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Hellen Monção de Carvalho Santana

Euler obstruction and generalizations

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Obstrução de Euler e generalizações

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutora em Ciências – Matemática. *VERSÃO REVISADA*

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À minha avó Idalina e ao meu esposo Paulo.

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“I could have missed the pain But I’d have had to miss the dance.”

Tony Arata

RESUMO

SANTANA, H. M. C. **Obstrução de Euler e generalizações**. 2020. 99 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2020.

Sejam $f, g : (X, 0) \to (\mathbb{C}, 0)$ germes de função analítica definidos sobre um espaço analítico complexo X. O número de Brasselet de uma função f descreve numericamente a topologia de sua fibra de Milnor generalizada. Neste trabalho, apresentamos fórmulas que comparam os números de Brasselet de f em X e de f restrita a $X \cap \{g = 0\}$ no caso em que g possui conjunto crítico estratificado de dimensão um. Se, adicionalmente, f possui singularidade isolada na origem, calculamos o número de Brasselet de g em X e o comparamos com o número de Brasselet de f em X. Como consequência, obtemos fórmulas para calcular a obstrução local de Euler de X e de $X \cap \{g = 0\}$ na origem, comparando esses números com invariantes locais associados a f e a g. Estudamos ainda a topologia local de uma deformação de g, $\tilde{g} = g + f^N$, para um número natural $N \gg 1$. Apresentamos uma relação entre os números de Brasselet de g e $\tilde{g}$ em $X \cap \{f = 0\}$, no caso em que f possui singularidade isolada na origem. Apresentamos também uma nova demonstração para a fórmula de Lê-Iomdine para o número de Brasselet.

Palavras-chave: Obstrução de Euler, número de Brasselet, pontos críticos de Morse estratificados, invariantes locais topológicos.

ABSTRACT

SANTANA, H. M. C. **Euler obstruction and generalizations**. 2020. 99 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2020.

Let $f,g:(X,0)\to(\mathbb{C},0)$ be germs of analytic functions defined over a complex analytic space X. The Brasselet number of a function f describes numerically the topology of its generalized Milnor fibre. In this thesis, we present formulas to compare the Brasselet numbers of f in X and of the restriction of f to $X\cap\{g=0\}$, in the case where g has a one-dimensional stratified critical set and f has an arbitrary critical set. If, additionally, f has isolated singularity at the origin, we compute the Brasselet number of g in X and compare it with the Brasselet number of f in X. As a consequence, we obtain formulas to compute the local Euler obstruction of X and of $X\cap\{g=0\}$ at the origin, comparing these numbers with local invariants associated to f and g. We also study the local topology of a deformation of g, $\tilde{g}=g+f^N$, for a positive integer number $N\gg 1$. We provide a relation between the Brasselet number of g and $\tilde{g}$ in $X\cap\{f=0\}$, in the case where f has isolated singularity at the origin. We also provide a new proof for the Lê-Iomdine formula for the Brasselet number.

Keywords: Euler obstruction, Brasselet number, Stratified Morse critical points, local topological invariants.

CONTENTS

INTRODUCTION

Let $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be an analytic function defined in a neighborhood of the origin and Σf the critical locus of f. Milnor studied the set $f^{-1}(\delta)\cap B_\varepsilon$, denoted by $F_{f,0}$ and later called Milnor fiber, where δ is a regular value of $f, 0<|\delta|\ll\varepsilon\ll 1$. In (MILNOR, 1968), Milnor proved that, if f has an isolated singularity, $F_{f,0}$ has the homotopy type of a wedge of $\mu(f)$ spheres of dimension $n-1$, where $\mu(f)$ is the Milnor number of f. This number also gives an important geometric information associated to the function f, which is the number of Morse points in a Morsification of f in a neighborhood of the origin.

In (HAMM, 1971), Hamm generalized Milnor's results for complete intersections with isolated singularity $F=(f_1,\dots,f_k):(\mathbb{C}^n,0)\to(\mathbb{C}^k,0), 1<k<n$, proving that the generalized Milnor fiber $F^{-1}(\delta)\cap B_\varepsilon, 0<|\delta|\ll\varepsilon\ll 1$, has the homotopy type of a wedge of $\mu(F)$ spheres of dimension $n-k$. In this context, Lê (LÊ, 1973) and Greuel (GREUEL, 1975) proved that $\mu(F)+\mu(F')=\dim_{\mathbb{C}}(\frac{\mathscr{O}_{\mathbb{C}^n,0}}{I})$, where $F':(\mathbb{C}^n,0)\to(\mathbb{C}^{k-1},0)$ is the map with components $f_1,\dots,f_{k-1}$ and I is the ideal generated by $f_1,\dots,f_{k-1}$ and the $(k\times k)$-minors $\frac{\partial(f_1,\dots,f_k)}{\partial(x_{i_1},\dots,x_{i_k})}$. Notice that the number $\dim_{\mathbb{C}}(\frac{\mathscr{O}_{\mathbb{C}^n,0}}{I})$ is the number of critical points of a Morsification of f_k appearing on the Milnor fibre of F'.

If f is defined over a complex analytic space X, the singular part of X should be considered in the study of the sets $X\cap f^{-1}(\delta)\cap B_\varepsilon$ and Σf, where $0<|\delta|\ll\varepsilon\ll 1$. A way to obtain a numerical information about the singular locus of X is using the local Euler obstruction, a singular invariant introduced by MacPherson, in (MACPHERSON, 1974), where he proved the Deligne-Grothendieck conjecture about characteristic classes of singular varieties. If the function f has an isolated singularity at the origin, a generalization for the Milnor number is the Euler obstruction of the function f, which is also a generalization of the local Euler obstruction and was introduced in (BRASSELET *et al.*, 2004), by Brasselet, Massey, Parameswaran and Seade. In (SEADE; TIBĂR; VERJOVSKY, 2005), Seade, Tibăr and Verjovsky proved that, up to sign, this number is the number of Morse critical points of a stratified Morsification of f appearing in the regular part of X in a neighborhood of the origin.

In a more general context, if f is defined over a complex analytic germ $(X,0)$ equipped with a good stratification $\mathscr{V}$ for f (see Definition 1.8.1) and the function f does not have isolated singularity at the origin, a way to describe the generalized Milnor fiber $X\cap f^{-1}(\delta)\cap B_\varepsilon$ is to use another generalization of the local Euler obstruction, the Brasselet number of f at the origin, $B_{f,X}(0)$, introduced by Dutertre and Grulha in (DUTERTRE; GRULHA, 2014). In that paper, the authors presented a Lê-Greuel type formula for the Brasselet number: if $g:X\to\mathbb{C}$ is prepolar with respect to $\mathscr{V}$ at the origin (see Definition 1.8.10) and $0<|\delta|\ll\varepsilon\ll 1$, then

$$B_{f,X}(0)-B_{f,X^g}(0)=(-1)^{d-1}n_q,$$

where n_q is the number of Morse critical points of a partial Morsification of $g|_{X\cap f^{-1}(\delta)\cap B_\varepsilon}$ appearing in the regular part of X, and $X^g=X\cap\{g=0\}$.

They also proved several results about the topology of functions with isolated singularity defined over an analytic complex Whitney stratified variety X. If X is equidimensional, let $f,g: X\to\mathbb{C}$ be analytic functions with isolated singularity at the origin, such that g is prepolar with respect to the good stratification induced by f at the origin (see Example 1.8.2) and f is prepolar with respect to the good stratification induced by g at the origin, then $B_{f,X^g}(0)=B_{g,X^f}(0)$, where $X^f=X\cap\{f=0\}$. Also, if n_q is the number of Morse critical points of a Morsification of $g|_{X\cap f^{-1}(\delta)\cap B_\varepsilon}$ appearing in the regular part of X and m_q is the number of Morse critical points of a Morsification of $f|_{X\cap g^{-1}(\delta)\cap B_\varepsilon}$ appearing in the regular part of X, for $0<|\delta|\ll\varepsilon\ll 1$, then

$$B_{f,X}(0)-B_{g,X}(0)=(-1)^{d-1}(n_q-m_q).$$

An interesting consequence of this last statement is a way to compare the local Euler obstruction $Eu_{X^f}(0)$ and the Brasselet number $B_{f,X\cap H}(0)$, given by the equality $Eu_{X^f}(0)=B_{f,X\cap H}(0)$, where H is a generic hyperplane passing through the origin.

In this work, we start considering, in Chapter 2, two function-germs $f,g:X\to\mathbb{C}$ and a good stratification $\mathscr{V}$ of X relative to f. We suppose that the critical locus $\Sigma_{\mathscr{V}}g$ of g is one-dimensional, that $\Sigma_{\mathscr{V}}g\cap\{f=0\}=\{0\}$ and we denote by $\mathscr{V}^f$ the collection of strata of $\mathscr{V}$ contained in $\{f=0\}$ and by $V_1,\ldots,V_q$ the strata of $\mathscr{V}$ not contained in $\{f=0\}$. Then, we prove (Lemma 2.1.1) that the refinement

$$\mathscr{V}'=\Big\{V_i\setminus\Sigma_{\mathscr{V}}g, V_i\cap\Sigma_{\mathscr{V}}g, i\in\{1,\ldots,q\}\Big\}\cup\mathscr{V}^f \tag{1}$$

is a good stratification of X relative to f and $\mathscr{V}'^{\{g=0\}}$ is a good stratification of $X\cap\{g=0\}$ relative to $f|_{X\cap\{g=0\}}$, where

$$\mathscr{V}'^{\{g=0\}}=\Big\{V_i\cap\{g=0\}\setminus\Sigma_{\mathscr{V}}g, V_i\cap\Sigma_{\mathscr{V}}g, i\in\{1,\ldots,q\}\Big\}\cup\Big(\mathscr{V}^f\cap\{g=0\}\Big)$$

and $\mathscr{V}^f\cap\{g=0\}$ denotes the collection of strata of type $V^f\cap\{g=0\}$, with $V^f\in\mathscr{V}^f$.

We write $\Sigma_{\mathscr{V}}g$ as a union of irreducible components (branches) $\Sigma_{\mathscr{V}}g=b_1\cup\ldots\cup b_r$, where $b_j\subseteq V_{i_j}$, for some $i_j\in\{1,\ldots,q\}$ and we take a regular value δ of $f, 0<|\delta|\ll 1$, and, for each $j\in\{1,\ldots,r\}$, we set $f^{-1}(\delta)\cap b_j=\{x_{i_1},\ldots,x_{i_{k(j)}}\}$. So, in this case, the local degree m_{f,b_j} of $f|_{b_j}$ is k. Let ε be sufficiently small such that the local Euler obstruction of X and X^g are constant on $b_j\cap B_\varepsilon$. In this case, we denote by $Eu_X(b_j)$ (respectively, $Eu_{X^g}(b_j)$) the local Euler obstruction of X (respectively, X^g) at a point of $b_j\cap B_\varepsilon$. If g is tractable at the origin with respect to $\mathscr{V}$ relative to f (see Definition 1.8.12) and $0 < |\delta| \ll \varepsilon \ll 1$, we prove (Theorem 2.1.2) that

$$B_{f,X}(0) - B_{f,X^g}(0) - m_{f,b_j}\sum_{j=1}^{r}(Eu_X(b_j) - Eu_{X^g}(b_j)) = (-1)^{d-1}m,$$

where m is the number of stratified Morse critical points of a partial Morsification of $g : X \cap f^{-1}(\delta) \cap B_\varepsilon \to \mathbb{C}$ appearing on $X_{reg} \cap f^{-1}(\delta) \cap \{g \neq 0\} \cap B_\varepsilon$.

We conclude that, in the case where g is not prepolar with respect to $\mathscr{V}$ relative to f, the Lê-Greuel type formula for the Brasselet number presents a type of defect. More precisely, this formula shows us that the number of Morse critical points m on the regular part of X does not contain all the topological information given by the difference $B_{f,X}(0) - B_{f,X^g}(0)$.

Then, after that, we suppose that f has an isolated singularity at the origin and we consider a Whitney stratification $\mathscr{W}$ of X. Let $\mathscr{V}$ be the good stratification of X induced by f and let us suppose that g is tractable at the origin with respect to $\mathscr{V}$ relative to f. We prove (Lemma 2.2.2) that the refinement $\mathscr{V}''$ of $\mathscr{V}$,

$$\mathscr{V}'' = \Big\{V_i \setminus \{g = 0\}, V_i \cap \{g = 0\} \setminus \Sigma_{\mathscr{W}} g, V_i \cap \Sigma_{\mathscr{W}} g, V_i \in \mathscr{V}\Big\} \cup \{0\} \tag{2}$$

is a good stratification of X relative to g such that $\mathscr{V}''^{\{f=0\}} = \{V_i'' \cap \{f = 0\}, V_i'' \in \mathscr{V}''\}$ is a good stratification of X^f relative to $g|_{X^f}$.

Using this stratification, we prove (Corollary 2.2.11) that,

$$B_{g,X^f}(0) = B_{f,X^g}(0) - \sum_{j=1}^{r} m_{f,b_j}(Eu_{X^g}(b_j) - B_{g,X\cap\{f=\delta\}}(b_j)),$$

where $B_{g,X\cap\{f=\delta\}}(b_j)$ is the Brasselet number of g at a point $x_{j_s} \in b_j \cap \{f = \delta\}$.

As a consequence of this result, we obtain a way to compare the local Euler obstruction $Eu_{X^g}(0)$ and the Brasselet number $B_{g,X\cap H}(0)$ in the case where g has a one-dimensional critical locus. Let l be a generic linear form over $\mathbb{C}^n$ and $H = l^{-1}(0)$. We prove (Corollary 2.2.13) that:

$$B_{g,X\cap H}(0) = Eu_{X^g}(0) - \sum_{j=1}^{r} m_{b_j}(Eu_{X^g}(b_j) - B_{g,X\cap l^{-1}(\delta)}(b_j)),$$

where m_{b_j} is the multiplicity of the branch b_j at the origin. In this same setting, we also prove (Corollary 2.2.22) that

$$B_{g,X}(0) - B_{f,X}(0) = (-1)^{d-1}(n_q - m_q) - \sum_{j=1}^{r} m_{f,b_j}(Eu_X(b_j) - B_{g,X\cap\{f=\delta\}}(b_j)).$$

For analytic functions defined over a nonsingular subspace of $(\mathbb{C}^n, 0)$ and with a s-dimensional singular locus, $s \geq 1$, Massey generalized the Milnor number with the Lê numbers (Definition 3.1.5), in (MASSEY, 1990). In this context, Massey provided (Theorem 4.3 of (MASSEY, 1988)) a (handle) decomposition of the Milnor fibre of f, where the number of

n-cells attached at each (dimensional) level is an appropriate Lê number. If we take a generic linear form l over $\mathbb{C}^n$, Iomdin (IOMDIN, 1974a), with an algebraic approach, and Lê (LÊ, 1980), with a geometric approach, proved that, for $s = 1$, it is possible to compare the Milnor fibres of the analytic function f with a one-dimensional critical set and of the function $f + l^N$, for $N \gg 1$ sufficiently large. In (MASSEY, 2003), Massey compared Lê numbers associated to the analytic functions f and $f + al^N$, for N sufficiently large, $a \in \mathbb{C}$ and $s \geq 1$, and obtained several Lê-Iomdin formulas for the Lê numbers. In Chapter 3, we suppose that g has a one-dimensional critical set and that f has an isolated singularity at the origin and we consider the analytic function-germ $\tilde{g} = g + f^N, N \gg 1$. We compare the Brasselet numbers $B_{f,X^g}(0)$ and $B_{f,X^{\tilde{g}}}(0)$ and we obtain (Theorem 3.2.9)

$$B_{f,X^g}(0) - B_{f,X^{\tilde{g}}}(0) = \textstyle\sum_{j=1}^{r} m_{f,b_j}(Eu_{X^g}(b_j) - B_{g,X\cap f^{-1}(\delta)}(b_j)).$$

After, we compare the Brasselet numbers $B_{g,X}(0)$ and $B_{\tilde{g},X}(0)$, and we obtain (Theorem 3.3.4) a Lê-Iomdin formula for the Brasselet number. If $0 < |\alpha| \ll 1$, then

$$B_{\tilde{g},X}(0) = B_{g,X}(0) + N\sum_{j=1}^{r} m_{f,b_j} Eu_{f,X\cap\tilde{g}^{-1}(\alpha)}(b_j),$$

that generalizes the Lê-Iomdin formula for the Euler characteristic of the Milnor fibre to the case of a function with isolated singularity.

In Chapter 1 we present definitions and results about objects we will need to develop this work, like the local Euler obstruction and the Brasselet number. In Chapter 2 we present results about functions with arbitrary singularities and about functions with isolated singularity at the origin. Chapter 3 is devoted to results about the local topology of the deformation of g, $\tilde{g} = g + f^N$, where $N \gg 1$ is a positive integer number.

CHAPTER

1

PRELIMINARIES

This chapter presents general definitions and properties that we will need in the development of this work.

1.1 Complex analytic spaces

In this section we present definitions and some results about complex analytic spaces in two different ways. We begin with an algebraic approach, using sheaf theory, which will be useful to understand the constructions in the forward sections, and then we present the classical geometric approach for these spaces.

The main reference for the first part of this section is (GREUEL; LOSSEN; SHUSTIN, 2007).

Definition 1.1.1. Let X be a topological space. A **sheaf** $\mathscr{F}$ of rings over X, consists of the following:

1. For each open subset U of X, there exists a ring $\mathscr{F}(U)$, whose elements are called **sections** of $\mathscr{F}$ over U.

2. For each pair of open subsets $V \subseteq U$ of X, there exists a map, called **restriction map**, $r_U^V : \mathscr{F}(U) \to \mathscr{F}(V)$, which could be denoted by $r_U^V(s) = s_{|_V}$, satisfying:

   i) $r_U^U = id_{\mathscr{F}(U)}$, for each open subset U of X;

   ii) $r_V^W \circ r_U^V = r_U^W$, for each open subsets $W \subseteq V \subseteq U$ of X.

3. For each open subset U of X and each covering $\cup_{i\in I} U_i$ of U by open subsets of X, we have:

   i) for all $s_1, s_2 \in \mathscr{F}(U)$, if $s_{1|_{U_i}} = s_{2|_{U_i}}$ $\forall\, i \in I$, then $s_1 = s_2$;

ii) for each $i \in I$, consider $s_i \in \mathscr{F}(U_i)$. If ${s_i}_{|U_i \cap U_j} = {s_j}_{|U_i \cap U_j}$ for all $i, j \in I$, there exists $s \in \mathscr{F}(U)$ such that $s_{|U_i} = s_i$ for all $i \in I$.

If $\mathscr{F}$ satisfies conditions 1 and 2, $\mathscr{F}$ is called **presheaf** of rings over X.

Example 1.1.2. Considering the standard topology on $\mathbb{C}^n$, the sheaf of holomorphic functions $\mathscr{O}_{\mathbb{C}^n}$ over $\mathbb{C}^n$ is defined as the following: for each open subset $U \subseteq \mathbb{C}^n$, define $\mathscr{O}_{\mathbb{C}^n}(U) := \{f : U \to \mathbb{C} \mid f \text{ is } holomorphic\}$. For open subsets $V \subseteq U$, consider the usual restrictions $r_V^U : \mathscr{O}_{\mathbb{C}^n}(U) \to \mathscr{O}_{\mathbb{C}^n}(V)$, with $r_V^U(s) = s_{|V}$.

Let $\mathscr{F}$ be a sheaf over a topological space X and $a \in X$. Consider pairs (U, s), where U is an open subset of X containing a and $s \in \mathscr{F}(U)$. Over the set of pairs (U, s), define the relation $\sim$ as in the following: $(U, s) \sim (V, t)$ if there exists an open neighborhood W of a such that $a \in W \subset U \cap V$ and $s|_W = t|_W$ or, equivalently, $r_U^W(s) = r_V^W(t)$. This is an equivalence relation and its equivalence classes are denoted by $[U, s]_a$, for each $a \in X$.

Definition 1.1.3. The equivalence class $[U, s]_a$ is called the **germ** of $\mathscr{F}$ at the point a. The set $\mathscr{F}_a$ of all germs of $\mathscr{F}$ is called the **stalk** of $\mathscr{F}$ at a.

Example 1.1.4. The stalk $\mathscr{O}_{\mathbb{C}^n,x}$ of the sheaf $\mathscr{O}_{\mathbb{C}^n}$ in a point $x \in \mathbb{C}^n$ is the set of germs of holomorphic functions of $\mathbb{C}^n$ at x.

Remark 1.1.5. (For the detailed description see 2.1 of (MANIN, 2018)) Given a presheaf $\mathscr{F}$ defined over a topological space X, it is possible to construct a sheaf $\mathscr{F}^+$ over X called the **sheaf associated to** $\mathscr{F}$, such that $\mathscr{F}_x = \mathscr{F}_x^+$, for all $x \in X$.

If $\mathscr{A}$ is a sheaf of rings on X, a sheaf $\mathscr{F}$ on X is called a **sheaf of $\mathscr{A}$-modules** if $\mathscr{F}(U)$ is an $\mathscr{A}(U)$-module and the restriction maps r_V^U are morphisms of $\mathscr{A}(V)$-modules. Let $\mathscr{F}_1, \mathscr{F}_2$ be sheaves of $\mathscr{A}$-modules on X. $\mathscr{F}_1$ is a subsheaf of $\mathscr{F}_2$ if for each open subset $U \subset X$, $\mathscr{F}_1(U)$ is an $\mathscr{A}(U)$-submodule of $\mathscr{F}_2(U)$ and the restriction maps of $\mathscr{F}_1$ are induced by the ones of $\mathscr{F}_2$. Sheaves of $\mathscr{A}$-submodules of $\mathscr{A}$ are called **sheaves of ideals** in $\mathscr{A}$. The sheaf $\mathscr{F}$ of $\mathscr{A}$-modules is **free** if it is isomorphic to the direct sum $\oplus_{i \in I} \mathscr{A}_i$, where $\mathscr{A}_i$ is a sheaf of $\mathscr{A}$-module, for all $i \in I$, and $\oplus_{i \in I} \mathscr{A}_i$ denotes the sheaf associated to the presheaf that maps each open subset U of X to the $\mathscr{A}(U)$-module $\oplus_{i \in I} \mathscr{A}_i(U)$. If I is a finite set, the number of elements in I is the **rank** of $\mathscr{F}$. More generally, a sheaf of $\mathscr{A}$-modules $\mathscr{F}$ is **locally free of rank** k if for each $x \in X$ there exists a neighborhood U such that $\mathscr{F}(U)$ is a free $\mathscr{A}(U)$-module of rank k.

Our goal now is to define the preimage and the direct image sheaves. For that we will need the notion of direct limit. Let the objects of a category be sets with an algebraic structure (such as groups, rings, modules over a ring) and the morphisms the morphisms of these structures. Let $\{A_i, i \in I\}$ be a family of objects enumerated by elements of an ordered set of index I and for all $i \leq j$, consider a family of homomorphisms $f_{ij} : A_i \to A_j$ with the following properties:

1. $f_{ii} = id|_{A_i}$;
2. $f_{ik} = f_{jk} \circ f_{ij}$, for all $i \le j \le k$.

The family of pairs (A_i, f_{ij}) is called a **directed system over** I. The **direct limit** $\varinjlim A_i$ of the directed system (A_i, f_{ij}) is defined as the quotient of the disjoint union of the sets A_i modulo an equivalence relation $\sim$:

$$\varinjlim A_i = \sqcup_i A_i / \sim,$$

where if $x_i \in A_i$ and $x_j \in A_j, x_i \sim x_j$ if, and only if, $f_{ik}(x_i) = f_{jk}(x_j)$, for some $k \in I$.

If $f : X \to Y$ is a continuous map of topological spaces and $\mathscr{A}$ is a sheaf of rings on X, the **direct image** $f_*\mathscr{A}$ of $\mathscr{A}$ by f is the sheaf of rings associated to the presheaf that maps each open subset V of Y to the element $\mathscr{A}(f^{-1}(V))$. If $\mathscr{F}$ is a sheaf of $\mathscr{A}$-modules, the direct image $f_*\mathscr{F}$ of $\mathscr{F}$ by f is the sheaf of $f_*\mathscr{A}$-modules associated to the presheaf that maps each open subset V of Y to the $\mathscr{A}(f^{-1}(V))$-module $\mathscr{F}(f^{-1}(V))$.

Moreover, if $\mathscr{G}$ is a sheaf of rings on Y, the **topological preimage sheaf** $f^{-1}\mathscr{G}$ is the sheaf of rings associated to the presheaf that maps each open subset U of X to the ring $\varinjlim_{f(U) \subset V} \mathscr{G}(V)$, the limit being taken over all open sets $V \subset Y$ containing $f(U)$ (the order is the inclusion). If $i : X \hookrightarrow Y$ is the inclusion map of a subspace X of Y, then $\mathscr{G}|_X := i^{-1}(\mathscr{G})$ is called **topological restriction** of $\mathscr{G}$ to X.

If $\mathscr{A}_X$ and $\mathscr{A}_Y$ are sheaves of rings on X and on Y respectively, it is also possible to define the **analytic preimage sheaf** of a sheaf of $\mathscr{A}_Y$-modules $\mathscr{G}$ as the sheaf of $\mathscr{A}_X$-modules associated to the presheaf that associates to each open subset U of X, the $\mathscr{A}_X(U)$-module

$$f^*\mathscr{G}(U) := f^{-1}\mathscr{G}(U) \otimes_{f^{-1}\mathscr{A}_Y(U)} \mathscr{A}_X(U).$$

A pair $(X, \mathscr{A}_X)$ given by a topological space X and a sheaf of rings $\mathscr{A}_X$ is called a **ringed space**. A **morphism of ringed spaces** is a pair of maps $(f, f^\sharp) : (X, \mathscr{A}_X) \to (Y, \mathscr{A}_Y)$, where $f : X \to Y$ is a continuous map of topological spaces and, for each open subset $U \subset Y$, $f^\sharp|_U : \mathscr{A}_Y(U) \to f_*\mathscr{A}_X(U)$ is a homomorphism of rings, where $f_*\mathscr{A}_X$ denotes the direct image of $\mathscr{A}_X$ by f. If f is a homeomorphism and $f^\sharp|_U$ is an isomorphism of rings, for each open subset U of X, $(f, f^\sharp)$ is an **isomorphism of ringed spaces**.

Definition 1.1.6. Let $D \subset \mathbb{C}^n$ be an open subset and $\mathscr{O}_D$ the sheaf of holomorphic functions over D. An ideal sheaf $\mathscr{I} \subset \mathscr{O}_D$ is called of **finite type** if, for every point $p \in D$, there exist an open neighborhood U of p in D and holomorphic functions $f_1, \ldots, f_k \in \mathscr{O}(U)$ such that $\mathscr{I}(U) = \langle f_1 \ldots, f_k \rangle$.

For an ideal sheaf of finite type $\mathscr{I}$, we can define the sheaf $\frac{\mathscr{O}_D}{\mathscr{I}}$ of quotient rings on D as the sheaf associated to the presheaf that maps each open subset $U \subset D$ to the quotient ring $\frac{\mathscr{O}_D(U)}{\mathscr{I}(U)}$.

Definition 1.1.7. Let $\mathscr{I}$ be an ideal sheaf of finite type. The **analytic set in D defined by $\mathscr{I}$** is given by

$$V(\mathscr{I}) = \left\{ p \in D; \left(\frac{\mathscr{O}_D}{\mathscr{I}} \right)_p \neq 0 \right\}.$$

Notice that $\left(\frac{\mathscr{O}_D}{\mathscr{I}}\right)_p \neq 0$ if, and only if, $f(p) = 0$ for all $f \in \mathscr{I}_p$. Therefore, for an open neighborhood U of p, since $\mathscr{I}$ is of finite type, there exist $f_1, \dots, f_k \in \mathscr{O}(U)$ such that $\mathscr{I}_p = \langle f_1 \dots, f_k \rangle$ and then

$$V(\mathscr{I}) \cap U = V(f_1, \dots, f_k).$$

Let us see the definition that gives the local structure of a complex analytic space.

Definition 1.1.8. Let D be an open subset of $\mathbb{C}^n$ and $\mathscr{I}$ be an ideal of finite type in the sheaf of holomorphic functions $\mathscr{O}_D$. A **complex model space defined by $\mathscr{I}$** is the pair $(X, \mathscr{O}_X)$ given by a topological space $X = V(\mathscr{I}) \subset D$ and the sheaf of rings $\mathscr{O}_X = \left(\frac{\mathscr{O}_D}{\mathscr{I}}\right)|_X$, given by the topological restriction of $\frac{\mathscr{O}_D}{\mathscr{I}}$ to X.

Definition 1.1.9. A **complex analytic space** is a pair $(X, \mathscr{O}_X)$ given by a Hausdorff topological space X and a sheaf of rings $\mathscr{O}_X$ such that, for every $p \in X$, there exists a neighborhood U of p, such that $(U, \mathscr{O}_X|_U)$ is isomorphic to a complex model space (as ringed spaces).

Definition 1.1.10. A **closed complex analytic subspace** of a complex space $(X, \mathscr{O}_X)$ is a ringed space $(Y, \mathscr{O}_Y)$ given by an ideal sheaf of finite type $\mathscr{I}_Y \subset \mathscr{O}_X$ such that $Y = V(\mathscr{I}_Y)$ and $\mathscr{O}_Y = (\frac{\mathscr{O}_X}{\mathscr{I}_Y})|_Y$ is the topological restriction of $\frac{\mathscr{O}_X}{\mathscr{I}_Y}$ to Y.

Example 1.1.11. Let $f_1(x,y,z) = x^4 - y^3$ and $f_2(x,y,z) = x^5 - z^3$ be holomorphic function germs in $\mathscr{O}_{\mathbb{C}^3,0}$ and consider the sheaf of ideals of finite type $\mathscr{I}$ over $\mathscr{O}_{\mathbb{C}^3}$ that associates to each open set U of $\mathbb{C}^3$ the ideal $\mathscr{I}(U) = \langle f_1, f_2 \rangle \mathscr{O}_{\mathbb{C}^3}(U)$. Let

$$X = \left\{ p \in \mathbb{C}^3; \left(\frac{\mathscr{O}_{\mathbb{C}^3}}{\langle f_1, f_2 \rangle} \right)_p \neq 0 \right\} = \{ p \in \mathbb{C}^3; f_1(p) = f_2(p) = 0 \}.$$

Hence,

$$X = \{ (x,y,z) \in \mathbb{C}^3; x = t^3, y = t^4, z = t^5, t \in \mathbb{C} \}.$$

Over X define the sheaf $\mathscr{O}_X$ of rings associated to the presheaf that associates to each open set $U \cap X$ the quotient ring $\frac{\mathscr{O}_{\mathbb{C}^3}(X \cap U)}{\mathscr{I}(X \cap U)}$. Therefore, $(X, \mathscr{O}_X)$ is a complex analytic space.

We will often work on a neighborhood of a point x of a complex analytic space $(X, \mathscr{O}_X)$, so we will need the following notion.

Definition 1.1.12. A **complex analytic space germ** is a pair (X, x) given by a complex space $(X, \mathscr{O}_X)$ and a point $x \in X$. A morphism of complex space germs $f : (X, x) \to (Y, y)$ is a morphism of ringed spaces $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ such that $f(x) = y$, which will be called **holomorphic map germ**.

We are interested in a complex space $(X,\mathscr{O}_X)$ such that, for each $x \in X$, the stalk $\mathscr{O}_{X,x}$ of $\mathscr{O}_X$ at x is reduced, that is, it has no nilpotent elements. In this case, $(X,\mathscr{O}_X)$ is called **reduced** and a **holomorphic function germ** $f : (X,x) \to (\mathbb{C},y)$ is given by a restriction of a holomorphic function $F : U \subset \mathbb{C}^n \to \mathbb{C}$, where U is a neighborhood of x in $\mathbb{C}^n$ (see page 38 of (GREUEL; LOSSEN; SHUSTIN, 2007)).

From now on we will always consider reduced complex analytic spaces.

If $U \subset X$ is an open neighborhood of $x \in X$, the germ (U,x) is identified with the germ (X,x) and U is called a **representative** of (X,x). Similarly, if $f : (X,x) \to (Y,y)$ is a holomorphic map germ, $U \subset X$ and $V \subset Y$ are representatives of X and Y, respectively, and $f(U) \subset V$, $f : U \to V$ is called a representative of the map germ f.

Definition 1.1.13. Let X be a complex analytic space, A be a closed complex analytic subspace of X given by the ideal sheaf of finite type $\mathscr{I}_A$ and $x \in X$. Then (A,x) is called **irreducible** if the stalk $\mathscr{I}_{A_x} \subset \mathscr{O}_{X,x}$ of $\mathscr{I}_A$ at x is a prime ideal. Otherwise, (A,x) is called **reducible**.

Proposition 1.1.14. (Proposition 1.51 (GREUEL; LOSSEN; SHUSTIN, 2007)) Let X be a complex analytic space, $A \subset X$ a closed complex analytic subspace and $x \in X$. There exists a decomposition

$$(A,x) = (A_1,x) \cup \ldots \cup (A_r,x),$$

where $(A_1,x),\ldots,(A_r,x) \subset (X,x)$ are irreducible germs of analytic sets such that $(A_i,x) \not\subseteq (A_j,x)$ for $i \neq j$. This decomposition is unique up to permutation of (A_i,x).

Definition 1.1.15. Let X be a complex analytic space and $p \in X$. Then the **dimension** $\dim_p X$ of X at p is define by the Krull dimension of the local ring $\mathscr{O}_{X,p}$. Also, the dimension $\dim X$ of X is defined as

$$\dim X := sup\{\dim_p X; p \in X\}.$$

We present now a geometric approach for complex analytic spaces. The main reference for the last part of this section is (LOJASIEWICZ, 1991).

Let M be a complex manifold (see page 133 of (LOJASIEWICZ, 1991)).

Definition 1.1.16. A **globally analytic subset** of the manifold M is a set of the form

$$V(f_1,\ldots,f_k) = \{z \in M; f_1(z) = \cdots = f_k(z) = 0\},$$

where $f_1,\ldots,f_k$ are holomorphic functions on M.

Definition 1.1.17. Let a be a point of M. An **analytic germ at** a is the germ at a of a globally analytic subset of an open neighborhood of a.

Definition 1.1.18. A subset Z of complex manifold M is called an **analytic subset** of M if every point of the manifold M has an open neighborhood U such that the set $Z \cap U$ is a globally analytic subset of U. In particular, any closed submanifold of the manifold M is an analytic subset.

Analytic subsets of open subsets of the manifold M are called **locally analytic subsets** of M.

Definition 1.1.19. A **(complex) analytic space** is a topological Hausdorff space X with an analytic atlas, i.e., with a family of homeomorphisms $\varphi_i : G_i \to V_i$, where $\{G_i\}$ is an open cover of X and V_i are locally analytic subsets of $\mathbb{C}^{n_i}$ such that the mappings

$$\varphi_k \circ \varphi_i^{-1} : \varphi_i(G_i \cap G_k) \to \varphi_k(G_i \cap G_k)$$

are holomorphic.

An analytic space X is an n-**dimensional manifold** if its structure is induced by the structure of an n-dimensional manifold.

Remark 1.1.20. Definitions 1.1.9 and 1.1.19 are equivalent. See page 39 of (GREUEL; LOSSEN; SHUSTIN, 2007).

1.2 On the properties about complex algebraic sets

In this section, we will see some properties proved by Iomdin, in (IOMDIN, 1974b), about the structure of complex algebraic sets in a neighborhood of nonisolated singular points. We note that the statements of this section are also valid over complex analytic sets.

The following abbreviated notation will be used: a certain linear relation holds between the vectors $w_1, \ldots, w_k$ *mod* $v_1, \ldots, v_s$ if this relations holds for vectors $w_1, \ldots, w_k$ in the quotient space formed from $\mathbb{C}^m$ by the subspace spanned by the vectors $v_1, \ldots, v_s$, where $w_i, v_i \in \mathbb{C}^m$.

Let $0 \in V \subset Y \subset Y^* \subset \mathbb{C}^m$ be complex algebraic sets, $g_1, \ldots, g_s$ generators of the ideal $I(Y^*)$ and $g_1, \ldots, g_s, f_1, \ldots, f_r$ generators of $I(Y)$. Assume that $Y^\star \setminus Y$ is regular.

