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Page-Wootters mechanism: the role of coherence and finite sized clocks

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"When you're thinking about something that you don't understand, you have a terrible, uncomfortable feeling called confusion. It's a very difficult and unhappy business. And so most of the time you're rather unhappy, actually, with this confusion. You can't penetrate this thing. Now, is the confusion's because we're all some kind of apes that are kind of stupid working against this, trying to figure out [how] to put the two sticks together to reach the banana and we can't quite make it, the idea? And I get this feeling all the time that I'm an ape trying to put two sticks together, so I always feel stupid. Once in a while, though, the sticks go together on me and I reach the banana."

Richard Feynman

ABSTRACT

MENDES, L.R.S **Page-Wootters mechanism:** the role of coherence and finite sized clocks. 2021. 98p. Thesis (Doctor in Science) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2021.

Among the many proposals to approach the concept of time in quantum theory, the Page-Wootters mechanism has attracted much attention in the last few years. Originally, this mechanism explored a stationary pure bipartite non-interacting global system, i.e., a system of interest together with an ancillary clock, to determine how the evolution in time can emerge and an equation of motion can be obtained for a quantum particle conditioned to the measurement of the state of the clock. In this mechanism, time is seen as an inaccessible coordinate and the apparent passage of time arises as a consequence of correlations between the subsystems of a global state. Here we propose a measure that captures the relational character of the mechanism, showing that the internal coherence is the necessary ingredient for the emergence of time in the Page–Wootters model. Connecting it to results in quantum thermodynamics, showing that it is directly related to the extractable work from quantum coherence. In a second step we analyze such a timeless approach to quantum theory but deriving an equation of motion for a mixed state system that evolves according to its gravitationally induced interaction with a non-ideal quantum clock. The interaction considered is known to describe the gravitational decoherence mechanism, and the clock model is the recently proposed quasi-ideal clock, i.e., one constructed to approximate the time-energy canonical commutation relation. As a result of our considerations, we obtained an equation of motion that is non-linear in nature, dependent on the system's initial conditions.

Keywords: Page-Wootters. Quasi-ideal clock. Internal coherence. Non-linear equation of motion.

RESUMO

MENDES, L.R.S **Mecanismo de Page-Wootters:** o papel da coerência e relógios finitos. 2021. 98p. Tese (Doutorado em Ciências) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2021.

Dentre as muitas propostas de como abordar o conceito de tempo na teoria quântica, o mecanismo de Page-Wootters tem atraído muita atenção nos últimos anos. Originalmente, este mecanismo se focou num sistema global bipartido puro, não-interagente e estacionário, constituído de um sistema de interesse junto com um relógio, e buscou determinar como a evolução no tempo pode emergir e uma equação de movimento ser obtida para uma partícula quântica condicionada a medição do estado do relógio nestas circunstâncias. Nesse mecanismo, o tempo é visto como uma coordenada inacessível e a aparente passagem do tempo surge como consequência das correlações entre os subsistemas de um estado global. Aqui propomos uma medida que captura o caráter relacional do mecanismo, mostrando que a coerência interna é o ingrediente necessário para o surgimento do tempo no modelo de Page-Wootters. Conectando-o a resultados em termodinâmica quântica, mostrando que está diretamente relacionada ao trabalho extraível da coerência quântica. Em uma segunda etapa, analisamos essa abordagem atemporal da teoria quântica, mas derivando uma equação de movimento para um sistema de interesse num estado misto que evolui de acordo com sua interação induzida gravitacionalmente com um relógio quântico não ideal. A interação considerada é conhecida por descrever o mecanismo de decoerência gravitacional, e o modelo do relógio é o relógio quase ideal recentemente proposto, construído para aproximar a relação de comutação canônica tempo-energia. Como resultado de nossas considerações, obtivemos uma equação de movimento que é de natureza não linear, dependente das condições iniciais do sistema.

Palavras-chave: Page-Wootters. Relógio quase ideal. Coerência interna. Equação não linear de movimento.

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1 INTRODUCTION

If asked about the nature of time, what would qualify as a correct answer? Due to the elusive nature of the object in question, probably mixed answer would be given. Perhaps something related to change, something associated with entropy or even just an illusion. In the context of quantum systems, some, or perhaps most, would state that time is nothing more than a parameter that appears in the time dependent Schrödinger's equation

$$i\,\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = H\left|\psi(t)\right\rangle\,, \tag{1.1}$$

which is representative of a "real time". In this way time is not a dynamical variable, it is something which exists irrespective of the rest, being external to what is being studied. This concept of time is reminiscent of the one advocated by Sir Isaac Newton of an "absolute, true and mathematical time". (1) Newton considered time to be something apart from the rest which passes independently of everything, something that we could only approximate with our calculations. It is intriguing that such a classical concept, permeates our fundamental understanding of what is non-classical. A readily example would be that of measurement: which depends intrinsically on the given depiction, at least for the most commonly found Copenhagen interpretation, where a measurement is defined for a specific instant of time. (2)

The pressure of old arguments dissuaded most (3) from pursuing a concept of time where we can accommodate its quantumness, where it is not simply a parameter, but an operator. Nonetheless, even if we accept that time should be a classical concept a problem arises when we realize that, within our classical theories, this is not the only way to interpret time. In fact one of the most successful theories of physics, the theory of relativity, has a very distinct view about time, not considering it to be absolute but dependent on perspective. Therefore, how could we keep this antiquated notion about time when dealing with quantum theory? In part this problem was already tackled. As is well known, the theory of relativity branches into two, the special and the general, for the former, we already have a method of uniting it to quantum mechanics, which can be found in quantum field theory, another successful step for Physics. It is easily seen why, in special relativity, even though we do not have an absolute time, which will be commonly referred as a coordinate, we have a notion of a special frame of reference, that of inertial frames. In this way, the difference does not seem so striking, absolute time is replaced by a set of inertial frames, however, for general relativity things are very different.

In its way to arriving at general relativity, Einstein debated whether or not the right path to describe gravity would be through the use of covariant field equations (4), or in other words, with a theory where the change of a system of coordinates in spacetime would not impact the physics of the problem. This was a deep question which is related to the physical meaning of such coordinates. We can summarize his struggle following the development of the *Lochbetrachtung* or *hole argument**. (4–5) In the argument a hole is a place in spacetime which is devoid of gravitational sources, i.e., matter, which is used to specify a particular field. If we consider $S(\chi)$ to be the solution for the field equations inside the hole, given a certain position of spacetime specified by a four-vector χ, we can define a change of coordinates with the particularity that outside of the hole we get the same solution in a different position $S'(\chi')$. Noticing that the second solution can be written in the previous coordinates $S'(\chi)$. If it is required that a change in coordinates preserve the field equations, the solutions $S'(\chi')$, $S(\chi)$ and $S'(\chi)$, are all valid, however, $S(\chi)$ and $S'(\chi)$ are distinct in the same coordinate and nonetheless equal outside the hole, since outside $S'(\chi) \to S'(\chi')$. For Einstein this was a demonstration that a covariant theory would not allow the complete definition of physical events.

The resolution of this argument only came later when Einstein realized that all events that have some physical meaning are events which can be observed, and that for these events he already had a covariant theory. More importantly for us, this came together with the realization that there is no intrinsic meaning on some coordinate, being their only purpose to relate observable events. As he said (6):

> *Our space-time verifications invariably amount to a determination of space-time coincidences. If, for example, events consisted merely in the motion of material points, then ultimately nothing would be observable but the meetings of two or more of these points. Moreover, the results of our measurements are nothing but verifications of such meetings of the material points of our measuring instruments with other material points, coincidences between the hands of a clock and points on the clock-dial, and observed point-events happening at the same place at the same time.*
>
> *The introduction of a system of reference serves no other purpose than to facilitate the description of the totality of such coincidences. We allot to the universe four space-time variables x^1, x^2, x^3, x^4 in such a way that for every point-event there is a corresponding system of values of the variables $x^1...x^4$. To two coincident point-events there corresponds one system of values of the variables $x^1...x^4$, i.e., coincidence is characterized by the identity of the co-ordinates. If, in place of the variables $x^1...x^4$, we introduce functions of them, x'^1, x'^2, x'^3, x'^4, as a new system of coordinates, so that the systems of values are made to correspond to one another without ambiguity, the equality of all four co-ordinates in the new system will also serve as an expression for the*

* There is some debate on the meaning and implications of the hole argument, we will not discuss those issues, instead we will follow a historic timeline for the problem.

> *space-time coincidence of the two point-events. As all our physical experience can be ultimately reduced to such coincidences, there is no immediate reason for preferring certain systems of coordinates to others, that is to say, we arrive at the requirement of general covariance.*

This means that for general relativity, time, which is one coordinate used to describe events in spacetime, is nothing more than that, a coordinate, and should not carry special meaning to it.

The problem in reconciliating these contrasting views about time, found in quantum theory and general relativity, in a possible theory of quantum gravity[†], is what is known as the *problem of time.* (9–11) It spans many questions of fundamental character growing in a multifaceted way to technical ones, including and not limited to

- If time is a fundamental concept in a theory of quantum gravity;
- If probabilities are conserved;
- How meaningful is the concept of spacetime;
- Is the concept of measurement well defined;
- How do we interpret the microcausality condition of quantum field theory;
- How different choices of time variables will impact the dynamics of the system
- Can we find functionals which respect the constrains which are expected in classical spacetime.

The manner in which these problem can be resolved depend on the possible ways to deal with the constraints of the canonical theory of general relativity. Isham (9) divides these ways into three broad categories

Tempus ante quantum - First we identify a classical internal time associated with the canonical variables, then we solve the constraints and only after, the system is quantized. The goal is to obtain a Schrödinger-like equation in respect to the chosen time. ex: Internal Schrödinger interpretation, unimodular gravity and matter clocks and reference fluids.

Tempus post quantum - First we quantize the system, then we impose constraints and finally identify an internal time. Here it is obtained a Wheeler–DeWitt-like equation for the functionals. ex: Klein-Gordon interpretation, third quantization and semi-classical interpretation;

[†] Although there are similar questions when dealing with thermodynamics in curved spacetime (7), or quantum cosmology (8)

Tempus nihil est- Forget about time! We should construct a coherent and consistent quantum theory without time as a concept and recover the dynamics from within our construction. ex: Naïve Schrödinger interpretation, conditional probability interpretation, consistent histories approach and frozen time formalism.

This thesis will be concerned with approaches within the *Tempus nihil est* category, which is not to say that our focus will be directed towards very high energies. In fact, we will only discuss systems with low energies, in the setting of non-relativistic quantum mechanics. As it is well known, we do not have a quantum theory of gravity, even though we have many attempts. (12–15) The central problem here is that we cannot distinguish which alternative is viable or correct because their domain is not yet testable and internal consistency or beauty are not (and should not) be sufficient criteria to decide what is physically acceptable, only experimentation can give us that. (16–17) Therefore, we are left with what works and can be tested, such as quantum mechanics, special and general relativity and quantum field theory. Here, we follow the notion given by our alternative of choice, of studying a formulation of quantum mechanics which carries a concept of time, that is akin to that of general relativity, in that it does not have a special, external, role. Specifically, we will investigate the Page-Wootters's conditional probability interpretation (18) (which may be referenced as PaW, PaW model, construction or mechanism), emphasizing, that even in the original presentation of this approach it is not mentioned high energies or relativistic systems[‡], it does however concerns itself with the quantization in a closed system, which for them is representative of the Universe.

Their argument to eliminate time starts when considering that the possible states for the theory must be eigenstates of the Hamiltonian of the Universe. If we utilize $|\Psi\rangle$ as the notation for the global state of the Universe and its Hamiltonian as H, the requirement above can be mathematically described as a Wheeler-DeWitt-like (19) constraint

$$H\,|\Psi\rangle = 0. \tag{1.2}$$

With this condition any notion of evolution will necessarily come from within the construction, as a relation between its parts. Hence, if inside our Universe we define a system which we want to evaluate, with Hamiltonian H_S, we should have another system, that in respect to which, the evolution happens. For obvious reasons this additional system is designated as the clock, described by the state $|\psi_C\rangle$ with Hamiltonian H_C. It is required that each clock state can be reached from an initial clock state, which for simplicity, can be assumed to be $|\psi_C(0)\rangle$, so that

$$|\psi_C(\tau)\rangle = e^{-iH_C\tau}\,|\psi_C(0)\rangle\,, \tag{1.3}$$

with τ being clock time. The state of the system of interest can be defined as

$$|\psi_S(\tau)\rangle = \langle\psi_C(\tau)|\Psi\rangle\,, \tag{1.4}$$

‡ Apart from presentation and context.

i.e., where the state of the system is obtained by conditioning the global state to a certain clock time. In the original formalism it is considered that both parts of the Universe will not interact with each other, so that the total Hamiltonian is

$$H = H_S + H_C. \tag{1.5}$$

Then, the evolution of the system of interest is given by

$$\begin{aligned} i\frac{d\,|\psi_S\rangle}{d\tau} &= i\frac{d\,\langle\psi_C(\tau)|\Psi\rangle}{d\tau} \\ &= -\,\langle\psi_C(\tau)|H_C|\Psi\rangle \\ &= -\,\langle\psi_C(\tau)|H - H_S|\Psi\rangle \\ &= H_S\,\langle\psi_C(\tau)|\Psi\rangle = H_S\,|\psi_S(\tau)\rangle\,, \end{aligned} \tag{1.6}$$

showing that, in this Universe, which runs on a internal clock time, we still have the usual equation of motion. A distinct feature of this mechanism lies in the way utilized to recover the dynamics, which allows then to postulate conditional probabilities connecting positions of the system, such as the system going to the right R with the "time" τ, a value of an observable associated to the clock

$$\text{Prob}(R \text{ when } \tau) = \frac{\langle\Psi|\Pi_S(R)\otimes\Pi_C(\tau)|\Psi\rangle}{\langle\Psi|\Pi_C(\tau)|\Psi\rangle}, \tag{1.7}$$

where $\Pi_R(\cdot)$ and $\Pi_C(\cdot)$ represents operators corresponding to reading of the indicated clock time or position of the system. Then, higher values for the probabilities will indicate stronger correlation between the position of the particle and the passage of time, which means that a conditional probability $\text{Prob}(R \text{ when } \tau) = 1$ would indicate perfect correlation. This gives a very operational approach for the problem of time.

We observe that Eq.(1.4) implies in a general state for the Universe with the following form

$$|\Psi\rangle \propto \int d\tau\,|\psi_C(\tau)\rangle\,|\psi_S(\tau)\rangle\,, \tag{1.8}$$

indicating that time will be perceived as arising from entanglement between the subsystems. From the description of the mechanism, from its definition of conditional probabilities to the form of the state above, it becomes evident the impact of this construction for the interpretation of quantum mechanics. It has a compatible (perhaps forcible) many-worlds interpretation (20) attached to it, with different "times" being associated to different outcomes to the system without invoking any collapse of the wave function.

