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Topics on the theory of Frobenius manifolds

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Thales Novelli Castro

Tópicos na teoria das variedades de Frobenius

Dissertação apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Mestre em Ciências – Matemática. *EXEMPLAR DE DEFESA*

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To my friends.

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[1] <https://icmc.usp.br>
[2] <https://www.gov.br/capes/pt-br>

"You picked flowers - well, so have I.
Let them be, then, combined;
Let us exchange our flowers fair,
And in the brightest wreath them bind."
(Max Planck)

RESUMO

CASTRO, T. N. **Tópicos na teoria das variedades de Frobenius**. 2024. 91 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2024.

Este trabalho se propõe a apresentar uma conexão entre as variedades de Frobenius, um conceito da geometria diferencial presente na teoria topológica de campos, e sistemas de equações de tipo hidrodinâmico. Formuladas por Dubrovin ma década de 1990, as variedades de Frobenius visam dar uma interpretação geométrica às chamadas equações de associatividade, ou equações WDVV, um sistema não linear cuja solução é uma função quasi-homogênea que descreve constantes de estrutura de uma álgebra associativa. Os sistemas de tipo hidrodinâmico surgem, como o nome sugere, em estudos sobre mecânica de fluidos, especialmente dinâmica de gases. A relação, do ponto de vista geométrico, entre essas duas entidades se dá por meio de uma representação hamiltoniana para essas equações, proveniente de um tipo específico de estrutura de Poisson. Especificamente, o trabalho apresenta uma visão geral dos principais aspectos geométricos da teoria, desencadeando num teorema segundo o qual o *loop-space* de uma variedade de Frobenius carrega uma chamada estrutura bi-hamiltoniana de tipo hidrodinâmico.

Palavras-chave: Variedades de Frobenius, Geometria diferencial, Estruturas de Poisson, Hamiltoniano, Sistemas de tipo hidrodinâmico.

ABSTRACT

CASTRO, T. N. **Topics on the theory of Frobenius manifolds**. 2024. 91 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2024.

This work aims to present a connection between Frobenius manifolds, a concept of differential geometry which shows up in topological field theory, and systems of differential equations of hydrodynamic type. Formulated by Dubrovin in the 1990s, Frobenius manifolds aim to give a geometric interpretation to the so-called associativity equations, or WDVV equations, a nonlinear system whose solution is a quasi-homogeneous function describing structure constants of an associative algebra. Hydrodynamic-type systems arise, as the name suggests, in studies on fluid mechanics, especially gas dynamics. From the geometric approach, the relation between these two entities is given by means of a Hamiltonian representation for these equations, arising from a specific type of Poisson structure. Specifically, the work presents an overview of the main geometric aspects of the theory, leading to a theorem according to which the loop-space of a Frobenius manifold carries a so-called bi-Hamiltonian structure of hydrodynamic type.

Keywords: Frobenius manifolds, Differential geometry, Poisson structures, Hamiltonian, Systems of hydrodyamic type.

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CHAPTER

1

INTRODUCTION

This thesis aims to review, at least partially, a remarkable relation between two, a priori, quite far apart areas of mathematical physics: two dimensional topological field theory and gas dynamics. In our narrative, the first is represented by the theory of Frobenius manifolds, the second is described by a class of systems of PDEs christened evolution equations of hydrodynamic type. A Frobenius manifold is a quite complicated differential geometric object, whose main ingredients are a (real or complex) manifold M endowed with a flat metric g and a commutative and associative product defined on $\mathfrak{X}(M)$ (the sheaf of vector fields on M) that is compatible with the metric g in such a way that each tangent space of M inherits a structure of a Frobenius algebra. Frobenius manifolds were introduced by Boris Dubrovin in the 90s of the last century to give a geometric interpretation to the so called WDVV-equations, a system of non-linear PDEs discovered by the physicists Robbert Dijkgraaf, Erik and Hermann Verlinde, (DIJKGRAAF; VERLINDE; VERLINDE, 1991), and Edward Witten, (WITTEN, 1990), in their work on two dimensional topological field theory. This system of equations plays a major role in the definition of the so called *quantum cohomology*, a suitable *deformation* of the standard cohomology ring, which has major applications in symplectic topology as a tool to define global symplectic invariants for closed symplectic manifolds. On the other hand, a system of (evolution) equations of hydrodynamic type, in its simplest form, can be written as

$$u^i_t = V^i_j u^j_x, \quad i,j = 1,\dots,n, \tag{1.1}$$

where $u^k = u^k(t,x)$ for all k and $V^i_j = V^i_j(u^1,\dots,u^n)$ for all $i,j = 1,\dots,n$. These equations play a relevant role in gas dynamics and their very definition makes clear their geometric nature: the V^i_j's are the components of a tensor of type $(1,1)$. The Hamiltonian formalism for this class of equations was studied in-depth during decade of the 80th by Sergey Novikov and (some of) the components of his school. The climax of this analysis was a geometric characterization, given in terms of a metric in the Euclidean space, of the class of PDEs of hydrodynamic type admitting a Hamiltonian formulation and of the so called Poisson structures of hydrodynamic type, see for example (DUBROVIN; NOVIKOV, 1989). We now explain the relations between Frobenius manifolds and systems of hydrodynamic type admitting a Hamiltonian representation with respect to a Poisson structure of hydrodynamic type. To this end, it worth recalling that the notion of (complete) integrability for a Hamiltonian system is nowadays very often associated to the existence of a biHamiltonian representation of the latter, i.e. the existence of a pair of Poisson structures on the underlying phase-space such that: 1) the system admits a Hamiltonian representation with respect to both, 2) every linear combination of the two Poisson structures is again a Poisson structure. As it will be explained in the last section of this work, the compatibility of two Poisson structures of hydrodynamic type is translated into a condition of compatibility between the metrics defining them, which, in turn, is enclosed into the notion of a flat pencils of metrics, see Definition 22. The link between Frobenius manifolds and Hamiltonian system of hydrodynamics type can be stated (in one direction) as follows: every Frobenius manifold carries a flat pencil of metrics, see Theorem 14. We now present the content of our work. In Chapter 2 the notions of a Frobenius algebra and of a Frobenius manifold are introduced. Several examples of Frobenius algebras are presented, among them, the one coming from a 2-dimensional topological field theory is discussed in details via a sample of (nice) pictures. In Chapter 3 we specialize our presentation to a particular class of Frobenius algebras, the semi-simple ones. A characterization theorem for this class of algebras is presented and the corresponding class of Frobenius manifolds is introduced. The relevance of this class of Frobenius manifold is due to the fact that every semi-simple Frobenius manifold carries two classes of distinguished coordinates: the flat and the canonical ones. We note that the relation between these two types of coordinates, which will not be investigated in these notes, plays a major role in the development of the theory. Chapter 4, the last chapter, will be devoted to an introduction to the systems of equations of hydrodynamic type and to their Hamiltonian representation. In the very last part of this chapter we will prove that on every Frobenius manifold can be defined a flat pencil of metrics, which entails the existence of a compatible pair of Poisson structures of hydrodynamic type on the so called loop space of the underlying Frobenius manifold, see Theorem 14. We close this introduction mentioning that the previous theorem has a converse. More precisely one can prove that a flat pencil of metrics satisfying three additional properties which we will not spell out here, see (DUBROVIN, 1998a), induces a Frobenius structure on the underlying manifold. It is difficult to underestimate the importance of this result. For example, it yields the existence of a Frobenius structure of the orbit-spaces of every Coxeter group, see (DUBROVIN, 1999).

CHAPTER

2

FROBENIUS MANIFOLDS

The concept of Frobenius algebras, named after Ferdinand Georg Frobenius, was originally introduced as a set of algebras with special symmetry conditions on their so called regular representations, notably in the works of Brauer and Nesbitt, with further developments by Nakayama (see (BRAUER; NESBITT, 1937), (NAKAYAMA, 1939), (NAKAYAMA, 1941)).

In modern days, Frobenius algebras spark interest in mathematics due to it's connections with Topological quantum Field Theories (TFT's). This, particularly, is what inspired Boris Dubrovin to construct the concept of Frobenius manifolds, which is the main focus of this project. In this chapter, we introduce the concept and some examples of Frobenius algebras, show their connection to 2-dimensional TFT's and finally arrive at the definition of a Frobenius manifold.

2.1 Frobenius algebras

Consider $\mathbb{K}$ to be a field of characteristic zero, usually either $\mathbb{R}$ or $\mathbb{C}$. All algebras over $\mathbb{K}$ are assumed to be associative, commutative and unital. The unit shall always be denoted by e.

Definition 1 (Frobenius algebras)**.** Let V be a finite dimensional algebra over $\mathbb{K}$ whose product will be denoted by $\cdot : V \times V \longrightarrow V$. A *Frobenius algebra structure* on V is defined by the existence of a symmetric bilinear non-degenerate form g such that

$$g(a\cdot b, c) = g(a, b\cdot c) \quad \forall a,b,c \in V. \tag{2.1}$$

The form g will be referred to as the *Frobenius metric* of the algebra.

The following proposition provides an alternative definition which shall be of great use in the following sections.

Proposition 1. A Frobenius algebra structure on V can be equivalently defined by the existence of a linear form $\theta \in V^*$ such that the mapping $g : V \times V \longrightarrow V$ given by

$$g(a,b) := \theta(a \cdot b) \tag{2.2}$$

is non-degenerate.

Demonstration. If a Frobenius structure on V is given, then we can define θ to be $\theta(a) := g(a,e)$.

Conversely, if the form θ is given, then we simply check that Equation 2.2 is satisfied. Indeed, for any $a,b,c \in V$, associativity of the product implies that

$$g(a \cdot b, c) = \theta((a \cdot b) \cdot c) = \theta(a \cdot (b \cdot c)) = g(a, b \cdot c).$$

□

We have now defined two objects that establish the structure of a Frobenius algebra: the form $\theta \in V^*$, and the metric $g \in \mathrm{Sym}^2 V^*$. We can introduce a third one, $c \in \mathrm{Sym}^3 V^*$, defined by

$$c(a,b,c) := \theta(a \cdot b \cdot c). \tag{2.3}$$

It is worth noting that the associativity condition on the product places strong algebraic constraints on these objects, as we shall see in future chapters. Furthermore, reconstructing the product in the context of Frobenius manifolds heavily relies on form c.

2.2 Examples of Frobenius algebras

We now present a series of examples coming from different fields of study. The main goal of this section is to familiarize ourselves with the concept of Frobenius algebras and emphasize their ubiquity in mathematics.

Example 1 (Trivial)**.** We begin with the trivial example. Let $V = \mathbb{K}$ and choose $\lambda \in \mathbb{K} \setminus \{0\}$. Then we simply define $\theta(a) := \lambda a$. More generally, any finite field extension $V \supseteq \mathbb{K}$ is equipped with a Frobenius algebra structure. In fact, we can take any non-trivial linear map $\theta : V \longrightarrow \mathbb{K}$. Since it is a field homomorphism, then $\ker(\theta) = \{0\}$.

Example 2 (Measure theory)**.** Let Ω be a finite set equipped with a positive measure μ. Set $V = \{f : \Omega \longrightarrow \mathbb{R}\}$ with $\theta(f) = \mu(f)$. This defines a Frobenius algebra over $\mathbb{R}$.

More generally, if we relax the finitness condition on the dimension of V, we can instead consider it to be the set of continuous functions defined on the unit interval, that is, $V = \mathscr{C}[0,1]$, with

$$\theta(f) = \int_{[0,1]} f \, \mathrm{d}x.$$

This is an instance of an infinite dimensional Frobenius algebra.

Example 3 (Representation theory)**.** Let $G = \{g_0, g_1, ..., g_n\}$ be a finite abelian group, where g_0 is the identity element. Define $V := \mathbb{K}G$ as the group ring of G over $\mathbb{K}$, that is, the set of formal linear combinations

$$\sum_{i=0}^{n} \alpha_i g_i, \quad \alpha_i \in \mathbb{K}.$$

with addition and multiplication defined by

$$\left(\sum_i \alpha_i g_i\right) + \left(\sum_i \beta_i g_i\right) = \sum_i (\alpha_i + \beta_i) g_i,$$
$$\left(\sum_i \alpha_i g_i\right) \cdot \left(\sum_i \beta_i g_i\right) = \sum_{i,j} \alpha_i \beta_j g_i g_j.$$

It is clear that V is a commutative algebra. This is naturally identified with $V \cong \{f : G \longrightarrow \mathbb{K}\}$ where the product is given by the convolution

$$(f * h)(g) = \int_G f(\alpha) \cdot h(\alpha^{-1} g) \, \mathrm{d}\mu(\alpha),$$

with μ begin the counting measure.

Take any element $u \in V$. We define the Frobenius structure on V by

$$\theta : V \longrightarrow \mathbb{K}$$
$$u = \sum_{i=0}^{n} \alpha_i g_i \longmapsto \alpha_0.$$

It follows that the aforementioned inner product is non-degenerate. Indeed,

$$\theta(u g_i^{-1}) = \theta\left(\sum_{j=0}^{n} a_j g_j g_i^{-1}\right) = \theta\left(a_0 g_0 g_i^{-1} + ... + a_i g_0 + ... + a_n g_n g_i^{-1}\right) = a_i,$$

and so,

$$\theta(ug) = 0 \quad \forall g \in G \iff a_i = 0 \quad \forall i \iff u = 0.$$

The hypothesis that G is abelian can be relaxed, in which case we take $V = Z(\mathbb{K}[G])$ as being the center of the group algebra, that is,

$$Z(\mathbb{K}[G]) = \{z \in \mathbb{K}[G] \quad | \quad \forall g \in \mathbb{K}[G], zg = gz\}.$$

In such a case, we have the natural identification with class functions of G, that is,

$$Z(\mathbb{K}[G]) \cong \{f : G \longrightarrow \mathbb{K} \quad | \quad f(g) = g(hgh^{-1})\}.$$

Example 4 (Polynomials 1)**.** Fix a natural number $n \in \mathbb{N}^*$, and let

$$V = \frac{\mathbb{K}[t]}{(t^n)}.$$

We claim that any linear form $\theta : V \longrightarrow \mathbb{K}$ such that $\theta(t^{n-1}) \neq 0$ defines a Frobenius structure on V.

To show this, we must prove that g is non-degenerate. Assume that $\exists a \in V$ such that $\theta(a \cdot p) = 0 \ \forall p \in V$. The most general form that a can have is

$$\begin{aligned} a(t) &= t^h \cdot \left(c_0 + c_1 t + \dots + c_{n-1} t^{n-1}\right) \\ &= t^h \cdot \varphi(t). \end{aligned}$$

We shall prove that, in these conditions, $h \geq n$. Take the Taylor expansion at $t = 0$ of $1/\varphi(t)$, which we shall denote by $\tilde{\varphi}(t)$, truncated at order n. It clearly defines the inverse of $\varphi(t)$ inside V. Then if $h < n$, we can take

$$p = \tilde{\varphi}(t) \cdot t^{n-1-h}.$$

Then $a \cdot p = t^{n-1}$ and, by assumption, $\theta(a \cdot p) = 0$. But this contradicts the hypothesis that $\theta(t^{n-1}) \neq 0$.

Example 5 (Polynomials 2)**.** We now present a generalized version of the example above. Let $f_1, \dots, f_m \in \mathbb{K}[[z_1, \dots, z_m]]$ be elements in the ring of formal power series without constant term. Assume that the quotient

$$V = \frac{\mathbb{K}[[z_1, \dots, z_n]]}{(f_1, \dots, f_m)}$$

is of finite dimension over $\mathbb{K}$. Consider the Jacobian determinant

$$\det\left(\frac{\partial f_i}{\partial z_j}\right)_{ij}.$$

Since it is not an element of the ideal $(f_1, \dots, f_m)$, it has a non-zero image in V. Therefore, it generates a 1-dimensional vector subspace of V, which is an ideal that we shall denote by I.

As before, one can prove that an arbitrary linear map $\theta : V \longrightarrow \mathbb{K}$ defines a Frobenius algebra structure if and only if

$$I \not\subseteq \ker\theta.$$

In the previous example, we had $I = (t^{n-1})$.

Let's choose a specific linear form with $\mathbb{K} = \mathbb{C}$. We set

$$\theta : V \longrightarrow \mathbb{C}$$
$$\theta(h) = \frac{1}{(2\pi i)^m} \int_B \frac{h(z)}{f_1(z)\ldots f_m(z)} \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_m$$

where

$$B = \{z \in \mathbb{C}^m : |f_i(z)| = \rho,\ i = 1,\ldots,m,\ \rho > 0 \text{ sufficiently small}\}.$$

It follows from Grothendieck's Local Duality Theorem (GRIFFITHS; HARRIS, 1994, p. 659) that the corresponding bilinear form g is non-degenerate, and this thus defines a Frobenius algebra structure.

Example 6 (Ring theory)**.** Let A be a commutative ring which is also

- Artinian, i.e, A satisfies the descending chain condition, that is, any sequence $I_1 \supseteq I_2 \supseteq \ldots$ of ideals must stabilize at I_n for some $n > 0$.
- Local, i.e, A has a unique maximal ideal $\mathscr{M}$.

We define the following:

Definition 2. We define the *socle* of A as the set $\mathrm{soc}(A) = \{a \in A : a\mathscr{M} = 0\}$.

Definition 3. We say A is a *Gorenstein ring* if and only if any of the following equivalent conditions are satisfied:

- $\mathrm{soc}(A)$ is a simple A-module, i.e, does not contain any non-zero ideal;
- $\mathrm{soc}(A) \cong A/\mathscr{M}$ as $\mathbb{K}$-vector spaces.

One can show that Example 5 is an instance of a Gorenstein ring. The Frobenius algebra structure is given by the following theorem:

Theorem 1. If A is a commutative local $\mathbb{K}$-algebra of finite dimension, then the following statements are equivalent:

1. A is Frobenius;
2. A has a unique minimal ideal;
3. A is Gorenstein;

Demonstration. (3) $\iff$ (2)**:** Assume first that A is Gorenstein. Then $\mathrm{soc}(A)$ is a simple A-submodule, and thus, a minimal ideal. We'll show that it is unique. Because A is artinian, $\mathrm{soc}(A)$ is an essential submodule. Thus, any non-trivial minimal ideal $J \subset A$ has a non-trivial intersection with $\mathrm{soc}(\mathrm{A})$. Since submodule intersections is again a submodule and $J \cap \mathrm{soc}(A) \subset \mathrm{soc}(A)$, then we must have $J = \mathrm{soc}(A)$. Conversely, if A has a unique minimal ideal, then, since it is artinian, a similar argument shows that this minimal ideal must indeed be the socle of A, and thus A is Gorenstein.

(3) $\implies$ (1): See (BEHNKE, 1981, p. 221).

(1) $\implies$ (3): We recall that a finite dimensional algebra over $\mathbb{K}$ is Frobenius if and only if for all ideals $I \subset A$, we have $\mathrm{ann}(\mathrm{ann}(I)) = I$ and $(I : \mathbb{K}) + (\mathrm{ann}(I) : \mathbb{K}) = (A : \mathbb{K})$, where where $(W : K)$ denotes the dimension over $\mathbb{K}$ of the vector space W. See Theorem (61.3) by Curtis and Reiner (2006, p. 414). Choosing $I = \mathrm{soc}(A)$, we have that

$$(\mathrm{soc}(A) : \mathbb{K}) = (A : \mathbb{K}) - (\mathrm{ann}(\mathrm{soc}(A)) : \mathbb{K}) = (A : \mathbb{K}) - (\mathscr{M} : \mathbb{K}),$$

that is, $\mathrm{soc}(A)$ is isomorphic, as a $\mathbb{K}$-vector space, to $A/\mathscr{M}$.

□

This means that, in the local case, Frobenius algebras and Gorenstein rings are the same thing.

2.3 2-dimensional Topological Field Theories

We now present an example of a Frobenius algebra arising from 2-dimensional Topological Field Theories. The material presented here, as well as further reading on this particular subject, is shown in great detail by Atiyah (1990).

Let Σ be a smooth oriented surface with boundary $\partial\Sigma$. We do not assume that the boundary is equipped with the orientation induced by Σ. We set

$$\partial\Sigma = C_1 \sqcup \ldots \sqcup C_k, \tag{2.4}$$

where each C_i is an oriented cycle. A 2-dimensional Topological Field Theory, or 2d-TFT, is a functor Z,

$$(\Sigma, \partial\Sigma) \xmapsto{Z} \left(v_{(\Sigma,\partial\Sigma)}, A_{(\Sigma,\partial\Sigma)}\right), \tag{2.5}$$

which assigns to each compact oriented 1-manifold $\partial\Sigma$ a finite dimensional $\mathbb{C}$-vector space $Z(\partial\Sigma) := A_{(\Sigma,\partial\Sigma)}$ and to each surface Σ with boundary $\partial\Sigma$ a vector $Z(\Sigma) := v_{(\Sigma,\partial\Sigma)} \in A_{(\Sigma,\partial\Sigma)}$. Such an assignment is assumed to depend only on the topology of $(\Sigma, \partial\Sigma)$.

To explicitly define the functor Z, first consider the distinguished pair $(\Delta, \partial\Delta)$ where

$$\Delta = \mathbb{D}^2 = \{z \in \mathbb{C} : |z| \leq 1\}, \quad \partial\Delta = \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}. \tag{2.6}$$

We denote the vector space associated with it by A:

$$Z(\partial\Delta) = A. \tag{2.7}$$

Now, each linear space $A_{(\Sigma,\partial\Sigma)}$ is set as

$$A_{(\Sigma,\partial\Sigma)} = \begin{cases} \mathbb{C} \text{ if } \partial\Sigma = \emptyset, \\ \bigotimes_{i=1}^{k} A_i \text{ if } \partial\Sigma = \bigsqcup_{i=1}^{k} C_i, \end{cases} \tag{2.8}$$

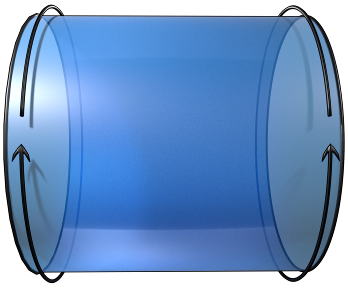
where each A_i is defined to be

$$A_i = \begin{cases} A \text{ if the orientation of } C_i \text{ is coherent with the one induced from } \Sigma, \\ A^* \text{ otherwise.} \end{cases} \tag{2.9}$$

Moreover, we require the following axioms:

1. Normalization The cylinder in Figure 1, along with the orientation on it's boundary,

Figure 1 – A cylinder with oriented boundary



Source: Elaborated by the author.

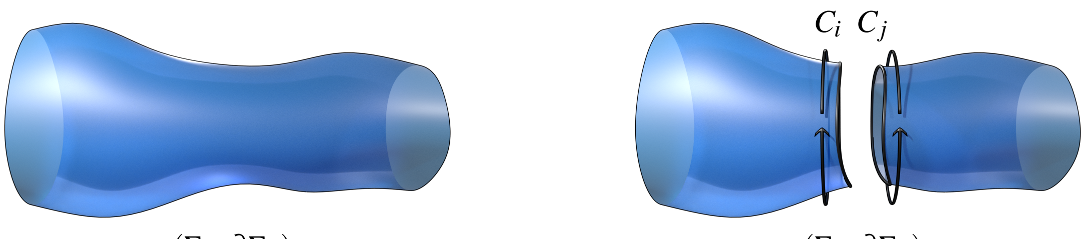
is associated to the pair $(\mathrm{id}_A, A^* \otimes A)$. [1]

2. Multiplicativity Assume we have a disjoint union $(\Sigma,\partial\Sigma) = (\Sigma_1,\partial\Sigma_1) \sqcup (\Sigma_2,\partial\Sigma_2)$. Then the assigned vector is a decomposible tensor and

$$A_{(\Sigma,\partial\Sigma)} = A_{(\Sigma_1,\partial\Sigma_1)} \otimes A_{(\Sigma_2,\partial\Sigma_2)},$$
$$v_{(\Sigma,\partial\Sigma)} = v_{(\Sigma_1,\partial\Sigma_1)} \otimes v_{(\Sigma_2,\partial\Sigma_2)}.$$

3. Factorization Assume $(\Sigma_1,\partial\Sigma_1)$ and $(\Sigma_2,\partial\Sigma_2)$ are equal outside a ball, inside of which they differ as shown in Figure 2:

Figure 2 – The manifold is split as shown, and left with two additional cycles on it's boundary



Source: Elaborated by the author.