Put $\varphi = \sum ||f_j||^2$ and let $h = \sum_l ||h_l||^2$ be a function such that $h|_V$ has a zero at the coordinate origin and h_l are polynomials in $\mathbb{C}^m$.

Corollary 1.2.1. (Corollary 1.7 of (IOMDIN, 1974b)) There is an $\varepsilon > 0$ and a neighborhood G of the set $V \setminus \{0\}$ in $Y^\star$ such that at points z of $D_\varepsilon \cap G \setminus Y$, the vectors *grad* $h(z)$ and *grad* $\varphi(z)$ are complex linearly independent *mod grad* $g_1(z), \ldots,$ *grad* $g_s(z)$, where D_ε is the closed ball with center at the coordinate origin and radius ε.

In the particular case where V is the set of singular points of Y in a neighborhood of the origin and V is one-dimensional, Iomdin proved, in (IOMDIN, 1974a), several interesting properties of algebraic complex sets intersected by a generic hyperplane.

Suppose that l is a linear form in $\mathbb{C}^m$ such that $l|_V$ has an isolated zero at the origin. Since V is one-dimensional, for a sufficiently small $w \neq 0$, the set $l^{-1}(w) \cap V$ consists of n points $z^i(w), i = 1, \ldots, n$.

Lemma 1.2.2. (Lemma 2.1 of (IOMDIN, 1974a)) If $w \neq 0$ is sufficiently small, the variety $Y \cap l^{-1}(w)$ has an isolated singularity at each point $z^i(w), i = 1, \ldots, n$.

For each sufficiently small $w \neq 0$, there are defined n smooth manifolds $\hat{\Sigma}^t_z(w)$ given by the intersection of $Y \cap l^{-1}(w)$ with spheres of sufficiently small radii centered at $z^i(w)$.

Lemma 1.2.3. (Lemma 2.2 of (IOMDIN, 1974a)) The manifolds $\hat{\Sigma}^t_z(w_1)$ and $\hat{\Sigma}^t_z(w_2)$ are diffeomorphic if the points $z^i(w_1)$ and $z^i(w_2)$ belong to the same branch of V at zero.

1.3 Analytic cycles and intersections multiplicity

In (MASSEY, 2003), Massey described the topology of complex analytic singularities using varieties associated to these singularities. We will use one of his approaches about intersection multiplicity of analytic cycles to understand the topology of the intersection of a polar curve with a variety. In this section, we will present the definition and properties we will need.

Let $(X, \mathscr{O}_X)$ be a complex analytic space and $\{V^i\}$ the collection of irreducible components of X. Let p be a point of V^i. Choose one of the irreducible germ components $(V^i_p)_j$ of V^i at p and let I_p be the prime ideal of $\mathscr{O}_{X,p}$ associated to $(V^i_p)_j$. Denote by m_{V^i} the length of the ring $(\mathscr{O}_{X,p})_{I_p}$. Notice that m_{V^i} does not depend on the point p or on the component $(V^i_p)_j$ chosen.

Definition 1.3.1. The **analytic cycle** of X is given by the formal sum

$$[X] = \sum_V m_V [V].$$

Remark 1.3.2. Notice that, if $f, g \in \mathscr{O}_X$, then $[V(fg)] = [V(f)] + [V(g)]$ and $[V(f^m)] = m[V(f)]$.

Let us now give some definitions and properties about intersection of cycles.

Definition 1.3.3. Let V and W be irreducible analytic subspaces of a connected complex manifold M and Z be an irreducible component of $V \cap W$. If $codim_M Z = codim_M V + codim_M W$, then we say that V **intersects** W **properly along** Z. If V and W intersect properly along each component of $V \cap W$, then V and W are said to **intersect properly in** M and the **intersection product** $[V][W]$ is defined by $[V][W] = [V \cap W]$.

Two cycles $\sum m_i[V_i]$ and $\sum m_j[W_j]$ are said to intersect properly if V_i and W_j intersect properly for all i and j. If this is the case, the intersection product is given by

$$\sum m_i[Vi]\sum n_j[Wj] = \sum m_i n_j([V_i][W_j]) = \sum m_i n_j([V_i \cap W_j]).$$

Definition 1.3.4. Let C_1 and C_2 be two cycles that intersect properly and let $\{Z_k, k = 1,\dots,n\}$ the irreducible components of the intersection product of C_1 and C_2, that is, $C_1.C_2 = \sum_{i=1}^n n_k[Z_k]$. The coefficient n_k is called **intersection number** of C_1 and C_2 at Z_k, that is, n_k is the number of times Z_k occurs in the intersection.

Let us see a practical way, presented by Massey in A.9 of (MASSEY, 2003), to compute the intersection number defined above.

Remark 1.3.5. Let M be a complex manifold, α be a coherent sheaf of ideals (see page 128, (MANIN, 2018)) in $\mathscr{O}_M$ and let us denote by $V(\alpha)$ the analytic subspace defined by the vanishing of α. Given a point p in M, suppose that $W = V(\alpha)$ is a curve in M, which is reduced and irreducible at p, and consider a hypersurface $V(f) \subseteq M, f \in \mathscr{O}_M$, which intersects W properly at p, where $V(f)$ is the analytic subspace defined by the vanishing of f. Let $\varphi(t)$ be a local parametrization of W such that $\varphi(0) = p$. The intersection number of $[W]$ and $[V(f)]$ at p is given by $mult_t f(\varphi(t))$, the degree of the lowest nonzero term of $f(\varphi(t))$.

1.4 Module of Kähler differentials

We aim to introduce the relative Nash modification. For that we will need to define the module of Kähler differentials (see (GREUEL; LOSSEN; SHUSTIN, 2007) for details).

Definition 1.4.1. Let B be a ring, A be a B-algebra and M an A-module. A **B-derivation with values in M** is a B-linear map $\delta : A \to M$ satisfying the Leibniz rule,

$$\delta(fg) = \delta(f)g + f\delta(g), f, g \in A.$$

The set $Der_B(A,M) := \{\delta : A \to M; \delta \text{ is a } B\text{-derivation}\} \subset Hom_B(A,M)$ is via $(a.\delta)(f) := a.\delta(f)$ an A-module called **module of B-derivations of A with values in M**. Notice that, for $B = \mathbb{C}$, each $\delta \in Der_{\mathbb{C}}(\mathbb{C}\{x_1,\dots,x_n\},M)$ has a unique expression

$$\delta = \textstyle\sum_{i=1}^n \delta(x_i).\frac{\partial}{\partial x_i}.$$

Theorem 1.4.2. (I.1.106 of (GREUEL; LOSSEN; SHUSTIN, 2007)) Let A be an analytic $\mathbb{C}$-algebra.

1. There exists a pair (Ω_A^1, d_A), called **module of Kähler differentials**, consisting of a finitely generated A-module Ω_A^1 and a derivation $d_A : A \to \Omega_A^1$ such that for each finitely generated A-module M, the A-linear morphism

$$\theta_M : Hom_A(\Omega^1_A, M) \to Der_{\mathbb{C}}(A,M),\ \varphi \mapsto \varphi \circ d_A,$$

is an isomorphism of A-modules.

2. The pair (Ω^1_A, d_A) is uniquely determined up to unique isomorphism.

3. If $A = \mathbb{C}\{x_1,\ldots,x_n\} := \mathbb{C}\{\mathbf{x}\}$ then $\Omega^1_{\mathbb{C}\{\mathbf{x}\}}$ is free of rank n with basis $dx_1,\ldots,dx_n$ and $d = d_{\mathbb{C}\{\mathbf{x}\}} : \mathbb{C}\{\mathbf{x}\} \to \Omega^1_{\mathbb{C}\{\mathbf{x}\}}$ is given by $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$.

Now let us extend the module of Kähler differentials initially defined over a $\mathbb{C}$-algebra to the case where it is defined over a complex space. Let X be a complex space, $x \in X$ and U be an open neighborhood of x which is isomorphic to a local model space Y defined by an ideal of finite type $\mathscr{I} \subset \mathscr{O}_D$, where D is an open subset of $\mathbb{C}^n$. The sheaf Ω^1_D is defined to be the free sheaf $\mathscr{O}_D dx_1 \oplus \cdots \oplus \mathscr{O}_D dx_n$ and the derivation $d : \mathscr{O}_D \to \Omega^1_D$ is defined by $df = \sum_{i=i}^n \frac{\partial f}{\partial x_i} dx_i$.

Definition 1.4.3. Let $\mathscr{O}_Y$ be the sheaf of quotients of rings $\mathscr{O}_D/\mathscr{I}$ and assume that $\mathscr{I} = \langle f_1,\ldots,f_k\rangle_{\mathscr{O}_D}$. Let $\mathscr{O}_D d\mathscr{I}$ be the subsheaf of Ω^1_D generated by $df_1,\ldots,df_k$ and $\mathscr{I}\Omega^1_D$ the subsheaf of Ω^1_D generated by $f_j dx_i, i = 1,\ldots,n, j = 1,\ldots,k$. The module of Kähler differentials on Y is defined by the topological restriction of the sheaf $\frac{\Omega^1_D}{\mathscr{I}\Omega^1_D + \mathscr{O}_D d\mathscr{I}}$ to Y, denoted by Ω^1_Y, with the induced derivation denoted by $d_Y : \mathscr{O}_Y \to \Omega^1_Y$.

If $\varphi : U \to Y$ is an isomorphism to the local space Y, one defines $\Omega^1_U := \varphi^*\Omega^1_Y$, where $\varphi^*\Omega^1_Y$ is the analytic preimage sheaf of Ω^1_Y by φ. In this case, by Theorem 1.4.2 (2), Ω^1_U is, up to a unique isomorphism, independent of the choice of φ. Hence, by gluing the locally defined sheaves Ω^1_U, it is possible to obtain a unique sheaf Ω^1_X on X, the **sheaf of holomorphic Kähler differentials** on X with a unique derivation $d_X : \mathscr{O}_X \to \Omega^1_X$. Using this sheaf, it is possible to obtain a regularity criterion for the complex space (X,x) (see Theorem 1.110 in (GREUEL; LOSSEN; SHUSTIN, 2007)). Aiming to extend this criterion to morphisms of complex spaces, the concept of relative differential module need to be introduced. We do not provide the complete description of this object (see (GREUEL; LOSSEN; SHUSTIN, 2007)), but only a characterization of it which is enough to understand the relative Nash modification.

Definition 1.4.4. Let X and S be complex spaces, where X is defined by the sheaf of ideals of finite type $\mathscr{I} \subset \mathscr{O}_D$, D is an open subset of $\mathbb{C}^n$ and $S \subset \mathbb{C}^k$. Consider a morphism of complex spaces $h : X \to S$, induced by the map $h = (h_1,\ldots,h_k) : D \to \mathbb{C}^k$. The **sheaf of relative holomorphic Kähler differential forms** of X over S is defined by

$$\Omega^1_{X/S} = \frac{\Omega^1_D}{(\mathscr{I}\Omega^1_D + \mathscr{O}_D d\mathscr{I} + \langle h_1,\ldots,h_k\rangle_{\mathscr{O}_D})|X},$$

where $(\mathscr{I}\Omega^1_D + \mathscr{O}_D d\mathscr{I} + \langle h_1,\ldots,h_k\rangle_{\mathscr{O}_D})|X$ denotes the topological restriction of the sheaf $(\mathscr{I}\Omega^1_D + \mathscr{O}_D d\mathscr{I} + \langle h_1,\ldots,h_k\rangle_{\mathscr{O}_D})$ to X.

1.5 Stratification of a complex analytic space

A way to study a complex space is considering a decomposition of this (possibly singular) space into a union of smooth components. In this section we will see different types of a special type of decomposition, called stratification.

Let X be a complex analytic space defined over an open subset U of $\mathbb{C}^N$.

Definition 1.5.1. A **complex analytic stratification** of X is a locally finite decomposition of X into complex analytic submanifolds (the **strata**) such that the closure of each stratum is complex analytic and a union of strata.

In the following, all stratifications considered are complex analytic.

A **refinement** of a stratification $\mathscr{V}$ of X is a stratification $\mathscr{R}$ of X such that each stratum of $\mathscr{V}$ is a union of strata of $\mathscr{R}$.

A useful type of complex stratification is the following.

Definition 1.5.2. A **Whitney stratification** of X is a stratification that satisfies the following conditions: for all pair of strata $(V_\alpha, V_\beta), V_\beta \subset \overline{V_\alpha}$, suppose $(x_i) \in V_\alpha$ is a sequence of points converging to some $y \in V_\beta$. Suppose $(y_i) \in V_\beta$ also converges to y, that (with respect to some local coordinate system of $\mathbb{C}^N$) the secant lines $l_i = \overline{x_i y_i}$ converge to some limit line l and that the sequence of tangent planes $T_{x_i}V_\alpha$ converges to a plane τ. Then

1. **Whitney's condition (a)**: $T_y V_\beta \subset \tau$ and
2. **Whitney's condition (b)**: $l \subset \tau$.

Let us see some properties about Whitney stratifications.

Remark 1.5.3.
1. Whitney's condition (b) implies the Whitney's condition (a) (see (MATHER, 2012)).
2. The transversal intersection of two Whitney stratified spaces is a Whitney stratified space, whose strata are the intersections of the strata of the two spaces (see (ORRO; TROTMAN, 2010)).
3. Suppose A is a subanalytic (resp. complex analytic, complex algebraic) subset of a real analytic (resp. complex analytic, resp. complex algebraic) smooth manifold M. Then there exists a Whitney stratification of A into subanalytic (resp. complex analytic, resp. complex algebraic) smooth manifolds. Furthermore, if $\mathscr{C}$ is a locally finite collection of subanalytic (resp. complex analytic, resp. complex algebraic) subsets of A, then the stratification may be chosen so that each element of $\mathscr{C}$ is a union of strata of the stratification. In this case, the stratification is called **adapted to** $\mathscr{C}$ (see (GORESKY; MACPHERSON, 1988)).

The strata of the next stratification satisfies a specific condition with respect to the fibres of a function-germ $f : X \to \mathbb{C}$.

Definition 1.5.4. A **Thom stratification** $\mathscr{V}$ of X with respect to f is a Whitney stratification of X such that each pair of strata (V_α, V_β) satisfies the (a_f)**-Thom condition** at a point $p \in V_\beta$, that is, the differential df has constant rank on V_α and for any sequence of points $(p_i) \in V_\alpha$ such that (p_i) converges to p and $Ker(d_{p_i}(f|_{V_\alpha}))$ converges to some T in the appropriate Grassmanian, $Ker(d_p(f|_{V_\beta})) \subset T$.

Remark 1.5.5. Thom stratifications of a complex analytic space X with respect to a function-germ $f : X \to \mathbb{C}$ always exist (see (HIRONAKA, 1976)).

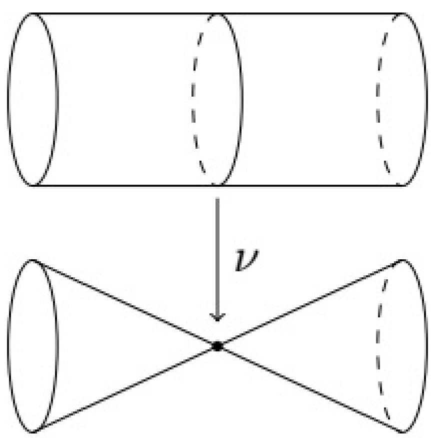
1.6 Relative local polar varieties and intersection multiplicity

In this section, we will describe the relative local polar varieties, introduced by Lê and Teissier in (LÊ; TEISSIER, 1981) and (TEISSIER, 1982). We begin with the Nash modification.

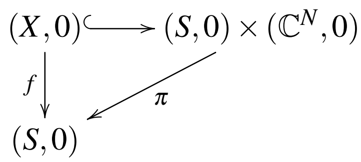
Let $(X,0) \subset (\mathbb{C}^N,0)$ be a complex analytic germ of dimension d in a open set $U \subset \mathbb{C}^N$. Suppose that $(X,0)$ is equidimensional, that is, all components of the regular part X_{reg} of X have same dimension, and reduced, that is, the local ring $\mathscr{O}_{X,0}$ has no nilpotent elements. Let $G(d,N)$ be the Grassmannian manifold of the vector subspaces of dimension d in $\mathbb{C}^N$, $x \in X_{reg}$ and consider the Gauss map $\phi : X_{reg} \to U \times G(d,N)$ given by $x \mapsto (x, T_x(X_{reg}))$.

Definition 1.6.1. The closure of the image of the Gauss map ϕ in $U \times G(d,N)$, denoted by $\tilde{X}$, is called the **Nash modification** of X. It is a complex analytic space endowed with an analytic projection map $\nu : \tilde{X} \to X$.

Example 1.6.2. The Nash modification of a cone is the cylinder:



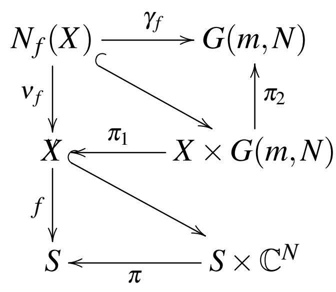
Let us now define the relative Nash modification (see chapter I.1.2 in (TEISSIER, 1982) for details). Let $f: X \to S$ be a morphism of reduced analytic spaces such that the relative Kähler differential module $\Omega^1_{X/S}$ is locally free of rank $m = \dim X - \dim S$ over $X \setminus Sing(X)$. Suppose that X is equipped with an embedding $(X,0) \subset (S,0) \times (\mathbb{C}^N, 0)$. For all commutative diagram



of a small enough representative of the germ of f at 0, where π is the natural projection from $(S,0) \times (\mathbb{C}^N, 0)$ to $(S,0)$, the relative Nash modification is described as follows. Consider the morphism $\phi_f : X \setminus Sing(X) \to G(m,N)$ defined by $\phi_f(x) = \lim_{x_k \to x} T_{x_k} f^{-1}(f(x_k))$.

Definition 1.6.3. The **relative Nash modification** $N_f(X) \subset X \times G(m,N)$ is the closure of the graph of the morphism ϕ_f in $X \times G(m,N)$.

Considering the inclusion $N_f(X) \hookrightarrow X \times G(m,N)$ and the natural projections $\pi_1 : X \times G(m,N) \to X$ and $\pi_2 : X \times G(m,N) \to G(m,N)$, one can define a relative Gauss map $\gamma_f : N_f(X) \to G(m,N)$ and the map $\nu_f : N_f(X) \to X$ in the following diagram (see page 418 of (TEISSIER, 1982))



Now we are ready to define the relative local polar varieties. Let $\mathscr{D}$ be a sequence of vector subspaces of $\mathbb{C}^N$

$$\mathscr{D} : (0) \subset D_{N-1} \subset D_{N-2} \subset \cdots \subset D_1 \subset D_0 = \mathbb{C}^N,$$

where the codimension of D_i is i.

If k and m are integers such that $0 \le k \le m \le N$, we consider in the Grassmanian manifold $G(m,N)$ the algebraic subvariety

$$\sigma_k(\mathscr{D}) = \{T \in G(m,N); \dim(T \cap D_{m-k+1}) \ge k\}.$$

Remark 1.6.4. The algebraic subvariety $\sigma_k(\mathscr{D})$ depends only on $D_{m-k+1} \subset \mathbb{C}^N$, hence we will also write $\sigma_k(D_{m-k+1})$.

Proposition 1.6.5. (1.3 Proposition 2 of (TEISSIER, 1982)) Let $f:(X,0)\to(S,0)$ be a morphism as described before. Suppose that S is nonsingular and that X is equipped with an embedding $(X,0)\subset(S,0)\times(\mathbb{C}^N,0)$. For all integer $k, 0\leq k\leq m:=\dim X-\dim S$ there exists an open Zariski set W_k of $G(N-m+k-1,N)$ (of subspaces of codimension $m-k+1$ of $\mathbb{C}^N$) such that for all $D_{m-k+1}\in W_k$,

1. $\gamma_f^{-1}(\sigma_k(D_{m-k+1}))\cap\nu_f^{-1}(X\setminus Sing(X))$ is reduced and dense in the set $\gamma_f^{-1}(\sigma_k(D_{m-k+1}))$ and $\gamma_f^{-1}(\sigma_k(D_{m-k+1}))$ is empty or purely of codimension $m-k+1$ in $N_f(X)$;
2. If $\gamma_f^{-1}(\sigma_k(D_{m-k+1}))\cap\nu_f^{-1}(0)$ is not empty, then
$$\dim\ \gamma_f^{-1}(\sigma_k(D_{m-k+1}))\cap\nu_f^{-1}(0)=\dim\ \nu_f^{-1}(0)-k.$$

As a consequence of this proposition, Teissier proved the following result about the structure of the set $P_k\langle f;D_{m-k+1}\rangle:=|\nu(\gamma_f^{-1}(\sigma_k(D_{m-k+1})))|$.

Corollary 1.6.6. (Corollary 1.3.2 (TEISSIER, 1982)) The closure $P_k\langle f;D_{m-k+1}\rangle$ of $P_k\langle f;D_{m-k+1}\rangle\cap X_{reg}$ in X is a closed reduced analytic subspace of X, empty or of pure codimension k.

Definition 1.6.7. $P_k\langle f;D_{m-k+1}\rangle$ is called the **relative local polar variety** of codimension k associated to f and D_{m-k+1}.

There is an invariant associated to varieties that is useful to describe the local geometry of the variety around some fixed point. Let us see the algebraic definition of this invariant. For that we follow (MATSUMURA, 1989).

Let $(A,\mathscr{M})$ be a Noetherian local ring of dimension d, M be a finite A- module and q be a $\mathscr{M}$- primary ideal of A. By §14 of (MATSUMURA, 1989), the Samuel function $l:\mathbb{N}\to\mathbb{N}$, given by $l(n)=length\left(\frac{M}{q^{n+1}M}\right)$, can be written, for n sufficiently large, as a polynomial in n, with rational coefficients and whose coefficient of higher degree is $\frac{e(q,M)}{d!}, e(q,M)\in\mathbb{Z}$.

Definition 1.6.8. The integer $e(q,A)$ is called the **multiplicity** of the ideal q. In the case where q is the maximal ideal $\mathscr{M}$ of A, $e(\mathscr{M},A)$ is called the multiplicity of the local ring A.

Definition 1.6.9. Let V be a complex variety and x be a point in V. The **multiplicity** of V in x is the multiplicity of the local ring $\mathscr{O}_{V,x}$.

In (TEISSIER, 1982), Teissier proved that the multiplicity of the local polar variety does not depend on the sequence of vector spaces chosen, but on the analytic type of the morphism f.

Theorem 1.6.10. (Theorem 3.1 of (TEISSIER, 1982)) For each integer $k, 0\leq k\leq m=\dim X-\dim S$, there exists an open Zariski set W_k in $G(N-m+k-1,N)$ such that the multiplicity $m_0(P_k\langle f;\mathscr{D}\rangle)$ of $P_k\langle f;\mathscr{D}\rangle$ in 0 does not depend on $\mathscr{D}\in W_k$, but only on the analytic type of the germ of the morphism f in 0.

Let us now describe the local polar varieties and the intersection multiplicities in the specific context we will use. For that, let us recall the definition of the *Tor* module.

Definition 1.6.11. Let A be a ring and M be an A- module. A **free resolution** $F_\bullet$ of M is an exact complex of free A-modules F_i

$$F_\bullet : \cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0,$$

such that $Coker(d_1) = M$.

Definition 1.6.12. Let M and N be A- modules and $F_\bullet$ be a free resolution of M. The i-th Tor-module $Tor_i^A(M,N)$ of M and N is the i-th homological group of the complex $F_\bullet \otimes_A N$,

$$F_\bullet \otimes_A N : \cdots \longrightarrow F_i \otimes_A N \xrightarrow{d_i} F_{i-1} \otimes_A N \longrightarrow \cdots \longrightarrow F_1 \otimes_A N \xrightarrow{d_1} F_0 \otimes_A N.$$

Remark 1.6.13. In the above definition, if $G_\bullet$ is a free resolution of the A-module N, the i-th Tor-module $Tor_i^A(M,N)$ of M and N can also be defined by the i-th homological group of the complex $G_\bullet \otimes_A M$.

Let $(X,0) \subset (\mathbb{C}^N,0)$ be an equidimensional complex analytic reduced space of dimension d in a neighborhood of the origin, $f : (X,0) \to (\mathbb{C},0)$ be a complex analytic function-germ and l be a generic linear form over $(X,0)$ Let $\mathscr{V} = \{V_i\}$ be a stratification of a small representative X of $(X,0)$ and $d_i = \dim V_i$. We denote by $\Gamma^0_{f|_{\overline{V_i}}}$ the relative local polar curve of codimension $d_i - 1$ associated to f and $\overline{V_i}$. Denoting $\overline{V_i} \cap \{f = 0\}$ by $\overline{V_i}^f$ and using (LOESER, 1984), the intersection multiplicity $I(\overline{V_i}^f, \Gamma^0_{f|_{\overline{V_i}}})$ of $\Gamma^0_{f|_{\overline{V_i}}}$ at 0 in $\overline{V_i}^f$ can be computed with the sum

$$I(\overline{V_i}^f, \Gamma^0_{f|_{\overline{V_i}}}) = \sum_{j\geq 0} (-1)^j long_{\mathscr{O}_{\overline{V_i},0}} \left(Tor_j^{\mathscr{O}_{\overline{V_i},0}} (\mathscr{O}_{\Gamma^0_{f|_{\overline{V_i}}}}, \mathscr{O}_{\overline{V_i}^f}) \right).$$

By Corollary 1.6.6, $\Gamma^0_{f|_{\overline{V_i}}}$ is a reduced curve, what implies that it is Cohen-Macaulay. Therefore, $\mathscr{O}_{\Gamma^0_{f|_{\overline{V_i}}}}$ is Cohen-Macaulay. Since $\mathscr{O}_{\Gamma^0_{f|_{\overline{V_i}}}}$ is Noetherian, it is finitely generated and then, by Corollary B.8.12 of (GREUEL; LOSSEN; SHUSTIN, 2007), this local ring is flat. Therefore, By Proposition B.3.2 (page 403 (GREUEL; LOSSEN; SHUSTIN, 2007)),

$$Tor_j^{\mathscr{O}_{(\overline{V_i},0)}} (\mathscr{O}_{\Gamma^0_{f|_{\overline{V_i}}}}, \mathscr{O}_{\overline{V_i}^f}) = 0, \text{ for all } j \geq 1.$$

So,

$$I(\overline{V_i}^f, \Gamma^0_{f|_{\overline{V_i}}}) = long_{\mathscr{O}_{\overline{V_i},0}} \left(Tor_0^{\mathscr{O}_{\overline{V_i},0}} (\mathscr{O}_{\Gamma^0_{f|_{\overline{V_i}}}}, \mathscr{O}_{\overline{V_i}^f}) \right).$$

In (LOESER, 1984), Loeser also shows (page 213) that, in fact,

$$I(\overline{V_i}^f, \Gamma^0_{f|_{\overline{V_i}}}) = dim_{\mathbb{C}} \left(\frac{\mathscr{O}_{\Gamma^0_{f|_{\overline{V_i}}},0}}{f\mathscr{O}_{\Gamma^0_{f|_{\overline{V_i}}},0}} \right).$$

Since $\Gamma^0_{f|_{\overline{V_i}}}$ is one-dimensional, for a regular value $0 < |\delta| \ll 1$ of $f, \Gamma^0_{f|_{\overline{V_i}}} \cap f^{-1}(\delta)$ is a finite number of points. So, the degree of the function $f : \Gamma^0_{f|_{\overline{V_i}}} \to \mathbb{C}$ is a way to compute the complex dimension above, which is in this case the number of points in $\Gamma^0_{f|_{\overline{V_i} \cap f^{-1}(\delta)}}$ (see (NUÑO-BALLESTEROS; TOMAZELLA, 2008)).

1.7 The local Euler obstruction and the Euler obstruction of a function

In this section, we will see the definition of the local Euler obstruction, a singular invariant defined by MacPherson in (MACPHERSON, 1974) and used as one of the main tools in his proof of the Deligne-Grothendieck conjecture about the existence and uniqueness of Chern classes for singular varities.

Let $(X,0) \subset (\mathbb{C}^n,0)$ be an equidimensional reduced complex analytic germ of dimension d in an open set $U \subset \mathbb{C}^n$. Consider a complex analytic Whitney stratification $\mathscr{V} = \{V_\lambda\}$ of U adapted to X such that $\{0\}$ is a stratum. We choose a small representative of $(X,0)$, denoted by X, such that 0 belongs to the closure of all strata. We write $X = \cup_{i=0}^{q} V_i$, where $V_0 = \{0\}$ and $V_q = X_{reg}$, where X_{reg} is the regular part of X. We suppose that $V_0, V_1, \dots, V_{q-1}$ are connected and that the analytic sets $\overline{V_0}, \overline{V_1}, \dots, \overline{V_q}$ are reduced. We write $d_i = dim(V_i),\ i \in \{1, \dots, q\}$. Note that $d_q = d$.

Let $G(d,N)$ be the Grassmannian manifold and $\tilde{X}$ the Nash modification of X. Consider the extension $\mathscr{T}$ of the tautological bundle over $U \times G(d,N)$. Since $\tilde{X} \subset U \times G(d,N)$, we consider $\tilde{\mathscr{T}}$ the restriction of $\mathscr{T}$ to $\tilde{X}$, called the **Nash bundle**, and $\pi : \tilde{\mathscr{T}} \to \tilde{X}$ the projection of this bundle.

In this context, denoting by φ the natural projection of $U \times G(d,N)$ at U, we have the following diagram:

$$\begin{array}{ccc} \tilde{\mathscr{T}} & \longrightarrow & \mathscr{T} \\ \pi \downarrow & & \downarrow \\ \tilde{X} & \longrightarrow & U \times G(d,N) \\ \nu \downarrow & & \downarrow \varphi \\ X & \longrightarrow & U \subseteq \mathbb{C}^N \end{array}$$

Considering $||z|| = \sqrt{z_1\overline{z_1} + \cdots + z_N\overline{z_N}}$, the 1-differential form $w = d||z||^2$ over $\mathbb{C}^N$ defines a section in $T^*\mathbb{C}^N$ and its pullback $\varphi^* w$ is a 1- form over $U \times G(d,N)$. Denote by $\tilde{w}$ the restriction of $\varphi^* w$ over $\tilde{X}$, which is a section of the dual bundle $\tilde{\mathscr{T}}^*$.

Choose ε small enough for $\tilde{w}$ be a nonzero section over $\nu^{-1}(z), 0 < ||z|| \leqslant \varepsilon$, let B_ε be the closed ball with center at the origin with radius ε and denote by

1. $Obs(\tilde{\mathscr{T}}^*,\tilde{w}) \in \mathbb{H}^{2d}(\nu^{-1}(B_\varepsilon),\nu^{-1}(S_\varepsilon),\mathbb{Z})$ the obstruction for extending $\tilde{w}$ from $\nu^{-1}(S_\varepsilon)$ to $\nu^{-1}(B_\varepsilon)$;

2. $O_{\nu^{-1}(B_\varepsilon),\nu^{-1}(S_\varepsilon)}$ the fundamental class in $\mathbb{H}_{2d}(\nu^{-1}(B_\varepsilon),\nu^{-1}(S_\varepsilon),\mathbb{Z})$.

Definition 1.7.1. The **local Euler obstruction** of X at 0, $Eu_X(0)$, is given by the evaluation

$$Eu_X(0) = \langle Obs(\tilde{\mathscr{T}}^*,\tilde{w}), O_{\nu^{-1}(B_\varepsilon),\nu^{-1}(S_\varepsilon)}\rangle.$$

Let us see some properties of the local Euler obstruction.

Remark 1.7.2. 1. The local Euler obstruction of X at a regular point of X is 1.

2. The local Euler obstruction at a point of a curve is the multiplicity of this point at the curve. (See (GONZÁLEZ-SPRINBERG, 1981))

3. The local Euler obstruction is constant on the strata of a Whitney stratification.(See (BRASSELET; SCHWARTZ, 1981))

In (BRASSELET; LÊ; SEADE, 2000), Brasselet, Lê and Seade proved a formula to calculate the Euler obstruction using generic linear forms.

Theorem 1.7.3. (Theorem 3.1 of (BRASSELET; LÊ; SEADE, 2000)) Let $(X,0)$ and $\mathscr{V}$ be given as before, then for each generic linear form l, there exists ε_0 such that for any ε with $0 < \varepsilon < \varepsilon_0$ and $\delta \neq 0$ sufficiently small, the Euler obstruction of $(X,0)$ is equal to

$$Eu_X(0) = \sum_{i=1}^{q} \chi(V_i \cap B_\varepsilon \cap l^{-1}(\delta)).Eu_X(V_i),$$

where χ is the Euler characteristic, $Eu_X(V_i)$ is the Euler obstruction of X at a point of V_i, $i = 1,\ldots,q$ and $0 < |\delta| \ll \varepsilon \ll 1$.

Let us give the definition of another invariant introduced by Brasselet, Massey, Parameswaran and Seade in (BRASSELET *et al.*, 2004). Let $f : X \to \mathbb{C}$ be a holomorphic function with isolated singularity at the origin given by the restriction of a holomorphic function $F : U \to \mathbb{C}$ and denote by $\overline{\nabla}F(x)$ the conjugate of the gradient vector field of F in $x \in U$,

$$\overline{\nabla}F(x) := \left(\overline{\frac{\partial F}{\partial x_1}},\ldots,\overline{\frac{\partial F}{\partial x_n}}\right).$$

Since f has isolated singularity at the origin, for all $x \in X\setminus\{0\}, T_x(V_i(x))$ is transverse to $Ker(d_xF)$, that is,

$$Ang\langle\overline{\nabla}F(x), T_x(V_i(x))\rangle < \frac{\pi}{2},$$

where $V_i(x)$ is a stratum containing x and $Ang\langle\cdot,\cdot\rangle$ denotes the angle between two nonzero vectors. Therefore, the projection $\hat{\zeta}_i(x)$ of $\overline{\nabla}F(x)$ over $T_x(V_i(x))$ is nonzero. Using this projection, the authors, in (BRASSELET *et al.*, 2004), constructed a stratified vector field over X, denoted by $\overline{\nabla}f(x)$. Let $\tilde{\zeta}$ be the lifting of $\overline{\nabla}f(x)$ as a section of the Nash bundle $\tilde{T}$ over $\tilde{X}$, without singularity over $\nu^{-1}(X\cap S_\varepsilon)$.

Let $\mathscr{O}(\tilde{\zeta})\in\mathbb{H}^{2n}(\nu^{-1}(X\cap B_\varepsilon),\nu^{-1}(X\cap S_\varepsilon))$ be the obstruction cocycle for extending $\tilde{\zeta}$ as a nonzero section of $\tilde{T}$ inside $\nu^{-1}(X\cap B_\varepsilon)$.

Definition 1.7.4. The **local Euler obstruction of a function** $f, Eu_{f,X}(0)$, is the evaluation of $\mathscr{O}(\tilde{\zeta})$ on the fundamental class $[\nu^{-1}(X\cap B_\varepsilon),\nu^{-1}(X\cap S_\varepsilon)]$.

The next theorem compares the Euler obstruction of a space X with the Euler obstruction of function defined over X.

Theorem 1.7.5. (Theorem 3.1 of (BRASSELET *et al.*, 2004)) Let $(X,0)$ and $\mathscr{V}$ be given as before and let $f:(X,0)\to(\mathbb{C},0)$ be a function with an isolated singularity at 0. For $0<|\delta|\ll\varepsilon\ll 1$, we have

$$Eu_{f,X}(0)=Eu_X(0)-\sum_{i=1}^{q}\chi(V_i\cap B_\varepsilon\cap f^{-1}(\delta)).Eu_X(V_i).$$

Let us now see an example that justifies why the Euler obstruction of a function is seen as a generalization of the Milnor number.

