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On spaces of special elliptic n-gons

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Sobre espaços de n-ágonos elípticos especiais

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências – Matemática. *VERSÃO REVISADA*

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"Moon river, wider than a mile
I'm crossing you in style someday
Oh, dream maker
You heartbreaker
Wherever you're going I'm going your way
Two drifters off to see the world
There's such a lot of world to see
We're after the same rainbow's end
Waiting round the bend
My huckleberry friend
Moon river and me."
— Johnny Mercer, *Moon River*

RESUMO

FRANCO, F. A. **Sobre espaços de n-ágonos elípticos especiais**. 2018. 77 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2018.

Neste trabalho, estudamos relações entre *isometrias elípticas especiais* no plano hiperbólico complexo. Uma isometria elíptica especial pode ser vista como uma "rotação" em torno de um eixo fixo (uma geodésica complexa) e é determinada especificando-se um ponto não-isotrópico p (o ponto *polar* do eixo fixo) bem como um número complexo unitário α (o *ângulo* da isometria). Qualquer relação entre isometrias elípticas especiais com ângulos racionais dá origem a uma representação $H^{(k_1,\dots,k_n)} \to \mathrm{PU}(2,1)$, onde $H^{(k_1,\dots,k_n)} := \left\langle r_1,\dots,r_n \,|\, r_n \dots r_1 = 1,\ r_i^{k_i} = 1 \right\rangle$ e $\mathrm{PU}(2,1)$ denota o grupo de isometrias que preservam a orientação do plano hiperbólico complexo.

Denotamos por R_α^p a isometria elíptica especial determinada pelo ponto não-isotrópico p e pelo complexo unitário α. Relações da forma $R_{\alpha_n}^{p_n} \dots R_{\alpha_1}^{p_1} = 1$ em $\mathrm{PU}(2,1)$, chamadas *n-ágonos elípticos especiais*, podem ser modificadas a partir de relações "curtas" conhecidas como *bendings*: dado um produto $R_\beta^q R_\alpha^p$, existe um subgrupo uniparamétrico $B : \mathbb{R} \to \mathrm{SU}(2,1)$ tal que $B(s)$ está no centralizador de $R_\beta^q R_\alpha^p$ e $R_\beta^{B(s)q} R_\alpha^{B(s)p} = R_\beta^q R_\alpha^p$ para todo $s \in \mathbb{R}$. Assim, para cada $i = 1,\dots,n-1$, podemos mudar $R_{\alpha_{i+1}}^{p_{i+1}} R_{\alpha_i}^{p_i}$ por $R_{\alpha_{i+1}}^{B(s)p_{i+1}} R_{\alpha_i}^{B(s)p_i}$ obtendo um novo n-ágono.

Provamos que a parte genérica do espaço de pentágonos com ângulos e sinais de pontos fixados é conexa por meio de bendings. Além disso, descrevemos certas relações de comprimento 4, chamadas *f-bendings,* e provamos que o espaço de pentágonos com produto de ângulos fixado é conexo por meio de bendings e f-bendings.

Palavras-chave: Geometria hiperbólica, geometria hiperbólica complexa, espaços de representações, bendings, espaços de Teichmüller.

ABSTRACT

FRANCO, F. A. **On spaces of special elliptic n-gons**. 2018. 77 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2018.

We study relations between *special elliptic isometries* in the complex hyperbolic plane. A special elliptic isometry can be seen as a 'rotation' around a fixed axis (a complex geodesic). Such an isometry is determined by specifying a nonisotropic point p (the *polar* point to the fixed axis) and a unitary complex number α, the *angle* of the isometry. Any relation between special elliptic isometries with rational angles gives rise to a representation $H^{(k_1,\dots,k_n)} \to \mathrm{PU}(2,1)$, where $H^{(k_1,\dots,k_n)} := \left\langle r_1,\dots,r_n \,|\, r_n \dots r_1 = 1,\ r_i^{k_i} = 1 \right\rangle$ and $\mathrm{PU}(2,1)$ stands for the group of orientation-preserving isometries of the complex hyperbolic plane.

We denote by R^p_α the special elliptic isometry determined by the nonisotropic point p and by the unitary complex number α. Relations of the form $R^{p_n}_{\alpha_n} \dots R^{p_1}_{\alpha_1} = 1$ in $\mathrm{PU}(2,1)$, called *special elliptic n-gons*, can be modified by 'short' relations known as *bendings*: given a product $R^q_\beta R^p_\alpha$, there exists a one-parameter subgroup $B : \mathbb{R} \to \mathrm{SU}(2,1)$ such that $B(s)$ is in the centralizer of $R^q_\beta R^p_\alpha$ and $R^{B(s)q}_\beta R^{B(s)p}_\alpha = R^q_\beta R^p_\alpha$ for every $s \in \mathbb{R}$. Then, for each $i = 1,\dots,n-1$, we can change $R^{p_{i+1}}_{\alpha_{i+1}} R^{p_i}_{\alpha_i}$ by $R^{B(s)p_{i+1}}_{\alpha_{i+1}} R^{B(s)p_i}_{\alpha_i}$ obtaining a new n-gon.

We prove that the generic part of the space of pentagons with fixed angles and signs of points is connected by means of bendings. Furthermore, we describe certain length 4 relations, called *f-bendings,* and prove that the space of pentagons with fixed product of angles is connected by means of bendings and f-bendings.

Keywords: Hyperbolic geometry, complex hyperbolic geometry, representation spaces, bendings, Teichmüller spaces.

CONTENTS

CHAPTER

1

INTRODUCTION

The central concept of this text is that of a *bending*. In what follows, we discuss bendings as they are introduced in (ANAN'IN, 2012). In that work, the author studies representations of the hyperelliptic group $H_n := \langle r_1, \dots, r_n \mid r_n \dots r_1 = 1, r_i^2 = 1 \text{ for every } i \rangle$ in the group $\mathrm{PU}(2,1)$ of orientation-preserving isometries of the complex hyperbolic plane. Since the involutions of $\mathrm{PU}(2,1)$ are precisely the reflections in nonisotropic points of the complex hyperbolic plane, we can assume that r_i is a reflection $R(p_i)$ in a nonisotropic point p_i. So, the remaining relation can be written in $\mathrm{SU}(2,1)$ in the form $R(p_n) \dots R(p_1) = \delta$, where δ is a cubic root of the unity. In this context, if we move the points p_{i-1} and p_i along a geodesic that joins them without altering their distance, we obtain new points q_{i-1} and q_i satisfying $R(q_i)R(q_{i-1}) = R(p_i)R(p_{i-1})$. This is a *bending relation.* It changes the original relation $R(p_n) \dots R(p_1) = \delta$ into the new one $R(p_n) \dots R(q_i)R(q_{i-1}) \dots R(p_1) = \delta$. So, bendings can be seen as deformations of representations $\rho : H_n \to \mathrm{PU}(2,1)$, i.e., they act on the $\mathrm{PU}(2,1)$-representation space of H_n modulo conjugation. Since every length 4 relation between reflections is essentially a consequence of bending relations (see (ANAN'IN, 2012, Corollary 3.3)), the first truly 'rich' case is that of *pentagons,* that is, relations $R(p_5)R(p_4)R(p_3)R(p_2)R(p_1) = \delta$ of length 5 (here, we also assume that at most one of the p_i's is positive). Curiously, the space of pentagons is connected by means of bendings (ANAN'IN, 2012) and, if $\delta = 1$, then the group generated by the reflections $R(p_1), \dots, R(p_5)$ is actually a Fuchsian group (ANAN'IN; GONÇALVES, 2007, Corollary 3.16). It is an important open conjecture that every pentagon is faithful and discrete (ANAN'IN, 2012, Conjecture 1.2).

We generalize the concept of bending by considering relations between *special elliptic isometries* of the complex hyperbolic plane. A special elliptic isometry is a 'rotation' around a fixed axis (a complex geodesic) which is determined by specifying a nonisotropic point p (the *polar* point to the fixed axis) and a unitary complex number α (the *angle* of the isometry). Let R^p_α stand for the special elliptic isometry determined by the nonisotropic point p and the unitary complex number α. Given a relation $R^{p_n}_{\alpha_n} \dots R^{p_1}_{\alpha_1} = \delta$ in $\mathrm{SU}(2,1)$, bendings are one-parameter changes in a product $R^{p_i}_{\alpha_i} R^{p_{i-1}}_{\alpha_{i-1}}$ that preserve the relation. More precisely, given a product $R^q_\beta R^p_\alpha$ of (nontrivial) special elliptic isometries with p and q distinct and nonorthogonal, there exists a one-parameter subgroup $B : \mathbb{R} \to \mathrm{SU}(2,1)$ such that $B(s)$ is in the centralizer of $R^q_\beta R^p_\alpha$ and $R^{B(s)q}_\beta R^{B(s)p}_\alpha = R^q_\beta R^p_\alpha$ for every $s \in \mathbb{R}$. In Section 4.2, we give a geometric description of these bendings.

The case of special elliptic isometries is actually quite different from that of reflections. For instance, there exist relations of the form $R^{q_2}_{\beta_2} R^{q_1}_{\beta_1} = R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ with $\alpha_i \neq \beta_i$. When $\alpha_1\alpha_2 = \beta_1\beta_2$, we call such a relation an *f-bending relation* (see Definition 32). Analogously to bending relations, these can be seen as one-parameter deformations of a given relation of special elliptic isometries. During an f-bending, however, both points and angles change.

We call relations of the form $R^{p_n}_{\alpha_n} \dots R^{p_1}_{\alpha_1} = \delta$ in $\mathrm{SU}(2,1)$ *special elliptic n-gons* (see Section 5.1 for the precise definition). So, we can deform an n-gon by a composition of bending relations, or by compositions of bending and f-bending relations. Each of these types of deformations has its own invariants (Proposition 42): bendings preserve angles while bendings plus f-bendings preserve only the product of angles (both preserve signatures of points as well as δ). Hence, according to the type(s) of deformations that are allowed, one must consider the space of n-gons with the suitable set of invariants.

Aiming to show that the space of special elliptic pentagons with fixed angles and signatures of points is connected by means of bendings, we consider the decompositions $F = R^{p_3}_{\alpha_3} R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ of an isometry $F \in \mathrm{SU}(2,1)$ in terms of products of three special elliptic isometries with fixed angles $\alpha_1, \alpha_2, \alpha_3$. In these decompositions, we ask the triples p_1, p_2, p_3 to be *strongly regular* with respect to the angles $\alpha_1, \alpha_2, \alpha_3$: roughly speaking, this means that p_1, p_2, p_3 do not lie in a same complex line and that the isometry $R^{p_3}_{\alpha_3} R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ is *regular* (it is neither special elliptic nor unipotent, see Definitions 43 and 45). We prove that the space of such decompositions is an algebraic surface (Theorem 50) that can have one or two connected components, and that each connected component is bending-connected (Theorem 57). In this way, we are able to establish that two special elliptic pentagons $R^{p_5}_{\alpha_5} R^{p_4}_{\alpha_4} R^{p_3}_{\alpha_3} R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} = \delta$ with the same signatures of points (angles and δ are fixed) and such that at least one of the isometries $R^{p_{i+1}}_{\alpha_{i+1}} R^{p_i}_{\alpha_i}$ is hyperbolic, can be connected by means of finitely many bendings (Proposition 61). Finally, we show that the space of pentagons with fixed product of angles is connected by means of finitely many bendings and f-bendings (Proposition 62).

This text is organized as follows. In Chapter 2 we review some elementary complex hyperbolic geometry and consider basic aspects of special elliptic isometries and their products. In Chapter 3, we discuss relations of length 2 and 3. In Chapter 4, we introduce bendings and f-bendings which are used, in Chapter 5, to study the deformations of

n-gons by compositions of bendings and f-bendings. We also describe in Chapter 5 the space of decompositions $F = R^{p_3}_{\alpha_3} R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ of a regular isometry as the product of three special elliptic isometries (with fixed angles) and apply it to prove the bending/f-bending connectedness of spaces of pentagons.

CHAPTER

2

COMPLEX HYPERBOLIC GEOMETRY

In this chapter we will briefly discuss the geometry of the complex hyperbolic plane. Our approach is coordinate-free and follows (ANAN'IN; GROSSI, 2011), (ANAN'IN; GROSSI; GUSEVSKII, 2011), and (GOLDMAN, 1999). In Sections 2.1, 2.2, and 2.3 we outline a construction of the projective model of the complex hyperbolic plane and present some concepts that we will need in this text, like geodesics, complex geodesics, metric circles, horocycles, and hypercycles. In Section 2.4, following (PARKER, 2003), (PARKER, 2012), (GOLDMAN, 1999), and (CANO; NAVARRETE; SEADE, 2015), we discuss the isometries of the complex hyperbolic plane. Finally, in Sections 2.5 and 2.6, we study regular elliptic isometries and their product.

2.1 Metric

Let V be a 3-dimensional $\mathbb{C}$-linear space equipped with a Hermitian form $\langle\cdot,\cdot\rangle$ of signature $++-$. We will frequently use the same letter to denote both a point in $\mathbb{P}_{\mathbb{C}}V$ and a representative of it in V, but no confusion should arise.

We say that a point $p\in V$ is *negative*, *null*, or *posive* if $\langle p,p\rangle<0$, $\langle p,,p\rangle=0$, or $\langle p,p\rangle>0$, respectively. Since $\langle\lambda p,\lambda p\rangle=|\lambda|^2\langle p,p\rangle$ for any $\lambda\in\mathbb{C}^*$, it makes sense to consider this definition for points $p\in\mathbb{P}_{\mathbb{C}}V$ as well. Null points in $\mathbb{P}_{\mathbb{C}}V$ are called *isotropic points*. The *signature* of a point $p\in\mathbb{P}_{\mathbb{C}}V$ is respectively $-$, 0, or $+$ if p is negative, isotropic, or positive. In this way, the projective space $\mathbb{P}_{\mathbb{C}}V$ is divided into negative, isotropic, and positive points:

$$\mathrm{B}V:=\{p\in\mathbb{P}_{\mathbb{C}}V\mid\langle p,p\rangle<0\},\quad \mathrm{S}V:=\{p\in\mathbb{P}_{\mathbb{C}}V\mid\langle p,p\rangle=0\},$$

$$\mathrm{E}V:=\{p\in\mathbb{P}_{\mathbb{C}}V\mid\langle p,p\rangle>0\}.$$

We denote the signature of $p\in\mathbb{P}_{\mathbb{C}}V$ by $\sigma p\in\{-1,0,+1\}$.

Given $p \in \mathbb{P}_{\mathbb{C}}V$, let f stand for (the germ of) a smooth $\mathbb{C}$-valued function defined on an open neighborhood of p. Let $\varphi \in \operatorname{Lin}_{\mathbb{C}}(\mathbb{C}p, V)$ be a $\mathbb{C}$-linear map from the subspace generated by p to V and define

$$t_\varphi f := \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \widehat{f}((1+\varepsilon\varphi)p),$$

where $\varepsilon \in \mathbb{R}$ and $\widehat{f}$ is the lift of f to an open neighborhood of $\mathbb{C}p \setminus \{0\}$ in V (the lift satisfies $\widehat{f}(\lambda p) = \widehat{f}(p)$ for every $\lambda \in \mathbb{C}^*$). Note that the derivation t_φ vanishes identically if and only if $\varphi p \in \mathbb{C}p$. Therefore, the tangent space $\mathrm{T}_p\mathbb{P}_{\mathbb{C}}V$ can be identified with the space of $\mathbb{C}$-linear maps $\operatorname{Lin}_{\mathbb{C}}(\mathbb{C}p, V/\mathbb{C}p)$.

If p is nonisotropic, i.e., $\langle p,p\rangle \neq 0$, we have a natural identification between $V/\mathbb{C}p$ and $p^\perp := \{q \in V \mid \langle p,q\rangle = 0\}$. So, in this case, $\mathrm{T}_p\mathbb{P}_{\mathbb{C}}V \simeq \operatorname{Lin}_{\mathbb{C}}(\mathbb{C}p, p^\perp)$, which provides a non-degenerate Hermitian form on $\mathrm{T}_p\mathbb{P}_{\mathbb{C}}V$ given by

$$\langle t_1, t_2\rangle := -\frac{\langle t_1(p), t_2(p)\rangle}{\langle p,p\rangle}.$$

Clearly, this Hermitian form varies smoothly with a nonisotropic p. We will consider $\mathbb{P}_{\mathbb{C}}V \setminus \mathrm{S}V$ with the metric

$$(t_1, t_2) := \operatorname{Re}\langle t_1, t_2\rangle\,. \tag{2.1}$$

It is not difficult to see that $\mathrm{B}V$ is a Riemannian open 4-ball called the *complex hyperbolic plane*. Its ideal boundary $\mathrm{S}V$ is a three-dimensional sphere known as the *absolute*. We call $\mathbb{P}_{\mathbb{C}}V$ the *extended complex hyperbolic plane* (note that $\mathrm{E}V$ is a pseudo-Riemannian manifold).

It will be useful to consider a linear map $t \in \operatorname{Lin}_{\mathbb{C}}(\mathbb{C}p, p^\perp)$ in the form $t = \langle -, p\rangle v$ for some $v \in p^\perp$, where $tx = \langle x, p\rangle v$. In this way, the Hermitian form on $\mathrm{T}_p\mathbb{P}_{\mathbb{C}}V$ introduced above can be written as

$$\langle t_1, t_2\rangle = -\langle p,p\rangle\langle v_1, v_2\rangle\,,$$

where $t_1 := \langle -, p\rangle v_1$, $t_2 := \langle -, p\rangle v_2$, and $v_1, v_2 \in p^\perp$.

Let $p \in \mathbb{P}_{\mathbb{C}}V$ be a nonisotropic point. We will write

$$\pi'[p]v := \frac{\langle v,p\rangle}{\langle p,p\rangle}p \in \mathbb{C}p \quad \text{and} \quad \pi[p]v := v - \frac{\langle v,p\rangle}{\langle p,p\rangle}p \in p^\perp$$

for the projections of $v \in V$ over $\mathbb{C}p$ and $p^\perp$, respectively. Note that these projections do not depend on the choice of representative for p. Therefore, for $v \in V$, we have the orthogonal decomposition $v = \pi'[p]v + \pi[p]v$.

We will apply the following lemma several times.

Lemma 1 (Lemma 4.1.4 (ANAN'IN; GROSSI; GUSEVSKII, 2011))**.** *Let $c : [a,b] \to \mathbb{P}_{\mathbb{C}}V$ be a smooth curve and let $c_0 : [a,b] \to V$ be a smooth lift of c. If $c(t_0) \notin \mathrm{S}V$, then*

$$\dot c(t_0) = \langle -, c_0(t_0)\rangle \frac{\pi[c(t_0)]\dot c_0(t_0)}{\langle c_0(t_0), c_0(t_0)\rangle}$$

is the tangent vector to c at $c(t_0)$.

2.2 Geodesics, complex geodesics and tance

For every projective line L in $\mathbb{P}_{\mathbb{C}}V$, there exists a unique point $c \in \mathbb{P}_{\mathbb{C}}V$ such that[1] $\mathbb{P}c^{\perp} = \mathrm{L}$. This point is the *polar point* of L. The *signature* of a projective line L is, by definition, the signature of the (restriction of the Hermitian form to the) $\mathbb{C}$-linear subspace of V corresponding to L. By Sylvester's criterion, a projective line $\mathrm{L} = \mathbb{P}c^{\perp}$ can only be of signatures $-+$, $++$, and $0+$. Indeed, a projective line in $\mathbb{P}_{\mathbb{C}}V$ satisfies one of the following:

- The signature of L is $-+$ iff c is positive; in this case, we say that L *hyperbolic*. Geometrically, L consists of two Poincaré discs $\mathrm{L} \cap \mathrm{B}V$ and $\mathrm{L} \cap \mathrm{E}V$ glued along their common boundary $\mathrm{L} \cap \mathrm{S}V$.
- The signature of L is $++$ iff c is negative; in this case, we say that L is *spherical.* Geometrically, L is the usual 2-sphere of constant curvature.
- The signature of L is $0+$ iff c is isotropic, and L is called *Euclidean.*

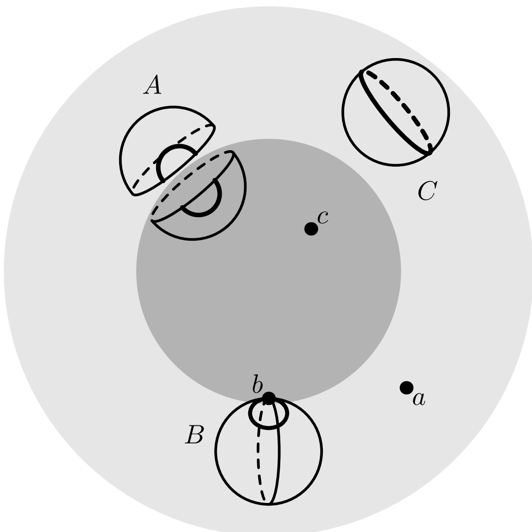


Figure 1 – Projective lines in $\mathbb{P}_{\mathbb{C}}V$ and their polar points

In Figure 1, the darker ball in the center is $\mathrm{B}V$ and the lighter is $\mathrm{E}V$. The points a, b, and c are, respectively, the polar points of the hyperbolic line A, the Euclidean line B, and the spherical line C.

The negative part $\mathrm{L} \cap \mathrm{B}V$ of a hyperbolic projective line L is often called a *complex geodesic*. We will sometimes refer to projective lines as *complex lines* or simply *lines*. We denote by $\mathrm{L}(p_1, p_2)$ the complex line $\mathbb{P}(\mathbb{C}p_1 + \mathbb{C}p_2)$ joining pairwise distinct $p_1, p_2 \in \mathbb{P}_{\mathbb{C}}V$.

[1] We often denote the projectivization $\mathbb{P}_{\mathbb{C}}S$ of a set $S \subset V \setminus \{0\}$ simply by $\mathbb{P}S$.

Remark 2. Two distinct complex lines L_1 and L_2 intersect at a single point. If p and q are, respectively, the polar points of L_1 and L_2, then the intersection point $\mathrm{L}_1 \cap \mathrm{L}_2$ is the polar point of the line $\mathrm{L}(p,q)$. If L_1 and L_2 are noneuclidean, i.e., p and q are nonisotropic, we say that they are *orthogonal* if $q \in \mathrm{L}_1$ and $p \in \mathrm{L}_2$ (this actually means that L_1 and L_2 are orthogonal, in the Hermitian sense, at their intersection point).

Take an $\mathbb{R}$-linear subspace $W \subset V$ with $\dim_{\mathbb{R}} W = 2$ such that the Hermitian form, being restricted to W, is real and does not vanish. By definition, $\mathrm{G}\,W := \mathbb{P}_{\mathbb{C}} W$ is a *geodesic* in $\mathbb{P}_{\mathbb{C}} V$. It is not difficult to verify that $\mathbb{C}W \simeq \mathbb{C} \otimes_{\mathbb{R}} W$; so, $\mathbb{P}_{\mathbb{C}} W = \mathbb{P}_{\mathbb{R}} W$. It follows that $\mathrm{G}\,W \simeq \mathbb{S}^1$, i.e., a geodesic is topologically a circle. The restrictions $\mathrm{G}W \cap \mathrm{B}V$ and $\mathrm{G}W \cap \mathrm{E}V$ consist exactly of all the (usual) geodesics of the (pseudo-)Riemannian metric (2.1) respectively on $\mathrm{B}V$ and $\mathrm{E}V$ (ANAN'IN; GROSSI, 2011, Corollary 5.5). A geodesic $\mathrm{G}\,W$ generates its projective line $\mathbb{P}_{\mathbb{C}}(\mathbb{C}W)$. We say that $\mathrm{G}\,W$ is *hyperbolic*, *spherical*, or *Euclidean* depending on the signature of its projective line. A hyperbolic geodesic has exactly two isotropic points (its *vertices*). Given two distinct and nonorthogonal points $p_1, p_2 \in \mathbb{P}_{\mathbb{C}} V$, there exists a unique geodesic containing both points. Such geodesic is the projectivization of the real space $W := \mathbb{R}p_1 + \mathbb{R}\langle p_1, p_2\rangle p_2$ and is denoted by $\mathrm{G}\wr p_1, p_2 \wr$.

Geodesics can be easily characterized in terms of the Hermitian form on V:

Proposition 3 ((ANAN'IN; GROSSI, 2011))**.** *Let* $\mathrm{G}\wr p_1, p_2\wr$ *be a geodesic in a noneuclidean projective line* L*. Then, a point* $x \in \mathrm{L}$ *belongs to* $\mathrm{G}\wr p_1, p_2\wr$ *if and only if*

$$b(x, p_1, p_2) := \langle x, p_1\rangle \langle p_1, p_2\rangle \langle p_2, x\rangle - \langle x, p_2\rangle \langle p_2, p_1\rangle \langle p_1, x\rangle = 0.$$

Let $p_1, p_2 \in \mathbb{P}_{\mathbb{C}} V \setminus \mathrm{S}V$ be nonisotropic points. The *tance* between p_1 and p_2 is defined by

$$\mathrm{ta}(p_1, p_2) := \frac{\langle p_1, p_2\rangle \langle p_2, p_1\rangle}{\langle p_1, p_1\rangle \langle p_2, p_2\rangle}.$$

If $p_1, p_2 \in \mathrm{B}V$, then $\mathrm{ta}(p_1, p_2) \geq 1$ and $\mathrm{ta}(p_1, p_2) = 1$ if and only if $p_1 = p_2$. We can also define the tance between subsets $X, Y \subset \mathbb{P}_{\mathbb{C}} V$ by

$$\mathrm{ta}(X, Y) := \inf\{\mathrm{ta}(x, y) \mid x \in X \text{ and } y \in Y\},$$

where we adopt the following convention: $\mathrm{ta}(p_1, p_2) = +\infty$ if at least one of the points p_1, p_2 is isotropic and p_1, p_2 are distinct and nonorthogonal; $\mathrm{ta}(p_1, p_2) = 1$ if $\langle p_1, p_2\rangle = 0$.

The distance is a monotonic function of the tance:

Proposition 4 (See 3.2 in (ANAN'IN; GROSSI, 2011))**.** *If* $p_1, p_2 \in \mathrm{B}V$ *(or* $p_1, p_2 \in \mathrm{E}V$ *and* p_1, p_2 *are in the same hyperbolic line* L*), then* $\cosh^2(\mathrm{dist}(p_1, p_2)) = \mathrm{ta}(p_1, p_2)$*. If* p_1 *and* p_2 *lie in the same spherical line* L*, then* $\cos^2(\mathrm{dist}(p_1, p_2)) = \mathrm{ta}(p_1, p_2)$*.*

Proof. Assume that $p_1, p_2 \in \mathrm{B}V$. Let v_1 and v_2 be the vertices of the geodesic $\mathrm{G}\wr p_1, p_2\wr$. Fix representatives $p_1, v_1, v_2 \in V$ such that $\langle v_1, v_2\rangle = -1/2$ and $p_1 = v_1 + v_2$. We can parametrize a lift of the segment of geodesic $c(t) : [0, a] \to \mathbb{P}_{\mathbb{C}}V$ connecting p_1 and p_2 as $c_0(t) = e^t v_1 + e^{-t} v_2$ with $c_0(a) = p_2$. Since $\langle \dot{c}_0(t), c_0(t) \rangle = 0$ and $\langle c_0(t), c_0(t) \rangle = -1$, by Lemma 1, $(\dot{c}(t), \dot{c}(t)) = 1$. Thus, the length of such segment is

$$\int_0^a \sqrt{(\dot{c}(t), \dot{c}(t))} = a.$$

But

$$\mathrm{ta}(p_1, p_2) = \mathrm{ta}(v_1 + v_2, e^a v_1 + e^{-a} v_2) = \cosh^2 a$$

and the proposition follows in this case.