As everything in physics (and other fields of knowledge) this approach is not without its faults and criticisms, with the biggest proponent of spread disinterest on the conditional probability interpretation being Karel V. Kuchař. (10) He had three main criticisms: The first is related to the constraint in Eq.(1.2), stating that any internal clock cannot commute with the Hamiltonian, if we wish it to be a clock. Therefore, as the clock is part of the state of the Universe, the global state should not be able to truly respect the constraint. The second, questions the generalization of the construction for Klein-Gordon systems, arguing that, for a relativistic particle, the prediction given by the conditional probabilities of finding the system at a certain position when the clock has a certain time differs from the accepted probability of localization. The last inquires the apparent impossibility to obtain the transition probabilities or the correct propagators. The question was: Okay, you can compute those conditional probabilities, however as soon as the measurement is performed, the conditional probability is acquired, the clock will cease to work and the information that was obtained is only in regards to a specific moment. Therefore, it would not be possible to derive a sense of history out of this formalism. However, recently, proposals that overcome these criticisms where presented.

We start presenting the resolution of the last criticism, given in the works of Gambini *et al.* (21) and Giovanetti *et al.*. (22) Gambini *et al.* utilized the concept of "evolving constants of the motion" proposed in (23) together with some parameter, that is not necessarily time, in a way that after averaging over all possible values of the parameter, it drops out, and the correct propagators can be obtained. Possibly, more relevant for the PaW proposal, was the solution of Giovanetti *et al.*. In their work, they showed that by correctly and carefully employing the von Neumann prescription of measurements, not only the correct propagators are obtained, they are obtained without making additional claims about the original proposal. Showing to be well suited to accommodate measurements performed at different times: if one finds the system in x_1 at clock time 12 o'clock what is the probability of finding the system in x_2 at clock time 6 o'clock. Giving new breath and renewed attention to the PaW mechanism.

Attention which brought resolution for the other two criticisms. (24–25) Both solutions have as basis the identification that can be made of the PaW formalism and other two relational constructions of quantum theory. In this way the first criticism can be addressed by demonstrating that the conditional probabilities are indeed gauge-invariant, coinciding with the expected for the other formalisms, which in turn implies that the probabilities comply with the constraints. The second can be dealt with by properly choosing a POVM, which can be used to obtain a localization probability that can be extended to Klein-Gordon systems.

From the time that the original PaW proposal was conceived, many advances were made (26–46), together with the creation of completely new fields of research, as is the case of the field of quantum information theory. With that, new tools and more refined models came to be, enabling us to connect certain aspects of previously unconnected areas or obtain new results from the refinements. Here, we aim to investigate the PaW mechanism under the light of these advances. For that purpose in Chapter 2 we introduce several tools which are going to be useful to connect the constraint in Eq.(1.2) with the areas of resource theories of asymmetry. Which will allow us to interpret this constraint as an imposed symmetry to the initial state of the mechanism. These tools come in the form of a brief exposition to relevant aspects of representation theory; we introduce the concept of what is a symmetry, how it can be enforced and what is its connection to quantum reference frames, an area of relevance for the theory of asymmetry. We also introduce the framework of resource theory of coherence, which, even though appearing distant from our proposed area of study is in fact connected by means of the resource theory of asymmetry. It is also in this section that we present monotones for these resource theories which constitute the basis for some of the results in later chapters.

In Chapter 3 we take a tour through the universe of quantum timekeeping. As a working example for the original PaW proposal it was employed a simple two tick clock, with two clock positions and two system positions. From that many quantum clock models of interest were presented. We start this chapter describing the ingredients which constitute the best case for a quantum clock, defining the ideal clock. Here it will be clear that this clock is just an idealization which cannot be realized in principle. Therefore, we presented different clock models which can be realized, highlighting a construction that approximates the ideal case so well that is named quasi-ideal clock. We end the section with some relevant examples.

Chapter 4 contains our first result, which utilizes the advances presented in Chapter 3 of resource theories of coherence (47) and asymmetry. (48) In the view that the conditional probabilities can be used to recover the dynamics in the PaW mechanism, i.e, we consider then to be the main indicative of better correlation between clock time and system. Different initial states will render better or worst correlation, for the mechanism, depending on this requirement we say that the initial state will be better or worse. In a way that if the initial state exhibits poor results, measured by the conditional probabilities, here we say that the clock is bad. With this definition in mind we introduce a monotone for coherence that appears to agree with the conditional probabilities given a certain constraint on the parameter of the Bell states, the states that we used to investigate the model. This monotone is representative of a certain type of coherence, internal coherence, that we argue to be the necessary ingredient for the model, differently from the conceived notion of entanglement as previous advocated. This monotone seems to be relevant for problems with similar constraints given in the PaW mechanism, such is the case for quantum thermodynamics, within the framework of thermal operations. We show how this connection can be understood and how it connects with other results in the literature.

Chapter 5 contains our second result. Here, we further expand the timeless formulation of quantum mechanics in a two-fold way: we employ the quasi-ideal clock model as the “timekeeper” in the timeless formulation obtaining a Schrödinger-like equation for the case where clock and system interact gravitationally. In doing that, we extend a previous result where it was obtained a time non-local Schrödinger equation when the system is in a pure state, and there is an interaction. (49) Additionally, we consider the system to be in a mixed state, and we derive an equation of motion that describes gravitational decoherence. The quasi-ideal clock can approximate the ideal case depending on its dimension, which alleviates its errors. Therefore, when we have high enough dimension the clock behaves similarly with the ideal clock. For this case we have a unitary evolution with a Hamiltonian that appears to be time dilated, similar to a time dilation effect mediated by gravity which was previously obtained. Non-unitarity in the equation of motion is achieved when we do account for those errors, finding terms that resemble decoherence. We also obtained two additional terms dependent on the system state's initial conditions, making the equation to be non-linear. These two additional terms are distinct, one resembling unitary dynamics in relation to the perturbation and the other as decoherence involving the time dilated Hamiltonian.

We end with Chapter 6 where we present our conclusions about the work and present future avenues of research.

Below we are the works connected with the results presented in this thesis

- Leandro R. S. Mendes and Diogo O. Soares-Pinto. Time as a consequence of internal coherence Proc. R. Soc. A. 475, 20190470 (2019);
- Leandro R. S. Mendes, Frederico Brito and Diogo O. Soares-Pinto. Non-linear equation of motion for Page-Wootters mechanism with interaction and quasi-ideal clocks preprint arXiv:2107.11452 (2021).

2 QUANTUM FRAMES OF REFERENCE

"*No verbal explanation can indicate a direction in space or the orientation of a coordinate system*"

Asher Peres and Petra F. Scudo

This is the first sentence on a paper by Asher Peres and Petra F. Scudo (50) entitled *Unspeakable quantum information*. This sentence represents the fundamental difference between the two types of information that exist. Any information that can be verbally communicated can ultimately be described in terms of a yes or no question, or, in other words, does not depend on a specific system to be completely understood. This information can be coded by a infinitude of systems, which includes a light switched on or off, sound and silence, a voltage or no voltage and, upon a predefined agreement on the meaning of the signals, it is completely understood in the sense that nothing else is required for the comprehension of the receiver: "I am in this house", "the light is on", "Bill's mom left", "there is no food", all clear messages with meaning that can be coded as yes or no answers. For this information it was given the name of *Speakable information*. Not all information can convey messages in this manner, being completely understood just by the message, there exists *Unspeakable information*, a clear example being that of a direction in space. If two parties Bill and Ted are distant and have no knowledge about each other orientation, there is no yes or no message that could make them align themselves: Imagine trying to say that you should look to the right without knowing in relation to what this right is defined. When dealing with unspeakable information the only way to transmit it is through the use of material objects. Those objects are necessary to make clear definitions about what is said up and down or left and right. If Bill and Ted are in different galaxies they could, as an example, orient themselves by locating a star that they both could see and establish a mutual reference.

The main concept to be understood is that in the second case, where Bill tries to indicate a direction in space to Ted, there is a lack of asymmetry. When trying to describe a direction, Bill encounters the problem that the right can be in any direction whatsoever, given that all directions are equivalent, we say that directions in space are symmetric. When using a star, planet, road, etc, we are establishing a reference and with respect to that point the symmetry of the right direction is broken, it cannot be in any direction, it is defined with respect to a chosen object, a reference shared between the parties. This object used to break a symmetry is what we call a *frame of reference* and it can take several forms, as stated previously: celestial bodies, houses, gyroscopes and so on.

The use of reference frames is ubiquitous in physics starting from its very basic definitions, for example, in non-relativistic quantum mechanics, when presenting the wave function in position representation it is assumed that this position, which is referred to, is relative to some previously established reference, even if it is not explicitly mentioned. When preparing a state in the lab, setting magnetic fields or quantifying entanglement in systems of indistinguishable particles, frames of reference are always there despite our forgetfulness. Here we will be primarily interested on situations where the objects that act as our frames of reference are quantum entities, therefore we are interested in quantum reference frames.

To this purpose we start by giving an introduction to key elements of representation theory, a mathematical construction that has the structure to deal with the lack of reference frames in quantum theory, which is going to be useful in understanding our results. There are several canonical texts addressing this topic, here, we choose to follow those that approach this topic within a quantum information theory context, which includes mainly the book of Ashok Das and Susumu Okubo (51) the thesis of Iman Marvian (52), Giulio Chiribela (53), and the review of Bartlett *et.al.* (54)

2.1 Introduction to representation theory

The core notion to be introduced, is that of a group:

Definition 1 *A set is going to be called a group G if its elements $g \in G$ obey associative multiplication,*

$$(g_1g_2)g_3 = g_1(g_2g_3)$$

with the set containing an identity element e such that

$$eg = ge = g$$

and an inverse of each element

$$g^{-1}g = gg^{-1} = e.$$

The elements g of the group are parameters associated with possible transformations within the group. Usually in quantum theory we are going to represent these transformations with unitary operators $U(g)$, which have a one-to-one correspondence with the parameters that represent the action of the group. If a quantum state $|\psi(x)\rangle$ is prepared, always relative to a coordinate system, the possible transformations, such as translation and rotation, will be connected to coordinate systems, then they can be represented by the unitary operators, as is the case of spatial translations, where we could write $U(g)|\psi(x)\rangle = |\psi(x+g)\rangle$.

Akin to the presented definition, if $U(g_1)$ and $U(g_2)$ are transformations then

$$U(g_1g_2) = U(g_1)U(g_2), \tag{2.1}$$

will also be a transformation. Further, they will be associative

$$U(g_1g_2)U(g_3) = U(g_1)U(g_2g_3), \tag{2.2}$$

contain an inverse

$$U(g)U(g^{-1}) = U(g^{-1})U(g) = \mathbb{1}, \tag{2.3}$$

and $U(e) = \mathbb{1}$, indicating that the unitary representation also forms a group. Although we are going to talk specifically about the representation $U(g)$, it should be said that in general there is not a unique representation for a group, in fact, there are possibly infinitely many representations. As result of that, given an invertible matrix S, being that the only required condition, the set composed of matrices

$$W(g) = S^{-1}U(g)S, \tag{2.4}$$

obey associative multiplication

$$\begin{aligned} W(g_1)W(g_2) &= S^{-1}U(g_1)SS^{-1}U(g_2)S \\ &= S^{-1}U(g_1)U(g_2)S \\ &= S^{-1}U(g_1g_2)S \\ &= W(g_1g_2), \end{aligned} \tag{2.5}$$

have an identity element

$$W(e) = S^{-1}U(e)S = S^{-1}S = \mathbb{1} \tag{2.6}$$

and

$$\begin{aligned} W(g_1g_2)W(g_3) &= S^{-1}U(g_1g_2)SS^{-1}U(g_3)S \\ &= S^{-1}U(g_1g_2)U(g_3)S \\ &= S^{-1}U(g_1)U(g_2g_3)S \\ &= S^{-1}U(g_1)SS^{-1}U(g_2g_3)S \\ &= W(g_1)W(g_2g_3) \end{aligned} \tag{2.7}$$

therefore also being a representation of the group. The transformation in Eq.(2.4) is called a *similarity transformation*. All other representations that can be obtained through a similarity transformation are understood to be equivalent to the representation $\{U(g)\}$, with the transformation being regarded as equivalent to a change of basis.

If performing a similarity transformation, we obtain a matrix which is written as

$$W(g) = \begin{cases} \begin{bmatrix} W_1(g) & V(g) \\ 0 & W_2(g) \end{bmatrix} \\ \text{or} \\ \begin{bmatrix} W_1(g) & 0 \\ \tilde{V}(g) & W_2(g) \end{bmatrix} \end{cases}, \tag{2.8}$$

where the sub-matrices $W_1(g)$ and $W_2(g)$ are representations of the group in lower dimensional spaces and the off-diagonal matrices respect $V(g_1g_2) = W_1(g_1)V(g_2) + V(g_1)W_2(g_2)$ and $\tilde{V}(g_1g_2) = W_2(g_1)\tilde{V}(g_2) + \tilde{V}(g_1)W_1(g_2)$, it is said that the representation $\{U(g)\}$ is reducible. We observe that if the off-diagonal elements are equal to zero the matrix becomes

$$W(g) = \begin{bmatrix} W_1(g) & 0 \\ 0 & W_2(g) \end{bmatrix}, \tag{2.9}$$

or, in other words, we can decompose the representation into a block diagonal form $W(g) = W_1(g) \oplus W_2(g)$. If we cannot find a similarity transformation that achieves such decomposition, it is said that the representation is irreducible. In the case that we can decompose a representation in a block diagonal form $W(g) = W_1(g) \oplus W_2(g) \oplus W_4(g) \oplus \cdots \oplus W_m(g)$, for a given m, in which each block is irreducible then the representation is said to be fully reducible. These concepts can be understood in a more intuitive manner, if the representation $U(g)$ can be broken down into smaller representations due to a unitary transformation, or similarity transformation, it means that it is a reducible representation. If in finding these transformations we notice that a representation cannot be decomposed further, this will be called an irreducible representation or simply *irreps*.

With the concept of irreducible representation it is possible to present an important result in the theory of representation, which is, sometimes, broken as two different results, called Schur's lemma

Lemma 1 *Let $U(g)$ be an irreducible representation of the group G on the Hilbert space $\mathcal{H}$. If there is an operator A satisfying*

$$AU(g) = U(g)A, \tag{2.10}$$

for all group elements then A is a multiple of the identity.

Proof: Let

$$A|\phi\rangle = \lambda|\phi\rangle, \tag{2.11}$$

then, the statement of the lemma allow us to write

$$AU(g)|\phi\rangle = U(g)A|\phi\rangle = \lambda U(g)|\phi\rangle. \tag{2.12}$$

Therefore, $U(g)|\phi\rangle$ is also in the eigenvector space of A. As we have seen in subsection 3.1, this means that the eigenvector space of A is an invariant space, which, here, means that it is equal to the entire Hilbert space, concluding the proof.