Example 1.7.6. Let f be a holomorphic function over a n-dimensional nonsingular complex analytic space $(X,0)$ with an isolated singularity in 0. In this case, $Eu_X(0)=1$, $Eu_X(X\setminus\{0\})=1$ and the fibre $X\setminus\{0\}\cap B_\varepsilon\cap f^{-1}(t_0)$ has the homotopy type of a bouquet of $\mu(f)$ $(n-1)$-dimensional spheres, where $0<|t_0|\ll\varepsilon\ll 1$. So,

$$\chi(X\setminus\{0\}\cap B_\varepsilon\cap f^{-1}(t_0))=1+(-1)^{dim_{\mathbb{C}}X-1}\mu(f).$$

By Theorem 1.7.5, considering $V_0=\{0\}$ and $V_1=X\setminus\{0\}$,

$$\begin{aligned} Eu_{f,X}(0) &= Eu_X(0)-\sum_{i=0}^{1}\chi(V_i\cap B_\varepsilon\cap f^{-1}(t_0)).Eu_X(V_i) \\ &= 1-\left(0+(1+(-1)^{dim_{\mathbb{C}}X-1}\mu(f))\right) \\ &= (-1)^n\mu(f). \end{aligned}$$

Let us now see a definition we will need to define a generic point of a function-germ. Let $\mathscr{V}=\{V_\lambda\}$ be a stratification of a reduced complex analytic space X.

Definition 1.7.7. Let p be a point in a stratum V_β of $\mathscr{V}$. A **degenerate tangent plane of $\mathscr{V}$ at** p is an element T of some Grassmanian manifold such that $T=\lim\limits_{p_i\to p}T_{p_i}V_\alpha$, where $p_i\in V_\alpha$, $V_\alpha\neq V_\beta$.

Definition 1.7.8. Let $(X,0) \subset (U,0)$ be a germ of complex analytic space in $\mathbb{C}^n$ equipped with a Whitney stratification and let $f : (X,0) \to (\mathbb{C},0)$ be an analytic function, given by the restriction of an analytic function $F : (U,0) \to (\mathbb{C},0)$. Then 0 is said to be a **generic point** of f if the hyperplane $Ker(d_0F)$ is transverse in $\mathbb{C}^n$ to all degenerate tangent planes of the Whitney stratification at 0.

Now, let us see the definition of a Morsification of a function.

Definition 1.7.9. Let $\mathscr{W} = \{W_0, W_1, \ldots, W_q\}$, with $0 \in W_0$, a Whitney stratification of the complex analytic space X. A function $f : (X,0) \to (\mathbb{C},0)$ is said to be **Morse stratified** if $\dim W_0 \geq 1, f|_{W_0} : W_0 \to \mathbb{C}$ has a Morse point at 0 and 0 is a generic point of f with respect to W_i, for all $i \neq 0$.

A **stratified Morsification** of a germ of analytic function $f : (X,0) \to (\mathbb{C},0)$ is a deformation $\tilde{f}$ of f such that $\tilde{f}$ is Morse stratified.

In (SEADE; TIBĂR; VERJOVSKY, 2005), Seade, Tibăr and Verjovsky proved that the Euler obstruction of a function f is also related to the number of Morse critical points of a stratified Morsification of f.

Proposition 1.7.10. (Proposition 2.3 of (SEADE; TIBĂR; VERJOVSKY, 2005)) Let $f : (X,0) \to (\mathbb{C},0)$ be a germ of analytic function with isolated singularity at the origin. Then,

$$Eu_{f,X}(0) = (-1)^d n_{reg},$$

where n_{reg} is the number of Morse points in X_{reg} in a stratified Morsification of f.

1.8 Brasselet number

In this section, we present definitions and results needed in the development of the results of this work. The main reference for this section is (MASSEY, 1996).

Let X be a reduced complex analytic space (not necessarily equidimensional) of dimension d in an open set $U \subseteq \mathbb{C}^n$ and let $f : (X,0) \to (\mathbb{C},0)$ be an analytic map. We write $V(f) = f^{-1}(0)$.

Let us specify the stratification we will need in this setting.

Definition 1.8.1. A **good stratification of** X **relative to** f is a stratification $\mathscr{V}$ of X which is adapted to $V(f)$ such that $\{V_\lambda \in \mathscr{V}, V_\lambda \nsubseteq V(f)\}$ is a Whitney stratification of $X \setminus V(f)$ and such that for any pair (V_λ, V_γ) such that $V_\lambda \nsubseteq V(f)$ and $V_\gamma \subseteq V(f)$, the (a_f)-Thom condition is satisfied. In this setting, that is equivalent to the following: if $p \in V_\gamma$ and $(p_i) \in V_\lambda$ are such that $(p_i) \to p$ and $T_{p_i}V(f|_{V_\lambda} - f|_{V_\lambda}(p_i))$ converges to some $\mathscr{T}$, then $T_pV_\gamma \subseteq \mathscr{T}$.

Example 1.8.2. If $f : (X,0) \to (\mathbb{C},0)$ has a stratified isolated critical point and $\mathscr{V}$ is a Whitney stratification of X, then

$$\{V_\lambda \setminus X^f, V_\lambda \cap X^f \setminus \{0\}, \{0\}, V_\lambda \in \mathscr{V}\} \tag{1.1}$$

is a good stratification of X relative to f, called **good stratification induced by** f, where $X^f = X \cap \{f = 0\}$.

Let $\mathscr{V}$ be a good stratification of X relative to f.

Definition 1.8.3. The **critical locus of** f **relative to** $\mathscr{V}$, $\Sigma_{\mathscr{V}} f$, is given by the union

$$\Sigma_{\mathscr{V}} f = \bigcup_{V_\lambda \in \mathscr{V}} \Sigma(f|_{V_\lambda}).$$

Proposition 1.8.4. (Proposition 1.3 of (MASSEY, 1996)) Given an analytic map $f : X \to \mathbb{C}$, a stratification $\mathscr{V}$ of X, and a point $p \in f^{-1}(0)$, there exists a neighborhood of p in which $\Sigma_{\mathscr{V}} f \subseteq f^{-1}(0)$.

Definition 1.8.5. Given an analytic map $f : X \to \mathbb{C}$ and a point $p \in f^{-1}(0)$, the **Milnor fiber of** f **at** p, $F_{f,p}$, is defined to be the (homeomorphism-type of the) space obtained by the intersection

$$F_{f,p} := B_\varepsilon(p) \cap X \cap f^{-1}(\xi),$$

where $0 < |\xi| \ll \varepsilon$ and $B_\varepsilon(p)$ is the closed ball with center in p and radius ε.

We remark that the previous definition is independent of all the choices.

Definition 1.8.6. Suppose that we are given two maps $f : X \to \mathbb{C}$ and $g : X \to \mathbb{C}$. Define the map $\Phi := (f,g) : X \to \mathbb{C}^2$. Let Y be an analytic subset of X. The **relative polar variety of** Y **with respect to** f **and** g, denoted by $\Gamma_{f,g}(Y)$, is the closure in X of the critical locus of $\Phi|_{Y_{reg} \setminus X^f}$, where Y_{reg} denotes the regular part of Y and $X^f = X \cap \{f = 0\}$.

Since each stratum V_λ of a stratification $\mathscr{V}$ is a complex analytic submanifold, the relative polar variety of V_λ with respect to f and g, $\Gamma_{f,g}(V_\lambda)$, is defined. Also, if $V_\lambda \subset V(f)$, then $\Gamma_{f,g}(V_\lambda) = \emptyset$.

Definition 1.8.7. The **relative polar variety of** f **and** g **with respect to** $\mathscr{V}$, $\Gamma_{f,g}(\mathscr{V})$, is given by the union $\cup_\lambda \Gamma_{f,g}(V_\lambda)$.

Definition 1.8.8. If Y is an analytic subset of X, the **symmetric relative polar variety of** Y **with respect to** f **and** g, $\tilde{\Gamma}_{f,g}(Y)$, is the closure in X of the critical locus of $\Phi|_{Y_{reg} \setminus (X^f \cup X^g)}$, where $X^f = X \cap \{f = 0\}$ and $X^g = X \cap \{g = 0\}$.

Definition 1.8.9. If $\mathscr{V} = \{V_\lambda\}$ is a stratification of X, the **symmetric relative polar variety of** f **and** g **with respect to** $\mathscr{V}$, $\tilde{\Gamma}_{f,g}(\mathscr{V})$, is the union $\cup_\lambda \tilde{\Gamma}_{f,g}(V_\lambda)$.

Definition 1.8.10. Let $\mathscr{V}$ be a good stratification of X relative to a function $f : (X,0) \to (\mathbb{C},0)$. A function $g : (X,0) \to (\mathbb{C},0)$ is **prepolar with respect to** $\mathscr{V}$ **at the origin** if the origin is a stratified isolated critical point, that is, 0 is an isolated point of $\Sigma_{\mathscr{V}} g$.

Remark 1.8.11. (See page 976 of (MASSEY, 1996)) The last definition is equivalent to the following statements:

1. For any analytic extension $\tilde{g}$ of g to an open neighbourhood of the origin in $\mathbb{C}^n$, $\Sigma_{\mathscr{V}} \tilde{g}$ is empty or has the origin as an isolated point;
2. $V(g)$ transversely intersects each stratum of $\mathscr{V}$ in a neighbourhood of the origin, except perhaps at the origin itself.

Definition 1.8.12. A function $g : (X,0) \to (\mathbb{C},0)$ is **tractable at the origin with respect to a good stratification** $\mathscr{V}$ **of** X **relative to** $f : (X,0) \to (\mathbb{C},0)$ if $dim_0\, \tilde{\Gamma}_{f,g}(\mathscr{V}) \leq 1$ and, for all strata $V_\alpha \subseteq X^f$, $g|_{V_\alpha}$ has no critical point in a neighbourhood of the origin except perhaps at the origin itself.

Proposition 1.8.13. (Proposition 1.12 of (MASSEY, 1996)) Suppose that $\mathscr{V}$ is a good stratification of X for f at the origin, and that g is prepolar with respect to $\mathscr{V}$ at the origin. Then, $dim_0 V(g) \cap \Gamma_{f,g}(\mathscr{V}) \leq 0$ (where < 0 indicates an empty germ at the origin). Hence, $\Gamma_{f,g}(\mathscr{V}) = \tilde{\Gamma}_{f,g}(\mathscr{V})$ and each of these sets is either one-dimensional or empty at the origin. In particular, g is tractable at the origin with respect to $\mathscr{V}$ relative to f.

Remark 1.8.14. (See page 974 of (MASSEY, 1996) or page 135 of (DUTERTRE; GRULHA, 2014)) If the symmetric relative polar variety $\tilde{\Gamma}_{f,g}(\mathscr{V})$ has dimension one, for each $V_i \in \mathscr{V}$, it is possible to associate a *multiplicity* $\mu^f(\tilde{\Gamma}_{f,g}(V_i))$ to each $\tilde{\Gamma}_{f,g}(V_i)$ in the following way: if the stratum V_i is one-dimensional, $\mu^f(\tilde{\Gamma}_{f,g}(V_i)) := 1$. If V_i is not one-dimensional, let ν be a component of $\tilde{\Gamma}_{f,g}(V_i)$ and p be a point of $\nu \setminus \{0\}$ close to the origin. The mapping $g : V_i \cap \{f = f(p)\} \to \mathbb{C}$ has an isolated singularity at p and let μ^ν the Milnor number of this singularity. Then $\mu^f(\tilde{\Gamma}_{f,g}(V_i))$ is the sum of Milnor numbers μ^ν over all components ν. Note that, if g is prepolar with respect to $\mathscr{V}$ at the origin, by Proposition 1.8.13, $\Gamma_{f,g}(\mathscr{V}) = \tilde{\Gamma}_{f,g}(\mathscr{V})$, and we can write $\mu^f(\tilde{\Gamma}_{f,g}(V_i))$ instead of $\mu^f(\Gamma_{f,g}(V_i))$.

Let us now see the definition of decent analytic function-germs. Let $\mathscr{V} = \{V_\lambda\}$ be a stratification of a reduced complex analytic space X.

Definition 1.8.15. Let $g : (X,0) \to (\mathbb{C},0)$ be a function-germ. For any analytic stratification $\mathscr{V}$ of X, $0 < |\delta| \ll 1$ and any function $f : (X,0) \to (\mathbb{C},0)$, g is **decent with respect to** $\mathscr{V}$ **relative to** f if there exists a neighborhood Ω of 0 such that $g : \Omega \cap X \cap f^{-1}(\delta) \setminus X^g \to \mathbb{C}$ has only generic points.

Proposition 1.8.16. (Proposition 1.14 of (MASSEY, 1996)) Let $\mathscr{V}$ be a good stratification of X relative to f at the origin. Then, for a generic choice of linear form, l, l is decent to $\mathscr{V}$ relative to f and, moreover, f is decent with respect to $\mathscr{V}$ relative to l.

Another concept useful for this work is the notion of constructible functions. Consider a Whitney stratification $\mathscr{W} = \{W_1, \ldots, W_q\}$ of X such that each stratum W_i is connected.

Definition 1.8.17. A constructible function with respect to the stratification $\mathscr{W}$ of X is a function $\beta : X \to \mathbb{Z}$ which is constant on each stratum W_i, that is, there exist integers $t_1, \ldots, t_q$, such that

$$\beta = \sum_{i=1}^{q} t_i.1_{W_i},$$

where 1_{W_i} is the characteristic function of W_i.

Definition 1.8.18. The Euler characteristic $\chi(X,\beta)$ of a constructible function $\beta : X \to \mathbb{Z}$ with respect to the stratification $\mathscr{W}$ of X, given by $\beta = \sum_{i=1}^{q} t_i.1_{W_i}$, is defined by

$$\chi(X,\beta) = \sum_{i=1}^{q} t_i.\chi(W_i).$$

Before we state the Dutertre and Grulha results, we need to introduce some definitions about normal Morse data. We cite as main references (GORESKY; MACPHERSON, 1988) and (SCHÜRMANN; TIBAR, 2010). The first concept we present is the complex link, an object analogous to the Milnor fibre, important in the study of complex stratified Morse theory.

Let V be a stratum of the stratification $\mathscr{V}$ of X and let x be a point of V. Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic complex function-germ such that the differential form $Dg(x)$ does not vanish on a degenerate tangent plane of $\mathscr{V}$ at x. Let N be a normal slice to V at x, that is, N is a closed complex submanifold of $\mathbb{C}^n$ which is transversal to V at x and $N \cap V = \{x\}$.

Definition 1.8.19. Let B_ε be the closed ball of radius ε centered at x. The **complex link** l_V of V is defined by

$$l_V = X \cap N \cap B_\varepsilon \cap \{g = \delta\},$$

where $0 < |\delta| \ll \varepsilon \ll 1$.

The **normal Morse datum** $NMD(V)$ of V is the pair of spaces

$$NMD(V) = (X \cap N \cap B_\varepsilon, X \cap N \cap B_\varepsilon \cap \{g = \delta\}).$$

In section 2.3 (Part II) of (GORESKY; MACPHERSON, 1988), the authors explained why this two notions are independent of all choices made.

Definition 1.8.20. Let $\beta : X \to \mathbb{Z}$ be a constructible function with respect to the stratification $\mathscr{V}$. Its normal Morse index $\eta(V,\beta)$ along V is defined by

$$\eta(V,\beta) = \chi(NMD(V),\beta) = \chi(X \cap N \cap B_\varepsilon, \beta) - \chi(l_V, \beta).$$

In the case where the constructible function is the local Euler obstruction, the following identities are valid ((SCHÜRMANN; TIBAR, 2010), page 34):

$$\eta(V', Eu_{\overline{V}}) = 1, \text{ if } V' = V \text{ and } \eta(V', Eu_{\overline{V}}) = 0, \text{ if } V' \neq V.$$

We present now the definition of the Brasselet number and the main theorems of (DUTERTRE; GRULHA, 2014), used as inspiration for this work.

Let $f : (X,0) \to (\mathbb{C},0)$ be a complex analytic function germ and let $\mathscr{V}$ be a good stratification of X relative to f. We denote by $V_1,\ldots,V_q$ the strata of $\mathscr{V}$ that are not contained in $\{f=0\}$ and we assume that $V_1,\ldots,V_{q-1}$ are connected and that $V_q = X_{reg} \setminus \{f=0\}$. Note that V_q could be not connected.

Definition 1.8.21. Suppose that X is equidimensional. Let $\mathscr{V}$ be a good stratification of X relative to f. The **Brasselet number** of f at the origin, $B_{f,X}(0)$, is defined by

$$B_{f,X}(0) = \textstyle\sum_{i=1}^{q} \chi(V_i \cap f^{-1}(\delta) \cap B_\varepsilon) Eu_X(V_i),$$

where $0 < |\delta| \ll \varepsilon \ll 1$.

Remark: If V_q^i is a connected component of V_q, $Eu_X(V_q^i) = 1$.

Notice that if f has a stratified isolated singularity at the origin, then $B_{f,X}(0) = Eu_X(0) - Eu_{f,X}(0)$ (see Theorem 1.7.5).

In (DUTERTRE; GRULHA, 2014), Dutertre and Grulha proved interesting formulas describing the topological relation between the Brasselet number and a number of certain critical points of a special type of deformation of functions. Let us now present some of these results. Fist we need the definition of a special type of Morsification, introduced by Dutertre and Grulha.

Definition 1.8.22. A **partial Morsification** of $g : f^{-1}(\delta) \cap X \cap B_\varepsilon \to \mathbb{C}$ is a function $\tilde{g} : f^{-1}(\delta) \cap X \cap B_\varepsilon \to \mathbb{C}$ (not necessarily holomorphic) which is a local Morsification of all isolated critical points of g in $f^{-1}(\delta) \cap X \cap \{g \neq 0\} \cap B_\varepsilon$ and which coincides with g outside a small neighborhood of these critical points.

Let $g : (X,0) \to (\mathbb{C},0)$ be a complex analytic function which is tractable at the origin with respect to $\mathscr{V}$ relative to f. Then $\tilde{\Gamma}_{f,g}$ is a complex analytic curve and for $0 < |\delta| \ll 1$ the critical points of $g|_{f^{-1}(\delta)\cap X}$ in B_ε lying outside $\{g=0\}$ are isolated. Let $\tilde{g}$ be a partial Morsification of $g : f^{-1}(\delta) \cap X \cap B_\varepsilon \to \mathbb{C}$ and, for each $i \in \{1,\ldots,q\}$, let n_i be the number of stratified Morse critical points of $\tilde{g}$ appearing on $V_i \cap f^{-1}(\delta) \cap \{g \neq 0\} \cap B_\varepsilon$.

Theorem 1.8.23. (Theorem 4.2 of (DUTERTRE; GRULHA, 2014)) Let $\beta : X \to \mathbb{Z}$ be a constructible function with respect to the stratification $\mathscr{V}$. Suppose that $g : (X,0) \to (\mathbb{C},0)$ is a complex analytic function tractable at the origin with respect to $\mathscr{V}$ relative to f. For $0 < |\delta| \ll \varepsilon \ll 1$, we have

$$\chi(X \cap f^{-1}(\delta) \cap B_\varepsilon, \beta) - \chi(X \cap g^{-1}(0) \cap f^{-1}(\delta) \cap B_\varepsilon, \beta) = \sum_{i=1}^{q} (-1)^{d_i - 1} n_i \eta(V_i, \beta).$$

In the case that $\beta = Eu_X$, the last theorem implies the following.

Corollary 1.8.24. (Corollary 4.3 of (DUTERTRE; GRULHA, 2014)) Suppose that X is equidimensional and that g is tractable at the origin with respect to $\mathscr{V}$ relative to f. For $0 < |\delta| \ll \varepsilon \ll 1$, we have

$$\chi(X \cap f^{-1}(\delta) \cap B_\varepsilon, Eu_X) - \chi(X \cap g^{-1}(0) \cap f^{-1}(\delta) \cap B_\varepsilon, Eu_X) = (-1)^{d-1} n_q.$$

If one supposes, in addition, that g is prepolar, a consequence of this result is a Lê-Greuel type formula for the Brasselet number.

Theorem 1.8.25. (Theorem 4.4 of (DUTERTRE; GRULHA, 2014)) Suppose that X is equidimensional and that g is prepolar with respect to $\mathscr{V}$ at the origin. For $0 < |\delta| \ll \varepsilon \ll 1$, we have

$$B_{f,X}(0) - B_{f,X^g}(0) = (-1)^{d-1} n_q,$$

where n_q is the number of stratified Morse critical points on the top stratum $V_q \cap f^{-1}(\delta) \cap B_\varepsilon$ appearing in a Morsification of $g : X \cap f^{-1}(\delta) \cap B_\varepsilon \to \mathbb{C}$.

Suppose that X is equipped with a Whitney stratification $\mathscr{V} = \{V_0, V_1, \ldots, V_q\}$ with $V_0 = \{0\}$, and $f, g : X \to \mathbb{C}$ have an isolated stratified singularity at the origin with respect to this stratification. We give now some results proved by Dutertre and Grulha in Section 6 of (DUTERTRE; GRULHA, 2014) in this setting.

Proposition 1.8.26. Suppose that g (resp. f) is prepolar with respect to the good stratification induced by f (resp. g) at the origin. Let $\beta : X \to \mathbb{Z}$ be a constructible function with respect to the Whitney stratification $\mathscr{V}$. For $0 < |\delta| \ll \varepsilon \ll 1$,

$$\chi(X^f \cap g^{-1}(\delta) \cap B_\varepsilon, \beta) = \chi(X^g \cap f^{-1}(\delta) \cap B_\varepsilon, \beta).$$

A corollary of this proposition is the following result.

Corollary 1.8.27. Suppose that X is equidimensional and that g (resp. f) is prepolar with respect to the good stratification induced by f (resp. g) at the origin. Then

$$B_{f,X^g}(0) = B_{g,X^f}(0).$$

In (DUTERTRE; GRULHA, 2014), the authors also related the topology of the generalized Minor fibres of f and g and some number of Morse points.

Theorem 1.8.28. Suppose that g (resp. f) is prepolar with respect to the good stratification induced by f (resp. g) at the origin. Let $\beta : X \to \mathbb{Z}$ be a constructible function with respect to the Whitney stratification $\mathscr{V}$. For $0 < |\delta| \ll \varepsilon \ll 1$,

$$\chi(X \cap f^{-1}(\delta) \cap B_\varepsilon, \beta) - \chi(X \cap g^{-1}(\delta) \cap B_\varepsilon, \beta) = \sum_{i=1}^{q} (-1)^{d_i - 1}(n_i - m_i)\eta(V_i, \beta),$$

where n_i (resp. m_i) is the number of stratified Morse critical points on the stratum $V_i \cap f^{-1}(\delta) \cap B_\varepsilon$ (resp. $V_i \cap g^{-1}(\delta) \cap B_\varepsilon$) appearing in a Morsification of $g : X \cap f^{-1}(\delta) \cap B_\varepsilon \to \mathbb{C}$ (resp. $f : X \cap g^{-1}(\delta) \cap B_\varepsilon \to \mathbb{C}$).

In the case where $\beta = Eu_X$, the last theorem implies the following result.

Corollary 1.8.29. Suppose that X is equidimensional and that g (resp. f) is prepolar with respect to the good stratification induced by f (resp. g) at the origin. Then

$$B_{f,X}(0) - B_{g,X}(0) = (-1)^{d-1}(n_q - m_q),$$

where n_q (resp. m_q) is the number of stratified Morse critical points on the top stratum $V_q \cap f^{-1}(\delta) \cap B_\varepsilon$ (resp. $V_q \cap g^{-1}(\delta) \cap B_\varepsilon$) appearing in a Morsification of $g : X \cap f^{-1}(\delta) \cap B_\varepsilon \to \mathbb{C}$ (resp. $f : X \cap g^{-1}(\delta) \cap B_\varepsilon \to \mathbb{C}$).

Applying Corollary 1.8.27 to the case where the function g is a generic linear form, one obtains the following result.

Corollary 1.8.30. Suppose that X is equidimensional. Let H be a generic hyperplane. Then

$$Eu_{X^f}(0) = B_{f,X \cap H}(0).$$

The Brasselet number $B_{f,X\cap H}(0)$ can also be compared to $B_{f,X}(0)$ using the dimension d of $(X,0)$ and the relative local polar curves.

Corollary 1.8.31. Suppose that X is equidimensional. Then

$$B_{f,X}(0) - B_{f,X\cap H}(0) = (-1)^{d-1} I(\Gamma^0_{f|_X}, X^f),$$

where H is a generic hyperplane.

At last, a consequence of Corollary 1.8.29 in the case where g is a generic linear form, is the following.

Corollary 1.8.32. Suppose that X is equidimensional. Let l be a generic linear form. Then

$$\mu^f(\Gamma_{f,l}(V_q)) - \mu^l(\Gamma_{f,l}(V_q)) = (-1)^d Eu_{f,X}(0).$$

CHAPTER

2

BRASSELET NUMBER AND FUNCTIONS WITH ONE-DIMENSIONAL SINGULAR SET

2.1 Some results for functions with arbitrary singularities

Let $(X,0)$ be a reduced equidimensional analytic germ of dimension d in an open set $U \subset \mathbb{C}^n$ and $f,g : (X,0) \to (\mathbb{C},0)$ be two germs of functions. Let $\mathscr{V}$ be a good stratification of X relative to f and suppose that the critical locus of g, $\Sigma_{\mathscr{V}} g$, is one-dimensional and that $\Sigma_{\mathscr{V}} g \cap \{f=0\} = \{0\}$.

Let $V_1,\ldots,V_q$ be the strata of $\mathscr{V}$ not contained in $\{f=0\}$. Suppose that $\{0\}$ is a stratum of $\{f=0\}$, that for each $i \in \{1,\ldots,q-1\}$, V_i is connected, V_q is equal to $X_{reg} \setminus \{f=0\}$ and that $d_i = \dim V_i$. In this case, we can construct a good stratification of X relative to f that gives us also a good stratification of $X \cap \{g=0\}$ relative to $f|_{X\cap\{g=0\}}$. We start this section with the construction of this stratification.

Lemma 2.1.1. (First stratification lemma) Let $\mathscr{V}$ be a good stratification of X relative to f and $\mathscr{V}^f$ the collection of strata of $\mathscr{V}$ contained in $\{f=0\}$ (including the stratum $\{0\}$). Then, the refinement

$$\mathscr{V}' = \Big\{V_i \setminus \Sigma_{\mathscr{V}} g, V_i \cap \Sigma_{\mathscr{V}} g, i \in \{1,\ldots,q\}\Big\} \cup \mathscr{V}^f \tag{2.1}$$

is a good stratification of X relative to f and $\mathscr{V}'^{\{g=0\}}$ is a good stratification of $X \cap \{g=0\}$ relative to $f|_{X\cap\{g=0\}}$, where

$$\mathscr{V}'^{\{g=0\}} = \Big\{V_i \cap \{g=0\} \setminus \Sigma_{\mathscr{V}} g, V_i \cap \Sigma_{\mathscr{V}} g, i \in \{1,\ldots,q\}\Big\} \cup \mathscr{V}^f \cap \{g=0\},$$

and $\mathscr{V}^f \cap \{g=0\}$ denotes the collection of strata of type $V^f \cap \{g=0\}$, with $V^f \in \mathscr{V}^f$. Moreover, if g is tractable at the origin with respect to $\mathscr{V}$ relative to f, then g is tractable at the origin with respect to $\mathscr{V}'$ relative to f.

Proof. Since $\Sigma_{\mathscr{V}} g \cap \{f=0\} = \{0\}$ and $V_1,\ldots,V_q$ are the strata of $\mathscr{V}$ not contained in $\{f=0\}$, we can write $\Sigma_{\mathscr{V}} g = \{0\} \cup (V_1 \cap \Sigma_{\mathscr{V}} g) \cup \ldots \cup (V_q \cap \Sigma_{\mathscr{V}} g)$. Let us show that the refinement of $\mathscr{V}$,

$$\mathscr{V}' = \Big\{V_i \setminus \Sigma_{\mathscr{V}} g, V_i \cap \Sigma_{\mathscr{V}} g, i \in \{1,\ldots,q\}\Big\} \cup \mathscr{V}^f,$$

is a good stratification of X relative to f. Since the collection of strata contained in $\{f=0\}$ was not refined, $\{f=0\}$ is a union of strata of $\mathscr{V}'$. Now we will show that

$$\Big\{V_\alpha \in \mathscr{V}'; V_\alpha \not\subseteq \{f=0\}\Big\} = \Big\{V_i \setminus \Sigma_{\mathscr{V}} g, V_i \cap \Sigma_{\mathscr{V}} g, i \in \{1,\ldots,q\}\Big\}$$

is a Whitney stratification of $X \setminus \{f=0\}$. We can refine this stratification to obtain a Whitney stratification. But since $\mathscr{V}$ is a good stratification of X relative to f, $\{V_\alpha \in \mathscr{V}; V_\alpha \not\subseteq \{f=0\}\}$ is a Whitney stratification of $X \setminus \{f=0\}$. Since $\Sigma_{\mathscr{V}} g$ is closed, $V_\alpha \setminus \Sigma_{\mathscr{V}} g$ is an open subset of V_α and then Whitney's condition (b) is verified over the strata of type $V_\alpha \setminus \Sigma_{\mathscr{V}} g$. So, the refinement should be done only over the stratum of type $\Sigma_{\mathscr{V}} g \cap V_i$. Since $\Sigma_{\mathscr{V}} g$ is one-dimensional, a refinement of $\Sigma_{\mathscr{V}} g \cap V_i$ would be done by taking off a finite number of points. So, in a sufficiently small neighborhood of the origin, Whitney's condition (b) is verified over $\{V_i \setminus \Sigma_{\mathscr{V}} g, V_i \cap \Sigma_{\mathscr{V}} g, i \in \{1,\ldots,q\}\}$.

At last, let us verify the Thom condition. Let p be a point in $V_\beta \subseteq \{f=0\}$ and (p_k) be a sequence of points in $V_\alpha \not\subseteq \{f=0\}$. Suppose that $\lim_{k\to\infty} p_k = p$ and that $\lim_{k\to\infty} T_{p_k} V(f|_{V_\alpha} - f|_{V_\alpha}(p_k)) = T$. We must show that $T_p V_\beta \subseteq T$.

Let $V_\alpha = V_i \cap \Sigma_{\mathscr{V}} g$ for some $i \in \{1,\ldots,q\}$. Since (p_k) is a sequence of points in $V_i \cap \Sigma_{\mathscr{V}} g$ and $\Sigma_{\mathscr{V}} g$ is one-dimensional, (p_k) must converge to the origin, that is, $p=0$ and $V_\beta = \{0\}$. So, $\{0\} = T_p V_\beta \subseteq T$.

Let us now verify the Thom condition for $V_\alpha = V_i \setminus \Sigma_{\mathscr{V}} g, i \in \{1,\ldots,q\}$. We have

$$T_{p_k} V(f|_{V_i \setminus \Sigma_{\mathscr{V}} g} - f|_{V_i \setminus \Sigma_{\mathscr{V}} g}(p_k)) = T_{p_k} V(f|_{V_i} - f|_{V_i}(p_k)),$$

and since $\mathscr{V}$ is a good stratification of X relative to f, the Thom condition is verified for (V_i, V_β). Hence, $T_p V_\beta \subseteq \lim_{k\to\infty} T_{p_k} V(f|_{V_i} - f|_{V_i}(p_k)) = \lim_{k\to\infty} T_{p_k} V(f|_{V_i \setminus \Sigma_{\mathscr{V}} g} - f|_{V_i \setminus \Sigma_{\mathscr{V}} g}(p_k)) = T$. Therefore, $\mathscr{V}'$ is a good stratification of X relative to f.

Let us now show that $\mathscr{V}'^{\{g=0\}}$ is a good stratification of $X \cap \{g=0\}$ relative to $f|_{X\cap\{g=0\}}$,

$$\mathscr{V}'^{\{g=0\}} = \Big\{V_i \cap \{g=0\} \setminus \Sigma_{\mathscr{V}} g, V_i \cap \Sigma_{\mathscr{V}} g, i \in \{1,\ldots,q\}\Big\} \cup \mathscr{V}^f \cap \{g=0\}.$$

Since $\Sigma_{\mathscr{V}} g \cap \{f=0\} = \{0\}, \{g=0\}$ intersects each stratum of $\mathscr{V}^f$ transversely. Therefore, for each $\tilde{V}_i \in \mathscr{V}^f$, $\{g=0\} \cap \tilde{V}_i$ is a complex analytic submanifold of $\tilde{V}_i$. Hence, $\{f=0\} \cap \{g=0\}$ is a union of strata contained in $\mathscr{V}^f \cap \{g=0\}$. Now, we will verify that

$$\{V_i \cap \{g=0\} \setminus \Sigma_{\mathscr{V}} g, V_i \cap \Sigma_{\mathscr{V}} g, i \in \{1,\ldots,q\}\}$$

stratification of $X \cap \{g = 0\} \setminus \{f = 0\}$. Consider a pair of strata of type $(V_i \cap \Sigma_{\mathscr{V}} g, V_j \cap \Sigma_{\mathscr{V}} g)$. If necessary, we can refine these strata to guarantee Whitney's condition (b). Since $\Sigma_{\mathscr{V}} g$ has dimension one, this refinement would be given by taking off a finite number of points. Therefore, in a sufficiently small neighborhood of the origin, Whitney's condition (b) is verified for this type of stratum. Now, let us verify this condition for pairs of strata of the type $(V_i \cap \{g=0\} \setminus \Sigma_{\mathscr{V}} g, V_j \cap \{g=0\} \setminus \Sigma_{\mathscr{V}} g)$ and $(V_i \cap \{g=0\} \setminus \Sigma_{\mathscr{V}} g, V_j \cap \Sigma_{\mathscr{V}} g)$.

1. Let us show that $(V_i \cap \{g=0\} \setminus \Sigma_{\mathscr{V}} g, V_j \cap \{g=0\} \setminus \Sigma_{\mathscr{V}} g)$ is Whitney regular. Since these strata contain no critical points of $g, V_i \cap \{g=0\}$ and $V_j \cap \{g=0\}$ are transverse intersections. Therefore, $(V_i \cap \{g=0\} \setminus \Sigma_{\mathscr{V}} g, V_j \cap \{g=0\} \setminus \Sigma_{\mathscr{V}} g)$ is Whitney regular, since (V_i, V_j) is Whitney regular. (See (ORRO; TROTMAN, 2010)).

2. Let us show that $(V_i \cap \{g=0\} \setminus \Sigma_{\mathscr{V}} g, V_j \cap \Sigma_{\mathscr{V}} g)$ is Whitney regular. The intersection $V_i \cap \{g=0\}$ is transverse, since it contains no critical points of g. Whitney's condition (b) could fail over $V_j \cap \Sigma_{\mathscr{V}} g$, but since $\Sigma_{\mathscr{V}} g$ is one-dimensional, we can refine this stratum by taking off a finite number of points and ensure that, in a sufficiently small neighborhood of the origin, $(V_i \cap \{g=0\} \setminus \Sigma_{\mathscr{V}} g, V_j \cap \Sigma_{\mathscr{V}} g)$ is Whitney regular.