[1] A simple reminder that if V,W are two finite dimensional vector spaces, then $V^* \otimes W \cong \mathrm{Hom}(V,W)$

Then, since $A_i = A_j^*$, we have a natural (i,j)-contraction map

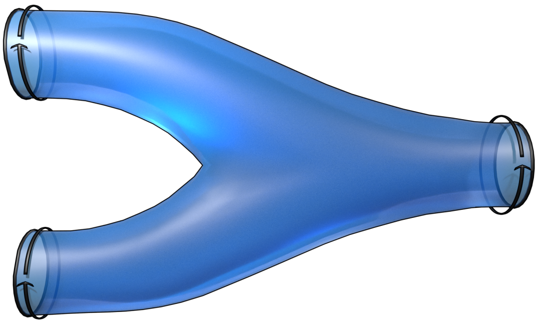
$$\begin{aligned} A_{(\Sigma_2,\partial\Sigma_2)} &\longrightarrow A_{(\Sigma_1,\partial\Sigma_1)} \\ A_1 \otimes \ldots \otimes A_i \otimes \ldots \otimes A_j \otimes \ldots \otimes A_k &\longrightarrow A_1 \otimes \ldots \otimes A_k \\ v_{(\Sigma_2,\partial\Sigma_2)} &\longmapsto v_{(\Sigma_1,\partial\Sigma_1)} \end{aligned} \tag{2.10}$$

where we require that $v_{(\Sigma_1,\partial\Sigma_1)}$ is the (i,j)-contraction of $v_{(\Sigma_2,\partial\Sigma_2)}$.

The following theorem shows that there is a Frobenius algebra behind the structure above.

Theorem 2. The space $A = A_{(\Delta,\partial\Delta)}$ carries a natural structure of a Frobenius algebra. The product is defined by mapping the manifold in Figure 3 to $c \in A^* \otimes A^* \otimes A \cong \mathrm{Hom}(A \otimes A, A)$.

Figure 3 – Product



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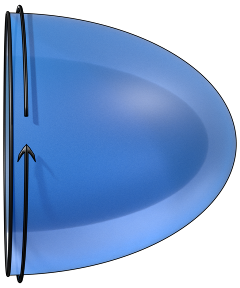
The Frobenius form is acquired by mapping the manifold in Figure 4 to $\theta \in A^*$.

Figure 4 – Frobenius form θ



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Demonstration. Firstly, commutativity of the product follows from the diffeomorphism shown in Figure 5. An analogous observation in Figure 6 shows associativity.

Figure 5 – Commutativity

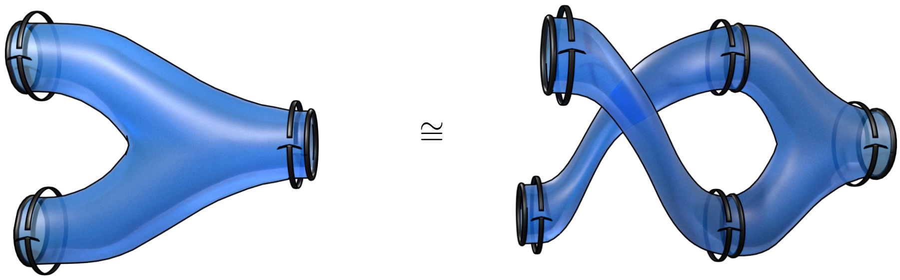

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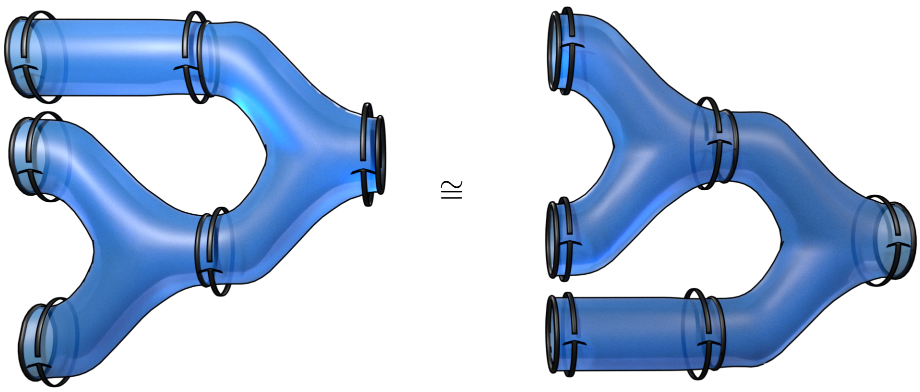
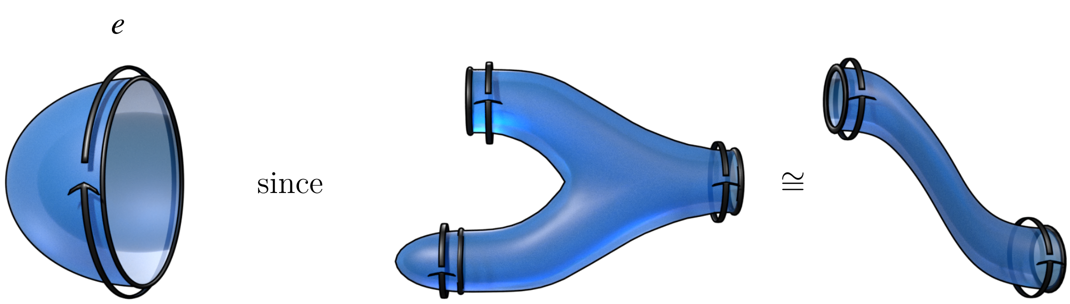
Figure 6 – Associativity



Source: Elaborated by the author.

The identity vector is given in Figure 7, as well as it's product with another arbitrary vector.

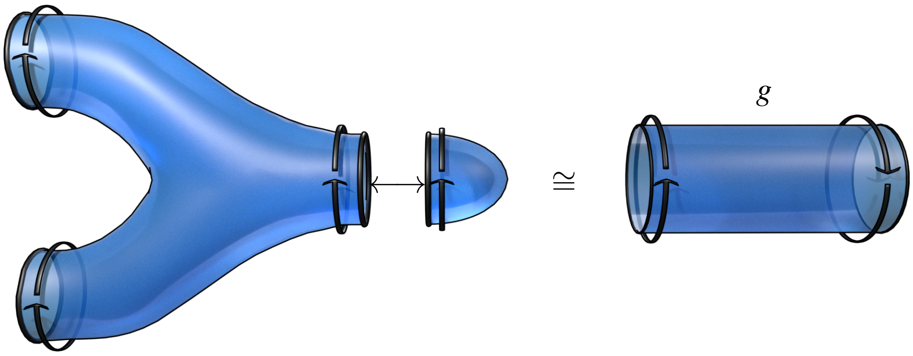
Figure 7 – Identity



Source: Elaborated by the author.

The Frobenius metric, given by $g(a,b) = \theta(a \cdot b)$, is represented by Figure 8.

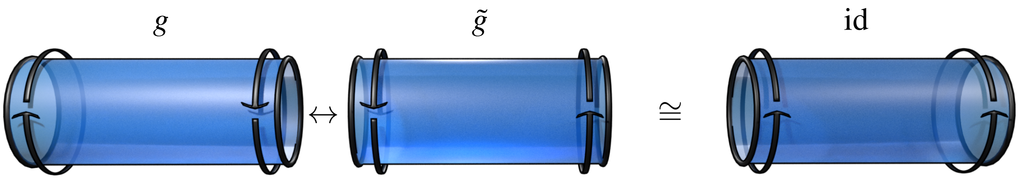
Figure 8 – Frobenius metric



Source: Elaborated by the author.

Finally, the Frobenius metric is non-degenerate, as shown in Figure 9.

Figure 9 – Non-degeneracy

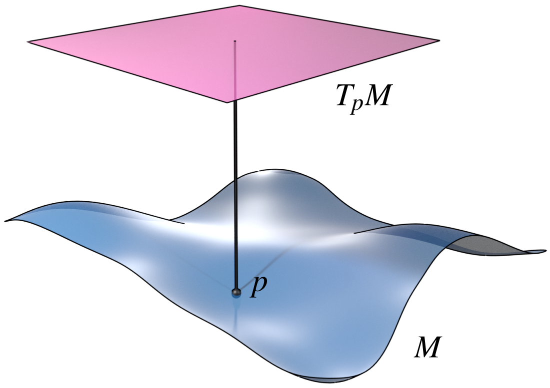


Source: Elaborated by the author.

□

This example is of particular interest since it is very close to the way in which Dubrovin arrived at the notion of Frobenius manifolds. In fact, he was studying the so called WDVV equations of associativity, which, as mentioned in (DUBROVIN, 1998b), is the problem of finding a quasi-homogeneous function such that certain combinations of it's third derivatives generate structure constants of an associative algebra. WDVV equations arise as defining relations of 2-dimensional Topological Field Theories. Frobenius manifolds were introduced as a geometric interpretation and a coordinate-free form for the WDVV equations. Further reading on this subject, as well as most of the math regarding Frobenius manifolds presented in this work, can be found in the aforementioned work, as well as in (DUBROVIN, 1996) and (DUBROVIN, 1992).

Figure 10 – A Frobenius manifold is a family of Frobenius algebras



Source: Elaborated by the author.

2.4 Frobenius manifolds

2.4.1 Definition and basic properties

We have now enough material to introduce the concept of Frobenius manifolds. Let M be a n-dimensional complex manifold [2] , and let TM and T^*M denote the holomorphic tangent and cotangent bundles, respectively. Assume that M has a smoothly varying Frobenius algebra structure on each tangent space. If V is a vector bundle over M, we denote by $\Gamma(M,V)$ the space of it's global sections.

We recall that our primal objective is to define a manifold such that it's tangent bundle's fibers are equipped with Frobenius algebra structures. We have seen in section 2.1 that the structure of a Frobenius algebra over a vector space was given by three algebraic objects (see Equation 2.1, Equation 2.2 and Equation 2.3). In the context of Frobenius manifolds, they shall be promoted to sections

$$\theta \in \Gamma(T^*M), \tag{2.11}$$

$$g \in \Gamma(\mathrm{Sym}^2 T^*M), \tag{2.12}$$

$$c \in \Gamma(\mathrm{Sym}^3 T^*M). \tag{2.13}$$

Let's see how these are constructed. Namely, they appear directly from defining a metric for the manifold and structure constants for the product of vector fields such that they are compatible in the sense of a Frobenius structure.

Definition 4 (Dubrovin-Frobenius manifold)**.** Let M be a n-dimensional complex manifold with a smoothly varying Frobenius algebra structure on each tangent space. A *Dubrovin-Frobenius structure* on M is given by:

[2] Although we use the field of complex numbers for convenience, the concept applies to real manifolds as well.

(FM1) A flat metric over M, that is, a symmetric non-degenerate $\mathscr{O}_M$-bilinear form $g \in \Gamma(\mathrm{Sym}^2 T^*M)$ whose Levi-Civita connection ∇ is flat. This defines a metric over M.

(FM2) A $(1,2)$-tensor $c \in \Gamma(TM \otimes \mathrm{Sym}^2 T^*M)$ such that the induced multiplication of vector fields defined by

$$X \cdot Y = c(\cdot, X, Y), \quad X, Y \in \Gamma(TM) \tag{2.14}$$

is associative. Moreover, by using the metric, we define a completely covariant tensor $c^\flat \in \Gamma(T^*M \otimes \mathrm{Sym}^2 T^*M)$, which is simply c with a lowered index, and we require that

$$c^\flat \in \Gamma(\mathrm{Sym}^3 T^*M), \tag{2.15}$$

$$\nabla c^\flat \in \Gamma(\mathrm{Sym}^4 T^*M). \tag{2.16}$$

Note that $c^\flat$ is the manifold equivalent of the algebraic operator defined in Equation 2.3. Throughout the text, both c and $c^\flat$ will be denoted simply by "c", letting context dictate which one is being used. Usually, c (with one upper index) shall be referred to as the *structure constants* of the product.

(FM3) A vector field $e \in \Gamma(TM)$, called *unit vector field*, such that the bundle isomorphism $c(\cdot, e, \cdot) : TM \longrightarrow TM$ is the identity morphism. We also require e to be covariant constant:

$$\nabla e = 0. \tag{2.17}$$

(FM4) A vector field $E \in \Gamma(TM)$, called *Euler vector field*, such that

$$\mathscr{L}_E g = (2-d)g, \tag{2.18}$$

$$\mathscr{L}_E c = c. \tag{2.19}$$

Here c denotes the structure constants and $d \in \mathbb{C}$ is called the *charge* of the manifold.

It is worth making a few comments about this definition. Firstly, since the Levi-Civita connection of g is assumed to be flat, there is a distinguished local system of coordinates $(t^1, \ldots, t^n)$ with respect to which the metric is constant and the Christoffel symbols all vanish. This set of coordinates will be referred to as *flat coordinates*.

Furthermore, the Euler vector field E is a conformal Killing field that scales the metric by a constant. In conjunction to the flatness of ∇, this implies that E is an affine vector field, that is,

$$\nabla\nabla E = 0. \tag{2.20}$$

This allows us to show that

Proposition 2. In flat coordinates, the Euler vector field is given by

$$E = \sum_{\alpha=1}^{n} ((1-q_\alpha)t^\alpha + r_\alpha)\frac{\partial}{\partial t^\alpha}, \quad q_\alpha, r_\alpha \in \mathbb{C}. \tag{2.21}$$

Moreover, the coordinates can be chosen in such a way that $r_\alpha \neq 0$ only if $q_\alpha = 1$.

Demonstration. The following argument is by Hitchin (1997, p. 80). Firstly, since E is a conformal Killing field that scales the metric by a constant, then, in flat coordinates, it is the sum of an infinitesimal translation, rotation, and dilation (see section 1.4 of the book by Schottenloher (2008, p. 15) for a demonstration of this fact.) It may thus assume the form

$$E = \sum_{\alpha,\beta} S_{\alpha\beta} t^\alpha \frac{\partial}{\partial t^\beta} + a\sum_\alpha t^\alpha \frac{\partial}{\partial t^\alpha} + \sum_\alpha r_\alpha \frac{\partial}{\partial t^\alpha},$$

where S is a skew-symmetric rotation matrix. To simplify the exposition, we will prove the statement only under the assumption that S has distinct eigenvalues λ_α. In this case, the expression above becomes

$$E = \sum_\alpha (\lambda_\alpha + a)t^\alpha \frac{\partial}{\partial t^\alpha} + \sum_\alpha r_\alpha \frac{\partial}{\partial t^\alpha}.$$

Now simply choose $q_\alpha := 1 - \lambda_\alpha - a$ and the first part of the proposition is proved.

For the second part, if $q_\alpha \neq 1$, then one can simply apply a translation by $\frac{r_\alpha}{1-q_\alpha}$ to transform the expression into the form

$$E = \sum_\alpha (1-q_\alpha)t^\alpha \frac{\partial}{\partial t^\alpha}.$$

□

Lastly, we note that flat coordinates are unique only up to an Euclidean transformation and e is covariant constant. This allows us to assume without loss of generality that the first flat coordinate is always chosen so that the corresponding coordinate vector field is always the unit vector field:

$$e = \partial t^1. \tag{2.22}$$

2.4.2 Potential function and WDVV equations

With respect to the system of flat coordinates, we can set the components of the metric tensor and the structure constants as

$$g_{\alpha\beta} = g\left(\frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial t^\beta}\right), \tag{2.23}$$

$$c^\gamma_{\alpha\beta} = \left(\mathrm{d}t^\gamma, \frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial t^\beta}\right). \tag{2.24}$$

Note that item (FM2) in Definition 4 sets both $c_{\alpha\beta\gamma}$ and $\partial_\delta c_{\alpha\beta\gamma}$ as completely symmetric tensors. This leads us to the following proposition:

Proposition 3. Locally, there exists a function F such that

$$c_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta . \partial t^\gamma}. \tag{2.25}$$

The function F is called the *potential function* of the Frobenius manifold.

Demonstration. The argument follows directly from Poincaré's Lemma. Indeed, since ∇c is completely symmetric and

$$\nabla c = \sum_\delta \frac{\partial c}{\partial t^\delta} \otimes \frac{\partial}{\partial t^\delta} = \sum_\delta \partial_\delta c_{\alpha\beta\gamma} \mathrm{d}t^\alpha \, \mathrm{d}t^\beta \, \mathrm{d}t^\gamma \otimes \frac{\partial}{\partial t^\delta},$$

we can write

$$\frac{\partial c_{\alpha\beta\gamma}}{\partial t^\delta} = \frac{\partial c_{\alpha\beta\delta}}{\partial t^\gamma}.$$

Now, define a 3-form $\xi = c_{\alpha\beta\gamma} \mathrm{d}t^\alpha \wedge \mathrm{d}t^\beta \wedge \mathrm{d}t^\gamma$. The symmetry of ∇c implies that ξ is a closed form since

$$\begin{aligned}
\mathrm{d}\xi &= \mathrm{d}c_{\alpha\beta\gamma} \mathrm{d}t^\alpha \wedge \mathrm{d}t^\beta \wedge \mathrm{d}t^\gamma \\
&= \partial_\delta c_{\alpha\beta\gamma} \mathrm{d}t^\delta \wedge \mathrm{d}t^\alpha \wedge \mathrm{d}t^\beta \wedge \mathrm{d}t^\gamma \\
&= \partial_\gamma c_{\alpha\beta\delta} \mathrm{d}t^\delta \wedge \mathrm{d}t^\alpha \wedge \mathrm{d}t^\beta \wedge \mathrm{d}t^\gamma \\
&= -\partial_\gamma c_{\alpha\beta\delta} \mathrm{d}t^\gamma \wedge \mathrm{d}t^\alpha \wedge \mathrm{d}t^\beta \wedge \mathrm{d}t^\delta \\
&= -\mathrm{d}\xi.
\end{aligned}$$

From this, we apply Poincaré's Lemma and deduce that there is a 2-form $\zeta = \zeta_{\alpha\beta} \, \mathrm{d}t^\alpha \wedge \mathrm{d}t^\beta$ such that $\xi = \mathrm{d}\zeta$. We then have

$$\mathrm{d}\zeta = \partial_\gamma \zeta_{\alpha\beta} \, \mathrm{d}t^\gamma \wedge \mathrm{d}t^\alpha \wedge \mathrm{d}t^\beta = \partial_\gamma \zeta_{\alpha\beta} \, \mathrm{d}t^\alpha \wedge \mathrm{d}t^\beta \wedge \mathrm{d}t^\gamma = c_{\alpha\beta\gamma} \mathrm{d}t^\alpha \wedge \mathrm{d}t^\beta \wedge \mathrm{d}t^\gamma.$$

This means that we found a set of functions $\zeta_{\alpha\beta}$ such that $c_{\alpha\beta\gamma} = \partial_\gamma \zeta_{\alpha\beta}$. Symmetry then implies that $\zeta_{\alpha\beta} = \zeta_{\beta\alpha}$. This argument can be repeated to find functions ω_α such that $\zeta_{\alpha\beta} = \partial_\beta \omega_\alpha$. Repeating the process one final time yields the potential function F. □

Corollary 1. The metric g in terms of the potential function is written as

$$g_{\alpha\beta} = \frac{\partial^3 F}{\partial t^1 \partial t^\alpha \partial t^\beta}. \tag{2.26}$$

Demonstration. Note from Equation 2.3 that for any $X, Y \in \Gamma(TM)$ we have

$$g(X,Y) = c(e,X,Y).$$

Combine this with Equation 2.22 and the result follows directly. □

It is worth noting that, even though we have written all of the mathematical objects defined so far in terms of a single function, the associativity condition of the algebra on the tangent bundle requires the potential to satisfy an overdetermined system of nonlinear partial differential equations. Recall the structure constants

$$\frac{\partial}{\partial t^{\alpha}} \cdot \frac{\partial}{\partial t^{\beta}} = c^{\gamma}_{\alpha\beta} \frac{\partial}{\partial t^{\gamma}}. \tag{2.27}$$

Then from the definition of c we have that that

$$c_{\alpha\beta\gamma} = g_{\gamma\delta} c^{\delta}_{\alpha\beta} \implies c^{\gamma}_{\alpha\beta} = g^{\gamma\delta} c_{\delta\alpha\beta}. \tag{2.28}$$

Now associativity is written in terms of the structure constants as

$$c^{\mu}_{\alpha\delta} c^{\delta}_{\beta\nu} - c^{\mu}_{\beta\delta} c^{\delta}_{\alpha\nu} = 0. \tag{2.29}$$

Now we simply apply Proposition 3 and Equation 2.28 to obtain

$$\boxed{\frac{\partial^3 F}{\partial t^{\alpha} \partial t^{\beta} \partial t^{\gamma}} \cdot g^{\gamma\delta} \cdot \frac{\partial^3 F}{\partial t^{\delta} \partial t^{\lambda} \partial t^{\mu}} = \frac{\partial^3 F}{\partial t^{\mu} \partial t^{\beta} \partial t^{\gamma}} \cdot g^{\gamma\delta} \cdot \frac{\partial^3 F}{\partial t^{\delta} \partial t^{\lambda} \partial t^{\alpha}}} \tag{2.30}$$

These are the so called associativity equations, or *WDVV equations*, short for Witten-Dijkgraaf-Verlinde-Verlinde, originally proposed by Dijkgraaf, Verlinde and Verlinde (1991) in the article "Topological strings in $d < 1$" , where it occupies an important place in topological field theories. This development is discussed in greater detail in the text by Magri (2016). For additional readings on the subject, see (VAŠÍČEK; VITOLO, 2021).

CHAPTER 3

ANALYTIC THEORY OF FROBENIUS MANIFOLDS

3.1 Semisimple Frobenius algebras

Let $\mathbb{K}$ be a field of characteristic zero, usually taken to be the field of complex numbers $\mathbb{C}$. Let A be an associative, commutative and unital algebra of finite dimension n over $\mathbb{K}$.

Definition 5. For each $a \in A$, we denote the endomorphism given by multiplication by a as

$$\begin{aligned} l_a : A &\longrightarrow A \\ x &\longmapsto a \cdot x. \end{aligned} \tag{3.1}$$

Our objective in this particular section is to prove the following theorem:

Theorem 3. In regards to the algebra A, the following conditions are equivalent:

i) A has no nilpotent elements;

ii) We can find idempotent elements $\sigma_1, ..., \sigma_n \in A$ such that $\sigma_i \cdot \sigma_j = \sigma_i \delta_{ij}$;

iii) $A \cong \mathbb{K}^n$, with multiplication defined component-wise;

iv) There is an element $\mathscr{E} \in A$ such that $l_{\mathscr{E}} : A \longrightarrow A$ has a simple spectrum.

There are a few observations that need to be made before a formal proof is presented. We note, though, that it will be based on a study of the minimal ideals of A, that is, ideals that don't contain any other non-trivial ideal.

Definition 6. An algebra A that satisfies the conditions of Theorem 3 is called an *semisimple algebra*.

Although the notion of minimal ideal is, in a sense, dual to the one of maximal ideal, there is a notable difference. While a theorem by Krull (1929) shows that every unital ring R has a maximal ideal [1], this is in general not true for their minimal counterparts.

Example 7. The ring of integers $\mathbb{Z}$ has no minimal ideal. Indeed, let $I_n = (n) \subset \mathbb{Z}$ be the ideal generated by $0 \neq n \in \mathbb{Z}$. This is without loss of generality, since $\mathbb{Z}$ is a principal ideal domain. Then it is clear that I_n contains the ideal $I_{n^2} = (n^2)$, and so it cannot be a minimal ideal.

However, if R is an Artinian ring, then Zorn's Lemma implies that it must have minimal ideals. In fact, all of the algebras considered here are Artinian:

Lemma 1. Every finite dimensional algebra A is Artinian.

Demonstration. Indeed, since A is a vector space, then every ideal $I \subset A$ is a $\mathbb{K}$-vector subspace of A. Since A is finite dimensional, every descending chain of ideals must stabilize in a 1-dimensional subspace of A. □

The following lemma offers an important description of the minimal ideals of A:

Lemma 2 (Brauer's Lemma)**.** If $I \subset A$ is a minimal ideal, then there are exactly two possibilities:

i) $I^2 = 0$;

ii) $I = e \cdot A$, where e is idempotent.

Demonstration. Let $I \subset A$ be a minimal ideal and assume that $I^2 \neq 0$. Then we can find elements $x, y \in I$ such that $x \cdot y \neq 0$. Denote by f the morphism l_x restricted to the ideal I, that is,

$$\begin{aligned} f = l_x|_I : I &\longrightarrow I \\ a &\longmapsto x \cdot a. \end{aligned}$$

It follows from hypothesis that f is non-zero. Moreover, both $\ker(f)$ and $\mathrm{Im}(f)$ are submodules of I. But then minimality of I implies that $\ker(f) = 0$ and $\mathrm{Im}(f) = I$, that is, f is an isomorphism. Because of this, we can find $e \in I$ such that $x = x \cdot e$. It follows that e is idempotent, since

$$x = x \cdot e \iff x \cdot e = x \cdot e^2 \iff f(e) = f(e^2) \iff e = e^2.$$

[1] In fact, Krull's theorem is equivalent to the Axiom of Choice. See (HODGES, 1979).

Finally, since $e \cdot A$ is a non-zero ideal contained in I, it follows from minimality that $I = e \cdot A$. □

We now have all the necessary tools to prove Theorem 3.