If $p_1, p_2 \in \mathrm{E}V$ lie in a spherical line L, let p_1' be the orthogonal point to p_1 in L. We can fix representatives $p_1, p_1' \in V$ such that $\langle p_1, p_1 \rangle = \langle p_1', p_1' \rangle = 1$ and parametrize a lift of the geodesic segment $c : [0, a] \to \mathbb{P}_{\mathbb{C}}V$ connecting p_1 and p_2 by $c_0(t) = \cos(t) p_1 + \sin(t) p_1'$, where $t \in [0, a]$ and $c_0(a) = p_2$. Again we have that the length of this segment will be a, but in this case $\mathrm{ta}(p_1, p_2) = \mathrm{ta}(p_1, \cos(a) p_1 + \sin(a) p_1') = \cos^2 a$. □

Proposition 5. *Let* G *be a geodesic in a hyperbolic complex line* L *and let* $p \in \mathrm{L} \cap \mathrm{B}V$ *be a negative point. Then,*

$$\mathrm{ta}(p, \mathrm{G}) = \left| \frac{\langle v_1, p \rangle \langle p, v_2 \rangle}{\langle p, p \rangle \langle v_1, v_2 \rangle} \right| + \frac{1}{2}.$$

Proof. Let v_1 and v_2 be the vertices of the geodesic G. We choose representatives $p, v_1, v_2 \in V$ such that their Gram matrix is given by

$$\begin{bmatrix} -1 & 1 & r\varepsilon \\ 1 & 0 & -\frac{1}{2} \\ r\overline{\varepsilon} & -\frac{1}{2} & 0 \end{bmatrix},$$

where $r > 0$ and $|\varepsilon| = 1$. Note that, since v_1, v_2 and p lie in a same projective line, the determinant of this Gram matrix is zero; this implies that $\mathrm{Re}\,\varepsilon = \frac{1}{4r}$. Consider the parametrization of the geodesic G given by $c(t) = e^t v_1 + e^{-t} v_2$, $t \in \mathbb{R}$. Then

$$\mathrm{ta}(p, c(t)) = \frac{\langle p, c(t) \rangle \langle c(t), p \rangle}{\langle p, p \rangle \langle c(t), c(t) \rangle} = e^{2t} + r^2 e^{-2t} + 2r\,\mathrm{Re}\,\varepsilon.$$

Solving

$$\frac{d}{dt} \mathrm{ta}(p, c(t)) = 0,$$

which is equivalent to $2e^{2t} - 2r^2 e^{-2t} = 0$, we find that $t = \ln(\sqrt{r})$. Therefore,

$$\mathrm{ta}(p, \mathrm{G}) = 2r + \frac{1}{2}.$$

The result now follows from

$$r = \frac{1}{2} \left| \frac{\langle p, v_2 \rangle \langle v_1, p \rangle}{\langle p, p \rangle \langle v_1, v_2 \rangle} \right|.$$

□

Suppose that $p_1,p_2\in\mathbb{P}_{\mathbb{C}}V$ are orthogonal and that they span a nonnull projective line L. Then every geodesic in L containing p_1 also contains p_2. In particular, every geodesic in an Euclidean line L contains the isotropic point that is the polar point of L. So, if two geodesics G_1 and G_2 in a same complex line L intersect at a nonisotropic point p, then they also intersect at the point $\overline{p}$ which is the orthogonal to p in L. Given geodesics G_1 and G_2 in a noneuclidean complex line that intersect at a nonisotropic point p, we will denote by $\angle_p\mathrm{G}_1\mathrm{G}_2$ the oriented angle from G_1 to G_2 at the point p. Note that $\angle_{\overline{p}}\mathrm{G}_1\mathrm{G}_2=\angle_p\mathrm{G}_2\mathrm{G}_1$.

2.3 Metric circles, hypercycles and horocycles

Besides geodesics, there are some other important 'linear' geometric objects in the hyperbolic plane that will be needed later. (We call a geometric object *linear* when it is of the form $\mathbb{P}_{\mathbb{K}}W$ for some real subspace $W\subset V$, where $\mathbb{K}=\mathbb{R},\mathbb{C}$.) These objects are the metric circles, the hypercycles, and the horocycles; in this section, we study them from a coordinate-free perspective.

Let L be a hyperbolic line in $\mathbb{P}_{\mathbb{C}}V$. The *metric circle* $C(p,r)$ in L centered at $p\in\mathrm{L}$ with radius r is the set of all points $x\in\mathrm{L}$ whose distance from p equals r,

$$C(p,r):=\{x\in\mathrm{L}\mid\mathrm{ta}(x,p)=\cosh^2 r\}.$$

We will consider that $\{p\}$ is a metric circle with radius zero centered at p.

Metric circles limit to *horocycles*: let $q\in\mathrm{L}$ be a nonisotropic point and let p_n be a sequence in L converging to an isotropic point v over the geodesic ray from q to v. If C_n is the unique metric circle with center p_n passing through q, we have that C_n converges to $C\cup\{v\}$, where C is a horocycle. We say that the horocycle C is centered at v. Note that C is orthogonal to every geodesic containing v as a vertex.

Finally, let $\mathrm{G}\subset\mathrm{L}$ be a geodesic and let $r>0$. Let $H(G,r)$ be the set of nonisotropic points $x\in\mathrm{L}$ whose distance from G equals r,

$$H(G,r):=\{x\in\mathrm{L}\setminus\mathrm{S}V\mid\mathrm{ta}(x,\mathrm{G})=\cosh^2 r\}.$$

In other words, $H(G,r)$ is the curve r-equidistant to G. It consists of four disjoint curves (two curves of positive points and two of negative points) each of which is called a *hypercycle*. The nonisotropic part of a geodesic G consists of the hypercycles that are 0-equidistant to G.

Proposition 6. *Let W be a 2-dimensional real subspace of V. If the form* $\mathrm{Re}\,\langle -,-\rangle|_W$ *is, respectively, definite, nondegenetate indefinite, or degenerate, then* (*the nonisotropic part of*) $\mathbb{P}_{\mathbb{C}}W$ *is respectively a metric circle, a couple of hypercycles or a horocycle in the hyperbolic complex line* $\mathbb{P}_{\mathbb{C}}(\mathbb{C}W)$.

Proof. Suppose that $\operatorname{Re}\langle -,-\rangle|_W$ is definite of signature (say) $--$. Take $q_1,q_2\in W$ with $\operatorname{Re}\langle q_1,q_2\rangle=0$, $\langle q_1,q_1\rangle=\langle q_2,q_2\rangle=-1$. We can assume that $\langle q_1,q_2\rangle=ri$ with $r>0$. Let $p:=q_1-iq_2$. Note that $\langle p,p\rangle=-2(1+r)<0$, that is, $p\in\mathrm{B}V$. Let us show that $\operatorname{ta}(x,p)$ is constant for every $x\in\mathbb{P}_{\mathbb{C}}W$. Every point x in the (topological) circle $\mathbb{P}_{\mathbb{C}}W$ different from q_1 is of the form $x=sq_1+q_2$ for some $s\in\mathbb{R}$. Then,

$$\operatorname{ta}(x,p)=\frac{|\langle sq_1+q_2,q_1-iq_2\rangle|^2}{2(1+r)(s^2+1)}=\frac{(s^2+1)(1+r)^2}{2(1+r)(s^2+1)}=\frac{1+r}{2}$$

which does not depend on s. Hence, $\mathbb{P}_{\mathbb{C}}W$ is a metric circle. The proof is analogous when signature of $\operatorname{Re}\langle -,-\rangle|_W$ is $++$.

Suppose that the signature of $\operatorname{Re}\langle -,-\rangle|_W$ is $-+$. Take $p,q\in W$ with $\langle p,p\rangle=1/2$, $\langle q,q\rangle=-1/2$, and $\operatorname{Re}\langle p,q\rangle=0$. We can assume that $\langle p,q\rangle=ir/2$ with $r>0$. Note that $v_1:=p+q$ and $v_2:=p-q$ are isotropic points. Furthermore, $\langle v_1,v_2\rangle=1-ir$. Then $U:=\mathbb{R}v_1+\mathbb{R}(1-ir)v_2$ determines the geodesic G through v_1 and v_2. By Proposition 5, the tance $\operatorname{ta}(sq+p,\mathrm{G})$ does not depend on $-1<s<1$ because

$$\frac{\langle sq+p,v_2\rangle\langle v_1,sq+p\rangle}{\langle sq+p,sq+p\rangle\langle v_1,v_2\rangle}=-\frac{1}{2}(ir-1).$$

Therefore, the nonisotropic part of the (topological) circle $\mathbb{P}_{\mathbb{C}}W$ is a hypercycle.

Finally, suppose that the signature of $\operatorname{Re}\langle -,-\rangle|_W$ is $0-$. Let v be the isotropic point in $\mathbb{P}_{\mathbb{C}}W$. We want to prove that every geodesic through v is orthogonal to $\mathbb{P}_{\mathbb{C}}W$. Let $p\in\mathbb{P}_{\mathbb{C}}W$ be a negative point and let G the geodesic through p and v. The geodesic G has another isotropic point w. We choose representatives for p, v, and w such that $\langle v,v\rangle=\langle w,w\rangle=0$, $\langle v,w\rangle=-1/2$, and $p=v+w$. The curves $c_1(t)=e^tv+e^{-t}w$ and $c_2(t)=(1+it)v+w$ parameterize the negative part of G and the negative part of $\mathbb{P}W$, respectively. Furthermore, $c_1(0)=c_2(0)=p$. Since $\dot c_1(t)=e^tv-e^{-t}w$, $\dot c_2(t)=iv$, $\langle\dot c_1(0),\dot c_1(0)\rangle=1$, $\langle\dot c_2(0),\dot c_2(0)\rangle=0$, and $\langle\dot c_1(0),\dot c_2(0)\rangle=-i/2$, by Lemma 1,

$$t_1:=\langle -,p\rangle\frac{\pi[p](v-w)}{\langle p,p\rangle}\quad\text{and}\quad t_2:=\langle -,p\rangle\frac{\pi[p](iv)}{\langle p,p\rangle}$$

are respectively the tangent vectors to the curves c_1 and c_2 at p. It follows that

$$(t_1,t_2)=\operatorname{Re}(-\langle t_1,t_2\rangle)=\operatorname{Re}(-\langle p,p\rangle\langle\pi[p](v-w),\pi[p](iv)\rangle)=\operatorname{Re}\frac{i}{2}=0.$$

In other words, the geodesic G and the (topological) circle $\mathbb{P}_{\mathbb{C}}W$ are orthogonal and $\mathbb{P}_{\mathbb{C}}W$ is a horocycle. □

The above Proposition shows that metric circles, hypercycles, geodesics, and horocycles in a hyperbolic complex line $\mathbb{P}_{\mathbb{C}}(\mathbb{C}W)$ are actually 'linear', that is, they come from the projectivizations $\mathbb{P}_{\mathbb{C}}W$ of 2-dimensional real subspaces $W\subset V$. Curiously, the absolute $\mathbb{P}_{\mathbb{C}}(\mathbb{C}W)\cap\mathrm{S}V$ of a hyperbolic disc is itself linear (because it is possible for $\operatorname{Re}\langle\cdot,\cdot\rangle|_W$ to be null).

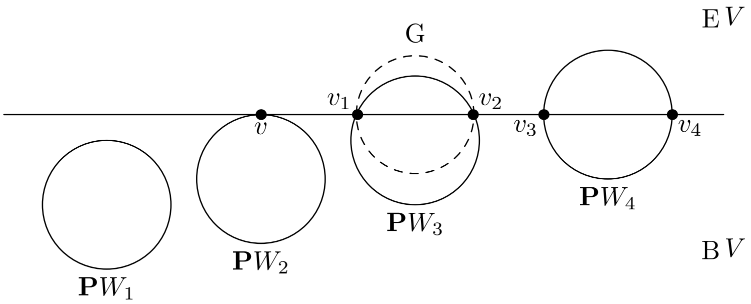


Figure 2 – Projectivizations of 2-dimensional real subspaces of V in a same hyperbolic complex line

Figure 2 illustrates these cases, assuming that all $\mathbb{P}W_i$'s are in the same hyperbolic complex line L (the straight line, the upper half-plane, and the lower half-plane represent, respectively, $\mathrm{L} \cap \mathrm{S}\,V$, $\mathrm{L} \cap \mathrm{E}\,V$, and $\mathrm{L} \cap \mathrm{B}\,V$): $\mathbb{P}W_1$ is a metric circle, $\mathbb{P}W_2$ is a horocycle centered at $v \in \mathrm{S}\,V$, $\mathbb{P}W_3$ is a hypercycle of the geodesic G with vertices v_1, v_2, and $\mathbb{P}W_4$ is a geodesic with vertices v_3, v_4.

2.4 Isometries

The group of orientation-preserving isometries of the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}} := \mathrm{B}\,V$ is $\mathrm{PU}\,V$ (GOLDMAN, 1999, Chapter 6), i.e., the projectivization of

$$\mathrm{U}V := \{I \in \mathrm{GL}(V) \mid \langle Iv, Iw\rangle = \langle v, w\rangle \text{ for every } v, w \in V\}.$$

As a matrix group, $\mathrm{PU}\,V$ (respectively, $\mathrm{U}V$) is denoted by $\mathrm{PU}(2,1)$ (respectively, $\mathrm{U}(2,1)$). Frequently, we will consider the 3-fold cover $\mathrm{SU}\,V \to \mathrm{PU}\,V$, where $\mathrm{SU}\,V$ (respectively, $\mathrm{SU}(2,1)$) consists of the elements in $\mathrm{U}V$ (respectively, in $\mathrm{U}(2,1)$) with determinant equal to 1. Clearly,

$$\mathrm{PU}(2,1) = \mathrm{SU}(2,1)/\{1, \omega, \omega^2\},$$

where $\omega = \exp(2\pi i/3)$ is a cubic root of unity.

By the Brouwer fixed point theorem, an isometry $I \in \mathrm{PU}(2,1)$ fixes a point in $\overline{\mathbb{H}}^2_{\mathbb{C}} := \mathrm{B}\,V \cup \mathrm{S}\,V$. We say that I is *elliptic* if it has a negative fixed point $c \in \mathrm{B}\,V$, *parabolic* if it fixes exactly one point in the absolute $\mathrm{S}\,V$, and *hyperbolic* if it fixes exactly two points in $\mathrm{S}\,V$. Fixed points of an isometry $I \in \mathrm{PU}(2,1)$ correspond to the eigenvectors of a lift of I to $\mathrm{SU}(2,1)$.

If no confusion is possible, we will refer to elements of $I \in \mathrm{SU}(2,1)$ as isometries of $\mathbb{H}^2_{\mathbb{C}}$ and will call its eigenvectors fixed points. Given an isometry $I \in \mathrm{SU}(2,1)$, one of the following occurs (PARKER, 2012, Thm. 3.6, pg. 16):

(i) I has two eigenvectors with eigenvalues μ and $\overline{\mu}^{-1}$ with $|\mu| \neq 1$. In this case I is hyperbolic;

(ii) I has a repeated eigenvalue μ, with $|\mu| = 1$, that correspond to a null eigenvector. In this case I is parabolic;

(iii) I has a negative eigenvector. In this case I is elliptic.

Let I be an elliptic isometry and let $c \in \mathrm{B}V$ be an I-fixed point. Then I stabilizes the projective line $\mathbb{P}c^{\perp}$ and, by the Brouwer fixed point theorem, the isometry I also has a fixed point $p \in \mathbb{P}c^{\perp}$. The point $q \in \mathbb{P}c^{\perp}$ that is orthogonal to p must also be fixed by I. Hence, we have an orthogonal basis for V given by eigenvectors of I. Let $\mu_1, \mu_2, \mu_3 \in \mathbb{C}$ with $\mu_1\mu_2\mu_3 = 1$ be the eigenvalues of c, p, q, respectively. Since none of c, p, q belongs to $\mathrm{S}V$, we have $|\mu_i| = 1$ for $i = 1, 2, 3$. An elliptic isometry is said to be *regular* if its eigenvectors have pairwise distinct eigenvalues; otherwise, it is called *special.*

A parabolic isometry is *unipotent* (or *screw parabolic*) if it can be lifted to a unipotent element of $\mathrm{SU}(2,1)$. A parabolic isometry that is not unipotent is called *ellipto-parabolic.*

If $I \in \mathrm{SU}(2,1)$ has exactly two repeated eigenvalues, then its eigenvalues are μ, μ, μ^{-2} with $|\mu| = 1$ and either I is a special elliptic isometry or I fixes an isotropic point and stabilizes a hyperbolic complex line through this point where its acts as a parabolic isometry (of the Poincaré disc), i.e., I is ellipto-parabolic (since we are assuming that $\mu^{-2} \neq \mu$, I cannot be unipotent). See (PARKER, 2012, Prop. 3.7, pg. 17) for a proof.

Summarizing, we have the following proposition:

Proposition 7. *Let $I \in \mathrm{SU}(2,1)$. Then I satisfies one of the following*

1. *I is elliptic and can be written, in a basis of nonisotropic pairwise orthogonal points, as*

$$\begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_1^{-1}\mu_2^{-1} \end{bmatrix},$$

where $|\mu_i| = 1$. In this case, I is regular elliptic if its eigenvalues are pairwise distinct and special elliptic otherwise;

2. *I is hyperbolic and can be written, in a basis consisting of two isotropic points v_1, v_2 and of a positive point c, as*

$$\begin{bmatrix} \mu & 0 & 0 \\ 0 & \overline{\mu}^{-1} & 0 \\ 0 & 0 & \overline{\mu}\mu^{-1} \end{bmatrix},$$

where $|\mu| \neq 1$ and $\mathbb{P}c^{\perp} = \mathrm{L}(v_1, v_2)$;

3. *I is ellipto-parabolic and can be written, in a basis v_1, v_2, c, as*

$$\begin{bmatrix} \mu & \xi & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \overline{\mu}^{-2} \end{bmatrix},$$

where $|\mu| = 1$, $\mu^3 \neq 1$, $v_1 \in \mathrm{S}V$, $c \in \mathrm{E}V$ is orthogonal to v_1, v_2, and $v_2 \neq v_1$.

4. *I is unipotent and can be written, in a basis v_1, c, v_2, as*

$$\begin{bmatrix} \delta & \xi_1 & \xi_2 \\ 0 & \delta & \xi_3 \\ 0 & 0 & \delta \end{bmatrix},$$

where $\delta^3 = 1$, $v_1 \in \mathrm{S}V$, $c \in \mathrm{E}V$ is orthogonal to v_1, v_2, and $v_1 \neq v_2$.

The classification of isometries of the complex hyperbolic plane can also be made through a trace function (GOLDMAN, 1999, pg. 204):

Proposition 8. *Let $f : \mathbb{C} \to \mathbb{R}$ be the real polynomial function given by $f(\tau) = |\tau|^4 - 8\,\mathrm{Re}(\tau^3) + 18|\tau|^2 - 27$ and let $I \in \mathrm{SU}(2,1)$. Then*

1. *I is hyperbolic if and only if $f(\operatorname{tr} I) > 0$;*

2. *I is regular elliptic if and only if $f(\operatorname{tr} I) < 0$;*

3. *I has repeated eigenvalues if and only if $f(\operatorname{tr} I) = 0$.*

Furthermore, if C_3 denotes the set of cubic roots of the unity,

4. *I is ellipto-parabolic if and only if I is not elliptic and $\operatorname{tr} I \in f^{-1}(0) \setminus 3C_3$;*

5. *I is special elliptic if and only if I is elliptic and $\operatorname{tr} I \in f^{-1}(0) \setminus 3C_3$;*

6. *I is unipotent if and only if $\operatorname{tr} I \in 3C_3$.*

2.5 Special elliptic isometries

In this section, we describe the geometry of special elliptic isometries.

Let $I \in \mathrm{SU}(2,1)$ be a special elliptic isometry. Then the eigenvalues of I are unitary complex numbers $\alpha, \alpha, \alpha^{-2}$. We assume that $\alpha \neq \alpha^{-2}$ (or, equivalently, $\alpha^3 \neq 1$) because, otherwise, I acts identically on $\mathbb{P}_{\mathbb{C}}V$. We take an orthogonal basis c, d, p consisting of eigenvectors of I corresponding, respectively, to $\alpha, \alpha, \alpha^{-2}$. The line $\mathbb{P}p^{\perp}$ is pointwise fixed by I. So, every complex line passing through p is I-stable. This leads to the following

Proposition 9. *If $p \in \mathrm{B}V$ then I acts on every complex geodesic passing through p as a rotation around p by the angle $\operatorname{Arg}(\alpha^3)$. If $p \in \mathrm{E}V$, then I acts on every complex geodesic orthogonal to $\mathbb{P}p^\perp$ at a negative point x as a rotation around x by the angle $\operatorname{Arg}(\alpha^{-3})$.*

Proof. Let $p \in \mathrm{B}V$, let L be a complex line passing through p, and let $p' \in \mathrm{L}$ be the orthogonal to p in L. Take representatives $p, p' \in V$ such that $\langle p, p\rangle = -\langle p', p'\rangle$. Every negative point in L has the form $p + kp'$ for some $k \in \mathbb{C}$ with $|k| \leq 1$. In this way, we identify the complex geodesic $\mathrm{L} \cap \mathrm{B}V$ with the closed disc in $\mathbb{C}$ centered at the origin (p corresponds to the origin). We have $I : p + kp' \mapsto \alpha^{-2}p + \alpha p' \cong p + \alpha^3 kp'$, where $\cong$ means $\mathbb{C}$-proportionality.

The case $p \in \mathrm{E}V$ is similar. Let L be a complex line orthogonal to $\mathbb{P}p^\perp = \mathrm{L}(c, d)$ at a negative point $x \in \mathbb{P}p^\perp$. It is easy to see that p lies in L and that it is exactly the point in L that is orthogonal to x. We take representatives $p, x \in \mathrm{L}$ such that $\langle x, x\rangle = -\langle p, p\rangle$ and proceed as above. □

We say that I is a *complex rotation* around the *center* p when $p \in \mathrm{B}V$ or around the *axis* $\mathbb{P}p^\perp$ when $p \in \mathrm{E}V$.

Remark 10. The special elliptic isometry $I \in \mathrm{SU}(2,1)$ with eigenvalues $\alpha, \alpha, \alpha^{-2}$ and eigenvectors c, d, p will be, from now on, denoted by R^p_α.

It is easy to see that the special elliptic isometry I is given by the expression

$$R^p_\alpha : x \mapsto (\alpha^{-2} - \alpha)\frac{\langle x, p\rangle}{\langle p, p\rangle}p + \alpha x.$$

Clearly, if $\alpha = -1$, then R^p_α is an involution (in terms of $\mathbb{H}^2_{\mathbb{C}}$, this means that R^p_α is the reflection in the point p if $p \in \mathrm{B}V$ or the reflection in the complex geodesic $\mathbb{P}p^\perp \cap \mathrm{B}V$ if $p \in \mathrm{E}V$). As observed above, R^p_α fixes the point p and also fixes pointwise the line $\mathbb{P}p^\perp$. It has no other fixed points.

Finally, a few observations to be quoted later. First, given a cubic root of the unity $\delta \in \mathbb{C}$ and a special elliptic isometry R^p_α, we have

$$R^p_{\delta\alpha} = \delta R^p_\alpha.$$

Then, in $\mathrm{PU}(2,1)$, $R^p_{\delta\alpha} = R^p_\alpha$. Moreover, $R^p_\delta = \delta$ which is the identity in $\mathrm{PU}(2,1)$. Finally, $R^p_\alpha = \delta$ in $\mathrm{SU}(2,1)$ if and only if $\alpha = \delta$.

2.6 Product of special elliptic isometries

In what follows, we will obtain the formula for the trace of the product of special elliptic isometries. This result is a first step in understanding the relations between special elliptic isometries.

The trace of the product of reflections was obtained in (ANAN'IN, 2012) and (PRATOUSSEVITCH, 2005). Since reflections are a particular case of special elliptic isometries, the formula that we obtain is a generalization of such results.

Lemma 11. *Let $p_1,\dots,p_n \in \mathbb{P}_{\mathbb{C}}V$ be nonisotropic points and let $\alpha_1,\dots,\alpha_n$ be unitary complex numbers. Let $R_i := R_{\alpha_i}^{p_i}$ (see Remark 10 in the previous section). Then*

$$R_n\dots R_2R_1x = \sum_{\ell=1}^{n}\left[\sum_{\substack{1\le i_1<\dots<i_t=\ell\\ 1\le t\le \ell}} \beta(i_1,\dots,i_t)\frac{\langle x,p_{i_1}\rangle\langle p_{i_1},p_{i_2}\rangle\dots\langle p_{i_{t-1}},p_{i_t}\rangle}{\langle p_{i_1},p_{i_1}\rangle\dots\langle p_{i_t},p_{i_t}\rangle}\right]p_\ell+$$
$$+\alpha_1\alpha_2\cdots\alpha_n x,$$

where

$$\beta(i_1,\dots,i_t) := \prod_{\ell=1}^{t}\left(\alpha_{i_\ell}^{-2}-\alpha_{i_\ell}\right)\prod_{k=1}^{n-t}\alpha_{j_k} \tag{2.2}$$

and $\{i_1,\dots,i_t,j_1,\dots,j_{n-t}\}=\{1,\dots,n\}$.