Let us now verify the Thom condition over the strata of $\mathscr{V}'^{\{g=0\}}$. Let $V_\alpha \nsubseteq \{f=0\}$ and $V_\beta \subseteq \{f=0\}$ be strata of $\mathscr{V}'^{\{g=0\}}$, p be a point in V_β and (p_i) be a sequence of points in V_α. Suppose that $\lim\limits_{i\to\infty}(p_i) = p$ and that $\lim\limits_{i\to\infty} T_{p_i}V(f|_{V_\alpha} - f|_{V_\alpha}(p_i)) = T$. We must show that $T_pV_\beta \subseteq T$. If $V_\beta = \{0\}$, $T_pV_\beta = \{0\}$ and then $T_pV_\beta \subseteq T$. Suppose now that $V_\beta \neq \{0\}$. Notice that, since $\Sigma_{\mathscr{V}} g \cap \{f=0\} = \{0\}$ and $\{0\} \neq V_\beta \subseteq \{f=0\}, p \in V_\beta$ implies that $p \notin \Sigma_{\mathscr{V}} g$. As we have seen above, it is sufficient verify the Thom condition for $V_\alpha = V_j \cap \{g=0\} \setminus \Sigma_{\mathscr{V}} g$. We have,

$$\begin{aligned} T_{p_i}V(f|_{V_\alpha} - f|_{V_\alpha}(p_i)) &= T_{p_i}V(f|_{V_j\cap\{g=0\}\setminus\Sigma_{\mathscr{V}} g} - f|_{V_j\cap\{g=0\}\setminus\Sigma_{\mathscr{V}} g}(p_i)) \\ &= T_{p_i}V(\tilde{f}|_{V_j} - \tilde{f}|_{V_j}(p_i)) \cap T_{p_i}V(\tilde{g}), \end{aligned}$$

which implies that

$$\begin{aligned} \lim_{i\to\infty} T_{p_i}V(f|_{V_\alpha} - f|_{V_\alpha}(p_i)) &= \lim_{i\to\infty} T_{p_i}V(\tilde{f}|_{V_j} - \tilde{f}|_{V_j}(p_i)) \cap T_{p_i}V(\tilde{g}) \\ &\subseteq \lim_{i\to\infty} T_{p_i}V(\tilde{f}|_{V_j} - \tilde{f}|_{V_j}(p_i)) \cap \lim_{i\to\infty} T_{p_i}V(\tilde{g}), \end{aligned}$$

where $\tilde{f}$ and $\tilde{g}$ denote analytic extensions of f and g to an open neighborhood of the origin in the ambient space $(U,0)$ of $(X,0)$. Since $p, p_i \notin \Sigma_{\mathscr{V}} g$, if $\lim\limits_{i\to\infty} T_{p_i}V(\tilde{f}|_{V_j} - \tilde{f}|_{V_j}(p_i)) = T_1$, we have that $T \subseteq T_1 \cap T_pV(\tilde{g})$. On the other hand, if $V_\beta = \tilde{V}_\lambda \cap \{g=0\}$, with $\tilde{V}_\lambda \in \mathscr{V}^f$ and $p \notin \Sigma_{\mathscr{V}} g$,

$$T_pV_\beta = T_p(\tilde{V_\lambda} \cap \{g=0\}) = T_p\tilde{V_\lambda} \cap T_pV(\tilde{g}).$$

Since the Thom condition is valid over $\mathscr{V}'$, we have that $T_p\tilde{V_\lambda} \subseteq T_1$. Since g is tractable at the origin with respect to $\mathscr{V}$ relative to f, for $p \notin \Sigma_{\mathscr{V}} g, T_pV(\tilde{g})$ intersects $T_p\tilde{V}_\lambda$ transversely. Therefore, $T_pV(\tilde{g})$ intersects T_1 transversely. This implies that

$$\begin{aligned} T = \lim_{i\to\infty} T_{p_i}V(\tilde{f}|_{V_j} - \tilde{f}|_{V_j}(p_i)) \cap T_{p_i}V(\tilde{g}) &= \lim_{i\to\infty} T_{p_i}V(\tilde{f}|_{V_j} - \tilde{f}|_{V_j}(p_i)) \cap \lim_{i\to\infty} T_{p_i}V(\tilde{g}) \\ &= T_1 \cap T_pV(\tilde{g}). \end{aligned}$$

Therefore,

$$T_pV_\beta = T_p(\tilde{V_\lambda} \cap \{g=0\}) = T_p\tilde{V_\lambda} \cap T_pV(\tilde{g}) \subseteq T_1 \cap T_pV(\tilde{g}) = T.$$

Suppose now that g is tractable at the origin with respect to $\mathscr{V}$ relative to f. Let us show that g is tractable at the origin with respect to $\mathscr{V}'$ relative to f. For that we should verify that (1): $\dim_0 \tilde{\Gamma}_{f,g}(\mathscr{V}') \leq 1$; and that (2): $g|_{V_\alpha}$ has no singularity in a neighborhood of the origin, except perhaps the origin itself, for $V_\alpha \in \mathscr{V}'$ contained in $\{f=0\}$. Condition (2) is valid, since we have not refined the strata contained in $\{f=0\}$. Let us verify condition (1). Since for each $V_i \nsubseteq \{f=0\}, \Sigma g|_{V_i} \subset \{g=0\}$, we have

$$\tilde{\Gamma}_{f,g}(\mathscr{V}') = \bigcup_{i=1}^{q} \overline{\Sigma(f,g)|_{(V_i \setminus \Sigma g|_{V_i}) \setminus \{f=0\} \cup \{g=0\}}} = \bigcup_{i=1}^{q} \overline{\Sigma(f,g)|_{V_i \setminus \{f=0\} \cup \{g=0\}}} = \tilde{\Gamma}_{f,g}(\mathscr{V}).$$

Then $\dim_0 \tilde{\Gamma}_{f,g}(\mathscr{V}') = \dim_0 \tilde{\Gamma}_{f,g}(\mathscr{V}) \leq 1$ and condition (1) is verified. Therefore, g is tractable at the origin with respect to $\mathscr{V}'$. ■

In the beginning we have supposed that $\Sigma_{\mathscr{V}} g$ is one-dimensional. Let us now give the description we will use for this set. By definition, $\Sigma_{\mathscr{V}} g = \bigcup_{V_\alpha \in \mathscr{V}} \Sigma g|_{V_\alpha}$, but since $\Sigma_{\mathscr{V}} \cap \{f=0\} = \{0\}$, we can write $\Sigma_{\mathscr{V}} g = \bigcup_{\alpha=1}^{q} \Sigma g|_{V_\alpha} \cup \{0\}$, where V_α is a stratum not contained in $\{f=0\}$. Since $\Sigma_{\mathscr{V}} g$ is one-dimensional at the origin, for each stratum $V_\alpha \in \mathscr{V}, \Sigma g|_{V_\alpha}$ is either one-dimensional or the origin itself, and $\overline{\Sigma g|_{V_\alpha}} = \Sigma g|_{V_\alpha} \cup \{0\}$. Let us verify that $\overline{\Sigma g|_{V_\alpha}}$ is an analytic set. Using the description of the critical space in Chapter 4 of (LOOIJENGA, 1984), suppose that $\dim_0 \overline{V_\alpha} = k+1$, let $f_1, \ldots, f_l$ be the defining functions of $\overline{V_\alpha}$ at 0 and $J_i(f_1, \ldots, f_l, g)$ denote the ideal in $\mathscr{O}_0^N$ generated by the $i \times i$-minors of the Jacobian matrix of the map $(f_1, \ldots, f_l, g)$. The critical space of $g|_{\overline{V_\alpha}}$ is the subspace of $\mathscr{O}_0^N$ defined at 0 by the vanishing of $f_1, \ldots, f_l$ and the $(N-k) \times (N-k)$-minors of the Jacobian matrix of the map $(f_1, \ldots, f_l, g)$. So,

$$\Sigma g|_{\overline{V_\alpha}} = V(f_1, \ldots, f_l) \cap V(J_{N-k}(f_1, \ldots, f_l, g)).$$

If the Jacobian matrix of the map $(f_1, \ldots, f_l)$ has maximal rank at a point x in $\overline{V_\alpha}$, then $x \in \Sigma g|_{\overline{V_\alpha}}$ if, and only if, $x \in V(J_{N-k}(f_1, \ldots, f_l, g))$, that is, x is a critical point of $g|_{V_\alpha}$. If the Jacobian matrix of $(f_1, \ldots, f_l)$ has no maximal rank at x, then x is automatically a point of $V(J_{N-k}(f_1, \ldots, f_l, g))$. Therefore, $\Sigma g|_{\overline{V_\alpha}} = \Sigma g|_{V_\alpha} \cup Sing(\overline{V_\alpha})$, where $Sing(\overline{V_\alpha})$ denote the set of singular points of $\overline{V_\alpha}$. Then, $\Sigma g|_{V_\alpha} = \Sigma g|_{\overline{V_\alpha}} \setminus Sing(\overline{V_\alpha})$ and $\overline{\Sigma g|_{V_\alpha}} = \overline{\Sigma g|_{\overline{V_\alpha}} \setminus Sing(\overline{V_\alpha})}$. By definition, $\Sigma g|_{\overline{V_\alpha}}$ is an analytic set and by Theorem IV. 2.4.1 of (LOJASIEWICZ, 1991), so is $Sing(\overline{V_\alpha})$. Then, by Proposition IV.8.3.5 (LOJASIEWICZ, 1991), the closure $\overline{\Sigma g|_{\overline{V_\alpha}} \setminus Sing(\overline{V_\alpha})}$ is an analytic set. So, by a consequence of the Remmert-Stein theorem (page 241, (LOJASIEWICZ, 1991)), $\overline{\Sigma g|_{V_\alpha}}$ has an irreducible decomposition into one-dimensional subvarieties, which will be called branches,

$$\overline{\Sigma g|_{V_\alpha}} = \Sigma g|_{V_\alpha} \cup \{0\} = b_{\alpha_1} \cup \ldots \cup b_{\alpha_t}.$$

Making this process for each stratum V_α, we can decompose $\Sigma_{\mathscr{V}} g$ into branches b_j,

$$\Sigma_{\mathscr{V}} g = \bigcup_{\alpha=1}^{q} \Sigma g|_{V_\alpha} \cup \{0\} = b_1 \cup \ldots \cup b_r,$$

where $b_j \subseteq V_\alpha$, for some $\alpha \in \{1,\ldots,q\}$. Notice that a stratum V_α can contain no branch and that a stratum V_j can contain more than one branch, but, the way we described, a branch can not be contained in two different strata. Let δ be a regular value of $f, 0 < |\delta| \ll 1$, and let us write, for each $j \in \{1,\ldots,r\}, f^{-1}(\delta) \cap b_j = \{x_{i_1},\ldots,x_{i_{k(j)}}\}$. So, in this case, the local degree m_{f,b_j} of $f|_{b_j}$ is $k(j)$. Let ε be sufficiently small such that the local Euler obstruction of X and of X^g are constant on $b_j \cap B_\varepsilon$. In this case, we denote by $Eu_X(b_j)$ (respectively, $Eu_{X^g}(b_j)$) the local Euler obstruction of X (respectively, X^g) at a point of $b_j \cap B_\varepsilon$.

The next theorem calculates, in our setting, the difference $B_{f,X}(0) - B_{f,X^g}(0)$ without the prepolarity of g with respect to the good stratification relative to f at the origin.

We fix the good stratification $\mathscr{V}'$ of X relative to f constructed in Lemma 2.1.1 given as a refinement of the initial good stratification $\mathscr{V}$ of X relative to f.

Theorem 2.1.2. Suppose that g is tractable at the origin with respect to $\mathscr{V}$ relative to f. Then, for $0 < |\delta| \ll \varepsilon \ll 1$,

$$B_{f,X}(0) - B_{f,X^g}(0) - \textstyle\sum_{j=1}^{r} m_{f,b_j}(Eu_X(b_j) - Eu_{X^g}(b_j)) = (-1)^{d-1}m,$$

where m is the number of stratified Morse critical points of a partial Morsification of $g : X \cap f^{-1}(\delta) \cap B_\varepsilon \to \mathbb{C}$ appearing on $X_{reg} \cap f^{-1}(\delta) \cap \{g \neq 0\} \cap B_\varepsilon$.

Proof. By Corollary 1.8.24, if $0 < |\delta| \ll \varepsilon \ll 1$,

$$\chi(X \cap f^{-1}(\delta) \cap B_\varepsilon, Eu_X) - \chi(X \cap g^{-1}(0) \cap f^{-1}(\delta) \cap B_\varepsilon, Eu_X) = (-1)^{d-1}m,$$

that is,

$$B_{f,X}(0) - \chi(X \cap g^{-1}(0) \cap f^{-1}(\delta) \cap B_\varepsilon, Eu_X) = (-1)^{d-1}m.$$

Let us now calculate $\chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon, Eu_X)$.

$$\begin{aligned}\chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon, Eu_X) &= \sum_{V_i\not\subseteq\Sigma_{\mathcal{V}}g}\chi(V_i\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon)Eu_X(V_i)\\ &+ \sum_{V_i\subseteq\Sigma_{\mathcal{V}}g}\chi(V_i\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon)Eu_X(V_i).\end{aligned}$$

If $V_i\not\subseteq\Sigma_{\mathcal{V}}g$, V_i intersects $\{g=0\}$ transversely and $Eu_X(V_i)=Eu_{X^g}(V_i\cap g^{-1}(0))$. Let us denote by $W_1,\ldots,W_s$ the strata contained in $\Sigma_{\mathcal{V}}g$. Then, we obtain

$$\begin{aligned}\chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon, Eu_X) &= \sum_{V_i\not\subseteq\Sigma_{\mathcal{V}}g}\chi(V_i\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon)Eu_{X^g}(V_i\cap g^{-1}(0))\\ &+ \sum_{l=1}^{s}\chi(W_l\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon)Eu_X(W_l).\end{aligned}$$

For each $W_l\subseteq\Sigma_{\mathcal{V}}g$, let k_l be the number of branches b_{l_t} containing in W_l. Then,

$$\chi(W_l\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon)=\sum_{b_{l_t}\subseteq W_l}\chi(b_{l_t}\cap f^{-1}(\delta)\cap B_\varepsilon)=\sum_{t=1}^{k_l}m_{f,b_{l_t}}$$

and then

$$\sum_{l=1}^{s}\chi(W_l\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon)Eu_X(W_l)=\sum_{j=1}^{r}m_{f,b_j}Eu_X(b_j).$$

Therefore,

$$\begin{aligned}\chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon, Eu_X) &= \sum_{V_i\not\subseteq\Sigma_{\mathcal{V}}g}\chi(V_i\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon)Eu_{X^g}(V_i\cap g^{-1}(0))\\ &+ \sum_{j=1}^{r}m_{f,b_j}Eu_X(b_j).\end{aligned}$$

On the other hand,

$$\begin{aligned}B_{f,X^g}(0) &= \sum\chi(V_i\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon)Eu_{X^g}(V_i\cap g^{-1}(0))\\ &= \sum_{V_i\not\subseteq\Sigma_{\mathcal{V}}g}\chi(V_i\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon)Eu_{X^g}(V_i\cap g^{-1}(0))\\ &+ \sum_{l=1}^{s}\chi(W_l\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon)Eu_{X^g}(W_l\cap g^{-1}(0)).\end{aligned}$$

Using the notation of branches again, we obtain

$$\begin{aligned} B_{f,X^g}(0) &= \sum_{V_i \not\subseteq \Sigma_{\mathscr{V}} g} \chi(V_i \cap g^{-1}(0) \cap f^{-1}(\delta) \cap B_\varepsilon) Eu_{X^g}(V_i \cap g^{-1}(0)) \\ &+ \sum_{j=1}^{r} m_{f,b_j} Eu_{X^g}(b_j). \end{aligned}$$

Therefore,

$$\begin{aligned} \chi(X \cap g^{-1}(0) \cap f^{-1}(\delta) \cap B_\varepsilon, Eu_X) &= \sum_{V_i \not\subseteq \Sigma_{\mathscr{V}} g} \chi(V_i \cap g^{-1}(0) \cap f^{-1}(\delta) \cap B_\varepsilon) Eu_{X^g}(V_i \cap g^{-1}(0)) \\ &+ \sum_{j=1}^{r} m_{f,b_j} Eu_X(b_j) \\ &= B_{f,X^g}(0) \\ &- \sum_{j=1}^{r} m_{f,b_j} (Eu_{X^g}(b_j) - Eu_X(b_j)) \end{aligned} \tag{2.2}$$

Hence,

$$B_{f,X}(0) - B_{f,X^g}(0) - \sum_{j=1}^{r} m_{f,b_j}(Eu_X(b_j) - Eu_{X^g}(b_j)) = (-1)^{d-1} m.$$

■

This result shows that, in this case, the Lê-Greuel formula for the Brasselet number of g presents a defect given by the sum of differences of Euler obstructions above.

Remark 2.1.3. In Theorem 2.1.2, if we suppose that g is prepolar, using the same previous notations, we have that the set $\Sigma_{\mathscr{V}} g \cap \{f = \delta\}$ is empty, which gives us the Lê-Greuel type formula

$$B_{f,X}(0) - B_{f,X^g}(0) = (-1)^{d-1} m,$$

proved by Dutertre and Grulha in (DUTERTRE; GRULHA, 2014).

A consequence of Theorem 2.1.2 is a relation between the differences of the Euler obstruction at the origin and at the branches. We need first the following lemma.

Lemma 2.1.4. Let $V \subset \mathbb{C}^N$ be an analytic complex subset of dimension d and $l : \mathbb{C}^N \to \mathbb{C}$ be a generic linear form. Then $l^{-1}(0)$ is transverse to $V \setminus \{0\}$.

Proof. We fix local coordinates $(x_1, \dots, x_N)$ in $\mathbb{C}^N$ and define, for each $a = (a_1, \dots, a_N) \in \mathbb{C}^N$, $l_a(x) = a_1 x_1 + \cdots + a_N x_N$. Let $W = \{(x,a) \in \mathbb{C}^N \times \mathbb{C}^N; x \in V \setminus \{0\}, l_a(x) = 0\}$. Then $\dim W =$

$2N - (N - d + 1) = N + d - 1$. Consider the projection $\pi : W \to \mathbb{C}^N$ given by $(x,a) \mapsto a$ and let $\Delta \subset \mathbb{C}^N$ be the discriminant of π.

If $a \in \mathbb{C}^N \setminus \Delta$, then $V \setminus \{0\}$ intersects $\{l_a = 0\}$ transversely and $\pi^{-1}(a)$, which is equal to $V \setminus \{0\} \cap \{l_a|_{V\setminus\{0\}} = 0\}$, is a submanifold of W with dimension $\dim W - N = N + d - 1 - N = d - 1$. ■

Let l be a generic linear form over $\mathbb{C}^n$, X a complex analytic space equipped with a Whitney stratification and $\mathscr{V}$ the good stratification of X induced by l. Then Lemma 2.1.1 can be applied to $\mathscr{V}$ and we obtain a good stratification $\mathscr{V}'$ of X relative to l such that $\mathscr{V}'^{\{g=0\}}$ is a good stratification of X^g relative to $l_{|\{g=0\}}$.

Corollary 2.1.5. Let l be a generic linear form over X and $\mathscr{V}$ the good stratification of X induced by l. Suppose that g is tractable at the origin with respect to $\mathscr{V}$ relative to l. For $0 < |\delta| \ll \varepsilon \ll 1$, we have,

$$Eu_X(0) - Eu_{X^g}(0) - \textstyle\sum_{j=1}^{r} m_{b_j}(Eu_X(b_j) - Eu_{X^g}(b_j)) = (-1)^{d-1}m,$$

where m is the number of stratified Morse critical points of a partial Morsification of $g : X \cap l^{-1}(\delta) \cap B_\varepsilon \to \mathbb{C}$ appearing on $X_{reg} \cap l^{-1}(\delta) \cap \{g \neq 0\} \cap B_\varepsilon$ and m_{b_j} is the multiplicity of the branch b_j at the origin.

Proof. Since l is a generic linear form over X and $\Sigma_{\mathscr{V}} g$ is one-dimensional, by Lemma 2.1.4, $\Sigma_{\mathscr{V}} g \cap \{l = 0\} = \{0\}$. Notice that, since l is generic, the local degree m_{l,b_j} of $l|_{b_j}$ at the origin is precisely the multiplicity of the branch b_j at the origin, which we will denote by m_{b_j}. Therefore, applying the previous theorem, since $B_{l,X}(0) = Eu_X(0)$ and $B_{l,X^g}(0) = Eu_{X^g}(0)$, we have the formula. ■

Remark 2.1.6. By Corollary 2.1.5, the number m of stratified Morse critical points of the Morsification of $g : X \cap l^{-1}(\delta) \cap B_\varepsilon \to \mathbb{C}$ appearing on $X_{reg} \cap l^{-1}(\delta) \cap \{g \neq 0\} \cap B_\varepsilon$ does not depends on the generic linear form l.

2.2 Some results for functions with isolated singularity

In Section 6 of (DUTERTRE; GRULHA, 2014), Dutertre and Grulha proved several relations between the Brasselet number of functions with isolated singularity and other invariants. In this section, we provide the generalization of some of their results to the context we describe in the following. Let X be an analytic complex space and $\mathscr{W} = \{W_0, \ldots, W_q\}$ be a Whitney stratification of X with $W_0 = \{0\}$. From now on, we consider f and g functions defined over X such that f has an isolated singularity at the origin, $\Sigma_{\mathscr{W}} g$ is a one-dimensional analytic set and $\Sigma_{\mathscr{W}} g \cap \{f = 0\} = \{0\}$. Let $\mathscr{V}$ be the good stratification of X induced by f and suppose that g is tractable at the origin with respect to $\mathscr{V}$ relative to f.

Remark 2.2.1. Notice that, in this setting, $\Sigma_{\mathscr{W}} g = \Sigma_{\mathscr{V}} g$. By definition, $\Sigma_{\mathscr{V}} g = \cup_{V_\alpha \in \mathscr{V}} \Sigma g|_{V_\alpha}$. Since g is tractable at the origin with respect to $\mathscr{V}$ relative to f, for all strata $V_\alpha = W_i \cap \{f = 0\}, W_i \in \mathscr{W}, g|_{V_\alpha}$ has no critical points, except perhaps the origin. On the other hand, $\Sigma_{\mathscr{W}} g \cap \{f = 0\} = \{0\}$ implies that $\Sigma g|_{W_i \setminus \{f=0\}} = \Sigma g|_{W_i}$, for all $W_i \in \mathscr{W}$. Therefore, $\Sigma_{\mathscr{W}} g = \Sigma_{\mathscr{V}} g$.

Let us describe the stratification we will use in this section.

Lemma 2.2.2. (**Second stratification lemma**) Let $\mathscr{V}$ be the good stratification of X induced by $f, \mathscr{V}^f$ the collection of strata of $\mathscr{V}$ contained in $\{f = 0\}$ and suppose that g is tractable at the origin with respect to $\mathscr{V}$ relative to f. Consider the refinement of $\mathscr{V}$,

$$\mathscr{V}'' = \{V_i \setminus \{g = 0\}, V_i \cap \{g = 0\} \setminus \Sigma_{\mathscr{W}} g, V_i \cap \Sigma_{\mathscr{W}} g, V_i \in \mathscr{V}\} \cup \{0\}. \tag{2.3}$$

Then $\mathscr{V}''$ is a good stratification of X relative to g such that $\mathscr{V}''^{\{f=0\}}$,

$$\mathscr{V}''^{\{f=0\}} = \{V_i \cap \{f = 0\} \setminus \{g = 0\}, V_i \cap \{f = 0\} \cap \{g = 0\} \setminus \Sigma_{\mathscr{W}} g, V_i \in \mathscr{V}^f\} \cup \{0\},$$

is a good stratification of X^f relative to $g|_{X^f}$.

Moreover, f is prepolar at the origin with respect to $\mathscr{V}''$ relative to g.

Proof. Let us first show that $\mathscr{V}''$ is a good stratification of X with respect to g.

1. $V(g)$ is a union of strata of type $V_i \cap \{g = 0\} \setminus \Sigma_{\mathscr{W}} g$ and $V_i \cap \Sigma_{\mathscr{W}} g$;

2. Let us show that

$$\{V_i \setminus \{g = 0\}, V_i \in \mathscr{V}\} = \{W_i \cap \{f = 0\} \setminus \{g = 0\}, W_i \setminus \big(\{f = 0\} \cup \{g = 0\}\big), W_i \in \mathscr{W}\}$$

   is a Whitney stratification of $X \setminus \{g = 0\}$. Since f has an isolated singularity at the origin, $\{f = 0\}$ intersects each strata W_i transversely. Therefore, since $\mathscr{W}$ is a Whitney stratification of X, $\{V_i \setminus \{g = 0\}, V_i \in \mathscr{V}\}$ satisfies Whitney's condition (b).

3. Let us verify the Thom condition. Let $V_\lambda \nsubseteq V(g), V_\gamma \subset V(g)$ be strata of $\mathscr{V}''$ and let (p_k) be a sequence of points of V_λ converging to a point $p \in V_\gamma$. Suppose that the sequence of tangent spaces $T_{p_k} V(g|_{V_\lambda} - g|_{V_\lambda}(p_i))$ converges to T. We must show that $T_p V_\gamma \subseteq T$. If $p = 0$, then $V_\gamma = \{0\}$ and $\{0\} = T_p V_\gamma \subseteq T$. Suppose now that $p \neq 0$ and consider $V_\gamma = W_j \cap \Sigma_{\mathscr{W}} g, W_j \in \mathscr{W}$. Since Thom stratifications always exist, one may take a refinement of $W_j \cap \Sigma_{\mathscr{W}} g$ that guarantees that the Thom condition is valid over this strata. Since $\Sigma_{\mathscr{W}} g$ is one-dimensional, this refinement would be given by taking off a finite number of points. Therefore, working on a sufficiently small neighborhood of the origin, Thom condition is verified over $W_j \cap \Sigma_{\mathscr{W}} g$. For $p \neq 0$, we have two options for $V_\lambda \nsubseteq V(g)$, which are $W_i \setminus \big(\{f = 0\} \cup \{g = 0\}\big)$ and $W_i \cap \{f = 0\} \setminus \{g = 0\}$, where $W_i \in \mathscr{W}$.

Suppose that $V_\lambda = W_i \setminus \big(\{f=0\}\cup\{g=0\}\big), W_i \in \mathscr{W}$, and let $\tilde{g}$ be an analytic extension of g to an open neighborhood of the origin in $\mathbb{C}^n$. Then

$$\begin{aligned}
\lim_{k\to\infty} T_{p_k}V(g|_{V_\lambda} - g|_{V_\lambda}(p_k)) &= \lim_{k\to\infty} T_{p_k}V(g|_{W_i\setminus\big(\{f=0\}\cup\{g=0\}\big)} - g|_{W_i\setminus\big(\{f=0\}\cup\{g=0\}\big)}(p_k)) \\
&= \lim_{k\to\infty} T_{p_k}V(\tilde{g}-\tilde{g}(p_k)) \cap T_{p_k}W_i \\
&\subseteq \lim_{k\to\infty} T_{p_k}V(\tilde{g}-\tilde{g}(p_k)) \cap \lim_{k\to\infty} T_{p_k}W_i
\end{aligned}$$

If $p \in V_\gamma = V_j \cap \{g=0\} \setminus \Sigma_{\mathscr{W}} g, V_j \in \mathscr{V}$ and writing $\lim_{k\to\infty} T_{p_k}W_i = T_1$, since $p \notin \Sigma_{\mathscr{W}} g$, the last limit is equal to

$$T_pV(\tilde{g}-\tilde{g}(p)) \cap T_1 = T_pV(\tilde{g}) \cap T_1.$$

Suppose now that $V_\gamma = W_j \cap \{g=0\} \setminus \big(\{f=0\} \cup \Sigma_{\mathscr{W}} g\big), W_j \in \mathscr{W}$. Then

$$T_pV_\gamma = T_p\big(W_j \cap \{g=0\} \setminus (\{f=0\} \cup \Sigma_{\mathscr{W}} g)\big) = T_pW_j \cap T_pV(\tilde{g}).$$

By Whitney's condition (a) over strata of $\mathscr{W}$, $T_pW_j \subseteq T_1$. Since g is tractable at the origin with respect to $\mathscr{V}$ relative to $f, T_pV(\tilde{g})$ intersects $T_p(W_j \setminus \{f=0\}) = T_pW_j$ transversely at $p \notin \Sigma_{\mathscr{W}} g$. Therefore, the intersection $T_pV(\tilde{g}) \cap T_1$ is transverse. Then we conclude that

$$\lim_{k\to\infty} T_{p_k}V(g|_{V_\lambda} - g|_{V_\lambda}(p_k)) = T_pV(\tilde{g}) \cap T_1.$$

Therefore,

$$T_pV_\gamma = T_pW_j \cap T_pV(\tilde{g}) \subseteq T_1 \cap T_pV(\tilde{g}).$$

Now, let $\tilde{f}$ be an analytic extension of f to the ambient space U of X. If $V_\gamma = W_j \cap \{g=0\} \cap \{f=0\} \setminus \Sigma_{\mathscr{W}} g, W_j \in \mathscr{W}$,

$$T_pV_\gamma = T_p(W_j \cap \{g=0\} \cap \{f=0\} \setminus \Sigma_{\mathscr{W}} g) = T_pW_j \cap T_pV(\tilde{g}) \cap T_pV(\tilde{f}).$$

Using Whitney's condition (a) over strata of $\mathscr{W}$ again,

$$T_pV_\gamma = T_pW_j \cap T_pV(\tilde{g}) \cap T_pV(\tilde{f}) \subseteq T_1 \cap T_pV(\tilde{g}).$$

Let us now analyze the case where $V_\lambda = W_i \cap \{f=0\} \setminus \{g=0\}, W_i \in \mathscr{W}$.

Then

$$\begin{aligned}
\lim_{k\to\infty} T_{p_k}V(g|_{V_\lambda} - g|_{V_\lambda}(p_k)) &= \lim_{k\to\infty} T_{p_k}V(g|_{W_i\cap\{f=0\}\setminus\{g=0\}} - g|_{W_i\cap\{f=0\}\setminus\{g=0\}}(p_k)) \\
&= \lim_{k\to\infty} T_{p_k}V(\tilde{g}-\tilde{g}(p_k)) \cap T_{p_k}W_i \cap T_{p_k}V(\tilde{f}) \\
&\subseteq \lim_{k\to\infty} T_{p_k}V(\tilde{g}-\tilde{g}(p_k)) \cap \lim_{k\to\infty} T_{p_k}W_i \cap \lim_{k\to\infty} T_{p_k}V(\tilde{f}).
\end{aligned}$$

Notice that p must be contained in $\{f=0\}$, since it is a point of the closure of $W_i \cap \{f=0\} \setminus \{g=0\}, W_i \in \mathscr{W}$. Therefore, the only option we have for V_γ is

$W_j \cap \{f=0\} \cap \{g=0\} \setminus \Sigma_{\mathscr{W}} g, W_j \in \mathscr{W}$. Then, writing $\lim_{k\to\infty} T_{p_k} W_i = T_1$, the last limit is equal to

$$T_pV(\tilde{g}-\tilde{g}(p)) \cap T_1 \cap T_pV(\tilde{f}) = T_pV(\tilde{g}) \cap T_1 \cap T_pV(\tilde{f}).$$

For $V_\gamma = W_j \cap \{g=0\} \cap \{f=0\} \setminus \Sigma_{\mathscr{W}} g, W_j \in \mathscr{W}$, we have

$$T_pV_\gamma = T_p(W_j \cap \{g=0\} \cap \{f=0\} \setminus \Sigma_{\mathscr{W}} g) = T_pW_j \cap T_pV(\tilde{g}) \cap T_pV(\tilde{f}).$$

Since f has an isolated singularity at the origin, $T_pV(\tilde{f})$ intersects T_pW_j transversely and since g is tractable at the origin with respect to $\mathscr{V}, T_pV(\tilde{g})$ intersects $T_pV(\tilde{f}) \cap T_pW_j$ transversely. Since, by Whitney's condition (a) over strata of $\mathscr{W}, T_pW_j \subseteq T_1$, the intersections on $T_pV(\tilde{g}) \cap T_1 \cap T_pV(\tilde{f})$ are transverse. Then we conclude that

$$\lim_{k\to\infty} T_{p_k} V(g|_{V_\lambda} - g|_{V_\lambda}(p_k)) = T_pV(\tilde{g}) \cap T_1 \cap T_pV(\tilde{f}).$$

Using Whitney's condition (a) over strata of $\mathscr{W}$ again,

$$T_pV_\gamma = T_pW_j \cap T_pV(\tilde{g}) \cap T_pV(\tilde{f}) \subseteq T_1 \cap T_pV(\tilde{g}) \cap T_pV(\tilde{f}).$$

Let us now verify that $\mathscr{V}''^{\{f=0\}}$ is a good stratification of X^f relative to $g|_{X^f}$. This is valid because

$$\mathscr{V}''^{\{f=0\}} = \{W_i \cap \{f=0\} \setminus \{g=0\}, W_i \cap \{f=0\} \cap \{g=0\} \setminus \Sigma_{\mathscr{W}} g, W_i \in \mathscr{W}\}$$

is given by strata of $\mathscr{V}''$.

At last, we will show that f is prepolar with respect to $\mathscr{V}''$ at the origin. For that, we need verify that for all $V_\alpha \in \mathscr{V}'', 0 \notin V_\alpha, f|_{V_\alpha}$ is nonsingular. If $V_\alpha = V_i \setminus \{g=0\}, V_i \in \mathscr{V}$, since f has an isolated singularity at the origin, $f|_{V_i\setminus\{g=0\}}$ has no singularity . Suppose now that $V_\alpha = V_i \cap \Sigma_{\mathscr{W}} g$, with $V_i \in \mathscr{V}$. Since, by Proposition 1.8.4, $\Sigma f|_{V_\alpha} \subset \{f=0\}$ and, by hypothesis, $\Sigma_{\mathscr{W}} g \cap \{f=0\} = \{0\}$, $f|_{V_\alpha}$ is nonsingular. Now, let $V_\alpha = V_i \cap \{g=0\} \setminus \Sigma_{\mathscr{W}} g, V_i \in \mathscr{V}$ and $x \in \Sigma f|_{V_\alpha}, x \neq 0$. Since $\Sigma f|_{V_\alpha} \subset \{f=0\}, V_i = W_i \cap \{f=0\}$, for some $W_i \in \mathscr{W}$. Then $x \in W_i \cap \{f=0\} \cap \{g=0\} \setminus \Sigma_{\mathscr{W}} g$. But g is tractable at the origin with respect to $\mathscr{V}$ relative to f, which implies that $W_i \cap \{f=0\}$ intersects $\{g=0\}$ transversely and gives us a contradiction. Therefore, f is prepolar at the origin with respect to $\mathscr{V}''$. ■

Let us see an adaptation of Theorem 3.9 of (MASSEY, 1996) to the case we are working on. For that we will need the following property.

Lemma 2.2.3. Let K and F be subspaces of a Hausdorff topological space X and $f: K \to F$ be a continuous map. Suppose that K is compact. If A is a closed subset of K, then

$$f| : K \cap f^{-1}(F - f(A)) \to F - f(A)$$

is a proper map.

Proof. Notice that f is proper. Let K' be a compact subset of F. Since X is Hausdorff, K' is closed. Therefore, $f^{-1}(K')$ is compact.

Let now K'' be a compact subset of $F - f(A)$. Since K is compact, F is Hausdorff and f is a proper map, f is a closed map (see page 125, (ENGELKING, 1989)), which implies that $f(A)$ is closed in F. Then K'' is a compact subset of F. Since f is proper, $f^{-1}(K'')$ is compact in $K - f^{-1}(f(A)) = K \cap f^{-1}(F - f(A))$. Therefore, $f|$ is proper. ■

Lemma 2.2.4. Let $f, g : X \to \mathbb{C}$ be holomorphic functions and $\mathscr{W}$ be a Whitney stratification of X. Suppose that f has an isolated singularity at the origin and let $\mathscr{V}$ be the good stratification of X induced by f. Suppose that g has a one-dimensional critical locus (with respect to $\mathscr{W}$), that g is tractable at the origin with respect to $\mathscr{V}$ relative to f and that $\Sigma_{\mathscr{W}} g \cap \{f = 0\} = \{0\}$. Then, for $0 < |\alpha| \ll |\delta| \ll \varepsilon < 1$ and a closed ball B_ε centered at the origin,

$$\chi(X \cap g^{-1}(\alpha) \cap f^{-1}(\delta) \cap B_\varepsilon) = \chi(X \cap g^{-1}(\alpha) \cap f^{-1}(0) \cap B_\varepsilon).$$

Proof. Let

$$\mathscr{V}'' = \{V_i \setminus \{g = 0\}, V_i \cap \{g = 0\} \setminus \Sigma_{\mathscr{W}} g, V_i \cap \Sigma_{\mathscr{W}} g, V_i \in \mathscr{V}\} \cup \{0\} \tag{2.4}$$

be the good stratification of X relative to g constructed in Lemma 2.2.2. By this lemma, f is prepolar at the origin with respect to $\mathscr{V}''$. So, $V(f)$ intersects each stratum of $\mathscr{V}''$ transversely in a neighborhood of the origin, except perhaps at the origin itself. Hence, we can choose a sufficiently small ε such that in an open ball containing B_ε, $V(f)$ intersects $\{V_\lambda \cap V(g) \setminus \Sigma_{\mathscr{W}} g, V_\lambda \in \mathscr{V}\}$ transversely and such that the sphere ∂B_ε intersects each $V_\lambda \cap V(g) \cap V(f)$ transversely.