Demonstration. Notice first that (iii) $\implies$ (ii) $\implies$ (i). Indeed, if $A \cong \mathbb{K}^n$ with multiplication defined component-wise, we can simply take the idempotents to be $\sigma_1 = (1,0,...,0)$, $\sigma_2 = (0,1,0,...)$, ..., $\sigma_n = (0,...,0,1)$, which gives us (ii). Furthermore, (i) follows directly from the fact that $\mathbb{K}$ is a field and thus has no nilpotent elements.

Let's now show that (i) $\implies$ (iii). This is done via induction on the dimension of A. The property is clearly valid for $n = 1$, since in that case $A \cong \mathbb{K}$. Now assume it is valid for $n-1$. Then by Lemma 1 and Lemma 2, we know that A has a minimal ideal $I = e \cdot A$ where $e \in A$ is idempotent. Define a new ideal $I' = (1-e) \cdot A$. Then the algebra is a direct sum $A = I \oplus I'$. Also, I' is a $(n-1)$-dimensional $\mathbb{K}$-algebra without nilpotent elements, and by induction hypothesis we have $I' \cong \mathbb{K}^{n-1}$, and the result follows.

Finally, we show that (ii) $\iff$ (iv). If (ii) is given, then we can simply take

$$\mathscr{E} = \sum_{k=1}^{n} k \cdot \sigma_k \tag{3.2}$$

and condition (iv) is readily satisfied. Conversely, if (iv) is given, then by commutativity, the family of operators $\{l_a\}_{a \in A}$ is a family of commuting operators. In particular, any operator l_a commutes with $l_{\mathscr{E}}$. This implies that l_a preserves the eigen-spaces of $\mathscr{E}$, and thus it must be diagonalizable. Now, since $\{l_a\}_{a \in A}$ is a family of commuting diagonilizable operators, they must be simultaneously diagonilizable, say by a basis $e_1, ..., e_n$. Now the idempotents σ_i can be constructed via suitable rescalings of e_i. Indeed, since any e_i is an eigenvector of eny l_{e_j}, then, for scalars $\lambda, \nu \in \mathbb{K}$,

$$\begin{cases} e_i \cdot e_j = \lambda e_j, \\ e_j \cdot e_i = \nu e_i. \end{cases}$$

But A is assumed to be a commutative algebra. Thus this can only happen if

$$\begin{cases} e_i \cdot e_j = 0 \text{ when } i \neq j; \\ e_i^2 = \lambda_i e_i \text{ for each } i. \end{cases}$$

We finally take $\sigma_i := \lambda_i^{-1} e_i$, and the proof is finished. □

Corollary 2. The unit of A is given by the sum of the idempotents:

$$1_A = \sum_{i=1}^{n} \sigma_i \tag{3.3}$$

Demonstration. Firstly, it's clear from the proof of Theorem 3 that the idempotents $\sigma_1,\dots,\sigma_n$ constitute a basis for the algebra A. We write, for a generic $u \in A$ and $a_1,\dots,a_n,b_1,\dots,b_n \in \mathbb{K}$,

$$u = \sum_{k=1}^{n} a_k\sigma_k, \quad 1_A = \sum_{k=1}^{n} b_k\sigma_k.$$

Solving for $u \cdot 1_A = u$, we have:

$$\begin{aligned}\left(\sum_{k=1}^{n} a_k\sigma_k\right)\cdot\left(\sum_{l=1}^{n} b_l\sigma_l\right) &= \sum_{k=1}^{n} a_k\sigma_k\\ \sum_{k,l=1}^{n} a_kb_l(\sigma_k\cdot\sigma_l) &= \sum_{k=1}^{n} a_k\sigma_k\\ \sum_{k,l=1}^{n} a_kb_l(\delta_{kl}\sigma_l) &= \sum_{k=1}^{n} a_k\sigma_k\\ \sum_{k=1}^{n} a_kb_k\sigma_k &= \sum_{k=1}^{n} a_k\sigma_k.\end{aligned}$$

Comparing each coefficient gives us $a_kb_k = a_k\forall k$ and, since a_k is generic, $b_k = 1\forall k$, and the proof is done. □

Definition 7. The basis consisting of the idempotents $\sigma_1,\dots,\sigma_n$ will be referred to as the *canonical basis* of the algebra A. It is unique up to reordering.

A few additional corollaries derive from the theorem when A has, additionally, a Frobenius structure attached to it. They all refer to the algebraic constraints imposed by the Frobenius metric g and the Frobenius form θ.

Corollary 3. The canonical basis elements are orthogonal in relation to the Frobenius metric, that is,

$$g\left(\sigma_i,\sigma_j\right) = 0 \text{ if } i \neq j. \tag{3.4}$$

Demonstration. This follows directly from computation:

$$g\left(\sigma_i,\sigma_j\right) = g\left(\sigma_i\cdot\sigma_i,\sigma_j\right) = g\left(\sigma_i,\sigma_i\cdot\sigma_j\right) = g\left(\sigma_i,0\right) = 0.$$

□

Corollary 4. The maps $\{l_a\}_{a\in A}$ from Definition 5 constitute a family of commuting self-adjoint endomorphisms of A.

Demonstration. Computation is analogous to the proof above. For any $u,v \in A$,

$$g\left(l_a(u),v\right) = g\left(a\cdot u,v\right) = g\left(u,a\cdot v\right) = g\left(u,l_a(v)\right).$$

□

Corollary 5. Let $\eta_1, ..., \eta_n \in A^*$ denote the basis of A^* dual to the canonical basis of A. Define functions $\mu_1, ..., \mu_n$ by $\mu_k = \theta(\sigma_k)$ for each k. Then

$$\theta = \sum_{k=1}^{n} \mu_k \eta_k, \tag{3.5}$$

$$g = \sum_{k=1}^{n} \mu_k \eta_k^2, \tag{3.6}$$

$$c = \sum_{k=1}^{n} \mu_k \eta_k^3. \tag{3.7}$$

Demonstration. This is done through direct computation. First, writing $\theta = \sum_i \theta_i \eta_i$ as a generic linear combination of the dual basis' elements, we evaluate it on the basis vectors:

$$\theta(\sigma_k) = \left(\sum_i \theta_i \eta_i\right)(\sigma_k) = \sum_i \theta_i \eta_i(\sigma_k) = \theta_k \implies \theta_k = \mu_k \text{ for any } k,$$

and this shows Equation 3.5.

The exact same procedure can be applied to g and c to show Equation 3.6 and Equation 3.7, keeping note that Corollary 3 implies that all cross terms automatically vanish. □

3.2 Semisimple Frobenius manifolds

In analogy to what we did in section 2.4, we now expand these algebraic concepts to the context of manifolds. Let (M, g, c, e, E) be a Dubrovin-Frobenius manifold, as described in Definition 4.

Definition 8 (Semisimple Frobenius manifold)**.** We say a point $p \in M$ is *semisimple* if the corresponding Frobenius algebra (T_pM, g_p, c_p, e_p) is a semisimple algebra. Moreover, we say that the Frobenius manifold M is *semisimple* if it has at least one semisimple point.

Item (iv) in Theorem 3 implies that semi-simplicity of points in a Frobenius manifold is an open property. We denote the set of semisimple points in M by

$$M_{SS} := \{p \in M : p \text{ is semisimple }\}, \tag{3.8}$$

and, of course, $M_{SS} \subset M$ is open.

Definition 9. The complement of M_{SS}, denoted by $K_M := M \setminus M_{SS}$, is called the *caustic* of M.

It is an interesting fact that the caustic K_M is either an empty set or a hypersurface. This is shown by Hertling (2002, p. 13).

Let's turn our attention to the idempotents introduced in item (ii) of Theorem 3. Here, they are promoted to local holomorphic vector fields $\sigma_1, ..., \sigma_n \in \Gamma(TM)$. The following result is very important for the construction of the theory. It's proof is shown at next section after the introduction of some additional auxiliary tools.

Theorem 4 (Dubrovin's Holonomicity Theorem)**.** The idempotent vector fields are holonomic, that is, for any i, j,

$$\left[\sigma_i, \sigma_j\right] = 0. \tag{3.9}$$

Corollary 6. There is a local holomorphic system of coordinates $u^1, ..., u^n$ such that it's partial derivatives contitute the canonical basis at each point of the coordinate system, that is,

$$\frac{\partial}{\partial u^i} = \sigma_i. \tag{3.10}$$

Demonstration. See Proposition 8.1 at (ILIEV, 2006, p. 70). □

It is worth noting that many formulas in the theory get considerably simplified in this system of coordinates. Namely, the tensor $c \in \Gamma\left(TM \otimes \mathrm{Sym}^3 T^*M\right)$ has it's components simply given by

$$c^c_{ab} = \delta^c_a \delta^c_b.$$

Definition 10. The set of coordinates $(u^1, ..., u^n)$ is called the set of *Dubrovin canonical coordinates*, or simply *canonical coordinates*.

Unfortunately, canonical coordinates are not uniquely defined. Firstly, they have an ambiguity of permutations: we could choose a different labelling for the idempotents, which would result in a different labelling for the coordinates. Furthermore, we have a "shift" (or translation) ambiguity: we can add a constant to any of the coordinates and Corollary 6 is still valid. The latter is resolved by the following proposition:

Proposition 4. Canonical coordinates can be chosen in such a way that the Euler vector field is given by

$$E = \sum_{i=1}^{n} u^i \frac{\partial}{u^i}. \tag{3.11}$$

Demonstration. It is sufficient to show that

$$\mathscr{L}_E\left(\frac{\partial}{\partial u^i}\right) = -\frac{\partial}{\partial u^i}. \tag{3.12}$$

Indeed, if this identity is true, then writing E as a sum of functions of the coordinates,

$$E = \sum_{i=1}^{n} E^i(u^1, ..., u^n) \frac{\partial}{\partial u^i},$$

we have

$$\mathscr{L}_E\left(\frac{\partial}{\partial u^i}\right)=\left[E,\frac{\partial}{\partial u^i}\right]=\sum_{j=1}^{n}\left(E\left(\frac{\partial}{\partial u^i}\right)^j-\frac{\partial}{\partial u^i}\left(E^j\right)\right)\frac{\partial}{\partial u^j}=-\sum_{j=1}^{n}\frac{\partial E^j}{\partial u^i}\frac{\partial}{\partial u^j}.$$

Then, Equation 3.12 leads us to the system of equations below,

$$\begin{cases}\frac{\partial E^i}{\partial u^j}=0 \text{ if } i\neq j,\\ \frac{\partial E^i}{\partial u^i}=1,\end{cases}$$

which is readily integrated to obtain Equation 3.11.

To prove Equation 3.12, we apply item (FM4) of Definition 4. Then we find that, given any two vector fields $X,Y\in\Gamma(TM)$, the Lie derivative of their product satisfies the following:

$$\mathscr{L}_E(c)=c \implies \mathscr{L}_E(X\cdot Y)-\mathscr{L}_E(X)\cdot Y-\mathscr{L}_E(Y)\cdot X=X\cdot Y.$$

We now specialize this formula for the canonical basis fields.

- **Step 1:** Take $X=\partial u^i$ and $Y=\partial u^j$ with $i\neq j$. Then

$$\mathscr{L}_E\left(\frac{\partial}{\partial u^i}\right)\cdot\frac{\partial}{\partial u^j}+\mathscr{L}_E\left(\frac{\partial}{\partial u^j}\right)\cdot\frac{\partial}{\partial u^i}=0 \implies \mathscr{L}_E\left(\frac{\partial}{\partial u^i}\right)\cdot\frac{\partial}{\partial u^j}=0 \text{ for } i\neq j.$$

  This implies that $\mathscr{L}_E(\partial u^i)=\lambda_i\partial u^i$ for some scalar λ_i.

- **Step 2:** Take $X=Y=\partial u^i$. Then the same procedure yields

$$\lambda_i\frac{\partial}{\partial u^i}-2\lambda_i\frac{\partial}{\partial u^i}=\frac{\partial}{\partial u^i} \implies \lambda_i=-1,$$

  and this concludes the proof.

□

Corollary 7. $\mathscr{L}_E e=-e$.

Demonstration. we recall from Corollary 2 that $e=\sum_{i=1}^{n}\partial/\partial u^i$, and so

$$\begin{aligned}\mathscr{L}_E e&=\sum_i\mathscr{L}_E\left(\frac{\partial}{\partial u^i}\right)\\&=\sum_i\left[E,\frac{\partial}{\partial u^i}\right]\\&=\sum_i\left(E\cdot\frac{\partial}{\partial u^i}-\frac{\partial}{\partial u^i}\cdot E\right).\end{aligned}$$

Now, the term inside the summation turns out to be

$$\begin{aligned} E\cdot\frac{\partial}{\partial u^i}-\frac{\partial}{\partial u^i}\cdot E &= \sum_j\left(u^j\frac{\partial}{\partial u^j}\frac{\partial}{\partial u^i}\right)-\frac{\partial}{\partial u^i}\left(\sum_j u^j\frac{\partial}{\partial u^j}\right)\\ &= \sum_j\left(u^j\frac{\partial^2}{\partial u^i\partial u^j}-\delta_{ij}\frac{\partial}{\partial u^j}-u^j\frac{\partial^2}{\partial u^i\partial u^j}\right)\\ &= -\frac{\partial}{\partial u^i}. \end{aligned}$$

and the result follows. □

Note that, if we consider the operator l_E of multiplication by the Euler vector field, this result implies that the spectrum is equal to the set of canonical coordinates. But according to item (iv) in Theorem 3, this means that if we fix canonical coordinates in this way, then all points at which canonical coordinates are pairwise distinct are semisimple points.

3.3 Extended deformed connection

We now proceed by introducing a family of connections $(\nabla^z)_{z\in\mathbb{C}}$ in TM labelled by complex numbers. These are called *deformed connections* and are defined by the formula

$$\nabla^z_X Y=\nabla_X Y+zX\cdot Y, \tag{3.13}$$

where $X,Y\in\Gamma(TM)$ and ∇ denotes the usual Levi-Civita connection. Note that

$$\nabla^0=\nabla.$$

Theorem 5. ∇^z is flat $\forall z\in\mathbb{C}$.

Demonstration. We first emphasize that whenever we have a connection on the tangent bundle, it induces connections on all the tensor algebras of the tangent bundle. In particular, we also have a connection on the cotangent bundle. As usual, they are denoted by the same symbol.

Let $\xi\in\Gamma(T^*M)$. As before, we denote by t^α the system of flat coordinates with relation to ∇, and write $\xi=\xi_\alpha\,\mathrm{d}t^\alpha$. The condition of ∇^z-flatness for ξ is given by the following system of differential equations:

$$\nabla^z\xi=0\iff\partial_\alpha\xi_\beta=zc^\gamma_{\alpha\beta}\xi_\gamma.$$

Now, proving that ∇^z is flat is equivalent to proving that the system above is integrable. For that, we need to check that $\partial_\alpha\partial_\varepsilon=\partial_\varepsilon\partial_\alpha$. We have

$$\begin{cases}\partial_\alpha\partial_\varepsilon\xi_\beta=\partial_\alpha\left[zc^\gamma_{\varepsilon\beta}\xi_\gamma\right]=z\partial_\alpha c^\gamma_{\varepsilon\beta}\xi_\gamma+z^2c^\gamma_{\varepsilon\beta}c^\delta_{\alpha\gamma}\xi_\delta,\\ \partial_\varepsilon\partial_\alpha\xi_\beta=\partial_\varepsilon\left[zc^\gamma_{\alpha\beta}\xi_\gamma\right]=z\partial_\varepsilon c^\gamma_{\alpha\beta}\xi_\gamma+z^2c^\gamma_{\alpha\beta}c^\delta_{\varepsilon\gamma}\xi_\delta.\end{cases}$$

We verify the equality by comparing the linear and quadratic terms in z. This means that we require that

$$\partial_\alpha c^\gamma_{\varepsilon\beta} = \partial_\varepsilon c^\gamma_{\alpha\beta},$$
$$c^\gamma_{\varepsilon\beta} c^\delta_{\alpha\gamma} = c^\gamma_{\alpha\beta} c^\delta_{\varepsilon\gamma}.$$

But this follows directly from the fact that $\nabla c \in \Gamma(\mathsf{Sym}^4 T^*M)$ and from the associativity condition of the product. □

We now have all the necessary tools to prove Theorem 4.

Demonstration. Firstly, a reminder that the curvature $R_\nabla \in \Gamma(\wedge^2 T^*M \otimes \mathsf{End}(TM))$ of a connection ∇ is given by

$$R_\nabla(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}, \quad X,Y \in \Gamma(TM).$$

Now, since by Theorem 5 ∇^z is flat for all $z \in \mathbb{C}$, we have that

$$R_{\nabla^z}(\sigma_i,\sigma_j) = \nabla^z_{\sigma_i}\nabla^z_{\sigma_j} - \nabla^z_{\sigma_j}\nabla^z_{\sigma_i} - \nabla^z_{[\sigma_i,\sigma_j]} = 0 \quad \forall i,j.$$

Applying the operator to σ_l and writing

$$\nabla_{\sigma_i}\sigma_j = \sum_k \Gamma^k_{ij}\sigma_k,$$
$$[\sigma_i,\sigma_j] = \sum_k f^k_{ij}\sigma_k,$$

we have the following:

$$\nabla^z_{\sigma_i}\left[\sum_k \Gamma^k_{jl}\sigma_k + z\sigma_j\delta_{lj}\right] - \nabla^z_{\sigma_j}\left[\sum_k \Gamma^k_{il}\sigma_k + z\sigma_i\delta_{li}\right] - \sum_{k,h} f^k_{ij}\Gamma^h_{kl}\sigma_h - z\sum_k f^k_{ij}\sigma_l\delta^l_k = 0.$$

Take the linear term in z. We have

$$\sum_k \Gamma^k_{jl}\delta_{ki}\sigma_k + \sum_h \Gamma^h_{ij}\sigma_h\delta_{lj} - \sum_k \Gamma^k_{il}\delta_{jk}\sigma_k - \sum_h \Gamma^h_{ji}\sigma_h\delta_{li} - \sum_k f^k_{ij}\sigma_k\delta^l_k = 0.$$

Taking the component of the generic idempotent σ_k, we have

$$\Gamma^k_{jl}\delta_{ki} + \Gamma^k_{ij}\delta_{lj} - \Gamma^k_{il}\delta_{jk} - \Gamma^k_{ji}\delta_{li} = f^k_{ij}\delta^l_k.$$

Finally, setting $l = k$, we have $f^k_{ij} = 0$, as we wished. □

Let's return our attention to the family of deformed connections

$$\nabla^z_X Y = \nabla_X Y + zX\cdot Y, \quad X,Y \in \Gamma(TM).$$

We have already shown that these connections are flat for any $z \in \mathbb{C}$. But actually, something stronger is true. We can take such a family of connections and construct a single connection which is also flat. First, we establish some notation.

Definition 11. We define the tensors $\mathscr{U}, \mu \in \Gamma(TM \otimes T^*M)$ by the formulas

$$\mathscr{U}(X) = E \cdot X, \tag{3.14}$$

$$\mu(X) = \frac{2-d}{2}X - \nabla_X E, \tag{3.15}$$

where d is the charge of the manifold, as defined in Definition 4.

We also add the assumption that $\nabla E \in \Gamma(TM \otimes T^*M)$ is diagonalizable. In particular, flat coordinates are chosen in such a way that the operator μ is diagonal, with

$$\mu = \mathrm{diag}(\mu_1, ..., \mu_n), \tag{3.16}$$

$$\mu_\alpha = q_\alpha - \frac{d}{2}, \tag{3.17}$$

where each q_α is given by the form of the Euler vector field in flat coordinates, as shown in Proposition 2.

Consider the following construction, where π denotes the projection map.

$$\begin{tikzcd} \pi^* TM \arrow[r] \arrow[d] & TM \arrow[d] \\ \mathbb{C}^* \times M \arrow[r, "\pi"'] & M \end{tikzcd}$$

We define the following objects:

$$\begin{aligned} \mathscr{T}_M &= \text{ Tangent sheaf of } M; \\ \pi^*\mathscr{T}_M &= \text{ Sheaf of sections of } \pi^*TM; \\ \pi^{-1}\mathscr{T}_M &= \text{ Sheaf of sections of } \pi^*TM \text{which are constant along the fibers of } \pi. \end{aligned}$$

We can interpret a section of $\pi^*\mathscr{T}_M$ as vector fields on M whose components depend on the complex time z.

Now, the Levi-Civita connection ∇ and the tensors that define the Frobenius manifold are lifted to the pullback bundle π^*TM in such a way that

$$\nabla_{\partial_z} Y = 0 \quad \forall Y \in \pi^{-1}\mathscr{T}_M. \tag{3.18}$$

In other words, we pullback the Levi-Civita connection in such a way that the derivative of vectors whose components don't depend on z with respect to ∂_z always vanishes. To deform the lifted connection, we need to introduce Christoffel symbols along tangential directions to M and along the tangential direction to $\mathbb{C}^*$. We do this by introducing a connection $\hat{\nabla}$ on π^*TM given by

$$\hat{\nabla}_X Y = \nabla_X Y + zX \cdot Y, \tag{3.19}$$

$$\hat{\nabla}_{\partial_z} Y = \nabla_{\partial_z} Y + \mathscr{U}(Y) - \frac{1}{z}\mu(Y), \tag{3.20}$$

where $X \in \mathscr{T}_M$, $Y \in \pi^* \mathscr{T}_M$. We call $\hat{\nabla}$ the *extended deformed connection* of the Dubrovin-Frobenius manifold.

Theorem 6. $\hat{\nabla}$ is flat. More precisely, the flatness is equivalent to the following conditions:

- ∇c is totally symmetric, i.e., $\nabla c \in \Gamma(\mathrm{Sym}^4 T^* M)$;
- Associativity of the product;
- $\nabla\nabla E = 0$;
- $\mathscr{L}_E c = c$.

Demonstration. We follow a procedure similar to the one in the proof of Theorem 5. We start with the connection $\hat{\nabla}$ induced on the dual of the tangent bundle of M. As before, we look for $\hat{\nabla}$-flat sections $\xi \in \Gamma(\pi^* T^* M)$, with $\xi = \xi_\alpha(z,t)\, \mathrm{d}t^\alpha$. Again, the flatness is equivalent to the following joint system of differential equations.

$$\begin{cases} \partial_\alpha \xi_\beta = z c^\gamma_{\alpha\beta} \xi_\gamma, \\ \partial_z \xi_\beta = E^\gamma c^\alpha_{\gamma\beta} \xi_\alpha - \frac{1}{z} \mu^\sigma_\beta \xi_\sigma. \end{cases}$$

Again, we check for integrability. First, we need to check that $\partial_\varepsilon \partial_\alpha = \partial_\alpha \partial_\varepsilon$. But this is exactly the same as in Theorem 5, which means it follows from the symmetry of ∇c and the associativity of the product.

Next, we must check that $\partial_\alpha \partial_z = \partial_z \partial_\alpha$. But this holds true if and only if the following expression vanishes identically in z:

$$\frac{1}{z} \partial_\varepsilon \mu^\sigma_\beta \xi_\sigma + \left(c^\sigma_{\varepsilon\beta} - c^\gamma_{\varepsilon\beta} \mu^\sigma_\gamma - \partial_\varepsilon E^\alpha c^\sigma_{\alpha\beta} - E^\alpha \partial_\varepsilon c^\sigma_{\alpha\beta} + \mu^\lambda_\beta c^\sigma_{\varepsilon\lambda} \right) \xi_\sigma + z \left(c^\gamma_{\varepsilon\beta} c^\sigma_{\alpha\gamma} - c^\gamma_{\alpha\beta} c^\sigma_{\varepsilon\gamma} \right) E^\alpha \xi_\sigma.$$

Firstly, it follows from associativity that the linear term in z always vanishes. Now recall from Equation 3.15 that

$$\mu^\sigma_\beta = \frac{2-d}{2} \delta^\sigma_\beta - \partial_\beta E^\sigma.$$

Then the first term in the equation above is zero if and only if the Euler vector field is an affine vector field, that is

$$\partial_\varepsilon \mu^\sigma_\beta = 0 \iff \partial_\varepsilon \partial_\beta E^\sigma = 0 \iff \nabla\nabla E = 0.$$

Moreover, looking at the middle term in the equation, we expand it by applying the definition of μ. We obtain the following:

$$c^\sigma_{\varepsilon\beta} - \frac{2-d}{2} c^\sigma_{\varepsilon\beta} + c^\gamma_{\varepsilon\beta} \partial_\gamma E^\sigma - \partial_\varepsilon E^\alpha c^\sigma_{\alpha\beta} - E^\alpha \partial_\varepsilon c^\sigma_{\alpha\beta} + \frac{2-d}{2} c^\sigma_{\varepsilon\beta} - \partial_\beta E^\lambda c^\sigma_{\varepsilon\lambda} = 0.$$

The two terms with the fractions cancel each other. Moreover, applying the symmetry of ∇c to obtain $E^\alpha \partial_\varepsilon c^\sigma_{\alpha\beta} = E^\alpha \partial_\alpha c^\sigma_{\varepsilon\gamma}$, we get

$$c^\sigma_{\varepsilon\beta} = E^\alpha \partial_\alpha c^\sigma_{\varepsilon\beta} - c^\gamma_{\varepsilon\beta} \partial_\gamma E^\sigma + \partial_\varepsilon E^\alpha c^\sigma_{\alpha\beta} + \partial_\beta E^\lambda c^\sigma_{\varepsilon\lambda}.$$

But these are just the components $(\mathscr{L}_E c)^{\sigma}_{\varepsilon\beta}$, and since $\mathscr{L}_E c = c$, this completes the proof. □

All in all, this means that we are able to attach to a Frobenius manifold a joint system of differential equations

$$\begin{cases} \partial_\alpha \xi = z\mathscr{C}_\alpha^T \xi, \\ \partial_z \xi = (\mathscr{U} + \frac{1}{z}\mu)\xi, \end{cases} \tag{3.21}$$

which are just the equations on the theorem above, but adopting matricial notation, that is,

$$\xi \mapsto \begin{pmatrix} \xi_1 \\ \cdot \\ \cdot \\ \cdot \\ \xi_n \end{pmatrix}, \quad (\mathscr{C}_\alpha)^\gamma_\beta = c^\gamma_{\alpha\beta}. \tag{3.22}$$

Note 1. The operator $\mathscr{U}$ is g-self-adjoint. This follows from the Frobenius property

$$g(\mathscr{U}(X),Y) = g(X\cdot E,Y) = g(X,Y\cdot E) = g(X,\mathscr{U}(Y)).$$

Furthermore, μ is g-anti-symmetric.