Lemma 2 *Let $U(g)$ and $\tilde{U}(g)$ be inequivalent irreducible representations of the group G on the finite dimensional Hilbert space $\mathcal{H}$ and $\tilde{\mathcal{H}}$, respectively. If there is an operator A which satisfies*

$$AU(g) = \tilde{U}(g)A, \tag{2.13}$$

for all group elements then either A will be zero or the irreducible representations are unitarily equivalent up to a constant factor.

Proof: From the lemma, we have

$$
\begin{aligned}
AU(g) &= \tilde{U}(g)A \\
U(g^{-1})A^\dagger &= A^\dagger \tilde{U}(g^{-1}) \\
U(g)U(g^{-1})A^\dagger \tilde{U}(g) &= U(g)A^\dagger \tilde{U}(g^{-1})\tilde{U}(g) \\
A^\dagger \tilde{U}(g) &= U(g)A^\dagger.
\end{aligned} \tag{2.14}
$$

Therefore

$$U(g)A^\dagger A = A^\dagger A U(g), \tag{2.15}$$

which implies, due to the previous lemma, that

$$A^\dagger A = AA^\dagger = c\mathbb{1}. \tag{2.16}$$

This allow us to introduce an unitary $T = A/\sqrt{c}$ satisfying the constraint above, also

$$TU(g) = \tilde{U}(g)T, \tag{2.17}$$

showing that the two representations are unitarily equivalent. Obviously, the only other case would be for $c = 0$ and $A = 0$.

2.2 Symmetries

We already gave a notion of what we mean by symmetry where in the case of Bill and Ted a symmetry existed in the direction on space. In the case that we have the action of a group on a density operator

$$\mathcal{U}_g(\rho) = U(g)\rho U^\dagger(g), \tag{2.18}$$

where we express the transformation with the super-operator notation $\mathcal{U}_g$, a symmetry transformation can be understood as a transformation which leaves some aspect of the quantum state unchanged, more specifically, it should not change observable properties of the state (55), if both state and measurement apparatus are transformed. In other words, translating a state and its apparatus should not change its physics. If symmetries should not change the physics, it means that the dynamics of the system should not change, hence if the evolution of a state can be represented by a map acting on the quantum state $\mathcal{V}(\rho)$, it is required that it commutes with the symmetry transformation $\mathcal{U}_g(\rho)$,

$$\mathcal{V}(\mathcal{U}_g(\rho)) = \mathcal{U}_g(\mathcal{V}(\rho)). \tag{2.19}$$

When it is the case, i.e., the time evolution commutes with the symmetric transformations we say that it is a covariant evolution or that the time evolution is *G-covariant*. In the same manner, if it is the case that the measurement apparatus obey the same commutation with the group elements, we call it a *covariant measurement.* It should be said that in general (56) these transformations can be expressed by either unitary or anti-unitary operators, e.g., when there is time reversal symmetry, but here we focus exclusively on unitary operators.

Not all quantum states will be affected by symmetric transformations, because the symmetries of a quantum state can make it *invariant* to these transformations. When this is the case, Eq.(2.18) becomes

$$\mathcal{U}_{g_1}(\rho) = \rho, \tag{2.20}$$

where the subscript introduced with the super-operator makes explicit the invariance of the quantum state over a specific transformation. If the state is invariant under the entire group, it will be referred as a *G-invariant* state

$$\forall g \in G \;\; \mathcal{U}_g(\rho) = \rho. \tag{2.21}$$

We can extend the notion of invariance to spaces. If we have a Hilbert space $\mathcal{H}$ with a representation $U(g)$, we call a subspace of this Hilbert space $\mathcal{M} \subseteq \mathcal{H}$ invariant if $U(g)\mathcal{M} = \mathcal{M}$, for the entire group. Which means that the action of any of the elements of the group will map elements of this space to elements of itself. If we have an invariant subspace of the Hilbert space we will have an irrep in that subspace if the only choice for subspace is the entire subspace or $\{0\}$. Using these concepts we can justify the decomposition of a Hilbert space as

$$\mathcal{H} = \bigoplus_i \mathcal{H}_i, \tag{2.22}$$

where in each subspace we have the action of an irrep

$$U(g) = \bigoplus_i U_i(g). \tag{2.23}$$

Not all irreps are going to be unique, therefore, we can have several possible equivalent irreps in each subspace, being the number of possible equivalent irreps called *multiplicity,* denoted by μ. Hence, we can rewrite the decompositions above as

$$\mathcal{H} = \bigoplus_i \bigoplus_j^{\mu} \mathcal{H}_i^j \tag{2.24}$$

and

$$U(g) = \bigoplus_i \bigoplus_j^{\mu} U_i^j(g). \tag{2.25}$$

In the quantum information literature the decomposition given in Eqs.(2.24) and (2.25) are usually replaced by dividing each subspace into virtual spaces (57), then it is defined a space that carry the irreps of the group $\mathcal{M}_i$ and a space that carries the multiplicities $\mathcal{N}_i$, in which the dimension of this space equals the number of multiplicities of each subspace and the most common notation

$$\mathcal{H} = \bigoplus_i \mathcal{M}_i \otimes \mathcal{N}_i, \tag{2.26}$$

and

$$U(g) = \bigoplus_i U_i(g) \otimes \mathbb{1}_{\mathcal{N}_i}. \tag{2.27}$$

To make a formal connection to how we are going to manipulate the information from a reference frame with the group representation we must introduce the notion of twirling states and therefore the haar measure.

2.3 Haar measure

As it was said the group representation has a one-to-one correspondence with the parameters of the group, and this correspondence implies that certain characteristics of the parameter space are carried by the representation. More specifically, this means that the representation is going to inherit its topology, in this way, the group manifold will depend on the parameter space. Given that groups are a manifold, we can separate the groups into compact and non-compact. A group is going to be called compact if it has a bounded and closed topology, such is the case for the groups $U(1)$ and $SU(2)$, since the parameter space associated to them have its elements associated to a circle and a sphere, respectively, they will have a finite number of elements. If the group does not have a bounded and closed topology it is called non-compact, with an example being the group $\mathbb{R}$ which is related to the real line, and therefore has elements ranging from $-\infty$ to ∞. For compact groups it is possible to define an integral on the volume of the group manifold using a measure $\mu(dg)$ known as Haar measure, where its volume will be finite and usually normalized so that

$$\int_G \mu(dg) = 1. \tag{2.28}$$

A Haar measure can always be defined for any Lie group, such that, for a fixed element $g_0 \in G$ it satisfies left-invariance

$$\mu_L(d(g_0 g)) = \mu_L(dg) \tag{2.29}$$

and right invariance

$$\mu_R(d(g g_0)) = \mu_R(dg). \tag{2.30}$$

Usually the left-invariant and right-invariant measures will not necessarily be equal to one another $\mu_L(dg) \neq \mu_R(dg)$, but for the groups used here and for almost all important groups for applications, they will be, therefore we will abandon the measure notation $\mu(dg)$ for the more compact dg. Meanwhile, for non-compact groups we may not be able to find a suitable Haar measure, unless this group is locally compact. With this being the case, it is possible to define a volume integral, however, the volume of integration will be infinite.

We are interested primarily in continuous groups, nevertheless all results can be recast for the case of discrete groups, with the only necessary change being the substitution of the integral in Eq.(2.28) by a sum over the elements of the group

$$\frac{1}{|G|}\sum_{g\in G}. \tag{2.31}$$

The Haar measure can be used to perform an average over the group elements on a function, vector or operators. Considering $|\psi\rangle \in \mathcal{H}$ this average will have the form

$$\langle|\psi\rangle\rangle = \int_G U(g)\,|\psi\rangle\, dg, \tag{2.32}$$

with the immediate generalization for the case that we have a density operator $\rho \in \mathcal{H}$, being

$$\langle\rho\rangle = \int_G U(g)\rho U^\dagger(g)dg. \tag{2.33}$$

2.4 The twirling map

Henceforth, we are going to use a special notation when referring to the average over a group, that is known as a twirling operation $\mathcal{G}(\cdot)$, defined in the same way as Eq.(2.33)

$$\mathcal{G}(\rho) = \int_G U(g)\rho U^\dagger(g)dg. \tag{2.34}$$

We note that the resulting state will be invariant under the group action $U(g)\mathcal{G}(\rho)U^\dagger(g) = \mathcal{G}(\rho)\ \forall g \in G$.

The use of this operation has been extensive, specially in the field of quantum computation in error correction, quantum algorithms generalizing the Fourier transform, randomized benchmarking estimating the average gate fidelity and fundamental problems like extending the conditions for Noether's theorem. (58–61) However, as stated previously, we are interested in its connection to frames of reference, which can be well understood continuing with the example that Bill and Ted are trying to communicate with each other.

Ted have just discovered an awesome guitar lick and wants to share it with Bill but, unfortunately, Bill is traveling with Rufus and Ted does not know exactly where. He remembers that Rufus left a device that allows him to communicate with them through the use of quantum states and he proceeds to prepare quantum states. He decides to encode the picking pattern for the lick in the quantum states orientations preparing the states ρ, obviously, relative to a local reference to him.

Upon receiving the quantum states Rufus tells Bill that it is a message and he needs to use some operation to transform these states into states in their local frame of reference.

Knowing that the frame of reference of Bill and Ted have a relative orientation dictated by a group element $g \in G$, Bill only needs to make the transformation $U(g)^\dagger \rho U(g)$. However, Bill has no idea of which group element connects both frames of reference, therefore the states that he is receiving are states that can be associated with any of the group elements. Such a state have the form

$$\int_G U(g)\rho U^\dagger(g)dg, \tag{2.35}$$

a mixture of possible states*. This tells us that we can represent the absence of a frame of reference with the twirling operation. With this notion in mind we can formalize protocols for changing quantum reference frames and measures to quantify certain properties of quantum states.

But what exactly is the effect of this operation, can we associate it with a more common phenomenon? The answer is yes. We can describe its effect when relating the action of the twirling operation to the decomposition in Eq.(2.26). Here, we follow the derivation presented in. (54) We start with the decomposition in Eq.(2.25) of the transformations, rewriting the twirling operations

$$\begin{aligned}
\mathcal{G}(\rho) &= \bigoplus_{i,j,i',j'} \int_G U_i^j(g)\rho U_{i'}^{j'\dagger}(g)dg \\
&= \bigoplus_{i,j,i'} \int_G U_i^j(g)\rho U_{i'}^{j\dagger}(g)dg \\
&= \sum_{i,j,i'} \int_G U_i^j(g)\Pi_j\rho\Pi_j U_{i'}^{j\dagger}(g)dg \\
&= \sum_j \sum_{i,i'} \int_G \mathcal{U}^j_{i,i'g}\left[\mathcal{P}_j(\rho)\right]dg \\
&= \sum_j \mathcal{G}_j \circ \mathcal{P}_j(\rho),
\end{aligned} \tag{2.36}$$

where in the second line we expressed the operator ρ in terms of its indices as, $\rho_{i,j,i',j'} = \int_G U_i^j(g)\rho U_{i'}^{j'\dagger}(g)dg$, which by invariance of the twirling i.e. $U_i^j(g)\rho_{i,j,i',j'}U_{i'}^{j'\dagger} = \rho_{i,j,i',j'}$, guarantees that if the irreps are inequivalent $j \neq j'$, then $\rho_{i,j,i',j'} = 0$, according to Schur's lemma. In the third line we introduced the projectors onto the subspaces $\Pi_j = \sum_\lambda \Pi_{j,\lambda}$. In the last line we used $\mathcal{G}_j(\rho_j) = \sum_{i,i'} \int_G U_i^j(g)\rho_j U_{i'}^{j\dagger}(g)dg$. Given the decomposition of Eq.(2.26), each transformation $U_i^j(g) = U_j(g) \otimes \Pi_i$, allowing us to express the twirling as

$$\mathcal{G}(\rho) = \sum_j (\mathcal{G}_{\mathcal{M}_j} \otimes \mathbb{1}_{\mathcal{N}_j}) \circ \mathcal{P}_j(\rho). \tag{2.37}$$

Hence, by Schur's lemma $\mathcal{G}_{\mathcal{M}_j}$ is a multiple of the identity on the space $\mathcal{M}_j$. This shows that this operation has the general form of decoherence (62), with the difference that this decoherence is related to the lack of a frame of reference.

* Fortunately, Ted totally remembers that he can also communicate with Bill classically using the guitar as a frame of reference and manages to describe the lick.

2.5 Quantum coherence

Among the useful properties that a quantum state can possess is quantum coherence, one of the fundamental elements that distinguish between what is quantum and what is classical. A lot of the focus given to quantum coherence came in the form of resource theories, through which, it can be shown a strong connection between framework of asymmetry and coherence. Before we show how this connection is made we are going to present the most usual approaches to deal with quantum coherence.

2.5.1 The framework of resource theories

Not everything, with some saying almost all thing is the world, is free. Usually there is some cost or restriction associated with what is of desire, desire that is going to imbue a certain price, which most times, sadly, also devalues what comes cheap e.g. air is free therefore we pollute it, truffles are hard to get therefore they are expensive. Resource theories have a similar maxim, separating what is free and what is prohibited, and given an identified resource, establishing what is possible to do with it.

Then a general resource theory, defined within the constraints of quantum theory, will consist of a set of free operations, defined as a subset $\mathcal{O}_{\mathcal{F}}$ of the set of all operations $\mathcal{O}$, and a set of free states, defined as a subset $\mathcal{F}$ of all possible states $\mathcal{B}$, where every state not belonging to this subset, in $\mathcal{B}\backslash\mathcal{F}$ is considered a resource.

The canonical example of a resource theory is that of entanglement, where the set of free operations are local operations and classical communication (LOCC) and free states are unentagled states, hence, defining the set of entangled states as the resource. These resources are represented by monotonic functions. If $f(\sigma)$ is a monotone then under any transformation

$$\sigma \to \hat{\sigma} \tag{2.38}$$

necessarily

$$f(\hat{\sigma}) \leq f(\sigma). \tag{2.39}$$

A monotone established within a resource theory framework must hold the property in Eq.(2.39), given that the transformation $\sigma \to \hat{\sigma}$ is executed by a free operation $\mathcal{E} \in \mathcal{O}_{\mathcal{F}}$. There are several proposals (63) for resource theories of coherence, establishing certain rules that any coherence measure has to obey to be considered as a proper measure of coherence.

2.5.2 Resource theories for coherence

Coherence is recognized by the presence of off-diagonal elements in the density matrix of a quantum state, consequently, coherence of a diagonal density will be zero and the state will be called incoherent. Defined in this way, we see that a change of basis will also change the coherence of the state, then it is important to first choose a suitable basis, which will depend on the physics that is being investigated and fix it. with respect to this basis we can agree on which states are incoherent. Setting $\{|i\rangle\}_{i=0,\ldots,d-1}$ as the orthonormal basis of choice, where d is the dimension of the associated finite Hilbert space $\mathcal{H}$, all incoherent states living in the set of free states $\delta \in \mathcal{F}$ will have the form

$$\delta = \sum_{i=0}^{d-1} \delta_i |i\rangle \langle i| , \tag{2.40}$$

with δ_i being a probability. The extension for the multipartite scenario can be obtained by defining suitable basis for each party and making the tensor product between then $|i\rangle^1 \otimes |j\rangle^2 \otimes \ldots$ with states

$$\delta^{1,2,\ldots} = \sum_k \delta_k \alpha_k^1 \otimes \beta_k^2 \otimes \ldots, \tag{2.41}$$

where each state α_k^1, $\beta_k^2, \ldots$ is incoherent as in Eq.(2.40).