Proof. The fact clearly holds for $n=1$. Assuming that it is also true for $n-1$, we have

$$R_n(R_{n-1}\dots R_1x) = (\alpha_n^{-2}-\alpha_n)\frac{\langle R_{n-1}\dots R_1x,\, p_n\rangle}{\langle p_n,p_n\rangle}p_n+\alpha_n x$$
$$= \left[\sum_{\ell=1}^{n-1}\sum_{\substack{1\le i_1<\dots<i_t=\ell\\ 1\le t\le \ell}}(\alpha_n^{-1}-\alpha_n)\beta_{n-1}(i_1,\dots,i_t)\frac{\langle x,p_{i_1}\rangle\dots\langle p_{i_{t-1}},p_{i_t}\rangle\langle p_\ell,p_n\rangle}{\langle p_{i_1},p_{i_1}\rangle\dots\langle p_{i_t},p_{i_t}\rangle\langle p_n,p_n\rangle}+\right.$$
$$\left.+\alpha_1\alpha_2\cdots\alpha_{n-1}(\alpha_n^{-1}-\alpha_n)\frac{\langle x,p_n\rangle}{\langle p_n,p_n\rangle}\right]p_n+$$
$$+\sum_{k=1}^{n-1}\left[\sum_{\substack{1\le j_1<\dots<j_s=k\\ 1\le s\le k}}\alpha_n\beta_{n-1}(j_1,\dots,j_s)\frac{\langle x,p_{j_1}\rangle\dots\langle p_{j_{s-1}},p_{j_s}\rangle}{\langle p_{j_1},p_{j_1}\rangle\dots\langle p_{j_s},p_{j_s}\rangle}\right]p_k+$$
$$+\alpha_1\alpha_2\cdots\alpha_{n-1}\alpha_n x$$
$$= \left[\sum_{\substack{1\le i_1<\dots<i_t=n\\ 1\le t\le n}}\beta_n(i_1,\dots,i_t)\frac{\langle x,p_{i_1}\rangle\dots\langle p_{i_{t-1}},p_{i_t}\rangle}{\langle p_{i_1},p_{i_1}\rangle\dots\langle p_{i_t},p_{i_t}\rangle}\right]p_n+$$
$$+\sum_{k=1}^{n-1}\left[\sum_{\substack{1\le j_1<\dots<j_s=k\\ 1\le s\le k}}\beta_n(j_1,\dots,j_s)\frac{\langle x,p_{j_1}\rangle\dots\langle p_{j_{s-1}},p_{j_s}\rangle}{\langle p_{j_1},p_{j_1}\rangle\dots\langle p_{j_s},p_{j_s}\rangle}\right]p_k+$$
$$+\alpha_1\alpha_2\cdots\alpha_n x.$$
$$= \sum_{\ell=1}^{n}\left[\sum_{\substack{1\le i_1<\dots<i_t=\ell\\ 1\le t\le \ell}}\beta_n(i_1,\dots,i_t)\frac{\langle x,p_{i_1}\rangle\langle p_{i_1},p_{i_2}\rangle\dots\langle p_{i_{t-1}},p_{i_t}\rangle}{\langle p_{i_1},p_{i_1}\rangle\dots\langle p_{i_t},p_{i_t}\rangle}\right]p_\ell+$$
$$+\alpha_1\alpha_2\cdots\alpha_n x,$$

as desired (here, β_{n-1} and β_n stand for the number β, defined as above, but for the product of $n-1$ and n isometries, respectively). $\square$

For instance, since the only strictly increasing lists of 1 and 2 are

$$(1),\ (2) \text{ and } (1,2),$$

in the expression of $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ there will be only three β terms:

$$\begin{aligned}
\beta(1) &= (\alpha_1^{-2}-\alpha_1)\alpha_2,\\
\beta(2) &= \alpha_1(\alpha_2^{-2}-\alpha_2),\\
\beta(1,2) &= (\alpha_1^{-2}-\alpha_1)(\alpha_2^{-2}-\alpha_2).
\end{aligned}$$

So,

$$R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}x = \beta(1)\frac{\langle x,p_1\rangle}{\langle p_1,p_1\rangle}p_1 + \left[\beta(2)\frac{\langle x,p_2\rangle}{\langle p_2,p_2\rangle} + \beta(1,2)\frac{\langle x,p_1\rangle\langle p_1,p_2\rangle}{\langle p_1,p_1\rangle\langle p_2,p_2\rangle}\right]p_2 + \alpha_1\alpha_2 x.$$

For the product of three special elliptic isometries $R^{p_3}_{\alpha_3}R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$, the strictly increasing lists of 1, 2, and 3 are

$$(1),\ (2),\ (3),\ (1,2),\ (1,3),\ (2,3),\ \text{and } (1,2,3).$$

Then the β terms are

$$\begin{aligned}
\beta(1) &= (\alpha_1^{-2}-\alpha_1)\alpha_2\alpha_3,\\
\beta(2) &= \alpha_1(\alpha_2^{-2}-\alpha_2)\alpha_3,\\
\beta(3) &= \alpha_1\alpha_2(\alpha_3^{-2}-\alpha_3),\\
\beta(1,2) &= (\alpha_1^{-2}-\alpha_1)(\alpha_2^{-2}-\alpha_2)\alpha_3,\\
\beta(1,3) &= (\alpha_1^{-2}-\alpha_1)\alpha_2(\alpha_3^{-2}-\alpha_3),\\
\beta(2,3) &= \alpha_1(\alpha_2^{-2}-\alpha_2)(\alpha_3^{-2}-\alpha_3),\\
\beta(1,2,3) &= (\alpha_1^{-2}-\alpha_1)(\alpha_2^{-2}-\alpha_2)(\alpha_3^{-2}-\alpha_3).
\end{aligned}$$

Therefore,

$$\begin{aligned}
R^{p_3}_{\alpha_3}R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}x = \beta(1)\frac{\langle x,p_1\rangle}{\langle p_1,p_1\rangle}p_1 &+ \left[\beta(2)\frac{\langle x,p_2\rangle}{\langle p_2,p_2\rangle} + \beta(1,2)\frac{\langle x,p_1\rangle\langle p_1,p_2\rangle}{\langle p_1,p_1\rangle\langle p_2,p_2\rangle}\right]p_2 +\\
&+ \left[\beta(3)\frac{\langle x,p_3\rangle}{\langle p_3,p_3\rangle} + \beta(1,3)\frac{\langle x,p_1\rangle\langle p_1,p_3\rangle}{\langle p_1,p_1\rangle\langle p_3,p_3\rangle} + \beta(2,3)\frac{\langle x,p_2\rangle\langle p_2,p_3\rangle}{\langle p_2,p_2\rangle\langle p_3,p_3\rangle}\right.\\
&\left. + \beta(1,2,3)\frac{\langle x,p_1\rangle\langle p_1,p_2\rangle\langle p_2,p_3\rangle}{\langle p_1,p_1\rangle\langle p_2,p_2\rangle\langle p_3,p_3\rangle}\right]p_3 + \alpha_1\alpha_2\alpha_3 x.
\end{aligned}$$

Lemma 11 easily provides a formula for the trace of the product of special elliptic isometries:

Proposition 12. *Let $R_{\alpha_i}^{p_i} \in \mathrm{SU}(2,1)$ be special elliptic isometries, $i=1,\dots,n$. If $[g_{ij}]$ is the Gram matrix of the nonisotropic points $p_1,\dots,p_n$, then*

$$\operatorname{tr} R_{\alpha_n}^{p_n}\dots R_{\alpha_1}^{p_1} = 3\alpha_1\cdots\alpha_n + \sum_{\substack{1\le i_1<\cdots<i_t\le n\\ 1\le t\le n}} \beta(i_1,\dots,i_t)\frac{g_{i_1i_2}\cdots g_{i_ti_1}}{g_{i_1i_1}\cdots g_{i_ti_t}},$$

assuming that, when $t=1$, the sum in the right side is simply[2] $\beta(i_1)$.

Proof. By Lemma 11,

$$\operatorname{tr} R_{\alpha_n}^{p_n}\dots R_{\alpha_1}^{p_1} = 3\alpha_1\cdots\alpha_n + \sum_{\ell=1}^{n}\sum_{\substack{1\le i_1<\cdots<i_t=\ell\\ 1\le t\le \ell}} \beta(i_1,\dots,i_t)\frac{g_{i_ti_1}g_{i_1i_2}\cdots g_{i_{t-1}i_t}}{g_{i_1i_1}\cdots g_{i_ti_t}}.$$

The proposition now follows from

$$\sum_{\ell=1}^{n}\sum_{\substack{1\le i_1<\cdots<i_t=\ell\\ 1\le t\le \ell}} \beta(i_1,\dots,i_t)\frac{g_{i_ti_1}g_{i_1i_2}\cdots g_{i_{t-1}i_t}}{g_{i_1i_1}\cdots g_{i_ti_t}} = \sum_{\substack{1\le i_1<\cdots<i_t\le n\\ 1\le t\le n}} \beta(i_1,\dots,i_t)\frac{g_{i_1i_2}\cdots g_{i_ti_1}}{g_{i_1i_1}\cdots g_{i_ti_t}}.$$

□

Again, in order to obtain the trace of a product of n special elliptic isometries, we first determine, for each $\ell=1,\dots,n$, all strictly increasing lists of elements of $\{1,\dots,n\}$ with length ℓ. Then we are able to obtain all β terms (given by (2.2)) that will appear in such expression.

As some important examples (that will be needed later), we calculate the trace of the product of 1, 2, and 3 special elliptic isometries:

$$\operatorname{tr} R_{\alpha_1}^{p_1} = 3\alpha_1 + \beta(1) = 2\alpha_1 + \alpha_1^{-2},$$

$$\begin{aligned}\operatorname{tr} R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1} &= 3\alpha_1\alpha_2 + \beta(1)+\beta(2)+\beta(1,2)\frac{g_{12}g_{21}}{g_{11}g_{22}}\\ &= 3\alpha_1\alpha_2 + (\alpha_1^{-2}-\alpha_1)\alpha_2 + \alpha_1(\alpha_2^{-2}-\alpha_2)\\ &\quad + (\alpha_1^{-2}-\alpha_1)(\alpha_2^{-2}-\alpha_2)\operatorname{ta}(p_1,p_2)\\ &= \alpha_1\alpha_2 + \alpha_1^{-2}\alpha_2 + \alpha_1\alpha_2^{-2} + (\alpha_1^{-2}-\alpha_1)(\alpha_2^{-2}-\alpha_2)\operatorname{ta}(p_1,p_2),\end{aligned}$$

$$\begin{aligned}\operatorname{tr} R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1} = {}&\alpha_1^{-2}\alpha_2\alpha_3 + \alpha_1\alpha_2^{-2}\alpha_3 + \alpha_1\alpha_2\alpha_3^{-2} + (\alpha_1^{-2}-\alpha_1)(\alpha_2^{-2}-\alpha_2)\alpha_3\operatorname{ta}(p_1,p_2)\\ &+ (\alpha_1^{-2}-\alpha_1)\alpha_2(\alpha_3^{-2}-\alpha_3)\operatorname{ta}(p_1,p_3) + \alpha_1(\alpha_2^{-2}-\alpha_2)(\alpha_3^{-2}-\alpha_3)\operatorname{ta}(p_2,p_3)\\ &+ (\alpha_1^{-2}-\alpha_1)(\alpha_2^{-2}-\alpha_2)(\alpha_3^{-2}-\alpha_3)\frac{g_{12}g_{23}g_{31}}{g_{11}g_{22}g_{33}}.\end{aligned}$$

[2] This means that we could also have written

$$\operatorname{tr} R_{\alpha_n}^{p_n}\dots R_{\alpha_1}^{p_1} = 3\alpha_1\cdots\alpha_n + \sum_{i=1}^{n}\beta(i) + \sum_{\substack{1\le i_1<\cdots<i_t\le n\\ 2\le t\le n}} \beta(i_1,\dots,i_t)\frac{g_{i_1i_2}\cdots g_{i_ti_1}}{g_{i_1i_1}\cdots g_{i_ti_t}}.$$

Finally, if $\alpha = -1$, then $\alpha^{-2} - \alpha = 2$. This implies that, in the particular case of a product of reflections (special elliptic isometries with angles $= -1$), the β terms in (2.2) are given by

$$\beta(i_1, \ldots, i_t) = (-1)^{n-t} 2^t.$$

Together with Proposition 12, this leads to the trace of a product of reflections:

Corollary 13. *If* $[g_{ij}]$ *is the Gram matrix of the nonisotropic points* $p_1, \ldots, p_n \in \mathbb{P}_{\mathbb{C}} V$, *then*

$$\operatorname{tr} R^{p_n}_{-1} \ldots R^{p_1}_{-1} = 3(-1)^n + 2n(-1)^{n-1} + \sum_{\substack{1 \leq i_1 < \cdots < i_t \leq n \\ 2 \leq t \leq n}} (-1)^{n-t} 2^t \frac{g_{i_1 i_2} \cdots g_{i_t i_1}}{g_{i_1 i_1} \cdots g_{i_t i_t}}.$$

CHAPTER

3

BASIC RELATIONS BETWEEN SPECIAL ELLIPTIC ISOMETRIES

3.1 Relations between reflections

In this section we discuss a few results in (ANAN'IN, 2012) that motivate the facts that we are going to establish in the remaining part of this work. These results concern relations between reflections in positive or negative points of the complex hyperbolic plane. Reflections are involutions of the complex hyperbolic plane; they are nothing but special elliptic isometries with angle -1, i.e., isometries of the form

$$R^p_{-1} : x \mapsto 2\frac{\langle x, p\rangle}{\langle p, p\rangle}p - x$$

where $p \in \mathbb{P}_{\mathbb{C}}V \setminus \mathrm{S}V$. Although $R^p_{-\delta}$ is also a reflection if $\delta^3 = 1$, here we will only consider relations of the form $R^{p_n}_{-1} \dots R^{p_1}_{-1} = 1$ in $\mathrm{PU}(2,1)$ with a lift to $\mathrm{SU}(2,1)$ of the form $R^{p_n}_{-1} \dots R^{p_1}_{-1} = \delta$, where $\delta \in \mathbb{C}$ is a cubic root of unity.

Short relations ($n \leq 4$) between reflections are easy to understand:

Proposition 14 (See Remark 3.2 in (ANAN'IN, 2012))**.** *Let $\delta \in \mathbb{C}$ be such that $\delta^3 = 1$ and let $p_1, p_2, p_3 \in \mathbb{P}_{\mathbb{C}}V \setminus \mathrm{S}V$. Then*

- $R^{p_2}_{-1} R^{p_1}_{-1} = \delta$ *if and only if* $p_1 = p_2$ *and* $\delta = 1$*;*
- $R^{p_3}_{-1} R^{p_2}_{-1} R^{p_1}_{-1} = \delta$ *if and only if* p_1, p_2, p_3 *are pairwise orthogonal and* $\delta = 1$*.*

These results follow directly from Corollary 13. For instance, since $\operatorname{tr} R^{p_2}_{-1} R^{p_1}_{-1} = 4\operatorname{ta}(p_1, p_2) - 1$, it follows from $R^{p_2}_{-1} R^{p_1}_{-1} = \delta$ that $\delta = 1$ and $\operatorname{ta}(p_1, p_2) = 1$.

Relations of the form $R^{p_2}_{-1} R^{p_1}_{-1} = \delta$ are called *cancellations*, and relations of the form $R^{p_3}_{-1} R^{p_2}_{-1} R^{p_1}_{-1} = \delta$ are called *orthogonal relations*. In the next sections, we generalize these results.

Proposition 15 (see Proposition 2.6 in (ANAN'IN, 2012)). *Let p_1 and p_2 be distinct and nonorthogonal points in $\mathbb{P}_{\mathbb{C}}V\setminus \mathrm{S}V$. There exists a one-parameter subgroup $B(s):\mathbb{R}\to \mathrm{SU}(2,1)$ such that $R_{-1}^{B(s)p_2}R_{-1}^{B(s)p_1}=R_{-1}^{p_2}R_{-1}^{p_1}$ for all $s\in\mathbb{R}$.*

The relations described in the above proposition are called *bendings*. All relations between reflections with $n\leq 4$ follow from cancellations, orthogonal relations, and bendings. In Section 4 we discuss some possible generalizations of such a fact.

Understanding the relations of length 5 between reflections involves studying the decompositions $F:=R_{-1}^{p_3}R_{-1}^{p_2}R_{-1}^{p_1}$ of an arbitrary regular isometry F into the product of three reflections. A triple $p_1,p_2,p_3\in\mathbb{P}_{\mathbb{C}}V\setminus\mathrm{S}V$ is *strongly regular* if at most one of p_1,p_2,p_3 is positive and if these points do not lie in the same complex line.[1] It is easy to see that, if p_1,p_2,p_3 is strongly regular, then $F:=R_{-1}^{p_3}R_{-1}^{p_2}R_{-1}^{p_1}$ is a regular isometry (this essentially means that the isometry is not special elliptic nor unipotent with an Euclidean fixed line, see Definition 43).

Understanding what happens to a strongly regular triple p_1,p_2,p_3 (or, equivalently, to a decomposition $F:=R_{-1}^{p_3}R_{-1}^{p_2}R_{-1}^{p_1}$ of a regular isometry F) when it is modified by composition of bendings involving p_1,p_2 and p_2,p_3 generates valuable information about length 5 relations between reflections:

Lemma 16 (see Lemma 4.1.1 in (ANAN'IN, 2012)). *Geometrically, all strongly regular triples p_1,p_2,p_3 with $\sigma p_i=\sigma_i$ and fixed conjugacy class of $R_{-1}^{p_3}R_{-1}^{p_2}R_{-1}^{p_1}$ are parameterized by the surface $S\subset\mathbb{R}(t_1,t_2,t)$ given by the equation*

$$(t_1-1)(t_2-1)=t_1t_2(t-1)^2+\frac{\alpha^2}{t_1t_2}+\beta$$

and by the inequalities

$$\sigma_1\sigma_2t_1>0,\quad \sigma_1\sigma_2t_1>\sigma_1\sigma_2,\quad \sigma_2\sigma_3t_2>0,\quad \sigma_2\sigma_3t_2>\sigma_2\sigma_3,$$

where $t_1:=\mathrm{ta}(p_1,p_2)$, $t_2:=\mathrm{ta}(p_2,p_3)$, $t:=\tau(p_1,p_2,p_3):=\mathrm{Re}\,\frac{g_{13}g_{22}}{g_{12}g_{23}}$, and $[g_{ij}]$ is the Gram matrix of p_1,p_2,p_3.

It follows that, geometrically, two strongly regular triples p_1,p_2,p_3 with fixed conjugacy class of $R_{-1}^{p_3}R_{-1}^{p_2}R_{-1}^{p_1}$ can be connected by means of finitely many bendings involving p_1,p_2 and p_2,p_3. As a consequence, we reach a complete description of *pentagons,* i.e., of length 5 relations $R_{-1}^{p_5}R_{-1}^{p_4}R_{-1}^{p_3}R_{-1}^{p_2}R_{-1}^{p_1}=\delta$ where at most one of $p_1,\dots,p_5$ is positive, $\delta^3=1$, and $\langle p_i,p_{i+1}\rangle\neq 0$ (the index i is taken modulo 5).

Theorem 17 (see Subsection 5.4 in (ANAN'IN, 2012)). *Two pentagons with the same δ and the same signature of points can be connected by means of finitely many bendings.*

[1] Beware that this definition of strongly regular triple differs from the one that we are going to give in Definition 45.

In Chapter 5 we present a generalization of the above theorem to the case of special elliptic isometries.

3.2 Special elliptic isometries: length 2 relations

In this section we determine when a couple of nonisotropic points $p_1, p_2 \in \mathbb{P}_{\mathbb{C}}V \setminus \mathrm{S}V$ and a pair of unitary complex numbers (angles) α_1, α_2 satisfy $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} = \delta$, where $\delta \in \mathbb{C}$, $\delta^3 = 1$. In what follows, we will denote the circle of unitary complex numbers by $\mathbb{S}^1$.

Clearly, $R^p_\beta R^p_\alpha = R^p_{\alpha\beta}$ for every $p \in \mathbb{P}_{\mathbb{C}}V \setminus \mathrm{S}V$ and every $\alpha, \beta \in \mathbb{S}^1$. Hence, $R^p_\alpha R^p_{\overline{\alpha}} = 1$ in $\mathrm{SU}(2,1)$. In the next proposition we show that the converse is also true: if a product of two special elliptic isometries is identical, then the centres of the isometries are equal and the angles are inverses. We remind that, if $R^{p_1}_{\alpha_1}$ and $R^{p_2}_{\alpha_2}$ are nonidentical special elliptic isometries (α_1 and α_2 are not cubic roots of the unity) with $p_1 \neq p_2$, then p_i is fixed by $R^{p_j}_{\alpha_j}$ if and only if $\langle p_i, p_j \rangle = 0$ for $i \neq j$.

Proposition 18. *Let $\delta \in \mathbb{C}$ be such that $\delta^3 = 1$ and let $\alpha_1, \alpha_2 \in \mathbb{S}^1$ be angles with $\alpha_1^3 \neq 1$ and $\alpha_2^3 \neq 1$. Then $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} = \delta$ if and only if $p_1 = p_2$ and $\alpha_1\alpha_2 = \delta$.*

Proof. Assume that p_1, p_2 are distinct and nonorthogonal. Then $R^{p_1}_{\alpha_1}$ fixes p_1 and this point is not fixed by $R^{p_2}_{\alpha_2}$. So, $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ does not fix p_1 and it does not act identically in $\mathbb{P}_{\mathbb{C}}V$.

Now, if p_1 and p_2 are orthogonal (which implies that the line $\mathrm{L}(p_1, p_2)$ is noneuclidean), let c be the intersection point between $\mathbb{P}p_1^\perp$ and $\mathbb{P}p_2^\perp$. In the basis p_1, p_2, c, the isometries $R^{p_1}_{\alpha_1}$ and $R^{p_2}_{\alpha_2}$ are given, respectively, by

$$\begin{bmatrix} \alpha_1^{-2} & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_1 \end{bmatrix} \text{ and } \begin{bmatrix} \alpha_2 & 0 & 0 \\ 0 & \alpha_2^{-2} & 0 \\ 0 & 0 & \alpha_2 \end{bmatrix}.$$

Then, in this basis, $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ is given by

$$\begin{bmatrix} \alpha_1^{-2}\alpha_2 & 0 & 0 \\ 0 & \alpha_1\alpha_2^{-2} & 0 \\ 0 & 0 & \alpha_1\alpha_2 \end{bmatrix}.$$

Therefore, $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} = \delta$ if and only if $\alpha_1^3 = \alpha_2^3 = 1$ which implies that $R^{p_1}_{\alpha_1}$ and $R^{p_2}_{\alpha_2}$ are both identical in $\mathrm{PU}(2,1)$. □

Assuming the hypothesis of Proposition 18, we have $\delta = 1$ if and only if $\alpha_1 = \overline{\alpha}_2$. If $\delta \neq 1$ and if one of the angles is α, then the other must be $\delta\overline{\alpha}$. It follows that a relation of length 2 in $\mathrm{SU}(2,1)$ is either of the form $R^p_\alpha R^p_{\overline{\alpha}} = 1$ or can be written as $R^p_\alpha R^p_{\delta\overline{\alpha}} = \delta$. These relations are called **cancellations**.

3.3 Length 3 relations

In order to obtain all possible solutions for length 3 relations, we study when the product of two special elliptic isometries is special elliptic.

Given distinct points $p_1, p_2 \in \mathbb{P}_{\mathbb{C}}V \setminus \mathrm{S}V$, the product $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ stabilizes the complex line $\mathrm{L}(p_1, p_2)$ because it fixes the polar point c of the line, $\mathbb{P}c^\perp = \mathrm{L}(p_1, p_2)$. Therefore, if $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ is special elliptic, then its pointwise fixed complex line L is either equal to $\mathrm{L}(p_1, p_2)$ or contains c.

Lemma 19. *Let $p_1, p_2 \in \mathbb{P}_{\mathbb{C}}V \setminus \mathrm{S}V$ be distinct points and let $\alpha_1, \alpha_2 \in \mathbb{S}^1$ be angles satisfying $\alpha_i^3 \neq 1$. The isometry $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ has a fixed point in the line $\mathrm{L}(p_1, p_2)$ with eigenvalue $\alpha_1\alpha_2$ if and only if $\mathrm{ta}(p_1, p_2) = 1$, i.e., if and only if $\mathrm{L}(p_1, p_2)$ is Euclidean.*

Proof. Let $p_3 \in \mathrm{L}(p_1, p_2)$ be a fixed point of $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ with eigenvalue $\alpha_1\alpha_2$. Suppose that $\langle p_1, p_2 \rangle \neq 0$. Then, $p_3 \neq p_2$ and we can write $p_3 = p_1 + \lambda p_2$. Thus,

$$R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} p_3 = \mu\alpha_2 p_1 + \left(\lambda\alpha_1\alpha_2^{-2} + \mu(\alpha_2^{-2} - \alpha_2)\frac{\langle p_1, p_2\rangle}{\langle p_2, p_2\rangle}\right) p_2,$$

where

$$\mu = \alpha_1^{-2} + \lambda(\alpha_1^{-2} - \alpha_1)\frac{\langle p_2, p_1\rangle}{\langle p_1, p_1\rangle}.$$

So, since $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} p_3 = \alpha_1\alpha_2 p_3 = \alpha_1\alpha_2 p_1 + \lambda\alpha_1\alpha_2 p_2$ by hypothesis, we have $\mu = \alpha_1$ and

$$\lambda\alpha_2^{-2} + (\alpha_2^{-2} - \alpha_2)\frac{\langle p_1, p_2\rangle}{\langle p_2, p_2\rangle} = \lambda\alpha_2.$$

It follows that

$$\lambda\frac{\langle p_2, p_1\rangle}{\langle p_1, p_1\rangle} = -1 \quad \text{and} \quad \lambda = -\frac{\langle p_1, p_2\rangle}{\langle p_2, p_2\rangle}$$

which implies $\mathrm{ta}(p_1, p_2) = 1$. Now, if $\langle p_1, p_2 \rangle = 0$, we have $\mu = \alpha_1^{-2}$, $\alpha_1^{-2}\alpha_2 = \alpha_1\alpha_2$, and $\lambda\alpha_1\alpha_2^{-2} = \lambda\alpha_1\alpha_2$; hence, $\alpha_1^3 = \alpha_2^3 = 1$, a contradiction.

Conversely, assuming that $\mathrm{ta}(p_1, p_2) = 1$, we define $p_3 := p_1 + \lambda p_2$ for $\lambda = -\frac{\langle p_1, p_2\rangle}{\langle p_2, p_2\rangle}$ and p_3 will be the desired fixed point. $\square$

Using the previous lemma we will show that, if $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ is special elliptic and the line $\mathrm{L}(p_1, p_2)$ is noneuclidean, then p_1 and p_2 are orthogonal and the angles are equal (up to multiplication by a cubic root of the unity).

Proposition 20. *Let $p_1, p_2 \in \mathbb{P}_{\mathbb{C}}V \setminus \mathrm{S}V$ be distinct points and let $\alpha_1, \alpha_2 \in \mathbb{S}^1$ be angles with $\alpha_i^3 \neq 1$. Assume that the line $\mathrm{L}(p_1, p_2)$ is noneuclidean. Then $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ is special elliptic if and only if $\langle p_1, p_2 \rangle = 0$ and $\alpha_1^3 = \alpha_2^3$, i.e., $\alpha_1 = \delta\alpha_2$ for some $\delta \in \mathbb{C}$ with $\delta^3 = 1$.*

Proof. Let c be the intersection point between $\mathbb{P}p_1^\perp$ and $\mathbb{P}p_2^\perp$, i.e., $\mathrm{L}(p_1, p_2) = \mathbb{P}c^\perp$. The point c is nonisotropic because the line generated by p_1 and p_2 is noneuclidean. Assume that $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ is special elliptic. Since $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ fixes c, we have one of the following cases:

1. The isometry $R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}$ fixes the line $\mathbb{P}c^\perp$ pointwise. This implies that $p_2 \in \mathbb{P}c^\perp$ is fixed by $R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}$. So, p_2 is also fixed by $R_{\alpha_1}^{p_1}$ and $\langle p_1, p_2\rangle = 0$ due to $p_1 \neq p_2$.

2. The line $\mathbb{P}c^\perp$ is stable under $R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}$ but not pointwise fixed. In this case, the line pointwise fixed by $R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}$ must pass through c. Hence, there must be a fixed point $p_3 \in \mathbb{P}c^\perp$ of $R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}$ with eigenvalue $\alpha_1\alpha_2$. But $\mathbb{P}c^\perp = \mathrm{L}(p_1,p_2)$ and, by the previous lemma, this cannot happen since the line $\mathrm{L}(p_1,p_2)$ is noneuclidean.