Fixing the appropriate ε, let us show that, for $0 < \eta, \nu \ll \varepsilon$, the map

$$\begin{array}{c} B_\varepsilon \cap X \cap \Phi^{-1}(int(D_\eta) \times int(D_\nu) - \Phi(\tilde{\Gamma}_{f,g}(\mathscr{V}) \cup \Sigma_{\mathscr{W}} g)) \\ \downarrow \Phi:=(f,g) \\ int(D_\eta) \times int(D_\nu) - \Phi(\tilde{\Gamma}_{f,g}(\mathscr{V}) \cup \Sigma_{\mathscr{W}} g) \end{array}$$

is a stratified proper submersion with respect to $\mathscr{V}$, where D_η and D_ν are small closed balls centered at the origin.

Since X is Hausdorff, $B_\varepsilon \cap X$ is compact and (f, g) is a continuous map, by Lemma 2.2.3, $(f, g) : B_\varepsilon \cap X \to int(D_\eta) \times int(D_\nu)$ is proper and so is the restriction Φ defined above.

Let us prove that Φ is a submersion. Since f has an isolated singularity at the origin and the symmetric relative polar curve $\tilde{\Gamma}_{f,g}(\mathscr{V})$ and the singular locus $\Sigma_{\mathscr{W}} g$ were excluded, Φ has no critical point inside $int(B_\varepsilon) \cap X$.

Let us now verify that Φ has no critical points on the boundary $\partial B_\varepsilon \cap X$. Let $\tilde{f}$ and $\tilde{g}$ be extensions of f and g to the ambient space, respectively. By contradiction, suppose that no matter how small we pick η, ν, Φ has a stratified critical point on the boundary $\partial B_\varepsilon \cap X$. Since the covering given by the stratification is locally finite, we can assume that all these critical points lie in some stratum V_λ. Then there exists a sequence of critical points (p_i) of $\partial B_\varepsilon \cap V_\lambda$ such that $p_i \to p$, $f(p_i) \to 0$, $g(p_i) \to 0$, and

$$T_{p_i}V(\tilde{f} - \tilde{f}(p_i)) \cap T_{p_i}V(\tilde{g} - \tilde{g}(p_i)) \cap T_{p_i}V_\lambda \subseteq T_{p_i}\partial B_\varepsilon.$$

Hence, $p \in V_\beta \subset \overline{V_\lambda}$, $f(p) = 0$, $p \notin \Sigma_{\mathscr{W}} g$ and $p \notin \tilde{\Gamma}_{f,g}(\mathscr{V})$. Then $V(\tilde{f})$, $V(\tilde{g})$ and T_pV_β intersect transversely at p and

$$T_{p_i}V(\tilde{f} - \tilde{f}(p_i)) \to T_pV(\tilde{f} - \tilde{f}(0)) = T_pV(\tilde{f}) \text{ and } T_{p_i}V(\tilde{g} - \tilde{g}(p_i)) \to T_pV(\tilde{g} - \tilde{g}(0)) = T_pV(\tilde{g}).$$

Therefore, if we suppose that $T_{p_i}V_\lambda \to \mathscr{T}$, applying the limit to

$$T_{p_i}V(\tilde{f} - \tilde{f}(p_i)) \cap T_{p_i}V(\tilde{g} - \tilde{g}(p_i)) \cap T_{p_i}V_\lambda \subseteq T_{p_i}\partial B_\varepsilon,$$

we obtain that

$$T_pV(\tilde{f}) \cap T_pV(\tilde{g}) \cap \mathscr{T} \subseteq T_p\partial B_\varepsilon.$$

By Whitney's condition (a), $T_pV_\beta \subseteq \mathscr{T}$, and then

$$T_pV(\tilde{f}) \cap T_pV(\tilde{g}) \cap T_pV_\beta \subseteq T_p\partial B_\varepsilon,$$

which is a contradiction, since we choose ε sufficiently small such that $V(g) \cap V(f) \cap V_\beta$ intersects ∂B_ε transversely.

Hence, Φ is a stratified proper submersion. By the Ehresmann Fibration Theorem, all fibres are homeomorphic.

Notice that, since g is tractable at the origin with respect to $\mathscr{V}$ relative to f, Φ has no critical points contained in $V(f)$, that is, Φ has no critical points of the type $(0, \alpha), \alpha \neq 0$.

Then, for $0 < |\alpha| \ll |\delta| \ll \varepsilon < 1$, with α being a regular value of g, the fibres $\Phi^{-1}(\delta, \alpha)$ and $\Phi^{-1}(0, \alpha)$ are homeomorphic, that is,

$$\chi(X \cap g^{-1}(\alpha) \cap f^{-1}(\delta) \cap B_\varepsilon) = \chi(X \cap g^{-1}(\alpha) \cap f^{-1}(0) \cap B_\varepsilon).$$

■

Another property we will need is the following version of Lemma 1.2.2 in our setting.

Lemma 2.2.5. Let $f,g : X \to \mathbb{C}$ be holomorphic functions and $\mathscr{W}$ be a Whitney stratification of X. Suppose that f has an isolated singularity at the origin and let $\mathscr{V}$ be the good stratification of X induced by f. Suppose that g has a one-dimensional critical locus (with respect to $\mathscr{W}$), that g is tractable at the origin with respect to $\mathscr{V}$ relative to f and that $\Sigma_{\mathscr{W}} g \cap \{f = 0\} = \{0\}$. If $0 < |\delta| \ll 1$, then

$$\Sigma_{\mathscr{W} \cap \{f=\delta\}} g \cap \{g = 0\} \cap B_\varepsilon = \Sigma_{\mathscr{W}} g \cap \{f = \delta\} \cap \{g = 0\} \cap B_\varepsilon.$$

Proof. Let $\tilde{g}$ and $\tilde{f}$ be analytic extensions of g and f to the ambient space U.

Let $p \in \Sigma_{\mathscr{W}} g \cap \{f = \delta\} \cap \{g = 0\} \cap B_\varepsilon$ and V_α the stratum of $\mathscr{V}$ that contains p. Then $d_p\tilde{g}|_{V_\alpha} = 0$, and $rk(d_p\tilde{g}|_{V_\alpha}, d_p\tilde{f}|_{V_\alpha}) \leq 1$. So, p is a critical point of $g|_{V_\alpha \cap \{f=\delta\}}$, that is, $p \in \Sigma_{\mathscr{W} \cap \{f=\delta\}} g| \cap \{g = 0\} \cap B_\varepsilon$.

Let us show that $\Sigma_{\mathscr{W} \cap \{f=\delta\}} g \cap \{g = 0\} \cap B_\varepsilon \subseteq \Sigma_{\mathscr{W}} g \cap \{f = \delta\} \cap \{g = 0\} \cap B_\varepsilon$.

Suppose that there exists in $\Sigma_{\mathscr{W} \cap \{f=f(p_i)\}} g \cap \{g = 0\} \cap B_\varepsilon \setminus \Sigma_{\mathscr{W}} g$ a sequence of points (p_i) converging to 0. Then, for all $i, p_i \in \Gamma_{f,g}(\mathscr{V}) \cap \{g = 0\} \setminus \Sigma_{\mathscr{W}} g$. Since $(p_i) \in \{g = 0\} \setminus \Sigma_{\mathscr{W}} g$, each p_i is a critical point of $f|_{\{g=0\} \setminus \Sigma_{\mathscr{W}} g}$. So, by Proposition 1.8.4, $p_i \in \{f = 0\}$, for all i, which is a contradiction. Therefore,

$$\Sigma_{\mathscr{W} \cap \{f=\delta\}} g \cap \{g = 0\} \cap B_\varepsilon = \Sigma_{\mathscr{W}} g \cap \{f = \delta\} \cap \{g = 0\} \cap B_\varepsilon.$$

■

If $V_1, \ldots, V_q$ are the strata not contained in $\{f = 0\}$, we can write $\Sigma_{\mathscr{W}} g = b_1 \cup \ldots \cup b_r$ as a union of branches b_j, where $b_j \subseteq V_{i(j)}$, for some $i(j) \in \{1, \ldots, q\}$, as we saw before. Let δ be a regular value of $f, 0 < |\delta| \ll 1$, and $f^{-1}(\delta) \cap b_j = \{x_{i_1}, \ldots, x_{i_{k(j)}}\}$. For each $x_\theta \in f^{-1}(\delta) \cap b_j$, let D_{x_θ} be the closed ball with center at x_θ and radius $0 < r_\theta \ll 1$. We choose r_θ sufficiently small such that the balls D_{x_θ} are pairwise disjoint and the union of balls $D_j = D_{x_{i_1}} \cup \ldots \cup D_{x_{i_{k(j)}}}$ is contained in B_ε, where $0 < |\delta| \ll \varepsilon \ll 1$ and ε is sufficiently small such that the local Euler obstruction of X is constant on $b_j \cap B_\varepsilon$. Notice that, in this case, we can choose $x_\theta \in b_j, j \in \{1, \ldots, r\}, \theta \in \{i_1, \ldots, i_{k(j)}\}$, and write $Eu_X(x_\theta) = Eu_X(b_j)$.

Before we prove the first theorem of this section, we will prove a useful regularity condition over the branches b_j. Notice that the next lemma is a version of Corollary 1.2.3 in our setting.

Lemma 2.2.6. Let $0 < |\alpha| \ll 1$ and $0 < |\delta| \ll 1$ be regular values of g and f, respectively. For all $\theta_1 \neq \theta_2, \theta_1, \theta_2 \in \{i_1, \ldots, i_{k(j)}\}$, and $|\alpha| \ll |\delta|$,

$$\chi(X \cap g^{-1}(\alpha) \cap f^{-1}(\delta) \cap D_{x_{\theta_1}}) = \chi(X \cap g^{-1}(\alpha) \cap f^{-1}(\delta) \cap D_{x_{\theta_2}}).$$

Proof. Consider the function φ over $b_j \setminus \{0\}$ given by $\varphi(x) = \chi(X \cap \{g = \alpha\} \cap \{f = f(x)\} \cap D_x)$, where $|\alpha| \ll |f(x)|$ and D_x the closed ball with center at x and radius $0 < r_x \ll 1$. We should prove that φ is constant. Since $b_j \setminus \{0\}$ is connected, it is sufficient to show that φ is locally constant, that is, given $x \in b_j \setminus \{0\}$, there must exist a neighborhood V_x of x such that for all $y \in b_j \setminus \{0\} \cap V_x$, $\varphi(x) = \varphi(y)$. It is enough to show that there exist $\varepsilon_x > 0$ and a neighborhood V_x such that for all $y \in b_j \setminus \{0\} \cap V_x$ and all $0 < \varepsilon \leq \varepsilon_x, S(y,\varepsilon)$ intersects $g^{-1}(0) \cap f^{-1}(f(y))$ transversely.

Let us denote by $N(\varepsilon)$ the tube $\{z \in U; d(\Sigma_{\mathscr{V}} g, z) = \varepsilon\}$. We can replace $S(y,\varepsilon)$ with $N(\varepsilon)$ and we have to show that there exist $\varepsilon_x > 0$ and a neighborhood V_x of x such that for all $y \in b_j \setminus \{0\} \cap V_x$ and all $\varepsilon \leq \varepsilon_x$, $N(\varepsilon)$ intersects $g^{-1}(0) \cap f^{-1}(f(y))$ transversely. We can also replace the distance function to $\Sigma_{\mathscr{W}} g$ with a real analytic function h such that $h^{-1}(0) = \Sigma_{\mathscr{W}} g$ and $h \geq 0$. Then we replace the tube $N(\varepsilon)$ with $\{z \in U; h(z) = \varepsilon\} = h^{-1}(\varepsilon)$.

By contradiction, suppose that there exists a point y such that $N(\varepsilon)$ does not intersect $g^{-1}(0) \cap f^{-1}(f(y))$ transversely. Let V_α be the stratum of $\mathscr{V}$ that contains y. Then, using the terminology of Iomdin in (IOMDIN, 1974a), since $y \in V_\alpha \cap \{g = 0\}$, the vectors $grad\ h|_{V_\alpha}(y)$ and $grad\ f|_{V_\alpha}(y) = (1/2f|_{V_\alpha}(y))grad\ ||f|_{V_\alpha}(y)||$ are complex linearly dependent $mod\ grad\ g|_{V_\alpha}(y)$. Hence, $grad\ f|_{V_\alpha}(y) = \lambda grad\ h|_{V_\alpha}(y), mod\ grad\ g|_{V_\alpha}(y)$.

Now,

$$\begin{aligned} grad\ ||f|_{V_\alpha}||^2(y) &= 2f|_{V_\alpha}(y) grad\ f|_{V_\alpha}(y) = 2f|_{V_\alpha}(y)\lambda grad\ h|_{V_\alpha}(y) \\ &= \frac{\lambda f|_{V_\alpha}(y)}{h|_{V_\alpha}(y)} 2h|_{V_\alpha}(y) grad\ h|_{V_\alpha}(y) = \gamma grad\ ||h|_{V_\alpha}||^2(y), \end{aligned}$$

with $\gamma = \frac{\lambda f|_{V_\alpha}(y)}{h|_{V_\alpha}(y)}$. The last equality means that the vectors $grad\ ||f|_{V_\alpha}||^2(y)$ and $grad\ ||h|_{V_\alpha}||^2(y)$ are complex linearly dependent $mod\ grad\ g|_{V_\alpha}(y)$. This contradicts Corollary 1.2.1: using the functions $||h|_{V_\alpha}||^2$ and $||f|_{V_\alpha}||^2$, since $\Sigma_{\mathscr{W}} g \cap \{f = 0\} = \{0\}$, this corollary implies that there exist $\varepsilon > 0$ and a neighborhood G of $\Sigma_{\mathscr{W}} g$ in $\{g = 0\}$ such that at points z of $D_\varepsilon \cap G \setminus \Sigma_{\mathscr{W}} g, grad\ ||f|_{V_\alpha}||^2(z)$ and $grad\ ||h|_{V_\alpha}||^2(z)$ are complex linearly independent $mod\ grad\ g|_{V_\alpha}(z)$.

Therefore, φ is locally constant. ■

Remark 2.2.7. The last lemma shows that, for $0 \leq |\alpha| \ll |\delta| \ll \varepsilon \ll 1$, the Euler characteristic of $X \cap g^{-1}(\alpha) \cap f^{-1}(\delta) \cap D_{x_\theta}$ is constant over $b_j \cap B_\varepsilon, j \in \{1, \ldots, r\}$ and $\theta \in \{i_1 \ldots, i_{k(j)}\}$. Then, for each stratum $V_1, \ldots, V_q$ of $\mathscr{V}$ not contained in $\{f = 0\}, \chi(\overline{V_i} \cap g^{-1}(\alpha) \cap f^{-1}(\delta) \cap D_{x_\theta})$ is constant over $b_j \cap B_\varepsilon$. Notice that $\chi(V_i \cap g^{-1}(\alpha) \cap f^{-1}(\delta) \cap D_{x_\theta})$ is also constant over $b_j \cap B_\varepsilon$. In fact, if V_i is closed, there is nothing to do. Suppose that $V_i \neq \overline{V_i}$ and write $\overline{V_i} = V_i \cup (\overline{V_i} \setminus V_i)$. By definition of complex analytic stratification, $\overline{V_i} \setminus V_i$ is analytic and union of strata of $\mathscr{V}$ of smaller dimension,

$$\overline{V_i} = V_i \cup V_{i_1} \cup \ldots \cup V_{i_n}.$$

For the stratum of smallest dimension, $V_0 = \{0\}$, we have $\overline{V_0} = V_0$. Reducing to the case

$\overline{V_i} = V_i \cup W_i$, with $W_i = \overline{W_i}$, by additivity of the Euler characteristic,

$$\begin{aligned}\chi(\overline{V_i}\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_{x_\theta}) &= \chi(V_i\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_{x_\theta})\\ &+ \chi(\overline{W_i}\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_{x_\theta}).\end{aligned}$$

Since both $\chi(\overline{V_i}\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_{x_\theta})$ and $\chi(\overline{W_i}\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_{x_\theta})$ are constant on $b_j\cap B_\varepsilon$, $\chi(V_i\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_{x_\theta})$ is constant over $b_j\cap B_\varepsilon$.

Let $\beta : X \to \mathbb{Z}$ be a constructible function with respect to $\mathscr{W}$. Using Remark 2.2.7, since each b_j is contained in one unique stratum of $\mathscr{W}$ and β is constant over each one of them, we can use the following notation:

1. $\beta(b_j) := \beta(x_\theta)$, for a chosen $x_\theta \in b_j$;
2. $\tilde{\beta}(b_j) = \chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_{x_\theta},\beta)$ for $j\in\{1,\dots,r\}$ and $\theta\in\{i_1,\dots,i_{k(j)}\}$.

Theorem 2.2.8. Let $\beta : X \to \mathbb{Z}$ be a constructible function with respect to the stratification $\mathscr{W}$. For $0<|\alpha|\ll|\delta|\ll\varepsilon\ll 1$, we have

$$\begin{aligned}\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(0)\cap B_\varepsilon,\beta) &= \chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon,\beta)\\ &- \sum_{j=1}^{r} m_{f,b_j}(\beta(b_j)-\tilde{\beta}(b_j)).\end{aligned}$$

Proof. We have

$$\chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon) =$$

$$\begin{aligned}&= \chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon\setminus\cup_{i=1}^{r}D_j)+\chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap\left(\cup_{j=1}^{r}D_j\right))\\ &= \chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon\setminus\cup_{j=1}^{r}D_j)+\sum_{j=1}^{r}\chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap D_j).\end{aligned}$$

For each $j\in\{1,\dots,r\}$, as we saw before, $D_j = D_{x_{i_1}}\cup\dots D_{x_{i_{k(j)}}}$, where D_{x_θ} is a closed ball with center at x_θ. Since f is an analytic function germ and, for each $\theta\in\{i_1,\dots,i_{k(j)}\}$, $X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap D_{x_\theta}$ is an analytic germ at x_θ, it is contractible. So, $\chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap D_{x_\theta}) = 1$. Therefore,

$$\chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap D_j) = m_{f,b_j}.$$

Hence,

$$\chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon) = \chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon\setminus(\cup_{j=1}^{r}D_j))+\sum_{j=1}^{r}m_{f,b_j}.$$

On the other hand, we have

$$
\begin{aligned}
&\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap B_\varepsilon)\\
= &\ \chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap B_\varepsilon\setminus\cup_{j=1}^{r}D_j)+\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap(\cup_{j=1}^{r}D_j))\\
= &\ \chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap B_\varepsilon\setminus\cup_{j=1}^{r}D_j)+\sum_{j=1}^{r}\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_j).
\end{aligned}
$$

Using again that $D_j=D_{x_{i_1}}\cup\ldots D_{x_{i_{k(j)}}}$, we can write

$$
\begin{aligned}
\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_j) &= \chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap(\cup_{q=1}^{k(j)}D_{x_q}))\\
&= \sum_{q=1}^{k(j)}\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_{x_q}).
\end{aligned}
$$

By Lemma 2.2.6, for all $l_1,l_2\in\{i_1,\ldots,i_{k(j)}\}$,

$$
\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_{x_{l_1}})=\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_{x_{l_2}}).
$$

Hence, fixing $\theta\in\{i_1,\ldots,i_{k(j)}\}$, we have that

$$
\begin{aligned}
\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_j) &= \sum_{q=1}^{k(j)}\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_{x_q})\\
&= m_{f,b_j}\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_{x_\theta})
\end{aligned}
$$

and then

$$
\begin{aligned}
\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap B_\varepsilon) &= \chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap B_\varepsilon\setminus\cup_{j=1}^{r}D_j)\\
&+ \sum_{j=1}^{r}m_{f,b_j}\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_{x_\theta}).
\end{aligned}
$$

By Lemma 2.2.5, $\Sigma_{\mathscr{W}}g\cap\{f=\delta\}\cap\{g=0\}\cap B_\varepsilon=\{x_1,\ldots,x_s\}$ is the set of critical points of $g|_{\{f=\delta\}\cap B_\varepsilon}$ appearing in $\{g=0\}$. Then, since $0<|\alpha|\ll|\delta|\ll\varepsilon\ll 1$,

$$
\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap B_\varepsilon\setminus(\cup_{j=1}^{r}D_j)) = \chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon\setminus(\cup_{j=1}^{r}D_j)).
$$

Because $(0,\alpha)$ and (δ,α) are regular values of (f,g), by Lemma 2.2.4,

$$
\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(0)\cap B_\varepsilon)=\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap B_\varepsilon).
$$

Then

$$
\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(0)\cap B_\varepsilon)=\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap B_\varepsilon)
$$

$$
\begin{aligned}
&= \chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap B_\varepsilon\setminus(\cup_{j=1}^{r}D_j)) + \sum_{j=1}^{r} m_{f,b_j}\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_{x_\theta})\\
&= \chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon\setminus(\cup_{j=1}^{r}D_j)) + \sum_{j=1}^{r} m_{f,b_j}\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_{x_\theta})\\
&= \chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon) - \sum_{j=1}^{r} m_{f,b_j} + \sum_{j=1}^{r} m_{f,b_j}\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_{x_\theta})\\
&= \chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon) - \sum_{j=1}^{r} m_{f,b_j}(1-\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_{x_\theta})).
\end{aligned}
$$

By additivity of the constructible function β, we obtain

$$
\begin{aligned}
\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(0)\cap B_\varepsilon,\beta) &= \chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon,\beta)\\
&- \sum_{j=1}^{r} m_{f,b_j}(\beta(b_j)-\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_{x_\theta},\beta))\\
&= \chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon,\beta)\\
&- \sum_{j=1}^{r} m_{f,b_j}(\beta(b_j)-\tilde{\beta}(b_j)).
\end{aligned}
$$

■

Remark 2.2.9. If we suppose that g has an isolated singularity, we obtain that $\Sigma_{\mathscr{W}} g\cap\{f=\delta\}=\emptyset$ and, in this case, the formula of Theorem 2.2.8 is

$$
\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(0)\cap B_\varepsilon,\beta) = \chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon,\beta),
$$

which is the equality proved in Proposition 6.2 of (DUTERTRE; GRULHA, 2014).

Remark 2.2.10. Let $\mathscr{W}$ be a Whitney stratification of X and $\mathscr{V}$ the good stratification of X induced by f. Suppose that g is tractable at the origin with respect to $\mathscr{V}$ relative to f, $\Sigma_{\mathscr{W}} g$ is one-dimensional and that $\Sigma_{\mathscr{W}} g\cap\{f=0\}=\{0\}$. The refinement $\mathscr{V}'$ of $\mathscr{V}$, constructed in Lemma 2.1.1, is a good stratification of X relative to f, such that $\mathscr{V}'^{\{g=0\}}$ is a good stratification of X^g relative to $f|_{X^g}$. On the other hand, the refinement $\mathscr{V}''$ of $\mathscr{V}$, constructed in Lemma 2.2.2, is a good stratification of X relative to g such that $\mathscr{V}''^{\{f=0\}}$ is a good stratification of X^f relative to $g|_{X^f}$. But, in fact, $\mathscr{V}''$ is also a refinement of $\mathscr{V}'$. Therefore, in this context, we can refine a Whitney stratification of X to obtain an appropriate stratification for which the Brasselet numbers $B_{f,X}(0), B_{f,X^g}(0), B_{g,X(0)}$ and $B_{g,X^f}(0)$ can be explicitly calculated.

Applying the previous theorem to the case where $\beta = Eu_X$, we can compare $B_{g,X^f}(0)$ and $B_{f,X^g}(0)$.

Corollary 2.2.11. Let $\mathscr{W}$ be a Whitney stratification of X and $\mathscr{V}$ the good stratification of X induced by f. Suppose that g is tractable at the origin with respect to $\mathscr{V}$ relative to f. Then, for $0 \ll |\delta| \ll \varepsilon \ll 1$,

$$B_{g,X^f}(0) = B_{f,X^g}(0) - \sum_{j=1}^{r} m_{f,b_j}(Eu_{X^g}(b_j) - B_{g,X\cap f^{-1}(\delta)}(b_j)).$$

Proof. Applying Theorem 2.2.8 to $\beta = Eu_X$, we obtain

$$\begin{aligned} \chi(X\cap g^{-1}(\alpha)\cap f^{-1}(0)\cap B_\varepsilon, Eu_X) &= \chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon, Eu_X) \\ &- \sum_{j=1}^{r} m_{f,b_j}(Eu_X(b_j) - \tilde{Eu}_X(b_j)). \end{aligned}$$

To compute $\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(0)\cap B_\varepsilon, Eu_X)$ we will use strata of the refinement $\mathscr{V}''$ of $\mathscr{V}$ not contained in $\{g=0\}$. Let $W_1,\dots,W_t$ be these strata. Since g is tractable at the origin with respect to $\mathscr{V}$ relative to f, by Lemma 2.2.2, f is prepolar a the origin with respect to $\mathscr{V}''$, that is, $\{f=0\}$ intersects each W_i transversely, for $i\in\{1,\dots,t\}$. So, $Eu_X(W_i) = Eu_{X^f}(S)$, for each connected component S of W_i^f. Then, for $0<|\alpha|\ll\varepsilon\ll 1$,

$$\begin{aligned} \chi(X\cap f^{-1}(0)\cap g^{-1}(\alpha)\cap B_\varepsilon, Eu_X) &= \sum_{i=1}^{t} \chi(W_i\cap f^{-1}(0)\cap g^{-1}(\alpha)\cap B_\varepsilon) Eu_X(W_i) \\ &= \sum_{i=1}^{t}\sum_{S} \chi(S\cap f^{-1}(0)\cap g^{-1}(\alpha)\cap B_\varepsilon) Eu_{X^f}(S) \\ &= B_{g,X^f}(0). \end{aligned}$$

To compute $\chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon, Eu_X)$ we will use strata of the refinement $\mathscr{V}''$ of $\mathscr{V}'$ not contained in $\{f=0\}$. Then, using Equation (2.2) of Theorem 2.1.2 with the notation above, we have that

$$\chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon, Eu_X) = B_{f,X^g}(0) + \sum_{j=1}^{r} m_{f,b_j}(Eu_X(b_j) - Eu_{X^g}(b_j)).$$

We will now compute $\tilde{Eu}_X(b_j) = \chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_{x_\theta}, Eu_X)$, where, as we describe before, $D_{x_\theta}\subset B_\varepsilon$, $j\in\{1,\dots,r\}$, $\theta\in\{i_1,\dots,i_{k(j)}\}$, is the closed ball with center at $x_\theta\in f^{-1}(\delta)\cap b_j$ and radius $0<r_l\ll 1$. For that computation we will use strata of $\mathscr{V}''$ not contained in $\{f=0\}$ or in $\{g=0\}$, that is

$$\{W_1\setminus\{f=0\}\cup\{g=0\},\dots,W_q\setminus\{f=0\}\cup\{g=0\}, W_i\in\mathscr{W}\}.$$

Since f is prepolar at the origin with respect to $\mathscr{V}''$, $f^{-1}(\delta)$ intersects each stratum $W_j\setminus\{f=0\}\cup\{g=0\}$ transversely and for all $W_j\setminus\{f=0\}\cup\{g=0\}$,

$$Eu_X(W_j\setminus\{f=0\}\cup\{g=0\}) = Eu_{X\cap f^{-1}(\delta)}((W_j\setminus\{f=0\}\cup\{g=0\})\cap f^{-1}(\delta)).$$

So, writing $W_j \setminus \{f=0\} \cup \{g=0\} = U_j$,

$$\begin{aligned}
\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_{x_\theta}, Eu_X) &= \sum_{i=1}^{q} \chi(U_i\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_{x_\theta}) Eu_X(U_i)\\
&= \sum_{i=1}^{q} \chi(U_i\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_{x_\theta}) Eu_{X\cap f^{-1}(\delta)}(U_i\cap f^{-1}(\delta))\\
&= B_{g,X\cap f^{-1}(\delta)}(x_\theta)\\
&= B_{g,X\cap f^{-1}(\delta)}(b_j),
\end{aligned}$$

where the last equality holds by Remark 2.2.7.

Therefore,

$$\begin{aligned}
B_{g,X^f}(0) &= \chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon, Eu_X)\\
&- \sum_{j=1}^{r} m_{f,b_j}(Eu_X(b_j) - \chi(X\cap g^{-1}(\alpha)\cap f^{-1}(\delta)\cap D_{x_\theta}, Eu_X))\\
&= B_{f,X^g}(0) + \sum_{j=1}^{r} m_{f,b_j}(Eu_X(b_j) - Eu_{X^g}(b_j)) - \sum_{j=1}^{r} m_{f,b_j}(Eu_X(b_j) - B_{g,X\cap f^{-1}(\delta)}(b_j))\\
&= B_{f,X^g}(0) - \sum_{j=1}^{r} m_{f,b_j}(Eu_{X^g}(b_j) - B_{g,X\cap f^{-1}(\delta)}(b_j)).
\end{aligned}$$

■

Remark 2.2.12. If g has an isolated singularity at the origin, then $\Sigma_{\mathscr{W}} g \cap \{f=\delta\} = \emptyset$. So, in this case, the formula of Corollary 2.2.11 is given by

$$B_{f,X^g}(0) = B_{g,X^f}(0),$$

which is the equality proved in Corollary 6.3 of (DUTERTRE; GRULHA, 2014).

Since a generic linear form l over $\mathbb{C}^n$ has an isolated singularity, we can consider the good stratification $\mathscr{V}$ of X induced by l and by Lemma 2.1.4, $\Sigma_{\mathscr{W}} g \cap \{l=0\} = \{0\}$. So, the construction made in Lemma 2.2.2 can be done in the case where f is a generic linear form. Applying Corollary 2.2.11 to this case, we obtain the following consequence.

Corollary 2.2.13. Let $l : \mathbb{C}^n \to \mathbb{C}$ be a generic linear form and $\mathscr{V}$ the good stratification of X induced by l. Suppose that g is tractable at the origin with respect to $\mathscr{V}$ relative to l. If $H = l^{-1}(0)$, we have that

$$B_{g,X\cap H}(0) = Eu_{X^g}(0) - \textstyle\sum_{j=1}^{r} m_{b_j}(Eu_{X^g}(b_j) - B_{g,X\cap l^{-1}(\delta)}(b_j)).$$

Proof. Applying Corollary 2.2.11 to $f = l$, we obtain

$$B_{g,X^l}(0) = B_{l,X^g}(0) - \sum_{j=1}^{r} m_{b_j}(Eu_{X^g}(b_j) - B_{g,X\cap l^{-1}(\delta)}(b_j)). \tag{2.5}$$

Since $B_{l,X^g}(0) = Eu_{X^g}(0)$ and $H = l^{-1}(0)$, Equation (2.5) can be written as

$$B_{g,X\cap H}(0) = Eu_{X^g}(0) - \sum_{j=1}^{r} m_{b_j}(Eu_{X^g}(b_j) - B_{g,X\cap l^{-1}(\delta)}(b_j)).$$

■

Remark 2.2.14. Notice that, by Theorem 5.1 of (DUTERTRE; GRULHA, 2014), $B_{g,X\cap H}(0)$ does not depend on $H = l^{-1}(0)$, so the sum of Brasselet numbers $\sum_{j=1}^{r} m_{b_j} B_{g,X\cap l^{-1}(\delta)}(b_j)$ does not depend on the the generic linear form l.

Remark 2.2.15. If g has an isolated singularity, then $\Sigma_{\mathscr{W}} g \cap \{l = \delta\} = \emptyset$. So, in this case, the formula of Corollary 2.2.13 is given by

$$B_{g,X\cap H}(0) = Eu_{X^g}(0),$$

which is the equality proved in Corollary 6.6 of (DUTERTRE; GRULHA, 2014).

Remark 2.2.16. If l is a generic linear form over $\mathbb{C}^n$, $l^{-1}(\delta)$ intersects $X \cap \{g = 0\}$ transversely and then

$$Eu_{X^g}(b_j) = Eu_{X^g \cap l^{-1}(\delta)}(b_j \cap l^{-1}(\delta)) = B_{g,X\cap l^{-1}(\delta)\cap L}(b_j \cap l^{-1}(\delta)),$$

where the last equality is justified by Corollary 6.6 of (DUTERTRE; GRULHA, 2014) and L is a generic hyperplane in $\mathbb{C}^n$ passing through $x_\theta \in l^{-1}(\delta) \cap b_j$, $j \in \{1, \ldots, r\}$ and $\theta \in \{i_1, \ldots, i_{k(j)}\}$.

Denoting $B_{g,X\cap l^{-1}(\delta)\cap L}(b_j \cap l^{-1}(\delta))$ by $B'_{g,X\cap l^{-1}(\delta)}(b_j)$, the formula obtained in 2.2.13 can be written as

$$B_{g,X\cap H}(0) = Eu_{X^g}(0) - \textstyle\sum_{j=1}^{r} m_{b_j}(B'_{g,X\cap l^{-1}(\delta)}(b_j) - B_{g,X\cap l^{-1}(\delta)}(b_j)).$$

This result allows us to compare the Brasselet number $B_{g,X\cap H}(0)$ and the Euler obstruction $Eu_{X^g}(0)$ in terms of the dimension of the analytic complex space $(X,0)$.

Let $I_0(X^f, \Gamma^0_{f|_X})$ be the intersection multiplicity of X^f and $\Gamma^0_{f|_X}$, where $\Gamma^0_{f|_X}$ is the general relative polar curve of f (see (LÊ; TEISSIER, 1981)). By Corollary 1.8.31, if $d = \dim(X)$, then

$$B_{f,X}(0) - B_{f,X\cap H}(0) = (-1)^{d-1} I_0(X^f, \Gamma^0_{f|_X}). \tag{2.6}$$

Corollary 2.2.17. Let $l : \mathbb{C}^n \to \mathbb{C}$ be a generic linear form, $\mathscr{V}$ the good stratification of X induced by l and $H = l^{-1}(0)$. Suppose that g is tractable at the origin with respect to $\mathscr{V}$. Then:

1. If d is even, $B_{g,X\cap H}(0) \geq Eu_{X^g}(0)$;

2. If d is odd, $B_{g,X\cap H}(0) \leq Eu_{X^g}(0)$.

Proof. For $0 < |\delta| \ll \varepsilon \ll 1$ and a generic linear form l defined over X, we apply Formula (2.6) to the space $X \cap l^{-1}(\delta)$, whose dimension is $d-1$. We fix a point $x_\theta \in l^{-1}(\delta) \cap b_j$, for each $j \in \{1,\dots,r\}, \theta \in \{i_1,\dots,i_{k(j)}\}$ and we calculate the difference of Brasselet numbers around the singular point x_θ. We have

$$B_{g,X\cap l^{-1}(\delta)}(x_\theta) - B'_{g,X\cap l^{-1}(\delta)}(x_\theta) = (-1)^{d-2} I_{x_\theta}((X \cap l^{-1}(\delta))^g, \Gamma^{x_\theta}_{g|_{X\cap l^{-1}(\delta)}}).$$

By Remark 2.2.16 ,

$$B_{g,X\cap H}(0) - Eu_{X^g}(0) = \textstyle\sum_{j=1}^r m_{b_j}(B_{g,X\cap l^{-1}(\delta)}(x_\theta) - B'_{g,X\cap l^{-1}(\delta)}(x_\theta)).$$

Hence,

1. If d is even, $B_{g,X\cap H}(0) - Eu_{X^g}(0) = \sum_{j=1}^r m_{b_j}(B_{g,X\cap l^{-1}(\delta)}(x_\theta) - B'_{g,X\cap l^{-1}(\delta)}(x_\theta)) \geq 0$, that is, $B_{g,X\cap H}(0) \geq Eu_{X^g}(0)$;
2. If d is odd, $B_{g,X\cap H}(0) - Eu_{X^g}(0) = \sum_{j=1}^r m_{b_j}(B_{g,X\cap l^{-1}(\delta)}(x_\theta) - B_{g,X\cap l^{-1}(\delta)}(x_\theta)) \leq 0$, that is, $B_{g,X\cap H}(0) \leq Eu_{X^g}(0)$.

■

If $g : \mathbb{C}^n \to \mathbb{C}$ has an isolated singularity at the origin, Lê and Teissier proved, in (LÊ; TEISSIER, 1981), that, for $0 < |\alpha| \ll \varepsilon \ll 1$, $Eu_{X^g}(0) = \chi(g^{-1}(\alpha) \cap H \cap B_\varepsilon)$, where $X = \mathbb{C}^n$, H is a generic hyperplane and α is a regular value of g. The next result is a generalization of this result to our setting.

