{Z}$ and $\beta^{-1}(V) \subseteq \mathcal{Z}$ are Noetherian formal affine open subsets. Then, $\mathcal{X} \sqcup_{\mathcal{Z}} \mathcal{Y}$ is Noetherian formal scheme.

Proof. By Theorem 5.36, since the gluing of formal schemes is also a scheme, for each i, j it sufficient to show that $\mathcal{O}_{\mathcal{X}\sqcup_{\mathcal{Z}}\mathcal{Y}}(U_i \sqcup_{W_{i,j}} V_j)$ is a Noetherian adic ring. By hypothesis $W_{i,j} := \alpha^{-1}(U_i) = \beta^{-1}(V_j)$ is Noetherian formal affine open subsets of $\mathcal{Z}$. Let's say $W_{i,j} = \mathrm{Spf}(T_{i,j})$, then by Propositon 5.41 follow that $T_{i,j}$ is Noetherian adic ring as well as R_i and S_j. By Proposition 2.16 and Lemma 5.26 one obtains that $R_i \widehat{\times_{T_{i,j}}} S_j$ is Noetherian adic ring. Since

$$U_i \sqcup_{W_{i,j}} V_j \cong \mathrm{Spf}(R_i \widehat{\times_{T_{i,j}}} S_j),$$

the desired conclusion follows. □

5.2.1 The gluing of k-formal schemes

Definition 5.43. Let k be an arbitrary field, and let $\mathcal{X} \to \mathrm{Spf}(k)$ be a formal k-scheme. We call $\mathcal{X}$ a k-formal scheme *locally of finite type* or that $\mathcal{X}$ is locally of finite type over k, if there is an affine formal open cover $\mathcal{X} = \cup_{i\in I} U_i$ such that for all i, $U_i = \mathrm{Spf}(R_i)$, R_i is isomorphic to a quotient of a power series ring over the field with a suitable ideal. We say that $\mathcal{X}$ is of *finite type over k* if $\mathcal{X}$ is locally of finite type and quasi-compact.

Question 5.44. Is the gluing of k-formal schemes (finite type or locally of finite type) a k-formal scheme (finite type or locally of finite type)?

Let $\mathcal{X} = \mathrm{Spf}(R)$, $\mathcal{Y} = \mathrm{Spf}(S)$ and $\mathcal{Z} = \mathrm{Spf}(T)$, where $(R, \mathfrak{m})$, $(S, \mathfrak{n})$ and $(T, \mathfrak{t})$ are $\mathfrak{m}$, $\mathfrak{n}$ and $\mathfrak{t}$-adic local Noetherian rings (equicharacteristic) with the same residue field k. In addition, assume that $R \xrightarrow{\pi_R} T \xleftarrow{\pi_S} S$ are surjective homomorphisms of rings.

Corollary 5.45. Consider $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ as above. Then, $\mathcal{X} \sqcup_{\mathcal{Z}} \mathcal{Y}$ is an k-affine formal scheme of finite type.

Proof. By Proposition 5.35, one has $\mathcal{X} \sqcup_{\mathcal{Z}} \mathcal{Y} = \mathrm{Spf}(\widehat{R \times_T S})$. The assumption that the maps $R \xrightarrow{\pi_R} T \xleftarrow{\pi_S} S$ are surjective homomorphisms of Noetherian local rings gives that $R \times_T S$ is Noetherian and local ring with maximal ideal $\mathfrak{m} \times_{\mathfrak{t}} \mathfrak{n}$ (ANANTHNARAYAN; AVRAMOV; MOORE, 2012, Lemma 1.2) (or (ENDO; GOTO; ISOBE, 2021, Lemma 2.1)). Also, since $\widehat{R \times_T S}$ is the $\mathfrak{m} \times_{\mathfrak{t}} \mathfrak{n}$-adic completion of $R \times_T S$, the Cohen-Structure Theorem provides that $\widehat{R \times_T S}$ is isomorphic to a quotient of a power series ring over the field k with a suitable ideal. $\square$

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