We conclude that $\langle p_1, p_2\rangle = 0$. The $R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}$-eigenvalues of p_1,p_2,c are respectively $\alpha_1^{-2}\alpha_2, \alpha_1\alpha_2^{-2}, \alpha_1\alpha_2$. Two of these eigenvalues have to be equal and this leads to $\alpha_1^3 = \alpha_2^3$.

The converse is immediate. □

Corollary 21. *Let $\delta \in \mathbb{C}$ be such that $\delta^3 = 1$ and let $\alpha_1, \alpha_2 \in \mathbb{S}^1$ be angles with $\alpha_i^3 \neq 1$. Assume that $p_1,p_2,p_3 \in \mathbb{P}_{\mathbb{C}}V \setminus \mathrm{S}V$ are pairwise distinct and do not lie in a same Euclidean line. Then $R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1} = \delta$ if and only if p_1,p_2,p_3 are pairwise orthogonal and*

$$\alpha_1^{-2}\alpha_2\alpha_3 = \alpha_1\alpha_2^{-2}\alpha_3 = \alpha_1\alpha_2\alpha_3^{-2} = \delta.$$

Proof. By the previous proposition, we know that the points p_1,p_2,p_3 must be pairwise orthogonal. As in the proof of Proposition 18, we write $R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}$ in the orthogonal basis p_1,p_2,p_3, obtaining

$$\begin{bmatrix} \alpha_1^{-2}\alpha_2\alpha_3 & 0 & 0 \\ 0 & \alpha_1\alpha_2^{-2}\alpha_3 & 0 \\ 0 & 0 & \alpha_1\alpha_2\alpha_3^{-2} \end{bmatrix}.$$

Thus, $R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1} = \delta$ means $\alpha_1^{-2}\alpha_2\alpha_3 = \alpha_1\alpha_2^{-2}\alpha_3 = \alpha_1\alpha_2\alpha_3^{-2} = \delta$. □

Suppose that $R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1} = \delta$. Note that $\delta = 1$ if and only if $\alpha_1 = \alpha_2 = \alpha_3$. If $\delta \neq 1$, then two of the angles α_i are equal (say, equal to α) and the other angle is $\delta\alpha$. These relations are called **orthogonal relations**.

From the proof of the previous proposition we also obtain the following result:

Corollary 22. *Two special elliptic isometries $R_{\alpha_1}^{p_1}$ and $R_{\alpha_2}^{p_2}$ commute if and only if their product is special elliptic.*

CHAPTER

4

BENDINGS AND f-BENDINGS

In Section 3.3 we obtained necessary and sufficient conditions for the product of two special elliptic isometries to be a special elliptic isometry. As a consequence, we characterized all length 3 relations between special elliptic isometries.

In this chapter we discuss length 4 relations between special elliptic isometries. We obtain length 4 relations by understanding when $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} = R^{q_2}_{\beta_2} R^{q_1}_{\beta_1}$. In other words, we look for different ways to write an isometry (that is already a product of two special elliptic isometries) as a product of two special elliptic isometries.

We focus our attention on two cases. The first is the case where the angles are equal, i.e., the relations are of the form $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} = R^{q_2}_{\alpha_2} R^{q_1}_{\alpha_1}$. The second is when the product of the angles is equal, i.e., $\alpha_1\alpha_2 = \beta_1\beta_2$.

In both cases, we will frequently consider that $\sigma p_1 = \sigma q_1$ and $\sigma p_2 = \sigma q_2$. With this condition, the relations in the first case are called *bendings*. For relations belonging to the second case, we define *angle components* (see page 48) and require that the angles α_i and β_i lie in the same components. We call *f-bendings* the relations that satisfy these conditions.

We describe bendings/f-bendings and prove that they are the only length 4 relations preserving the corresponding mentioned invariants.

In the end of the chapter we briefly discuss some other possible solutions (for the sake of curiosity).

4.1 Bendings

We start by obtaining conditions over points $p_i, q_i \in \mathbb{P}V$ and angles $\alpha_i, \beta_i \in \mathbb{S}^1$ such that $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} = \delta R^{q_2}_{\beta_2} R^{q_1}_{\beta_1}$, where $\delta^3 = 1$.

Let $p_1,p_2,q_1,q_2 \in \mathbb{P}_{\mathbb{C}}V \setminus \mathrm{S}V$ and $\alpha_1,\alpha_2,\beta_1,\beta_2 \in \mathbb{S}^1$ be such that $p_1 \neq p_2$, $q_1 \neq q_2$, and $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1} = \delta R^{q_2}_{\beta_2}R^{q_1}_{\beta_1}$, where $\delta^3 = 1$. Note that, if $\mathrm{L} := \mathrm{L}(p_1,p_2) = \mathrm{L}(q_1,q_2)$ (i.e., the points p_1,p_2,q_1,q_2 lie in a same complex line), then $\alpha_1\alpha_2 = \delta\beta_1\beta_2$. Indeed, since both $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ and $\delta R^{q_2}_{\beta_2}R^{q_1}_{\beta_1}$ stabilize the line L, they must have the same value on the polar point c of L.

The following lemma shows that the converse also holds.

Lemma 23. *Let $p_1,p_2,q_1,q_2 \in \mathbb{P}_{\mathbb{C}}V \setminus \mathrm{S}V$ be such that the p_i's (as well as the q_i's) are distinct and nonorthogonal. Assume that*

$$R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1} = \delta R^{q_2}_{\beta_2}R^{q_1}_{\beta_1} \tag{4.1}$$

for some $\alpha_i,\beta_j \in \mathbb{S}^1$ with $\alpha_i^3 \neq 1$, $\beta_j^3 \neq 1$, $i,j = 1,2$, and $\delta \in \mathbb{C}$ with $\delta^3 = 1$. Then

- *$\alpha_1\alpha_2 = \delta\beta_1\beta_2$ implies $\mathrm{L}(p_1,p_2) = \mathrm{L}(q_1,q_2)$;*
- *$\alpha_1\alpha_2 \neq \delta\beta_1\beta_2$ implies that the lines $\mathrm{L}(p_1,p_2)$ and $\mathrm{L}(q_1,q_2)$ are orthogonal (in the sense of Remark 2).*

Remark 24. In particular, if p_1,p_2,q_1,q_2 satisfy $I := R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1} = \delta R^{q_2}_{\beta_2}R^{q_1}_{\beta_1}$ with $\alpha_1\alpha_2 \neq \delta\beta_1\beta_2$, $\alpha_i^3 \neq 1$, and $\beta_i^3 \neq 1$, then I is a regular elliptic isometry.

Proof. Assume that the lines $\mathrm{L}(p_1,p_2)$ and $\mathrm{L}(q_1,q_2)$ are distinct and let c and d be, respectively, their polar points. If $\alpha_1\alpha_2 = \delta\beta_1\beta_2$, then the line $\mathrm{L}(c,d)$ is pointwise fixed by $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ with eigenvalue $\alpha_1\alpha_2$. So, $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ is special elliptic and, by Proposition 20, $p_1 = p_2$ or $\langle p_1,p_2\rangle = 0$. This contradiction shows us that $\alpha_1\alpha_2 = \delta\beta_1\beta_2$, implies $\mathrm{L}(p_1,p_2) = \mathrm{L}(q_1,q_2)$.

Now, let us prove that, if the lines $\mathrm{L}(p_1,p_2)$ and $\mathrm{L}(q_1,q_2)$ are nonorthogonal, i.e., $d \notin \mathrm{L}(p_1,p_2)$ (or, equivalently, $c \notin \mathrm{L}(q_1,q_2)$), then $\alpha_1\alpha_2 = \delta\beta_1\beta_2$. By Lemma 11,

$$\delta\beta_1\beta_2 d = R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}d = \alpha_1\alpha_2 d + \beta(1)\frac{\langle d,p_1\rangle}{\langle p_1,p_1\rangle}p_1 + \beta(2)\frac{\langle d,p_2\rangle}{\langle p_2,p_2\rangle}p_2 + \beta(1,2)\frac{\langle d,p_1\rangle\,\langle p_1,p_2\rangle}{\langle p_1,p_1\rangle\,\langle p_2,p_2\rangle}p_2.$$

Since $d \notin \mathrm{L}(p_1,p_2)$, the points p_1,p_2,d are linearly independent, and the β coefficients do not vanish, we have $\langle d,p_1\rangle = \langle d,p_2\rangle = 0$ and $\alpha_1\alpha_2 = \delta\beta_1\beta_2$. □

There is a more geometrical approach for the last part of this proof: if $\alpha_1\alpha_2 \neq \delta\beta_1\beta_2$, then obviously the lines $\mathrm{L}(p_1,p_2) = \mathbb{P}c^\perp$ and $\mathrm{L}(q_1,q_2) = \mathbb{P}d^\perp$ are distinct. This imples that the isometry $I := R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1} = \delta R^{q_2}_{\beta_2}R^{q_1}_{\beta_1}$ has three distinct stable complex lines, $\mathbb{P}c^\perp$, $\mathbb{P}d^\perp$ and $\mathbb{P}e^\perp$, where e is the polar point of $\mathrm{L}(c,d)$. By the classification of isometries, c, d, and e are pairwise orthogonal and, hence, the three stable lines are pairwise orthogonal.

We will now focus our attention in the case of *nonorthogonal relations* of length 4 (meaning that we are in the case $\alpha_1\alpha_2 = \delta\beta_1\beta_2$) and, more particularly, in the case $\alpha_1 = \beta_1, \alpha_2 = \beta_2$. In the last section of this chapter we briefly discuss the case $\alpha_1\alpha_2 \neq \delta\beta_1\beta_2$.

Lemma 25. *Let R^p_α be a special elliptic isometry and let $I \in \mathrm{SU}(2,1)$ be an isometry. Then $IR^p_\alpha I^{-1} = R^{Ip}_\alpha$.*

Proof. Let $x \in \mathbb{P}_{\mathbb{C}}V$. Then

$$R^p_\alpha(I^{-1}x) = (\alpha^{-2}-\alpha)\frac{\left\langle I^{-1}x,p\right\rangle}{\langle p,p\rangle}p + \alpha I^{-1}x,$$

(for the formula of a special elliptic isometry, see Section 2.5). Since I is a unitary map, $\left\langle I^{-1}x,p\right\rangle = \langle x, Ip\rangle$ and

$$I(R^p_\alpha I^{-1}x) = (\alpha^{-2}-\alpha)\frac{\langle x,Ip\rangle}{\langle p,p\rangle}Ip + \alpha x = R^{Ip}_\alpha x,$$

as desired. □

It follows from the above lemma that, if C is an isometry in the centralizer of a product $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$, then

$$R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1} = (R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1})^C = (CR^{p_2}_{\alpha_2}C^{-1})(CR^{p_1}_{\alpha_1}C^{-1}) = R^{Cp_2}_{\alpha_2}R^{Cp_1}_{\alpha_1}.$$

In the next theorem we describe all relations that can be obtained by this method.

Theorem 26. *Let $p_1,p_2 \in \mathbb{P}_{\mathbb{C}}V \setminus \mathrm{S}V$ be distinct and nonorthogonal and suppose that $\mathrm{L}(p_1,p_2)$ is noneuclidean. There exists a one-parameter subgroup $B : \mathbb{R} \to \mathrm{SU}(2,1)$ such that $B(s)$ commutes with $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ and $R^{B(s)p_2}_{\alpha_2}R^{B(s)p_1}_{\alpha_1} = R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ for every $s \in \mathbb{R}$. Furthermore, for every isometry C in the centralizer of $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$, there exists $s \in \mathbb{R}$ such that $Cp_i = B(s)p_i$, $i = 1,2$.*

Definition 27. A relation $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1} = R^{q_2}_{\alpha_2}R^{q_1}_{\alpha_1}$ with $\sigma q_i = \sigma p_i$, $i = 1,2$, is called a **bending** relation.

Proof of Theorem 26. Two isometries in $\mathrm{SU}(2,1)$ commute if and only if they have the same stable sets (see (CAO; GONGOPADHYAY, 2011)). Denote by c the polar point of the line $\mathrm{L}(p_1,p_2)$.

Assume that $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ is regular elliptic; then, $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ fixes two points $p,q \in \mathrm{L}(p_1,p_2)$ as well as the point c. An isometry $I \in \mathrm{SU}(2,1)$ commutes with $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ if and only if I is an elliptic isometry that fixes c, p, and q. It follows that the centralizer $C(R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1})$ of $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ has real dimension 2. In fact, we can write any isometry $I \in C(R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1})$ in the basis c,p,q as

$$I = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_1^{-1}\mu_2^{-1} \end{bmatrix}$$

with $|\mu_i| = 1$. So, any isometry in $C(R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1})$ is determined by the choice of two unitary complex numbers.

Let

$$I_1 = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_1^{-1}\mu_2^{-1} \end{bmatrix} \quad \text{and} \quad I_2 = \begin{bmatrix} \mu_1' & 0 & 0 \\ 0 & \mu_2' & 0 \\ 0 & 0 & \mu_1'^{-1}\mu_2'^{-1} \end{bmatrix}$$

be elements of $C(R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1})$ written in the basis c,p,q. The actions of I_1 and I_2 on an arbitrary point different from p in the line $\mathbb{P}c^{\perp}$ are given by $I_1 : p + \lambda q \mapsto \mu_2 p + \lambda\mu_1^{-1}\mu_2^{-1} q \cong p + \lambda\mu_1^{-1}\mu_2^{-2} q$ and $I_2 : p + \lambda q \mapsto \mu_2' p + \lambda\mu_1'^{-1}\mu_2'^{-1} q \cong p + \lambda\mu_1'^{-1}\mu_2'^{-2} q$, where $\simeq$ means $\mathbb{C}$-proportionality. Thus, if these isometries satisfy $\mu_1\mu_2^2 = \mu_1'\mu_2'^2$, we have that $I_1 p_i = I_2 p_i$, $i = 1,2$, and by Lemma 25 $I_1 R^{p_i}_{\alpha_i} I_1^{-1} = I_2 R^{p_i}_{\alpha_i} I_2^{-1}$, $i = 1,2$. Then, if we consider the one-parameter subgroup $B : \mathbb{R} \to \mathrm{SU}(2,1)$ given in terms of the ordered basis c,p,q by

$$B(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{s}{2}i} & 0 \\ 0 & 0 & e^{-\frac{s}{2}i} \end{bmatrix}$$

we obtain the desired one-parameter subgroup of $\mathrm{SU}(2,1)$. Such subgroup satisfies that $B(\pi)p_i$ is the antipodal point of p_i in the metric circle with center p (or q, depending on the signal σp_i).

Now, suppose that $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ is hyperbolic. Then, it fixes two isotropic points $v_1, v_2 \in \mathbb{P}c^{\perp}$. An isometry $I \in \mathrm{SU}(2,1)$ commutes with $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ if and only if, in the basis c, v_1, v_2,

$$I = \begin{bmatrix} \overline{\mu}\mu^{-1} & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \overline{\mu}^{-1} \end{bmatrix}$$

for some $\mu \in \mathbb{C}$. If

$$I_1 = \begin{bmatrix} \overline{\mu}_1\mu_1^{-1} & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \overline{\mu}_1^{-1} \end{bmatrix} \quad \text{and} \quad I_2 = \begin{bmatrix} \overline{\mu}_2\mu_2^{-1} & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \overline{\mu}_2^{-1} \end{bmatrix}$$

are isometries in $C(R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1})$ written in the basis c, v_1, v_2, the actions of such isometries on a point in $\mathbb{P}c^{\perp}$ different from v_1 are given by

$$I_j : v_1 + \lambda v_2 \mapsto \mu_j v_1 + \lambda\overline{\mu}_j^{-1} v_2 \cong v_1 + \lambda\mu_j^{-1}\overline{\mu}_j^{-1} v_2.$$

Thus, if I_1 and I_2 satisfy $\mu_1\overline{\mu}_1 = \mu_2\overline{\mu}_2$, then $I_1 R^{p_i}_{\alpha_i} I_1^{-1} = I_2 R^{p_i}_{\alpha_i} I_2^{-1}$, $i = 1,2$. Then, we consider the one-parameter subgroup given, in the basis c, v_1, v_2, by

$$B(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{s} & 0 \\ 0 & 0 & e^{-s} \end{bmatrix}.$$

It remains to consider the parabolic case. Since, by hypothesis, the line $\mathrm{L}(p_1,p_2)$ is noneuclidean, it must be hyperbolic. In this case, $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ fixes an isotropic point v_1 in the line $\mathrm{L}(p_1,p_2)$ as well as the polar point c of $\mathrm{L}(p_1,p_2)$ (c is a positive point). Therefore, an isometry $I\in\mathrm{SU}(2,1)$ is in the centralizer of $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ if and only if it can be written in the basis v_1,v_2,c, where v_2 is a point in $\mathbb{P}c^{\perp}$, $v_2\neq v_1$, as

$$I=\begin{bmatrix}\mu & \xi & 0\\ 0 & \mu & 0\\ 0 & 0 & \overline{\mu}^{-2}\end{bmatrix}.$$

Choosing representatives for v_1 and v_2 such that $\langle v_1,v_2\rangle=1/2$ and $\langle v_2,v_2\rangle=\sigma_2$, we note that $\mathrm{Re}(\xi\mu^{-1})=0$ because

$$\sigma_2=\langle v_2,v_2\rangle=\langle Iv_2,Iv_2\rangle=\langle \xi v_1+\mu v_2,\xi v_1+\mu v_2\rangle=\sigma_2+\mathrm{Re}(\xi\mu^{-1}).$$

So, $\xi=it\mu$, where $t\in\mathbb{R}$. Given isometries

$$I_1=\begin{bmatrix}\mu_1 & it_1\mu_1 & 0\\ 0 & \mu_1 & 0\\ 0 & 0 & \overline{\mu_1}^{-2}\end{bmatrix}\quad\text{and}\quad I_2=\begin{bmatrix}\mu_2 & it_2\mu_2 & 0\\ 0 & \mu_2 & 0\\ 0 & 0 & \overline{\mu_2}^{-2}\end{bmatrix}$$

written in the basis v_1,v_2,c with $|\mu_i|=1$ and $t_i\in\mathbb{R}$, we have

$$I_j:v_1+\tau v_2\mapsto \mu_j v_1+\tau\left(it_j\mu_j v_1+\mu_j v_2\right)\cong v_1+\frac{\tau}{1+it_j\tau}v_2.$$

Therefore, if $t_1=t_2$, then $I_1R^{p_i}_{\alpha_i}I_1^{-1}=I_2R^{p_i}_{\alpha_i}I_2^{-1}$, $i=1,2$. So, we can consider the one-parameter subgroup

$$B(s)=\begin{bmatrix}1 & is & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},$$

also written in the basis v_1,v_2,c.

The last statement of the theorem follows by construction. □

After Section 4.3 we will be able to prove that Theorem 26 gives us all bending relations:

Proposition 28. *Given a bending relation* $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}=R^{q_2}_{\alpha_2}R^{q_1}_{\alpha_1}$*, there exists* $s\in\mathbb{R}$ *such that* $B(s)p_1=q_1$ *and* $B(s)p_2=q_2$*.*

In this way, we call $B(s)$ a *bending involving* p_1 *and* p_2. We also say that the points q_1 and q_2 are obtained by bending p_1 and p_2 (with respect to the isometry $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$) if there exists $s\in\mathbb{R}$ such that $B(s)p_i=q_i$.

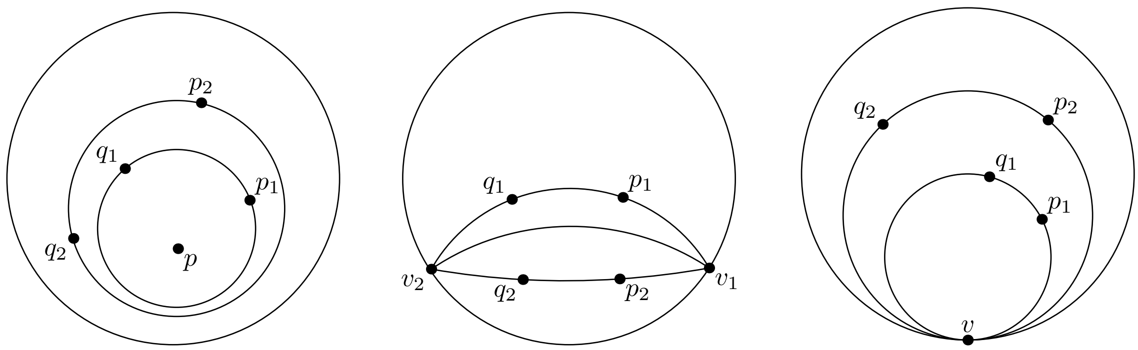
4.2 Geometrical description of bendings

Suppose that p_1 and p_2 lie in a complex geodesic $\mathrm{L} := \mathrm{L}(p_1, p_2) \cap \mathbb{B}V$. Let us consider the cases where $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ is (a) regular elliptic; (b) hyperbolic; or (c) parabolic. The bendings $B(s)$ of $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ given by Theorem 26 change the points p_1 and p_2 as follows: p_1 and p_2 move keeping the distance between them constant, along

(a) their respective metric circles centered at p, the point in L fixed by $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ (in other words, p_1 and p_2 rotate around p by the same angle);

(b) their respective hypercycles, corresponding to the geodesic G in L stabilized by $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$;

(c) their respective horocycles, centered at the isotropic point in L fixed by $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$.



Conversely, if points q_1 and q_2 are obtained by moving p_1 and p_2 as described in (a), (b), or (c), we have that the points obtained satisfy $R^{q_2}_{\alpha_2} R^{q_1}_{\alpha_1} = R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$.

It could happen that the metric circle, hypercycle, or horocycle (as discussed above) that contains p_1 also contains p_2. In fact, if we consider $\alpha_1 = \alpha_2 = -1$, then $R^{p_2}_{-1} R^{p_1}_{-1}$ is hyperbolic and stabilize the geodesic $\mathrm{G} := \mathrm{G}\wr p_1, p_2 \wr$ which is the hypercycle of points with distance 0 to G.

Proposition 29. *Points p_1 and p_2 with the same signature are in a same orbit of bendings of $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ if and only if $\alpha_1 = \delta\alpha_2$ for some $\delta \in \mathbb{C}$ with $\delta^3 = 1$. In this case, we can assume that the bendings $B(s)$ of $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ satisfy $B(1)p_1 = p_2$.*

To prove this proposition, we will need some of the considerations that are made in the next section.

4.3 f-bendings

One could ask if Theorem 26 gives us all length 4 relations. As we will see, this is indeed true if we take into account some invariants concerning angles and signs of points (in general, the fact does not hold). Lemma 23 provides some hints on what could be the length 4 relations that are not bendings. For instance, we could have a relation where $\alpha_1\alpha_2 = \beta_1\beta_2$ or where $\alpha_1\alpha_2 \neq \beta_1\beta_2$ but $\alpha_1 \neq \beta_1$ and $\alpha_2 \neq \beta_2$ (not to mention that we could also have relations where $\sigma p_i \neq \sigma q_i$).

In this section we obtain length 4 relations of the form $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} = R^{p'_2}_{\alpha'_2} R^{p'_1}_{\alpha'_1}$ with $\alpha_1\alpha_2 = \alpha'_1\alpha'_2$ and $\sigma p_i = \sigma p'_i$ (we also require that the angles α_i and α'_i are in the *same component,* see Definition 31). They will be constructed as follows: given a product $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$, we find a (one-parametric) family of products $R^{p'_2}_{\alpha'_2} R^{p'_1}_{\alpha'_1}$ that act like $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ on the line $\mathrm{L}(p_1, p_2)$ and on the polar point c of such line.

The next lemma shows that this does not quite guarantee that the obtained products are equal to $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ in $\mathrm{PU}(2,1)$ but, as we will see in Theorem 34, the construction will work.

Lemma 30. *Let $F, G \in \mathrm{SU}(2,1)$ be isometries that stabilize a noneuclidean line $\mathrm{L} = \mathbb{P}c^\perp$. If $F|_\mathrm{L} = G|_\mathrm{L}$ and $Fc = Gc$, then $F = G$ or $F = R^c_{-1} G$.*

Proof. Suppose that F and G are regular elliptic with eigenvalues λ_i and μ_i, respectively, and with $\mu_1 = \lambda_1$. Let $\eta \in \mathbb{C}$ be such that $\eta\lambda_2 = \mu_2$. Since $F, G \in \mathrm{SU}(2,1)$, we have $\mu_3 = \overline{\eta}\lambda_3$. However, by hypothesis, $\lambda_2\overline{\lambda_3} = \mu_2\overline{\mu_3}$, and it follows that $\eta^2 = 1$ which concludes the proof in this case.

If F and G are hyperbolic, we can write

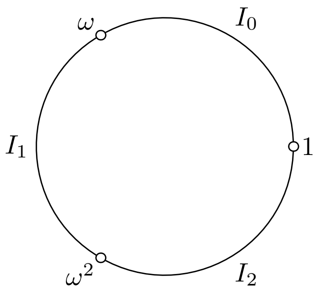
$$F = \begin{bmatrix} \lambda^{-1}\overline{\lambda} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \overline{\lambda}^{-1} \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} \mu^{-1}\overline{\mu} & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \overline{\mu}^{-1} \end{bmatrix},$$

where $|\lambda| \neq 1$, $|\mu| \neq 1$ and $\lambda^{-1}\overline{\lambda}$ and $\mu^{-1}\overline{\mu}$ are, respectively, the eigenvalues related to the eigenvector c. Let $\eta \in \mathbb{C}$ be such that $\mu = \eta\lambda$. By hypothesis, we have $\lambda^{-1}\overline{\lambda} = \mu^{-1}\overline{\mu}$, implying $\eta^{-1} = \overline{\eta}$, that is, $|\eta| = 1$. On the other hand, since $F, G \in \mathrm{SU}(2,1)$, we have $\det F = \det G = 1$ which implies $\lambda^{-1}\overline{\lambda}\eta\lambda\overline{\eta}^{-1}\overline{\lambda}^{-1} = \eta\overline{\eta}^{-1} = 1$. Hence, $\eta \in \mathbb{R}$ and $\eta = \pm 1$.

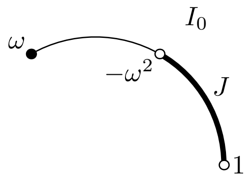
In the case where F and G are parabolic, we write (see the proof of Theorem 26)

$$F = \begin{bmatrix} \lambda^{-2} & 0 & 0 \\ 0 & \lambda & is\lambda \\ 0 & 0 & \lambda \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} \mu^{-2} & 0 & 0 \\ 0 & \mu & it\mu \\ 0 & 0 & \mu \end{bmatrix},$$

where $|\lambda| = 1$, $|\mu| = 1$, $s, t \in \mathbb{R}$, and λ^{-2} and μ^{-2} are, respectively, the eigenvalues related to the eigenvector c. As before, $\lambda^{-2} = \mu^{-2}$ implies $\mu = \pm\lambda$. Let $r \in \mathbb{R}$ be such that $s = rt$. Since the above matrices are written in a basis c, v_1, v_2, where $v_1, v_2 \in \mathrm{L}$ and v_1 is the isotropic fixed point of both F and G, the action of F on the complex line L is given by $v_1 + \tau v_2 \mapsto v_1 + \left(\frac{\tau}{1+is\tau}\right) v_2$. By hypothesis, F and G acts in the same way on the line L; so, $r = 1$. □



In this section we will denote by ω the specific cubic root of the unity $\omega := \exp(2\pi i/3)$. Consider the open arcs I_0, I_1, and I_2 in $\mathbb{S}^1 \subset \mathbb{C}$ delimited by the cubic roots of unity 1, ω and ω^2, i.e., I_i is the open arc delimited by ω^i and ω^{i+1} (the indices are modulo 3) and $I_i := \omega^i I_0$. We will also consider the open arc $J \subset I_0$ limited by the points 1 and $-\omega^2$. Note that $J^2 = I_0$ and $J^6 = I_0^3 = \mathbb{S}^1 \setminus \{1\}$.