Let l be a generic linear form over $\mathbb{C}^n$, $\{\mathbb{C}^n \setminus \{0\}, \{0\}\}$ a Whitney stratification of $\mathbb{C}^n$ and $\{\mathbb{C}^n \setminus \{l=0\}, \{l=0\}, \{0\}\}$ the good stratification of $\mathbb{C}^n$ induced by l. Consider a point $x_\theta \in \{l=\delta\} \cap b_j$, for each $j \in \{1,\dots,r\}, \theta \in \{i_1,\dots,i_{k(j)}\}$ and let D_{x_θ} the closed ball with center at x_θ and radius $r_l, 0 < |\alpha| \ll |\delta| \ll r_l \ll \varepsilon \ll 1$, sufficiently small such that the balls D_{x_θ} are pairwise disjoint and the union of balls $D_j = D_{x_{i_1}} \cup \ldots \cup D_{x_{i_{k(j)}}}$ is contained in B_ε, where $0 < |\delta| \ll \varepsilon \ll 1$.

Corollary 2.2.18. Let $H = l^{-1}(0)$ be a generic hyperplane through the origin and suppose that g is tractable at the origin with respect to the good stratification of $\mathbb{C}^n, \{\mathbb{C}^n \setminus \{l=0\}, \{l=0\}\}$, induced by l. Then, for $x_\theta \in \{l=\delta\} \cap b_j$, $j \in \{1,\dots,r\}, \theta \in \{i_1,\dots,i_{k(j)}\}$, chosen as before,

$$Eu_{\{g=0\}}(0) = \chi(g^{-1}(\alpha) \cap H \cap B_\varepsilon) + \textstyle\sum_{j=1}^r (-1)^{n-1} m_{b_j}(\mu(g|_{l^{-1}(\delta)}, x_\theta) + \mu'(g|_{l^{-1}(\delta)}, x_\theta)).$$

Proof. Applying Remark 2.2.16, we obtain

$$B_{g,H}(0) = Eu_{\{g=0\}}(0) - \textstyle\sum_{j=1}^r m_{b_j}(B_{g,l^{-1}(\delta)\cap L}(x_\theta) - B_{g,l^{-1}(\delta)}(x_\theta)),$$

where L is a generic hyperplane in $\mathbb{C}^n$ passing through x_θ. Using the definition of the Brasselet number, we have

$$\begin{aligned} B_{g,H}(0) &= \chi(H\cap g^{-1}(\alpha)\cap B_\varepsilon) \\ B_{g,l^{-1}(\delta)}(x_\theta) &= \chi(g^{-1}(\alpha)\cap l^{-1}(\delta)\cap D_{x_\theta}) = 1+(-1)^{n-2}\mu(g|_{l^{-1}(\delta)},x_\theta) \\ B_{g,l^{-1}(t_0)\cap L}(x_\theta) &= \chi(g^{-1}(\alpha)\cap l^{-1}(\delta)\cap L\cap D_{x_\theta}) = 1+(-1)^{n-3}\mu(g|_{l^{-1}(\delta)\cap L},x_\theta) \\ &= 1+(-1)^{n-3}\mu'(g|_{l^{-1}(\delta)},x_\theta). \end{aligned}$$

Therefore,

$$\begin{aligned} \chi(g^{-1}(\alpha)\cap H\cap B_\varepsilon) &= Eu_{\{g=0\}}(0)-\sum_{j=1}^{r} m_{b_j}((-1)^{n-3}\mu'(g|_{l^{-1}(\delta)},x_\theta)-(-1)^{n-2}\mu(g|_{l^{-1}(\delta)},x_\theta)) \\ &= Eu_{\{g=0\}}(0)-\sum_{j=1}^{r}(-1)^{n-3}m_{b_j}(\mu'(g|_{l^{-1}(\delta)},x_\theta)+\mu(g|_{l^{-1}(\delta)},x_\theta)) \\ &= Eu_{\{g=0\}}(0)-(-1)^{n-1}\sum_{j=1}^{r} m_{b_j}(\mu'(g|_{l^{-1}(\delta)},x_\theta)+\mu(g|_{l^{-1}(\delta)},x_\theta)). \end{aligned}$$

Hence,

$$Eu_{\{g=0\}}(0) = \chi(g^{-1}(\alpha)\cap H\cap B_\varepsilon)+(-1)^{n-1}\textstyle\sum_{j=1}^{r} m_{b_j}(\mu'(g|_{l^{-1}(\delta)},x_\theta)+\mu(g|_{l^{-1}(\delta)},x_\theta)).$$

■

Remark 2.2.19. Using the previous corollary, we can compare the difference $Eu_{X^g}(0)-\chi(g^{-1}(\alpha)\cap H\cap B_\varepsilon)$ using the dimension of $\mathbb{C}^n$.

1. If n is even,

   $$Eu_{X^g}(0)-\chi(g^{-1}(\alpha)\cap H\cap B_\varepsilon) = (-1)^{n-1}\sum_{j=1}^{r} m_{b_j}(\mu'(g|_{l^{-1}(\delta)},x_\theta)+\mu(g|_{l^{-1}(\delta)},x_\theta)) \leq 0$$

   So, $Eu_{X^g}(0) \leq \chi(g^{-1}(\alpha)\cap H\cap B_\varepsilon)$.

2. If n is odd,

   $$Eu_{X^g}(0)-\chi(g^{-1}(\alpha)\cap H\cap B_\varepsilon) = (-1)^{n-1}\sum_{j=1}^{r} m_{b_j}(\mu'(g|_{l^{-1}(\delta)},x_\theta)+\mu(g|_{l^{-1}(\delta)},x_\theta)) \geq 0$$

   So, $Eu_{X^g}(0) \geq \chi(g^{-1}(\alpha)\cap H\cap B_\varepsilon)$.

Let $\mathscr{W} = \{\{0\},W_1\ldots,W_q\}$ be a Whitney stratification of X, $\mathscr{V}$ the good stratification of X induced by $f,\mathscr{V}'$ the good stratification of X relative to f obtained as a refinement of $\mathscr{V}$ in Lemma 2.1.1 and $\mathscr{V}''$ the good stratification of X relative to g obtained as a refinement of $\mathscr{V}$ in Lemma 2.2.2. Suppose that $\Sigma_{\mathscr{W}}g\cap\{f=0\}=\{0\}$. Let $T_1,\ldots,T_q$ be the strata of $\mathscr{V}''$ not contained in $\{g=0\}$ and $V_1,\dots,V_q$ the strata of $\mathscr{V}'$ not contained in $\{f=0\}$. Let n_s (resp. m_t) be the number of stratified Morse critical points of a Morsification of $f : X\cap g^{-1}(\alpha)\cap B_\varepsilon \to \mathbb{C}$ (resp. $g : X\cap f^{-1}(\delta)\cap B_\varepsilon\to\mathbb{C}$) appearing on $T_s\cap g^{-1}(\alpha)\cap\{f\neq 0\}\cap B_\varepsilon$ (resp. $V_t\cap f^{-1}(\delta)\cap\{g\neq 0\}\cap B_\varepsilon$), where $0<|\delta|\ll 1$ is a regular value of f and $0<|\alpha|\ll 1$ is a regular value of g. We write $\Sigma_{\mathscr{W}}g$ as a union of branches $b_1\cup\ldots\cup b_r$, where $b_j\subseteq V_{i(j)}$, for $i(j)\in\{1,\dots,q\}$. Suppose that $\{f=\delta\}\cap b_j=\{x_{i_1},\dots,x_{i_{k(j)}}\}$. For each $\theta\in\{i_1,\dots,i_{k(j)}\}$, let D_{x_θ} be the closed ball with center at x_θ and radius $r_l, 0<|\alpha|\ll|\delta|\ll r_l\ll\varepsilon\ll 1$, sufficiently small for the balls D_{x_θ} to be pairwise disjoint and the union of balls $D_j=D_{x_{i_1}}\cup\ldots\cup D_{x_{i_{k(j)}}}$ to be contained in B_ε, where $0<|\delta|\ll\varepsilon\ll 1$, and ε is sufficiently small such that the local Euler obstruction of X at a point of $b_j\cap B_\varepsilon$ is constant.

Theorem 2.2.20. Let $\beta : X\to\mathbb{Z}$ be a constructible function with respect to $\mathscr{W}$ and suppose that g is tractable at the origin with respect to $\mathscr{V}$ relative to f. For $0<|\alpha|\ll|\delta|\ll\varepsilon\ll 1$,

$$\chi(X\cap g^{-1}(\alpha)\cap B_\varepsilon,\beta)-\chi(X\cap f^{-1}(\delta)\cap B_\varepsilon,\beta)=$$
$$\textstyle\sum_{s=0}^{q}(-1)^{\dim T_s-1}n_s\eta(T_s,\beta)-\sum_{t=0}^{q}(-1)^{\dim V_t-1}m_t\eta(V_t,\beta)-\sum_{j=1}^{r}m_{f,b_j}\big(\beta(b_j)-\tilde\beta(b_j)\big).$$

Proof. By Lemma 2.2.2, since g is tractable at the origin with respect to $\mathscr{V}$ relative to f and $\Sigma_{\mathscr{W}}g\cap\{f=0\}=\{0\}$, f is prepolar at the origin with respect to $\mathscr{V}''$ and, therefore, tractable at the origin with respect to $\mathscr{V}''$ relative to g. By Theorem 1.8.23,

$$\chi(X\cap g^{-1}(\alpha)\cap B_\varepsilon,\beta)-\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(0)\cap B_\varepsilon,\beta)=\sum_{s=0}^{q}(-1)^{\dim T_s-1}n_s\eta(T_s,\beta).$$

Since g is tractable at the origin with respect to $\mathscr{V}$ relative to f, also by Theorem 1.8.23,

$$\chi(X\cap f^{-1}(\delta)\cap B_\varepsilon,\beta)-\chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon,\beta) \;=\; \sum_{t=0}^{q}(-1)^{\dim V_t-1}m_t\eta(V_t,\beta).$$

By Theorem 2.2.8,

$$\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(0)\cap B_\varepsilon,\beta) \;=\; \chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon,\beta)-\sum_{j=1}^{r}m_{f,b_j}(\beta(b_j)-\tilde\beta(b_j))$$

which gives that

$$\begin{aligned}
&\chi(X\cap g^{-1}(\alpha)\cap B_\varepsilon,\beta)-\chi(X\cap f^{-1}(\delta)\cap B_\varepsilon,\beta)\\
=\;&\chi(X\cap g^{-1}(\alpha)\cap f^{-1}(0)\cap B_\varepsilon,\beta)-\chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon,\beta)\\
+\;&\sum_{s=0}^{q}(-1)^{\dim T_s-1}n_s\eta(T_s,\beta)-\sum_{t=0}^{q}(-1)^{\dim V_t-1}m_t\eta(V_t,\beta)\\
=\;&\chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon,\beta)-\sum_{j=1}^{r}m_{f,b_j}(\beta(b_j)-\tilde\beta(b_j))\\
-\;&\chi(X\cap g^{-1}(0)\cap f^{-1}(\delta)\cap B_\varepsilon,\beta)+\sum_{s=0}^{q}(-1)^{\dim T_s-1}n_s\eta(T_s,\beta)-\sum_{t=0}^{q}(-1)^{\dim V_t-1}m_t\eta(V_t,\beta))\\
=\;&\sum_{s=0}^{q}(-1)^{\dim T_s-1}n_s\eta(T_s,\beta)-\sum_{t=0}^{q}(-1)^{\dim V_t-1}m_t\eta(V_t,\beta)-\sum_{i=1}^{r}m_{f,b_j}(\beta(b_j)-\tilde\beta(b_j)).
\end{aligned}$$

■

Remark 2.2.21. If g has an isolated singularity and $\mathscr{W}$ is a Whitney stratification of X, the stratification $\mathscr{V}'$ obtained in Lemma 2.1.1 is equal to the good stratification $\mathscr{V}$ of X induced by f and the good stratification $\mathscr{V}''$ obtained in Lemma 2.2.2 is equal to the good stratification of X induced by g. Therefore, keeping the notation $T_1,\dots,T_q$ for the strata of $\mathscr{V}''$ not contained in $\{g=0\}$ and $V_1,\dots,V_q$ for the strata of $\mathscr{V}'$ not contained in $\{f=0\}$, we obtain that $T_i = W_i \setminus \{g=0\}$ and $V_i = W_i \setminus \{f=0\}, W_i \in \mathscr{W}$, for all $i \in \{1,\dots,q\}$. Nevertheless, $\Sigma_{\mathscr{W}} g \cap \{f=\delta\} = \emptyset$. So, in this case, the formula of Theorem 2.2.20 is given by

$$\chi(X \cap f^{-1}(\alpha) \cap B_\varepsilon, \beta) - \chi(X \cap g^{-1}(\delta) \cap B_\varepsilon, \beta) = \textstyle\sum_{i=1}^{q} (-1)^{\dim W_i - 1}(n_i - m_i)\eta(W_i, \beta),$$

which is the formula proved in Theorem 1.8.28.

If we apply Theorem 2.2.20 to the case where $\beta = Eu_X$, we obtain the following consequence.

Corollary 2.2.22. Suppose that g is tractable at the origin with respect to $\mathscr{V}$ relative to f. For $0 < |\alpha| \ll |\delta| \ll \varepsilon \ll 1$, in the setting of the previous statement,

$$B_{g,X}(0) - B_{f,X}(0) = (-1)^{d-1}(n_{reg} - m_{reg}) - \textstyle\sum_{j=1}^{r} m_{f,b_j}(Eu_X(b_j) - B_{g,X\cap\{f=\delta\}}(b_j)),$$

where $n_{reg} = n_q$ and $m_{reg} = m_q$ in the previous notation.

Proof. First we have $\eta(T_s, Eu_X) = 0$, for $s \in \{1,\dots,q-1\}$, $\eta(V_t, Eu_X) = 0$, for $t \in \{1,\dots,q'-1\}$, where $V_t \in \mathscr{V}'$ are the strata not contained in $\{f=0\}$ and $T_s \in \mathscr{V}''$ are the strata not contained in $\{g=0\}$. Also, since the local Euler obstruction is constant over $b_j \cap B_\varepsilon$, we can write $Eu_X(x_\theta) = Eu_X(b_j)$ and, by Lemma 2.2.6, $B_{g,X\cap\{f=\delta\}}(x_\theta) = B_{g,X\cap\{f=\delta\}}(b_j)$, $\theta \in \{i_1,\dots,i_{k(j)}\}$. Therefore, we have the formula.

■

Remark 2.2.23. If g has an isolated singularity, this last formula is given by

$$B_{g,X}(0) - B_{f,X}(0) = (-1)^{d-1}(n_{reg} - m_{reg}),$$

which is the formula proved in Corollary 1.8.29.

In Corollary 1.8.32, the authors showed that if f has an isolated singularity and l is a generic linear form, then, denoting $\Gamma_{f,l}(V_q) = \Gamma^q_{f,l}$, where V_q is the top stratum of the good stratification of X induced by f,

$$\mu^f(\Gamma^q_{f,l}) - \mu^l(\Gamma^q_{f,l}) = (-1)^d Eu_{f,X}(0) = (-1)^{d-1}(B_{f,X}(0) - Eu_X(0)),$$

where $\mu^f(\Gamma_{f,l}(V_q))$ and $\mu^l(\Gamma_{f,l}(V_q))$ are defined as in Remark 1.8.14.

In the following, we use Corollary 2.2.22 to present a generalization of this result to a function-germ $g : X \to \mathbb{C}$ with a one-dimensional critical locus.

Corollary 2.2.24. Let l be a generic linear form over X and $\mathscr{V}$ the good stratification of X induced by l. If g is tractable at the origin with respect to $\mathscr{V}$ relative to l, then

$$\mu^g(\Gamma^q_{g,l}) - \mu^l(\Gamma^q_{g,l}) = (-1)^{d-1}\left(B_{g,X}(0) - Eu_X(0) + \textstyle\sum_{j=1}^r m_{b_j}(Eu_X(b_j) - B_{g,X\cap\{l=\delta\}}(b_j))\right).$$

Proof. Applying Corollary 2.2.22 to $f = l$, we obtain

$$B_{g,X}(0) - B_{l,X}(0) = (-1)^{d-1}(n_{reg} - m_{reg}) - \sum_{j=1}^r m_{b_j}(Eu_X(b_j) - B_{g,X\cap\{l=\delta\}}(b_j)). \tag{2.7}$$

By Proposition 1.8.16 of (MASSEY, 1996), l is decent with respect to $\mathscr{V}$ relative to g and g is decent with respect to $\mathscr{V}$ relative to l. Then, we can replace n_{reg} with $\mu^g(\Gamma^q_{g,l})$ and m_{reg} with $\mu^l(\Gamma^q_{g,l})$. Also, $B_{l,X}(0) = Eu_X(0)$. Hence, Formula (2.7) is given by

$$B_{g,X}(0) - Eu_X(0) = (-1)^{d-1}(\mu^g(\Gamma^q_{g,l}) - \mu^l(\Gamma^q_{g,l})) - \textstyle\sum_{j=1}^r m_{b_j}(Eu_X(b_j) - B_{g,X\cap\{l=\delta\}}(b_j)),$$

that is,

$$B_{g,X}(0) - Eu_X(0) + \textstyle\sum_{j=1}^r m_{b_j}(Eu_X(b_j) - B_{g,X\cap\{l=\delta\}}(b_j)) = (-1)^{d-1}(\mu^g(\Gamma^q_{g,l}) - \mu^l(\Gamma^q_{g,l})).$$

Hence,

$$\mu^g(\Gamma^q_{g,l}) - \mu^l(\Gamma^q_{g,l}) = (-1)^{d-1}\left(B_{g,X}(0) - Eu_X(0) + \textstyle\sum_{j=1}^r m_{b_j}(Eu_X(b_j) - B_{g,X\cap\{l=\delta\}}(b_j))\right).$$

■

Using the last two results, we obtain another way to calculate the Brasselet number $B_{g,X}(0)$.

Proposition 2.2.25. Let l be a generic linear form over X and $\mathscr{V}$ the good stratification of X induced by l. Suppose that g is tractable at the origin with respect to $\mathscr{V}$ relative to l. Then, for $0 < |\delta| \ll \varepsilon \ll 1$,

$$B_{g,X}(0) = (-1)^{d-1} n_{reg} + Eu_{X^g}(0) - \textstyle\sum_{j=1}^r m_{b_j}(Eu_{X^g}(b_j) - B_{g,X\cap\{l=\delta\}}(b_j)),$$

where n_{reg} is the number of stratified Morse critical points of the Morsification of $l : X \cap g^{-1}(\delta) \cap B_\varepsilon \to \mathbb{C}$ appearing on $X_{reg} \cap g^{-1}(\delta) \cap \{l \neq 0\} \cap B_\varepsilon$.

Proof. Applying Corollary 2.2.22 to the case where f is the generic linear form l,

$$B_{g,X}(0)-B_{l,X}(0)=(-1)^{d-1}(n_{reg}-m_{reg})-\sum_{j=1}^{r} m_{b_j}(Eu_X(b_j)-B_{g,X\cap\{l=\delta\}}(b_j)).$$

Since $B_{l,X}(0)=Eu_X(0)$, this means that

$$B_{g,X}(0)=(-1)^{d-1}(n_{reg}-m_{reg})+Eu_X(0)-\sum_{j=1}^{r} m_{b_j}(Eu_X(b_j)-B_{g,X\cap\{l=\delta\}}(b_j)). \tag{2.8}$$

But, by Corollary 2.1.5,

$$Eu_X(0)-Eu_{X^g}(0)-\sum_{j=1}^{r} m_{b_j}(Eu_X(b_j)-Eu_{X^g}(b_j))=(-1)^{d-1}m_{reg},$$

that is,

$$Eu_X(0)-\sum_{j=1}^{r} m_{b_j}Eu_X(b_j)=(-1)^{d-1}m_{reg}+Eu_{X^g}(0)-\sum_{j=1}^{r} m_{b_j}Eu_{X^g}(b_j). \tag{2.9}$$

So, using equations (2.8) and (2.9), we obtain that,

$$\begin{aligned}
B_{g,X}(0) &= (-1)^{d-1}(n_{reg}-m_{reg})+Eu_X(0)-\sum_{j=1}^{r} m_{b_j}Eu_X(b_j)+\sum_{j=1}^{r} B_{g,X\cap\{l=\delta\}}(b_j)\\
&= (-1)^{d-1}(n_{reg}-m_{reg})+(-1)^{d-1}m_{reg}+Eu_{X^g}(0)-\sum_{j=1}^{r} m_{b_j}Eu_{X^g}(b_j)\\
&+ \sum_{j=1}^{r} B_{g,X\cap\{l=\delta\}}(b_j)\\
&= (-1)^{d-1}n_{reg}+Eu_{X^g}(0)-\sum_{j=1}^{r} m_{b_j}(Eu_{X^g}(b_j)-B_{g,X\cap\{l=\delta\}}(b_j)).
\end{aligned}$$

■

Corollary 2.2.26. Let l be a generic linear form over X and $\mathscr{V}$ the good stratification of X induced by l. Suppose that g is tractable at the origin with respect to $\mathscr{V}$ relative to l. Then, for $0<|\delta|\ll\varepsilon\ll 1$,

$$B_{g,X}(0)=(-1)^{d-1}n_{reg}+Eu_{X^g}(0)-\sum_{j=1}^{r} m_{b_j}(B'_{g,X\cap l^{-1}(\delta)}(b_j)-B_{g,X\cap l^{-1}(\delta)}(b_j)),$$

where n_{reg} is the number of stratified Morse critical points of the Morsification of $l: X\cap g^{-1}(\delta)\cap B_\varepsilon\to\mathbb{C}$ appearing on $X_{reg}\cap g^{-1}(\delta)\cap\{l\neq 0\}\cap B_\varepsilon$.

Proof. We have the formula since, by Remark 2.2.16, $Eu_{X^g}(b_j)=B'_{g,X\cap l^{-1}(\delta)}(b_j)$.

■

Remark 2.2.27. Let l be a generic linear form over X and $\mathscr{V}$ the good stratification of X induced by l. Suppose that g is tractable at the origin with respect to $\mathscr{V}$ relative to l and let $H=l^{-1}(0)$. By Theorem 1.8.25, $(-1)^{d-1}n_{reg}=B_{g,X}(0)-B_{g,X\cap H}(0)$. Using this equality in the formula

$$B_{g,X}(0) = (-1)^{d-1} n_{reg} + Eu_{X^g}(0) - \sum_{j=1}^{r} m_{b_j} (B'_{g,X\cap l^{-1}(\delta)}(b_j) - B_{g,X\cap l^{-1}(\delta)}(b_j)),$$

we obtain

$$B_{g,X\cap H}(0) = Eu_{X^g}(0) - \sum_{j=1}^{r} m_{b_j} (B'_{g,X\cap l^{-1}(\delta)}(b_j) - B_{g,X\cap l^{-1}(\delta)}(b_j)),$$

which is the formula obtained in Remark 2.2.16.

CHAPTER

3

LÊ-IOMDIN FORMULA FOR THE BRASSELET NUMBER

The Milnor number is a very useful invariant associated to a complex function f with isolated singularity defined over an open neighborhood of the origin in $\mathbb{C}^N$. It gives numerical information about the local topology of the hypersurface $V(f)$ and computes the Euler characteristic of the Milnor fibre of f at the origin.

If we consider a function with a one-dimensional critical set defined over an open subset of $\mathbb{C}^n$ and a generic linear form l over $\mathbb{C}^n$, Iomdin gave an algebraic proof (Theorem 3.2), in (IOMDIN, 1974a), of a relation between the Euler characteristic of the Milnor fibre of f and the Euler characteristic of the Milnor fibre of $f+l^N, N \gg 1$ and $N \in \mathbb{N}$, using properties of algebraic sets with one-dimensional critical locus. In (LÊ, 1980), Lê proved (Theorem 2.2.2) this same relation in a more geometric approach and with a way to obtain the Milnor fibre of f by attaching a certain number of n-cells to the Milnor fibre of $f|_{\{l=0\}}$.

In (MASSEY, 2003), Massey worked with a function f with critical locus of higher dimension defined over a nonsingular space and defined the Lê numbers and cycles, which provides a way to numerically describe the Milnor fibre of this function with nonisolated singularity. Massey compared (Theorem II.4.5), using appropriate coordinates, the Lê numbers of f and $f+l^N$, where l is a generic linear form over $\mathbb{C}^n$ and N is sufficiently large, obtaining a Lê-Iomdin type relation between these numbers. He also gave (Theorem II.3.3) a handle decomposition of the Milnor fibre of f, where the number of attached cells is some Lê number. Massey extended the concept of Lê numbers to the case of functions with nonisolated singularities defined over complex analytic spaces, introducing the Lê-Vogel cycles, and proved the Lê-Iomdin-Vogel formulas: the generalization of the Lê-Iomdin formulas in this more general sense.

The Brasselet number also describes the local topological behavior of a function with nonisolated singularities defined over an arbitrarily singular analytic space. The Lê-Iomdin formula for the Brasselet number follows from (BRASSELET *et al.*, 2004), with an algebraic approach, using Lê-Vogel-cycles and vanishing cycles.

In this chapter we provide a new proof for this formula, with a topological approach, in the case of an analytic function g with a one-dimensional critical set and defined over an analytic complex space.

3.1 Classical Lê-Iomdin formulas

In this section we present formulas proved by Iomdin, Lê and Massey.

Let $(Y^*,0)$ be a complete intersection germ with isolated singularity at the origin of codimension $k-1$ defined in $(\mathbb{C}^n,0)$ by $f_1 = \cdots = f_{k-1} = 0, U$ be an open subset of $\mathbb{C}^n$, $g : (U,0) \to (\mathbb{C},0)$ an analytic function-germ and $(Y,0)$ be a complete intersection germ defined by $f_1 = \cdots = f_{k-1} = g = 0$. Let S_ε be a sphere with center at 0 and radius ε, $\partial Y_\varepsilon = Y \cap S_\varepsilon$ the link associated to $(Y,0)$ and $\partial Y^*_\varepsilon = Y^* \cap S_\varepsilon$ the one associated to $(Y^*,0)$. Let $\varepsilon > 0$ be sufficiently small and define $\xi_k(0) : \partial Y^*_\varepsilon \to V(k,n)$, where $V(k,n)$ is the manifold of orthonormal k-planes in $\mathbb{C}^n$, the map that associates to each $z \in \partial Y^*_\varepsilon$, the k-plane obtained by the orthonormalization of Gram-Schmidt of $(grad\ f_1(z),\ldots,grad\ f_{k-1}(z),grad\ g(z))$.

We begin with the Lê and Iomdin's theorem and we follow the notation used by Lê in (LÊ, 1980). Suppose that $Y^* \setminus Y$ is nonsingular and that g has a one-dimensional critical set $\Sigma g \subset (Y,0)$. Let l be a generic linear form over U. If $(z_1,\ldots,z_n)$ are the local coordinates of $\mathbb{C}^n$ in U, without loss of generality, we can suppose that $l = z_1$. We consider a decomposition of Σg into branches Γ_v.

Theorem 3.1.1. (Theorem 2.2.2 (LÊ, 1980)) Let F be the Milnor fibre at 0 of the restriction of g to $(Y^*,0)$. If N is an integer sufficiently large, then

$$\chi(F) = \chi(F_N) - N\sum_v n_v \delta(\xi_k(x_i(t))),$$

where F_N is the Milnor fibre of the restriction of $g+z_1^N$ to $(Y^*,0), n_v$ is the degree of the restriction of z_1 to the branch Γ_v of $\Sigma g, x_i(t)$ is a singular point on $\Gamma_v \cap z_1^{-1}(t)$, $\delta(\xi_k(x_i(t)))$ is the degree of the map $\xi_k(x_i(t)) : \partial(Y^* \cap z_1^{-1}(t))_\varepsilon \to V(k,n)$, with $\partial(Y^* \cap z_1^{-1}(t))_\varepsilon = Y \cap z_1^{-1}(t) \cap S_\varepsilon(x_i(t))$, $S_\varepsilon(x_i(t))$ is a sphere with center at $x_i(t)$ and a sufficiently small radius ε.

Let us now see the Lê-Iomdin formula proved by Massey. We present here the case for functions defined over a nonsingular subspace of $\mathbb{C}^n$, and we recommend Part I of (MASSEY, 2003) for the general case. Let $h : (U,0) \subseteq (\mathbb{C}^n,0) \to (\mathbb{C},0)$ be an analytic function such that its critical locus Σh is a s-dimensional set. We will need some auxiliary concepts before we see the definition of Lê numbers.

Definition 3.1.2. For $0 \leq k \leq n$, the k**-th relative polar variety** $\Gamma^k_{h,z}$ of h with respect to z is the scheme (see page 118, (MANIN, 2018))

$$V\left(\frac{\partial h}{\partial z_k}, \dots, \frac{\partial h}{\partial z_n}\right)/\Sigma h,$$

where $z = (z_1, \dots, z_n)$ are fixed local coordinates.

Also, the k**-th polar cycle** of h with respect to z is the analytic cycle $[\Gamma^k_{h,z}]$.

Remark 3.1.3. If $f_1, \dots, f_r$ are holomorphic function germs in $\mathbb{C}^{n+1}$ and Y is an irreducible component of $V(f_1, \dots, f_r)$, then $\dim(Y) \geq \dim(\mathbb{C}^{n+1}) - r = n+1-r$. So, since $\Gamma^k_{h,z} = V(\frac{\partial h}{\partial z_k}, \dots, \frac{\partial h}{\partial z_n})/\Sigma h$, each irreducible component of $\Gamma^k_{h,z}$ has dimension, at least, $n+1-(n-k+1) = k$.

Definition 3.1.4. For $0 \leqslant k \leqslant n$, the k**-th Lê cycle** $[\Lambda^k_{h,z}]$ of h with respect to z is the difference of cycles $\left[\Gamma^{k+1}_{h,z} \cap V(\frac{\partial h}{\partial z_k})\right] - [\Gamma^k_{h,z}]$.

Definition 3.1.5. The k**-th Lê number** of h in p with respect to z, $\lambda^k_{h,z}$, is the intersection number

$$(\Lambda^k_{h,z}.V(z_0 - p_0, \dots, z_{k-1} - p_{k-1}))_p,$$

provided this intersection is purely zero-dimensional at p.

If this intersection is not purely zero-dimensional, the k-th Lê number of h at p with respect to z is said to be undefined.

Example 3.1.6. (Example II.1.10 of (MASSEY, 2003))

Consider the Whitney umbrella given by $h = y^2 - x^3 - tx^2$ and fix the coordinate system $z = (t, x, y)$.

We have $\Sigma h = V(\langle -x^2, -3x^2 - 2tx, 2y\rangle) = V(x, y)$, that is, the singular locus of h is the t-axis. Now, $V\left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right) = \{x = y = 0\} \cup \{-3x^2 - 2t = y = 0\}$. Hence, $\Gamma^1_{h,z} = V(-3x^2 - 2t, y)$.

Also, $V\left(\frac{\partial h}{\partial y}\right) = V(y)$ and $\Gamma^2_{h,z} = V(y)$.

By the definition of Lê cycles, $\Lambda^1_{h,z} = \left[\Gamma^2_{h,z} \cap V\left(\frac{\partial h}{\partial x}\right)\right] - [\Gamma^1_{h,z}] = [V(y,x)]$, which is the t-axis. Therefore, the underlying space of $\Lambda^1_{h,z}$ is the t-axis and this component occurs with multiplicity 1. The 0-th Lê cyle, on the other hand, is given by

$$\Lambda^0_{h,z} = \left[\Gamma^1_{h,z} \cap V\left(\frac{\partial h}{\partial t}\right)\right] - [\Gamma^0_{h,z}] = 2[V(x,y,t)] = 2[(0,0,0)].$$

Hence, the underlying space of $\Lambda^0_{h,z}$ is the origin, with multiplicity 2.

We can now compute the Lê numbers:

$$\lambda^0_{h,z} = (\Lambda^0_{h,z})_0 = 2 \text{ and } \lambda^1_{h,z} = (\Lambda^1_{h,z}.V(t))_0 = (V(y,x).V(t))_0 = 1$$

Corollary 3.1.7. (Corollary 1.19 (MASSEY, 2003)) Let $k \geq 0$. Suppose that Σh is a s-dimensional set. Then the Lê numbers $\lambda^i_{h,z}(p)$ is defined for $0 \leq i \leq k$.

If s is the dimension of Σh and $\lambda^s_{h,z}(p)$ exists, an interesting characterization for this number, given by Massey (page 49), in (MASSEY, 2003), is the following:

$$\lambda^s_{h,z}(p) = \sum_v n_v \mu_v,$$

where v runs over all s-dimensional components of Σh at p, n_v is the local degree of the map $(z_0, \ldots, z_{s-1})$ restricted to v at p and μ_v is the generic transverse Milnor number.

We still need one more definition before we state the Lê-Iomdin formulas.

Definition 3.1.8. Suppose that Γ^1_{h,z_0} is purely one-dimensional at the origin. Let η be an irreducible component of Γ^1_{h,z_0} (with its reduced structure) such that $\eta \cap V(z_0)$ is zero-dimensional at the origin. The **polar ratio** of η (for h at 0 with respect to z_0) is the ratio of intersection numbers $\frac{(\eta.V(h))_0}{(\eta.V(z_0))_0}$. If $\eta \cap V(z_0)$ is not zero-dimensional at the origin, then the polar ratio of η is equal to 1.

A **polar ratio** (of h at 0 with respect to z_0) is any one of the polar ratios of any component of the polar curve.

We are now ready to state the Lê- Iomdin formulas for Lê numbers.

Theorem 3.1.9. (Theorem II.4.5 of (MASSEY, 2003)) Let $j \geq 2$, $h : (U, 0) \subseteq (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic function, and $s \geq 1$ the dimension of Σh in 0. Let $z = (z_0, \ldots, z_n)$ be a linear choice of coordinates such that $\lambda^i_{h,z}(0)$ is defined for all $i \leq s$. Let a be a nonzero complex number, and let us consider the coordinates $\tilde{z} = (z_1, \ldots, z_n, z_0)$ to define the Lê numbers associated to $h + az_0^j$.

If j is greater than the maximum polar ratio for h then, for all complex number a, $\Sigma(h + az_0^j) = \Sigma h \cap V(z_0)$ as germs of sets at $0, \dim_0 \Sigma(h + az_0^j) = s - 1, \lambda^i_{h+az_0^j,\tilde{z}}(0)$ exists for all $i \leq s - 1$, and

$$\lambda^0_{h+az_0^j,\tilde{z}}(0) = \lambda^0_{h,z}(0) + (j-1)\lambda^1_{h,z}(0),$$

and, for $1 \leq i \leq s-1$,

$$\lambda^i_{h+az_0^j,\tilde{z}}(0) = (j-1)\lambda^{i+1}_{h,z}(0).$$

3.2 Local topology of a deformation of a function-germ with one-dimensional critical set

Let $f, g : (X, 0) \to (\mathbb{C}, 0)$ be complex analytic function-germs such that f has an isolated singularity at the origin. Let $\mathscr{W}$ be the Whitney stratification of X and $\mathscr{V}$ the good stratification of X induced by f. Suppose that $\Sigma_{\mathscr{W}} g$ is one-dimensional and that $\Sigma_{\mathscr{W}} g \cap \{f = 0\} = \{0\}$.

In this section we study the local topology of the deformation $\tilde{g}$ of g given by $\tilde{g} = g + f^N$, where $N \gg 1$ is a positive integer number. We begin with a discussion about the singular locus of $\tilde{g}$ and a description of the appropriate stratification with which we can compute explicitly the Brasselet numbers we will use.

By First stratification lemma (Lemma 2.1.1), if $\mathscr{V}^f$ denotes the set of strata of $\mathscr{V}$ contained in $\{f = 0\}$,

$$\mathscr{V}' = \{V_i \setminus \Sigma_{\mathscr{W}} g, V_i \cap \Sigma_{\mathscr{W}} g, V_i \in \mathscr{V}\} \cup \mathscr{V}^f$$

is a good stratification of X relative to f, such that $\mathscr{V}'^{\{g=0\}}$ is a good stratification of X^g relative to $f|_{X^g}$. In this whole section, we will use this good stratification of X relative to f. Suppose that g is tractable at the origin with respect to $\mathscr{V}$.