Note 2. Note that the application μ defined in Equation 3.15 is $C^\infty(M)$-linear and for this reason it defines an endomorphism of TM, i.e. a tensor of type $(1,1)$. Since $\mathscr{L}_E g = (2-d)g$, for all X,Y vector fields, one has

$$\mathscr{L}_E(g(X,Y)) = (2-d)g(X,Y) + g([E,X],Y) + g(X,[E,Y]). \tag{3.23}$$

Since $\nabla g = 0$ and ∇ is torsion-free, i.e. $\nabla_X Y - \nabla_Y X = [X,Y]$ for all vector fields X,Y,

$$\begin{aligned} g(\mu(X),Y) &= g\Big(\frac{2-d}{2}X - \nabla_X E, Y\Big) = g\Big(\frac{2-d}{2}X,Y\Big) - g(\nabla_X E,Y) \\ &= g\Big(\frac{2-d}{2}X,Y\Big) - g(\nabla_E X,Y) - g([X,E],Y) \\ &= g\Big(\frac{2-d}{2}X,Y\Big) - \mathscr{L}_E(g(X,Y)) + g(X,\nabla_E Y) - g([X,E],Y) \\ &\overset{\text{Equation 3.23}}{=} g\Big(X, -\frac{2-d}{2}Y\Big) + g(X,[Y,E]) + g(X,\nabla_E Y) \\ &= -g(X,\mu(Y)), \end{aligned}$$

i.e. μ is skew-symmetric with respect to g.

Note 3 (See Corollary 1.1, pag. 8, in (DUBROVIN, 1996))**.** We close this section noticing that the existence of an Euler vector field is the infinitesimal analogue of the quasi-homogeneity condition that one imposes on the the potential function F, that is,

$$F(k^{d_1}x^1,\dots,k^{d_n}x^n) = k^{d_F}F(x^1,\dots,x^n), \tag{3.24}$$

where k is a non-zero real number and $d_1,\dots,d_n$ and d_F are suitable complex numbers, see (DUBROVIN, 1996). In the same reference, the author shows that if $g_{11} \neq 0$ and E is semi-simple, i.e. the linear map $[E,\cdot]$ is diagonalizable with distinct eigenvalues, one can choose coordinates on M such that F (abusing notation) assumes the following form

$$F(t^1,\dots,t^n) = \frac{1}{2}(t^1)^2 t^n + \frac{1}{2}t^1 \sum_{i=2}^{n-1} t^i t^{n-i+1} + f(t^2,\dots,t^n), \tag{3.25}$$

where f is a suitable function. In these coordinates the *weights* d_i, see (3.24), are such that

1. $d_i + d_{n-i+1}$ does not depend on i;
2. $d_F = 2d_1 + d_n$.

Moreover, if one further normalizes the weights d_i so that $d_1 = 1$, then they can be represented un such a way that

$$d_i = 1 - q_i \quad \text{and} \quad d_F = 3 - d, \tag{3.26}$$

where d is as in item 2.18. The numbers d and q_i satisfy the following conditions

$$q_1 = 0, \quad q_n = d \quad \text{and} \quad q_i + q_{n-i+1} = d, \forall i. \tag{3.27}$$

These are the same coordinates used in Proposition 2. Finally, if the hypothesis on g_{11} is not satisfied, i.e. if $g_{11} = 0$, then one can prove the existence of a set of coordinates on M such that

$$F(t^1,\dots,t^n) = \frac{c}{6}(t^1)^3 + \frac{1}{2}t^1 \sum_{i=2}^{n-1} t^i t^{n-i+1} + f(t^2,\dots,t^n), \tag{3.28}$$

as shown in section 3.4.

3.4 2-dimensional Frobenius Manifolds

Let us provide an example in the lowest possible non-trivial dimension, $n = 2$. Since for every point of the manifold the respective tangent space has only two generators, one of them being the identity $e \in T_pM$, associativity is trivial. In fact, if we denote the other generator by v, then the only associativity conditions are

$$(e\cdot v)\cdot v = v\cdot v = e\cdot(v\cdot v) \text{ and } (e\cdot e)\cdot v = e\cdot v = e\cdot(e\cdot v),$$

both of which are automatically satisfied. Thus, here we won't have the need to go through the WDVV equations (Equation 2.30) in order to find the potential function F.

Lemma 3. Regarding the function $g(e,e)$, only one of the following alternatives happen:

i) $d = 0$;

ii) e is everywhere a null vector, i.e., $g(e,e)$ is identically zero.

Demonstration. We know that the Euler vector field is such that $\mathscr{L}_E g = (2-d)g$. Corollary 7 gives us $\mathscr{L}_E e = -e$. Then

$$\begin{aligned}\mathscr{L}_E\,(g(e,e)) &= (\mathscr{L}_E g)(e,e) + 2g\,(\mathscr{L}_E e, e)\\ &= (2-d)g(e,e) - 2g(e,e)\\ &= -dg(e,e).\end{aligned}$$

Now, since $e = (1,0,\ldots,0)$ via Equation 2.22, $g(e,e) = g_{11}$, which is constant in flat coordinates. Since it is constant, $\mathscr{L}_E\,(g(e,e)) = 0$. Then either the charge d or the function $g(e,e)$ must vanish. □

This leads us to the possibility of classifying every 2-dimensional semisimple Frobenius manifold. The result is stated in the following theorem:

Theorem 7 (Classification of 2-dimensional semisimple Frobenius manifolds)**.** Let a potential function $F(t_1,t_2)$ generate a 2-dimensional semisimple Frobenius manifold of charge d. Let

$$k = \frac{d-3}{d-1}. \tag{3.29}$$

Then F is equivalent to one of the following:

1. $\frac{1}{2}t_1^2 t_2 + c t_2^k$;
2. $\frac{1}{2}t_1^2 t_2 + c e^{t_2}$;
3. $\frac{1}{2}t_1^2 t_2 + c t_2^2 \ln(t_2)$;
4. $\frac{1}{2}t_1^2 t_2 + c \ln(t_2)$;
5. $\frac{1}{6}t_1^3 + \frac{1}{2}t_1 t_2^2 + c t_2^3$.

Note 4. We shall lower the indexes on the coordinates t^i to t_i to avoid confusion with exponents.

Demonstration. We apply Lemma 3 and split the proof in the two possible cases.

Assume first that the identity e is null, i.e., $g(e,e) = 0$. Then the metric can be written as $g = \mathrm{d}t^1\,\mathrm{d}t^2$. Recall from Corollary 1 that $g_{\alpha\beta} = \partial t_1 \partial t_\alpha \partial t_\beta F$. Combining these two, we reach at the following system of partial differential equations:

$$\begin{cases}\partial_{t_1}^3 F = 0,\\ \partial_{t_1}^2 \partial_{t_2} F = 1,\\ \partial_{t_1} \partial_{t_2}^2 F = 0.\end{cases}$$

The system is easily solvable through straight forward integration. Since the potential function is unique up to polynomial terms of order 2, the result can be simplified to

$$F(t_1,t_2) = \frac{1}{2}t_1^2 t_2 + f(t_2), \tag{3.30}$$

and now the problem boils down to the computation of the form of the function $f(t_2)$.

We now proceed to finding the Euler vector field. Write $E = E^\alpha \partial t_\alpha$. Since $\mathscr{L}_E g = (2-d)g$, we have for any vector fields $X = X^\alpha \partial t_\alpha, Y = Y^\alpha \partial t_\alpha \in \Gamma(TM)$,

$$\begin{aligned}
\mathscr{L}_E\left(g(X,Y)\right) &= \left(\mathscr{L}_E\right)(X,Y) + g\left(\mathscr{L}_E X, Y\right) + g\left(X, \mathscr{L}_E Y\right) \\
0 &= (2-d)X^1Y^2 + [E,X]^1\, Y^2 + X^1\,[E,Y]^2 \\
0 &= (2-d)X^1Y^2 + \sum_\alpha \left(E^\alpha \partial t_\alpha X^1 - x^\alpha \partial t_\alpha E^1\right) Y^2 + X^1 \sum_\alpha \left(E^\alpha \partial t_\alpha Y^2 - Y^\alpha \partial t_\alpha E^2\right).
\end{aligned}$$

Since X and Y will be specialized to ∂t_1 and ∂t_2 in the next few steps, we already considered the left-hand side of the equation above to be equal to zero. In fact,

$$X = Y = \partial t_1 = (1,0) \implies \frac{\partial E^2}{\partial t_1} = 0 \implies E^2 = \varphi(t_2), \tag{3.31}$$

$$X = Y = \partial t_2 = (0,1) \implies \frac{\partial E^1}{\partial t_2} = 0 \implies E^1 = \psi(t_1), \tag{3.32}$$

$$X = \partial t_1 \text{ and } Y = \partial t_2 \implies \frac{\partial E^1}{\partial t_1} + \frac{\partial E^2}{\partial t_2} = 2-d \implies \varphi'(t_2) + \psi'(t_1) = 2-d. \tag{3.33}$$

The final combination $X = \partial t_2$ and $Y = \partial t_1$ just returns $0 = 0$. To find the last needed equation to solve the system, we use the fact that $\mathscr{L}_E e = -e = (-1,0)$. This yields

$$\begin{aligned}
\left(\mathscr{L}_E e\right)^1 &= [E,e]^1 \\
&= \sum_\alpha \left(E^\alpha \partial t_\alpha e^1 - e^\alpha \partial t_\alpha E^1\right) \\
&= -\frac{\partial E^1}{\partial t_1},
\end{aligned}$$

which implies that

$$\frac{\partial E^1}{\partial t_1} = 1.$$

The term $\left(\mathscr{L}_E e\right)^2$ returns the same result as the case $X = Y = \partial t_1$ above. We now have

$$\frac{\partial E^1}{\partial t_1} = \psi(t_1) = 1 \implies \psi(t_1) = t_1,$$

where we note that the integration constant can be ignored due to the choice of coordinates being invariant under translations. Combining this result with Equation 3.33 yields

$$\varphi'(t_2) = 1-d.$$

Two scenarios are possible now. If $d \neq 1$, then the right-hand side is a non-zero constant, and we have $\varphi(t_2) = (1-d)t_2$. With this, the Euler vector field is given by

$$E = t_1 \frac{\partial}{\partial t_1} + (1-d)t_2 \frac{\partial}{\partial t_2}. \tag{3.34}$$

The final tool we haven't used yet is the tensor c. We'll use it to find an equation involving $\partial_{t_2}^3 F$, the sole partial derivative of the potential function that has not appeared until this point. Since $\mathscr{L}_E c = (3-d)c$, we have,

$$\mathscr{L}_E \left(c(\partial t_2, \partial t_2, \partial t_2)\right) = (3-d)c(\partial t_2, \partial t_2, \partial t_2) + 3c\left(\mathscr{L}_E \partial t_2, \partial t_2, \partial t_2\right), \tag{3.35}$$

$$\mathscr{L}_E \left(\partial_{t_2}^3 F\right) = (3-d)\partial_{t_2}^3 F + 3c\left(\mathscr{L}_E \partial t_2, \partial t_2, \partial t_2\right). \tag{3.36}$$

Now, to calculate the second term on the right-hand side, direct computation of the Lie derivative shows that

$$\mathscr{L}_E \partial t_2 = [E, \partial t_2] = (d-1)\partial t_2,$$

and applying linearity, we conclude that

$$c\left(\mathscr{L}_E \partial t_2, \partial t_2, \partial t_2\right) = (d-1)c(\partial t_2, \partial t_2, \partial t_2) = (d-1)\partial_{t_2}^3 F.$$

Returning to Equation 3.36 and already substituting $\partial_{t_2}^3 F$ for $f^{(3)}(t_2)$ and expanding the Lie derivative on the left-hand side, we arrive at

$$t_1 \frac{\partial}{\partial t_1}\left(f^{(3)}(t_2)\right) + (1-d)t_2 \frac{\partial}{\partial t_2}\left(f^{(3)}(t_2)\right) = (3-d)\frac{\partial^3}{\partial t_2^3}\left(f(t_2)\right) + 3(d-1)\frac{\partial^3}{\partial t_2^3}\left(f(t_2)\right).$$

Define the temporary constant $\lambda = 2d/(1-d)$, and the expression above simplifies to

$$t_2 f^{(4)}(t_2) = \lambda f^{(3)}(t_2).$$

We now integrate the equation above to arrive at a first order ODE for f:

$$\begin{aligned}
\int t_2 f^{(4)}(t_2)\, \mathrm{d}t_2 &= \int \lambda f^{(3)}(t_2)\, \mathrm{d}t_2, \\
t_2 f^{(3)}(t_2) - \int f^{(3)}(t_2)\, \mathrm{d}t_2 &= \lambda f^{(2)}(t_2) + \gamma, \\
t_2 f^{(3)}(t_2) - f^{(2)}(t_2) &= \lambda f^{(2)}(t_2) + \gamma, \\
t_2 f^{(3)}(t_2) &= (\lambda+1) f^{(2)}(t_2) + \gamma.
\end{aligned}$$

The same process is repeated a couple more times. Noting that

$$\lambda + 3 = \frac{2d}{1-d} + 3 = \frac{3-d}{1-d} = k,$$

which is how we arrive at the constant k in the statement of the theorem, this finally yields

$$t_2 \frac{\mathrm{d}f}{\mathrm{d}t_2} - kf(t_2) = \gamma t_2^2 + \beta t_2 + \alpha. \tag{3.37}$$

This differential equation has an integrating factor $\mu(t_2) = t_2^{-k}$, and it integrates to

$$t_2^{-k} f(t_2) = \alpha \int \frac{1}{t_2^{k+1}} \mathrm{d}t_2 + \beta \int \frac{1}{t_2^{k}} \mathrm{d}t_2 + \gamma \int \frac{1}{t_2^{k-1}} \mathrm{d}t_2. \tag{3.38}$$

This has three possible results, depending on the value of the charge d. Namely, modulo quadratic terms,

$$d \neq 3, -1 \implies f(t_2) = c t_2^k, \tag{3.39}$$

$$d = -1 \implies f(t_2) = c t_2^2 \ln(t_2), \tag{3.40}$$

$$d = 3 \implies f(t_2) = c \ln(t_2). \tag{3.41}$$

These are the potentials 1, 3 and 4 in the statement of the theorem. We have two more to go.

Right before Equation 3.34, we left behind the case $d = 1$. Here, we have $\varphi'(t_2) = 0$ which implies $\varphi(t_2) = c$. This returns the following Euler vector field:

$$E = t_1 \frac{\partial}{\partial t_1} + c \frac{\partial}{\partial t_2}. \tag{3.42}$$

Applying the Lie derivative again as in Equation 3.36 yields the following ODE for f:

$$f^{(4)}(t_2) = c f^{(3)}(t_2),$$

which integrates directly to

$$f(t_2) = c\,\mathrm{e}^t. \tag{3.43}$$

Finally, we now go to the case where e is not a null vector. Here, the metric can be written as $g = (\mathrm{d}t^1)^2 + (\mathrm{d}t^2)^2$. In this case, via Lemma 3, we must have $d = 0$. The exact same steps are followed. The system of PDE's for the potential function is

$$\begin{cases} \partial_{t_1}^3 F = 1, \\ \partial_{t_1} \partial_{t_2}^2 F = 1, \\ \partial_{t_1}^2 \partial_{t_2} F = 0, \end{cases}$$

which integrates to

$$F(t_1, t_2) = \frac{1}{6} t_1^3 + \frac{1}{2} t_1 t_2^2 + f(t_2).$$

The Euler vector field will be

$$E = t_1 \frac{\partial}{\partial t_1} + t_2 \frac{\partial}{\partial t_2},$$

and the ODE for the function f is

$$t_2 f^{(4)}(t_2) = 0,$$

which easily (ignoring quadratic terms) integrates to

$$f(t_2) = c t_2^3, \tag{3.44}$$

giving us the final for of the potential we were looking for. □

CHAPTER

4

SYSTEMS OF HYDRODYNAMIC TYPE

4.1 General properties of Poisson brackets on infinite dimensional phase spaces

It is a known fact that a Poisson structure allows us to define Hamilton equations on a manifold M (see section A.1). To define a Hamiltonian system, say we have a base manifold equipped with a Poisson structure, for instance given by a bivector $\Pi \in \Gamma\left(\wedge^2 TM\right)$. Namely, if $X \in \mathfrak{X}(M)$ is a Hamiltonian vector field, we have

$$X = X_{\mathscr{H}} = \iota_{\mathrm{d}\mathscr{H}}\Pi,$$
$$\dot{x}^i = X^i = X^i_{\mathscr{H}} = \Pi^{ij}(x)\frac{\partial \mathscr{H}}{\partial x^j} = (\iota_{\mathrm{d}\mathscr{H}}\Pi)^i.$$

In this chapter, we wish to define a similar structure in the case where the base manifold M consists of smooth vector functions $u = \left(u^1(x), \dots, u^N(x)\right)$, where $x = \left(x^1, \dots, x^d\right)$ is treated as an extra index the formulae and, for reasons linked with integration which we shall see in the following section, is assumed to run through a compact d-dimensional manifold with no boundary. Ideally, we would like to have a Poisson structure with similar properties to the usual finite dimensional case. Namely, we'd like to understand how evolution equations on $u = u(x,t) = u(x)(t)$ work in this context. We'll use the following notation for derivatives:

$$u_x = \frac{\mathrm{d}}{\mathrm{d}x}u;$$
$$u_t = \frac{\mathrm{d}}{\mathrm{d}t}u;$$
$$u_{(k)} = \frac{\mathrm{d}^{(k)}}{\mathrm{d}x^{(k)}}u.$$

Denote by $\mathscr{P}(u)$ the set of polynomials on $x, u, u_{(1)}, u_{(2)}, \dots$ of finite degree. Then here we call an *evolution equation* a system of the form

$$u_t = f(u), \quad f \in \mathscr{P}(u). \tag{4.1}$$

To extend the theory of Poisson structures and Hamiltonian systems to this infinite dimensional case, first we'll need:

1. An analogue of $\mathbf{d}\mathscr{H}$;
2. An analogue of the Poisson tensor $\mathbf{\Pi}$.

This construction is shown in great detail by Dubrovin and Novikov (1989), whose work we shall follow for the most part in this chapter. Initially, let's delve ourselves a little into how functions work in the context of these types of manifolds.

4.1.1 Functionals and variational derivatives

Remind ourselves of the set of polynomials $\mathscr{P}(u)$. Then the functionals considered here will generally have the form

$$F[u] = \int f(x, u(x), u_{(1)}(x), ..., u_{(k)}(x))\,\mathrm{d}x, \tag{4.2}$$

where $f \in \mathscr{P}(u)$ is called be the *density* of the functional F. We will often write it as just $f(u)$ for simplicity. A functional of this type is called a *local functional*. We can consider the case of a polynomial density since the base manifold is assumed to be compact and with no boundary. We wish to have an analogue of derivations in the set of these functionals.

Definition 12. The *variational derivative* of a functional $F[u]$, denoted by

$$\frac{\delta F}{\delta u(x)},$$

is defined by the quantity

$$F[u+h] - F[u] = \int \frac{\delta F}{\delta u(x)} \cdot h(x)\,\mathrm{d}x + o(h^2). \tag{4.3}$$

Proposition 5. The variational derivative of $F[u]$ is given by

$$\frac{\delta F}{\delta u} = \left(\frac{\delta F}{\delta u^1}, \dots, \frac{\delta F}{\delta u^N}\right),$$
$$\frac{\delta F}{\delta u^i(x)} := \sum_{l=1}^{\infty} (-1)^l \frac{\mathrm{d}^{(l)}}{\mathrm{d}x^{(l)}} \frac{\partial f}{\partial u^i_{(l)}} = \frac{\partial f}{\partial u} - \frac{\partial}{\partial x^\alpha}\frac{\partial f}{\partial u^i_\alpha} + \frac{\partial^2}{\partial x_\alpha x_\beta}\frac{\partial f}{\partial u^i_{\alpha\beta}} - \dots, \tag{4.4}$$
$$u^i_\alpha = \partial_{x^\alpha} u^i, \quad u^i_{\alpha\beta} = \partial_{x^\alpha}\partial_{x^\beta} u^i, \quad \dots$$

Demonstration. We start with 1-fields $u(x)$, and then proceed to the N-dimensional case. Firstly,

$$\begin{aligned} F[u+h] - F[u] &= \int [f(u+h) - f(u)]\,\mathrm{d}x \\ &= \int \left(\frac{\partial f}{\partial u}h + \frac{\partial f}{\partial u_{(1)}}h_x + \dots + \frac{\partial f}{\partial u_{(k)}}h_{(k)}\right)\mathrm{d}x + o\left(h^2\right). \end{aligned}$$

Now,

$$\int \frac{\partial f}{\partial u_{(1)}} h_x \, \mathrm{d}x = -\int \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial u_{(1)}}\right) h \, \mathrm{d}x \implies \int \frac{\partial f}{\partial u_{(k)}} h_{(k)} \, \mathrm{d}x = (-1)^k \int \frac{\mathrm{d}^{(k)}}{\mathrm{d}x^{(k)}}\left(\frac{\partial f}{\partial u_{(k)}}\right) h \, \mathrm{d}x,$$

and so the integral above results in

$$\int \sum_{l=0}^{\infty} (-1)^l \frac{\mathrm{d}^{(l)}}{\mathrm{d}x^{(l)}}\left(\frac{\partial f}{\partial u_{(l)}}\right) h \, \mathrm{d}x.$$

Finally, the variational derivative is written as

$$\frac{\delta F}{\delta u} := \sum_{l=0}^{\infty} (-1)^l \frac{\mathrm{d}^{(l)}}{\mathrm{d}x^{(l)}}\left(\frac{\partial f}{\partial u_{(l)}}\right).$$

Now, for the case of n-fields $u = (u^1, ..., u^N)$, $F[u+h] - F[u] = \int [f(u+h) - f(u)] \, \mathrm{d}x$ will evaluate to

$$\begin{aligned}\int \left(\frac{\partial f}{\partial u^1} h^1 + \frac{\partial f}{\partial u^2} h^2 + ... + \frac{\partial f}{\partial u^N} h^N + \frac{\partial f}{\partial u^1_{(1)}} h^1_{(1)} + ... + \frac{\partial f}{\partial u^N_{(N)}} h^N_{(1)} + ...\right) \mathrm{d}x + o\left(h^2\right) = \\ = \int \sum_{i=1}^{N} \left(\sum_{l=0}^{\infty} (-1)^l \frac{\mathrm{d}^{(l)}}{\mathrm{d}x^{(l)}} \frac{\partial f}{\partial u^i_{(l)}}\right) h^i \, \mathrm{d}x + o\left(h^2\right).\end{aligned}$$

□

Classical mechanics, in it's Lagrangian formulation, provide an notable example of the application of variational derivatives, as shown below:

Example 8 (Lagrangian formulation of classical mechanics)**.** Let's do a quick overview of the variational principals in Lagrange's formulation of mechanics. A more detailed approach can be found in (ARNOLD, 1989, p. 53). In this context, the states of a mechanical system are given by a differentiable manifold called *configuration space*. The Lagrangian function, defined as the difference between the kinetic and potential energies of the system,

$$\mathfrak{L} = T - U,$$

acts on this manifold's tangent bundle. Coordinates on this manifold are called generalized coordinates, and a curve in the configuration space should ,upon certain conditions, represent an evolution of said system. Let $u(x) =: \gamma(x) : [a,b] \longrightarrow \mathbb{R}^N$ be one of such curves with Lagrangian $\mathfrak{L} = \mathfrak{L}(x, u, u_{(1)})$, where

$$\begin{aligned} u &= (u^1, \dots, u^N), \\ u_{(1)} &= (u^1_{(1)}, \dots, u^N_{(1)}), \\ \frac{\partial \mathfrak{L}}{\partial u} &= \left(\frac{\partial \mathfrak{L}}{\partial u^1}, \dots, \frac{\partial \mathfrak{L}}{\partial u^N}\right). \end{aligned}$$

We define a functional S, called *action* of the system, by

$$S := \int_a^b \mathfrak{L}\left(x, u, u_{(1)}\right) \mathrm{d}x.$$

According to the stationary-action principle, the solutions to the equations of motion for the system must be stationary points of the action functional. In other words, it's variation should vanish along mechanically plausible curves. Calculating this explicitly, we have (omitting the dependence of $\mathfrak{L}$ in x for simplicity):

$$\begin{aligned}
S(u+h) - S(u) &= \int_a^b \left[\mathfrak{L}(u+h, u_{(1)} + h_{(1)}) - \mathfrak{L}(u, u_{(1)})\right] \mathrm{d}x \\
&= \int_a^b \left(\frac{\partial \mathfrak{L}}{\partial u} h + \frac{\partial \mathfrak{L}}{\partial u_{(1)}} h_{(1)} \right) \mathrm{d}x + o\left(h^2\right) \\
&= \int_a^b \frac{\partial \mathfrak{L}}{\partial u} h \,\mathrm{d}x + \int_a^b \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial \mathfrak{L}}{\partial u_{(1)}} h \right) \mathrm{d}x - \int_a^b \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial \mathfrak{L}}{\partial u_{(1)}} \right) h \,\mathrm{d}x + o\left(h^2\right) \\
&= \int_a^b \left(\frac{\partial \mathfrak{L}}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial \mathfrak{L}}{\partial u_{(1)}} \right) h \,\mathrm{d}x + \cancel{\left(\frac{\partial \mathfrak{L}}{\partial u_{(1)}} h \right)} \Bigg|_a^b \\
&= \int_a^b \left(\frac{\partial \mathfrak{L}}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial \mathfrak{L}}{\partial u_{(1)}} \right) h \,\mathrm{d}x,
\end{aligned}$$

where the canceled term vanishes because h should be fixed at the curve's end-points. The integrand written in components is

$$\frac{\partial \mathfrak{L}}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial \mathfrak{L}}{\partial u_{(1)}} = \left(\frac{\partial \mathfrak{L}}{\partial u^1} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial \mathfrak{L}}{\partial u^1_{(1)}}, \dots, \frac{\partial \mathfrak{L}}{\partial u^n} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial \mathfrak{L}}{\partial u^N_{(1)}} \right).$$

With this, the stationary-action principle is satisfied if and only if

$$\frac{\partial \mathfrak{L}}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial \mathfrak{L}}{\partial u_{(1)}} = 0 \iff \frac{\partial \mathfrak{L}}{\partial u^i} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial \mathfrak{L}}{\partial u^i_{(1)}} \quad \forall i = 1, \dots, N.$$

These are exactly the famous Euler-Lagrange equations found in classical mechanics. The quantity

$$\frac{\partial \mathfrak{L}}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial \mathfrak{L}}{\partial u_{(1)}} = \frac{\delta \mathfrak{L}}{\delta u}$$

is the Lagrangian's variational derivative.