Definition 31. For $\alpha, \beta \in \mathbb{S}^1 \setminus \{1, \omega, \omega^2\}$, we will write $\alpha \sim \beta$ to denote that α and β lie in the same arc I_j. In this case, we say that α and β are in the **same component**.

Being on a same component is an equivalence relation that is invariant under the product of powers of ω and complex conjugation, i.e., if $\alpha \sim \beta$, then $\omega^k \alpha \sim \omega^k \beta$ for every $k \in \mathbb{Z}$ and $\overline{\alpha} \sim \overline{\beta}$.

Given a product $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ with distinct and nonorthogonal p_1, p_2 and with $\alpha_i^3 \neq 1$, $i = 1, 2$, let $a_i \in J$ be such that $(a_i^2)^3 = \alpha_i^3$ (such a_i exists due to $J^6 = \mathbb{S}^1 \setminus \{1\}$). Since $a_i \in J$, we have $a_1 a_2 \in I_0$ and $0 < \operatorname{Arg} a_i < \pi/3$, implying that $0 < \operatorname{Arg} a_i^3 < \pi$. So, it makes sense to write that $\operatorname{Arg} a_1^3 + \operatorname{Arg} a_2^3 = \operatorname{Arg}(a_1 a_2)^3 > \operatorname{Arg} a_i^3$, $i = 1, 2$ (note that $a_1 a_2 \in I_0 \implies 0 < \operatorname{Arg} a_1 a_2 < 2\pi/3 \implies 0 < \operatorname{Arg}(a_1 a_2)^3 < 2\pi$).

Let G stand for the geodesic $\mathrm{G}\wr p_1, p_2 \wr$. Consider the geodesic G_1 through p_1 with $\angle_{p_1} \mathrm{G}_1 \mathrm{G} = \operatorname{Arg} a_1^3$ and the geodesic G_2 through p_2 with $\angle_{p_2} \mathrm{G} \mathrm{G}_2 = \operatorname{Arg} a_2^3$. Denote by r_i the reflection in the geodesic G_i, $i = 1, 2$, and by r the reflection in the geodesic G. Then $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ acts on the line L as the product of reflections $r_2 r_1$. (Indeed, the action of $R^{p_1}_{\alpha_1}$ on the line L equals $r r_1$ and that of $R^{p_2}_{\alpha_2}$ equals $r_2 r$; so, their product acts on L as $r_2 r r r_1 = r_2 r_1$.)

Now, given points $p_i' \in \mathrm{G}_i$ with $\sigma p_i' = \sigma p_i$, we denote by G' the geodesic $\mathrm{G}\wr p_1', p_2' \wr$ and write $\angle_{p_1'} \mathrm{G}_1 \mathrm{G}' = \operatorname{Arg} a_1'^3$ and $\angle_{p_2'} \mathrm{G}' \mathrm{G}_2 = \operatorname{Arg} a_2'^3$ with $a_1', a_2' \in J$. Taking $\alpha_i' \in \mathbb{S}^1$ such that $(a_i'^2)^3 = \alpha_i'^3$, the product $R^{p_2'}_{\alpha_2'} R^{p_1'}_{\alpha_1'}$ also acts on L as $r_2 r_1$.

Definition 32. A relation $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}=R^{p'_2}_{\alpha'_2}R^{p'_1}_{\alpha'_1}$ with $p'_i\in \mathrm{G}_i$, $\sigma p'_i=\sigma p_i$ and $\alpha_i\sim\alpha'_i$, $i=1,2$, is called an **f-bending** relation. In the case when $F:=R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ is regular elliptic, there are extra requirements: $p_i\in\mathrm{G}_i$, $\sigma p_i=\sigma p'_i$ and, if $q_i\in\mathrm{L}(p_1,p_2)$ stand for the F-fixed points with $\sigma q_i=\sigma p_i$, then q_i is not between p_i and p'_i, $i=1,2$.

These extra requirements in the regular elliptic case are intended to deal with the following situation. Assume that $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ is regular elliptic. A relation $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}=R^{p'_2}_{\alpha'_2}R^{p'_1}_{\alpha'_1}$ with q_i between p_i and p'_i satisfying $p'_i\in\mathrm{G}_i$ and $\sigma p_i=\sigma p'_i$ is a composition of a bending and an f-bending. Indeed, by bending $R^{p'_2}_{\alpha'_2}R^{p'_1}_{\alpha'_1}$ by $B(\pi)$ (more precisely, conjugating this product by a regular elliptic isometry that acts on the line $\mathrm{L}(p_1,p_2)$ as a rotation by π around q_i, see Section 4.2), we arrive at a configuration where q_i does not lie between p_i and p'_i, $i=1,2$. Now, $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}=R^{p'_2}_{\alpha'_2}R^{p'_1}_{\alpha'_1}$ is an f-bending relation.

In order to characterize f-bendings in Theorem 34, we need the following technical lemma.

Lemma 33. *Let $\alpha_i,\beta_i\in\mathbb{S}^1\setminus\{1,\omega,\omega^2\}$ with $\alpha_i\sim\beta_i$ and let $a_i,b_i\in J$ be such that $(a_i^2)^3=\alpha_i^3$ and $(b_i^2)^3=\beta_i^3$, $i=1,2$. Then $a_1a_2=b_1b_2$ if and only if $\alpha_1\alpha_2=\beta_1\beta_2$.*

Proof. Suppose that $a_1a_2=b_1b_2$. Let $\widetilde\alpha_i,\widetilde\beta_i\in I_0$ be such that $\widetilde\alpha_i^3=\alpha_i^3$ and $\widetilde\beta_i^3=\beta_i^3$. Then $\widetilde\alpha_i=a_i^2$ and $\widetilde\beta_i=b_i^2$ and, by hypothesis, $\widetilde\alpha_1\widetilde\alpha_2=\widetilde\beta_1\widetilde\beta_2$. Take $k_1,k_2\in\{0,1,2\}$ such that $\alpha_i=\omega^{k_i}\widetilde\alpha_i$ and $\beta_i=\omega^{k_i}\widetilde\beta_i$ (here we are using the fact that $\alpha_i\sim\beta_i$ to obtain the k_i's). Thus

$$\alpha_1\alpha_2=\omega^{k_1}\widetilde\alpha_1\omega^{k_2}\widetilde\alpha_2=\omega^{k_1}\omega^{k_2}\widetilde\alpha_1\widetilde\alpha_2=\omega^{k_1}\omega^{k_2}\widetilde\beta_1\widetilde\beta_2=\omega^{k_1}\widetilde\beta_1\omega^{k_2}\widetilde\beta_2=\beta_1\beta_2.$$

Conversely, assume that $\alpha_1\alpha_2=\beta_1\beta_2$. Let $\widetilde\alpha_i$, $\widetilde\beta_i$, and k_1,k_2 be as above. We have

$$(a_1a_2)^2=\widetilde\alpha_1\widetilde\alpha_2=\omega^{-k_1}\alpha_1\omega^{-k_2}\alpha_2=\omega^{-k_1}\beta_1\omega^{-k_2}\beta_2=\widetilde\beta_1\widetilde\beta_2=(b_1b_2)^2.$$

Since a_1a_2 and b_1b_2 lie in I_0, we conclude that $a_1a_2=b_1b_2$. ☐

Theorem 34. *Given a product $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$, let $a_i\in J$ be such that $(a_i^2)^3=\alpha_i^3$. Take points $p'_1\in\mathrm{G}_1,p'_2\in\mathrm{G}_2$ with $\sigma p'_i=\sigma p_i$ and angles $\alpha'_1,\alpha'_2\in\mathbb{S}^1$ with $\alpha_i\sim\alpha'_i$. Then $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}=R^{p'_2}_{\alpha'_2}R^{p'_1}_{\alpha'_1}$ if and only if $a_1a_2=a'_1a'_2$, where the $a'_i\in J$ are such that $\angle_{p'_1}\mathrm{G}_1\mathrm{G}'=\operatorname{Arg}a_1'^3$ and $\angle_{p'_2}\mathrm{G}'\mathrm{G}_2=\operatorname{Arg}a_2'^3$.*

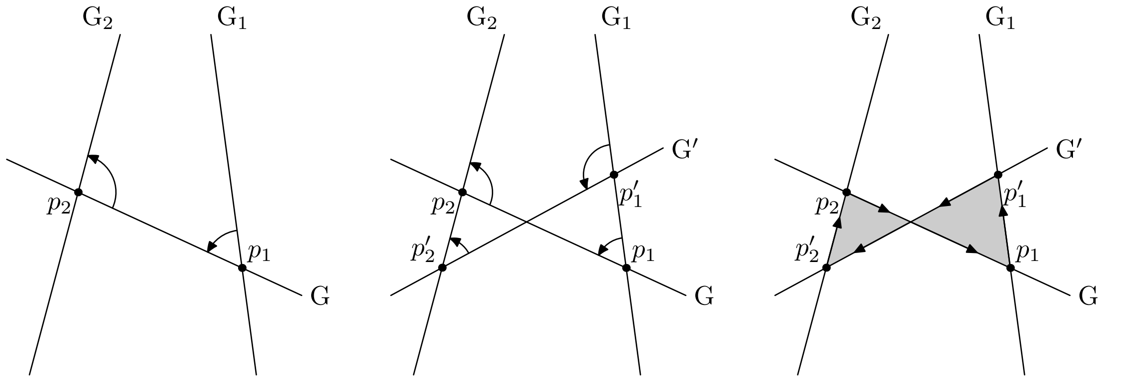
Proof. We already know that, if $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}=R^{p'_2}_{\alpha'_2}R^{p'_1}_{\alpha'_1}$, then $a_1a_2=a'_1a'_2$. On the other hand, by the construction in page 48, the actions of $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ and $R^{p'_2}_{\alpha'_2}R^{p'_1}_{\alpha'_1}$ over the complex line $\mathrm{L}:=\mathrm{L}(p_1,p_2)$ coincide. Furthermore, since $a_1a_2=a'_1a'_2$ and $\alpha_i\sim\alpha'_i$, by Lemma 33, we have $\alpha_1\alpha_2=\alpha'_1\alpha'_2$. Thus, if c is the polar point of L, we have that $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}c=R^{p'_2}_{\alpha'_2}R^{p'_1}_{\alpha'_1}c$. Lemma 30 shows that this is not enough to prove the proposition.

However, note that, starting with $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$, we can continuously make p_1 and p_2 move, respectively, over the geodesics G_1 and G_2, obtaining new angles $\alpha'_1\sim\alpha_1$

and $\alpha'_2 \sim \alpha_2$ (that vary continuously with the change of p_i's) and a product $R^{p'_2}_{\alpha'_2} R^{p'_1}_{\alpha'_1}$ that satisfies the properties listed in the previous paragraph. Therefore, by continuity, $R^{p'_2}_{\alpha'_2} R^{p'_1}_{\alpha'_1} = R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$. $\square$

Remark 35. When $\sigma p_1 = \sigma p_2$, we have the following geometric interpretation for f-bendings: nonisotropic points $p'_1 \in \mathrm{G}_1$ and $p'_2 \in \mathrm{G}_2$, and angles $\alpha'_1, \alpha'_2 \in \mathbb{S}^1$ with $\alpha_i \sim \alpha'_i$, satisfy $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} = R^{p'_2}_{\alpha'_2} R^{p'_1}_{\alpha'_1}$ if and only if the following conditions holds:

- $\operatorname{Area}(p_1 p'_1 p'_2 p_2 p_1) = 0$;
- Let $a'_1, a'_2 \in J$ be such that $\angle_{p'_1} \mathrm{G}_1 \mathrm{G}' = \operatorname{Arg} a'^3_1$ and $\angle_{p'_2} \mathrm{G}' \mathrm{G}_2 = \operatorname{Arg} a'^3_2$, where $\mathrm{G}' := \mathrm{G} \wr p'_1, p'_2 \wr$. Then $(a'^2_i)^3 = \alpha'^3_i$.



In fact, given $p'_i \in \mathrm{G}_i$, let G' stand for the geodesic $\mathrm{G} \wr p'_1, p'_2 \wr$. Let $a_i \in J$ be such that $(a^2_i)^3 = \alpha^3_i$ and write the angles $\angle_{p'_1} \mathrm{G}_1 \mathrm{G}' = \operatorname{Arg} a'^3_1$ and $\angle_{p'_2} \mathrm{G}' \mathrm{G}_2 = \operatorname{Arg} a'^3_2$ with $a'_i \in J$. Then

$$\operatorname{Area}(p_1 p'_1 p'_2 p_2 p_1) = \pm(\operatorname{Arg}(a'_1 a'_2)^3 - \operatorname{Arg}(a_1 a_2)^3)$$

and $\operatorname{Area}(p_1 p'_1 p'_2 p_2 p_1) = 0$ if and only if $a'_1 a'_2 = a_1 a_2$. Now, if $\alpha'_i \in I_{j_i}$ is such that $(a'^2_i)^3 = \alpha'^3_i$, then $a'_1 a'_2 = a_1 a_2$ is equivalent to $\alpha'_1 \alpha'_2 = \alpha_1 \alpha_2$.

When $\sigma p_1 \neq \sigma p_2$, such geometric interpretation makes no sense. In fact, in this case, we will consider the figure $p_1 p'_1 \overline{p'_2} \overline{p_2} p_1$. We have that the points p'_i and the angles α'_i satisfy $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} = R^{p'_2}_{\alpha'_2} R^{p'_1}_{\alpha'_1}$ if and only if $\operatorname{Area}(p_1 p'_1 \overline{p'_2} \overline{p_2} p_1) = \pm 2 \operatorname{Arg} b^3$, where $b \in J$ is such that $a'_1 = b a_1$ (if $\operatorname{Arg} a'_1 > \operatorname{Arg} a_1$) or, respectively, $a_1 = b a'_1$ (if $\operatorname{Arg} a'_1 < \operatorname{Arg} a_1$).

Theorem 34 gives a criterion to decide whether $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} = R^{p'_2}_{\alpha'_2} R^{p'_1}_{\alpha'_1}$ when p_i, p'_i are of the same signature and α_i, α'_i are in a same component. This naturally leads to the following question: requiring the necessary condition $\alpha_1 \alpha_2 = \alpha'_1 \alpha'_2$, does the equation $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} = R^{p'_2}_{\alpha'_2} R^{p'_1}_{\alpha'_1}$ actually have a solution in p'_1, p'_2 with $\sigma p_i = \sigma p'_i$ and $\alpha_i \sim \alpha'_i$? The answer is affirmative when $\sigma p_1 = \sigma p_2$:

Proposition 36. *Given $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ with $\sigma p_1 = \sigma p_2$ and a pair $\alpha'_1, \alpha'_2 \in \mathbb{S}^1$ with $\alpha_1 \alpha_2 = \alpha'_1 \alpha'_2$ and $\alpha_i \sim \alpha'_i$, there exists an f-bending relation $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} = R^{p'_2}_{\alpha'_2} R^{p'_1}_{\alpha'_1}$.*

Proof. Let $a_1, a_2 \in J$ be the unique angles satisfying $(a_i^2)^3 = \alpha_i^3$. Consider the following cases: (1) $\mathrm{Arg}(a_1a_2)^3 \leq \pi$ and (2) $\mathrm{Arg}(a_1a_2)^3 > \pi$, or, equivalently, (1) a_1a_2 lie in the arc $(1, -\omega^2]$ and (2) a_1a_2 lie in the arc $(-\omega^2, \omega)$ (if $a_1a_2 = \omega$, then $a_1 = a_2 = -\omega^2$). We will use Remark 35 to prove that, in the first case, given any $a_1' \in (0, a_1a_2)$ (respectively, $a_2' \in (0, a_1a_2)$), there exists an f-bending of $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ such that $\alpha_1'^3 = (a_1'^2)^3$ (respectively, $\alpha_2'^3 = (a_2'^2)^3$). In the second case, note that $-a_1a_2 \in I_2$ and, therefore, $\omega(-a_1a_2) \in I_1$ (actually, it is easy to see that $\omega(-a_1a_2) \in J$). So, in this case, we will show that, given $a_1' \in (-\omega a_1a_2, -\omega^2)$ (respectively, $a_2' \in (-\omega a_1a_2, -\omega^2)$), there exists an f-bending of $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ such that $\alpha_1'^3 = (a_1'^2)^3$ (respectively, $\alpha_2'^3 = (a_2'^2)^3$).

Assume that the first case holds. The geometric interpretation in Remark 35 implies that we can make p_1' (respectively, p_2') tend to the absolute in the case where the line $\mathrm{L}(p_1, p_2)$ is hyperbolic or to the point $\overline{p_1}$ (respectively, $\overline{p_2}$) in the case where $\mathrm{L}(p_1, p_2)$ is spherical, where $\overline{p_1}$ (respectively, $\overline{p_2}$) is the point in $\mathrm{L}(p_1, p_2)$ orthogonal to p_1 (respectively, p_2). This makes $\mathrm{Arg}\, a_1'^3$ (respectively, $\mathrm{Arg}\, a_2'^3$) tend to 0 and $\mathrm{Arg}\, a_2'^3$ (respectively, $\mathrm{Arg}\, a_1'^3$) tend to $\mathrm{Arg}(a_1a_2)^3$. So, a_1' (respectively, a_2') tend to 1 and a_2' (respectively, a_1') tend to a_1a_2.

In the second case, the same procedure makes $\mathrm{Arg}\, a_1'^3$ (respectively, $\mathrm{Arg}\, a_2'^3$) tend to π and $\mathrm{Arg}\, a_2'^3$ (respectively, $\mathrm{Arg}\, a_1'^3$) tend to $\mathrm{Arg}(a_1a_2)^3 - \pi = \mathrm{Arg}(-a_1a_2)^3 = \mathrm{Arg}(-\omega a_1a_2)^3$ (note that (2) implies that $\mathrm{Arg}(a_1a_2)^3 > \pi$) and, therefore, a_1' (respectively, a_2') tend to $-\omega^2$ and a_2' (respectively, a_1') tend to $-\omega a_1a_2$.

It remains to prove that, given $\alpha_i' \sim \alpha_i$ satisfying $\alpha_1'\alpha_2' = \alpha_1\alpha_2$, then a_i' lies in the arc $(1, a_1a_2)$ in the first case or in the arc $(-\omega\alpha_1\alpha_2, -\omega^2)$ in the second case. If we have (1) and $\alpha_1' \in (a_1a_2, -\omega^2)$, then $a_1'a_2' \neq a_1a_2$; by Lemma 33, this contradicts $\alpha_1'\alpha_2' = \alpha_1\alpha_2$. If we have (2) and $a_1' \in (-\omega a_1a_2, -\omega^2)$, then $a_1'a_2' \in (1, a_1a_2)$ which implies that $a_1a_2 \neq a_1'a_2'$. Again, a contradiction. $\square$

If $\sigma p_1 \neq \sigma p_2$, an analogous argument shows that the above proposition is true if $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ is a hyperbolic isometry (see Proposition 37 below). But it is not true in the other cases: if $\sigma p_1 \neq \sigma p_2$ and $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ is not hyperbolic, then one of the p_i's cannot be made as close as desired to the absolute, and we cannot make $\mathrm{ta}(p_1, p_2)$ arbitrarily big.

Proposition 37. *Given $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$, with $\sigma p_1 \neq \sigma p_2$, and a pair $\beta_1, \beta_2 \in \mathbb{S}^1$ with $\beta_1\beta_2 = \alpha_1\alpha_2$ and $\alpha_i, \beta_i \in I_{i_j}$, if $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ is hyperbolic, then there exists an f-bending relation $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1} = R^{q_2}_{\beta_2}R^{q_1}_{\beta_1}$.*

Simultaneously changing the components of corresponding angles does not alter the nature of a bending/f-bending relation:

Remark 38. If $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1} = R^{p_2'}_{\alpha_2}R^{p_1'}_{\alpha_1}$ (respectively, $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1} = R^{p_2'}_{\alpha_2'}R^{p_1'}_{\alpha_1'}$) is a bending (respectively, an f-bending) relation, then

$$R^{p_2}_{\alpha_2}R^{p_1}_{\delta\alpha_1}=R^{p'_2}_{\alpha_2}R^{p'_1}_{\delta\alpha_1} \quad\text{and}\quad R^{p_2}_{\delta\alpha_2}R^{p_1}_{\alpha_1}=R^{p'_2}_{\delta\alpha_2}R^{p'_1}_{\alpha_1},$$

$$\left(\text{resp.}\quad R^{p_2}_{\alpha_2}R^{p_1}_{\delta\alpha_1}=R^{p'_2}_{\alpha'_2}R^{p'_1}_{\delta\alpha'_1} \quad\text{and}\quad R^{p_2}_{\delta\alpha_2}R^{p_1}_{\alpha_1}=R^{p'_2}_{\delta\alpha'_2}R^{p'_1}_{\alpha'_1}\right)$$

are bending (resp. f-bending) relations for any cubic root of the unity δ.

Theorem 39. *All length 4 relations $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}=R^{q_2}_{\beta_2}R^{q_1}_{\beta_1}$ with $\sigma p_i=\sigma q_i$ and $\alpha_1\alpha_2=\beta_1\beta_2$ and $\alpha_i\sim\beta_i$ follow from bending and f-bending relations.*

Proof. We consider the construction of page 48 for both products $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ and $R^{q_2}_{\beta_2}R^{q_1}_{\beta_1}$ obtaining geodesics G_i and G'_i through p_i and q_i, $i=1,2$. Either G_1 and G_2 intersect at a point p (possibly isotropic), or they do not intersect and we can consider a geodesic H that is orthogonal to both G_i's. Since $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}=R^{q_2}_{\beta_2}R^{q_1}_{\beta_1}$, the geodesics G'_i also intersect at p or H is orthogonal to the geodesics G'_i, respectively. From Section 4.2, we have that by bending one of these products we can make $\mathrm{G}_i=\mathrm{G}'_i$, $i=1,2$, arriving at a f-bending relation. □

4.4 Proofs of Propositions 28 and 29

Proof of Proposition 28. Consider the construction of page 48 for both products $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ and $R^{q_2}_{\alpha_2}R^{q_1}_{\alpha_1}$. We obtain, respectively, geodesics G, G_1, G_2 through p_1 and p_2, and H, H_1, H_2 through q_1 and q_2.

Suppose that $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}=R^{q_2}_{\alpha_2}R^{q_1}_{\alpha_1}$ is regular elliptic. In this case, G_1 and G_2 intersect in a nonisotropic point, and H_1 and H_2 also intersect in the same nonisotropic point. Moreover, the angle of both intersections is the same. Then, we can rotate q_1 and q_2 around this intersection point obtaining points q'_1 and q'_2 that lie, respectively, in the geodesic G_1 and G_2. Since, by hypothesis, $\angle_{q_1}\mathrm{H}_1\mathrm{H}=\angle_{p_1}\mathrm{G}_1\mathrm{G}$ and $\angle_{q_2}\mathrm{H}\mathrm{H}_2=\angle_{p_2}\mathrm{G}\mathrm{G}_2$, we have $q'_1=p_1$ and $q'_2=p_2$.

By the considerations made in Section 4.2, we conclude that $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}=R^{q_2}_{\alpha_2}R^{q_1}_{\alpha_1}$ is a bending relation. The proof is analogous in the other cases. □

Proof of Proposition 29. Consider the geodesics G, G_1, and G_2 as in the construction in page 48. Suppose that $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ is regular elliptic and let p be the fixed point of this isometry, that is, the intersection between G_1 and G_2 with $\sigma p=\sigma p_i$. Then, $\operatorname{dist}(p_1,p)=\operatorname{dist}(p_2,p)$ if and only if $\angle_{p_1}\mathrm{G}_1\mathrm{G}=\operatorname{Arg}a_1^3=\operatorname{Arg}a_2^3=\angle_{p_2}\mathrm{G}\mathrm{G}_2$ (again, we follow the notation in page 48). If follows that p_1 and p_2 are in the same metric circles centered in p if and only if there exists $\delta\in\mathbb{C}$ with $\delta^3=1$ such that $\alpha_1=\delta\alpha_2$.

When $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ is parabolic or hyperbolic (in particular, the complex line through p_1 and p_2 is hyperbolic), the proof is analogous. If G_1 and G_2 intersect at an isotropic point v, then p_1 and p_2 lie in the same horocycle centered at v if and only if $\alpha_1 = \delta\alpha_2$. If G_1 and G_2 do not intersect, then p_1 and p_2 lie in the same hypercycle of the geodesic orthogonal to both G_1 and G_2 if and only if $\alpha_1 = \delta\alpha_2$. $\square$

4.5 Other length 4 relations

Given a product $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$, note that

$$R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} = R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} R^{p_2}_{\bar{\alpha}_2} R^{p_2}_{\alpha_2} = R^{R^{p_2}_{\alpha_2} p_1}_{\alpha_1} R^{p_2}_{\alpha_2},$$

where we are using a cancellation (see Section 3.2) in the first equality and the last equality follows from Lemma 25. If $\sigma p_1 = \sigma p_2$ and $\alpha_1 \sim \alpha_2$, the relation $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} = R^{R^{p_2}_{\alpha_2} p_1}_{\alpha_1} R^{p_2}_{\alpha_2}$ is a composition of a bending and an f-bending. However, if $\sigma p_1 \neq \sigma p_2$ or $\alpha_1 \not\sim \alpha_2$, such a relation, being written in the form $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} = R^{q_2}_{\beta_2} R^{q_1}_{\beta_1}$, satisfies $\sigma p_i \neq \sigma q_i$ or $\alpha_i \not\sim \beta_i$. Hence, in this case, it is not a composition of bendings and f-bendings.

Proposition 40. *The relation $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} = R^{R^{p_2}_{\alpha_2} p_1}_{\alpha_1} R^{p_2}_{\alpha_2}$ is not a bending nor an f-bending relation. Moreover, when $\sigma p_1 \neq \sigma p_2$ or $\alpha_1 \not\sim \alpha_2$, this relation cannot be obtained by compositions of bendings and f-bendings of $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$.*

A relation of the form $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} = R^{R^{p_2}_{\alpha_2} p_1}_{\alpha_1} R^{p_2}_{\alpha_2}$ with $\sigma p_1 \neq \sigma p_2$ or $\alpha_1 \not\sim \alpha_2$ is called a **change of orientation**.

Another way of obtaining relations that are not bendings or f-bendings is to note that, if $\delta^3 = 1$ and $\delta \neq 1$, then $R^{p_2}_{\delta\alpha_2} R^{p_1}_{\alpha_1} = R^{p_2}_{\alpha_2} R^{p_1}_{\delta\alpha_1} = R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$. Since $\alpha_i \not\sim \delta\alpha_i$ for $\delta \neq 1$, these relations are neither bendings nor f-bendings. We call them **change of components**.

The last possible length 4 relation is hinted by Lemma 23. These are the relations $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} = R^{q_2}_{\beta_2} R^{q_1}_{\beta_1}$ with $\alpha_1\alpha_2 \neq \beta_1\beta_2$, i.e., the lines $\mathrm{L}(p_1, p_2)$ and $\mathrm{L}(q_1, q_2)$ are orthogonal. We will call such relations **change to orthogonal line**.