Proposition 3.2.1. For a sufficiently large $N, \tilde{g}$ has an isolated singularity at the origin with respect to the Whitney stratification $\mathscr{W}$ of X.

Proof. Let x be a critical point of $\tilde{g}$, U_x be a neighborhood of x and G and F be analytic extensions of g and f to U_x, respectively. If $V(x)$ is a stratum of $\mathscr{W}$ containing $x \neq 0$,

$$d_x\tilde{G}|_{V(x)} = 0 \Leftrightarrow d_xG|_{V(x)} + N(F(x))^{N-1} d_xF|_{V(x)} = 0$$

If $d_xG|_{V(x)} = 0$, then $N(F(x))^{N-1} d_xF|_{V(x)} = 0$, hence $x \in \{F = 0\}$. Then $x \in \Sigma_{\mathscr{W}} g \cap \{f = 0\} = \{0\}$. If $d_xG|_{V(x)} \neq 0$, we have $G \neq 0$. Since $d_x\tilde{G}|_{V(x)} = 0$, by Proposition 1.8.4, $\tilde{G} = 0$, which implies that $F \neq 0$. On the other hand, if $d_xG|_{V(x)} \neq 0$, $d_xG|_{V(x)} = -N(F(x))^{N-1} d_xF|_{V(x)}$, and then $x \in \tilde{\Gamma}_{f,g}(V(x))$. Suppose that x is arbitrarily close to the origin. Since f has an isolated singularity at the origin, we can define for the stratum $V(x)$, the function $\beta : (0, \varepsilon) \to \mathbb{R}, 0 < \varepsilon \ll 1$,

$$\beta(u) = \inf \left\{ \frac{||d_zg|_{V(x)}||}{||d_zf|_{V(x)}||}; z \in \tilde{\Gamma}_{f,g}(V(x)) \cap \{|f|_{V(x)}(z)| = u, u \neq 0\} \right\},$$

where $||.||$ denotes the operator norm, (defined, for each linear transformation $T : V \to W$ between normed vector fields, by $sup_{v \in V, ||v||=1} ||T(v)||$). Notice that, for each stratum $W_i \in \mathscr{W}, \tilde{\Gamma}_{f,g}(W_i) = \tilde{\Gamma}_{f,g}(W_i \setminus \{f = 0\})$. Since g is tractable at the origin with respect to $\mathscr{V}, \dim_0 \tilde{\Gamma}_{f,g}(\mathscr{V}) \leq 1$. Therefore, $\dim_0 \tilde{\Gamma}_{f,g}(W_i) = \dim_0 \tilde{\Gamma}_{f,g}(W_i \setminus \{f = 0\}) \leq 1$. Hence $\tilde{\Gamma}_{f,g}(V(z)) \cap \{|f| = u, u \neq 0\}$ is a finite number of points and β is well defined.

Since the function β is subanalytic, $\alpha(R) = \beta(1/R)$, for $R \gg 1$, is subanalytic,. Since by Proposition 2.2 of (LOI *et al.*, 2010) composition of subanalytic functions is a subanalytic function and the real function $h(x) = 1/x, x \neq 0$, is subanalytic, $1/\alpha(R)$ is subanalytic. Then, by page 135 of (LOI, 2003), there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{\alpha(R)} < R^{n_0}$, which implies $\beta(1/R) > (1/R)^{n_0}$, which implies, $\beta(u) > u^{n_0}$. Hence, for $z \in \tilde{\Gamma}_{f,g}(V(x)) \cap \{f|_{V(x)}(z)| = u\}$, $u \ll 1$, we have

$\frac{||d_z g|_{V(x)}||}{||d_z f|_{V(x)}||} \geq \beta(u) > u^{n_0}$, which implies, $||d_z g|_{V(x)}|| > |f|_{V(x)}(z)|^{n_0} ||d_z f|_{V(x)}||$.

On the other hand, since N is sufficiently large, we can suppose $N > n_0$. Since $\tilde{g}(z) = g(z) + f^N(z)$, we obtain using the previous inequality that for the critical point x of $\tilde{g}$,

$$N|f|_{V(x)}(x)|^{N-1} ||d_x f|_{V(x)}|| = ||d_x g|_{V(x)}|| > |f|_{V(x)}(x)|^{n_0} ||d_x f|_{V(x)}||,$$

which implies that $N|f|_{V(x)}(x)|^{N-1-n_0} > 1$.

Since x was taken sufficiently close to the origin, $f|_{V(x)}(x)$ is close to zero. Hence, $|f|_{V(x)}(x)| \ll 1$, which implies that $N - 1 - n_0 < 0$. Therefore, $N \leq n_0$, which is a contradiction. So, there is no x sufficiently close to the origin such that $d_x\tilde{g} = 0$. Therefore, $\tilde{g}$ has an isolated singularity at the origin. ■

We will now see how $\tilde{g}$ behaves with respect to the good stratification $\mathscr{V}$ of X induced by f.

Proposition 3.2.2. Let $\mathscr{V}$ be the good stratification of X induced by f. Then $\tilde{g}$ is prepolar at the origin with respect to $\mathscr{V}$.

Proof. By Proposition 3.2.1, $\tilde{g}$ is prepolar at the origin with respect to $\mathscr{W}$. So it is enough verify that $\tilde{g}|_{W_i \cap \{f=0\}}$ is nonsingular or has an isolated singularity at the origin, where W_i is a stratum from the Whitney stratification $\mathscr{W}$ of X. Suppose that $x \in \Sigma\tilde{g}|_{W_i \cap \{f=0\}}$. Then $d_x\tilde{g} = d_x g + Nf(x)^{N-1} d_x f = 0$, which implies that $d_x g = 0$. But g has no critical point on $W_i \cap \{f = 0\}$, since g is tractable at the origin with respect to $\mathscr{V}$. Therefore, $\tilde{g}$ is prepolar at the origin with respect to $\mathscr{V}$. ■

Corollary 3.2.3. Let $\tilde{\mathscr{V}}$ be the good stratification of X induced by $\tilde{g}$. Then f is prepolar at the origin with respect to $\tilde{\mathscr{V}}$.

Proof. By Proposition 3.2.2, $\tilde{g}$ is prepolar at the origin with respect to the good stratification $\mathscr{V}$ of X induced by f. Hence, by Lemma 6.1 of (DUTERTRE; GRULHA, 2014), f is prepolar at the origin with respect to $\tilde{\mathscr{V}}$. ■

Using the previous results, we can relate the relative symmetric polar varieties $\tilde{\Gamma}_{f,\tilde{g}}(\mathscr{V})$ and $\tilde{\Gamma}_{f,g}(\mathscr{V})$.

Remark 3.2.4. Let us describe $\tilde{\Gamma}_{f,\tilde{g}}(\mathscr{V})$. Let $\Sigma(\tilde{g}, f) = \{x \in X; rk(d_x\tilde{g}, d_x f) \leq 1\}$. Since f is prepolar at the origin with respect to the good stratification induced by $\tilde{g}$, $f|_{W_i \cap \{\tilde{g}=0\}}$ is nonsingular, for all $W_i \in \mathscr{W}, i \neq 0$. Also $\tilde{g}$ is prepolar at the origin with respect to the good stratification induced by f, which implies that $\tilde{g}|_{W_i \cap \{f=0\}}$ is nonsingular, for all $W_i \in \mathscr{W}, i \neq 0$. Nevertheless, since f and $\tilde{g}$ have a stratified isolated singularity at the origin, $\Sigma_{\mathscr{W}}\tilde{g} \cup \Sigma_{\mathscr{W}} f = \{0\}$. Therefore, the map (f, g) has no singularities in $\{g = 0\}$ or in $\{f = 0\}$. Hence, $\Sigma(\tilde{g}, f) = \tilde{\Gamma}_{f,\tilde{g}}(\mathscr{V})$. So, it is sufficient to describe $\Sigma(\tilde{g}, f)$. Let $x \in \Sigma(\tilde{g}, f)$, then

$$
\begin{aligned}
rk(d_x\tilde{g}, d_x f) \leq 1 \quad &\Leftrightarrow \quad \big(d_x\tilde{g} = 0\big) \text{ or } \big(d_x f = 0\big) \text{ or } \big(d_x\tilde{g} = \lambda d_x f\big)\\
&\Leftrightarrow \quad \big(d_x\tilde{g} = 0\big) \text{ or } \big(d_x f = 0\big) \text{ or } \big(d_x g = (-Nf(x)^{N-1} + \lambda) d_x f\big)
\end{aligned}
$$

Since $x \notin \{f = 0\}, d_x f \neq 0$. And since $\tilde{g}$ has an isolated singularity at the origin, $d_x\tilde{g} \neq 0$. If $-Nf(x)^{N-1} + \lambda = 0$, then $d_x g = 0$, that is, $x \in \Sigma_{\mathscr{W}} g$. If $-Nf(x)^{N-1} + \lambda \neq 0$, then $d_x g$ is a nonzero multiple of $d_x f$, that is, $x \in \tilde{\Gamma}_{f,g}(\mathscr{V})$. Therefore,

$$\Sigma(\tilde{g}, f) \subseteq \Sigma_{\mathscr{W}} g \cup \tilde{\Gamma}_{f,g}.$$

On the other hand, if $x \in \Sigma_{\mathscr{W}} g$, then $d_x g = 0$, and

$$d_x\tilde{g} = d_x g + Nf(x)^{N-1} d_x f = Nf(x)^{N-1} d_x f.$$

So, $x \in \Sigma(\tilde{g}, f)$. If $x \in \tilde{\Gamma}_{f,g}(\mathscr{V}), d_x g = \lambda d_x f$, and

$$d_x\tilde{g} = d_x g + Nf(x)^{N-1} d_x f = (\lambda + N) f(x)^{N-1} d_x f,$$

which implies $x \in \Sigma(\tilde{g}, f)$. Therefore, $\tilde{\Gamma}_{f,\tilde{g}}(\mathscr{V}) = \Sigma(\tilde{g}, f) = \Sigma_{\mathscr{W}} g \cup \tilde{\Gamma}_{f,g}(\mathscr{V})$.

Proposition 3.2.5. Let $\mathscr{V}$ be the good stratification of X induced by f and suppose that g is tractable at the origin with respect to $\mathscr{V}$. Then, for $N \gg 1$,

$$B_{g,X^f}(0) = B_{\tilde{g},X^f}(0) = B_{f,X^{\tilde{g}}}(0).$$

Proof. Since $\tilde{g} = g + f^N$, over $\{f = 0\}, \tilde{g} = g$. Therefore, $B_{g,X^f}(0) = B_{\tilde{g},X^f}(0)$. On the other hand, by Corollary 3.2.3, f is prepolar at the origin with respect to the good stratification $\tilde{\mathscr{V}}$ of X induced by $\tilde{g}$ and so is $\tilde{g}$ with respect to $\mathscr{V}$, by Proposition 3.2.2. Hence, by Corollary 1.8.27, $B_{f,X^{\tilde{g}}}(0) = B_{\tilde{g},X^f}(0)$. ■

Corollary 3.2.6. Let l be a generic linear form over X and $\mathscr{V}$ the good stratification of X induced by l. Denote $l^{-1}(0)$ by H and suppose that g is tractable at the origin with respect to $\mathscr{V}$ relative to l. Then

$$B_{g,X\cap H}(0) = B_{\tilde{g},X\cap H}(0) = Eu_{X^{\tilde{g}}}(0).$$

Proof. It follows directly by Proposition 3.2.5, using the equality $B_{g,X^l}(0) = B_{\tilde{g},X^l}(0)$, and Corollary 1.8.30. ■

Remark 3.2.7. Since, by Remark 2.2.14, the sum of Brasselet numbers $\sum_{j=1}^{r} m_{b_j} B_{g,X\cap l^{-1}(\delta)}(b_j)$ is independent of the choice of a sufficiently generic linear form l and so are all the other therms in Formula 3.1, we conclude that $Eu_{X^{\tilde{g}}}(0)$, where $\tilde{g} = g + l^N$ and $N \gg 1$, does not depend on the generic linear form l.

Corollary 3.2.8. Let N be a sufficiently large positive integer number.

1. If d is even, $Eu_{X^{\tilde{g}}}(0) \geq Eu_{X^g}(0)$;
2. If d is odd, $Eu_{X^{\tilde{g}}}(0) \leq Eu_{X^g}(0)$.

Proof. By Corollary 2.2.17, we have that

1. If d is even, $B_{g,X\cap H}(0) \geq Eu_{X^g}(0)$;
2. If d is odd, $B_{g,X\cap H}(0) \leq Eu_{X^g}(0)$.

Since $Eu_{X^{\tilde{g}}}(0) = B_{g,X\cap H}(0)$, by Corollary 3.2.6, we have the proof. ■

Proposition 3.2.9. Let $\mathscr{V}$ be the good stratification of X induced by f and $\tilde{\mathscr{V}}$ the good stratification of X induced by $\tilde{g}$. Suppose that g is tractable at the origin with respect to $\mathscr{V}$. Then, for $0 < |\delta| \ll \varepsilon \ll 1$,

$$B_{f,X^g}(0) - B_{f,X^{\tilde{g}}}(0) = \sum_{j=1}^{r} m_{f,b_j}(Eu_{X^g}(b_j) - B_{g,X\cap f^{-1}(\delta)}(b_j)).$$

Proof. By Corollary 2.2.11,

$$B_{f,X^g}(0) - B_{g,X^f}(0) = \sum_{j=1}^{r} m_{f,b_j}(Eu_{X^g}(b_j) - B_{g,X\cap f^{-1}(\delta)}(b_j)).$$

Since, by Proposition 3.2.5, $B_{g,X^f}(0) = B_{f,X^{\tilde{g}}}(0)$, we have the formula. ■

Corollary 3.2.10. Let l be a generic linear form over X, $\mathscr{V}$ the good stratification of X induced by l and $\tilde{\mathscr{V}}$ the good stratification of X induced by $\tilde{g}$. Suppose that g is tractable at the origin with respect to $\mathscr{V}$. Then, for $0 < |\delta| \ll \varepsilon \ll 1$,

$$Eu_{X^g}(0) - Eu_{X^{\tilde{g}}}(0) = \sum_{j=1}^{r} m_{b_j}(Eu_{X^g}(b_j) - B_{g,X\cap l^{-1}(\delta)}(b_j)). \tag{3.1}$$

Proof. It follows directly from Proposition 3.2.9, using that $B_{l,X^g}(0) = Eu_{X^g}(0)$ and that $B_{l,X^{\tilde{g}}}(0) = Eu_{X^{\tilde{g}}}(0)$. ■

Remark 3.2.11. By Remark 2.2.16, $Eu_{X^g}(b_j) = B'_{g,X\cap l^{-1}(\delta)}(b_j)$, where $B'_{g,X\cap l^{-1}(\delta)}(b_j)$ denotes the Brasselet number $B_{g,X\cap l^{-1}(\delta)\cap L}(b_j \cap l^{-1}(\delta))$ and L is a generic hyperplane in $\mathbb{C}^n$ passing through $x_\theta \in\in l^{-1}(\delta) \cap b_j, j \in \{1,\ldots,r\}$ and $\theta \in \{i_1,\ldots,i_{k(j)}\}$. So, the formula obtained in Corollary 3.2.10 can be written as

$$Eu_{X^g}(0) - Eu_{X^{\tilde{g}}}(0) = \sum_{j=1}^{r} m_{b_j}(B'_{g,X\cap l^{-1}(\delta)}(b_j) - B_{g,X\cap l^{-1}(\delta)}(b_j)).$$

Let m be the number of stratified Morse points of a partial Morsification of $g|_{X\cap f^{-1}(\delta)\cap B_\varepsilon}$ appearing on $X_{reg} \cap f^{-1}(\delta) \cap \{g \neq 0\} \cap B_\varepsilon$ and $\tilde{m}$ the number of stratified Morse points of a Morsification of $\tilde{g}|_{X\cap f^{-1}(\delta)\cap B_\varepsilon}$ appearing on $X_{reg} \cap f^{-1}(\delta) \cap \{\tilde{g} \neq 0\} \cap B_\varepsilon$. The next lemma shows how to compare m and $\tilde{m}$. In the following we keep the same description of $\Sigma_{\mathscr{W}} g$ given in the last chapter. Let us recall it: we write the one-dimensional set $\Sigma_{\mathscr{W}} g$ as a union of branches $b_1 \cup \ldots \cup b_r$, where $b_j \subseteq W_{i_j} \in \mathscr{W}$. Let δ be a regular value of $f, 0 < |\delta| \ll 1$, and let us write, for each $j \in \{1,\ldots,r\}, f^{-1}(\delta) \cap b_j = \{x_{i_1},\ldots,x_{i_{k(j)}}\}$. So, in this case, the local degree m_{f,b_j} of $f|_{b_j}$ is k. Let ε be sufficiently small such that the local Euler obstruction of X is constant on $b_j \cap B_\varepsilon$.

Remark 3.2.12. From the beginning we work in a sufficiently small neighborhood B_ε where the local Euler obstruction of X is constant on $b_j \cap B_\varepsilon$ and so is the local Euler obstruction of $X \cap f^{-1}(\delta)$, since, as we saw above, $Eu_X(b_j) = Eu_{X \cap f^{-1}(\delta)}(b_j)$. We also know that, by Lemma 2.2.6, the Brasselet number $B_{g,X\cap f^{-1}(\delta)}(b_j)$ is constant on the branch b_j. Hence, since

$$Eu_{g,X\cap f^{-1}(\delta)}(b_j) = Eu_X(b_j) - B_{g,X\cap f^{-1}(\delta)}(b_j),$$

so is the Euler obstruction of a function $Eu_{g,X\cap f^{-1}(\delta)}(b_j)$.

Corollary 3.2.13. In the context described above, we have

$$\tilde{m} = (-1)^{d-1} \sum_{j=1}^{r} m_{f,b_j} Eu_{g,X\cap f^{-1}(\delta)}(b_j) + m.$$

Proof. Lemma 2.1.1 ensures that $\mathscr{V}'^{\{g=0\}}$ is a good stratification of X^g relative to $f|_{X^g}$. Since g is tractable at the origin with respect to $\mathscr{V}$, by Theorem 2.1.2,

$$B_{f,X}(0) - B_{f,X^g}(0) - \textstyle\sum_{j=1}^{r} m_{f,b_j}(Eu_X(b_j) - Eu_{X^g}(b_j)) = (-1)^{d-1} m,$$

where m is the number of stratified Morse points of a Morsification of $g|_{X\cap f^{-1}(\delta)\cap B_\varepsilon}$ appearing in $X_{reg} \cap f^{-1}(\delta) \cap \{g = 0\} \cap B_\varepsilon$.

By Proposition 3.2.2, $\tilde{g}$ is prepolar at the origin with respect to $\mathscr{V}$, by Theorem 1.8.25,

$$B_{f,X}(0) - B_{f,X^{\tilde{g}}}(0) = (-1)^{d-1}\tilde{m},$$

where $\tilde{m}$ is the number of stratified Morse points of a Morsification of $\tilde{g}|_{X\cap f^{-1}(\delta)\cap B_\varepsilon}$ appearing in $X_{reg} \cap f^{-1}(\delta) \cap \{\tilde{g} = 0\} \cap B_\varepsilon$.

Using Proposition 3.2.9,

$$B_{f,X^g}(0) - B_{f,X^{\tilde{g}}}(0) = \textstyle\sum_{j=1}^{r} m_{f,b_j}(Eu_{X^g}(b_j) - B_{g,X\cap f^{-1}(\delta)}(b_j)),$$

we obtain that

$$\begin{aligned}
\sum_{j=1}^{r} m_{f,b_j}(Eu_{X^g}(b_j) - B_{g,X\cap f^{-1}(\delta)}(b_j)) &= B_{f,X^g}(0) - B_{f,X^{\tilde{g}}}(0) \\
&= B_{f,X}(0) - \sum_{j=1}^{r} m_{f,b_j}(Eu_X(b_j) - Eu_{X^g}(b_j)) - (-1)^{d-1} m \\
&- B_{f,X}(0) + (-1)^{d-1}\tilde{m},
\end{aligned}$$

which implies

$$\tilde{m} = m + (-1)^{d-1} \sum_{j=1}^{r} m_{f,b_j} (Eu_X(b_j) - B_{g,X\cap f^{-1}(\delta)}(b_j)).$$

Since f has an isolated singularity at the origin, $f^{-1}(\delta)$ intersects each stratum out of $\{f = 0\}$ transversely. So, $Eu_X(V_i) = Eu_{X\cap f^{-1}(\delta)}(S)$, for each connected component of $V_i \cap f^{-1}(\delta)$. In particular, $Eu_X(b_j) = Eu_{X\cap f^{-1}(\delta)}(b_j)$.

So, by Theorem 1.7.5,

$$Eu_X(b_j) - B_{g,X\cap f^{-1}(\delta)}(b_j) = Eu_{g,X\cap f^{-1}(\delta)}(b_j).$$

Therefore,

$$\tilde{m} = m + (-1)^{d-1} \textstyle\sum_{j=1}^{r} m_{f,b_j} Eu_{g,X\cap f^{-1}(\delta)}(b_j).$$

■

Proposition 3.2.14. Let $\tilde{\alpha}$ be a regular value of $\tilde{g}$ and α_t a regular value of f, $0 \ll |\tilde{\alpha}| \ll |\alpha_t| \ll 1$. If g is tractable at the origin with respect to $\mathscr{V}$ relative to f, then $B_{g,X\cap f^{-1}(\alpha_t)}(b_j) = B_{f,X\cap \tilde{g}^{-1}(\tilde{\alpha})}(b_j)$.

Proof. Let $x_t \in \{f = \alpha_t\} \cap b_j$, D_{x_t} the closed ball with center at x_t and radius r_t, $0 < |\alpha - \delta| \ll |\alpha_t| \ll r_t \ll 1$. We have

$$\begin{aligned} B_{g,X\cap f^{-1}(\alpha_t)}(x_t) &= \sum \chi(W_i \cap f^{-1}(\alpha_t) \cap g^{-1}(\alpha - \delta) \cap D_{x_t}) Eu_{X\cap f^{-1}(\alpha_t)}(W_i \cap f^{-1}(\alpha_t)) \\ &= \sum \chi(W_i \cap f^{-1}(\alpha_t) \cap g^{-1}(\alpha - \delta) \cap D_{x_t}) Eu_X(W_i). \end{aligned}$$

Let $g(x_t) = \alpha, \tilde{g}(x_t) = \alpha'$ and $f(x_t) = \alpha_t$. Then

$$\begin{aligned} p \in f^{-1}(\alpha_t) \cap g^{-1}(\alpha - \delta) &\Leftrightarrow g(p) = \alpha - \delta \text{ and } f(p) = \alpha_t \\ &\Leftrightarrow g(p) = g(x_t) - \delta \text{ and } f(p) = \alpha_t \\ &\Leftrightarrow g(p) + \alpha_t^N = \alpha + \alpha_t^N - \delta \text{ and } f(p) = \alpha_t \\ &\Leftrightarrow g(p) + f^N(p) = g(x_t) + f^N(x_t) - \delta \text{ and } f(p) = \alpha_t \\ &\Leftrightarrow \tilde{g}(p) = \tilde{g}(x_t) - \delta \text{ and } f(p) = \alpha_t \\ &\Leftrightarrow \tilde{g}(p) = \alpha' - \delta \text{ and } f(p) = \alpha_t. \end{aligned}$$

Therefore, denoting $\tilde{\alpha} = \alpha' - \delta$,

$$\begin{aligned} B_{g,X\cap f^{-1}(\alpha_t)}(x_t) &= \sum \chi(W_i \cap f^{-1}(\alpha_t) \cap g^{-1}(\alpha - \delta) \cap D_{x_t}) Eu_X(W_i) \\ &= \sum \chi(W_i \cap f^{-1}(\alpha_t) \cap \tilde{g}^{-1}(\tilde{\alpha}) \cap D_{x_t}) Eu_{X\cap \tilde{g}^{-1}(\tilde{\alpha})}(W_i \cap \tilde{g}^{-1}(\tilde{\alpha})) \\ &= B_{f,X\cap \tilde{g}^{-1}(\tilde{\alpha})}(x_t). \end{aligned}$$

■

An immediate consequence of the last proposition is the following.

Corollary 3.2.15. Let $\tilde{\alpha}$ be a regular value of $\tilde{g}$ and α_t a regular value of f, $0 \ll |\tilde{\alpha}| \ll |\alpha_t| \ll 1$. If g is tractable at the origin with respect to $\mathscr{V}$, then $Eu_{g,X\cap f^{-1}(\alpha_t)}(b_j) = Eu_{f,X\cap \tilde{g}^{-1}(\tilde{\alpha})}(b_j)$.

Proof. Let $x_t \in \{f = \alpha_t\} \cap b_j$, D_{x_t} the closed ball with center at x_t and radius r_t, $0 < |\alpha'| \ll |\alpha_t| \ll r_t \ll 1$. We have, by Proposition 3.2.14,

$$\begin{aligned}
Eu_{g,X\cap f^{-1}(\alpha_t)}(x_t) &= Eu_{X\cap f^{-1}(\alpha_t)}(x_t) - B_{g,X\cap f^{-1}(\alpha_t)}(x_t) \\
&= Eu_X(x_t) - B_{f,X\cap \tilde{g}^{-1}(\tilde{\alpha})}(x_t) \\
&= Eu_{X\cap \tilde{g}^{-1}(\tilde{\alpha})}(x_t) - B_{f,X\cap \tilde{g}^{-1}(\tilde{\alpha})}(x_t) \\
&= Eu_{f,X\cap \tilde{g}^{-1}(\tilde{\alpha})}(x_t).
\end{aligned}$$

■

3.3 Lê-Iomdin formula for the Brasselet number

Let $f, g : (X,0) \to (\mathbb{C},0)$ be complex analytic function-germs such that f has an isolated singularity at the origin. Let $\mathscr{W}$ be the Whitney stratification of X and $\mathscr{V}$ the good stratification of X induced by f. Suppose that $\Sigma_{\mathscr{W}} g$ is one-dimensional and that $\Sigma_{\mathscr{W}} g \cap \{f = 0\} = \{0\}$.

Let $\mathscr{V}'$ be the good stratification of X relative to f constructed in Lemma 2.1.1, $\mathscr{V}''$ the good stratification of X relative to g, constructed in Lemma 2.2.2 as a refinement of $\mathscr{V}$ and $\tilde{\mathscr{V}}$ the good stratification of X induced by $\tilde{g} = g + f^N, N \gg 1$.

Let α be a regular value of g, α' a regular value of $\tilde{g}$, $0 < |\alpha|, |\alpha'| \ll \varepsilon \ll 1$, n be the number of stratified Morse points of a Morsification of $f|_{X\cap g^{-1}(\alpha)\cap B_\varepsilon}$ appearing on $X_{reg} \cap g^{-1}(\alpha) \cap \{f \neq 0\} \cap B_\varepsilon$ and $\tilde{n}$ the number of stratified Morse points of a Morsification of $f|_{X\cap \tilde{g}^{-1}(\alpha')\cap B_\varepsilon}$ appearing on $X_{reg} \cap \tilde{g}^{-1}(\alpha') \cap \{f \neq 0\} \cap B_\varepsilon$.

Proposition 3.3.1. Suppose that g is tractable at the origin with respect to $\mathscr{V}$. Then, for $0 < |\delta| \ll \varepsilon \ll 1$,

$$B_{g,X}(0) - B_{\tilde{g},X}(0) = (-1)^{d-1}(n - \tilde{n}).$$

Proof. By Corollary 2.2.22,

$$B_{g,X}(0) - B_{f,X}(0) = (-1)^{d-1}(n-m) - \textstyle\sum_{j=1}^{r} m_{f,b_j}(Eu_X(b_j) - B_{g,X\cap\{f=\delta\}}(b_j)),$$

where m is the number of stratified Morse points of a Morsification of $g|_{X\cap f^{-1}(\delta)\cap B_\varepsilon}$ appearing on $X_{reg} \cap f^{-1}(\delta) \cap \{g \neq 0\} \cap B_\varepsilon$.

By Lemma 3.2.2, $\tilde{g}$ is prepolar at the origin with respect to $\mathscr{V}$. So, by Corollary 1.8.29,

$$B_{\tilde{g},X}(0) - B_{f,X}(0) = (-1)^{d-1}(\tilde{n} - \tilde{m}),$$

where $\tilde{m}$ is the number of stratified Morse points of a Morsification of $\tilde{g}|_{X\cap f^{-1}(\delta)\cap B_\varepsilon}$ appearing on $X_{reg} \cap f^{-1}(\delta) \cap \{\tilde{g} \neq 0\} \cap B_\varepsilon$. By Corollary 3.2.13,

$$\tilde{m} = (-1)^{d-1} \sum_{j=1}^{r} m_{f,b_j} Eu_{g,X\cap f^{-1}(\delta)}(b_j) + m.$$

So,

$$\begin{aligned} B_{g,X}(0) - B_{\tilde{g},X}(0) &= (-1)^{d-1}(n-\tilde{n}) - (-1)^{d-1}(m-\tilde{m}) \\ &- \sum_{j=1}^{r} m_{f,b_j}(Eu_X(b_j) - B_{g,X\cap\{f=\delta\}}(b_j)) \\ &= (-1)^{d-1}(n-\tilde{n}) + \sum_{j=1}^{r} m_{f,b_j} Eu_{g,X\cap f^{-1}(\delta)}(b_j) \\ &- \sum_{j=1}^{r} m_{f,b_j}(Eu_X(b_j) - B_{g,X\cap\{f=\delta\}}(b_j)) \\ &= (-1)^{d-1}(n-\tilde{n}), \end{aligned}$$

since, by Theorem 1.7.5,

$$Eu_X(b_j) - B_{g,X\cap f^{-1}(\delta)}(b_j) = Eu_{g,X\cap f^{-1}(\delta)}(b_j).$$

■

Our next goal is give another proof for the Lê-Iomdin formula for the Brasselet number. For that we need to compare n and $\tilde{n}$. We keep the same description of $\Sigma_{\mathscr{W}} g$ as before: we write $\Sigma_{\mathscr{W}} g$ as a union of branches $b_1 \cup \ldots \cup b_r$, where $b_j \subseteq W_{i(j)} \in \mathscr{W}$, where $\mathscr{W}$ is a Whitney stratification of X. Let δ be a regular value of f, α be a regular value of g, $0 < |\delta|, |\alpha| \ll 1$, and let us write, for each $j \in \{1, \ldots, r\}, f^{-1}(\delta) \cap b_j = \{x_{i_1}, \ldots, x_{i_{k(j)}}\}$ and we denote by m_{f,b_j} the local degree of $f|_{b_j}$. Let ε be sufficiently small such that the local Euler obstruction of X is constant on $b_j \cap B_\varepsilon$.

Lemma 3.3.2. Suppose that g is tractable at the origin with respect to $\mathscr{V}$ relative to f. If N is bigger than the maximum gap ratio of all components of the symmetric relative polar curve $\tilde{\Gamma}_{f,g}(\mathscr{V})$, then

$$\left([\tilde{\Gamma}_{f,g}(\mathscr{V})].[V(g)]\right)_0 = \left([\tilde{\Gamma}_{f,g}(\mathscr{V})].[V(\tilde{g})]\right)_0.$$

Proof. Since g is tractable at the origin with respect to $\mathscr{V}, \tilde{\Gamma}_{f,g}(\mathscr{V})$ is a curve. Let us write $[\tilde{\Gamma}_{f,g}(\mathscr{V})] = \sum_v m_v[v]$, where each component v of $\tilde{\Gamma}_{f,g}(\mathscr{V})$ is a reduced irreducible curve at the origin. Let $\alpha_v(t)$ be a parametrization of v such that $\alpha_v(0) = 0$. By Remark 1.8.14, each component v intersects $V(g-g(p))$ at a point $p \in v, p \neq 0$, sufficiently close to the origin and such that $g(p) \neq 0$. So,

$$codim_X\{0\} = codim_X V(g) + codim_X v.$$

Also, each component (reduced irreducible curve at the origin) $\tilde{v}$ of $\tilde{\Gamma}_{f,\tilde{g}}(\mathscr{V})$ intersects $V(\tilde{g}-\tilde{g}(p))$ at $p \in \tilde{v}, p \neq 0$ and $\tilde{g}(p) \neq 0$. Since $\tilde{\Gamma}_{f,\tilde{g}}(\mathscr{V}) = \Sigma_{\mathscr{W}} g \cup \tilde{\Gamma}_{f,g}(\mathscr{V})$, we also have that v intersects $V(\tilde{g}-\tilde{g}(p))$ at the point p, so

$$codim_X\{0\} = codim_X V(\tilde{g}) + codim_X v.$$

Therefore, by Remark 1.3.5 ,

$$\begin{aligned} ([v].[V(g)])_0 &= mult_t g(\alpha_v(t)) \\ ([v].[V(\tilde{g})])_0 &= mult_t \tilde{g}(\alpha_v(t)) = mult_t (g+f^N)(\alpha_v(t)) \\ &= min\{mult_t g(\alpha_v(t)), mult_t f^N(\alpha_v(t))\} \end{aligned}$$

Now,

$$mult_t f^N(\alpha_v(t)) = N([v].[V(f)])_0 \text{ and } mult_t g(\alpha_v(t)) = ([v].[V(g)])_0.$$

The gap ratio of v at the origin for g with respect to f is the ratio of intersection numbers $\frac{([v].[V(g)])_0}{([v].[V(f)])_0}$. So, if $N > \frac{([v].[V(g)])_0}{([v].[V(f)])_0}$, then $mult_t f^N(\alpha_v(t)) > mult_t g(\alpha_v(t))$.

Making the same procedure over each component v of $\tilde{\Gamma}_{f,g}(\mathscr{V})$ and using that N is bigger then the maximum gap ratio of all components v of $\tilde{\Gamma}_{f,g}(\mathscr{V})$, we conclude that

$$\left([\tilde{\Gamma}_{f,g}(\mathscr{V})].[V(g)]\right)_0 = \left([\tilde{\Gamma}_{f,g}(\mathscr{V})].[V(\tilde{g})]\right)_0.$$

■

Lemma 3.3.3. Let α and α' be regular values of g and $\tilde{g}$, respectively, with $0 < |\alpha|, |\alpha'| \ll \varepsilon \ll 1$. If $N \gg 1$ is bigger than the maximum gap ratio of all components of the symmetric relative polar curve $\tilde{\Gamma}_{f,g}(\mathscr{V})$ and large enough such that Proposition 3.2.1 is satisfied, then

$$\tilde{n} = n + (-1)^{d-1} N \sum_{j=1}^{r} m_{f,b_j} Eu_{f,X\cap \tilde{g}^{-1}(\alpha')}(b_j).$$

Proof. We start describing the critical points of $f|_{g^{-1}(\alpha)\cap B_\varepsilon}$. We have

$$\begin{aligned} x \in \Sigma f|_{g^{-1}(\alpha)\cap B_\varepsilon} &\Leftrightarrow x \in g^{-1}(\alpha) \cap B_\varepsilon \text{ and } rk(d_x g, d_x f) \leq 1 \\ &\Leftrightarrow x \in g^{-1}(\alpha) \cap B_\varepsilon \text{ and } \big(d_x g = 0\big) \text{ or } \big(d_x f = 0\big) \text{ or } \big(d_x g = \lambda d_x f, \lambda \neq 0\big). \end{aligned}$$

Since f has an isolated singularity at the origin and, by Proposition 1.8.4, $\Sigma_{\mathscr{W}} g \subset \{g = 0\}$, we have that $\Sigma f|_{g^{-1}(\alpha)\cap B_\varepsilon} = g^{-1}(\alpha) \cap B_\varepsilon \cap \tilde{\Gamma}_{f,g}(\mathscr{V})$. Therefore, n counts the number of Morse points of a Morsification of $f|_{g^{-1}(\alpha)\cap B_\varepsilon}$ coming from $g^{-1}(\alpha) \cap B_\varepsilon \cap \tilde{\Gamma}_{f,g}(\mathscr{V})$.