Definition 13. Define a product in $\mathscr{P}$ by

$$\begin{aligned}
(\cdot,\cdot) : \mathscr{P} \times \mathscr{P} &\longrightarrow \mathbb{R} \\
(f, g) &\longmapsto \int f g \,\mathrm{d}x.
\end{aligned} \tag{4.5}$$

Then we refer by *first variation* the quantity

$$(\mathscr{F}_F)_u(h) = \left(\frac{\delta f}{\delta u}, h \right) = \sum_{i=1}^N \left(\frac{\delta f}{\delta u^i} h^i \right) = \sum_{i=1}^N \int \left(\frac{\delta f}{\delta u^i} h^i \right) \mathrm{d}x. \tag{4.6}$$

It is worth noting that the variational derivative $\delta f/\delta u$ works as an analogue of a gradient in relation to the product in Equation 4.5. For this reason, we shall often write

$$\frac{\delta f}{\delta u} = \operatorname{grad}(F(u)). \tag{4.7}$$

In this sense, the first variation becomes the directional derivative of F in relation to u in the direction of h:

$$(\mathscr{F}_F)_u(h) = \left(\frac{\delta f}{\delta u}, h\right) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} F(u+\varepsilon h). \tag{4.8}$$

Example 9. Let $F[u] = \frac{1}{2}\int u^2\,\mathrm{d}x$. Then

$$(\mathscr{F}_F)_u(h) = \frac{1}{2}\int \frac{\delta u^2}{\delta u} h\,\mathrm{d}x = \int uh\,\mathrm{d}x.$$

4.1.2 The analogous Poisson structure

We now wish to define an analogue of a Poisson structure in the space of these functionals. Let's remind ourselves of the usual case where $M = \mathbb{R}^N$. If we fix $\Pi^{ij} = \{x^i, x^j\}$, then

$$\begin{aligned}
\{f,g\}_P = \Pi^{ij}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j} &= \begin{bmatrix} \frac{\partial f}{\partial x^1} & \dots & \frac{\partial f}{\partial x^N} \end{bmatrix} \begin{bmatrix} \cancel{\{x^1,x^1\}} & \{x^1,x^2\} & \dots & \{x^1,x^N\} \\ \{x^2,x^1\} & \ddots & \dots & \{x^2,x^N\} \\ \vdots & \vdots & \ddots & \vdots \\ \{x^N,x^1\} & \{x^N,x^2\} & \dots & \cancel{\{x^N,x^N\}} \end{bmatrix} \begin{bmatrix} \frac{\partial g}{\partial x^1} \\ \vdots \\ \frac{\partial g}{\partial x^N} \end{bmatrix} \\
&= (\operatorname{grad}(f))(x)^T \cdot M_P(x) \cdot (\operatorname{grad}(g))(x) \\
&= \Big\langle \operatorname{grad}(f)(x),\ M_P(x)\cdot \operatorname{grad}(g)(x) \Big\rangle.
\end{aligned}$$

Here, we are denoting by $M_P(x)$ the skew-symmetric matrix whose coefficients are $\{x^i, x^j\}$. It's easy to see that

$$\Big\langle \operatorname{grad}(f)(x),\ M_P(x)\cdot \operatorname{grad}(g)(x) \Big\rangle = -\Big\langle \operatorname{grad}(g)(x),\ M_P(x)\cdot \operatorname{grad}(f)(x) \Big\rangle.$$

To extend this operation to the functionals described in the previous section, by analogy, we use the L^2-style product given in Definition 13, where the "grad" operation was given by variational derivatives:

$$\{F,G\}[u] = \left(\frac{\delta F}{\delta u},\ M_P(u)\cdot\frac{\delta G}{\delta u}\right). \tag{4.9}$$

In the case of N-fields, where

$$\frac{\delta f}{\delta u} = \left(\frac{\delta f}{\delta u^1}, \dots, \frac{\delta f}{\delta u^N}\right), \quad \frac{\delta g}{\delta u} = \left(\frac{\delta g}{\delta u^1}, \dots, \frac{\delta g}{\delta u^N}\right),$$

this yields

$$\{F,G\}[u] = \sum_{i,j=1}^{N} \left(\frac{\delta f}{\delta u^i},\ M_P(u)^{ij} \cdot \frac{\delta g}{\delta u^j} \right) = \sum_{i,j=1}^{N} \int M_P(u)^{ij} \frac{\delta f}{\delta u^i} \frac{\delta g}{\delta u^j}\, \mathrm{d}x. \tag{4.10}$$

As with the usual Poisson brackets, we require that it satisfy the skew-symmetry and Jacobi identities:

$$\{F,G\} = -\{G,F\}, \tag{4.11}$$

$$\{\{F,G\},H\} + \{\{H,F\},G\} + \{\{G,H\},F\} = 0. \tag{4.12}$$

Now, skew-symmetry follows directly from the matrix $M_P(u)$ being skew-symmetric. The Jacobi identity, however, is not so easily obtained. The Leibniz rule is usually dropped since there isn't a generally defined way of multilying functionals.

It will be useful to define a quantity analogous to a "second derivative" of a functional F.

Definition 14. We define the *second variation* of F as

$$(\mathscr{S}_F)_u(h) := \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \operatorname{grad}(F)(u+\varepsilon h). \tag{4.13}$$

Lemma 4. The second variation is symmetric with respect to the scalar product, that is,

$$\big((\mathscr{S}_F)_u(w),\ h\big) = \big(w,\ (\mathscr{S}_F)_u(h)\big) \quad \forall u,v,h. \tag{4.14}$$

Demonstration. We simply note that

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \left(\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} F(u+\eta w+\varepsilon h) \right) &= \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \Big(\operatorname{grad}(F)(u+\eta w),\ h \Big) \\ &= \left(\left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \operatorname{grad}(F)(u+\eta w),\ h \right), \end{aligned}$$

and, similarly,

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \left(\left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} F(u+\varepsilon h+\eta w) \right) &= \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \Big(w,\ \operatorname{grad}(F)(u+\varepsilon h) \Big) \\ &= \left(w,\ \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \operatorname{grad}(F)(u+\varepsilon h) \right). \end{aligned}$$

□

Definition 15. We define the *Gardner* bracket as

$$\{F,G\}_0[u] = \big(\operatorname{grad}(F)(u),\ \partial \cdot \operatorname{grad}(G)(u)\big), \tag{4.15}$$

where $\partial = \mathrm{d}/\mathrm{d}x$.

It's easy to see that this bracket is, indeed, skew-symmetric.

Lemma 5. The bracket satisfies the identity

$$\operatorname{grad}\{F,H\}_0(u) = (\mathscr{S}_F)_u(\partial\cdot\operatorname{grad}(H)(u)) - (\mathscr{S}_H)_u(\partial\cdot\operatorname{grad}(F)(u)). \tag{4.16}$$

Demonstration.

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\{F,H\}_0(u+\varepsilon v) &= \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\big(\operatorname{grad}(F)(u+\varepsilon v),\ \partial\cdot\operatorname{grad}(H)(u+\varepsilon v)\big)\\
&= \left(\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\operatorname{grad}(F)(u+\varepsilon v),\ \partial\cdot\operatorname{grad}(H)(u+\varepsilon v)\right)\\
&\qquad+\left(\operatorname{grad}(F)(u+\varepsilon v),\ \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\partial\cdot\operatorname{grad}(H)(u+\varepsilon v)\right)\\
&= \Big((\mathscr{S}_F)_u(v),\ \partial\cdot\operatorname{grad}(H)(u)\Big)+\Big(\operatorname{grad}(F)(u),\ \partial\cdot(\mathscr{S}_H)_u(v)\Big)\\
&= \Big((\mathscr{S}_F)_u(\partial\cdot\operatorname{grad}(H)(u)),\ v\Big)-\Big((\mathscr{S}_H)_u(\partial\cdot\operatorname{grad}(F)(u)),\ v\Big)\\
&= \Big((\mathscr{S}_F)_u(\partial\cdot\operatorname{grad}(H)(u))-(\mathscr{S}_H)_u(\partial\cdot\operatorname{grad}(F)(u)),v\Big).
\end{aligned}$$

□

Proposition 6. $\{,\}_0$ satisfies Jacobi's identity.

Demonstration. Firstly, by Definition 15 and Lemma 5, we have

$$\begin{aligned}
\{\{F,H\}_0,G\}_0[u] &= \Big(\operatorname{grad}\{F,H\}_0(u),\ \partial\cdot\operatorname{grad}(G)(u)\Big)\\
&= \Big((\mathscr{S}_F)_u(\partial\cdot\operatorname{grad}(H)(u))-(\mathscr{S}_H)_u(\partial\operatorname{grad}(F)(u)),\ \partial\cdot\operatorname{grad}(G)(u)\Big),
\end{aligned}$$

and thus the cyclic sum $\{\{F,H\}_0,G\}_0+\{\{G,F\}_0,H\}_0+\{\{H,G\}_0,F\}_0$ will be

$$\begin{gathered}
\Big((\mathscr{S}_F)_u(\partial\cdot\operatorname{grad}(H)(u))-(\mathscr{S}_H)_u(\partial\operatorname{grad}(F)(u)),\ \partial\cdot\operatorname{grad}(G)(u)\Big)+\\
\Big((\mathscr{S}_G)_u(\partial\cdot\operatorname{grad}(F)(u))-(\mathscr{S}_F)_u(\partial\operatorname{grad}(G)(u)),\ \partial\cdot\operatorname{grad}(H)(u)\Big)+\\
\Big((\mathscr{S}_H)_u(\partial\cdot\operatorname{grad}(G)(u))-(\mathscr{S}_G)_u(\partial\operatorname{grad}(H)(u)),\ \partial\cdot\operatorname{grad}(G)(u)\Big).
\end{gathered}$$

Now we simply note that

$$\begin{aligned}
&\Big((\mathscr{S}_F)_u(\partial\cdot\operatorname{grad}(H)(u)),\ \partial\cdot\operatorname{grad}(G)(u)\Big)-\Big((\mathscr{S}_F)_u(\partial\cdot\operatorname{grad}(G)(u)),\ \partial\cdot\operatorname{grad}(H)(u)\Big)=\\
&=\cancel{\Big((\mathscr{S}_F)_u(\partial\cdot\operatorname{grad}(H)(u)),\ \partial\cdot\operatorname{grad}(G)(u)\Big)}-\cancel{\Big(\partial\cdot\operatorname{grad}(G)(u),\ (\mathscr{S}_F)_u(\partial\cdot\operatorname{grad}(H)(u))\Big)},
\end{aligned}$$

and so the entire expression vanishes. □

Corollary 8. If we define $\Pi_0=\partial$, then it is a Hamiltonian operator (analogous to Π_0 in the finite dimensional case).

4.1.3 An important example: The Korteweg-De Vries equation

Let's now define the following operators:

$$\Pi_1 := \partial^3 + \frac{2}{3}u\cdot\partial + \frac{1}{3}u_x, \tag{4.17}$$

$$\{F,G\}_1(u) := \int \frac{\delta f}{\delta u}\cdot\Pi_1\left(\frac{\delta g}{\delta u}\right)\mathrm{d}x. \tag{4.18}$$

Lemma 6. $\{\cdot,\cdot\}_1$ is both bilinear and skew-symmetric.

Demonstration. Bilinearity follows directly from each factor in Π_1 being bilinear. Skew symmetry follows from a straight forward calculation:

$$\begin{aligned}
\{F,G\}_1(u) &= \int \frac{\delta f}{\delta u}\left(\partial^3 + \frac{2}{3}u\partial + \frac{1}{3}u_x\right)\frac{\delta g}{\delta u}\mathrm{d}x\\
&= \int \frac{\delta f}{\delta u}\cdot\partial^3\left(\frac{\delta g}{\delta u}\right)\mathrm{d}x + \frac{2}{3}\int\frac{\delta f}{\delta u}\cdot u\partial\frac{\delta g}{\delta u}\mathrm{d}x + \frac{1}{3}\int\frac{\delta f}{\delta u}u_x\frac{\delta g}{\delta u}\mathrm{d}x\\
&= -\int \frac{\delta g}{\delta u}\cdot\partial^3\left(\frac{\delta f}{\delta u}\right)\mathrm{d}x - \frac{2}{3}\int\frac{\delta g}{\delta u}\cdot\partial\left(\frac{\delta f}{\delta u}u\right)\mathrm{d}x + \frac{1}{3}\int\frac{\delta f}{\delta u}u_x\frac{\delta g}{\delta u}\mathrm{d}x\\
&= -\int \frac{\delta g}{\delta u}\cdot\partial^3\left(\frac{\delta f}{\delta u}\right)\mathrm{d}x - \frac{2}{3}\int\frac{\delta g}{\delta u}u\partial\frac{\delta f}{\delta u}\mathrm{d}x - \frac{2}{3}\int\frac{\delta g}{\delta u}\frac{\delta f}{\delta u}u_x\mathrm{d}x + \frac{1}{3}\int\frac{\delta g}{\delta u}u_x\frac{\delta f}{\delta u}\mathrm{d}x\\
&= -\int \frac{\delta g}{\delta u}\cdot\partial^3\left(\frac{\delta f}{\delta u}\right)\mathrm{d}x - \frac{2}{3}\int\frac{\delta g}{\delta u}u\partial\frac{\delta f}{\delta u}\mathrm{d}x - \frac{1}{3}\int\frac{\delta g}{\delta u}u_x\frac{\delta f}{\delta u}\mathrm{d}x\\
&= -\int \frac{\delta g}{\delta u}\cdot\Pi_1\left(\frac{\delta f}{\delta u}\right)\mathrm{d}x\\
&= -\{G,F\}_1(u).
\end{aligned}$$

□

Lemma 7. This bracket satisfies a property similar to $\{\cdot,\cdot\}_0$, namely

$$\begin{aligned}
\mathrm{grad}\{F,H\}_1(u) =&(\mathscr{S}_F)_u(\Pi_1(u)\mathrm{grad}\,(H(u))) - (\mathscr{S}_H)_u(\Pi_1(u)\mathrm{grad}\,(F(u)))+\\
&+\frac{1}{3}\left(\mathrm{grad}\,(F(u))\cdot\partial\mathrm{grad}\,(H(u)) - \mathrm{grad}\,(H(u))\cdot\partial\mathrm{grad}\,(F(u))\right).
\end{aligned} \tag{4.19}$$

Demonstration. In this case, the Poisson structure Π_1 depends on u. First, we note that

$$\Pi_1(u+\varepsilon v) = \partial^3 + \frac{2}{3}(u+\varepsilon v)\partial + \frac{1}{3}(u_x+\varepsilon v_x),$$

and thus

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\Pi_1(u+\varepsilon v) = \frac{2}{3}v\cdot\partial + \frac{1}{3}v_x.$$

Now, following with the calculation,

$$\begin{aligned}
\operatorname{grad}\{F,H\}_1(u) &= \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \{F,H\}_1(u+\varepsilon v) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} (\operatorname{grad}(F(u+\varepsilon v)), \Pi_1(u+\varepsilon v)\operatorname{grad}(H(u+\varepsilon v))) \\
&= ((\mathscr{S}_F)_u(v), \Pi_1(u)\operatorname{grad}(H(u))) + \left(\operatorname{grad}(F(u)), \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \Pi_1(u+\varepsilon v)\cdot \operatorname{grad}(H)\right) \\
&\quad + (\operatorname{grad}(F(u)), \Pi_1(u)(\mathscr{S}_H)_u(v)) \\
&= ((\mathscr{S}_F)_u(\Pi_1(u)\operatorname{grad}(H(u))) - (\mathscr{S}_H)_u(\Pi_1(u)\operatorname{grad}(F(u))), v) \\
&\quad + \left(\operatorname{grad}(F(u)), \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \Pi_1(u+\varepsilon v)\cdot \operatorname{grad}(H)\right) \\
&= ((\mathscr{S}_F)_u(\Pi_1(u)\operatorname{grad}(H(u))) - (\mathscr{S}_H)_u(\Pi_1(u)\operatorname{grad}(F(u))), v) \\
&\quad + \left(\operatorname{grad}(F(u)), \left(\frac{2}{3}v\cdot\partial + \frac{1}{3}v_x\right)\operatorname{grad}(H(u))\right).
\end{aligned}$$

The first two terms in the proposition are already apparent. For the final term, we compute:

$$\begin{aligned}
\left(\operatorname{grad}(F(u)), \left(\frac{2}{3}v\cdot\partial + \frac{1}{3}v_x\right)\operatorname{grad}(H(u))\right) &= \int \frac{\delta f}{\delta u}\left(\frac{2}{3}v\cdot\partial + \frac{1}{3}v_x\right)\frac{\delta h}{\delta u}\,\mathrm{d}x \\
&= \frac{2}{3}\int \frac{\delta f}{\delta u} v\cdot\partial\left(\frac{\delta h}{\delta u}\right)\mathrm{d}x + \frac{1}{3}\int \frac{\delta f}{\delta u} v_x \frac{\delta h}{\delta u}\,\mathrm{d}x \\
&= -\frac{2}{3}\int \frac{\delta f}{\delta u} v_x \frac{\delta h}{\delta u}\,\mathrm{d}x - \frac{2}{3}\int \frac{\delta h}{\delta u} v\cdot\partial\left(\frac{\delta f}{\delta u}\right)\mathrm{d}x + \frac{1}{3}\int \frac{\delta f}{\delta u} v_x \frac{\delta h}{\delta u}\,\mathrm{d}x \\
&= -\frac{2}{3}\int \frac{\delta h}{\delta u} v\cdot\partial\left(\frac{\delta f}{\delta u}\right)\mathrm{d}x - \frac{1}{3}\int \frac{\delta f}{\delta u} v_x \frac{\delta h}{\delta u}\,\mathrm{d}x \\
&= -\frac{2}{3}\int \frac{\delta h}{\delta u} v\cdot\partial\left(\frac{\delta f}{\delta u}\right)\mathrm{d}x + \frac{1}{3}\int \partial\left(\frac{\delta f}{\delta u}\frac{\delta h}{\delta u}\right) v\,\mathrm{d}x \\
&= \frac{1}{3}\int \left(\frac{\delta f}{\delta u}\partial\frac{\delta h}{\delta u} - \frac{\delta h}{\delta u}\partial\frac{\delta f}{\delta u}\right) v\,\mathrm{d}x \\
&= \frac{1}{3}\left(\operatorname{grad}(F(u)\partial\operatorname{grad}(H(u)) - \operatorname{grad}(H(u))\partial\operatorname{grad}(F(u)), v\right).
\end{aligned}$$

□

Proposition 7. The operator Π_1 is a Hamiltonian operator, that is,

$$\{\{F,H\}_1,K\}_1 + \{\{K,F\}_1,H\}_1 + \{\{H,K\}_1,F\}_1 = 0 \quad \forall F,H,K. \tag{4.20}$$

Demonstration. Firstly, we expand:

$$\{\{F,H\}_1,K\} = (\operatorname{grad}\{F,H\}_1(u), \Pi_1(u)\operatorname{grad}(K(u)))$$

$$\begin{aligned}
&= \Big((\mathscr{S}_F)_u(\Pi_1(u)\operatorname{grad}(H(u))) - (\mathscr{S}_H)_u(\Pi_1(u)\operatorname{grad}(F(u))), \Pi_1(u)\operatorname{grad}(K(u))\Big) \\
&\quad + \frac{1}{3}\Big(\operatorname{grad}(F(u))\,\partial\operatorname{grad}(H(u)) - \operatorname{grad}(H(u))\,\partial\operatorname{grad}(F(u)), \Pi_1(u)\operatorname{grad}(K(u))\Big).
\end{aligned}$$

Now, considering all the cyclic permutations of this quantity after the last equality, we note that the contribution from the first term vanishes by means of a similar argument to the one used to prove Proposition 6. Indeed, Π_1, just like Π_0, is skew-symmetric while the second variation terms are symmetric. Now, for the second term, we need to evaluate

$$\sum_{\mathrm{Cyc.}\{F,H,K\}} \Big(\operatorname{grad}(F(u))\,\partial \operatorname{grad}(H(u)) - \operatorname{grad}(H(u))\,\partial \operatorname{grad}(F(u)) \,, \Pi_1(u) \operatorname{grad}(K(u)) \Big),$$

where the summation runs over the cyclic permutations of $\{F,H,K\}$. For visual simplicity, let's momentarily use the notation

$$\begin{aligned} \operatorname{grad}(F(u)) &= \alpha, \\ \operatorname{grad}(H(u)) &= \beta, \\ \operatorname{grad}(K(u)) &= \gamma, \\ \partial X &= X'. \end{aligned}$$

Then the sum above expands to

$$\begin{aligned} &\int (\alpha\beta' - \beta\alpha')\,\Pi_1(\gamma)\,\mathrm{d}x + \int (\gamma\alpha' - \alpha\gamma')\,\Pi_1(\beta)\,\mathrm{d}x + \int (\beta\gamma' - \gamma\beta')\,\Pi_1(\alpha)\,\mathrm{d}x = \\ &= \int \Big[\Pi_1 (\beta\alpha' - \alpha\beta') + \alpha'\Pi_1(\beta) + (\alpha\Pi_1(\beta))' - \beta'\Pi_1(\alpha) - (\beta\Pi - 1(\alpha))' \Big] \gamma\,\mathrm{d}x, \end{aligned}$$

and simple expansion by the definition of Π_1 shows that the quantity inside the square brackets vanishes. □

Proposition 8. The pair $(\{\cdot,\cdot\}_0, \{\cdot,\cdot\}_1)$ is a bihamiltonian structure, that is, the operator

$$\{\cdot,\cdot\} := \{\cdot,\cdot\}_0 + \{\cdot,\cdot\}_1$$

satisfies Jacobi's identity.