Theorem 41. *Let $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} = R^{q_2}_{\beta_2} R^{q_1}_{\beta_1}$ be a length 4 relation. Assume that $\alpha_i^3 \neq 1$ and $\beta_i^3 \neq 1$. If $\alpha_1\alpha_2 = \beta_1\beta_2$, we also require that the collection of signatures of the points p_1, p_2 and q_1, q_2 are the same. Then the relation is a composition of bendings, f-bendings, change of orientation, change of components, and change to orthogonal lines.*

Proof. If $\alpha_1\alpha_2 \neq \beta_1\beta_2$, then the relation is a change to orthogonal line. If $\alpha_1\alpha_2 = \beta_1\beta_2$, we can use a change of components to ensure that $\alpha_i \sim \beta_i$; then, possibly applying a change of orientation, we make $\sigma p_i = \sigma q_i$. Now the theorem follows from Theorem 39. $\square$

CHAPTER

5

CONNECTING N-GONS

In chapter 4, we saw that bendings are a continuous one-parametric way of changing points p_1 and p_2 preserving the product $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$. Such a change, unlike f-bendings, preserves the geometry of the pair p_1, p_2 (that is, preserves the tance between these points). Now, given a product $R^{p_3}_{\alpha_3}R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ with p_2 distinct of p_1, p_3 and nonorthogonal to p_1, p_3, bending p_1, p_2 (or p_2, p_3) we preserve the product $R^{p_3}_{\alpha_3}R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ but the geometric configuration of the triple p_1, p_2, p_3 changes. In fact, after a bending involving p_1, p_2, for instance, the tance between p_2 and p_3 may alter.

In this chapter, we study these changes.

5.1 Modifying n-gons

A **special elliptic n-gon** is a configuration of n points $p_1, \dots, p_n \in \mathbb{P}_{\mathbb{C}}V \setminus \mathrm{S}V$ along with n angles $\alpha_1, \dots, \alpha_n \in \mathbb{S}^1$ such that

(SN1) at most one point p_i is positive;

(SN2) p_i is not equal nor orthogonal to p_{i+1} for $i = 1, \dots, n-1$

(SN3) $\alpha_i^3 \neq 1$ for $i = 1, \dots, n$;

(SN4) $R^{p_n}_{\alpha_n} \dots R^{p_1}_{\alpha_1} = \delta$ for some $\delta \in \mathbb{C}$ with $\delta^3 = 1$;

(SN5) $\Pi\alpha_i \neq \delta$.

We will sometimes say that a relation $R^{p_n}_{\alpha_n} \dots R^{p_1}_{\alpha_1} = \delta$ is a special elliptic n-gon (or just an n-gon) if the points $p_1, \dots, p_n$ and angles $\alpha_1, \dots, \alpha_n$ satisfy the items above.

Given an n-gon $R^{p_n}_{\alpha_n} \dots R^{p_1}_{\alpha_1} = \delta$, we can use the length 4 relations, bending and f-bending relations that were obtained in Chapter 4 to modify it into a different relation.

In fact, if $2 \leq i \leq n$, we have

$$R_{\alpha_n}^{p_n} \dots \left(R_{\alpha_i}^{p_i} R_{\alpha_{i-1}}^{p_{i-1}}\right)^{B(s)} \dots R_{\alpha_1}^{p_1} = \delta,$$

where B is given by Theorem 26, i.e, $\left(R_{\alpha_i}^{p_i} R_{\alpha_{i-1}}^{p_{i-1}}\right)^{B(s)} = R_{\alpha_i}^{B(s)p_i} R_{\alpha_{i-1}}^{B(s)p_{i-1}} = R_{\alpha_i}^{p_i} R_{\alpha_{i-1}}^{p_{i-1}}$ is a bending relation.

It could happen that, after some of these bendings, we arrive at a relation that is not an n-gon (it does not satisfy (SN2)). However, if after finitely many such modifications, we arrive at an n-gon, then such n-gon has the same angles and same signatures of points (geometrically, it may be a different n-gon).

We can also modify an n-gon using f-bendings, arriving at

$$R_{\alpha_n}^{p_n} \dots R_{\alpha_i'}^{p_i'} R_{\alpha_{i-1}'}^{p_{i-1}'} \dots R_{\alpha_1}^{p_1} = \delta,$$

where $R_{\alpha_i'}^{p_i'} R_{\alpha_{i-1}'}^{p_{i-1}'} = R_{\alpha_i}^{p_i} R_{\alpha_{i-1}}^{p_{i-1}}$ is an f-bending relation. In this case, if we arrive at an n-gon, this n-gon does not have the same angles (assuming that the f-bending is not the trivial one) but it has the same product of angles, same components of angles (see page 48) and same sign of points. This is also true if we consider the n-gon modified by the composition of finitely many bendings **and** f-bendings.

Proposition 42. *Suppose that the special elliptic n-gon $R_{\beta_n}^{q_n} \dots R_{\beta_1}^{q_1} = \delta$ is obtained by modifying $R_{\alpha_n}^{p_n} \dots R_{\alpha_1}^{p_1} = \delta$ by the composition of finitely many bendings (resp. bendings and f-bendings). Then, $\beta_i = \alpha_i$ and $\sigma q_i = \sigma p_i$ (resp. $\Pi\beta_i = \Pi\alpha_i$, $\beta_i \sim \alpha_i$, and $\sigma q_i = \sigma q_i$).*

We are interested in the following problem: given an n-gon, can we obtain, by bending (resp. bending and f-bending) such n-gon, every other n-gon with the same invariants (as in the above proposition)?

5.2 Composition of bendings

Let $F \in \mathrm{SU}(2,1)$ be an isometry and assume that we have a decomposition $F := R_{\alpha_3}^{p_3} R_{\alpha_2}^{p_2} R_{\alpha_1}^{p_1}$ of F into the product of special elliptic isometries, where $p_1, p_2, p_3 \in \mathrm{B}V$. If p_2 is distinct from and nonorthogonal to p_1 and p_3, we can modify the triple p_1, p_2, p_3 by composition of bendings involving p_1, p_2 and p_2, p_3, obtaining a new decomposition for the isometry F. In this section, we determine all such decompositions for fixed angles $\alpha_1, \alpha_2, \alpha_3$.

In order to do this, we will determine all triples $p_1, p_2, p_3 \in \mathrm{B}V$ such that $R_{\alpha_3}^{p_3} R_{\alpha_2}^{p_2} R_{\alpha_1}^{p_1}$ has the same trace. Then, we will prove that we can connect any two such triples by means of compositions of bendings. Since we want the trace to determine an isometry (up to conjugacy), we need to make some assumptions on $R_{\alpha_3}^{p_3} R_{\alpha_2}^{p_2} R_{\alpha_1}^{p_1}$.

Definition 43 (See Definition 3.6 in (ANAN'IN, 2012))**.** An isometry $F \in \mathrm{SU}(2,1)$ is **regular** if, for each $\lambda \in \mathbb{C}$, we have

$$\dim_{\mathbb{C}}\{v \in V : Fv = \lambda v\} \leq 1.$$

In other words, an isometry in $\mathrm{SU}(2,1)$ is regular if it is neither special elliptic nor unipotent with a fixed Euclidean line.

If $p_1, p_2 \in \mathbb{P}_{\mathbb{C}}V \setminus \mathrm{S}V$ are distinct, nonorthogonal, and $\mathrm{ta}(p_1,p_2) \neq 1$, then $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ is regular. Indeed, by Proposition 20, the isometry $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ cannot be special elliptic. Now, suppose that the line $\mathrm{L}(p_1,p_2)$ is hyperbolic and $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ is parabolic, with isotropic fixed point $v \in \mathrm{L}(p_1,p_2)$. By Lemma 19, the eigenvalue of v is not $\alpha_1\alpha_2$ and then the isometry $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ has two distinct eigenvalues (that of the isotropic point v and that of the polar point of $\mathrm{L}(p_1,p_2)$) with distinct eigenvectors, and then, it cannot be an unipotent isometry. It follows that $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ is regular. We arrive at the following proposition.

Proposition 44. *Let $p_1, p_2 \in \mathbb{P}_{\mathbb{C}}V \setminus \mathrm{S}V$ be distinct nonorthogonal points. Let $\alpha_1, \alpha_2 \in \mathbb{S}^1$ be angles satisfying $\alpha_i^3 \neq 1$, $i = 1,2$. Then $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ is a regular isometry. More precisely, it is either regular elliptic, hyperbolic, or ellipto-parabolic.*

If p_1, p_2, p_3 are points in the same complex geodesic $\mathrm{L} \cap \mathrm{B}V$, it could happen that, after finitely many bendings involving p_1, p_2 and p_2, p_3, we arrive at a situation where $p_1 = p_2$ or $p_2 = p_3$. In this way, we focus our attention on triples that do not lie in the same complex line. It could also happen that, for given angles $\alpha_1, \alpha_2, \alpha_3$, there exists a triple p_1, p_2, p_3 that does not lie in the same complex line, but such that $R^{p_3}_{\alpha_3}R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ is special elliptic (see Section 4.5).

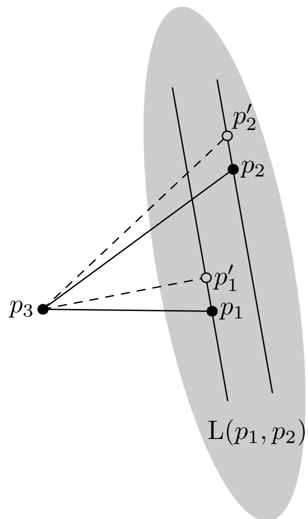
Definition 45. Given angles $\alpha_i \in \mathbb{S}^1$ with $\alpha_i^3 \neq 1$, $i = 1,2,3$, we say that a triple $p_1, p_2, p_3 \in \mathbb{P}_{\mathbb{C}}V \setminus \mathrm{S}V$ is **strongly regular** if at most one point p_i is positive, the points p_i do not lie in the same complex line and $R^{p_3}_{\alpha_3}R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ is a regular isometry. In this case, we will sometimes say that $R^{p_3}_{\alpha_3}R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ is a strongly regular triple.

In the previous definition, we require that at most one of the points p_1, p_2, p_3 is positive because, otherwise, it could happen that, after some suitable bendings involving p_1, p_2 and p_2, p_3, we arrive at the situation where one of the lines $\mathrm{L}(p_1,p_2)$ or $\mathrm{L}(p_2,p_3)$ is Euclidean.

Remark 46 (Existence of strongly regular triples)*.* Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{S}^1$ be angles with $\alpha_i^3 \neq 1$. Consider a triple $p_1, p_2, p_3 \in \mathbb{P}_{\mathbb{C}}V \setminus \mathrm{S}V$ whose points do not lie in a same complex line and such that at most one point is positive. By Lemma 23, if c is the polar point of $\mathrm{L}(p_1,p_2)$ and $\mathrm{L}(p_3,c)$ is not orthogonal to $\mathrm{L}(p_1,p_2)$ (that is, the polar point of $\mathrm{L}(p_3,c)$ does not line on $\mathrm{L}(p_1,p_2)$), then $R^{p_3}_{\alpha_3}R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ is a regular isometry.

Remark 47 (Regularity of triples depends on the angles)*.* It may happen that a triple $p_1, p_2, p_3 \in \mathbb{P}_{\mathbb{C}}V \setminus \mathrm{S}V$ is not strongly regular for given angles $\alpha_1, \alpha_2, \alpha_3$ but it becomes strongly regular when we change the angles. In fact, starting with a relation $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} = R^{q_2}_{\beta_2} R^{q_1}_{\beta_1}$ such that $\alpha_1\alpha_2 \neq \beta_1\beta_2$ and the points p_1 and p_2 are negative (see Lemma 23), the triple p_1, p_2, q_2 has at most one positive point and these points do not lie in the same complex line. The triple p_1, p_2, q_2 is not strongly regular with respect to the angles $\alpha_1, \alpha_2, \overline{\beta}_2$. But it is easy to see that this triple is strongly regular with respect to the angles $-1, -1, -1$, i.e., the product of reflections $R^{p_2}_{-1} R^{p_1}_{-1} R^{q_2}_{-1}$ is regular.

Proposition 48. *Composition of bendings (or bendings and f-bendings) preserve strong regularity of triples. More precisely, after the composition of finitely many bendings, a strongly regular triple p_1, p_2, p_3 will still be strongly regular with respect to the angles $\alpha_1, \alpha_2, \alpha_3$. If the change is made by compositions of bendings and f-bendings, the obtained new triple will be strongly regular with respect to the new angles $\alpha_1', \alpha_2', \alpha_3'$.*



Proof. Suppose that we have a strongly regular triple $R^{p_3}_{\alpha_3} R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$. Bending p_1, p_2 keeps the complex line through p_1 and p_2 invariant and, by hypothesis, p_3 does no lie in such line. Moreover, this bending does not change the isometry $F := R^{p_3}_{\alpha_3} R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ and, therefore, the triple remains strongly regular after a bending involving p_1, p_2. By symmetry, the same is true for bendings involving p_2, p_3. An analogous argument proves the result since the same is true for f-bendings of p_1, p_2. $\square$

Our objective is to obtain all strongly regular triples (geometrically, i.e., all geometric configurations of three points that correspond to strongly regular triples) with respect to fixed angles $\alpha_1,\alpha_2,\alpha_3$, and with fixed $\operatorname{tr} R^{p_3}_{\alpha_3} R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$. By Proposition 12,

$$\begin{aligned}\frac{\operatorname{tr} R^{p_3}_{\alpha_3} R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} - \alpha_1^{-2}\alpha_2\alpha_3 - \alpha_1\alpha_2^{-2}\alpha_3 - \alpha_1\alpha_2\alpha_3^{-2}}{(\alpha_1^{-2}-\alpha_1)(\alpha_2^{-2}-\alpha_2)(\alpha_3^{-2}-\alpha_3)} &= \\ = \frac{\alpha_3}{\alpha_3^{-2}-\alpha_3}\operatorname{ta}(p_1,p_2) + \frac{\alpha_1}{\alpha_1^{-2}-\alpha_1}\operatorname{ta}(p_2,p_3) + \frac{\alpha_2}{\alpha_2^{-2}-\alpha_2}\operatorname{ta}(p_1,p_3) + \frac{g_{12}g_{23}g_{31}}{g_{11}g_{22}g_{33}}.\end{aligned}$$

where $[g_{ij}]$ is the Gram matrix of p_1,p_2,p_3.

To continue this analysis, we will consider the pair of equations that we obtain by taking the real and imaginary parts of the above equation. The next proposition contains a simple property about unitary complex numbers that we will help us in this process.

Proposition 49. *Let $\alpha\in\mathbb{C}$ be a unitary complex number, $|\alpha|=1$. Then*

$$\operatorname{Re}\frac{\alpha}{\alpha^{-2}-\alpha} = -\frac{1}{2} \quad \textit{and} \quad \operatorname{Im}\frac{\alpha}{\alpha^{-2}-\alpha} = \frac{1}{2}\cot\frac{3\operatorname{Arg}\alpha}{2}.$$

Proof. Write $\alpha=\cos\theta+i\sin\theta$. Then

$$\frac{\alpha}{\alpha^{-2}-\alpha} = \frac{\alpha^3-1}{|\alpha^{-2}-\alpha|} = \frac{\cos 3\theta + i\sin 3\theta - 1}{2(1-\cos 3\theta)}.$$

Thus,

$$\operatorname{Re}\frac{\alpha}{\alpha^{-2}-\alpha} = \operatorname{Re}\frac{\cos 3\theta - 1}{2(1-\cos 3\theta)} = -\frac{1}{2}$$

and

$$\operatorname{Im}\frac{\alpha}{\alpha^{-2}-\alpha} = \operatorname{Im}\frac{i\sin 3\theta}{2(1-\cos 3\theta)} = \frac{1}{2}\cot\frac{3\theta}{2},$$

as desired. □

Let $[g_{ij}]$ stand for the Gram matrix of the triple p_1,p_2,p_3. We have

$$\begin{aligned}\frac{\det[g_{ij}]}{g_{11}g_{22}g_{33}} =& \frac{1}{g_{11}g_{22}g_{33}}\left(g_{11}g_{22}g_{33} + 2\operatorname{Re}(g_{12}g_{23}g_{31}) - g_{13}g_{31}g_{22} - g_{12}g_{21}g_{33} - g_{23}g_{32}g_{11}\right) \\ =& 1 + 2\operatorname{Re}\left(\frac{g_{12}g_{23}g_{31}}{g_{11}g_{22}g_{33}}\right) - \operatorname{ta}(p_1,p_2) - \operatorname{ta}(p_2,p_3) - \operatorname{ta}(p_3,p_1).\end{aligned}$$

Let $\alpha_1,\alpha_2,\alpha_3$ be angles with $\alpha_i^3\neq 1$. If $p_1,p_2,p_3\in \mathrm{B}V$ is strongly regular with respect to $\alpha_1,\alpha_2,\alpha_3$, then the isometry $R^{p_3}_{\alpha_3} R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ satisfies

$$\operatorname{Re}\left(\frac{\operatorname{tr} R^{p_3}_{\alpha_3} R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} - \alpha_1^{-2}\alpha_2\alpha_3 - \alpha_1\alpha_2^{-2}\alpha_3 - \alpha_1\alpha_2\alpha_3^{-2}}{(\alpha_1^{-2}-\alpha_1)(\alpha_2^{-2}-\alpha_2)(\alpha_3^{-2}-\alpha_3)}\right) = \frac{1}{2}(\beta-1)$$

where

$$\beta := \frac{\det[g_{ij}]}{g_{11}g_{22}g_{33}}.$$

Indeed, Proposition 49 implies that

$$\begin{aligned}
\operatorname{Re}\left(\frac{\operatorname{tr} R^{p_3}_{\alpha_3}R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}-\alpha_1^{-2}\alpha_2\alpha_3-\alpha_1\alpha_2^{-2}\alpha_3-\alpha_1\alpha_2\alpha_3^{-2}}{(\alpha_1^{-2}-\alpha_1)(\alpha_2^{-2}-\alpha_2)(\alpha_3^{-2}-\alpha_3)}\right) &= \\
&= -\frac{1}{2}\left(\operatorname{ta}(p_1,p_2)+\operatorname{ta}(p_2,p_3)+\operatorname{ta}(p_1,p_3)\right)+\operatorname{Re}\eta \\
&= -\frac{1}{2}(1+2\operatorname{Re}\eta-\beta)+\operatorname{Re}\eta \\
&= \frac{1}{2}(\beta-1)
\end{aligned}$$

where $\eta := \frac{g_{12}g_{23}g_{31}}{g_{11}g_{22}g_{33}}$. Since $p_1,p_2,p_3 \in \mathrm{B}V$ and these points do not lie in a same complex line (the triple is strongly regular), we have $\beta > 0$ by Sylvester's Criterion. Furthermore, if $t_1 := \operatorname{ta}(p_1,p_2)$, $t_2 := \operatorname{ta}(p_2,p_3)$, and $t_3 := \operatorname{ta}(p_1,p_3)$, we have

$$\begin{aligned}
&\operatorname{Im}\left(\frac{\operatorname{tr} R^{p_3}_{\alpha_3}R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}-\alpha_1^{-2}\alpha_2\alpha_3-\alpha_1\alpha_2^{-2}\alpha_3-\alpha_1\alpha_2\alpha_3^{-2}}{(\alpha_1^{-2}-\alpha_1)(\alpha_2^{-2}-\alpha_2)(\alpha_3^{-2}-\alpha_3)}\right) = \\
&= \operatorname{Im}\left(\frac{\alpha_3}{\alpha_3^{-2}-\alpha_3}\right)t_1+\operatorname{Im}\left(\frac{\alpha_1}{\alpha_1^{-2}-\alpha_1}\right)t_2+\operatorname{Im}\left(\frac{\alpha_2}{\alpha_2^{-2}-\alpha_2}\right)t_3+\operatorname{Im}\eta \\
&= \operatorname{Im}\left(\frac{\alpha_3}{\alpha_3^{-2}-\alpha_3}\right)t_1+\operatorname{Im}\left(\frac{\alpha_1}{\alpha_1^{-2}-\alpha_1}\right)t_2 \\
&\quad +\operatorname{Im}\left(\frac{\alpha_2}{\alpha_2^{-2}-\alpha_2}\right)(1-\beta+2\operatorname{Re}\eta-t_1-t_2)+\operatorname{Im}\eta \\
&= \left(\operatorname{Im}\left(\frac{\alpha_3}{\alpha_3^{-2}-\alpha_3}\right)-\operatorname{Im}\left(\frac{\alpha_2}{\alpha_2^{-2}-\alpha_2}\right)\right)t_1+\left(\operatorname{Im}\left(\frac{\alpha_1}{\alpha_1^{-2}-\alpha_1}\right)-\operatorname{Im}\left(\frac{\alpha_2}{\alpha_2^{-2}-\alpha_2}\right)\right)t_2 \\
&\quad +2\operatorname{Im}\left(\frac{\alpha_2}{\alpha_2^{-2}-\alpha_2}\right)\operatorname{Re}\eta+\operatorname{Im}\left(\frac{\alpha_2}{\alpha_2^{-2}-\alpha_2}\right)(1-\beta)+\operatorname{Im}\eta.
\end{aligned}$$

Therefore,

$$\operatorname{Im}\eta = \operatorname{Im}\tau+(\chi_2-\chi_3)t_1+(\chi_2-\chi_1)t_2-2\chi_2\operatorname{Re}\eta+2\chi_2\operatorname{Re}\tau,$$

where

$$\tau := \frac{\operatorname{tr} R^{p_3}_{\alpha_3}R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}-\alpha_1^{-2}\alpha_2\alpha_3-\alpha_1\alpha_2^{-2}\alpha_3-\alpha_1\alpha_2\alpha_3^{-2}}{(\alpha_1^{-2}-\alpha_1)(\alpha_2^{-2}-\alpha_2)(\alpha_3^{-2}-\alpha_3)}$$

and

$$\chi_i := \operatorname{Im}\left(\frac{\alpha_i}{\alpha_i^{-2}-\alpha_i}\right). \tag{5.1}$$

So,

$$|\eta|^2 = c_1(\operatorname{Re}\eta)^2+c_2\operatorname{Re}\eta+c_3t_1\operatorname{Re}\eta+c_4t_2\operatorname{Re}\eta+c_5t_1^2+c_6t_2^2+c_7t_1t_2+c_8t_1+c_9t_2+c_{10},$$

where

$$
\begin{aligned}
c_1 &:= 1+4\chi_2^2\\
c_2 &:= -8\chi_2^2\operatorname{Re}\tau-4\chi_2\operatorname{Im}\tau\\
c_3 &:= 4\left(\chi_2\chi_3-\chi_2^2\right)\\
c_4 &:= 4\left(\chi_1\chi_2-\chi_2^2\right)\\
c_5 &:= \chi_3^2-2\chi_2\chi_3+\chi_2^2\\
c_6 &:= \chi_1^2-2\chi_1\chi_2+\chi_2^2\\
c_7 &:= 2\left(\chi_1\chi_3-\chi_2\chi_3-\chi_1\chi_2+\chi_2^2\right)\\
c_8 &:= 4\left(\chi_2^2-\chi_2\chi_3\right)\operatorname{Re}\tau+2\left(\chi_2-\chi_3\right)\operatorname{Im}\tau\\
c_9 &:= 4\left(\chi_2^2-\chi_1\chi_2\right)\operatorname{Re}\tau+2\left(\chi_2-\chi_1\right)\operatorname{Im}\tau\\
c_{10} &:= 4\chi_2^2(\operatorname{Re}\tau)^2+4\chi_2\operatorname{Re}\tau\operatorname{Im}\tau+(\operatorname{Im}\tau)^2
\end{aligned}
\tag{5.2}
$$

In this way, given angles α_i with $\alpha_i^3\neq 1$ and given a conjugacy class of $R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}$, a strongly regular triple $p_1,p_2,p_3\in\mathrm{B}V$ (with respect to the angles α_i) is such that $R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}$ is in the fixed conjugacy class if $t_1=\operatorname{ta}(p_1,p_2)$, $t_2=\operatorname{ta}(p_2,p_3)$, and $t=\operatorname{Re}\eta$ satisfy the equality

$$
\begin{aligned}
\beta-1=2t-t_1-t_2&\\
&-\frac{c_1t^2+c_2t+c_3t_1t+c_4t_2t+c_5t_1^2+c_6t_2^2+c_7t_1t_2+c_8t_1+c_9t_2+c_{10}}{t_1t_2}
\end{aligned}
\tag{5.3}
$$

and the inequalities

$$
t_1>1,\quad t_2>1,\quad \text{and}\quad t>1. \tag{5.4}
$$

Conversely, assume that (t_1,t_2,t) satisfies (5.3) and (5.4). We take $g_{ii}=-1$, $g_{12}=g_{21}=\sqrt{t_1}$, and $g_{23}=g_{32}=\sqrt{t_2}$. Hence,

$$
g_{13}=-\frac{\overline{\eta}}{\sqrt{t_1}\sqrt{t_2}}=-\frac{t-i\left[\operatorname{Im}\tau+(\chi_2-\chi_3)t_1+(\chi_2-\chi_1)t_2-2\chi_2t+2\chi_2\operatorname{Re}\tau\right]}{\sqrt{t_1}\sqrt{t_2}}
$$

and $g_{31}=\overline{g_{13}}$. We arrive at a Gram matrix $[g_{ij}]$ that satisfies $\frac{\det[g_{ij}]}{g_{11}g_{22}g_{33}}=\beta$. By Sylvester's Criterion, there exists a strongly regular triple p_1,p_2,p_3 with Gram matrix equal to $[g_{ij}]$; this triple will be such that $R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}$ is in the fixed conjugacy class.

We have just proved the following theorem.

Theorem 50. *Geometrically, all strongly regular triples $p_1,p_2,p_3\in\mathrm{B}V$ with respect to the angles $\alpha_1,\alpha_2,\alpha_3\in\mathbb{S}^1$ such that the conjugacy class of $R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}$ is prescribed, are parameterized by the surface $S\subset\mathbb{R}^3(t_1,t_2,t)$ given by the equation*

$$
\begin{aligned}
t_1^2t_2+t_1t_2^2-2t_1t_2t+c_5t_1^2+c_6t_2^2+c_1t^2+&\\
&+(\beta-1+c_7)t_1t_2+c_3t_1t+c_4t_2t+c_8t_1+c_9t_2+c_2t+c_{10}=0
\end{aligned}
\tag{5.5}
$$

and by inequalities (5.4).

5.3 The equal angles case

Let $R_\alpha^{p_3}R_\alpha^{p_2}R_\alpha^{p_1}$ be a strongly regular triple with $p_1,p_2,p_3\in \mathrm{B}V$ and $\alpha^3\neq 1$. In this case, we have $\chi_1=\chi_2=\chi_3=:\chi$, where χ_i is the number introduced in (5.1). Thus, the constants c_i in (5.2) satisfy $c_3=\cdots=c_9=0$, $c_1=1+4\chi$, $c_2=-4\chi(2\chi\operatorname{Re}\tau+\operatorname{Im}\tau)$, and $c_{10}=(2\chi\operatorname{Re}\tau+\operatorname{Im}\tau)^2$. In particular, the surface S in Theorem 50 is given by the equality

$$t_1^2t_2+t_1t_2^2-2t_1t_2t+c_1t^2+(\beta-1)t_1t_2+c_2t+c_{10}=0,\tag{5.6}$$

and by the inequalities (5.4).