This way of constructing the incoherent states is quite universal, apart from a single framework that is going to be discussed later. However, determining the set of free operations is quite circumstantial, existing several classes of operations which are related to each other through relaxing or changing constraints.

2.5.3 Maximally incoherent operations

This is the class of *maximally incoherent operations* (MIO) (64), the largest class among the ones related by the incoherent states defined in Eq.(2.40). For this class it is only demanded that the operations do not create coherence, hence it is composed of all non-selective completely positive trace-preserving (CPTP) quantum operations† Π, where we have

$$\Pi(\mathcal{F}) \subseteq \mathcal{F}. \tag{2.42}$$

2.5.4 Incoherent operations

If we demand that the CPTP operations admit a Kraus decomposition

$$\Pi(\rho) = \sum_n K_n \rho K_n^\dagger, \tag{2.43}$$

such that $\sum_n K_n K_n^\dagger = \mathbb{1}$ and that for all elements n and free states $\delta \in \mathcal{F}$ we have

$$\frac{K_n \delta K_n^\dagger}{\mathrm{Tr}[K_n \delta K_n^\dagger]} \in \mathcal{F}, \tag{2.44}$$

we obtain the class of *incoherent operations* (IO). (63) Noticing that $p_n = \mathrm{Tr}[K_n \delta K_n^\dagger]$ is a probability the equation above is simply the post measurement state, therefore, even if access to the outcome of a measurement is granted, we are guaranteed to remain with a state that is incoherent if the input was incoherent.

† A selective quantum operation is one for which we keep record of the measurement outcomes.

2.5.5 Strictly incoherent operations

The class of incoherent operations demands incoherence only of the Kraus operators K_n i.e. that

$$K_n |i\rangle \propto |\pi_n(i)\rangle\,, \tag{2.45}$$

for an invertible function π_n that acts as a permutation over the possible values of the basis labels‡. We can add a constraint to the operators $K_n^\dagger$ (65), by noting that they can be written as

$$K_n^\dagger |i\rangle = \sum_j p_{ij}^*(n)\, |j\rangle\,, \tag{2.46}$$

then an incoherent operation $\mathcal{O}_n : \rho \to K_n \rho K_n^\dagger$ is

$$\langle i|\, \mathcal{O}_n(\rho)\, |i\rangle = \sum_{j,j'} p_{ij}(n) p_{ij'}^*(n)\, \langle j|\rho|j'\rangle\,, \tag{2.47}$$

which will have all off-diagonal elements being zero if the operators $K_n^\dagger$ also cannot produce coherence. With this additional requirement we get the class of *strictly incoherent operations* (SIO). (66)

There is another way of defining the incoherence of the Kraus operators, which is established through the dephasing operation

$$\Delta(\rho) = \sum_{i=0}^{d-1} |i\rangle\langle i|\, \rho\, |i\rangle\langle i|\,, \tag{2.48}$$

where here is going to be called *fully dephasing*§, in a way that the operation $\mathcal{O}_n$ will commute with it

$$[\mathcal{O}_n, \Delta] = 0, \tag{2.49}$$

meaning that the order in which we operate does not alter the coherence of the state.

2.5.6 Physically incoherent operations

It is well known that every CPTP map admits a Stinespring dilation, where we evolve the system in a larger Hilbert space and then trace out the unwanted degrees of freedom, mathematically expressed as

$$T(\rho) = \mathrm{Tr}_E[U(\rho \otimes \sigma_E)U^\dagger], \tag{2.50}$$

for $\rho \in \mathcal{H}_A$, $\sigma \in \mathcal{H}_E$, $U \in \mathcal{H}_{AE}$ and T as a CPTP map. If we restrict the operation to incoherent operations, so that the state that is acted upon and the unitaries are incoherent, we can say that we have a *free dilation* in our resource theory, which is not the case for

‡ Permutation works well here because incoherent operations are defined with respect to its basis, and a permutation of this reference basis will not create coherence.

§ This nomenclature is going to become clear in later chapters.

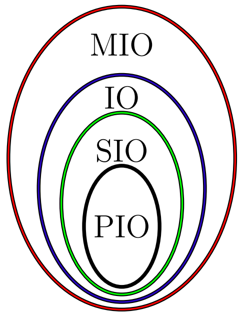

Figure 1 – Connection between the different classes of free operations for the resource theories of quantum coherence. Here are displayed: incoherent operations (IO), strictly incoherent operations (SIO), maximally incoherent operations (MIO) and physically incoherent operations (PIO).

Source: By the author.

any of the previous introduced classes. To address this problem it was introduced a new class of *physically incoherent operations* (PIO). (67–68) A quantum operation will belong to this class if its Kraus decomposition can be express as

$$K_n = U_n P_n = \sum_a e^{i\theta_a} |\pi_n(a)\rangle \langle a| P_n, \tag{2.51}$$

with an orthogonal and complete set of incoherent projectors P_n acting on system $\mathcal{H}_A$ and π_n being the same as in Eq.(2.45). The relation among the different frameworks is described in Fig.1, where they are divided from least to most inclusive, in regards to the allowed operations.

2.6 Quantifying the resources

It is not enough to characterize the possible operations for a given set of problems, it is also necessary to establish an axiomatic approach to quantify the resource in question. Although a framework to this end was introduced a while ago, together with the MIO class (64), only more recently an approach (63) aroused interest in the quantification of coherence. This approach which is similar to the quantification of entanglement established a few postulates that any measure of coherence should follow. Then, according to this approach for a candidate measure C be a valid measure of coherence it must be:

I. Non-negative:

$$C(\rho) \geq 0, \tag{2.52}$$

where we get equality if and only if the state in question is incoherent $\rho \in \mathcal{I}$, with $\mathcal{I}$ being the space of incoherent states, the space of free states in the resource theory of coherence.

IIa. Monotonic:

$$C(\rho) \geq C(\mathcal{O}_n(\rho)), \tag{2.53}$$

the measure cannot increase under an incoherent operation $\mathcal{O}_n$.

IIb. Strongly monotonic:

$$C(\rho) \geq \sum_n p_n C(\rho_n), \tag{2.54}$$

the measure cannot increase under incoherent operations, on average, even if they are selective. In the equation above $p_n = \mathrm{Tr}[K_n \rho K_n^\dagger]$ and $\rho_n = K_n \rho K_n^\dagger / p_n$.

III. Convex:

$$\sum_n p_n C(\rho_n) \geq C\left(\sum_n p_n \rho_n\right). \tag{2.55}$$

These conditions are not always considered sufficient to refer to C as a coherence measure. For those that demand more stringent criteria it is also required:

IV. Uniqueness: For any pure state ρ_ψ the measure must equal the von Neumann entropy of the diagonal pure state

$$C(\rho_\psi) = S(\Delta(\rho_\psi)). \tag{2.56}$$

V. Additivity:

$$C(\rho \otimes \sigma) = C(\rho) + C(\sigma). \tag{2.57}$$

In this manner if a quantifier only meet conditions I and either IIa or IIb it will be called a coherence monotone. These conditions are not unique and can be replaced by others. (69)

2.6.1 A distance based quantifier

Given the presented approach it is possible to propose several coherence quantifiers (47), here we are going to restrict our discussion to distance based quantifiers. These are based on distance measures between quantum states with possible measures constructed evaluating the distance of a quantum state ρ and its nearest incoherent state δ as

$$C_D(\rho) = \min_{\delta \in \mathcal{I}} D(\rho, \delta). \tag{2.58}$$

A notable quantifier is the relative entropy of coherence¶

$$C_r(\rho) = \min_{\delta \in \mathcal{I}} S(\rho || \delta), \tag{2.59}$$

where $S(\rho||\delta) = \mathrm{Tr}[\rho \log \rho] - \mathrm{Tr}[\rho \log \sigma]$ is the quantum relative entropy. By definition it satisfies I, it is jointly convex and non-increasing under CPTP maps fulfilling IIa and III,

¶ We remark here that the relative entropy is not a distance in its strict sense.

this for all presented classes. It can be shown that the stronger requirement IIb is also satisfied if we restrict the operations to the IO class (63), demonstrated by first choosing an incoherent $\delta' \in \mathcal{I}$ which achieves the minimum for the relative entropy of coherence and assuming operations as in Eq.(2.44)

$$\begin{aligned} C_r(\rho) &= S(\rho||\delta') \\ &\geq \sum_n S\left(K_n\rho K_n^\dagger||K_n\delta' K_n^\dagger\right) \\ &\geq \sum_n p_n S\left(K_n\rho K_n^\dagger/\mathrm{Tr}[K_n\rho K_n^\dagger]||K_n\delta' K_n^\dagger/\mathrm{Tr}[K_n\delta' K_n^\dagger]\right) \\ &= \sum_n p_n S(\rho_n||\delta) \\ &\geq \sum_n p_n \min_{\delta\in\mathcal{I}} S(\rho_n||\delta) \\ &= \sum_n p_n C_r(\rho_n), \end{aligned} \tag{2.60}$$

where the second line was obtained by monotonicity of the quantum relative entropy and the third from the fact that we can write the quantum relative entropy as $\sum_n S(\rho_n||\sigma_n) = \sum_n p_n S(\rho_n/p_n||\sigma_n/q_n) + \sum_n p_n \ln p_n/q_n$. It also has a very simple closed form, using the operation in Eq.(2.48) we can expand the relative entropy

$$\begin{aligned} C_r(\rho) &= \min_{\delta\in\mathcal{I}} S(\rho||\delta) \\ &= \min_{\delta\in\mathcal{I}}[S(\Delta(\rho)) - S(\rho) + S(\Delta(\rho)||\delta)] \\ &= S(\Delta(\rho)) - S(\rho), \end{aligned} \tag{2.61}$$

where for the last line we notice that the closest incoherent state to ρ is $\Delta(\rho)$. Looking to Eq.(2.61) we also see that the relative entropy of coherence obeys conditions IV and V, making it a measure of coherence in regards to anyone.

2.7 Connections between asymmetry and coherence

We saw that asymmetry can be characterized in relation to the action of group elements in a way that a state will contain asymmetry if it is not invariant under the elements of the chosen group. It turns out that we can connect the asymmetry of a quantum state to quantum coherence of that state by restricting the group to the group of translations. More specifically, the group of phase shifts and time translations which will be usually taken to be those which are isomorphic to the $U(1)$ group, i.e., which are compact. Then, asymmetry in relation to the group of translations is interpreted as coherence relative to the eigenbasis of an observable. Here we will consider that this observable is given by a Hamiltonian that will be associated to a group generating a representation

$$U = e^{-iHt}. \tag{2.62}$$

Hence, the set of incoherent states, the free states, is defined as those that are symmetric in relation to the group action

$$\mathcal{U}(\rho) = \rho, \tag{2.63}$$

for all group elements. This happens to be a valid condition for those states which are diagonal in the Hamiltonian eigenbasis.

2.7.1 Translationally-covariant operations

There is also a designation for this framework based on asymmetry, which is called *translationally-covariant operations* (TIO)‖. As the name suggests the set of free operations $\mathcal{O}$ will be characterized by all operations which are covariant in regards to the action of any element of the group

$$\mathcal{U} \circ \mathcal{O} = \mathcal{O} \circ \mathcal{U}. \tag{2.64}$$

From this equation we can also note that these operations will take incoherent states of the framework into incoherent states, which is done by observing that the state $\mathcal{O}(\rho)$ is indeed invariant

$$\mathcal{U}(\mathcal{O}(\rho)) = \mathcal{O}(\mathcal{U}(\rho)) = \mathcal{O}(\rho), \tag{2.65}$$

for all group elements.

Another important property of this set is that it admits a free dilation. Let σ_E be an incoherent ancilla state to which the system is going to be coupled, $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$ the composite Hilbert space, $\mathcal{V}$ a covariant unitary quantum operation and E an invariant measurement effect. Before proceeding we note that an operation can map a state which belongs to an input space to a different output space. Here, for simplicity, we consider that the input and output spaces are equal with the proof for different spaces being completely analogous. (70) Then,

$$\begin{aligned}
\mathcal{O}(\mathcal{U}_S(\rho)) &= \mathrm{Tr}_e \left[E\mathcal{V}\left(\mathcal{U}_S(\rho) \otimes \sigma\right)\right] \\
&= \mathrm{Tr}_e \left[E\mathcal{V}\left(\mathcal{U}_S(\rho) \otimes \mathcal{U}_e(\sigma)\right)\right] \\
&= \mathrm{Tr}_e \left[E\mathcal{U}_S \otimes \mathcal{U}_e\left(\mathcal{V}\left(\rho \otimes \sigma\right)\right)\right] \\
&= \mathcal{U}_S\left(\mathrm{Tr}_e \left[\mathcal{U}_e^{\dagger}(E)\left(\mathcal{V}\left(\rho \otimes \sigma\right)\right)\right]\right) \\
&= \mathcal{U}_S\left(\mathrm{Tr}_e \left[E\mathcal{V}\left(\rho \otimes \sigma\right)\right]\right) \\
&= \mathcal{U}_S\left(\mathcal{O}(\rho)\right),
\end{aligned} \tag{2.66}$$

‖ Here we choose to describe this set of operations with the acronym TIO, because sometimes it can be presented as translationally-invariant operations, and may be more known that way. However we described as translationally-covariant operations in the main body since operations are covariant and not invariant, as it is with states or effects.

where it was used the invariance of the operations during this derivation and in the fourth line the cyclic property of the trace.

A contrast is noticed between the way in which a state with coherence is defined. In the previously established framework a state defined as in Eq.(2.40), with the set $\{|i\rangle\}_{i=0,\ldots,d-1}$, can be seen as having labels that can be changed without changing the coherence content of the state, with an example operation being the permutation as in Eq.(2.45). When defining an observable and with respect to which we define coherence, each element of the set $\{|i\rangle\}_{i=0,\ldots,d-1}$ becomes eigenstates of said observable and the same permutation cannot be considered a free operation anymore, since it must cost something to make this change. In fact these states which are composed by the eigenstates of the observable can be said to contain different modes of asymmetry in relation to each other.