Demonstration. We simply expand

$$\begin{aligned} &\{\{F,H\}_0,K\}_1 + \{\{F,H\}_1,K\}_0 = \\ &= (\operatorname{grad}\{F,H\}_0, \Pi_1(u)\operatorname{grad}(K)) + (\operatorname{grad}\{F,H\}_1, \Pi_0(u)\operatorname{grad}(K)) = \\ &= \Big((\mathscr{S}_F)_u(\partial \operatorname{grad}(H(u))) - (\mathscr{S}_H)_u(\partial \operatorname{grad}(F(u))), \Pi_1(u)\operatorname{grad}(K(u)) \Big) + \\ &+ \Big((\mathscr{S}_F)_u(\Pi_1(u)\operatorname{grad}(H(u))) - (\mathscr{S}_H)_u(\Pi_1(u)\operatorname{grad}(F(u))), \Pi_0(u)\operatorname{grad}(K(u)) \Big) + \\ &+ \frac{1}{3}\Big(\operatorname{grad}(F(u))\partial \operatorname{grad}(H(u)) - \operatorname{grad}(H(u))\partial \operatorname{grad}(F(u)), \Pi_0(u)\operatorname{grad}(K(u)) \Big) \end{aligned}$$

and notice that it vanishes when summed over the cyclic permutations of $\{F,H,K\}$. □

Theorem 8 (Magri)**.** The equation

$$u_t = u_{(3)} + u_x u, \tag{4.21}$$

called *Korteweg-De Vries* equation, is a bi-Hamiltonian evolution equation with respect to the compatible pair (Π_0, Π_1).

Demonstration. Define the functionals

$$F_0 = \frac{1}{2}\int u^2\,\mathrm{d}x, \quad F_1 = \int \left(\frac{1}{6}u^3 - \frac{1}{2}u_x^2\right)\mathrm{d}x.$$

Then

$$\frac{\delta f_0}{\delta u} = u, \quad \frac{\delta f_1}{\delta u} = \frac{\partial f_1}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f_1}{\partial u_x} = \frac{1}{2}u^2 + u_{(2)}.$$

Therefore, each Hamiltonian system will be:

$$\begin{aligned}
\Pi_1\left(\frac{\delta f_0}{\delta u}\right) &= u_{(3)} + \frac{2}{3}uu_x + \frac{1}{3}u_x u = u_{(3)} + u_x u,\\
\Pi_0\left(\frac{\delta f_1}{\delta u}\right) &= \partial\left(\frac{u^2}{2} + u_{(2)}\right) = u_{(3)} + u_x u,
\end{aligned}$$

that is,

$$u_t = u_x u + u_{(3)} = \Pi_1\left(\frac{\delta f_0}{\delta u}\right) = \Pi_0\left(\frac{\delta f_1}{\delta u}\right).$$

$\square$

Theorem 9 (Gardner-Lenard-Miura-Kruskal-Lax)**.** There exists an infinite family of differential polynomials $\{f_n\}_{n\geq 1}$, $f_0 = \frac{1}{2}u^2$, $f_1 = \frac{1}{6}u^3 - \frac{1}{2}u_x^2$, ... where f_n contains derivatives of u up to order n such that

$$\Pi_0\left(\frac{\delta f_{n+1}}{\delta u}\right) = \Pi_1\left(\frac{\delta f_n}{\delta u}\right). \tag{4.22}$$

Demonstration. Writing $\partial^{-1} = \int$, we define the operator

$$R := \left(\partial^3 + \frac{2}{3}u\partial + \frac{1}{3}u_x\right)\circ\partial^{-1} = \partial^2 + \frac{2}{3}u + \frac{1}{3}u_x\partial^{-1}.$$

Notice that

$$\begin{aligned}
R(u_x) &= u_{(3)} + u_x u,\\
R(u_{(3)} + u_x u) &= u_{(5)} + \frac{10}{3}u_x u_{(2)} + \frac{5}{3}uu_{(3)} + \frac{5}{6}u_x u^2,\\
&\ \vdots
\end{aligned}$$

This way, we can define

$$
\begin{aligned}
v_0 &:= u_x, \\
v_1 &:= u_x u + u_{(3)}, \\
v_2 &:= u_{(5)} + \frac{10}{3} u_x u_{(2)} + \frac{5}{3} u u_{(3)} + \frac{5}{6} u_x u^2, \\
&\vdots \\
v_n &:= \Pi_0 \left(\frac{\delta f_n}{\delta u} \right) = \partial \left(\frac{\delta f_n}{\delta u} \right),
\end{aligned}
$$

and notice that

$$v_{n+1} = R(v_n).$$

Then

$$
\begin{aligned}
n = 0 &\implies \Pi_0 \left(\frac{\delta f_1}{\delta u} \right) = \Pi_1 \left(\frac{\delta f_0}{\delta u} \right) = u_{(3)} + u u_x, \quad \text{(KdV 1)} \\
&\vdots \\
n \text{ arbitrary} &\implies \Pi_0 \left(\frac{\delta f_{n+1}}{\delta u} \right) = \Pi_1 \left(\frac{\delta f_n}{\delta u} \right) = v_{n+1}. \quad \text{(KdV } n+1\text{)}
\end{aligned}
$$

□

Corollary 9. If $F_n = \int f_n \, \mathrm{d}x$, then

$$\{F_i, F_j\}_0 = 0 = \{F_i, F_j\}_1 \quad \forall i, j. \tag{4.23}$$

4.2 Systems of hydrodynamic type

In this section we will be concerned with systems of PDE's of the following type

$$u_t^k = \sum_{\alpha, j} V_j^{k\alpha}(\mathbf{u}) u_\alpha^j, \; j,k = 1, \ldots, n, \; \alpha = 1, \ldots, d, \tag{4.24}$$

where $\mathbf{u} := (u^1, \ldots, u^n)$ is a n-dimensional vector whose entries are functions of the (real) variables $(x^1, \ldots, x^d)$ and t, $u_\alpha^j = \frac{\partial u^j}{\partial x^\alpha}$ and $u_t^k = \frac{\partial u^k}{\partial t}$. The Equation 4.24 represents a dynamical system on the space of the u^k's which are, commonly, called *fields* and which will be thought as (smooth) maps between some open subset of $\mathbb{R}^d$ with values in a open subset of $\mathbb{R}^n$. Systems of PDE's like Equation 4.24 are named *d-dimensional homogeneous systems* of *hydrodynamical type*. The origin of the name comes from classical gas dynamics as the following remarks will make clear. Consider a thermodynamical system composed by an ideal gas described by the so called ideal gas law

$$PV = RT, \tag{4.25}$$

and recall that conservation of energy, in the form expressed by the first principle of thermodynamics, states that the changes of internal energy of the system equals the amount of work done by the system plus the amount of heat put in the systems. In formulas

$$\mathrm{d}Q = \mathrm{d}E + P\,\mathrm{d}V \tag{4.26}$$

Denoting by c_P and c_V the specific heat at constant pressure and, respectively, constant volume, defined by $c_V\,\mathrm{d}T = \mathrm{d}Q = \mathrm{d}E$ and $c_P\,\mathrm{d}T = \mathrm{d}Q = \mathrm{d}E + P\,\mathrm{d}V$, one can write $c_P\,\mathrm{d}T = c_V\,\mathrm{d}T + R\,\mathrm{d}T$, which entail the fundamental relation

$$c_P = c_V + R. \tag{4.27}$$

Writing now the infinitesimal variation of heat in terms of the infinitesimal variation of entropy as $\mathrm{d}Q = T\,\mathrm{d}S$, one can write

$$\mathrm{d}S = \frac{\mathrm{d}Q}{T} = \frac{c_V\,\mathrm{d}T + P\,\mathrm{d}V}{T} = c_V\frac{\mathrm{d}T}{T} + R\frac{\mathrm{d}V}{V} = c_V\frac{\mathrm{d}T}{T} - R\frac{\mathrm{d}\rho}{\rho},$$

where ρ is the density of the gas. Writing $\gamma = \frac{c_P}{c_V}$, which implies $\frac{R}{c_V} = \gamma - 1$, and assuming that the gas is undergoing a transformation at constant entropy (isentropic), one gets $T = ke^{\gamma-1}$, which, writing $P = \frac{RT}{V} = \frac{R}{m}\frac{T}{\rho}$ where m is the mass of the gas, becomes

$$P = ke^{\gamma}, \tag{4.28}$$

where k is a constant. Writing the (density) of total energy (kinetic+potential) of the system as $F = \frac{1}{2}\rho u^2 + k\rho^{\gamma}$, where u is the velocity, and calling $t_1 = u$ and $t_2 = \rho$, one arrives to

$$F = \frac{1}{2}t_2 t_1^2 + kt_2^{\gamma},$$

which is the potential for a 2-dimensional Frobenius manifold, section 3.4. One of the goals of this chapter is clarify this mysterious connection via the class of evolution equations introduced at the beginning of this chapter. To this end, we will first explain how the isentropic transformations of a perfect gas can be described by a system of PDE of the type of Equation 4.24. To this end, observe that the dynamics of such a system is described by the following system of evolutionary equations:

$$\begin{cases} \rho_t + (u\rho)_x & = 0 \\ u_t + uu_x + \frac{P_x}{\rho} & = 0, \end{cases} \tag{4.29}$$

the first being the equation describing the conservation of mass, i.e. the continuity equation, the second being the equation describing the motion of the gas according to Newton's second law of dynamics. Then one can prove

Proposition 9. The system Equation 4.29 can be represented as a 1-dimensional homogeneous system of hydrodynamic type.

The proof of this statement hinges on the following observation: the system in Equation 4.29 is equivalent to

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ u\end{bmatrix} = \begin{bmatrix}-u & -\rho \\ -k\gamma\rho^{\gamma-2} & -u\end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix}\rho \\ u\end{bmatrix},$$

which is of the same form as Equation 4.24. Note that in this case we have two fields, ρ and u, both functions of x and t.

Lemma 8. The coefficients of the right-hand side of Equation 4.24 transforms as a tensor of type $(1,1)$, i.e. as an endomorphism of the tangent bundle of $\mathbb{R}^n$.

4.3 Poisson structures of hydrodynamic type and a theorem of Dubrovin and Novikov

Hereafter we focus on 1-dimensional homogeneous systems of hydrodynamic type

$$u^i_t = V^i_k(\mathbf{u})u^k_x, \tag{4.30}$$

where $u^k_x = \frac{\partial u^j}{\partial x}$ for all k.

Note 5. In what follows the Einstein index convention on the summation over repeated indexes will be systematically used.

Hereafter, the phase-space of such a dynamical system will be identified with a space of smooth maps from the cylinder $S^1\times\mathbb{R}$ to a (suitable open) subset of some $\mathbb{R}^n$ (or more generally of a manifold M). The components of these maps are the fields $u^k(x,t): S^1\times\mathbb{R}\to\mathbb{R}$, $k=1,\ldots,n$. The t-dependence will be, in general, omitted. In other words, the phase-space of the dynamical systems which will be considered here is a space of maps which will be denoted by $L(M)$, the *loop-space* of M.

Definition 16. A Poisson bracket on the phase-space of Equation 4.30 is called of *hydrodynamic type*, or simply PSHT from now on, if it has the following form:

$$\{u^i(x),u^j(y)\} = g^{ij}(u(x))\delta'(x-y)+b^{ij}_k(u(x))u^k_x\delta(x-y). \tag{4.31}$$

Once a PSHT is given, then for any functionals I,J defined on the phase-space we have

$$\{I,J\} = \int \frac{\delta I}{\delta u^i}A^{ij}\frac{\delta J}{\delta u^j}\,\mathrm{d}x \quad \text{where} \quad A^{ij} = g^{ij}(u(x))\frac{\mathrm{d}}{\mathrm{d}x}+b^{ij}_k(u(x))u^k_x. \tag{4.32}$$

Moreover,

Definition 17. A functional $\mathscr{H}$ will be called a *Hamiltonian of hydrodynamic type*, or HHT from now on, if it is described by a density h depending only on the fields, but not on their x-derivatives, i.e. $\mathscr{H} = \int h(u(x))\,\mathrm{d}x$, where h does not depend on $u^i_{(n)}$ for all $n \geq 1$ and all i. The systems which can be described by a PSHT and a HHT, i.e. by an evolutionary equation of the type

$$u^i_t = \{u^i(x), \mathscr{H}\} = A^{ij}\frac{\delta\mathscr{H}}{\delta u^j(x)}, \tag{4.33}$$

with $\mathscr{H}$ and $\{\cdot,\cdot\}$ as above, are called *Hamiltonian systems of hydrodynamic type*, or simply HSHT from now on.

First we prove that if $\mathscr{H}$ is a HHT and $\{\cdot,\cdot\}$ is a PSHT, Equation 4.33 assumes the form of Equation 4.30. More precisely, we prove the following:

Lemma 9. If $\mathscr{H} = \int h\,\mathrm{d}x$ is a HHT and $\{\cdot,\cdot\}$ is a PSHT, then

$$\{u^i, \mathscr{H}\} = \Big[\frac{\partial^2 h}{\partial u^k \partial u^l} g^{il} + b^{il}_k \frac{\partial h}{\partial u^l}\Big] u^k_x. \tag{4.34}$$

In other words, under these assumptions, Equation 4.30 becomes

$$u^i_t = V^i_k u^k_x \quad \text{with} \quad V^i_k = \frac{\partial^2 h}{\partial u^k \partial u^l} g^{il} + b^{il}_k \frac{\partial h}{\partial u^l}. \tag{4.35}$$

Demonstration. It suffices to compute

$$\begin{aligned}
\{u^i, \mathscr{H}\} &= \iint \frac{\delta u^i(x)}{\delta u^k(\xi)} \{u^k(\xi), u^l(y)\} \frac{\delta h}{\delta u^l}(y)\,\mathrm{d}\xi\,\mathrm{d}y \\
&= \iint \delta^i_k \delta(x-\xi)\{u^k(\xi), u^l(y)\}\frac{\delta h}{\delta u^l}(y)\,\mathrm{d}\xi\,\mathrm{d}y \\
&= \iint \delta(x-\xi)\{u^i(\xi), u^l(y)\}\frac{\delta h}{\delta u^l}(y)\,\mathrm{d}\xi\,\mathrm{d}y \\
&= \iint \delta(x-\xi)[g^{il}(\xi)\delta'(\xi-y) + b^{il}_k(\xi)u^k_\xi\delta(\xi-y)]\frac{\delta h}{\delta u^l}(y)\,\mathrm{d}\xi\,\mathrm{d}y \\
&= \int \delta(x-\xi)g^{il}(\xi)\,\mathrm{d}\xi\left(\int \delta'(\xi-y)\frac{\delta h}{\delta u^l}(y)\,\mathrm{d}y\right) + \\
&\qquad + \int \delta(x-\xi)b^{il}_k(\xi)u^k_\xi\,\mathrm{d}\xi\left(\int \delta(\xi-y)\frac{\delta h}{\delta u^l}(y)\,\mathrm{d}y\right) \\
&= \int \delta(x-\xi)g^{il}(\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\delta h}{\delta u^l}(\xi)\,\mathrm{d}\xi + \int \delta(x-\xi)b^{il}_k(\xi)u^k_\xi\frac{\delta h}{\delta u^l}(\xi)\,\mathrm{d}\xi.
\end{aligned}$$

Since $\mathscr{H}$ is a HHT, $\frac{\delta h}{\delta u^l} = \frac{\partial h}{\partial u^l}$ and $\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\delta h}{\delta u^l} = \frac{\partial^2 h}{\partial u^k \partial u^l}u^k_\xi$, which can be inserted in the last term of the previous equalities to give

$$\begin{aligned}
\{u^i, \mathscr{H}\} &= \int \delta(x-\xi)u^k_\xi\,[g^{il}(\xi)\frac{\partial^2 h}{\partial u^k \partial u^l}(\xi) + b^{il}_k(\xi)\frac{\partial h}{\partial u^l}(\xi)]\,\mathrm{d}\xi \\
&= [g^{il}(x)\frac{\partial^2 h}{\partial u^k \partial u^l}(x) + b^{il}_k(x)\frac{\partial h}{\partial u^l}(x)]u^k_x,
\end{aligned}$$

which is what we wanted to prove. □

Before moving to the main result of this section, we need to recall a few notions from (pseudo)Riemanniann geometry.

4.3.1 Contravariant metrics and their associated connections

Recall that a metric on a smooth manifold is a non-degenerate, symmetric, covariant 2-tensor, or a non-degenerate symmetric tensor of type $(0,2)$, described in a local coordinate patch $x^1,\dots,x^n$ by the formula $g = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$. Under the change of (local) coordinates $y = y(x)$, the components of g change accordingly with the following formula

$$g_{kl}(x) = \frac{\partial y^i}{\partial x^k}\frac{\partial y^j}{\partial x^l} g_{ij}(y). \tag{4.36}$$

Symmetry is equivalent to $g_{ij} = g_{ji}$, for all i,j and non-degeneracy means that if given any $m \in M$ there is $v \in T_mM$ is such that $g(v,u) = 0$ for all $u \in T_mM$, then u is the null-vector. The components g_{ij} are obtained in the given coordinate system simply contracting g with the pair of holonomic vector fields $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}$, i.e. $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, for all i,j. A linear connection ∇ is called compatible with g if $\nabla g = 0$, i.e., if $\nabla_X g = 0$ for any vector field X on M. Note that a linear connection is completely determined by the set of its *Christoffel's symbols* Γ^k_{ij}, which are functions defined on every local coordinate patch by the formula

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \Gamma^k_{ij}\frac{\partial}{\partial x^k}, \tag{4.37}$$

for all i,j,k. Under the (local) change of coordinates $y = y(x)$, they undergo the following transformation law

$$\Gamma^p_{ij}(y) = \frac{\partial y^p}{\partial x^s}\frac{\partial^2 x^s}{\partial y^i \partial y^j} + \frac{\partial y^p}{\partial x^s}\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}\Gamma^s_{kl}(x), \tag{4.38}$$

which explicits their non-tensorial nature.

Note 6. The non-tensorial nature of the Christoffel's symbols, i.e. of the connections they represent, is a consequence of ∇ being $C^\infty(M)$-linear only in one of its entries. More precisely, while $\nabla_{fX}Y = f\nabla_X Y$, we have that $\nabla_X(fY) = X(f)Y + f\nabla_X Y$, for all $X,Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$.

Note 7. A given linear connection defines its Christoffel's symbols via Equation 4.37. As already recalled, these are functions defined locally on M (on every coordinate patch), satisfying Equation 4.38. On the other hand, given a set of local functions $\{\Gamma^k_{ij}\}$, each of them defined on a coordinate patch and satisfying the transformation law given in Equation 4.38, it defines a unique linear connection whose Christoffel's symbols are the Γ^k_{ij}. In this sense the datum of a linear connection is equivalent to the one of its Christoffel's symbols.

The above mentioned compatibility between ∇ and g reads as follows

$$\frac{\partial g_{ij}}{\partial x^k} = g_{il}\Gamma^l_{jk} + g_{jl}\Gamma^l_{ik},\ \forall i,j,k, \tag{4.39}$$

as one can check computing

$$\nabla_{\frac{\partial}{\partial x^k}} g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}).$$

Recall now that if ∇ is a linear connection, one can define its torsion $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ and curvature $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ via the following formulas

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \quad \text{and} \quad R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{4.40}$$

$\forall X,Y,Z \in \mathfrak{X}(M)$. Both T and R are $C^\infty(M)$-linear applications, i.e. they define two tensors on M. Then one can prove the following fundamental theorem:

Theorem 10 (Levi-Civita)**.** If g is a metric on M, there exist a unique linear connection ∇ compatible with g and whose torsion vanishes identically.

The connection mentioned in the previous theorem is called the *Levi-Civita*'s connection of the metric g. One can prove that the Levi-Civita's connection is characterized as the unique connection whose Christoffel's symbols, in every coordinate patch, are described by the following expressions

$$\Gamma^k_{ij} = \frac{g^{km}}{2}\Big(\frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m}\Big). \tag{4.41}$$

The latter are obtained (wisely) summing up the right-hand sides of Equation 4.39 after cyclically permuting (i,j,k) and using the property of Christoffel's symbols of the Levi-Civita's of being symmetric in its lower indices. In fact, since the torsion of the Levi-Civita'a connection is identically zero, in every coordinate patch and for all i,j one has

$$0 = T(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \Gamma^k_{ij} - \Gamma^k_{ji}.$$

Note 8. Note that given a metric g on M, one can define, on every coordinate patch, the symbol Γ^k_{ij} using Equation 4.41. Once these functions are defined, one can check that:

1. They transform as the Christoffel's symbols of a connection, i.e. they satisfy Equation 4.38;
2. they are symmetric in the lower indices;
3. They satisfy Equation 4.39.

Item (1) is a tedious, but straightforward, computation, (2) is obvious and (3) is verified noticing that Equation 4.41 is equivalent to

$$2g_{kr}\Gamma^k_{ij} = \frac{\partial g_{ir}}{\partial x^j} + \frac{\partial g_{jr}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^r},$$

which, after trading r with i, becomes

$$2g_{ki}\Gamma^k_{rj} = \frac{\partial g_{ri}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^r} - \frac{\partial g_{rj}}{\partial x^i}.$$

Summing these two identities term by term and using the symmetry of the components of the metric in the lower indices, one arrives at

$$2(g_{kr}\Gamma^k_{ij} + g_{ki}\Gamma^k_{rj}) = 2\frac{\partial g_{ri}}{\partial x^j},$$

which is Equation 4.39. In this way we proved Theorem 10, which can be restated as follows:

Theorem 11. If g is a metric on M whose coefficients are g_{ij}, then there exists a unique connection whose Christoffel's symbols are Γ^k_{lm}s and such that

$$\Gamma^k_{lm} = \Gamma^k_{ml}, \tag{4.42}$$

$$\frac{\partial g_{ij}}{\partial x^k} = g_{il}\Gamma^l_{jk} + g_{jl}\Gamma^l_{ik}. \tag{4.43}$$

where the Γ^l_{ml}'s defining this connection are defined by the formula Equation 4.41.

Note that Equation 4.42 says that the connection is torsion free, while Equation 4.43 says that it is compatible with the metric.

We now move to discuss, briefly, the contravariant side of the previous comments. First of all, a contravariant metric is symmetric, non-degenerate tensor of type $(2,0)$, i.e. one can written in a local patch assumes the following form $g = g^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}$. Note that to every covariant metric described locally by the coefficient g_{ij} corresponds a contravariant metric described, on the same coordinate patch, by the coefficient g^{lk}. The relation between covariant and contravariant metric tensors is enclosed by the formula

$$g^{ik}g_{kj} = \delta^i_j, \tag{4.44}$$

for all i, j. In other words, the matrix describing locally a contravariant metric is the inverse of the matrix describing, on the same coordinate patch, the associate covariant metric. First we prove the following:

Lemma 10. Under the local change of coordinates $y = y(x)$ a contravariant metric g transforms as follows

$$g^{ij}(x) = \frac{\partial x^i}{\partial y^k}\frac{\partial x^j}{\partial y^r}g^{kr}(y). \tag{4.45}$$

Demonstration. It suffices to compute

$$g^{ij}(x) = g(\mathrm{d}x^i, \mathrm{d}x^j) = g\Big(\frac{\partial x^i}{\partial y^k}\mathrm{d}y^k, \frac{\partial x^j}{\partial y^r}\mathrm{d}y^r\Big) = \frac{\partial x^i}{\partial y^k}\frac{\partial x^j}{\partial y^r}g(\mathrm{d}y^k, \mathrm{d}y^r) = \frac{\partial x^i}{\partial y^k}\frac{\partial x^j}{\partial y^r}g^{kr}(y).$$

□

For all i,j,k, set

$$\Gamma^{ij}_k = -g^{is}\Gamma^j_{sk}, \tag{4.46}$$

where g^{ij} are the coefficients of a contravariant metric and the Γ^j_{sk}'s are the Christoffel's symbols defining the Levi-Civita's connection of the (corresponding) metric g_{ij}.