Given $r_1,r_2>1$, let V_{r_1} and H_{r_2} stand for the vertical and horizontal lines determined by $t_1=r_1$ and $t_2=r_2$. Note that the vertical lines V_r are given by

$$d_1t_2^2+d_2t_2t+d_3t^2+d_4t_2+d_5t+d_6=0\tag{5.7}$$

where

$$\begin{aligned}
d_1&:=r;\\
d_2&:=-2r;\\
d_3&:=c_1;\\
d_4&:=r^2+(\beta-1)r;\\
d_5&:=c_2;\\
d_6&:=c_{10}.
\end{aligned}\tag{5.8}$$

It follows from $d_2^2-4d_1d_3=4r(r-c_1)$ that, if V_r is nonempty, then it is an ellipse (or a single point) when $r<c_1$, a parabola when $r=c_1$, or a hyperbola when $r>c_1$. Since the equation of the horizontal line H_r is the same as the equation of the vertical line V_r (we just replace t_2 by t_1 in (5.7) and maintain the coefficients), every statement about V_r will also be valid for H_r.

Theorem 51. *Vertical and horizontal lines correspond, respectively, to bendings involving p_1,p_2 and p_2,p_3. In particular, if a vertical/horizontal line is nonempty, it is smooth and connected.*

Proof. We will prove the theorem for vertical lines; the horizontal case follows by symmetry. Let $p_1,p_2,p_3\in\mathrm{B}V$ be a strongly regular triple in the vertical line V_r. Then $\operatorname{ta}(p_1,p_2)=r$. Bendings involving p_1,p_2 form a subset of V_r corresponding to the strongly regular triples $B(s)p_1,B(s)p_2,p_3$ (see Theorem 26). We introduce functions $t_2,t:\mathbb{R}\to\mathbb{R}$, $t_2=t_2(s)$, $t=t(s)$, such that the point $\big(r,t_2(s),t(s)\big)\in V_r$ correspond to the triple $B(s)p_1,B(s)p_2,p_3$. Let $\widetilde{B}:\mathbb{R}\to V_r$ be defined by $\widetilde{B}(s)=\big(r_1,t_2(s),t(s)\big)$. In what follows, we will find explicit expressions for $t_2(s),t(s)$.

Case 1: the isometry $R_\alpha^{p_2}R_\alpha^{p_1}$ is regular elliptic. Let v_1 and v_2 be its negative and positive fixed points, respectively. Since we are in the equal angles case, p_1 and p_2 lie in the same $R_\alpha^{p_2}R_\alpha^{p_1}$-stable metric circle in the line $\mathrm{L}(p_1,p_2)$. So, we can choose representatives for v_1,v_2,p_1,p_2 such that $\langle v_1,v_1\rangle=-1$, $\langle v_2,v_2\rangle=1$, $p_1=v_1+\lambda v_2$, $p_2=v_1+\mu v_2$ with $\mu=e^{-ai}\lambda$ and $|\lambda|^2<1$. We choose a representative for p_3 such that $\langle p_3,p_3\rangle=-1$ and write $z_1:=\langle v_1,p_3\rangle$, $z_2:=\langle v_2,p_3\rangle$. It follows that

$$t_2(s)=\mathrm{ta}(B(s)p_2,p_3)=\frac{|z_1|^2+2\operatorname{Re}(w_2)\cos(s)-\operatorname{Im}(w_2)\sin(s)+|\mu|^2|z_2|^2}{|\mu|^2-1}$$

and

$$\begin{aligned}t(s)=&\operatorname{Re}\frac{\langle p_1,p_2\rangle\,\langle p_2,p_3\rangle\,\langle p_3,p_1\rangle}{\langle p_1,p_1\rangle\,\langle p_2,p_2\rangle\,\langle p_3,p_3\rangle}=\\ =&\frac{1}{(|\lambda|^2-1)(|\mu|^2-1)}\Bigg(\Big(\operatorname{Re}w_2-|\mu|^2\operatorname{Re}w_1\Big)e^{si}+\Big(\operatorname{Re}w_1-|\lambda|^2\operatorname{Re}w_2\Big)e^{-si}+\\ &+|z_2|^2\operatorname{Re}(\overline{\lambda}\mu)-|\lambda|^2|\mu|^2|z_2|^2+|z_1|^2\Bigg),\end{aligned}$$

where $w_1=\lambda\overline{z}_1z_2$ and $w_2=\mu\overline{z}_1z_2$.

Now, since

$$t_2'(s)=-\frac{2\operatorname{Re}w_2\sin s+2\operatorname{Im}w_2\cos s}{|\mu|^2-1},$$

we have $t_2'(s)=0$ if and only if

$$s=\arctan\left(-\frac{\operatorname{Im}w_2}{\operatorname{Re}w_2}\right).$$

Moreover,

$$\begin{aligned}t'(s)=&\frac{1}{(|\lambda|^2-1)(|\mu|^2-1)}\Bigg(\Big((|\mu|^2-1)\operatorname{Re}w_1+(|\lambda|^2-1)\operatorname{Re}w_2\Big)\sin s\\ &-\Big((|\mu|^2-1)\operatorname{Im}w_1-(|\lambda|^2-1)\operatorname{Im}w_2\Big)\cos s\Bigg),\end{aligned}$$

implies that $t'(s)=0$ if and only if

$$s=\arctan\left(\frac{(|\mu|^2-1)\operatorname{Im}w_1-(|\lambda|^2-1)\operatorname{Im}w_2}{(|\mu|^2-1)\operatorname{Re}w_1+(|\lambda|^2-1)\operatorname{Re}w_2}\right).$$

It follows that there exists $s\in\mathbb{R}$ such that $t_2'(s)=t'(s)=0$ if and only if $\operatorname{Im}w_1\operatorname{Re}w_2+\operatorname{Re}w_1\operatorname{Im}w_2=\operatorname{Im}(w_1w_2)=0$. But since $\operatorname{Im}(w_1w_2)=|z_1|^2|z_2|^2\operatorname{Im}\overline{\lambda}\mu$, this is equivalent to $\overline{\lambda}\mu\in\mathbb{R}$, i.e., to $a=k\pi$ for some $k\in\mathbb{Z}$ (since $\mu=e^{-ai}\lambda$). Note that $a=k\pi$ implies by Proposition 3 that p_1, p_2, v_1, and v_2 lie in the same geodesic and this cannot happen.

Case 2: the isometry $R_\alpha^{p_2}R_\alpha^{p_1}$ is hyperbolic. Let v_1 and v_2 be its isotropic fixed points. We choose representatives for v_1,v_2,p_1,p_2 such that $\langle v_1,v_2\rangle=-\frac{1}{2}$, $p_1=v_1+\lambda v_2$, $p_2=v_1+\mu v_2$, where $\mu=e^{-2a}\lambda$ and $\operatorname{Re}\lambda>0$. We choose a representative for p_3 such that $\langle p_3,p_3\rangle=-1$ and write $z_1:=\langle v_1,p_3\rangle$ and $z_2:=\langle v_2,p_3\rangle$. Then

$$t_2(s)=\mathrm{ta}(B(s)p_2,p_3)=\frac{|z_1|^2e^{2s}+2\operatorname{Re}(\mu\overline{z}_1z_2)+|\mu|^2|z_2|^2e^{-2s}}{\operatorname{Re}\mu}$$

and

$$t(s)=\frac{1}{\operatorname{Re}\lambda\operatorname{Re}\mu}\bigg((\operatorname{Re}\lambda+\operatorname{Re}\mu)|z_1|^2e^{2s}+(|\lambda|^2\operatorname{Re}\mu+|\mu|^2\operatorname{Re}\lambda)|z_2|^2e^{-2s}$$
$$+(|\lambda|^2+|\mu|^2)\operatorname{Re}(\overline{z}_1z_2)+2\operatorname{Re}(\lambda\mu\overline{z}_1z_2)\bigg).$$

Note that $\lim\limits_{s\to\pm\infty}t_2(s)=+\infty$ and $\lim\limits_{s\to\pm\infty}t(s)=+\infty$ and that both $t_2(s)$ and $t(s)$ assume a minimum in exactly one point. Let us prove that $\widetilde{B}$ is locally injective.

We have

$$t_2'(s)=\frac{2|z_1|^2e^{2s}-2|\mu|^2|z_2|^2e^{-2s}}{\operatorname{Re}\mu}$$

and

$$t'(s)=\frac{2(\operatorname{Re}\lambda+\operatorname{Re}\mu)|z_1|^2e^{2s}-2(|\lambda|^2\operatorname{Re}\mu+|\mu|^2\operatorname{Re}\lambda)|z_2|^2e^{-2s}}{\operatorname{Re}\lambda\operatorname{Re}\mu}.$$

It follows that $t_2'(s)=0$ if and only if

$$s=\frac{1}{4}\ln\left(\frac{|\mu|^2|z_2|^2}{|z_1|^2}\right)$$

and $t'(s)=0$ if and only if

$$s=\frac{1}{4}\ln\left(\frac{(|\lambda|^2\operatorname{Re}\mu+|\mu|^2\operatorname{Re}\lambda)|z_2|^2}{(\operatorname{Re}\lambda+\operatorname{Re}\mu)|z_1|^2}\right).$$

Therefore, there exists $s\in\mathbb{R}$ such that $t_2'(s)=t'(s)=0$ if and only if $|\lambda|=|\mu|$. But $|\lambda|=|\mu|$ implies $a=0$, that is, $p_1=p_2$, a contradiction.

Case 3: the isometry $R_\alpha^{p_2}R_\alpha^{p_1}$ is ellipto-parabolic. Let v_1 be its isotropic fixed point and let v_2 be any other isotropic point in $\mathrm{L}(p_1,p_2)$. We choose representatives for v_1,v_2,p_1,p_2 such that $\langle v_1,v_2\rangle=-\frac{1}{2}$, $p_1=v_1+\lambda v_2$, $p_2=v_1+\mu v_2$, where $\mu=\frac{\lambda}{1+i\lambda a}$ and $\operatorname{Re}\lambda>0$. Hence, $\langle p_1,p_1\rangle=\langle p_2,p_2\rangle=-1$ and

$$t_2(s)=\frac{|\mu|^2|z_1|^2s^2-\Big(2\operatorname{Im}(\mu)|z_1|^2+2\operatorname{Im}(z_1\overline{z}_2)|\mu|^2\Big)s+|z_1|^2+|\mu|^2|z_2|^2+2\operatorname{Re}(\mu\overline{z}_1z_2)}{\operatorname{Re}\mu}$$

and

$$t(s)=\frac{1}{2\operatorname{Re}\lambda\operatorname{Re}\mu}\bigg(\Big(|\lambda|^2\operatorname{Re}\mu+|\mu|^2\operatorname{Re}\lambda\Big)|z_1|^2s^2-$$
$$-2\Big(|z_1|^2\operatorname{Im}(\lambda\mu)+|\lambda|^2\operatorname{Im}(z_1\overline{z}_2)\operatorname{Re}\mu+|\mu|^2(\operatorname{Im}(z_1\overline{z}_2)\operatorname{Re}\lambda\Big)s+$$
$$+|z_1|^2\operatorname{Re}\lambda+|\lambda|^2\operatorname{Re}(\overline{z}_1z_2)+2\operatorname{Re}(\lambda\mu\overline{z}_1z_2)+|\lambda|^2|z_2|^2\operatorname{Re}\mu+$$
$$+|z_1|^2\operatorname{Re}\mu+|\mu|^2\operatorname{Re}(\overline{z}_1z_2)+|\mu|^2|z_2|^2\operatorname{Re}\lambda\bigg).$$

Again, $\lim\limits_{s\to\pm\infty}t_2(s)=+\infty$ and $\lim\limits_{s\to\pm\infty}t(s)=+\infty$, and both $t_2(s)$ and $t(s)$ assume a minimum in exactly one point. Let us show that $\widetilde{B}$ is locally injective.

Since

$$t_2'(s) = 2\frac{|\mu|^2|z_1|^2 s - |z_1|^2 \operatorname{Im}\mu - |\mu|^2 \operatorname{Im}(z_1\overline{z}_2)}{\operatorname{Re}\mu}$$

and

$$t'(s) = \frac{1}{\operatorname{Re}\lambda \operatorname{Re}\mu}\Bigg(\Big(|\lambda|^2 \operatorname{Re}\mu + |\mu|^2 \operatorname{Re}\lambda\Big)|z_1|^2 s$$
$$- \Big(|z_1|^2 \operatorname{Im}(\lambda\mu) + |\lambda|^2 \operatorname{Im}(z_1\overline{z}_2)\operatorname{Re}\mu + |\mu|^2 \operatorname{Im}(z_1\overline{z}_2)\operatorname{Re}\lambda\Big)\Bigg),$$

we have $t_2'(s) = 0$ if and only if

$$s = \frac{|z_1|^2 \operatorname{Im}\mu + |\mu|^2 \operatorname{Im}(z_1\overline{z}_2)}{|\mu|^2|z_1|^2}.$$

Moreover, $t'(s) = 0$ if and only if

$$s = \frac{|z_1|^2 \operatorname{Im}(\lambda\mu) + |\lambda|^2 \operatorname{Im}(z_1\overline{z}_2)\operatorname{Re}\mu + |\mu|^2 \operatorname{Im}(z_1\overline{z}_2)\operatorname{Re}\lambda}{(|\lambda|^2 \operatorname{Re}\mu + |\mu|^2 \operatorname{Re}\lambda)|z_1|^2}.$$

Therefore, there exists $s \in \mathbb{R}$ such that $t_2'(s) = t'(s) = 0$ if and only if $|\lambda|^2 \operatorname{Im}\mu = |\mu|^2 \operatorname{Im}\lambda$. This also implies that $a = 0$, that is, $p_1 = p_2$, a contradiction.

In all cases, we conclude that $\widetilde{B}$ is locally injective; its image is a (topologically) closed curve in V_{r_1}. Therefore, $\widetilde{B}$ is also surjective. □

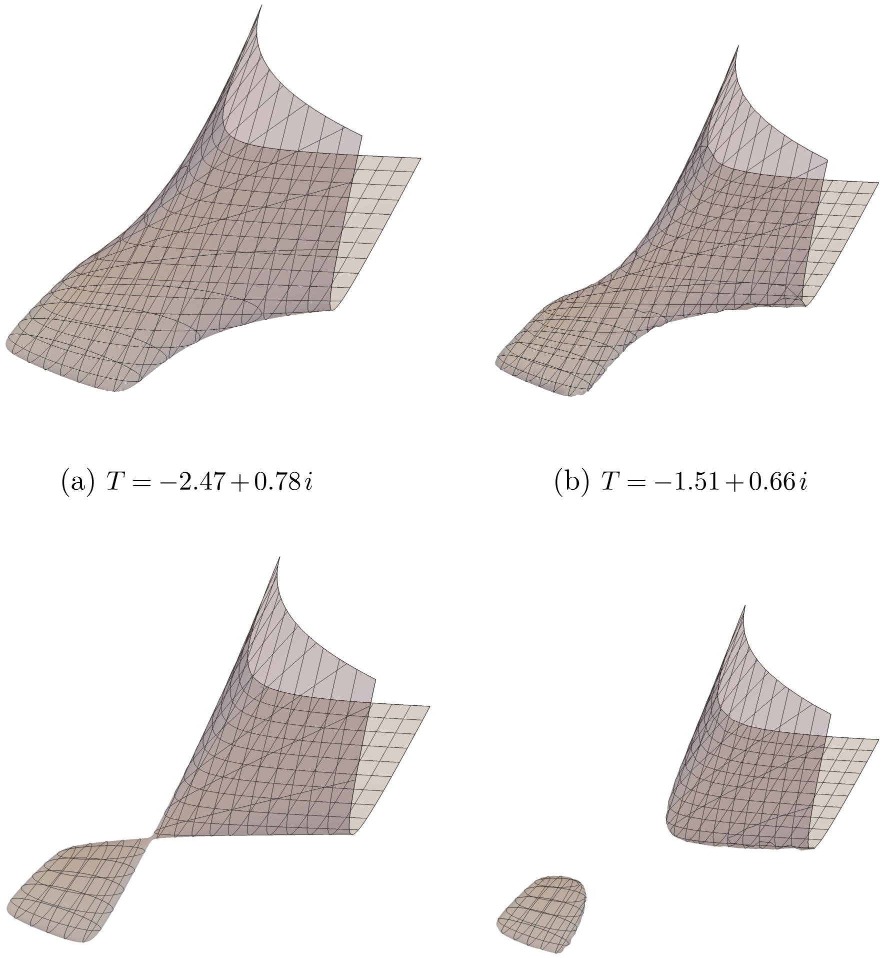
Remark 52. Assume that V_s is a single point for some $s > 1$. In this case, H_s is also a single point. The point V_s (resp. H_s) in S correspond to the Gram matrix of triples p_1, p_2, p_3 where $R_\alpha^{p_2} R_\alpha^{p_1}$ (resp. $R_\alpha^{p_3} R_\alpha^{p_2}$) is regular elliptic and p_3 (resp. p_1) lie in the line $\mathrm{L}(p, c)$, where c is the polar point to $\mathrm{L}(p_1, p_2)$ (resp. $\mathrm{L}(p_2, p_3)$) and p is the negative point fixed by $R_\alpha^{p_2} R_\alpha^{p_1}$ (resp. $R_\alpha^{p_3} R_\alpha^{p_2}$). In such a configuration, bending p_1, p_2 (resp. p_2, p_3) does not change the tance between p_1 and p_2 (resp. p_2 and p_3) and, moreover, does not change the geometric configuration of these points.

We have the following cases for the vertical line V_{c_1}:

(*i*) V_{c_1} is empty and there exist $1 < s_1 < s_2 < c_1$ such that V_{s_1} and V_{s_2} are single points while V_r is an ellipse for every $s_1 < r < s_2$; also, there exists $s > c_1$ such that V_r is a hyperbola for every $r > s$.

(*ii*) V_{c_1} is empty and there exists $s \geq c_1$ such that V_r is empty for every $1 < r \leq s$ while V_r is a hyperbola for every $r > s$.

(*iii*) V_{c_1} is a parabola and there exists $1 < s < c_1$ such that V_s is a single point, V_r is empty for every $1 < r < s$, V_r is an ellipse for every $s < r < c_1$, and V_r is a hyperbola for every $r > c_1$.



(a) $T = -2.47 + 0.78\,i$ (b) $T = -1.51 + 0.66\,i$

(c) $T = -1.02 + 0.59\,i$ (d) $T = -0.69 + 0.54\,i$

Figure 3 – Surface S for $\alpha = \exp\left(\frac{\pi i}{40}\right)$.

The numbers s_1 and s_2 in item (i) and the number s in item (iii) are roots of the equation

$$\left(c_1 y^2 + (c_2 + c_1(\beta - 1))y\right)^2 - 4(-c_1 y + y^2)\left(\frac{c_2^2}{4} - c_1 c_{10}\right) = 0.$$

We conclude that the surface S is either connected or has two connected components, one compact and one noncompact. There is a limiting case when S is connected but has a singular point p. Excluding the singular point, we get two connected components $S_1, S_2 \subset S$. One of these components, say S_1, together with the singular point, is compact, while $S_2 \cup \{p\}$ is noncompact (see item (c) in the picture above). We call S_1 and $S_2 \cup \{p\}$ **pieces** of S. In the nonsingular cases, we will also call each connected component of S a **piece** of S.

In Figure 3 we list some examples of S (drawing the vertical lines as well) for $\operatorname{tr} R_\alpha^{p_3} R_\alpha^{p_2} R_\alpha^{p_1} = T$ and $\alpha = e^{i\pi/40}$. Items (a), (b), and (c) illustrate the case (i); item (d) illustrates the case (*iii*). Figure 4 illustrates the case (*ii*).

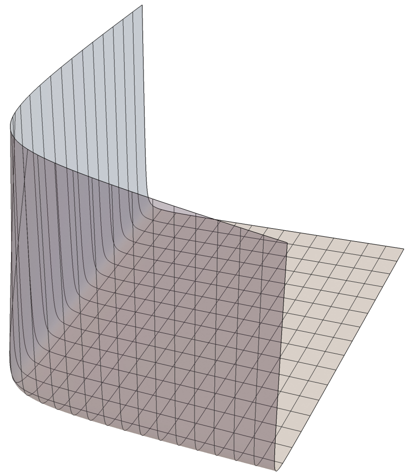


Figure 4 – Surface S for $\alpha = \exp\left(\frac{\pi i}{3}\right)$ and $T = 17.67 + 277.13\,i$.

We arrive at the following corollary:

Corollary 53. *Strongly regular triples $F = R_\alpha^{p_3} R_\alpha^{p_2} R_\alpha^{p_1}$ that lie in the same piece of S can be connected by means of finitely many bendings involving p_1, p_2 and p_2, p_3.*

Corollary 54. *Assume that $p_1, p_2, p_3 \in \mathrm{B}V$ and $q_1, q_2, q_3 \in \mathrm{B}V$ are two strongly regular triples with respect to the angles α, α, α and suppose that $R_\alpha^{p_3} R_\alpha^{p_2} R_\alpha^{p_1}$ and $R_\alpha^{q_3} R_\alpha^{q_2} R_\alpha^{q_1}$ are in the same conjugacy class (have the same trace). If at least one of $R_\alpha^{p_3} R_\alpha^{p_2}, R_\alpha^{p_2} R_\alpha^{p_1}$ and at least one of $R_\alpha^{q_3} R_\alpha^{q_2}, R_\alpha^{q_2} R_\alpha^{q_1}$ is hyperbolic, or if all of them are regular elliptic, then these triples can be connected by means of finitely many bendings.*

Consider the region $Q \subset \mathbb{R}(t_1, t_2)$ given by inequalities $t_1 > 1$ and $t_2 > 1$. Fixing $t_1, t_2 \in Q$, we have t_1 and t_2 in the projection of S into Q if and only if

$$\frac{2t_1t_2 - c_2 \pm \sqrt{(c_2 - 2t_1t_2)^2 - 4c_1(t_1^2 t_2 + t_1 t_2^2 + (\beta - 1)t_1 t_2 + c_{10})}}{2c_1} > 1.$$

So, the projection $R \subset \mathbb{R}(t_1, t_2)$ of the surface S into Q is given by the inequality

$$(c_2 - 2t_1t_2)^2 - 4c_1\left(t_1^2 t_2 + t_1 t_2^2 + (\beta - 1)t_1 t_2 + c_{10}\right) \geq 0.$$

Let C be the curve in S given by $t = \frac{2t_1t_2 - c_2}{2c_1}$. The projection $S \to R$ is a double covering ramified along C.

Corollary 55. *A vertical or horizontal line intersects C in exactly two points if it is an ellipse and in exactly one point otherwise (if it is a single point, a parabola, or a hyperbola). The intersection between a vertical line and a horizontal line can be empty, a single point, or two distinct points. If it is a single point, then they intersect over C.*

5.4 The general case

We now return to the general case of strongly regular triples $R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}$ with $p_1,p_2,p_3 \in \mathrm{B}V$. The results that we obtain in this section are analogous to those in the previous section. Consider then the surface S given by Theorem 50, where $c_1,\dots,c_{10}$ are defined by the equalities (5.2) and by the inequalities (5.4).

The vertical lines V_r are given by the equation

$$d_1t_2^2+d_2t_2t+d_3t^2+d_4t_2+d_5t+d_6=0, \tag{5.9}$$

where

$$\begin{aligned} d_1 &:= r+c_6;\\ d_2 &:= -2r+c_4;\\ d_3 &:= c_1;\\ d_4 &:= r^2+(\beta-1+c_7)r+c_9;\\ d_5 &:= c_3r+c_2;\\ d_6 &:= c_5r^2+c_8r+c_{10}. \end{aligned} \tag{5.10}$$

The horizontal lines H_r are given by

$$e_1t_2^2+e_2t_2t+e_3t^2+e_4t_2+e_5t+e_6=0, \tag{5.11}$$

where

$$\begin{aligned} e_1 &:= r+c_5;\\ e_2 &:= -2r+c_3;\\ e_3 &:= c_1;\\ e_4 &:= r^2+(\beta-1+c_7)r+c_8;\\ e_5 &:= c_4r+c_2;\\ e_6 &:= c_6r^2+c_9r+c_{10}. \end{aligned} \tag{5.12}$$

Unlike the previous case, here the equations defining the vertical lines V_r and the horizontal lines are distinct, i.e., the surface S will not necessarily be 'symmetric' when the angles $\alpha_1,\alpha_2,\alpha_3$ are not equal.

Define

$$k_1 := \frac{c_1+c_4+\sqrt{c_1^2+2c_1c_4+4c_1c_6}}{2} \quad \text{and} \quad k_2 := \frac{c_1+c_3+\sqrt{c_1^2+2c_1c_3+4c_1c_5}}{2}.$$

From these equations, we obtain that if V_{r_1} (resp. H_{r_2}) is nonempty, then it is an ellipse (or a single point) if $r_1<k_1$ (resp. $r_2<k_2$), a parabola if $r_1=k_1$ (resp. $r_2=k_2$), or a hyperbola if $r_1>k_1$ (resp. $r_2>k_2$). In other words, we have the following proposition.

Proposition 56. *An isometry $R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}$ with $p_1,p_2\in\mathrm{B}V$ is regular elliptic if $\mathrm{ta}(p_1,p_2)<k$, ellipto-parabolic if $\mathrm{ta}(p_1,p_2)=k$, and hyperbolic if $\mathrm{ta}(p_1,p_2)>k$, where*

$$k=\frac{c_1+c_4+\sqrt{c_1^2+2c_1c_4+4c_1c_6}}{2}$$

and c_1, c_4, c_6 are defined in (5.2).

Consider the region $Q \subset \mathbb{R}(t_1,t_2)$ given by inequalities $t_1 > 1$ and $t_2 > 1$. The projection $R \subset \mathbb{R}(t_1,t_2)$ of the surface S into Q is given by the inequality

$$(-2t_1t_2 + c_3t_1 + c_4t_2 + c_2)^2 - \\ - 4c_1\left(t_1^2t_2 + t_1t_2^2 + c_5t_1^2 + c_6t_2^2 + (\beta - 1 + c_7)t_1t_2 + c_8t_1 + c_9t_2 + c_{10}\right) \geq 0.$$

Let C be the curve in S given by $t = \frac{2t_1t_2 - c_3t_1 - c_4t_2 - c_2}{2c_1}$. The projection $S \to R$ is a double covering ramified along C.

Theorem 57. *Nonempty vertical and horizontal lines are smooth, connected, and correspond, respectively, to bendings involving p_1, p_2 and p_2, p_3. A nonempty vertical/horizontal line intersect C in exactly two points when it is an ellipse and in exactly one point otherwise (if it is a single point, a parabola, or a hyperbola). The intersection between a vertical line and a horizontal line can be empty, a single point, or two distinct points. When it is a single point, such an intersection occurs over C.*

Proof. We will prove the theorem only for vertical lines (the same reasoning works for horizontal lines). Let p_1, p_2, p_3 be a strongly regular triple in the vertical line V_r. Then $\operatorname{ta}(p_1,p_2) = r$. Bendings involving p_1, p_2 form a subset of the vertical line V_r that correspond to the strongly regular triples $B(s)p_1, B(s)p_2, p_3$. Consider the functions $t_2, t : \mathbb{R} \to \mathbb{R}$, $t_2 = t_2(s)$, $t = t(s)$, such that the point $\big(r, t_2(s), t(s)\big) \in V_r$ corresponds to the triple $B(s)p_1, B(s)p_2, p_3$. Denote by $\widetilde{B} : \mathbb{R} \to V_r$ the function $\widetilde{B}(s) := \big(r_1, t_2(s), t(s)\big)$. Let us find explicit expressions for the functions $t_2(s)$ and $t(s)$.