We can explain this better with the particular example of $U(1)$ group (52), which has the representation $e^{i\theta}$, of unit vectors on the complex plane. With this representation it is possible to decompose the representations with an orthonormal basis $\{|n,\mu\rangle\langle n,\mu|\}$ into an irrep

$$U(\theta) = \sum_{n,\mu} e^{in\theta} |n,\mu\rangle\langle n,\mu| \, , \tag{2.67}$$

where $n \in Z$ and μ is the multiplicity index. Given the irrep the space of linear operators is spanned by $\{|n,\mu\rangle\langle m,\nu|\}$ which has a subspace spanned by $\{|n+k,\mu\rangle\langle n,\nu|\}$ with operators

$$A^k = \sum_{n,\mu,\nu} \mathrm{Tr}(A\,|n+k,\mu\rangle\langle n,\nu|)\,|n+k,\mu\rangle\langle n,\nu| \, . \tag{2.68}$$

The action of the irreps of the group on the operators A^k is

$$\begin{aligned}
U(\theta)A^kU^\dagger(\theta) &= \sum_{n,n',n'',\mu,\nu} \mathrm{Tr}(A\,|n''+k,\mu\rangle\langle n'',\nu|)e^{in\theta}\,|n,\mu\rangle\langle n,\mu|n''+k,\mu\rangle\langle n'',\nu|\,e^{-in'\theta}\,|n',\mu\rangle\langle n',\mu| \\
&= \sum_{n,n',n'',\mu,\nu} \mathrm{Tr}(A\,|n''+k,\mu\rangle\langle n'',\nu|)e^{i(n-n')\theta}\,|n,\mu\rangle\langle n,\mu|n''+k,\mu\rangle\langle n'',\nu|n',\mu\rangle\langle n',\mu| \\
&= \sum_{n',\mu,\nu} \mathrm{Tr}(A\,|n'+k,\mu\rangle\langle n',\nu|)e^{i(n'+k-n')\theta}\,|n'+k,\mu\rangle\langle n',\nu| \\
&= \sum_{n',\mu,\nu} \mathrm{Tr}(A\,|n'+k,\mu\rangle\langle n',\nu|)e^{ik\theta}\,|n'+k,\mu\rangle\langle n',\nu| \\
&= e^{ik\theta}A^k.
\end{aligned} \tag{2.69}$$

Then, each operator in this subspace is called a mode k operator with each k interpreted as a different mode of asymmetry, with any operator being decomposable in terms of its modes as $A = \sum_k A^k$.

Let $\mathcal{E}$ be a $U(1)$-covariant operation, then

$$\begin{aligned}
U(\theta)\mathcal{E}(A^k)U^\dagger(\theta) &= \mathcal{E}(U(\theta)A^kU^\dagger(\theta)) \\
&= \mathcal{E}(e^{ik\theta}A^k) \\
&= e^{ik\theta}\mathcal{E}(A^k)
\end{aligned} \tag{2.70}$$

therefore, just like a Fourier transform of a time invariant system cannot change the frequency of a signal after being applied, a $U(1)$-covariant operation is going to map one operator in some mode k to another operator on the same mode k. Hence, if we want to transform a state ρ into a state σ we must use some resource which contain asymmetry in the same modes.

This motivates a distinction between the frameworks of coherence based on the type of information needed. If a certain task is one that requires asymmetry relative to a group generated by an observable, based on TIO, only operations which are covariant with respect to the chosen group will be allowed, which by the example above, will prohibit the transformation of states in states with different modes. Since this type of task depends on the use unspeakable information we say that the framework will be of unspeakable coherence. Otherwise, if an operation allows the transformation described, it will require the use of speakable information and the framework will be of speakable coherence. (70)

2.7.2 The Holevo asymmetry

Following the given definition of coherence, we can search on asymmetry quantifiers to be interpreted as coherence quantifiers. One important quantifier, first presented in (71) as just *the asymmetry*, can be considered as the natural measure[**] of asymmetry with respect to a group G

$$A_G(\rho) = S(\mathcal{G}(\rho)) - S(\rho), \tag{2.71}$$

where $S(\cdot)$ is the von Neumann entropy. This measure was studied in the context of measuring the quality and resource theory of reference frames (72–74) and it is sometimes called the Holevo asymmetry due to its connection with the Holevo χ quantity[††] (75)

$$\chi = S(\rho) - \sum_i p_i S(\rho_i). \tag{2.72}$$

Considering continuous groups we can write this quantity as

$$\begin{aligned} \chi_G &= S(\mathcal{G}(\rho)) - \int_G S(\mathcal{U}(\rho))dg \\ &= S(\mathcal{G}(\rho)) - \int_G S(\rho)dg \\ &= S(\mathcal{G}(\rho)) - S(\rho) = A_G(\rho), \end{aligned} \tag{2.73}$$

where in the second line we used the invariance under unitary of operations of the von Neumann entropy.

Interestingly the Holevo asymmetry also coincides with a studied measure of asymmetry, the relative entropy of asymmetry

$$A_r(\rho) = \min_{\sigma \in G\text{-inv}} S(\rho||\sigma), \tag{2.74}$$

** Here the word measure is used more loosely.

†† A useful quantity that upper bounds the maximum information between two random variables.

where G-inv denotes the set of states which are group invariant. Specifically, the Holevo asymmetry can be seen as a closed form of the relative entropy of asymmetry, in a similar way that we did in subsection 2.6.1,

$$\begin{aligned}\min_{\sigma\in G\text{-inv}} S(\rho)||\sigma) &= \min_{\sigma\in G\text{-inv}} \{\mathrm{Tr}[\rho\log\rho] - \mathrm{Tr}[\rho\log\sigma]\} \\ &= \mathrm{Tr}[\rho\log\rho] - \max_{\sigma\in G\text{-inv}} \mathrm{Tr}[\rho\log\sigma] \\ &= \mathrm{Tr}[\rho\log\rho] - \max_{\sigma\in G\text{-inv}} \mathrm{Tr}[\rho\log\mathcal{G}(\sigma)] \\ &= \mathrm{Tr}[\rho\log\rho] - \mathrm{Tr}[\mathcal{G}(\rho)\log\mathcal{G}(\rho)] \\ &= S(\mathcal{G}(\rho)) - S(\rho) = S(\rho||\mathcal{G}(\rho)),\end{aligned} \tag{2.75}$$

where in the fourth line we used $\max_{\sigma_2}\mathrm{Tr}(\sigma_1\log\sigma_2) = \mathrm{Tr}(\sigma_1\log\sigma_1)$, an equality that comes from Klein inequality $S(\sigma_1||\sigma_2)\geq 0$ and in the last line we used the G-invariance of $\mathcal{G}(\rho)$. We observe that the Holevo asymmetry, also by definition, obeys condition I of subsection 2.6, and can be shown to obey condition IIa, making it a true monotone[‡‡].

We should note here that the distinguishing factor between the measures that we presented, which are based on the relative entropy, is the set of incoherent states, therefore, if there is any condition in which the sets coincide the measures must also coincide. This is the case when the chosen observable is represented by a non-degenerate matrix, then the set of G-inv will be precisely the set of incoherent states $\mathcal{I}$, and the relative entropy of coherence will be equal to the Holevo asymmetry. A similar occurrence happens to a measure introduced as *relative entropy of superposition* (64), defined in relation to a channel $\Pi(\cdot)$ that maps the states into orthogonal subspaces using the projectors P_k, as

$$\Pi(\rho) = \sum_k P_k\rho P_k, \tag{2.76}$$

effectively removing off-diagonal blocks. The associated measure is given by

$$A_S(\rho) = S(\Pi(\rho)) - S(\rho), \tag{2.77}$$

which will coincide with the Holevo asymmetry in the case that the group of translations is isomorphic to the $U(1)$ group. (72) Finally, the relative entropy of superposition will be equal to the relative entropy of coherence in the case that all projectors P_k have rank one. (47)

‡‡ We do not show the proof here because it is completely analogous the one presented for the relative entropy of coherence, but using G-invariant operations, i.e., we start we $\rho_i = \mathcal{O}_i(\rho)/\mathrm{Tr}[\mathcal{O}_i(\rho)]$.

3 QUANTUM CLOCKS

" *Time is a created thing. To say "I don't have time" is to say "I don't want to".*"

Lao tzu

There is a particular example of reference frame that we did not address, one that gives time reference, a clock. If we turn back to our example of preparing a quantum state in a laboratory, any kind of timed operation performed on these states or in which these states are used, are not executed with respect to an ethereal Time, an absolute time that exists and passes irrespective of the rest, having distinct values which will mark the presence of events, such as 5 or 36, but with respect to the time that is shown on a clock.

A clock will be considered as a system which undergoes a set of states in approximate constant intervals. If we think about a clock, the object that comes to our mind will be of a wall clock or a watch, a device which outputs time information without the need of our interaction. This motivates a distinction in the nomenclature that is commonly used when talking about clocks. Frequently, any system that is used to measure time is considered as a clock, even if after a single use it must be reset, or in other words, it can only be used to measure a time interval. Then what is usually defined as a clock in the literature is more similar to what we would consider a stopwatch, just registering the time difference between events, e.g., the sunrise and the sunset. And as stated a clock should be considered as a system that gives time information in a autonomous fashion, which can be achieved by coupling the said clock to another system working as register, which maintain the information given by the clock and a certain engine which fuels this process. With this in mind here we will adopt the nomenclature of *ticking clock* (76), for clock constructions which work in an autonomous fashion, and we will refer to non-autonomous systems as clocks, in order to maintain consistency with the literature. Noting that most clocks models can be recast as ticking clocks given appropriate changes.

Every clock is going to be associated with a dynamical variable, in a way that measuring time equals getting information about that variable, which has a uniform motion* with respect to time, which means that this variable will covary with what we expect of the value of time. Therefore, we expect a clock variable $\gamma(t)$ to be given by

$$\gamma(t) = \gamma(0) + t, \tag{3.1}$$

so that any difference between two distinct times t_2 and t_1 will be equal to the difference between $\gamma(t_2)$ and $\gamma(t_1)$. (3) If we have a time observable $\mathcal{T}(t)$ associated to this clock we

* At least in most cases.

also demand covariance in relation to *time translations* in the same way defined in the previous section. If we have unitaries U which are associated with a Hamiltonian H, then

$$\mathcal{T}(t) = U(t)\mathcal{T}U^{\dagger}(t) = \mathcal{T} + t. \tag{3.2}$$

From the covariance we get

$$\begin{aligned} U(t)\mathcal{T} &= \mathcal{T}U(t) + tU(t) \\ [U(t), \mathcal{T}] &= tU(t) \end{aligned} \tag{3.3}$$

implying the canonical commutation relation

$$[H, \mathcal{T}] = -i. \tag{3.4}$$

This can be seen in the Stone-von Neumann theorem[†,‡]. (3, 77–78)

Theorem 1 *If we have a self adjoint operator Q on a Hilbert space $\mathcal{H}$, then there is a one-parameter (strongly) continuous unitary group $U(g)$, such that*

$$\lim_{g \to 0} \frac{(U(g) - \mathbb{1})\phi}{ig} = Q\phi, \tag{3.5}$$

where $\phi \in \mathcal{H}$ if the limit exists.

Therefore,

$$\begin{aligned} [H, \mathcal{T}] &= H\mathcal{T} - \mathcal{T}H \\ &= \lim_{t \to 0} \frac{(U(t) - \mathbb{1})\mathcal{T}}{it} - \lim_{t \to 0} \frac{\mathcal{T}(U(t) - \mathbb{1})}{it} \\ &= \lim_{t \to 0} \left(U(t)\mathcal{T} - \mathcal{T}U(t) - \mathcal{T} + \mathcal{T}\right) \frac{1}{it} \\ &= \lim_{t \to 0} \left(tU(t)\right) \frac{1}{it} = -i. \end{aligned} \tag{3.6}$$

It turns out that this is a strong condition to require for quantum clocks with not many of them being able to satisfy it.

† The theorem that we show was first presented by Marshall Stone, however, without the presence of a formal proof, which was latter accomplished by von Neumann. It is good to point out that the main objective of von Neumann was intrinsically connected to the canonical commutation relation.

‡ We omit the proof of the theorem because it is necessary to introduce a few concepts which are not relevant for the remainder of the thesis. For the interested reader the proof can be found in (77)

3.1 Ideal clock

What is the best clock? Some authors consider as a defining characteristic to obey Eq.(3.4), or in other words, to have a time observable which covaries (41) while others seem to focus in a distinguishable basis and continuity (79), or the ability to never go backwards in time. (80) The amalgamation of properties allows us to give the following definition

Definition 2 *An ideal clock is a system, with Hamiltonian H, that goes through a succession of eigenstates $|t_1\rangle, |t_2\rangle, |t_3\rangle, \ldots$ of an associated time observable $\mathcal{T}$ such that:*

I) It has a distinguishable basis of states $\langle t_i|t_j\rangle = \delta(t_i - t_j)$;

II) Each eigenstate has the property that $t_1 < t_2 < t_3 < \ldots$;

III) For $\tilde{t} > 0$ there is a nonzero probability that the clock will move forward in time: For each i there is an j such that $j > i$ and $|\langle t_j|e^{-iH\tilde{t}}|t_i\rangle \neq 0$;

IV) For $\tilde{t} > 0$ there is a zero probability of the clock going backwards in time: For each i there is an j such that $j > i$ and $|\langle t_i|e^{-iH\tilde{t}}|t_j\rangle| = 0$;

V) The Hamiltonian and time observable obey the canonical commutation relation

$$[H, \mathcal{T}] = -i. \tag{3.7}$$

Then, if we want the best clock we should search for candidate observables that can satisfy these requirements. A famous pair of observables, which can be used to construct the clock, are given by the position X and momentum P operators of a free particle in one dimension. Associating the momentum operator with the Hamiltonian of the system and the position operator as the time observable we get what can be called an *ideal momentum clock*. We can verify that this clock complies with the requirements above: the position operator has a distinguishable basis $\langle x|x'\rangle = \delta(x - x')$, which implies in having a vanishing probability of going backwards in time (3), but still has a non-vanishing probability of going forward in time, the position in the real line can satisfy *II*) and it is well known that the pair canonically commute

$$[P, X] = -i. \tag{3.8}$$

However, there is a problem with its construction, which can be stated in the form of an argument[§] laid down by Wolfgang Pauli. (81) He reasoned that it is not possible to construct a clock associated with a time operator satisfying the canonical commutation

§ Strictly speaking it only appeared as a footnote.

with a Hamiltonian since this would imply an unphysical requirement on the spectrum of the Hamiltonian. The argument goes like this: If we are going to represent time with an observable $\mathcal{T}$[¶] its spectrum will be in $\mathbb{R}$, going from $-\infty$ to $+\infty$, hence, making this observable unbounded. It follows, by defining a unitary operator as

$$U = e^{-i\lambda\mathcal{T}} = \sum_{j=0}^{\infty} \frac{(-i\lambda)^j \mathcal{T}^j}{j!} \tag{3.9}$$

with $\lambda \in \mathbb{R}$. And using

$$[H, \mathcal{T}^j] = ij\mathcal{T}^j, \ \forall j \in \mathbb{N}^*,$$

to obtain $[H, U] = \lambda U$ that

$$\begin{aligned} UHU^\dagger &= (HU - [H, U])U^\dagger \\ &= (HU - \lambda U)U^\dagger \\ &= H - \lambda \mathbb{1}. \end{aligned} \tag{3.10}$$

This shows that the spectrum of the clock Hamiltonian will also be unbounded, since for an energy eigenstate $|E\rangle$ with eigenvalue E we have

$$\begin{aligned} HU|E\rangle &= (UH + [H, U])|E\rangle \\ &= (E + \lambda\mathbb{1})\, U|E\rangle \end{aligned} \tag{3.11}$$

which, according to Pauli, is unphysical because all physical Hamiltonians must be bounded from below. Even though the momentum clock does satisfy the canonical relation it suffers from the same problem, prohibiting its construction. The thing is that there are examples of pairs of operators which do satisfy the canonical relation in Eq.(3.4) with Hamiltonians for which its spectrum is not unbounded. (3)

Therefore, the way to ideal clocks seems to be in restricting our attention to systems with bounded Hamiltonians, which here means bounding them from below, i.e., guaranteeing a ground state. Unruh and Wald (80) considered this case realizing that, even if we restrict our search in this way, the ideal clock properties cannot be simultaneously satisfied. They argued that given a bounded Hamiltonian H the function in *III*), $|\langle t_j|e^{-iH\tilde{t}}|t_i\rangle|$, is analytic in the lower half plane, which means that it is completely defined in the specified complex plane and can be completely defined from any point in that plane i.e. it cannot be equal to zero unless it is equal to zero everywhere. Therefore, requirement *IV*) is in contradiction with requirement *III*), since

$$|\langle t_j|e^{-iH\tilde{t}}|t_i\rangle| = |\langle t_i|e^{iH\tilde{t}}|t_j\rangle^*| = 0, \tag{3.12}$$

¶ We use here a general notation for a time observable and Hamiltonian because this argument does not applies only for the position and momentum operators.

the left hand side can not be nonzero while the middle function is zero, or, in other words, we cannot have a clock that move forward in time without it having a probability of going backwards in time.