Now, let us describe $\Sigma f|_{\tilde{g}^{-1}(\alpha')\cap B_\varepsilon}$.

$$x \in \Sigma f|_{\tilde{g}^{-1}(\alpha')\cap B_\varepsilon} \Leftrightarrow x \in \tilde{g}^{-1}(\alpha')\cap B_\varepsilon \text{ and } rk(d_x\tilde{g}, d_x f) \leq 1$$
$$\Leftrightarrow x \in \tilde{g}^{-1}(\alpha')\cap B_\varepsilon \text{ and } \big(d_x\tilde{g} = 0\big) \text{ or } \big(d_x f = 0\big) \text{ or } \big(d_x\tilde{g} = \lambda' d_x f, \lambda' \neq 0\big).$$

Since f and $\tilde{g}$ have an isolated singularity at the origin, we have that

$$\Sigma f|_{\tilde{g}^{-1}(\alpha')\cap B_\varepsilon} = \tilde{g}^{-1}(\alpha')\cap B_\varepsilon \cap \tilde{\Gamma}_{f,\tilde{g}}(\mathscr{V}).$$

Since $\tilde{\Gamma}_{f,\tilde{g}}(\mathscr{V}) = \tilde{\Gamma}_{f,g}(\mathscr{V}) \cup \Sigma_{\mathscr{W}} g$,

$$\Sigma f|_{\tilde{g}^{-1}(\alpha')\cap B_\varepsilon} = \big(\Sigma_{\mathscr{W}} g \cap \tilde{g}^{-1}(\alpha')\cap B_\varepsilon\big) \cup \big(\tilde{\Gamma}_{f,g}(\mathscr{V}) \cap \tilde{g}^{-1}(\alpha')\cap B_\varepsilon\big).$$

Notice that, since $\Sigma_{\mathscr{W}} g \cap \{f = 0\} = \{0\}, \Sigma_{\mathscr{W}} g \cap \tilde{g}^{-1}(\alpha')\cap B_\varepsilon \subset \{f \neq 0\}$. Also, by definition, $\tilde{\Gamma}_{f,g}(\mathscr{V}) \setminus \{0\} \subset \{f \neq 0\}$ Therefore, $\tilde{n}$ counts the number of Morse points of a Morsification of $f|_{\tilde{g}^{-1}(\alpha')\cap B_\varepsilon}$ coming from $\tilde{g}^{-1}(\alpha')\cap B_\varepsilon \cap \Sigma_{\mathscr{W}} g \cap \{f \neq 0\} \cap \{g = 0\}$ and from $\tilde{g}^{-1}(\alpha')\cap B_\varepsilon \cap \tilde{\Gamma}_{f,g}(\mathscr{V}) \cap \{f \neq 0\} \cap \{g \neq 0\}$.

By Lemma 3.3.2, the number of Morse points of a Morsification of $f|_{\tilde{g}^{-1}(\alpha')\cap B_\varepsilon}$ appearing on $\tilde{g}^{-1}(\alpha')\cap B_\varepsilon \cap \tilde{\Gamma}_{f,g}(\mathscr{V}) \cap \{f \neq 0\} \cap \{g \neq 0\}$ is precisely n. Let us describe the number of Morse points of a Morsification of $f|_{\tilde{g}^{-1}(\alpha')\cap B_\varepsilon}$ appearing on $\tilde{g}^{-1}(\alpha')\cap B_\varepsilon \cap \Sigma_{\mathscr{W}} g \cap \{f \neq 0\} \cap \{g = 0\}$. Using that $\Sigma_{\mathscr{W}} g \subset \{g = 0\}$,

$$x \in \tilde{g}^{-1}(\alpha')\cap B_\varepsilon \cap \Sigma_{\mathscr{W}} g \Leftrightarrow \tilde{g}(x) = \alpha' \text{ and } d_x g = 0$$
$$\Leftrightarrow g(x) + f(x)^N = \alpha' \text{ and } d_x g = 0$$
$$\Leftrightarrow f(x)^N = \alpha' \text{ and } d_x g = 0$$
$$\Leftrightarrow f(x) \in \{\alpha_0, \dots, \alpha_{N-1}\} \text{ and } d_x g = 0,$$

where $\{\alpha_0, \dots, \alpha_{N-1}\}$ are the N-th roots of α'. Therefore,

$$\tilde{g}^{-1}(\alpha')\cap B_\varepsilon \cap \Sigma_{\mathscr{W}} g = \bigcup_{i=0}^{N-1} f^{-1}(\alpha_i) \cap B_\varepsilon \cap \Sigma_{\mathscr{W}} g.$$

Since $\Sigma_{\mathscr{W}} g$ is one-dimensional, $f^{-1}(\alpha_i) \cap \Sigma_{\mathscr{W}} g$ is a finite set of critical points of $f|_{\tilde{g}^{-1}(\alpha')\cap B_\varepsilon}$. Since $\tilde{\Gamma}_{f,\tilde{g}}(\mathscr{V}) = \Sigma_{\mathscr{W}} g \cup \tilde{\Gamma}_{f,g}(\mathscr{V})$, each branch b_j of $\Sigma_{\mathscr{W}} g$ is a component of $\tilde{\Gamma}_{f,\tilde{g}}(\mathscr{V})$. If $V_{i(j)}$ is the stratum of $\mathscr{V}''$ containing b_j, then $f|_{V_{i(j)}\cap \tilde{g}^{-1}(\alpha')}$ has an isolated singularity at each point $x_\theta \in b_j \cap f^{-1}(\alpha_i) \cap \tilde{g}^{-1}(\alpha'), j \in \{1, \dots, r\}$ and $\theta \in \{i_1, \dots, i_{k(j)}\}$ (page 974, (MASSEY, 1996)). Using Proposition 1.7.10, we can count the number n_l of Morse points of a Morsification of $f|_{\tilde{g}^{-1}(\alpha')\cap B_\varepsilon}$ in a neighborhood of each x_θ,

$$Eu_{f,X\cap\tilde{g}^{-1}(\alpha')}(x_\theta) = (-1)^{d-1} n_l.$$

Since the Euler obstruction of a function is constant on each branch b_j, by Remark 3.2.12, we can denote $Eu_{f,X\cap\tilde{g}^{-1}(\alpha')}(x_\theta)$ by $Eu_{f,X\cap\tilde{g}^{-1}(\alpha')}(b_j)$, for all $x_\theta \in b_j \cap f^{-1}(\alpha_i) \cap \tilde{g}^{-1}(\alpha')$.

Therefore, if $b_j \cap f^{-1}(\alpha_i) \cap \tilde{g}^{-1}(\alpha') = \{x_{j_1}, \dots, x_{j_{m_{f,b_j}}}\}$, the number of Morse points of a Morsification of $f|_{\tilde{g}^{-1}(\alpha')\cap B_\varepsilon}$ appearing on $\big(X_{reg} \setminus \{\tilde{g}=0\}\big) \cap b_j \cap \{\tilde{g}=\alpha'\} \cap B_\varepsilon \cap \{f=\alpha_i\}$ is

$$n_{j_1} + \cdots + n_{j_{m_{f,b_j}}} \;=\; (-1)^{d-1} m_{f,b_j} Eu_{f,X\cap \tilde{g}^{-1}(\alpha')}(x_\theta).$$

Making the same analysis over each $\alpha_i \in \sqrt[N]{\alpha'}$, the number of Morse points of a Morsification of $f|_{\tilde{g}^{-1}(\alpha')\cap B_\varepsilon}$ appearing in $X_{reg} \setminus \{\tilde{g}=0\} \cap \{g=0\} \cap \{\tilde{g}=\alpha'\} \cap B_\varepsilon$ is

$$(-1)^{d-1} N \sum_{j=1}^{r} m_{f,b_j} Eu_{f,X\cap \tilde{g}^{-1}(\alpha')}(b_j).$$

Therefore,

$$\tilde{n} = n + (-1)^{d-1} N \sum_{j=1}^{r} m_{f,b_j} Eu_{f,X\cap \tilde{g}^{-1}(\alpha')}(b_j).$$

■

Theorem 3.3.4. Suppose that g is tractable at the origin with respect to the good stratification $\mathscr{V}$ of X induced by f. If α and α' are regular values of g and $\tilde{g}$, respectively, with $0 < |\alpha|, |\alpha'| \ll \varepsilon$, and $N \gg 1$ is bigger than the maximum gap ratio of all components of the symmetric relative polar curve $\tilde{\Gamma}_{f,g}(\mathscr{V})$ and large enough such that Proposition 3.2.1 is satisfied, then

$$B_{\tilde{g},X}(0) = B_{g,X}(0) + N \sum_{j=1}^{r} m_{f,b_j} Eu_{f,X\cap \tilde{g}^{-1}(\alpha')}(b_j).$$

Proof. It follows by Proposition 3.3.1 and Lemma 3.3.3. ■

This formula gives a way to compare the numerical data associated to the generalized Milnor fibre of a function g with a one-dimensional singular locus and to the generalized Milnor fibre of the deformation $\tilde{g} = g + f^N$, for $N \gg 1$ sufficiently large. This is what Lê (LÊ, 1980) and Iomdin (IOMDIN, 1974a) have done in the case where g is defined over a complete intersection in $\mathbb{C}^n$, g has a one-dimensional critical locus and f is a generic linear form over $\mathbb{C}^n$. Therefore, Theorem 3.3.4 generalizes this Lê-Iomdin formula.

For $X = \mathbb{C}^n$, let us consider $\mathscr{W} = \{\mathbb{C}^n \setminus \{0\}, \{0\}\}$ the Whitney stratification of $\mathbb{C}^n$. If f has an isolated singularity at the origin, the good stratification $\mathscr{V}$ of $\mathbb{C}^n$ induced by f is given by $\mathscr{V} = \{\mathbb{C}^n \setminus \{f=0\}, \{f=0\} \setminus \{0\}, \{0\}\}$.

Corollary 3.3.5. Suppose that g is tractable at the origin with respect to $\mathscr{V}$ relative to f. If α and α' are regular values of g and $\tilde{g}$, respectively, with $0 < |\alpha|, |\alpha'| \ll \varepsilon$, then

$$\chi(\tilde{g}^{-1}(\alpha') \cap B_\varepsilon) = \chi(g^{-1}(\alpha) \cap B_\varepsilon) + (-1)^{n-1} N \sum_{j=1}^{r} m_{f,b_j} \mu(g|_{f^{-1}(\delta_{j_i})}, b_j),$$

where $\mu(g|_{f^{-1}(\delta_{j_i})}, b_j)$ denotes the Milnor number of $g|_{X\cap f^{-1}(\delta_{j_i})\cap B_\varepsilon}$ at a point x_{j_i} of the branch b_j, with $f(x_{j_i}) = \delta_{j_i}$.

Proof. Keeping the same description of $\Sigma_{\mathscr{W}} g$, the good stratification of $\mathbb{C}^n$ relative to f constructed in Lemma 2.1.1 is $\mathscr{V}' = \{\mathbb{C}^n \setminus \{f=0\} \cup \Sigma_{\mathscr{W}} g, \{f=0\} \setminus \{0\}, \Sigma_{\mathscr{W}} g, \{0\}\}$.

Applying Lemma 2.2.2, we obtain that $\mathscr{V}''$, given by

$$\{\mathbb{C}^n \setminus \{f=0\} \cup \{g=0\}, \{f=0\} \setminus \{g=0\}, \{g=0\} \setminus \{f=0\} \cup \Sigma_{\mathscr{W}} g, \{f=0\} \cap \{g=0\} \setminus \Sigma_{\mathscr{W}} g, \Sigma_{\mathscr{W}} g, \{0\}\},$$

is a good stratification of $\mathbb{C}^n$ relative to g.

By definition of the Brasselet number, if $0 < |\alpha| \ll \varepsilon \ll 1$,

$$\begin{aligned} B_{g,X}(0) &= \sum_{V_i \in \mathscr{V}''} \chi(V_i \cap g^{-1}(\alpha) \cap B_\varepsilon) Eu_{\mathbb{C}^n}(V_i) \\ &= \chi\big((\mathbb{C}^n \setminus \{f=0\} \cup \{g=0\}) \cap g^{-1}(\alpha) \cap B_\varepsilon\big) Eu_{\mathbb{C}^n}(\mathbb{C}^n \setminus \{f=0\} \cup \{g=0\}) \\ &+ \chi\big((\{f=0\} \setminus \{g=0\}) \cap g^{-1}(\alpha) \cap B_\varepsilon\big) Eu_{\mathbb{C}^n}(\{f=0\} \setminus \{g=0\}) \\ &= \chi\big((\mathbb{C}^n \setminus \{g=0\}) \cap g^{-1}(\alpha) \cap B_\varepsilon\big) \\ &= \chi(g^{-1}(\alpha) \cap B_\varepsilon). \end{aligned}$$

The good stratification of $\mathbb{C}^n$ induced by $\tilde{g}$ is $\tilde{\mathscr{V}} = \{\{\tilde{g}=0\}, \mathbb{C}^n \setminus \{\tilde{g}=0\}, \{0\}\}$ and then, if $0 < |\alpha'| \ll \varepsilon \ll 1$,

$$B_{\tilde{g},X}(0) = \chi(\mathbb{C}^n \setminus \{\tilde{g}=0\} \cap g^{-1}(\alpha) \cap B_\varepsilon) Eu_{\mathbb{C}^n}(\mathbb{C}^n \setminus \{0\}) = \chi(\tilde{g}^{-1}(\alpha') \cap B_\varepsilon).$$

Since $f|_{\tilde{g}^{-1}(\alpha') \cap B_\varepsilon}$ is defined over $\mathbb{C}^n$ and has an isolated singularity at each $x_{j_i} \in b_j$, considering a small ball $B_\varepsilon(x_{j_i})$ with radius ε and center at x_{j_i}, by Example 1.7.6, for $0 < |\delta| \ll \varepsilon \ll 1$,

$$\begin{aligned} Eu_{f,\tilde{g}^{-1}(\alpha')}(x_{j_i}) &= (-1)^{n-1} \mu\big(f|_{\tilde{g}^{-1}(\alpha')}, x_{j_i}\big) \\ &= (-1)^{n-1}(-1)^{n-1}\big[\chi\big((f|_{\tilde{g}^{-1}(\alpha')})^{-1}(\delta) \cap B_\varepsilon(x_{j_i})\big) - 1\big] \\ &= \chi\big(f^{-1}(\delta_{j_i} - \delta) \cap \tilde{g}^{-1}(\alpha') \cap B_\varepsilon(x_{j_i})\big) - 1,\ f(x_{j_i}) = \delta_{j_i} \\ &\overset{*}{=} \chi\big(f^{-1}(\delta_{j_i}) \cap \tilde{g}^{-1}(\alpha' - \delta) \cap B_\varepsilon(x_{j_i})\big) - 1 \\ &= \chi\big(f^{-1}(\delta_{j_i}) \cap g^{-1}(\alpha' - \delta_{j_i}^N - \delta) \cap B_\varepsilon(x_{j_i})\big) - 1, g(x_{j_i}) = \alpha' - \delta_{j_i}^N \\ &= \chi\big((g|_{f^{-1}(\delta_{j_i})})^{-1}(\delta) \cap B_\varepsilon(x_{j_i})\big) - 1 \\ &= (-1)^{n-1} \mu\big(g|_{f^{-1}(\delta_{j_i})}, x_{j_i}\big), \end{aligned}$$

where the equality $(*)$ is justified by Proposition 1.8.26. Therefore, applying Theorem 3.3.4, we obtain

$$\chi(\tilde{g}^{-1}(\alpha') \cap B_\varepsilon) = \chi(g^{-1}(\alpha) \cap B_\varepsilon) + (-1)^{n-1} N \sum_{j=1}^{r} m_{f,b_j} \mu(g|_{f^{-1}(\delta_{j_i})}, b_j).$$

■

Another consequence of Theorem 3.3.4 is a different proof for the Lê-Iomdin formula proved by Massey in (MASSEY, 2003) in the case of a function with a one-dimensional singular locus.

Corollary 3.3.6. Let $\mathscr{V}$ be the good stratification of an open set $(U,0) \subseteq (\mathbb{C}^{n+1},0)$ induced by a generic linear form l defined over $\mathbb{C}^{n+1}$ and suppose that $g : (U,0) \subseteq (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ is tractable at the origin with respect to $\mathscr{V}$. Let $N \geq 2, \mathbf{z} = (z_0 \ldots, z_n)$ be a linear choice of coordinates such that $\lambda^i_{g,\mathbf{z}}(0)$ is defined for $i = 0,1$, and $\tilde{\mathbf{z}} = (z_1 \ldots, z_n, z_0)$ the coordinates for $\tilde{g} = g + l^N$ such that $\lambda^0_{\tilde{g},\tilde{\mathbf{z}}}$ is defined. If N is greater then the maximum gap ratio of each component of the symmetric relative polar curve $\tilde{\Gamma}_{f,g}$, then

$$\lambda^0_{\tilde{g},\tilde{\mathbf{z}}}(0) = \lambda^0_{g,\mathbf{z}}(0) + (N-1)\lambda^1_{g,\mathbf{z}}(0).$$

Proof. Without loss of generality, we can suppose that $l = z_0$. Let $F_{g,0}$ be the Milnor fibre of g at the origin and $F_{\tilde{g},0}$ the Milnor fibre of $\tilde{g}$ at the origin. Since g has a one-dimensional critical set, the possibly nonzero Lê numbers are $\lambda^0_{g,\mathbf{z}}(0)$ and $\lambda^1_{g,\mathbf{z}}(0)$ and, since $\tilde{g}$ has an isolated singularity at the origin, the only possibly nonzero Lê number is $\lambda^0_{\tilde{g},\tilde{\mathbf{z}}}(0)$. By Theorem 4.3 of (MASSEY, 1988),

$$\chi(F_{g,0}) = 1 + (-1)^n \lambda^0_{g,\mathbf{z}}(0) + (-1)^{n-1}\lambda^1_{g,\mathbf{z}}(0)$$

and

$$\chi(F_{\tilde{g},0}) = 1 + (-1)^n \lambda^0_{\tilde{g},\tilde{\mathbf{z}}}(0).$$

In (MASSEY, 2003), on page 49, Massey remarked that for $0 < |\delta| \ll \varepsilon \ll 1$,

$$\lambda^1_{g,\mathbf{z}}(0) = \sum_{j=1}^{r} m_{b_j} \mu(g|_{l^{-1}(\delta)}, b_j).$$

Therefore, by Corollary 3.3.5, we obtain that

$$1 + (-1)^n \lambda^0_{\tilde{g},\tilde{\mathbf{z}}}(0) \quad = \quad 1 + (-1)^n \lambda^0_{g,\mathbf{z}}(0) + (-1)^{n-1}\lambda^1_{g,\mathbf{z}}(0) + (-1)^n N \lambda^1_{g,\mathbf{z}}(0),$$

that is,

$$\lambda^0_{\tilde{g},\tilde{\mathbf{z}}}(0) = \lambda^0_{g,\mathbf{z}}(0) + (N-1)\lambda^1_{g,\mathbf{z}}(0).$$

■

3.4 Applications for generic linear forms

Let $g : (X,0) \to (\mathbb{C},0)$ be a complex analytic function-germ and l be a generic linear form in $\mathbb{C}^n$. Let $\mathscr{W} = \{\{0\}, W_1, \ldots, W_q\}$ be a Whitney stratification of X and $\mathscr{V}$ the good stratification of X induced by l. Suppose that $\Sigma_{\mathscr{W}} g$ is one-dimensional.

Let $\mathscr{V}'$ be the good stratification of X relative to l constructed in Lemma 2.1.1, $\mathscr{V}''$ the good stratification of X relative to g, constructed in Lemma 2.2.2 as a refinement of $\mathscr{V}$ and $\tilde{\mathscr{V}}$ the good stratification of X induced by $\tilde{g} = g + l^N, N \gg 1$.

Let α be a regular value of g, α' a regular value of $\tilde{g}$, $0 < |\alpha|, |\alpha'| \ll \varepsilon \ll 1$, n the number of stratified Morse points of a Morsification of $l|_{X \cap g^{-1}(\alpha) \cap B_\varepsilon}$ appearing on $X_{reg} \cap g^{-1}(\alpha) \cap \{l \neq 0\} \cap B_\varepsilon$, n_i the number of stratified Morse points of a Morsification of $l|_{W_i \setminus (\{g=0\} \cup \{l=0\}) \cap g^{-1}(\alpha) \cap B_\varepsilon}$ appearing on $W_i \cap g^{-1}(\alpha) \cap \{l \neq 0\} \cap B_\varepsilon$, $\tilde{n}$ the number of stratified Morse points of a Morsification of $l|_{X \cap \tilde{g}^{-1}(\alpha') \cap B_\varepsilon}$ appearing on $X_{reg} \cap \tilde{g}^{-1}(\alpha') \cap \{l \neq 0\} \cap B_\varepsilon$ and $\tilde{n}_i$ the number of stratified Morse points of a Morsification of $l|_{W_i \setminus \{\tilde{g}=0\} \cap \tilde{g}^{-1}(\alpha') \cap B_\varepsilon}$ appearing on $W_i \cap \tilde{g}^{-1}(\alpha') \cap \{l \neq 0\} \cap B_\varepsilon$, for each $W_i \in \mathscr{W}$.

As before, we write $\Sigma_{\mathscr{W}} g$ as a union of branches $b_1 \cup \ldots \cup b_r$ and we suppose that $\{l = \delta\} \cap b_j = \{x_{i_1}, \ldots, x_{i_{k(j)}}\}$. For each $t \in \{i_1, \ldots, i_{k(j)}\}$, let D_{x_t} be the closed ball with center at x_t and radius $r_t, 0 < |\alpha|, |\alpha'| \ll |\delta| \ll r_t \ll \varepsilon \ll 1$, sufficiently small for the balls D_{x_t} be pairwise disjoint and the union of balls $D_j = D_{x_{i_1}} \cup \ldots \cup D_{x_{i_{k(j)}}}$ be contained in B_ε and ε is sufficiently small such that the local Euler obstruction of X at a point of $b_j \cap B_\varepsilon$ is constant.

In (TIBĂR, 1998), Tibăr gave a bouquet decomposition for the Milnor fibre of $\tilde{g}$ in terms of the Milnor fibre of g. Let us denote by F_g the local Milnor fibre of g at the origin, $F_{\tilde{g}}$ the local Milnor fibre of $\tilde{g}$ at the origin and F_j the local Milnor fibre of $g|_{\{l=\delta\}}$ at a point of the branch b_j. Then there is a homotopy equivalence

$$F_{\tilde{g}} \overset{ht}{\simeq} (F_g \cup E) \bigvee_{j=1}^{r} \bigvee_{M_j} S(F_j),$$

where $\bigvee$ denotes the wedge sum of topological spaces (see (HATCHER, 2001), page 10) $M_j = Nm_{b_j} - 1, S(F_j)$ denotes the topological suspension (see (HATCHER, 2001), page 8) over $F_j, E := \cup_{j=1}^{r} Cone(F_j)$ and $F_g \cup E$ is the attaching to F_g of one cone over $F_j \subset F_g$ for each $j \in \{1, \ldots, r\}$. As a consequence of this theorem, Tibăr proved a Lê-Iomdin formula for the Euler characteristic of these Milnor fibres.

In the following, we present a new proof for this formula using our previous results.

Proposition 3.4.1. Suppose that g is tractable at the origin with respect to $\mathscr{V}$. If $0 < |\alpha|, |\alpha'| \ll |\delta| \ll \varepsilon \ll 1$, then

$$\chi(X \cap \tilde{g}^{-1}(\alpha') \cap B_\varepsilon) - \chi(X \cap g^{-1}(\alpha) \cap B_\varepsilon) \quad = \quad N \sum_{j=1}^{r} m_{b_j} \big(1 - \chi(F_j)\big),$$

where $F_j = X \cap g^{-1}(\alpha) \cap H_j \cap D_{x_t}$ is the local Milnor fibre of $g|_{\{l=\delta\}}$ at a point of the branch b_j and H_j denotes the generic hyperplane $l^{-1}(\delta)$ passing through $x_t \in b_j$, for $t \in \{i_1, \ldots, i_{k(j)}\}$.

Proof. For a stratum $V_i = W_i \setminus (\{g = 0\} \cup \{l = 0\})$ in $\mathscr{V}''$, $W_i \in \mathscr{W}$, let N_i be a normal slice to V_i at $x_t \in b_j$, for $t \in \{i_1, \ldots, i_{k(j)}\}$ and D_{x_t} a closed ball of radius r_t centered at x_t. Considering the constructible function $\mathbf{1}_X$, the normal Morse index along V_i is given by

$$\begin{aligned}\eta(V_i,\mathbf{1}_X) &= \chi(W_i\setminus(\{g=0\}\cup\{l=0\})\cap N_i\cap D_{x_t}) \\ &- \chi(W_i\setminus(\{g=0\}\cup\{l=0\})\cap N_i\cap\{g=\alpha\}\cap D_{x_t}) \\ &= \chi(W_i\cap N_i\cap D_{x_t})-\chi(W_i\cap N_i\cap\{g=\alpha\}\cap D_{x_t}) \\ &= 1-\chi(l_{W_i}).\end{aligned}$$

For a stratum $\tilde{V}_i = W_i\setminus(\{\tilde{g}=0\}\in\tilde{\mathscr{V}}$, $W_i\in\mathscr{W}$, let $\tilde{N}_i$ be a normal slice to $\tilde{V}_i$ at $x_t\in b_j$, for $t\in\{i_1,\dots,i_{k(j)}\}$. Considering the constructible function $\mathbf{1}_X$, the normal Morse index along $\tilde{V}_i$ is given by

$$\begin{aligned}\eta(\tilde{V}_i,\mathbf{1}_X) &= \chi((W_i\setminus\{\tilde{g}=0\})\cap\tilde{N}_i\cap D_{x_t})-\chi((W_i\setminus\{\tilde{g}=0\})\cap\tilde{N}_i\cap\{\tilde{g}=\alpha'\}\cap D_{x_t}) \\ &= \chi(W_i\cap\tilde{N}_i\cap D_{x_t})-\chi(W_i\cap\tilde{N}_i\cap\{\tilde{g}=\alpha'\}\cap D_{x_t}) \\ &= 1-\chi(l_{W_i}).\end{aligned}$$

Then applying Theorem 1.8.23 for $\mathbf{1}_X$, we obtain that

$$\chi(X\cap\tilde{g}^{-1}(\alpha')\cap B_\varepsilon)-\chi(X\cap\tilde{g}^{-1}(\alpha')\cap l^{-1}(0)\cap B_\varepsilon)=\textstyle\sum_{i=1}^{q}(-1)^{d_i-1}\tilde{n}_i(1-\chi(l_{W_i}))$$

and that

$$\chi(X\cap g^{-1}(\alpha)\cap B_\varepsilon)-\chi(X\cap g^{-1}(\alpha)\cap l^{-1}(0)\cap B_\varepsilon)=\textstyle\sum_{i=1}^{q}(-1)^{d_i-1}n_i(1-\chi(l_{W_i})),$$

where $d_i=\dim W_i$.

Therefore, since $\chi(X\cap\tilde{g}^{-1}(\alpha')\cap l^{-1}(0)\cap B_\varepsilon)=\chi(X\cap g^{-1}(\alpha)\cap l^{-1}(0)\cap B_\varepsilon)$,

$$\chi(X\cap\tilde{g}^{-1}(\alpha')\cap B_\varepsilon)-\chi(X\cap g^{-1}(\alpha)\cap B_\varepsilon)=\sum_{i=1}^{q}(-1)^{d_i-1}(\tilde{n}_i-n_i)(1-\chi(l_{W_i})).$$

Applying Lemma 3.3.3 and Corollary 3.2.15, we obtain, for each i,

$$\begin{aligned}\tilde{n}_i &= n_i+(-1)^{d_i-1}N\sum_{j=1}^{r}m_{b_j}Eu_{l,\overline{W_i}\cap\tilde{g}^{-1}(\alpha')}(b_j) \\ &= n_i+(-1)^{d_i-1}N\sum_{j=1}^{r}m_{b_j}Eu_{g,\overline{W_i}\cap H_j}(b_j),\end{aligned}$$

where H_j denotes the generic hyperplane $l^{-1}(\delta)$ passing through $x_t\in b_j$, for $t\in\{i_1,\dots,i_{k(j)}\}$.

Hence

$$\begin{aligned}\chi(X\cap\tilde{g}^{-1}(\alpha')\cap B_\varepsilon)-\chi(X\cap g^{-1}(\alpha)\cap B_\varepsilon) &= N\sum_{i=1}^{q}\left(\sum_{j=1}^{r} m_{b_j} Eu_{g,\overline{W_i}\cap H_j}(b_j)\right)\big(1-\chi(l_{W_i})\big)\\ &= N\sum_{j=1}^{r} m_{b_j}\Big(1-\chi\big(X\cap g^{-1}(\alpha)\cap H_j\cap D_{x_t}\big)\Big)\\ &= N\sum_{j=1}^{r} m_{b_j}\big(1-\chi(F_j)\big),\end{aligned}$$

for $t\in\{i_1,\dots,i_{k(j)}\}$. ■

BIBLIOGRAPHY

BRASSELET, J.-P.; LÊ, D. T.; SEADE, J. Euler obstruction and indices of vector fields. **Topology**, v. 39, n. 6, p. 1193 – 1208, 2000. ISSN 0040-9383. Citation on page 36.

BRASSELET, J.-P.; MASSEY, D.; PARAMESWARAN, A.; SEADE, J. Euler obstruction and defects of functions on singular varieties. **Journal of the London Mathematical Society**, Cambridge University Press, v. 70, n. 1, p. 59–76, 2004. Citations on pages 17, 36, 37, and 76.

BRASSELET, J.-P.; SCHWARTZ, M.-H. Sur les classes de Chern d'un ensemble analytique complexe. **Astérisque**, v. 82, p. 93–147, 1981. Citation on page 36.

DUTERTRE, N.; GRULHA, N. G. Lê–Greuel type formula for the Euler obstruction and applications. **Advances in Mathematics**, Elsevier, v. 251, p. 127–146, 2014. Citations on pages 17, 40, 42, 43, 44, 53, 54, 64, 66, 67, and 80.

ENGELKING, R. **General topology**. [S.l.]: Heldermann Verlag, Berlin, 1989. Citation on page 58.

GONZÁLEZ-SPRINBERG, G. L'obstruction locale d'Euler et le théoreme de MacPherson. **Astérisque**, v. 82, n. 83, p. 7–32, 1981. Citation on page 36.

GORESKY, M.; MACPHERSON, R. Stratified Morse theory. In: **Stratified Morse Theory**. [S.l.]: Springer, 1988. p. 3–22. Citations on pages 30 and 41.

GREUEL, G.-M. Der gauß-manin-zusammenhang isolierter singularitäten von vollständigen durchschnitten. **Mathematische Annalen**, Springer, v. 214, n. 3, p. 235–266, 1975. Citation on page 17.

GREUEL, G.-M.; LOSSEN, C.; SHUSTIN, E. I. **Introduction to singularities and deformations**. [S.l.]: Springer-Verlag Berlin Heidelberg, 2007. Citations on pages 21, 25, 26, 28, 29, and 34.

HAMM, H. Lokale topologische Eigenschaften komplexer räume. **Mathematische Annalen**, Springer, v. 191, n. 3, p. 235–252, 1971. Citation on page 17.

HATCHER, A. **Algebraic topology**. [S.l.]: Cambridge University Press, 2001. Citation on page 92.

HIRONAKA, H. Stratification and flatness. In: **Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976)**. [S.l.: s.n.], 1976. p. 199–265. Citation on page 31.

IOMDIN, I. Complex surfaces with a one-dimensional set of singularities. **Siberian Mathematical Journal**, Springer, v. 15, n. 5, p. 748–762, 1974. Citations on pages 20, 27, 61, 75, and 89.

____. Local topological properties of complex algebraic sets. **Siberian Mathematical Journal**, Springer, v. 15, n. 4, p. 558–572, 1974. Citation on page 26.

LÊ, D.-T. **Calcul du nombre de Milnor d'une singularité isolée d'intersection compléte**. [S.l.]: Centre de Mathematiques de l'Ecole Polytechnique, 1973. Citation on page 17.

LÊ, D. T. Ensembles analytiques complexes avec lieu singulier de dimension un (d'après Iomdine). **Seminaire sur les Singularités, Publications Matematiqués de l'Université Parıs VII**, p. 87–95, 1980. Citations on pages 20, 75, 76, and 89.

LÊ, D. T.; TEISSIER, B. Variétés polaires locales et classes de Chern des variétés singulieres. **Annals of Mathematics**, JSTOR, v. 114, n. 3, p. 457–491, 1981. Citations on pages 31, 67, and 68.

LOESER, F. Formules intégrales pour certains invariants locaux des espaces analytiques complexes. **Commentarii Mathematici Helvetici**, Springer, v. 59, n. 1, p. 204–225, 1984. Citation on page 34.

LOI, T. Tame topology and tarski-type systems. **Vietnam J. Math**, v. 31, n. 2, p. 127–136, 2003. Citation on page 79.

LOI, T. L. *et al.* Lecture 1: O-minimal structures. In: **The Japanese-Australian Workshop on Real and Complex Singularities: JARCS III**. [S.l.: s.n.], 2010. p. 19–30. Citation on page 79.

LOJASIEWICZ, S. **Introduction to complex analytic geometry**. [S.l.]: Birkhäuser, 1991. Citations on pages 25 and 51.

LOOIJENGA, E. J. **Isolated singular points on complete intersections**. [S.l.]: Cambridge University Press, 1984. Citation on page 50.

MACPHERSON, R. D. Chern classes for singular algebraic varieties. **Annals of Mathematics**, JSTOR, p. 423–432, 1974. Citations on pages 17 and 35.

MANIN, Y. I. **Introduction to the Theory of Schemes**. [S.l.]: Springer, 2018. Citations on pages 22, 28, and 77.

MASSEY, D. The Lê varieties, i. **Inventiones mathematicae**, Springer, v. 99, n. 1, p. 357–376, 1990. Citation on page 19.

MASSEY, D. B. The Lê-Ramanujam problem for hypersurfaces with one-dimensional singular sets. **Mathematische Annalen**, Springer, v. 282, n. 1, p. 33–49, 1988. Citations on pages 19 and 91.

____. Hypercohomology of Milnor fibres. **Topology**, Elsevier, v. 35, n. 4, p. 969–1003, 1996. Citations on pages 38, 39, 40, 41, 57, 72, and 88.

____. **Numerical control over complex analytic singularities**. [S.l.]: American Mathematical Soc., 2003. Citations on pages 20, 27, 28, 75, 76, 77, 78, 90, and 91.

MATHER, J. Notes on topological stability. **Bulletin of the American Mathematical Society**, v. 49, n. 4, p. 475–506, 2012. Citation on page 30.

MATSUMURA, H. **Commutative ring theory**. [S.l.]: Cambridge university press, 1989. Citation on page 33.

MILNOR, J. **Singular points of complex hypersurfaces**. [S.l.]: Tokyo, Princeton University Press and the University of Tokyo Press, 1968. Citation on page 17.

NUÑO-BALLESTEROS, J.; TOMAZELLA, J. N. The Milnor number of a function on a space curve germ. **Bulletin of the London Mathematical Society**, Oxford University Press, v. 40, n. 1, p. 129–138, 2008. Citation on page 35.

ORRO, P.; TROTMAN, D. Regularity of the transverse intersection of two regular stratifications. **Real and complex singularities**, v. 380, p. 298–304, 2010. Citations on pages 30 and 49.

SCHÜRMANN, J.; TIBAR, M. Index formula for MacPherson cycles of affine algebraic varieties. **Tohoku Mathematical Journal, Second Series**, Mathematical Institute, Tohoku University, v. 62, n. 1, p. 29–44, 2010. Citations on pages 41 and 42.

SEADE, J.; TIBĂR, M.; VERJOVSKY, A. Milnor numbers and Euler obstruction. **Bulletin of the Brazilian Mathematical Society**, Springer, v. 36, n. 2, p. 275–283, 2005. Citations on pages 17 and 38.

TEISSIER, B. Variétés polaires (ii) multiplicités polaires, sections planes, et conditions de Whitney. In: **Algebraic geometry**. [S.l.]: Springer, 1982. p. 314–491. Citations on pages 31, 32, and 33.

TIBĂR, M. Embedding nonisolated singularities into isolated singularities. In: **Singularities**. [S.l.]: Springer, 1998. p. 103–115. Citation on page 92.

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