Case 1: the isometry $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ is regular elliptic. Let v_1 and v_2 be its negative and positive fixed points, respectively. Choose representatives for v_1, v_2, p_1, p_2 such that $\langle v_1, v_1\rangle = -1$, $\langle v_2, v_2\rangle = 1$, $p_1 = v_1 + \lambda_1 v_2$, $p_2 = v_1 + \lambda_2 v_2$, where $|\lambda_1|^2 < 1$ and $|\lambda_2|^2 < 1$. We also choose a representative for p_3 such that $\langle p_3, p_3\rangle = -1$ and define $z_1 := \langle v_1, p_3\rangle$ and $z_2 := \langle v_2, p_3\rangle$. It follows that

$$t_2(s) = \operatorname{ta}(B(s)p_2, p_3) = \frac{|z_1|^2 + 2\operatorname{Re}(w_2)\cos(s) - \operatorname{Im}(w_2)\sin(s) + |\lambda_2|^2|z_2|^2}{|\lambda_2|^2 - 1}$$

and

$$\begin{aligned} t(s) =& \operatorname{Re}\frac{\langle p_1,p_2\rangle\langle p_2,p_3\rangle\langle p_3,p_1\rangle}{\langle p_1,p_1\rangle\langle p_2,p_2\rangle\langle p_3,p_3\rangle} = \\ =& \frac{1}{(|\lambda_1|^2-1)(|\lambda_2|^2-1)}\Bigg(\Big(\operatorname{Re} w_2 - |\lambda_2|^2\operatorname{Re} w_1\Big)e^{si} + \Big(\operatorname{Re} w_1 - |\lambda_1|^2\operatorname{Re} w_2\Big)e^{-si} + \\ & + |z_2|^2\operatorname{Re}(\overline{\lambda}_1\lambda_2) - |\lambda_1|^2|\lambda_2|^2|z_2|^2 + |z_1|^2\Bigg), \end{aligned}$$

where $w_1 = \lambda_1\overline{z}_1 z_2$ and $w_2 = \lambda_2\overline{z}_1 z_2$.

Now, since

$$t_2'(s) = -\frac{2\operatorname{Re} w_2 \sin s + 2\operatorname{Im} w_2 \cos s}{|\lambda_2|^2 - 1},$$

we have $t_2'(s) = 0$ if and only if

$$s = \arctan\left(-\frac{\operatorname{Im} w_2}{\operatorname{Re} w_2}\right).$$

Moreover,

$$\begin{aligned} t'(s) =& \frac{1}{(|\lambda_1|^2-1)(|\lambda_2|^2-1)}\Bigg(\Big((|\lambda_2|^2-1)\operatorname{Re} w_1 + (|\lambda_1|^2-1)\operatorname{Re} w_2\Big)\sin s \\ & - \Big((|\lambda_2|^2-1)\operatorname{Im} w_1 - (|\lambda_1|^2-1)\operatorname{Im} w_2\Big)\cos s\Bigg), \end{aligned}$$

and, therefore, $t'(s) = 0$ if and only if

$$s = \arctan\left(\frac{(|\lambda_2|^2-1)\operatorname{Im} w_1 - (|\lambda_1|^2-1)\operatorname{Im} w_2}{(|\lambda_2|^2-1)\operatorname{Re} w_1 + (|\lambda_1|^2-1)\operatorname{Re} w_2}\right).$$

It follows that there exists $s \in \mathbb{R}$ such that $t_2'(s) = t'(s) = 0$ if and only if $\operatorname{Im} w_1 \operatorname{Re} w_2 + \operatorname{Re} w_1 \operatorname{Im} w_2 = \operatorname{Im}(w_1 w_2) = 0$. But since $\operatorname{Im}(w_1 w_2) = |z_1|^2 |z_2|^2 \operatorname{Im} \overline{\lambda}_1 \lambda_2$, this is equivalent to $\overline{\lambda}_1 \lambda_2 \in \mathbb{R}$.

By Proposition 3, p_1, p_2, v_1 lie in the same geodesic if and only if $b(v_1, p_1, p_2) = 0$, i.e., if and only if

$$\begin{aligned} b(v_1,p_1,p_2) =& \langle v_1, p_1\rangle \langle p_1, p_2\rangle \langle p_2, v_1\rangle - \langle v_1, p_2\rangle \langle p_2, p_1\rangle \langle p_1, v_1\rangle \\ =& -1 + \lambda_1\overline{\lambda}_2 + 1 - \overline{\lambda}_1\lambda_2 \\ =& -2i\operatorname{Im}(\overline{\lambda}_1\lambda_2) = 0. \end{aligned}$$

In other words, p_1 and p_2 lie in a geodesic through v_1 if and only if $\overline{\lambda}_1\lambda_2 \in \mathbb{R}$. Since this cannot happen, we conclude that $\widetilde{B}$ is locally injective and its image is a (topologically) closed curve in V_{r_1}. Therefore, $\widetilde{B}$ is also surjective.

Case 2: the isometry $R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1}$ is hyperbolic. Let v_1 and v_2 be its isotropic fixed points. We choose representatives for v_1, v_2, p_1, p_2 such that $\langle v_1, v_2\rangle = -\frac{1}{2}$, $p_1 = v_1 + \lambda_1 v_2$, $p_2 = v_1 + \lambda_2 v_2$. We choose a representative for p_3 such that $\langle p_3, p_3\rangle = -1$ and define $z_1 := \langle v_1, p_3\rangle$ and $z_2 := \langle v_2, p_3\rangle$. Then

$$t_2(s) = \operatorname{ta}(B(s)p_2, p_3) = \frac{|z_1|^2 e^{2s} + 2\operatorname{Re}(\lambda_2 \overline{z}_1 z_2) + |\lambda_2|^2 |z_2|^2 e^{-2s}}{\operatorname{Re}\lambda_2}$$

and

$$\begin{aligned} t(s) = \frac{1}{\operatorname{Re}\lambda_1 \operatorname{Re}\lambda_2}\Bigg(& (\operatorname{Re}\lambda_1 + \operatorname{Re}\lambda_2)|z_1|^2 e^{2s} + (|\lambda_1|^2 \operatorname{Re}\lambda_2 + |\lambda_2|^2 \operatorname{Re}\lambda_1)|z_2|^2 e^{-2s} \\ & + (|\lambda_1|^2 + |\lambda_2|^2)\operatorname{Re}(\overline{z}_1 z_2) + 2\operatorname{Re}(\lambda_1\lambda_2\overline{z}_1 z_2)\Bigg). \end{aligned}$$

Note that $\lim_{s\to\pm\infty} t_2(s)=+\infty$ and $\lim_{s\to\pm\infty} t(s)=+\infty$, that both $t_2(s), t(s)$ assume a minimum in exactly one point, and that the image of $\widetilde{B}$ is a (topologically) closed curve in V_{r_1}. Let us prove that $\widetilde{B}$ is locally injective.

We have

$$t_2'(s)=\frac{2|z_1|^2e^{2s}-2|\lambda_2|^2|z_2|^2e^{-2s}}{\operatorname{Re}\lambda_2}$$

and

$$t'(s)=\frac{2\left(\operatorname{Re}\lambda_1+\operatorname{Re}\lambda_2\right)|z_1|^2e^{2s}-2\left(|\lambda_1|^2\operatorname{Re}\lambda_2+|\lambda_2|^2\operatorname{Re}\lambda_1\right)|z_2|^2e^{-2s}}{\operatorname{Re}\lambda_1\operatorname{Re}\lambda_2}.$$

It follows that $t_2'(s)=0$ if and only if

$$s=\frac{1}{4}\ln\left(\frac{|\lambda_2|^2|z_2|^2}{|z_1|^2}\right)$$

and that $t'(s)=0$ if and only if

$$s=\frac{1}{4}\ln\left(\frac{(|\lambda_1|^2\operatorname{Re}\lambda_2+|\lambda_2|^2\operatorname{Re}\lambda_1)|z_2|^2}{(\operatorname{Re}\lambda_1+\operatorname{Re}\lambda_2)|z_1|^2}\right).$$

Therefore, there exists $s\in\mathbb{R}$ is such that $t_2'(s)=t'(s)=0$ if and only if $|\lambda_1|=|\lambda_2|$.

We will show that this implies that the geodesic through p_1 and p_2 is orthogonal to the geodesic $\mathrm{G}\wr v_1,v_2\wr$. Since this cannot happen, we conclude that $\widetilde{B}$ is locally injective and, therefore, surjective (actually, bijective).

Assume $|\lambda_1|=|\lambda_2|$ and consider the curves $\gamma,\sigma:\mathbb{R}\to\mathbb{P}_{\mathbb{C}}V$ with lifts $\gamma_0,\sigma_0:\mathbb{R}\to V$ given by $\gamma_0(t)=e^tv_1+e^{-t}v_2$ and $\sigma_0(s)=e^{ti}v_1+e^{-ti}rv_2$, where $r=|\lambda_1|=|\lambda_2|$. Note that γ parameterizes the negative part of the geodesic with vertices v_1,v_2 and σ parameterizes the geodesic through p_1,p_2. Furthermore, these two curves intersect at $q=\gamma(a)=\sigma(0)$, where $e^{-2a}=r$. By Lemma 1, we have $\dot\gamma(a)=\langle -,q\rangle\left(e^{-a}v_2-e^av_1\right)$ and $\dot\sigma(0)=\langle -,q\rangle\left(2iv_2-\frac{1}{r}v_1\right)$. It follows that

$$(\dot\gamma(a),\dot\sigma(0))=\operatorname{Re}\left(-\langle q,q\rangle\left\langle e^{-a}v_2-e^av_1,2iv_2-\frac{1}{r}v_1\right\rangle\right)=\operatorname{Re}\left(\frac{i}{r}\langle q,q\rangle\left(e^{-a}+re^a\right)\right)=0,$$

that is, γ and σ are orthogonal.

Case 3: the isometry $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ is ellipto-parabolic. Let v_1 be its isotropic fixed point and let v_2 be any other isotropic point in $\mathrm{L}(p_1,p_2)$. We can choose representatives for v_1,v_2,p_1,p_2 such that $\langle v_1,v_2\rangle=-\frac{1}{2}$, $p_1=v_1+\lambda_1v_2$, $p_2=v_1+\lambda_2v_2$. Therefore, $\langle p_1,p_1\rangle=\langle p_2,p_2\rangle=-1$,

$$t_2(s)=\frac{|\lambda_2|^2|z_1|^2s^2-\left(2\operatorname{Im}(\lambda_2)|z_1|^2+2\operatorname{Im}(z_1\overline{z}_2)|\lambda_2|^2\right)s+|z_1|^2+|\lambda_2|^2|z_2|^2+2\operatorname{Re}(\lambda_2\overline{z}_1z_2)}{\operatorname{Re}\lambda_2},$$

and

$$t(s)=\frac{1}{2\operatorname{Re}\lambda_1\operatorname{Re}\lambda_2}\Big(\Big(|\lambda_1|^2\operatorname{Re}\lambda_2+|\lambda_2|^2\operatorname{Re}\lambda_1\Big)|z_1|^2s^2$$
$$-2\Big(|z_1|^2\operatorname{Im}(\lambda_1\lambda_2)+|\lambda_1|^2\operatorname{Im}(z_1\overline{z}_2)\operatorname{Re}\lambda_2+|\lambda_2|^2(\operatorname{Im}(z_1\overline{z}_2)\operatorname{Re}\lambda_1\Big)s$$
$$+|z_1|^2\operatorname{Re}\lambda_1+|\lambda_1|^2\operatorname{Re}(\overline{z}_1z_2)+2\operatorname{Re}(\lambda_1\lambda_2\overline{z}_1z_2)+|\lambda_1|^2|z_2|^2\operatorname{Re}\lambda_2$$
$$+|z_1|^2\operatorname{Re}\lambda_2+|\lambda_2|^2\operatorname{Re}(\overline{z}_1z_2)+|\lambda_2|^2|z_2|^2\operatorname{Re}\lambda_1\Big).$$

Again, $\lim\limits_{s\to\pm\infty}t_2(s)=+\infty$, $\lim\limits_{s\to\pm\infty}t(s)=+\infty$, and both $t_2(s),t(s)$ assume a minimum in exactly one point. Let us prove that $\widetilde{B}$ is locally injective.

Since
$$t_2'(s)=2\frac{|\lambda_2|^2|z_1|^2s-|z_1|^2\operatorname{Im}\lambda_2-|\lambda_2|^2\operatorname{Im}(z_1\overline{z}_2)}{\operatorname{Re}\lambda_2}$$

and

$$t'(s)=\frac{1}{\operatorname{Re}\lambda_1\operatorname{Re}\lambda_2}\Big[\Big(|\lambda_1|^2\operatorname{Re}\lambda_2+|\lambda_2|^2\operatorname{Re}\lambda_1\Big)|z_1|^2s$$
$$-\Big(|z_1|^2\operatorname{Im}(\lambda_1\lambda_2)+|\lambda_1|^2\operatorname{Im}(z_1\overline{z}_2)\operatorname{Re}\lambda_2+|\lambda_2|^2\operatorname{Im}(z_1\overline{z}_2)\operatorname{Re}\lambda_1\Big)\Big],$$

we have $t_2'(s)=0$ if and only if

$$s=\frac{|z_1|^2\operatorname{Im}\lambda_2+|\lambda_2|^2\operatorname{Im}(z_1\overline{z}_2)}{|\lambda_2|^2|z_1|^2}$$

and $t'(s)=0$ if and only if

$$s=\frac{|z_1|^2\operatorname{Im}(\lambda_1\lambda_2)+|\lambda_1|^2\operatorname{Im}(z_1\overline{z}_2)\operatorname{Re}\lambda_2+|\lambda_2|^2\operatorname{Im}(z_1\overline{z}_2)\operatorname{Re}\lambda_1}{(|\lambda_1|^2\operatorname{Re}\lambda_2+|\lambda_2|^2\operatorname{Re}\lambda_1)|z_1|^2}.$$

Therefore, there exists $s\in\mathbb{R}$ such that $t_2'(s)=t'(s)=0$ if and only if $|\lambda_1|^2\operatorname{Im}\lambda_2=|\lambda_2|^2\operatorname{Im}\lambda_1$.

But $|\lambda_1|^2\operatorname{Im}\lambda_2=|\lambda_2|^2\operatorname{Im}\lambda_1$ means that p_1 and p_2 lie in the same geodesic through v_1. Indeed, by Proposition 3, we have p_1, p_2, and v_1 in a same geodesic if and only if $b(v_1,p_1,p_2)=0$, where

$$\begin{aligned}b(v_1,p_1,p_2)&=\langle v_1,p_1\rangle\langle p_1,p_2\rangle\langle p_2,v_1\rangle-\langle v_1,p_2\rangle\langle p_2,p_1\rangle\langle p_1,v_1\rangle\\&=-\frac{1}{8}\Big(\lambda_2|\lambda_1|^2+\overline{\lambda}_1|\lambda_2|^2\Big)+\frac{1}{8}\Big(\overline{\lambda}_2|\lambda_1|^2+\lambda_1|\lambda_2|^2\Big)\\&=\frac{i}{4}\Big(|\lambda_2|^2\operatorname{Im}\lambda_1-|\lambda_1|^2\operatorname{Im}\lambda_2\Big).\end{aligned}$$

This cannot happen, and we conclude that $\widetilde{B}$ is surjective (actually, bijective). □

Analogously to Remark 52, if a vertical line V_s (resp. horizontal line H_s) is a single point for some $s>1$, then V_s (resp. H_s) in S correspond to the Gram matrix of triples p_1,p_2,p_3 where $R^{p_2}_{\alpha_2}R^{p_1}_{\alpha_1}$ (resp. $R^{p_3}_{\alpha_3}R^{p_2}_{\alpha_2}$) is regular elliptic and p_3 (resp. p_1) lie in the line

$\mathrm{L}(p,c)$, where c is the polar point to $\mathrm{L}(p_1,p_2)$ (resp. $\mathrm{L}(p_2,p_3)$) and p is the negative point fixed by $R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}$ (resp. $R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}$). In such a configuration, bending p_1,p_2 (resp. p_2,p_3) does not change the tance between p_1 and p_2 (resp. p_2 and p_3), and moreover, does not change the geometric configuration of these points.

Surfaces S in the general case look exactly like those in the equal angles case. In particular, we apply to them the definition of **piece** that is presented in page 66.

The next two corollaries follow directly from Theorems 50 and 57.

Corollary 58. *Strongly regular triples $F=R_{\alpha}^{p_3}R_{\alpha}^{p_2}R_{\alpha}^{p_1}$ that lie in the same piece of S can be connected by means of finitely many bendings involving p_1,p_2 and p_2,p_3.*

Corollary 59. *Assume that $p_1,p_2,p_3\in\mathrm{B}V$ and $q_1,q_2,q_3\in\mathrm{B}V$ are two strongly regular triples with respect to the angles $\alpha_1,\alpha_2,\alpha_3$ and suppose that $R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}$ and $R_{\alpha_3}^{q_3}R_{\alpha_2}^{q_2}R_{\alpha_1}^{q_1}$ are in the same conjugacy class (have the same trace). If at least one of $R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2},R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}$ and at least one of $R_{\alpha_3}^{q_3}R_{\alpha_2}^{q_2},R_{\alpha_2}^{q_2}R_{\alpha_1}^{q_1}$ is hyperbolic, or if all of them are regular elliptic, then these triples can be connected by means of finitely many bendings.*

5.5 Special elliptic pentagons

In this section we focus on special elliptic pentagons of negative points, i.e., relations of the form

$$R_{\alpha_5}^{p_5}R_{\alpha_4}^{p_4}R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}=\delta \tag{5.13}$$

in $\mathrm{SU}(2,1)$ with $\delta^3=1$ and $p_1,p_2,p_3,p_4,p_5\in\mathrm{B}V$, or, equivalently, a configuration of 5 points $p_1,p_2,p_3,p_4,p_5\in\mathrm{B}V$ plus 5 angles $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\in\mathbb{S}^1$, $\alpha_i^3\neq 1$, satisfying (5.13). We also assume that p_i is not equal to p_{i+1}, $i=1,2,3,4$.

Proposition 60. *If $R_{\alpha_5}^{p_5}R_{\alpha_4}^{p_4}R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}=\delta$ is a pentagon, then $R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}$ is a strongly regular triple.*

Proof. If $R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}$ is special elliptic, p_4 and p_5 are orthogonal by Corollary 21 and this cannot happen because $p_4,p_5\in\mathrm{B}V$. If p_1,p_2,p_3 lie on the same complex line L, then $R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}$ stabilizes the hyperbolic complex line L and fixes its polar point c. Therefore, $\delta R_{\overline{\alpha_4}}^{p_4}R_{\overline{\alpha_5}}^{p_5}$ also fixes c, implying that $p_4,p_5\in\mathrm{L}$ and $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5=\delta$. □

Let $R_{\alpha_5}^{p_5}R_{\alpha_4}^{p_4}R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}=\delta$ be a special elliptic pentagon of negative points. Assume that at least one of the isometries $R_{\alpha_{i+1}}^{p_{i+1}}R_{\alpha_i}^{p_i}$, $i=1,2,3,4$, is hyperbolic. Then, by means of bendings, we can arrive at the situation where $R_{\alpha_{i+1}}^{p_{i+1}}R_{\alpha_i}^{p_i}$ is hyperbolic for every $i=1,2,3,4$. In particular, we can make $\mathrm{ta}(p_4,p_5)$ as big as we want. Indeed, suppose (without loss of generality) that the hyperbolic product is in the triple $R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}$. Since, by Proposition 60, $R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}$ is a strongly regular triple, we can consider the surface S associated to this triple. The triple $R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}$ lies in a noncompact component of S and Proposition 56 implies that, by finitely many bendings involving p_3,p_2 and p_2,p_1, we can reach the situation where both $R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}$ and $R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}$ are hyperbolic. Now one just has to repeat this argument for other triples.

Let $R_{\alpha_5}^{q_5}R_{\alpha_4}^{q_4}R_{\alpha_3}^{q_3}R_{\alpha_2}^{q_2}R_{\alpha_1}^{q_1}=\delta$ be another special elliptic pentagon of negative points such that at least one of the isometries $R_{\alpha_{i+1}}^{q_{i+1}}R_{\alpha_i}^{q_i}$, $i=1,2,3,4$ is hyperbolic. Then these two pentagons can be connected by means of finitely many bendings. Indeed, we can assume that $\operatorname{ta}(p_4,p_5)=\operatorname{ta}(q_4,q_5)$ and that $R_{\alpha_{i+1}}^{p_{i+1}}R_{\alpha_i}^{p_i}$ and $R_{\alpha_{i+1}}^{q_{i+1}}R_{\alpha_i}^{q_i}$ are hyperbolic. Now, since $R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}$ and $R_{\alpha_3}^{q_3}R_{\alpha_2}^{q_2}R_{\alpha_1}^{q_1}$ are strongly regular triples and $\operatorname{tr}\delta R_{\overline{\alpha}_4}^{p_4}R_{\overline{\alpha}_5}^{p_5}=\operatorname{tr}\delta R_{\overline{\alpha}_4}^{q_4}R_{\overline{\alpha}_5}^{q_5}$, they can be connected by means of finitely many bendings.

Summarizing:

Proposition 61. *Two special elliptic pentagons of negative points $R_{\alpha_5}^{p_5}R_{\alpha_4}^{p_4}R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}=\delta$ and $R_{\alpha_5}^{q_5}R_{\alpha_4}^{q_4}R_{\alpha_3}^{q_3}R_{\alpha_2}^{q_2}R_{\alpha_1}^{q_1}=\delta$ (with same angles and δ) such that at least one of the isometries $R_{\alpha_{i+1}}^{p_{i+1}}R_{\alpha_i}^{p_i}$ (and $R_{\alpha_{i+1}}^{q_{i+1}}R_{\alpha_i}^{q_i}$) is hyperbolic can be connected by means of finitely many bendings.*

Finally, let us show that we can connect, by means of finitely many bendings and f-bendings, two special elliptic pentagons of negative points with the same δ, with angles in the same component, and with the same product of angles.

Proposition 62. *Let $R_{\alpha_5}^{p_5}R_{\alpha_4}^{p_4}R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}=\delta$ and $R_{\beta_5}^{q_5}R_{\beta_4}^{q_4}R_{\beta_3}^{q_3}R_{\beta_2}^{q_2}R_{\beta_1}^{q_1}=\delta$ be special elliptic pentagons of negative points such that the corresponding angles are in same components, $\alpha_i\sim\beta_i$, and such that the products of angles are the same, $\Pi\alpha_i=\Pi\beta_i$. These pentagons can be connected by means of finitely many bendings and f-bendings.*

Proof. We will show that there exists a sequence of finitely many bendings and f-bendings leading the pentagons to pentagons that have same corresponding angles; then, we just need to apply the previous proposition.

Without loss of generality, we can assume that every product of consecutive isometries $R_{\alpha_{i+1}}^{p_{i+1}}R_{\alpha_i}^{p_i}$ (resp., $R_{\beta_{i+1}}^{q_{i+1}}R_{\beta_i}^{q_i}$) that appear in the pentagons is hyperbolic; we simply apply an f-bending in order to make at least one of the isometries $R_{\alpha_{i+1}}^{p_{i+1}}R_{\alpha_i}^{p_i}$ (resp., $R_{\beta_{i+1}}^{q_{i+1}}R_{\beta_i}^{q_i}$) hyperbolic and then, as in the proof of Proposition 61, we can reach the desired situation by means finitely many bendings.

Now, proceeding as in the proof of Proposition 36, we can make $\alpha_i=\beta_i$. $\square$

In (ANAN'IN, 2012) it is shown that, if $F\in\mathrm{SU}(2,1)$ is a regular isometry with $\operatorname{tr}F\neq-1$, then there exists a strongly regular triple with respect to angles $\alpha_1=\alpha_2=\alpha_3=-1$ such that $F=R_{-1}^{p_3}R_{-1}^{p_2}R_{-1}^{p_1}$. We use this result to prove the following proposition.

Proposition 63 (Existence of pentagons)**.** *Given angles α_i with $\alpha_i^3\neq1$, $i=1,2,3,4,5$, and given $\delta\in\mathbb{C}$ with $\delta^3=1$, there exists a special elliptic pentagon $R_{\alpha_5}^{p_5}R_{\alpha_4}^{p_4}R_{\alpha_3}^{p_3}R_{\alpha_2}^{p_2}R_{\alpha_1}^{p_1}=\delta$.*

Proof. Define $\alpha := \Pi\alpha_i$. For $i = 3, 4, 5$ let $k_i \in \mathbb{N}$ be such that $\alpha_i \sim -\delta^{k_i}$ and let $k := k_3 + k_4 + k_5 + 1$. Let $\beta_1, \beta_2 \in \mathbb{S}^1$ be angles with $\beta_i^3 \neq 1$, $\beta_i \sim \alpha_i$, and $\beta_1\beta_2 = -\delta^{k_3+k_4+k_5}\overline{\alpha}$. Take distinct points $q_1, q_2 \in \mathrm{B}V$ such that $R^{q_1}_{\beta_1} R^{q_2}_{\beta_2}$ is regular (see Definition 43) and $\mathrm{tr}\left(R^{q_1}_{\beta_1} R^{q_2}_{\beta_2}\right) \neq -1$.

The fact that $\delta^k R^{q_1}_{\beta_1} R^{q_2}_{\beta_2}$ is a regular isometry implies that there exists a strongly regular triple $q_3, q_4, q_5 \in \mathrm{B}V$, with respect to the angles $\beta_3 = \beta_4 = \beta_5 = -1$ such that $R^{q_5}_{-1} R^{q_4}_{-1} R^{q_3}_{-1} = \delta^k R^{q_1}_{\beta_1} R^{q_2}_{\beta_2}$. So,

$$\delta^{k_3}\delta^{k_4}\delta^{k_5} R^{q_5}_{-\delta^{k_5}} R^{q_4}_{-\delta^{k_4}} R^{q_3}_{-\delta^{k_3}} R^{q_2}_{\overline{\beta}_2} R^{q_1}_{\overline{\beta}_1} = \delta^k$$

and we obtain a pentagon

$$R^{q_5}_{-\delta^{k_5}} R^{q_4}_{-\delta^{k_4}} R^{q_3}_{-\delta^{k_3}} R^{q_2}_{\overline{\beta}_2} R^{q_1}_{\overline{\beta}_1} = \delta.$$

The product of the angles in this pentagon is $-\delta^{-(k_3+k_4+k_5)}\overline{\beta}_1\overline{\beta}_2 = \alpha$. Therefore, by Proposition 62, it can be connected, by means of finitely many bendings and f-bendings, to a pentagon $R^{p_5}_{\alpha_5} R^{p_4}_{\alpha_4} R^{p_3}_{\alpha_3} R^{p_2}_{\alpha_2} R^{p_1}_{\alpha_1} = \delta$. $\square$

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