Hence, it seems, that ideal clocks are simply an idealization for which the construction is barred by its definition. That is not to say that it is an useless concept, one may use it to draw intuition about the physics of a problem and later employ more realistic models which may even bring new concepts for these very same problems.

3.2 Salecker-Wigner-Peres clock

The Salecker-Wigner-Peres (SWP) clock model is a finite dimensional clock based on the periodic system of a harmonic oscillator. It was first proposed by Salecker and Wigner (82), which argued that the measurement of space-time distances should be performed using only quantum clocks and not classical objects as measuring rods, with their interest lying on the impact of the clock's mass for this task. Latter on, Peres (83) improved its construction demonstrating some of its properties and proposing a time operator associated with the clock model. His interest was different of Salecker and Wigner, where he utilized the clock to actively measure an event, such as the time of flight of a quantum particle, concerned about the impact that this coupling would had on the system.

The clock construction start with a Hamiltonian with evenly spaced energy levels

$$H_C = \sum_{j=0}^{d-1} \omega j \left|E_j\right\rangle \left\langle E_j\right| , \tag{3.13}$$

where d is the dimension of the Hilbert space. The SWP clock states are constructed in a way that is connected to the energy eigenstates through a discrete Fourier transform

$$|\theta_k\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{-i2\pi jk/d} \left|E_j\right\rangle , \tag{3.14}$$

it is easy to observe that they form an orthonormal set of states $\{|\theta_k\rangle\}$

$$\begin{aligned} \langle\theta_k|\theta_{k'}\rangle &= \frac{1}{d} \sum_{j,j'=0}^{d-1} e^{i2\pi jk/d} e^{-i2\pi j'k'/d} \langle E_j|E_{j'}\rangle \\ &= \frac{1}{d} \sum_{j=0}^{d-1} e^{-i2\pi j(k'-k)/d} \\ &= \delta_{k,k'} \end{aligned} \tag{3.15}$$

for $k = 0, 1, \ldots, d-1$, which is dubbed the time basis. As mentioned, this is a good sign for the SWP clock, since it means that it will have a vanishing probability of going backwards in time while running forward. They evolve according to

$$
\begin{aligned}
e^{-iH_CT/d}|\theta_k\rangle &= e^{-iH_CT/d}\frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}e^{-i2\pi jk/d}|E_j\rangle \\
&= \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}e^{-i2\pi jk/d}e^{-i\omega jT/d}|E_j\rangle \\
&= \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}e^{-i2\pi j(k+1)/d}|E_j\rangle \\
&= |\theta_{k+1}\rangle_{\text{mod } d}\,,
\end{aligned}
\tag{3.16}
$$

where we used that the time period of the clock is related to its frequency as $T = 2\pi/\omega$ and that

$$\hbar = 1$$

, which is going to be the case for the rest of this thesis. Its evolution is given through regular, integer, time intervals $t_k = (T/d)k$, associated to the time operator

$$\mathcal{T}_C = \sum_k t_k |\theta_k\rangle\langle\theta_k|\,. \tag{3.17}$$

We should also note that after a time T

$$
\begin{aligned}
e^{-iH_CT}|\theta_k\rangle &= \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}e^{-i2\pi jk/d}e^{-i2\pi j}|E_j\rangle \\
&= \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}e^{-i2\pi j(k+d)/d}|E_j\rangle = |\theta_{k+d}\rangle_{\text{mod } d} = |\theta_k\rangle_{\text{mod } d}\,,
\end{aligned}
\tag{3.18}
$$

the states will repeat themselves.

Unfortunately this clock model is one of the examples of clock which fail to adhere to condition V), of canonical commutation. It was first noted by Peres (83), that the Hamiltonian and time observable defined in Eq.(3.13) and Eq.(3.17) give

$$
\begin{aligned}
\langle\theta_k|[H_C,\mathcal{T}_C]|\theta_k\rangle &= \langle\theta_k|H_C\mathcal{T}_C|\theta_k\rangle - \langle\theta_k|\mathcal{T}_CH_C|\theta_k\rangle \\
&= \langle\theta_k|H_C|\theta_k\rangle t_k - t_k\langle\theta_k|H_C|\theta_k\rangle \\
&= 0.
\end{aligned}
\tag{3.19}
$$

3.2.1 Pegg-Barnett phase states and the Hilgevoord construction

We should note here that there are at least two different constructions that are very similar, being one completely analogous, to the construction of the SWP clock. The first one pertain to Pegg and Barnett (84–86) where they studied and proposed a phase operator. They considered a $s+1$ dimensional Hilbert space spanned by energy eigenstates that, in this section only, will be denoted as $\{|n\rangle\}_{0,\ldots,s}$ following the original presentation, and phase states

$$|\theta_m\rangle = \frac{1}{\sqrt{s+1}}\sum_{n=0}^{s}e^{in\theta_m}|n\rangle\,, \tag{3.20}$$

with $\theta_m = \theta_0 + 2\pi m/(s+1)$ for $m = 0, 1, ..., s$, implying the phase operator

$$\begin{aligned}\hat{\phi} &= \sum_{m=0}^{s} \theta_m \left|\theta_m\right\rangle \left\langle\theta_m\right| \\ &= \theta_0 + \sum_{m=0}^{s} \frac{2\pi m}{s+1} \left|\theta_m\right\rangle \left\langle\theta_m\right| .\end{aligned} \tag{3.21}$$

Obviously, given the previous section we can observe that these phase states will share the same properties of the SWP clock states.

The second was described by Hilgevoord (87–88), where he tries to establish an ideal periodic clock with an unbounded spectra. His construction can be seen as a dilation of the SWP clock, where instead the finite sum of Eq.(3.20), for intended purposes, it is used an infinite sum. Although Hilgevoord regards this to be a construction of an ideal clock, there are serious doubts about the physicality of this clock, where it is said in (3):

> “ *I introduced my doubts that such an ideal clock could be regarded as an idealization at all, and it is difficult to see just what kind of physical system is being described here: what target system could be usefully described in there terms? The lack of a Schrödinger representation is particularly distressing given that it also makes the classical limit difficult to describe. It is all very well saying that the corresponding classical system is one with action-angle variables, but that is akin to saying that a given curve in three dimensional space can be described in spherical co-ordinates: no information about the Hamiltonian function is thus conferred. Worse than this, the system is not one with a description in terms of action-angle co-ordinates, but rather a system whose Hamiltonian is just the action. Given its manifest unphysicality, I would rather maintain that this so-called ideal quantum clock results more from an unfortunate mutilation of the phase observable of the quantum harmonic oscillator (i.e. its Naimark dilation), rather than vice versa*”

3.3 Quasi-ideal clock

Periodic clocks like the SWP have some problems if we expect to measure time as a continuous variable, SWP clock states are discontinuous being well defined only for integer time steps. Also as shown before, they cannot obey the canonical commutation relation. Given that, can we classify the SWP clock as being a good clock? Even though ideal clocks seem to be an impossibility, certainly we could hope to build clock models which can work closer to what we expect to be ideal.

It turns out that with some changes on the SWP clock we can fix some of these problems. This is accomplished by considering a coherent superposition of SWP states and with this creating the Quasi-ideal clock states (79):

Definition 3 *The Quasi-ideal clock (QIC) states are considered a Gaussian superposition of the form*

$$|\psi_{QI}(k_0)\rangle = \sum_{k\in S_d(k_0)} A e^{-\pi(k-k_0)^2/\sigma^2} e^{i2\pi j_0(k-k_0)/d} |\theta_k\rangle \,, \tag{3.22}$$

where σ is identified with the variance of QIC states, A is a normalization constant, $S_d(k_0)$ is a set of d consecutive integers centered about k_0, the initial position of the clock, given by

$$S_d(k_0) = \left\{ k : k \in \mathbb{Z} \text{ and } -\frac{d}{2} < k_0 - k < \frac{d}{2} \right\} \tag{3.23}$$

and having ωj_0 as its average energy.

It should be said that the range within the set of integers given by $S_d(k_0)$ is chosen in a way that the QIC state is centered in the SWP basis. Hence, the dimension of the clock, which indicates the number of consecutive integers in $S_d(k_0)$, will impact the changes on the initial position of the clock. If d is even it will change in integer k_0 and if d is odd it will change at half integer k_0. This difference will not alter the results in a significant way, therefore, we will assume even dimension for the clock unless stated otherwise.

Depending on the relation between the dimension of the clock and its variance, the Quasi-ideal clock states can either approximate an energy eigenstate or the SWP clock, motivating the following

Definition 4 *Given the clock dimension d and the variance $\sigma \in (0, d)$ we call the Quasi-ideal clock states as*

i) symmetric if $\sigma = \sqrt{d}$

ii) time squeezed if $\sigma < \sqrt{d}$

iii) energy squeezed if $\sigma > \sqrt{d}$

being called 'completely' if the Quasi-ideal clock has mean energy centered in the middle of the spectrum.

The QIC gets its name from its ability of approximating certain properties of the ideal case, such as continuity and a quasi-canonical commutation, which will be presented in the next couple of sections. Because of the relevance of this clock model to our results and the non-intuitive nature of the deductions about continuity we are going to explore it in detail in the next subsection, following closely Ref. (79).

3.3.1 Continuity

If x denotes a certain distance in space and t a certain interval in time, continuity is expressed as an equivalence between space and time translations for arbitrary values of x and t, which for a state $|\psi\rangle$ can be expressed as

$$\langle x|e^{-iHt}|\psi\rangle = \langle x - t|\psi\rangle \,. \tag{3.24}$$

Before proceeding we introduce a definition which will help to follow the several lemmas

Definition 5 *The parameter $\alpha_0 \in (0,1]$ is a measure of how close the parameter j_0 is to the edge of the energy spectrum, given by*

$$\alpha_0 = 1 - \left|1 - j_0\left(\frac{2}{d-1}\right)\right| \in (0,1]. \tag{3.25}$$

The QIC states depend on the parameter k_0, used both to find the range of values in $S_d(k_0)$ and to set the mean position for the clock. Therefore, to find a continuity relation for it, it is needed to consider both aspects, asking: what happens when we change the mean position of the clock and what happens when we change the set of integers which define the clock

Lemma 3 *Let the analytic Gaussian function be given by*

$$\psi(k_0;k) = Ae^{-\pi(k-k_0)^2/\sigma^2} e^{i2\pi j_0(k-k_0)/d}, \tag{3.26}$$

then, the action of the clock Hamiltonian on a QIC state for infinitesimal time of order δ may be approximated by an infinitesimal translation of the same order on the analytic Gaussian function as

$$e^{i\delta H_c(T/d)}\left|\psi_{QI}(k_0)\right\rangle = \sum_{k\in S_d(k_0)} \psi(k_0+\delta;k)\left|\theta_k\right\rangle + \left|\varepsilon\right\rangle, \tag{3.27}$$

where the norm of the error $|\varepsilon\rangle$ is bounded by $\|\,|\varepsilon\rangle\,\|_2 \leq \delta\varepsilon_{to} + \delta^2 C$ with

$$\frac{\varepsilon_{to}}{2\pi A d} < \begin{cases} \left(2\sqrt{d}\left(\frac{1}{2}+\frac{1}{2\pi d}+\frac{1}{1-e^{\pi}}\right)e^{-\frac{\pi d}{4}} + \frac{1}{2}+\frac{1}{2\pi d}+\frac{2}{1-e^{\pi}}\right)e^{-\frac{\pi d}{4}} & \textit{if } \sigma = \sqrt{d} \\ \left(2\sigma\left(\frac{\alpha_0}{2}+\frac{1}{2\pi\sigma^2}+\frac{1}{1-e^{\pi\sigma^2\alpha_0}}\right)e^{-\frac{\pi\sigma^2\alpha_0}{4}} + \left(\frac{1}{2\pi d}+\frac{d}{2\sigma^2}+\frac{1}{1-e^{-\frac{\pi d}{\sigma^2}}}+\frac{1}{1-e^{-\frac{\pi d^2}{\sigma^2}}}\right)e^{-\frac{\pi d^2}{4\sigma^2}}\right) & \textit{otherwise.} \end{cases} \tag{3.28}$$

Proof: The proof of this lemma consists in computing the difference between what we would expect from an infinitesimal evolution and the desired approximation in terms of the analytic Gaussian function. We start with the desired state after the infinitesimal evolution

$$e^{i\delta H_C(T/d)}\left|\psi_{QI}(k_0)\right\rangle = \sum_{k\in S_d(k_0)} \psi(k_0+\delta;k)\left|\theta_k\right\rangle \tag{3.29}$$

and the exact state

$$\begin{aligned} e^{i\delta H_C(T/d)} |\psi_{QI}(k_0)\rangle &= e^{i\delta(T/d)\sum \omega j |E_j\rangle\langle E_j|} \sum_{k\in S_d(k_0)} \psi(k_0;k) |\theta_k\rangle \\ &= \sum_{k\in S_d(k_0)} \psi(k_0;k) \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{-i2\pi j(k+\delta)/d} |E_j\rangle \\ &= \sum_{k,l\in S_d(k_0)} \psi(k_0;k) \frac{1}{d} \sum_{j=0}^{d-1} e^{-i2\pi j(k+\delta-l)/d} |\theta_l\rangle . \end{aligned} \tag{3.30}$$