We will now show how the Γ^{ij}_ks transform under (local) change of coordinates. More precisely, we will prove that

Lemma 11. Under change of (local) coordinates $u = u(p)$

$$\Gamma^{fd}_s(u) = g^{ki}(p)\frac{\partial u^f}{\partial p^k}\frac{\partial p^j}{\partial u^s}\frac{\partial^2 u^d}{\partial p^i \partial p^j} + \Gamma^{kl}_j(p)\frac{\partial u^f}{\partial p^k}\frac{\partial u^d}{\partial p^l}\frac{\partial p^j}{\partial u^s} \tag{4.47}$$

Demonstration. Equation 4.38 entails

$$\begin{aligned}
g^{ik}(p)\frac{\partial u^f}{\partial p^k}\frac{\partial u^d}{\partial p^l}\Gamma^l_{ij}(p)\,\mathrm{d}p^j &= g^{ki}(p)\frac{\partial u^f}{\partial p^k}\frac{\partial u^d}{\partial p^l}\frac{\partial p^l}{\partial u^c}\frac{\partial^2 u^c}{\partial p^i \partial p^j}\,\mathrm{d}p^j + \\
&\quad + g^{ki}(p)\frac{\partial u^f}{\partial p^k}\frac{\partial u^d}{\partial p^l}\frac{\partial u^a}{\partial p^i}\frac{\partial u^b}{\partial p^j}\Gamma^c_{ab}(u)\frac{\partial p^l}{\partial u^c}\,\mathrm{d}p^j \\
&= g^{ki}(p)\frac{\partial u^f}{\partial p^k}\delta^d_c\frac{\partial^2 u^c}{\partial p^i \partial p^j}\,\mathrm{d}p^j + g^{ki}(p)\frac{\partial u^f}{\partial p^k}\delta^d_c\frac{\partial u^a}{\partial p^i}\frac{\partial u^b}{\partial p^j}\Gamma^c_{ab}(u)\,\mathrm{d}p^j \\
&= g^{ki}(p)\frac{\partial u^f}{\partial p^k}\frac{\partial^2 u^d}{\partial p^i \partial p^j}\,\mathrm{d}p^j + g^{ki}(p)\frac{\partial u^f}{\partial p^k}\frac{\partial u^a}{\partial p^i}\frac{\partial u^b}{\partial p^j}\Gamma^d_{ab}(u)\,\mathrm{d}p^j \\
&\overset{\text{Equation 4.45}}{=} g^{ki}(p)\frac{\partial u^f}{\partial p^k}\frac{\partial^2 u^d}{\partial p^i \partial p^j}\,\mathrm{d}p^j + g^{fa}(u)\frac{\partial u^b}{\partial p^j}\Gamma^d_{ab}(u)\,\mathrm{d}p^j \\
&= g^{ki}(p)\frac{\partial u^f}{\partial p^k}\frac{\partial^2 u^d}{\partial p^i \partial p^j}\,\mathrm{d}p^j - \frac{\partial u^b}{\partial p^j}\Gamma^{fd}_b(u)\,\mathrm{d}p^j,
\end{aligned}$$

which, after multiplying the first and the last terms of the chain of equalities by $\frac{\partial p^j}{\partial u^s}$ and summing up over the repeated indices, yields the thesis. □

Note 9. Note that if the Γ^{ij}_k's and the Γ^l_{rs}'s relate to each other via Equation 4.46, where $\det g^{lm} \neq 0$, then the Γ^{ij}_k's transform as in Equation 4.47 *if and only if* the Γ^l_{rs}'s transform as in Equation 4.38.

Definition 18. The Γ^{ij}_k's defined in Equation 4.46 are called the *contravariant Christoffel's symbols* of the metric g_{ij}.

One can prove the following important fact:

Proposition 10. The symbols Γ^{ij}_k are determined uniquely by the following set of conditions:

$$g^{is}\Gamma^{jk}_s = g^{js}\Gamma^{ik}_s, \tag{4.48}$$

$$\Gamma^{ij}_k + \Gamma^{ji}_k = \frac{\partial g^{ij}}{\partial x^k}. \tag{4.49}$$

Demonstration. We need to prove, first, that if the Γ^{ij}_k's are defined as in Equation 4.46, where the g^{is}'s and the Γ^j_{sk}'s are the coefficients of a contravariant metric and, respectively, the Christoffel's symbols of the corresponding covariant metric, then they satisfy Equation 4.48 and Equation 4.49. Second, that if the g^{ij}'s are the coefficients of a contravariant metric and the Γ^{lk}_m's transform as in Equation 4.47, then the $\Gamma^k_{ij} = -g_{is}\Gamma^{sk}_j$ are the Christoffel's symbols of the covariant metric defined (locally) by the g_{ij}'s, see Equation 4.44, i.e. they transform as in Equation 4.38 and satisfy Equation 4.42 and Equation 4.43. We start by observing that using Equation 4.46, one can compute

$$g^{is}\Gamma^{jk}_s = -g^{is}g^{jl}\Gamma^k_{ls} = -g^{jl}g^{is}\Gamma^k_{sl} = g^{jl}\Gamma^{ik}_l,$$

proving Equation 4.48. On the other hand,

$$\begin{aligned}
0 = \frac{\partial(g^{is}g_{sj})}{\partial x^k} &= \frac{\partial g^{is}}{\partial x^k}g_{sj} + g^{is}\frac{\partial g_{sj}}{\partial x^k} \\
&\overset{\text{Equation 4.39}}{=} \frac{\partial g^{is}}{\partial x^k}g_{sj} + g^{is}\left(g_{sm}\Gamma^m_{jk} + g_{jm}\Gamma^m_{sk}\right) \\
&= \frac{\partial g^{is}}{\partial x^k}g_{sj} + \Gamma^i_{jk} + g^{is}g_{jm}\Gamma^m_{sk} \\
&= \frac{\partial g^{is}}{\partial x^k}g_{sj}g^{jr} + \Gamma^i_{jk}g^{jr} + g^{is}g_{jm}g^{jr}\Gamma^m_{sk} \\
&= \frac{\partial g^{ir}}{\partial x^k} - \Gamma^{ri}_k + g^{is}\Gamma^r_{sk} \\
&= \frac{\partial g^{ir}}{\partial x^k} - \Gamma^{ri}_k - \Gamma^{ir}_k.
\end{aligned}$$

It was already observed that if the Γ^{ij}_k's satisfy the transformation law in Equation 4.47, then $\Gamma^j_{sk} = -g_{si}\Gamma^{ij}_k$ satisfy Equation 4.38. We are left to prove that Equation 4.48 implies Equation 4.42, and Equation 4.49 implies Equation 4.43. Starting from Equation 4.48, one can observe that

$$\begin{aligned}
g^{is}\Gamma^{jk}_s = g^{jt}\Gamma^{ik}_t &\implies -g^{is}g^{il}\Gamma^k_{ls} = g^{jt}\Gamma^{ik}_t \\
&\implies -g^{is}g_{rj}g^{il}\Gamma^k_{ls} = g_{rj}g^{jt}\Gamma^{ik}_t \\
&\implies -g^{is}\Gamma^k_{rs} = \Gamma^{ik}_r \\
&\implies -g^{is}g_{si}\Gamma^k_{rs} = g_{is}\Gamma^{ik}_r \\
&\implies \Gamma^k_{rs} = \Gamma^k_{sr}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
0 = \frac{\partial(g^{ij}g_{jl})}{\partial x^k} &= \frac{\partial g^{ij}}{\partial x^k} g_{jl} + g^{ij}\frac{\partial g_{jl}}{\partial x^k} \\
&\overset{\text{Equation 4.49}}{=} (\Gamma^{ij}_k + \Gamma^{ji}_k) g_{jl} + g^{ij}\frac{\partial g_{jl}}{\partial x^k} \\
&\implies -g^{ij}\frac{\partial g_{jl}}{\partial x^k} = g_{lj}\Gamma^{ji}_k + g_{jl}\Gamma^{ij}_k \\
&\implies -g^{ij}\frac{\partial g_{jl}}{\partial x^k} = -\Gamma^i_{lk} + g_{jl}\Gamma^{ij}_k \\
&\implies -g_{si}g^{ij}\frac{\partial g_{jl}}{\partial x^k} = -g_{si}\Gamma^i_{lk} + g_{si}g_{jl}\Gamma^{ij}_k \\
&\implies \frac{\partial g_{sl}}{\partial x^k} = g_{si}\Gamma^i_{lk} + g_{li}\Gamma^i_{sk},
\end{aligned}$$

which is what we wanted to prove. □

Note 10. It should be clear now that the properties of the Levi-Civita's connection of the metric g_{ij} described in Equation 4.42 and Equation 4.43 correspond to the properties of contravariant Christoffel's symbols described in Equation 4.48 and, respectively, Equation 4.49.

Before moving on, in the following remark, we will try to justify the name contravariant Christoffel's symbols given to the Γ^{ij}_k's.

Note 11. Recall that, locally, a connection is described by its Christoffel's symbols via Equation 4.37. The latter implies that every linear connection extends to a connection to the $C^\infty(M)$-module $\Omega^1(M)$, via the following formula

$$\nabla_{\frac{\partial}{\partial x^i}} \mathrm{d}x^j = -\Gamma^j_{si}\,\mathrm{d}x^s. \tag{4.50}$$

In fact,

$$\langle \nabla_{\frac{\partial}{\partial x^i}} \mathrm{d}x^j, \frac{\partial}{\partial x^s}\rangle = -\langle \mathrm{d}x^j, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^s}\rangle = -\langle \mathrm{d}x^j, \Gamma^l_{si}\frac{\partial}{\partial x^l}\rangle = -\Gamma^j_{si}.$$

Suppose now that ∇ is the Levi-Civita's connection of a metric g_{ij}, whose contravariant metric is g^{ij}. Then

$$g^{li}\nabla_{\frac{\partial}{\partial x^i}} \mathrm{d}x^j = -g^{li}\Gamma^j_{si}\,\mathrm{d}x^s \overset{\nabla \text{ is symmetric}}{=} -g^{li}\Gamma^j_{is}\,\mathrm{d}x^s = \Gamma^{lj}_s\,\mathrm{d}x^s,$$

i.e. if one sets $\nabla^l = g^{li}\nabla_{\frac{\partial}{\partial x^i}}$, then

$$\nabla^l \mathrm{d}x^j = \Gamma^{lj}_s\,\mathrm{d}x^s. \tag{4.51}$$

More in general, if $u = u_i\,\mathrm{d}x^i$ and $v = v_j\,\mathrm{d}x^j$, then

$$\nabla^u v = \big(g^{ij}u_i\frac{\partial v_k}{\partial x^j} + \Gamma^{ij}_k u_i v_j\big)\,\mathrm{d}x^k. \tag{4.52}$$

This formula is the *contravariant analogue* of the more usual one

$$\nabla_{X^i\frac{\partial}{\partial x^i}}(v_j\,\mathrm{d}x^j) = X^i\frac{\partial v_j}{\partial x^i}\mathrm{d}x^j + X^i v_j \nabla_{\frac{\partial}{\partial x^i}}\mathrm{d}x^j \overset{\text{Equation 4.50}}{=} \big(X^i\frac{\partial v_s}{\partial x^i} - X^i v_j\Gamma^j_{is}\big)\,\mathrm{d}x^s.$$

Before moving on it is worth making the following long comment about the curvature tensor of the Levi-Civita's connection of a metric g.

4.3.1.1 Meaning of the curvature tensor.

Suppose a metric g is expressed in the local coordinates $(x^1,\dots,x^n)$ as $g_{ij}(x)\,\mathrm{d}x^i\,\mathrm{d}x^j$. One can ask if there exists a change of coordinates $x=x(y)$ such that, in the new coordinates $(y^1,\dots,y^n)$, g assumes a *diagonal form*, i.e. $g=\delta_{kl}\,\mathrm{d}y^k\,\mathrm{d}y^l=\mathrm{d}y^k\,\mathrm{d}y^k$, where the last term is meant to be a sum for k between 1 and n. In other words, one can ask if there is a change of coordinates $x=x(y)$ such that

$$g_{ij}(x(y))\frac{\partial x^i}{\partial y^k}\frac{\partial x^j}{\partial y^l}=\delta_{kl}\iff g_{ij}=\delta_{kl}\frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j}=\sum_k\frac{\partial y^k}{\partial x^i}\frac{\partial y^k}{\partial x^j}. \tag{4.53}$$

Deriving the right-hand side with respect to x^l one gets

$$\frac{\partial g_{ij}}{\partial x^l}=\sum_k\frac{\partial^2 y^k}{\partial x^l\partial x^i}\frac{\partial y^k}{\partial x^j}+\sum_k\frac{\partial y^k}{\partial x^i}\frac{\partial^2 y^k}{\partial x^l\partial x^j}.$$

Writing now the corresponding formulas for $\frac{\partial g_{il}}{\partial x^j}$, $\frac{\partial g_{lj}}{\partial x_i}$ using the symmetry of the second partial derivatives and (wisely) summing up one obtains

$$\sum_k\frac{\partial^2 y^k}{\partial x^i\partial x^j}\frac{\partial y^k}{\partial x^l}=\frac{1}{2}\left(\frac{\partial g_{il}}{\partial x^j}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^l}\right). \tag{4.54}$$

Multiplying both sides of the previous identity by $\frac{\partial y^s}{\partial x^r}g^{rl}$, Equation 4.41 entails

$$\sum_k\frac{\partial^2 y^k}{\partial x^i\partial x^j}g^{rl}\frac{\partial y^k}{\partial x^l}\frac{\partial y^s}{\partial x^r}=\frac{\partial y^s}{\partial x^r}\frac{g^{rl}}{2}\left(\frac{\partial g_{il}}{\partial x^j}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^l}\right)=\frac{\partial y^s}{\partial x^r}\Gamma^r_{ij},$$

where $g^{rl}\frac{\partial y^k}{\partial x^l}\frac{\partial y^s}{\partial x^r}=\delta^{ks}$, see the first identity in Equation 4.53. Then one arrives to the following systems of PDE's

$$\frac{\partial^2 y^s}{\partial x^i\partial x^j}=\frac{\partial y^s}{\partial x^r}\Gamma^r_{ij}. \tag{4.55}$$

Setting $g^s_i=\frac{\partial y^s}{\partial x^i}$ and $\phi^s_{ij}=\frac{\partial y^s}{\partial x^r}\Gamma^r_{ij}$, the previous system becomes

$$\frac{\partial g^s_i}{\partial x^j}=\phi^s_{ij},$$

whose solution is guaranteed once, for all i,s, the following integrability condition is satisfied

$$\frac{\partial\phi^s_{ij}}{\partial x^k}=\frac{\partial\phi^s_{ik}}{\partial x^j}, \tag{4.56}$$

for all j,k. Computing the derivatives and using Equation 4.56, one obtains

$$\begin{gathered}
\frac{\partial}{\partial x^k}\left(\frac{\partial y^s}{\partial x^r}\Gamma^r_{ij}\right)=\frac{\partial}{\partial x^j}\left(\frac{\partial y^s}{\partial x^r}\Gamma^r_{ik}\right)\\
\frac{\partial^2 y^s}{\partial x^k\partial x^r}\Gamma^r_{ij}+\frac{\partial y^s}{\partial x^r}\frac{\partial\Gamma^r_{ij}}{\partial x^k}=\frac{\partial^2 y^s}{\partial x^j\partial x^r}\Gamma^r_{ik}+\frac{\partial y^s}{\partial x^r}\frac{\partial\Gamma^r_{ik}}{\partial x^j}\\
\frac{\partial y^s}{\partial x^m}\Gamma^m_{kr}\Gamma^r_{ij}+\frac{\partial y^s}{\partial x^r}\frac{\partial\Gamma^r_{ij}}{\partial x^k}=\frac{\partial y^s}{\partial x^m}\Gamma^m_{jr}\Gamma^r_{ik}+\frac{\partial y^s}{\partial x^r}\frac{\partial\Gamma^r_{ik}}{\partial x^j}.
\end{gathered}$$

Changing the summation indices ($r \leftrightarrow m$) in the first summands of both sides, the previous identity becomes

$$\frac{\partial y^s}{\partial x^r}\Gamma^r_{km}\Gamma^m_{ij} + \frac{\partial y^s}{\partial x^r}\frac{\partial \Gamma^r_{ij}}{\partial x^k} = \frac{\partial y^s}{\partial x^r}\Gamma^r_{jm}\Gamma^m_{ik} + \frac{\partial y^s}{\partial x^r}\frac{\partial \Gamma^r_{ik}}{\partial x^j},$$

which yields to

$$\frac{\partial \Gamma^r_{ij}}{\partial x^k} - \frac{\partial \Gamma^r_{ik}}{\partial x^j} + \Gamma^r_{km}\Gamma^m_{ij} - \Gamma^r_{jm}\Gamma^m_{ik} = 0. \tag{4.57}$$

In other words, the system of PDE's in Equation 4.55, in the unknowns $\frac{\partial y^s}{\partial x^i}$, has a (local in a neighborhood of each point) solution, if and only if the condition Equation 4.57 holds true. On the other hand, computing $R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^k}$, one concludes immediately that the left-side of Equation 4.57 are the components of the curvature of the connection defined by the Γ^i_{jk}'s in the local patch whose whose coordinates are $x^1, \dots, x^n$. In this way it was proven that

Theorem 12 (Riemann)**.** The only obstruction to the existence of a (local) change of coordinates $y = y(x)$ which reduces the metric $g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ to a diagonal form $\delta_{kl}\,\mathrm{d}y^k\,\mathrm{d}y^l$ is the curvature tensor being non-zero.

Note that once a local solution of Equation 4.55 is given, i.e. given the set of functions $\frac{\partial y^s}{\partial x^i}$ for all $k, l = 1, \dots, n$, one can define for all $s = 1, \dots, n$, the 1-form $\mathrm{d}y^s = \frac{\partial y^s}{\partial x^i}\,\mathrm{d}x^i$, and from this the function y^s, which will be one of the local coordinate functions. It is worth noticing that for all i, j, k, $\Gamma^i_{jk}(y) = 0$. In fact,

$$\begin{aligned}
\Gamma^p_{ij}(x) &\overset{\text{Equation 4.38}}{=} \frac{\partial x^p}{\partial y^s}\frac{\partial^2 y^s}{\partial x^i \partial x^j} + \frac{\partial x^p}{\partial y^s}\frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j}\Gamma^s_{kl}(y) \\
&\overset{\text{Equation 4.55}}{=} \frac{\partial x^p}{\partial y^s}\frac{\partial y^s}{\partial x^r}\Gamma^r_{ij}(x) + \frac{\partial x^p}{\partial y^s}\frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j}\Gamma^s_{kl}(y) \\
&= \Gamma^p_{ij}(x) + \frac{\partial x^p}{\partial y^s}\frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j}\Gamma^s_{kl}(y),
\end{aligned}$$

which yields

$$\frac{\partial x^p}{\partial y^s}\frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j}\Gamma^s_{kl}(y) = 0,$$

implying that, for all s, k, l, $\Gamma^s_{kl}(y) = 0$ as desired. The latter has the following consequence. For all $i, j = 1, \dots, n$,

$$\nabla_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} = 0,$$

which, by the $C^\infty(M)$-linearity of the operator ∇ in the first entry, implies that for all vector fields X and for all $i = 1, \dots, n$,

$$\nabla_X \frac{\partial}{\partial y^i} = 0 \qquad \Longleftrightarrow \qquad \nabla \frac{\partial}{\partial y^i} = 0. \tag{4.58}$$

Definition 19. A (local) vector field X such that $\nabla X = 0$ is called *flat* or *parallel* with respect to ∇. In particular, the (local) holonomic basis $\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}$ induced by the local set of coordinates $y^1, \ldots, y^n$ diagonalizing the metric g is a *flat* holonomic basis of vector fields with respect to the Levi-Civita's connection defined by g.

Finally,

Definition 20. A metric g is called *flat* if the curvature of the corresponding Levi-Civita connection is identically equal to zero.

Note 12. In particular a metric is flat if and only if every point of the underlying manifold belongs to a coordinate patch where the metric assumes the diagonal form.

In general, if $\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}$ is a flat (local) holonomic basis of vector fields, the elements of its dual basis of 1-forms $\mathrm{d}y^1, \ldots, \mathrm{d}y^n$ satisfy $\nabla_X \mathrm{d}y^j = 0$, see Equation 4.50, for all vector fields X and all $j = 1, \ldots, n$. In other words the 1-forms $\mathrm{d}y^j$'s are flat or parallel with respect to ∇ as well.

Definition 21. A (locally defined) function f is called *flat* with respect to a connection ∇ if $\nabla \mathrm{d}f = 0$. In particular the local coordinates $y^1, \ldots, y^n$ diagonalizing g are flat with respect to the Levi-Civita's connection defining g.

We close this long comment with the following remarks:

Note 13.

1. Theorem 12 can be extended to the non-metric context. More precisely, on a neighborhood U of a given point $m \in M$, one can find $X_1, \ldots, X_n$, (generically) linearly independent and such that $\nabla X_i = 0$ for all i, *if and only if* the (restriction to U of the) curvature tensor of ∇ is identically zero. If the (restriction to U of the) torsion of ∇ vanishes, such a system of flat vector fields will induce a set of (generically) independent flat functions.

2. The previous comments can be applied verbatim to the case of contravariant metric and to their corresponding connections.

In particular, a contravariant metric is flat if every point of the underlying manifold belongs to a coordinate patch where its contravariant curvature tensor vanishes, i.e. such that for all i, j, k, l

$$R_l^{ijk} := g^{is}\Big(\frac{\partial \Gamma_l^{jk}}{\partial x^s} - \frac{\partial \Gamma_s^{jk}}{\partial x^l}\Big) + \Gamma_s^{ij}\Gamma_l^{sk} - \Gamma_s^{ik}\Gamma_l^{sj} \equiv 0. \tag{4.59}$$

We now close this part introducing the following important notion:

Definition 22. Two contravariant metrics g_1, g_2 on M form a *flat pencil* if:

1. For all $\lambda \in \mathbb{R}$, $g_\lambda := g_1 - \lambda g_2$ is a contravariant metric on M.
2. If the Γ^{ij}_{1k}'s and Γ^{ij}_{2k}'s are the Christoffel's symbols of the Levi-Civita's connections of g_1 and, respectively, g_2, then the $\Gamma^{ij}_{\lambda k} := \Gamma^{ij}_{1k} - \lambda \Gamma^{ij}_{2k}$ are the Christoffel's symbols of the Levi-Civita's connection of g_λ for all λ.
3. The contravariant metric g_λ is flat for all λ.

Now we can go back to the systems of hydrodynamic type. Suppose we are given a PSHT whose elementary brackets, if written with respect to the local fields $p^1, \dots, p^n$, are as in Equation 4.31, i.e. for all a, b,

$$\{p^a(x), p^b(y)\} = g^{ab}(p)\delta'(x-y) + b^{ab}_c(p)p^c_x\delta(x-y). \tag{4.60}$$

Hereafter we will prove the following fundamental result

Theorem 13 (Dubrovin-Novikov)**.**

1. Under the change of coordinates $u = u(p)$, the g^{ab}'s transform as a tensor of type $(2,0)$. Moreover, if $\det g^{ab} \neq 0$, the Γ^b_{dc}'s defined by $b^{ab}_c = -g^{ad}\Gamma^b_{dc}$ satisfy Equation 4.38, i.e. they are the Christoffel symbols of a (contravariant) connection.
2. The skew-symmetry of Equation 4.60 implies that $g^{ab} = g^{ba}$ and that the g^{ab}'s satisfy Equation 4.49. In particular, if $\det g^{ab} \neq 0$, given the previous two points, one can conclude that the g^{ab}'s define a (contravariant) metric and the b^{ab}_c's a connection compatible with it.
3. The Jacobi identity of Equation 4.60 implies that the curvature and the torsion of the connection defined by the b^{ab}_c's are identically zero, i.e. the latter is the (contravariant) Levi-Civita's connection of g_{ab}, where $g_{ab}g^{bc} = \delta^c_a$.