Writing both equations in the time basis we get

$$g_l(\delta) = \sum_{k,l\in S_d(k_0)} \psi(k_0;k) \frac{1}{d} \sum_{j=0}^{d-1} e^{-i2\pi j(k+\delta-l)/d} \tag{3.31}$$

and

$$g_l'(\delta) = \psi(k_0+\delta;k), \tag{3.32}$$

we can compute their difference using a Taylor series expansion around $\delta = 0$

$$g_l(\delta) - g_l'(\delta) = g_l(0) - g_l'(0) + \delta \left[\frac{\partial g_l(\delta)}{\partial \delta} - \frac{\partial g_l'(\delta)}{\partial \delta} \right]\Bigg|_{\delta=0} + C\delta^2, \tag{3.33}$$

where C is bounded and independent of δ. Computing the derivatives we get

$$\begin{aligned} \frac{\partial g_l(\delta)}{\partial \delta}\Bigg|_{\delta=0} &= \left[\frac{\partial}{\partial \delta} \sum_{k\in S_d(k_0)} \psi(k_0;k) \frac{1}{d} \sum_{j=0}^{d-1} e^{-i2\pi j(k+\delta-l)/d} \right]\Bigg|_{\delta=0} \\ &= -\frac{i2\pi}{d^2} \sum_{k\in S_d(k_0)} \psi(k_0;k) \sum_{j=0}^{d-1} e^{-i2\pi j(k+\delta-l)/d} \\ &= -\frac{i2\pi}{d^2} \sum_{k\in\mathbb{Z}} \psi(k_0;k) \sum_{j=0}^{d-1} e^{-i2\pi j(k+\delta-l)/d} + \frac{i2\pi}{d^2} \sum_{k\in\mathbb{Z}/S_d(k_0)} \psi(k_0;k) \sum_{j=0}^{d-1} e^{-i2\pi j(k+\delta-l)/d} \\ &= -\frac{i2\pi}{d^2} \sum_{k\in\mathbb{Z}} \psi(k_0;k) \sum_{j=0}^{d-1} e^{-i2\pi j(k+\delta-l)/d} + \varepsilon_1, \end{aligned} \tag{3.34}$$

where in the third line the finite sum was replaced by an infinite sum and in the fourth line we associate the sum without the integer set of interest with the error, bounded ‖ as

$$|\varepsilon_1| < \begin{cases} 2\pi A \frac{e^{-\pi d/4}}{1-e^{-\pi}} & \text{if } \sigma = \sqrt{d} \\ 2\pi A \frac{e^{-\pi d^2/4\sigma^2}}{1-e^{-\pi d/\sigma^2}} & \text{otherwise.} \end{cases} \tag{3.35}$$

Changing from a finite sum to an infinite sum carries the advantage of allowing the use of the Poisson summation formula

$$\sum_{k\in\mathbb{Z}} f(k) = \sqrt{d} \sum_{m\in\mathbb{Z}} \tilde{f}(md), \tag{3.36}$$

‖ See Appendix A.

where $\tilde{f}$ is the continuous Fourier transform of f and the identity

$$\sum_{j=0}^{d-1}\sum_{m\in\mathbb{Z}} f(j+md) = \sum_{s\in\mathbb{Z}} f(s), \tag{3.37}$$

leading to

$$\begin{aligned}
\left.\frac{\partial g_l(\delta)}{\partial\delta}\right|_{\delta=0} &= -\frac{i2\pi}{d^2}\sum_{k\in\mathbb{Z}}\psi(k_0;k)\sum_{j=0}^{d-1} e^{-i2\pi j(k+\delta-l)/d} + \varepsilon_1\\
&= -\frac{i2\pi}{d^{3/2}}\sum_{j=0}^{d-1}\sum_{m\mathbb{Z}}\tilde{\psi}(k_0;j+md)je^{i2\pi jl/d} + \varepsilon_1\\
&= -\frac{i2\pi}{d^{3/2}}\sum_{j=0}^{d-1}\sum_{m\mathbb{Z}}\tilde{\psi}(k_0;j+md)(s-md)e^{i2\pi jl/d} + \varepsilon_1\\
&= -\frac{i2\pi}{d^{3/2}}\left[\sum_{j=0}^{d-1}\sum_{m\mathbb{Z}}\tilde{\psi}(k_0;j+md)se^{i2\pi jl/d} - \sum_{j=0}^{d-1}\sum_{m\mathbb{Z}}\tilde{\psi}(k_0;j+md)mde^{i2\pi jl/d}\right] + \varepsilon_1\\
&= -\frac{i2\pi}{d^{3/2}}\left[\sum_{j=0}^{d-1}\sum_{m\mathbb{Z}}\tilde{\psi}(k_0;s)se^{i2\pi sl/d}e^{-i2\pi ml} - \sum_{j=0}^{d-1}\sum_{m\mathbb{Z}}\tilde{\psi}(k_0;j+md)mde^{i2\pi jl/d}\right] + \varepsilon_1\\
&= -\frac{i2\pi}{d^{3/2}}\left[\sum_{m\mathbb{Z}}\tilde{\psi}(k_0;s)se^{i2\pi sl/d} - \sum_{j=0}^{d-1}\sum_{m\mathbb{Z}}\tilde{\psi}(k_0;j+md)mde^{i2\pi jl/d}\right] + \varepsilon_1\\
&= -\frac{i2\pi}{d^{3/2}}\sum_{m\mathbb{Z}}\tilde{\psi}(k_0;s)se^{i2\pi sl/d} + \varepsilon_2 + \varepsilon_1,
\end{aligned} \tag{3.38}$$

where the second sum was considered an error that is bounded as

$$|\varepsilon_2| < \begin{cases} 4\pi A\sqrt{d}\left(\frac{1}{2}+\frac{1}{2\pi d}+\frac{1}{1-e^{-\pi d}}\right)e^{-\pi d/4} & \text{if } \sigma=\sqrt{d}\\ 4\pi A\sigma\left(\frac{\alpha_0}{2}+\frac{1}{2\pi\sigma^2}+\frac{1}{1-e^{-\pi\sigma^2\alpha_0}}\right)e^{-\pi\sigma^2\alpha_0^2/4} & \text{otherwise.}\end{cases} \tag{3.39}$$

Using the Poisson formula for $\tilde{f}$ ** and remembering that the Fourier transform $\mathcal{F}$ obeys the following relation

$$\mathcal{F}^{-1}[p\tilde{f}(p)] = \left(-\frac{id}{2\pi}\right)\frac{df(x)}{dx}, \tag{3.40}$$

we arrive at

$$\begin{aligned}
\frac{\partial g_l(\delta)}{\partial\delta} &= \sum_{m\in\mathcal{Z}}\left.\frac{\partial}{\partial y}\psi(k_0+y;l)\right|_{y=md} + \varepsilon_2 + \varepsilon_1\\
&= \left.\frac{\partial}{\partial y}\psi(k_0+y;l)\right|_{y=0} + \sum_{m\neq 0\in\mathcal{Z}}\left.\frac{\partial}{\partial y}\psi(k_0+y;l)\right|_{y=md} + \varepsilon_2 + \varepsilon_1\\
&= \frac{\partial g_l'(\delta)}{\partial\delta} + \varepsilon_3 + \varepsilon_2 + \varepsilon_1
\end{aligned} \tag{3.41}$$

where it was once again used the remaining sum as an error term bounded as

$$|\varepsilon_3| < \begin{cases} 4\pi A\left(\frac{1}{2}+\frac{1}{2\pi d}+\frac{1}{1-e^{-\pi}}\right)e^{-\pi d/4} & \text{if } \sigma=\sqrt{d}\\ 4\pi A\left(\frac{d}{2\sigma^2}+\frac{1}{2\pi d}+\frac{1}{1-e^{-\pi d^2/\sigma^2}}\right)e^{-\pi d^2/4\sigma^2} & \text{otherwise.}\end{cases} \tag{3.42}$$

** $\sum_{k\in\mathbb{Z}}\tilde{f}(k) = \sqrt{d}\sum_{m\in\mathbb{Z}} f(md)$

Hence

$$g_l(\delta) - g_l'(\delta) = \delta(\varepsilon_3 + \varepsilon_2 + \varepsilon_1) + C\delta^2, \tag{3.43}$$

with the resulting error $\varepsilon' = \varepsilon_3 + \varepsilon_2 + \varepsilon_1$ bounded as

$$|\varepsilon_3| < \begin{cases} 2\pi A \left(2\sqrt{d}\left(\frac{1}{2} + \frac{1}{2\pi d} + \frac{1}{1-e^{-\pi}}\right) e^{-\frac{\pi d}{4}} + \frac{1}{2} + \frac{1}{2\pi d} + \frac{1}{1-e^{-\pi}}\right) e^{-\frac{\pi d}{4}} & \text{if } \sigma = \sqrt{d} \\ 2\pi A \left(2\sigma\left(\frac{\alpha_0}{2} + \frac{1}{2\pi\sigma^2} + \frac{1}{1-e^{-\pi\sigma^2\alpha_0}}\right) e^{-\frac{\pi\sigma^2\alpha_0}{4}} + \left(\frac{1}{2\pi d} + \frac{d}{2\sigma^2} + \frac{1}{1-e^{-\frac{\pi d}{\sigma^2}}} + \frac{1}{1-e^{-\frac{\pi d^2}{\sigma^2}}}\right) e^{-\frac{\pi d^2}{4\sigma^2}}\right) & \text{otherwise.} \end{cases} \tag{3.44}$$

Lemma 4 *Given an initial k_0 and a finite translation Δ such that $S_d(k_0) = S_d(k_0 + \Delta)$, then*

$$e^{-iT\Delta H_C/d} |\psi_{QI}(k_0)\rangle = |\psi_{QI}(k_0 + \Delta)\rangle + |\varepsilon\rangle, \tag{3.45}$$

with the norm of the error being upper bounded as $\| \, |\varepsilon\rangle \, \|_2 < \Delta\varepsilon_t$.

Proof: The next step is to extend the previous lemma to finite translations with the particularity that they are sufficiently small so that the range $S_d(k_0)$ does not change. We observe that in the previous lemma the result is not written in terms of the evolved state $|\psi(k_0 + \delta)\rangle$ but the state $\sum_{k \in S_d(k_0)} \psi_{QI}(k_0 + \delta; k) |\theta_k\rangle$ which, can differ, from the evolved state depending on what happens with the range. Here it is wanted to use lemma 3, justifying this particularity. Then, if we split the translation into equal steps of size M we can write

$$e^{-iT(\Delta/M)H_C/d} |\psi_{QI}(k_0 + n\Delta/M)\rangle = |\psi_{QI}(k_0 + (n+1)\Delta/M)\rangle + |\varepsilon\rangle, \tag{3.46}$$

for $n \in \{0, 1, .., M-1\}$. Now it is only needed to find the error associated with the M steps from the desired state. Using the fact that the error adds linearly†† again from lemma 3

$$\| \, |\psi_{QI}(k_0 + \Delta)\rangle - \left(e^{-iT(\Delta/M)H_C/d}\right)^M |\psi_{QI}(k_0)\rangle \, \|_2 < M\left(\frac{\Delta\varepsilon_t}{M} + \frac{\Delta^2 C}{M^2}\right) = \Delta\varepsilon_t + \frac{\Delta^2 C}{M}, \tag{3.47}$$

taking the limit of M going to infinity we reach lemma 4.

Lemma 5 *For even dimension d and integer position k_0 we have*

$$\left\| \sum_{k \in S_d(k_0 - 1)} \psi(k_0; k) |\theta_k\rangle - \sum_{k \in S_d(k_0)} \psi(k_0; k) |\theta_k\rangle \right\|_2 < \varepsilon_{step}, \tag{3.48}$$

where

$$|\varepsilon_{step}| < \begin{cases} 2Ae^{-\pi d/4} & \textit{if } \sigma = \sqrt{d} \\ 2Ae^{-\pi d^2/4\sigma^2} & \textit{otherwise} \end{cases} \tag{3.49}$$

†† See Appendix B

Proof: Lastly we show the error from changing only the range $S_d(k_0)$. Since the range is comprised of d integers if the initial set goes through a translation of 1 integer, to a new set of integers, the only difference between both sets will be the first and last integers. Using the bounds in Definition 3 we know that these integers will be $k_0 - d/2$ and $k_0 + d/2$, hence, if we define

$$N_\varepsilon := \left\| \sum_{k \in S_d(k_0-1)} \psi(k_0;k) \left|\theta_k\right\rangle - \sum_{k \in S_d(k_0)} \psi(k_0;k) \left|\theta_k\right\rangle \right\|_2, \tag{3.50}$$

we get

$$\begin{aligned} N_\varepsilon &= \| \psi(k_0;k_0-d/2) \left|\theta_{k_0-d/2}\right\rangle - \psi(k_0;k_0+d/2) \left|\theta_{k_0+d/2}\right\rangle \|_2 \\ &= |\psi(k_0;k_0-d/2) - \psi(k_0;k_0+d/2)| \\ &= |Ae^{-\pi(k_0-d/2-k_0)^2/\sigma^2} e^{i2\pi j_0(k_0-d/2-k_0)/d} - Ae^{-\pi(k_0+d/2-k_0)^2/\sigma^2} e^{i2\pi j_0(k_0+d/2-k_0)/d}| \\ &= |Ae^{-\pi(-d/2)^2/d} e^{i2\pi j_0(-d/2)/d} - Ae^{-\pi(d/2)^2/d} e^{i2\pi j_0(d/2)/d}| \\ &= Ae^{-\frac{\pi d^2}{4\sigma^2}} |e^{-i\pi j_0} - e^{i\pi j_0}| = 2Ae^{-\frac{\pi d^2}{4\sigma^2}}. \end{aligned} \tag{3.51}$$

Using the lemmas presented, sequentially, we finally reach continuity

Theorem 2 *Let k_0, $t \in \mathbb{R}$. Then the effect of the Hamiltonian H_C for time t on the QIC state is approximated by*

$$e^{-itH_c} \left|\psi_{QI}(k_0)\right\rangle = \left|\psi_{QI}(k_0 + td/T)\right\rangle + \left|\varepsilon_{con}\right\rangle, \tag{3.52}$$

where

$$\| \left|\varepsilon_{con}\right\rangle \|_2 < |t|\frac{d}{T}\varepsilon_{to} + \left(|t|\frac{d}{T} + 1\right)\varepsilon_{step} + \varepsilon_{nor}(t). \tag{3.53}$$

with $\varepsilon_{nor}(t)$ being the error associated with the normalization of the state respecting the bounds

$$\varepsilon_{nor} \leq \begin{cases} \frac{40e^{-\pi d/2}}{3(1-e^{-\pi})} & \forall t \in \mathbb{R} \text{ if } \sigma = \sqrt{d} \\ \frac{40\sqrt{2}}{3\sigma}\left(\frac{e^{-\pi d^2/2\sigma^2}}{(1-e^{-2\pi d/\sigma^2})} + \frac{\sigma e^{-\pi\sigma^2/2}}{\sqrt{2}(1-e^{-\pi\sigma^2})}\right) & \forall t \in \mathbb{R} \text{ otherwise} \end{cases} \tag{3.54}$$

being assumed $\sigma \geq 1$ and $d = 2, 3, 4...$ to obtain the right side of Eq.(3.54).