Demonstration. First we will prove (1). To this end, we will find out how Equation 4.60 transforms under the change of coordinates $u = u(p)$. To this end we first recall the following (distributional) identity

$$f(y)\delta'(x-y) = f'(x)\delta(x-y) + f(x)\delta'(x-y), \tag{4.61}$$

and then we compute

$$\begin{aligned}
\{u^i(x), u^j(y)\} &= \iint \frac{\delta u^i}{\delta p^a}(x)\{p^a(x), p^b(y)\}\frac{\delta u^j}{\delta p^b}(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint \frac{\delta u^i}{\delta p^a}(x)(g^{ab}(p)\delta'(x-y) + b^{ab}_c(p)p^c_x\delta(x-y))\frac{\delta u^j}{\delta p^b}(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint \frac{\delta u^i}{\delta p^a}(x)g^{ab}(p)\delta'(x-y)\frac{\delta u^j}{\delta p^b}(y)\,\mathrm{d}x\,\mathrm{d}y + \iint \frac{\delta u^i}{\delta p^a}(x)b^{ab}_c(p)p^c_x\delta(x-y)\frac{\delta u^j}{\delta p^b}(y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Applying Equation 4.61 to the first summand of the last term of the previous computation, one continues by

$$\iint \frac{\delta u^i}{\delta p^a}(x) g^{ab}(p)\big(\delta'(x-y)\frac{\delta u^j}{\delta p^b}(x) + \frac{\mathrm{d}}{\mathrm{d}x}\frac{\delta u^j}{\delta p^b}(x)\delta(x-y)\big)\,\mathrm{d}x\,\mathrm{d}y + \iint \frac{\delta u^i}{\delta p^a}(x) b^{ab}_c(p) p^c_x \delta(x-y)\frac{\delta u^j}{\delta p^b}(y)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \iint \Big(\frac{\partial u^i}{\partial p^a}(x) g^{ab}(p)\frac{\partial u^j}{\partial p^b}(x)\delta'(x-y) + \big(\frac{\partial u^i}{\partial p^a}(x)\frac{\partial^2 u^j}{\partial p^c \partial p^b}(x) + \frac{\partial u^i}{\partial p^a}(x)\frac{\partial u^j}{\partial p^b}(x) b^{ab}_c(p)\big) p^c_x \delta(x-y)\Big)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \iint g^{ij}(u)\delta'(x-y) + b^{ij}_k(u) u^k_x \delta(x-y)\,\mathrm{d}x\,\mathrm{d}y,$$

where we set

$$\begin{cases} g^{ij}(u) & = \frac{\partial u^i}{\partial p^a}(x) g^{ab}(p)\frac{\partial u^j}{\partial p^b}(x), \\ b^{ij}_k(u) u^k_x & = \frac{\partial u^i}{\partial p^a}(x)\frac{\partial u^j}{\partial p^b}(x) b^{ab}_c(p) p^c_x. \end{cases}$$

Note now that since $u^k_x = \frac{d}{dx}u^k = \frac{\partial u^k}{\partial p^c}p^c_x$, the second identity can be rewritten as

$$b^{ij}_k(u)\frac{\partial u^k}{\partial p^c}p^c_x = \frac{\partial u^i}{\partial p^a}\frac{\partial u^j}{\partial p^b} b^{ab}_c(p) p^c_x \Longrightarrow b^{ij}_l(u) = \frac{\partial p^c}{\partial u^l}\frac{\partial u^i}{\partial p^a}\frac{\partial u^j}{\partial p^b} b^{ab}_c(p) + \frac{\partial p^c}{\partial u^l}\frac{\partial u^i}{\partial p^a}\frac{\partial^2 u^j}{\partial p^b \partial p^c} g^{ab}(p).$$

In other words we proved that under the change of coordinates $u = u(p)$, the components of a PSHT as in Equation 4.31 undergo the following transformation law

$$\begin{cases} g^{ij}(u) & = \frac{\partial u^i}{\partial p^a} g^{ab}(p)\frac{\partial u^j}{\partial p^b}, \\ b^{ij}_l(u) & = \frac{\partial p^c}{\partial u^l}\frac{\partial u^i}{\partial p^a}\frac{\partial u^j}{\partial p^b} b^{ab}_c(p) + \frac{\partial p^c}{\partial u^l}\frac{\partial u^i}{\partial p^a}\frac{\partial^2 u^j}{\partial p^b \partial p^c} g^{ab}(p). \end{cases} \tag{4.62}$$

These identities prove respectively the first statement and the second statement in (1), for the latter see Note 9. We will now prove (2), i.e. that

$$\{u^i(x), u^j(y)\} = -\{u^j(y), u^i(x)\} \Longrightarrow g^{ij} = g^{ji} \quad \text{and} \quad \frac{\partial g^{ij}}{\partial x^k} = b^{ij}_k + b^{ji}_k.$$

To this end we compute

$$\begin{aligned} \{u^j(y), u^i(x)\} &= g^{ji}(u(y))\delta'(y-x) + b^{ji}_k(u(y)) u^k_y \delta(y-x) \\ &= -g^{ji}(u(y))\delta'(x-y) + b^{ji}_k(u(y)) u^k_y \delta(y-x) \\ &\overset{\text{Equation 4.61}}{=} -\frac{d}{dx}(g^{ji}(u(x)))\delta(x-y) - g^{ji}(u(x))\delta'(x-y) + b^{ji}_k(u(y)) u^k_y \delta(x-y) \\ &= -\frac{\partial g^{ji}}{\partial u^k}u^k_x \delta(x-y) + b^{ji}_k(u(x)) u^k_x \delta(x-y) - g^{ji}(u(x))\delta'(x-y). \end{aligned}$$

On the other hand

$$-\{u^i(x), u^j(y)\} = -g^{ij}(u(x))\delta'(x-y) - b^{ij}_k(u(x)) u^k_x \delta(x-y),$$

which compared with the last term of the previous computation yields

$$\begin{cases} g^{ij}(u) & = g^{ji}(u) \\ -b^{ij}_k(u) & = -\frac{\partial g^{ji}}{\partial u^k} + b^{ji}(u) \iff \frac{\partial g^{ij}}{\partial u^k} = b^{ij}_k + b^{ji}_k. \end{cases}$$

Part (3) is a computation much more involved than the ones presented above and, for this reason, will not be presented in detail. Here we just comment that for a Poisson bracket (of local type, i.e.) of the form

$$\{u^i(x), u^j(y\} = \sum_p A^p_{ij}(\vec{u}, \vec{u}_x, \vec{u}_{xx}, \ldots)\delta^{(p)}(x-y), \tag{4.63}$$

where $\vec{u} = (u_1, u_2, \ldots)$, the Jacobi identity is equivalent to fact that the distribution

$$S^{ijk}(x,y,z) := \frac{\partial P^{ij}_{xy}}{\partial u^l_{(s)}(x)}\partial^s_x P^{lk}_{xz} + \frac{\partial P^{ki}_{zx}}{\partial u^l_{(s)}(z)}\partial^s_z P^{lj}_{zy} + \frac{\partial P^{jk}_{yz}}{\partial u^l_{(s)}(y)}\partial^s_y P^{li}_{yx},$$

is identically equal to zero, where $P^{ij}_{xy} := \{u^i(x), u^j(y)\}$. This means that for any choice of test functions $a(x), b(y)$ and $c(z)$,

$$\iiint S^{ijk}(x,y,z)a(x)b(y)c(z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \equiv 0. \tag{4.64}$$

Using the properties of the delta-Dirac, one can reduce the right-hand side of the previous identity, involving x, y and z, to an integral involving only one variable, say x, of the following type

$$\int \mathrm{d}x \sum_{\sigma,\tau} A^{ijk}_{\sigma\tau} a_i b^{(\sigma)}_j c^{(\tau)}_k, \tag{4.65}$$

where σ and τ denote the order of the derivative, which depend on the order of the derivative present in Equation 4.63. This is the same as reducing S^{ijk} to the *normal* form

$$\sum_{\sigma,\tau} A^{ijk}_{\sigma\tau}(x)\delta^{(\sigma)}(x-y)\delta^{(\tau)}(x-z), \tag{4.66}$$

and then Equation 4.64 becomes

$$A^{ijk}_{\sigma\tau} = 0, \tag{4.67}$$

for all possible choices of the indices. In our case, Equation 4.63 is Equation 4.60 and for this reason i, j, k vary between 1 and n while σ, τ can be $0, 1$ or 2. If one spell out these conditions, which are equivalent to the Jacobi identity for the Poisson brackets in Equation 4.60, then one arrives at

$$\begin{cases} b^{ij}_s g^{sk} - b^{kj}_s g^{si} & = 0 \\ (\frac{\partial b^{jk}_t}{\partial x^s} - \frac{\partial b^{jk}_s}{\partial x^t})g^{ti} + b^{ij}_t b^{tk}_s - b^{ik}_t b^{tj}_s & = 0, \end{cases} \tag{4.68}$$

which entail that torsion and, respectively, curvature of the connection defined by the b^{ij}_k's are identically zero. □

Note 14. It was shown that given a PSHT as in Equation 4.60, if $\det g^{ab} \neq 0$, then these coefficients represent a contravariant metric and the b^{ab}_c's the Christoffel's symbols of the corresponding Levi-Civita's connection. It is easier to prove the converse. If g^{ab}

is a contravariant metric and the b_c^{ab}'s are the Christoffel's symbols of its Levi-Civita's connection, then if the latter has zero torsion and zero curvature,

$$\{u^i(x), u^j(y)\} = g^{ij}(u)\delta'(x-y) + b_k^{ij}(u)u_x^k\delta(x-y) \tag{4.69}$$

defines a PSHT, i.e. the brackets so defined are skew-symmetric and satisfy the Jacobi's identity. The proof of these statements becomes trivial if one remembers that if the Levi-Civita's connection of a metric g is flat, i.e. it has zero curvature, one can find local coordinates $p^1, \dots, p^n$ (in a neighborhood of every point) diagonalizing the metric and such that, if written in these coordinates, its Christoffel's symbols are zero, see the comment following Theorem 12. In these coordinates the brackets in Equation 4.69 assume the following *constant* form

$$\{p^i(x), p^j(y)\} = \delta^{ij}\delta'(x-y),$$

proving their skew-symmetry and that they satisfy the Jacobi's identity.

Now we go back to the notion of flat pencil of metrics, see Definition 22, to link it with the one of compatible Poisson structures, see Definition 26. In particular we prove that

Corollary 10. Two metrics g_1, g_2 form a flat pencil if and only if $\{\cdot,\cdot\}_1, \{\cdot,\cdot\}_2$ form a pair of compatible PSHT.

Demonstration. If g_1, g_2 form a flat pencil of metrics and $\Gamma_{1k}^{ij}, \Gamma_{2k}^{ij}$ are the Christoffel's symbols of their Levi-Civita's connections, then for every λ the brackets

$$\{u^i(x), u^j(y)\}_\lambda = (g_1^{ij} - \lambda g_2^{ij})(u)\delta'(x-y) + (\Gamma_{1k}^{ij} - \lambda\Gamma_{2k}^{ij})(u)u_x^k\delta(x-y) \tag{4.70}$$

define a PSHT, see the Note 14. This implies that $\{\cdot,\cdot\}_1, \{\cdot,\cdot\}_2$ is a pair of compatible PSHT, where

$$\{u^i(x), u^j(y)\}_a = g_a^{ij}(u)\delta'(x-y) + \Gamma_{ak}^{ij}(u)u_x^k\delta(x-y),\ a = 1, 2. \tag{4.71}$$

The converse follows from Theorem 13, noticing that if Equation 4.71 defines two compatible PSHT, then, for every λ, Equation 4.70 is a PSHT. $\square$

Definition 23. A *biHamiltonian structure of hydrodynamic type* is a pair of compatible PSHT.

Finally, we want to make a comment about the link of the theory of PSHT with the one of Frobenius manifold. In particular we will prove the following:

Theorem 14. If M is a Frobenius manifold, $L(M)$, see the comment above Definition 16, carries a biHamiltonian structure of hydrodynamic type.

Demonstration. The result will follows once we are able to prove that every Frobenius manifold carries a flat pencil of metrics, which, to simplify the notation, we will denote by η and g. More precisely, we will denote by g the flat metric whose datum is part of the definition of a Dubrovin-Frobenius manifold, see Definition 4. On the other hand we introduce the tensor η via the formula

$$\eta^{ij} = g^{il} c^j_{lk} E^k. \tag{4.72}$$

Note that the previous formula defines a symmetric $(2,0)$ tensor on M (the symmetry follows after a little computation from the right-hand side in Equation 2.28). A more intrinsic way to see the previous formula goes, in words, as follows. The associative and commutative product defined by the totally symmetric tensor c induces a commutative and associative product on the cotangent space at every point of M, which in turn defines a product $\cdot$ on the (sheaf of the) 1-forms on M. The tensor η is a $(2,0)$-tensor on M such that $\eta(\omega_1, \omega_2) := \iota_E(\omega_1 \cdot \omega_2)$, for all (locally defined) 1-forms, where ι_E is the contraction operator (on the sheaf of forms on M) defined by the Euler vector field. One can prove that the symmetric $(2,0)$-tensor so defined is (generically) non-degenerate and, for this reason, defines a (contravariant) metric on M, which is called the *intersection form* of the Frobenius manifold. The content of the theorem can be rephrased now as follows: η and g form a flat pencil of metrics on M. Here, without entering in many details, we will present the main idea(s) behind the proof the this remarkable result. To this end we will need to prove that (η, g) fulfills the conditions in Definition 22. The first one, i.e. that $g_\lambda := \eta - \lambda g$ is a contravariant metric, can be proven after choosing suitable flat g-coordinates $t^1, \ldots, t^n$, with respect to which $E^i = d_i t^i + r_i$ (no sum over repeated indices!), where d_i, r_i are constants. The other two points to be checked are resumed as follows:

1. If the Γ^{ij}_k's and b^{ij}_k's are the Christoffel's symbols of the Levi-Civita's connections of g and, respectively, η then, for all λ, the $b^{ij}_{\lambda k} := b^{ij}_k - \lambda \Gamma^{ij}_k$ are the Christoffel's symbols of the Levi-Civita's connection of $g_\lambda = \eta - \lambda g$

2. For all λ, the metric $g_\lambda = \eta - \lambda g$ is flat, i.e. the curvature of its Levi-Civita's connection is zero.

3. In this case, we say that Π_1 and Π_0, as well as $\{\}_1$ and $\{\}_0$ are compatible.

Note that, in spite of the fact that the Christoffel's symbols are not tensorial, the first condition holds true if and only if it holds in a given system of coordinates, see Equation 4.38. For this reason, one can check the two points above (note that the second is coordinate independent) choosing a set of flat g-coordinates. Since on such coordinates the Γ^{ij}_k's are identically zero, one has that $b^{ij}_{\lambda k} = b^{ij}_k$, for all λ and, accordingly, the curvature tensor of the pencil will depend on λ only via the components of the metric g_λ, see Equation 4.59.

Once settled these points, the crucial step in the proof is the following statement. The functions

$$b_k^{ij} := \left(1 + d_j - \frac{d_F}{2}\right) c_k^{ij} \tag{4.73}$$

satisfy Equation 4.48 and Equation 4.49 for (in that formulas) $g = \eta$ defined as in Equation 4.72. Proving this statement entails to prove that the b^{ij}'s defined in Equation 4.73 are the Christoffel's symbols of the metric η. To prove the first item in the list above one uses again Proposition 10. More precisely, the point will follow if one is able to show that, if the b_k^{ij}'s are as in Equation 4.73, then, for all λ,

$$g_\lambda^{is} b_s^{jk} = g_\lambda^{js} b_s^{ik},$$
$$b_k^{ij} + b_k^{ji} = \frac{\partial g_\lambda^{ij}}{\partial x^k},$$

where the $x^1, \ldots, x^n$ are the set of g-flat coordinates chosen as above mentioned. The second point will follow from a (more or less) direct computation. $\square$

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APPENDIX

A

POISSON STRUCTURES ON MANIFOLDS

A.1 Poisson bracket from a symplectic structure

Definition 24 (Poisson structure)**.** Let M be a smooth manifold. Then a *Poisson structure* in M is a non-degenerate $\mathbb{R}$-bilinear map $\{\cdot,\cdot\} : \mathrm{C}^\infty(M) \times \mathrm{C}^\infty(M) \longrightarrow \mathrm{C}^\infty(M)$ such that, $\forall f, g, h \in \mathrm{C}^\infty(M)$, it satisfies the following:

(i) Skew symmetry: $\{f,g\} = -\{g,f\}$;

(ii) Jacobi identity: $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$;

(iii) Leibniz's rule: $\{fg,h\} = f\{g,h\} + g\{f,h\}$

The map $\{\cdot,\cdot\}$ is called a *Poisson bracket*.

Example 10 (Classical Poisson bracket)**.** Choosing local coordinates $(q^1,...,q^n,p^1,...,p^n)$ in $M = \mathbb{R}^{2n}$, the canonical Poisson bracket is given by

$$\{f,g\} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q^i}\right). \tag{A.1}$$

Conditions (i) and (ii) ensure that $\{\cdot,\cdot\}$ defines the structure of a Lie algebra on $\mathrm{C}^\infty(M)$. Now let $f \in \mathrm{C}^\infty(M)$ and consider the map

$$\begin{aligned} \{f,\cdot\} : \mathrm{C}^\infty(M) &\longrightarrow \mathrm{C}^\infty(M) \\ g &\longmapsto \{f,g\}. \end{aligned} \tag{A.2}$$

Then condition (iii) ensures that this is a derivation of $\mathrm{C}^\infty(M)$. Thus, it may be identified with a vector field $X_f \in \mathfrak{X}(M)$.

Definition 25. The vector field $X_f \in \mathfrak{X}(M)$ associated with the function $f \in \mathrm{C}^{\infty(M)}$ via the map $\{f,\cdot\}$ is called the *Hamiltonian vector field* of f.

It's worth noting that, in classical mechanics, if $\mathscr{H}$ is the Hamiltonian of a system and (x_i) are the generalized coordinates, then Hamilton's equations of motion for the system are given by

$$\dot{x}_i = \{\mathscr{H}, x_i\}. \tag{A.3}$$

The following example is very important.

Example 11 (Poisson structure from symplectic form)**.** Let N be a smooth manifold and let T^*N be it's cotangent bundle. We denote by π the canonical projection

$$\begin{aligned} \pi : T^*N &\longrightarrow N \\ \alpha &\longmapsto \pi(\alpha). \end{aligned}$$

We'd like to define a form $\Theta \in \Omega^1(T^*N)$. Let $\alpha \in T^*N$ and $v \in T_\alpha(T^*N)$, and denote by $\Theta_\alpha \in T^*_\alpha(T^*N)$ the form Θ at the point α. Let

$$\begin{aligned} (\pi_*)_\alpha : T_\alpha(T^*N) &\longrightarrow T_{\pi(\alpha)}N \\ v &\longmapsto (\pi_*)_\alpha(v). \end{aligned}$$

Then we define Θ point-wise by

$$\langle \Theta_\alpha, v\rangle \coloneqq \langle \alpha, (\pi_*)_\alpha(v)\rangle. \tag{A.4}$$

The form Θ is called *canonical Liouville form.* If $q^1,...,q^n$ are local coordinates in $U \subset N$ and $p_1,...,p_n$ are fibered coordinates in $\pi^{-1}(U) \subset T^*N$, then we have

$$\Theta|_{\pi^{-1}(U)} = \sum_{i=1}^{n} p_i \,\mathrm{d}q^i. \tag{A.5}$$

The exterior derivative of Θ is denoted by Ω,

$$\Omega \coloneqq \mathrm{d}\Theta = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q^i, \tag{A.6}$$

which is a symplectic form on N. Letting $X,Y \in \mathfrak{X}(T^*N)$, we have contractions

$$\begin{aligned} \iota_X\Omega &= \sum_{i=1}^{n} \left(\langle \mathrm{d}p_i, X\rangle \,\mathrm{d}q^i - \langle \mathrm{d}q^i, X\rangle \,\mathrm{d}p_i\right) \\ \iota_Y(\iota_X\Omega) &= \sum_{i=1}^{n} \left(\langle \mathrm{d}p_i, X\rangle\langle \mathrm{d}q^i, Y\rangle - \langle \mathrm{d}q^i, X\rangle\langle \mathrm{d}p_i, Y\rangle\right), \end{aligned}$$

that is,

$$\iota_Y(\iota_X\Omega) = \Omega(X,Y) \in \mathrm{C}^\infty(T^*N). \tag{A.7}$$

Now, since Ω is non-degenerate, then the map

$$\begin{aligned} \Omega^\flat : \mathfrak{X}(N) &\longrightarrow \Omega^1(N) \\ X &\longmapsto \iota_X\Omega \end{aligned}$$

is an isomorphism, and therefore admits in inverse map. Then, if $f \in \mathrm{C}^\infty(N)$, $\mathrm{d}f \in \Omega^1(N)$, we can find a vector field $X_f \in \mathfrak{X}(N)$ such that $\iota_{X_f}\Omega = \mathrm{d}f$.

Let $f, g \in \mathrm{C}^\infty(N)$ with vector fields $X_f, X_g \in \mathfrak{X}(N)$ as described above. Then

$$\{f,g\} := \iota_{X_g}\left(\iota_{X_f}\Omega\right) = \Omega\left(X_f, X_g\right) \tag{A.8}$$

is a Poisson bracket. Indeed, via Equation A.6, we have

$$\{f,g\} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q^i}\right). \tag{A.9}$$

The example above shows that a symplectic structure on a manifold implies the existence of a Poisson bracket. However, the converse is not generally true.

Example 12 (Linear Poisson structure)**.** Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie algebra of finite dimension over $\mathbb{R}$. Let $\mathfrak{g}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathbb{R})$ be it's dual space and and consider elements $f, g \in \mathrm{C}^\infty(\mathfrak{g}^*)$ and $\alpha \in \mathfrak{g}^*$. Notice that

$$\mathrm{d}f_\alpha \in \mathrm{Hom}_{\mathbb{R}}\left(T_\alpha(\mathfrak{g}^*), \mathbb{R}\right) \cong \mathrm{Hom}_{\mathbb{R}}\left(\mathfrak{g}^*, \mathbb{R}\right) \cong (\mathfrak{g}^*)^* \cong \mathfrak{g}$$

where the last identifications follows directly from the fact that the dimension of $\mathfrak{g}$ is finite. We may thus define a Poisson bracket

$$\{f,g\}(\alpha) := \langle \alpha, [\mathrm{d}f_\alpha, \mathrm{d}g_\alpha]\rangle. \tag{A.10}$$

All in all, for any elements $x, y \in \mathfrak{g} \cong \mathrm{Hom}_{\mathbb{R}}\left(\mathfrak{g}^*, \mathbb{R}\right)$ and for any $\alpha \in \mathfrak{g}^*$, we can define a Poisson bracket via

$$\{x,y\}(\alpha) = \langle \alpha, [\mathrm{d}x_\alpha, \mathrm{d}y_\alpha]\rangle = \langle \alpha, [x,y]\rangle. \tag{A.11}$$

Indeed, since $\mathrm{d}x_\alpha \in \mathfrak{g}$, we have $\mathrm{d}x_\alpha = x$, which is readily verified by evaluating the derivative

$$\mathrm{d}x_\alpha(v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} x(\alpha + tv) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(x(\alpha) + tx(v)\right) = x(v).$$

A.2 Poisson bracket from a bivector

As we have seen, a Poisson structure on a smooth manifold M is a non-degenerate bilinear map $\{\cdot,\cdot\} : \mathrm{C}^\infty(M) \times \mathrm{C}^\infty(M) \longrightarrow \mathrm{C}^\infty(M)$ defined on the space of smooth functions on M. Now Leibniz's rule implies that it is a biderivation. Let $\pi \in \Gamma\left(\bigwedge^2 TM\right)$ be a bivector. If $x_1, \ldots, x_n$ are local coordinates in a neighborhood $U \subset M$, then π generally has the form

$$\pi = \sum_{i<j} \pi_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \quad \pi_{ij} \in \mathrm{C}^\infty(U),$$

and we can define for $f,g \in \mathbf{C}^\infty(U)$ the map

$$\{f,g\} := \pi(\mathrm{d}f,\mathrm{d}g) = \iota_{\mathrm{d}g}\left(\iota_{\mathrm{d}f}\pi\right). \tag{A.12}$$

Now, it is not yet a Poisson bracket. The skew-symmetry and Leibniz's properties are readily verified directly from the definition, but Jacobi's identity is not satisfied in general. For this requirement to take place, we need further algebraic information on the bivector π. Let us be reminded of the exterior algebra of TM,

$$\bigwedge TM = \bigoplus_{j=0}^{\infty} \bigwedge^j TM,$$

which is a graded algebra with respect to the wedge product. Now, $\Gamma(\bigwedge^0 TM)$ is simply $\mathbf{C}^\infty(M)$. On the other hand, $\Gamma(\bigwedge^1 TM) = \mathfrak{X}(M)$, and so it is equipped with the usual Lie bracket

$$\begin{aligned}[\cdot,\cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(M)\\ (X,Y) &\longmapsto [X,Y](f) := X(Y(f)) - Y(X(f))\end{aligned}$$

where $f \in \mathbf{C}^\infty(M)$ and $X(f)(m) = \langle \mathrm{d}f_m, X_m\rangle$, $m \in M$.

Theorem 15 (Schouten-Nijenhuis)**.** The Lie bracket $[\cdot,\cdot]$ defined in $\mathfrak{X}(M)$ admits an extension to $\Gamma(\bigwedge TM)$ that satisfies

i) Graded anti-commutativity;

ii) Graded Leibniz rule;

iii) Graded Jacobi identity;

iv) Coincides with the Lie derivative.

This bracket is called *Schouten bracket*, and here is is denoted by $[\cdot,\cdot]_S$.

Demonstration. See Theorem 1.8.1 in (DUFOUR; ZUNG, 2005, p. 28). □

Corollary 11. The bracket $\{\cdot,\cdot\}$ defined via the bivector π satisfies Jacobi's identity if and only if $[\pi,\pi]_S = 0$.

Definition 26. Assume we have a triple (M,π_1,π_0) where M is a smooth manifold and π_1,π_0 are bivectors such that

(i) $[\pi_1,\pi_2]_S = [\pi_0,\pi_0]_S = 0$

(ii) $[\pi_1,\pi_0]_S = 0$.

Then we say (M, π_1, π_0) is a *Bihamiltonian manifold.*

Let $\{\cdot,\cdot\}_i$ denote the Poisson bracket generated by the bivector π_i. Then it is possible to show that condition (ii) is satisfied if and only if

$$\{\{f,g\}_0,h\}_1+\{\{f,g\}_1,h\}_0+\{\{h,f\}_0,g\}_1+\{\{h,f\}_1,g\}_0+\{\{g,h\}_0,f\}_1+\{\{g,h\}_1,f\}_0=0$$

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