With this we see that the QIC approximates the continuity observed for the ideal case pretty well, being the error exponentially small with dimension.

We also expect an ideal clock to remain continuous if under the influence of potential which is position dependent, which is translated to

$$\langle x|e^{-i(H_C+V(x))t}|\psi\rangle = e^{-i\int_{x-t}^{x} V(x')dx'} \langle x-t|\psi\rangle. \tag{3.55}$$

As it can be imagined it is possible to demonstrate that the QIC can also approximate this relation really well, however, for this specific case its error can only be bounded to be exponentially small in dimension for the case were the QIC is symmetric. For the squeezed case the associated errors will decrease in d^η where $0 < \eta < 1$, since this type of continuity can be associated to quantum control, we see that squeezed states will be less resilient for this task.

3.3.2 Quasi-canonical Commutation

Further it can be shown that the QIC can almost satisfy the canonical commutation relation between a constructed time operator as in Eq.(3.17) and the clock Hamiltonian H_C.

Theorem 3 *Let d be odd and both parameters $k_0, j_0 \in (-d/2, d/2)$. For all normalized states $|\psi_{QI}(k_0)\rangle$ the time operator $\mathcal{T}_C$ and the clock Hamiltonian H_C satisfy*

$$[\mathcal{T}_C, H_C]\,|\psi_{QI}(k_0)\rangle = i\,|\psi_{QI}(k_0)\rangle + |\varepsilon_{comm}\rangle\,, \tag{3.56}$$

where

$$\|\varepsilon_{comm}\|_2 \leq \begin{cases} \mathcal{O}(d^{9/4})e^{-\pi d/4} & \text{if } \sigma = \sqrt{d} \quad \text{and } k_0, j_0 = 0 \\ \mathcal{O}(d\sigma^{5/2})e^{-\pi\sigma^2(1-2j_0/d)^2/4} + \Big(\mathcal{O}(d^2/\sigma^{5/2}) & \\ +\mathcal{O}(d\sigma^{1/2})\Big)\, e^{-\pi d^2(1-2k_0/d)^2/4\sigma^2} + \Big(\mathcal{O}(d^3/\sigma^{5/2}) + \mathcal{O}(d^2\sigma^{3/2})\Big)\, e^{-\pi d^2/4\sigma^2} & \text{otherwise.} \end{cases} \tag{3.57}$$

Where the bound for the general case is obtained in the limit of $d, \sigma \to \infty$.

In the proof of this theorem it is computed the action of each operator on the QIC states bounding the summations and applying the Poisson summation formula as performed in the previous section. Apart from some technicalities, knowing the previous deductions, makes it more straightforward and therefore we choose to skip it.

Apart from being a demonstration that the QIC can almost manifest the covariance expected, making it a good clock, this result indicates interesting aspects of the QIC. The best possible case, where we have the smaller error is given when the clock is completely symmetric, where the QIC is up to a small correction, a minimum uncertainty state, i.e., given the energy and time basis we have $\Delta E \Delta t = 1/2 +$ (exponentially small value). With this, we see that the previous definition given for the QIC in terms of its variance and dimension is justified by the change in time and energy uncertainties. For time squeezed, when the variance is smaller than the square root of the dimension, the states will have less uncertainty in time $\Delta E > \Delta t$ and for energy squeezed states, when the variance is greater than the square root of the dimension, we will have less uncertainty in energy $\Delta t > \Delta E$.

4 TIME AND COHERENCE

On the original presentation Page and Wootters considered as a justification for the absence of time the existence of a superselection rule for energy. (89) Selection rules form a set of constraints on the possible transitions that can happen for a certain state. A clear example is seen in spontaneous transitions in the hydrogen atom. These transitions will only occur from a state with quantum numbers n, l and m to states n', l' and m' if the quantum numbers obey the relations

$$l' = l \pm 1 \text{ and } m' = m, m \pm 1.$$

These constraints are for an electric dipole transition, transitions of other orders, e.g., electric quadrupole transition, will have other restrictions. Superselection rules (SSR), are an extension of the usual selection rules. Meanwhile, the constraints of selection rules are valid for an order of approximation, in the case of a SSR, they are valid for all possible observables*. (89) That means that if we have two states $|\phi_1\rangle$ and $|\phi_2\rangle$, subjected to a SSR and an observable $\mathcal{B}$, it is true that

$$\langle\phi_1|\mathcal{B}|\phi_2\rangle = 0, \tag{4.1}$$

for any observable inside the universe. Then, when there is a SSR for energy, a time translation that connects two coherent eigenstates of energy would not be observable, entailing the constraint

$$H\left|\Psi\right\rangle = 0. \tag{4.2}$$

Here we present our first results about the PaW mechanism, which will be constantly referred just as the model.

4.1 Page-Wootters revised

For us, the important consideration is that this *unknown coordinate* time can be represented using group theoretical notions. More specifically using asymmetry theory, we understand that any state ρ wich acts as the global state in the PaW mechanism should inevitably be a symmetric state and perceived as

$$\rho' = \int d\theta p(\theta) U(\theta) \rho U^\dagger(\theta). \tag{4.3}$$

However, as described, the group of time translations is given by groups of the same form as the $U(1)$ which makes it straightforward to note, that this being the case, the state above is equivalent to the action of a dephasing operation (70) acting on a state ρ

$$\rho' \equiv \mathcal{D}(\rho) = \frac{1}{T}\int_0^T e^{-iHt}\rho e^{iHt} dt, \tag{4.4}$$

* Here the word possible makes reference to physically possible.

where T will be the period of the Hamiltonian used, which for the PaW mechanism is the two-spin non-interacting Zeeman Hamiltonian

$$H_z = h\left(\sigma_z^1 \otimes \mathbb{1}^2 + \mathbb{1}^1 \otimes \sigma_z^2\right), \tag{4.5}$$

where σ_z^i is the Pauli z-matrix for the ith qubit and h is a constant representing the magnetic field. Being its action to take a state ρ to a state $\mathcal{D}(\rho)$ that is block-diagonal in the energy basis. Here we note the introduced nomenclature for the fully dephasing operation $\Delta(\cdot)$, which differs from the dephasing operation above for it leaves any state in a diagonal form, instead of only block-diagonal. Hence, if we consider the space of physical states, defined as the allowed states in the universe, they all must be of the form in Eq.(4.4), i.e., for a density matrix ρ_Ψ associated to the state $|\Psi\rangle$ satisfying the constraint in Eq.(4.2) we have $\rho_\Psi = \mathcal{D}(\rho)$.

As it was described, the distinct characteristic of the model is that it makes use of conditional probabilities as a way to show the correlation that arises between clock and system. We will utilize global states which are composed by two qubits. Hence, defining the most simple clock, as a clock with two ticks, a qubit, where we adopt the "right", or 12 o'clock position being given by the state $|+\rangle$ and the "left", or 6 o'clock, position being the state $|-\rangle$. Therefore the other qubit is going to be labeled as the system. We define the probability of agreement as given by the conditional probability that, if one qubit is measured to be pointing in one direction the other qubit will be pointing in the same direction, either clock right and system right or clock left and system left, to be

$$\text{Prob(R|R)} = \frac{\text{Tr}(\mathcal{D}(\rho)E_{R_1R_2})}{\text{Tr}(\mathcal{D}(\rho)E_{R_2})} \tag{4.6}$$

and

$$\text{Prob(L|L)} = \frac{\text{Tr}(\mathcal{D}(\rho)E_{L_1L_2})}{\text{Tr}(\mathcal{D}(\rho)E_{L_2})}, \tag{4.7}$$

respectively. $E_{R_1R_2}$ and $E_{L_1L_2}$ are the optimal projectors to distinguish between $|+\rangle$ and $|-\rangle$, belonging to the set of operators $\{E_{\eta\mu} : \eta \in \{R_1, L_1\}, \mu \in \{R_2, L_2\}\}$ where each η and μ represents a right or left. In the same manner the probabilities for opposite directions will be denoted by

$$\text{Prob(R,L)} = \frac{\text{Tr}(\mathcal{D}(\rho)E_{R_1L_2})}{\text{Tr}(\mathcal{D}(\rho)E_{L_2})} \tag{4.8}$$

and

$$\text{Prob(L,R)} = \frac{\text{Tr}(\mathcal{D}(\rho)E_{L_1R_2})}{\text{Tr}(\mathcal{D}(\rho)E_{R_2})}. \tag{4.9}$$

It should be noted that we only take probabilities of agreement into consideration for conciseness, the model does not impose restrictions on which probabilities should be used and a perfectly good clock could be built with the probabilities for opposite directions†.

† Changing this convention does not alter the result obtained in this Chapter.

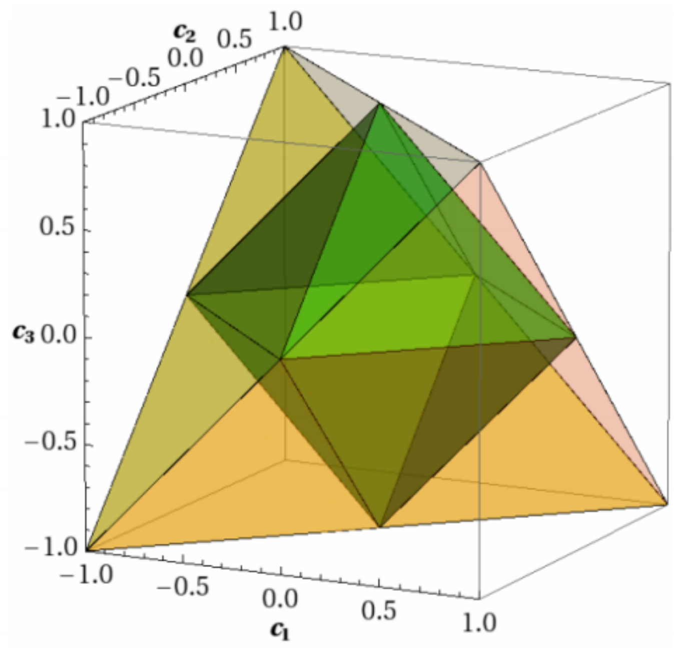

Figure 2 – Geometrical representation of the Bell diagonal states. The tetrahedron is composed of the valid Bell-diagonal states being the octahedron inside it, composed of the set of separable states. There are four regions, for each vertex of the tetrahedron, with the biggest eigenvalue in each corresponding to a Bell state. Source: LANG; CAVES (90).

We consider two-qubit Bell-diagonal states (90) as initial states. These states have maximally mixed reduced density operators and can be represented as

$$\rho = \frac{1}{4}\left(\mathbb{1} + \sum_{i=1}^{3} c_i \sigma_i \otimes \sigma_i\right), \tag{4.10}$$

where the parameters c_i, with $-1 \leq c_i = \text{Tr}(\rho\sigma_i \otimes \sigma_i) \leq 1$, form a triplet that determine whose states are physically acceptable i.e., with non-negative eigenvalues, $\lambda_{\gamma\nu} \geq 0$. These states have a very interesting geometric description, where all valid states are contained in a tetrahedron with its vertices being the four Bell states and all separable states contained in an octahedron inside the tetrahedron as shown in Figure 4.1.

Their eigenvalues can be obtained using

$$\lambda_{\gamma\nu} = \frac{1}{4}\left[1 + (-1)^{\gamma} c_1 - (-1)^{\gamma+\nu} c_2 + (-1)^{\nu} c_3\right], \tag{4.11}$$

where γ, $\nu = \{0,1\}$. In the σ_z-basis they take an X form

$$\rho = \frac{1}{4}\begin{bmatrix} 1+c_3 & 0 & 0 & c_1-c_2 \\ 0 & 1-c_3 & c_1+c_2 & 0 \\ 0 & c_1+c_2 & 1-c_3 & 0 \\ c_1-c_2 & 0 & 0 & 1+c_3 \end{bmatrix}. \tag{4.12}$$

Therefore the action of the dephasing operation results in

$$\mathcal{D}(\rho) = \frac{1}{4}\begin{bmatrix} 1+c_3 & 0 & 0 & 0 \\ 0 & 1-c_3 & c_1+c_2 & 0 \\ 0 & c_1+c_2 & 1-c_3 & 0 \\ 0 & 0 & 0 & 1+c_3 \end{bmatrix}. \tag{4.13}$$

Since we are writing the Bell-diagonal states in the basis of the Hamiltonian, the probability of agreement for the two qubit Bell-diagonal state is

$$\text{Prob(R|R)} = \frac{1}{4}\left(2 + c_1 + c_2\right). \tag{4.14}$$

Thus we can study the impact on the probability of agreement of the two qubits for several states just controlling the parameters $\{c_1, c_2, c_3\}$. A few interesting sets of parameters are those where: (*i*) $c_1 + c_2 = 1$ with $c_3 = 0$, (*ii*) $c_1 = -c_2$ and $c_1 = c_2 = 0$, both for any c_3, and (*iii*) $c_1 = c_2 = 1$ and $c_3 = -1$. The first set is composed of all dephased states that have the same form as the ones studied in Ref. (31). They act as a paradigm for the results. Therefore, we should be able to find the same probabilities; indeed the probability of agreement, Prob(R|R) $= 0.75$, is the same as that found in Ref. (31). The second set has conditional probabilities Prob(R|R) = Prob(L|L) = Prob(R,L) = Prob(L,R) $= 0.5$, which tells us that there is an equal chance that, upon measuring the clock, the system can be found in any of the two directions. It follows that there is no correlation between the clock and the system; in other words, there is no sense of time given by the conditional probabilities. This is because the flow of time in the model is represented as clock time. If the clock does not correlate with the position of the system there can be no established causal connection, and we cannot say that the clock is measuring time and therefore that the system is moving with respect to clock time. In the opposite direction the last set, that in fact is composed of only one element, given by one of the vertices of the tetrahedron formed by the Bell-diagonal states, gives a probability of agreement Prob(R|R) $= 1$, which is perfect agreement. The qubits are always going to be pointing in the same direction.

It is very interesting to note the different outcomes in relation to the conditional probabilities for the parameters $\{1, 1, -1\}$ and $\{1, -1, 1\}$. Respectively, these parameters correspond to two Bell states: $|\psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$ and $|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. These are two pure maximally entangled states, that result in two drastically different results. One gives the best possible probability and the other the worst possible probability. This indicates that entanglement between the subsystems in the initial state, before the dephasing operation, is not responsible for the working of the PaW clock. This does not indicate that entanglement of the dephased state is not required for the mechanism. In the next section we will see that in fact neither entanglement before nor entanglement after the dephasing operation can be connected to the emergence of time in the PaW model.

4.2 Coherence: Internal and External

We can now present our first result. The relative entropy of coherence presented in subsection 2.6.1 can be understood as measuring how distinguishable a general state ρ is from an incoherent state τ. However, the model clearly specify states which are symmetric in relation to the group of translations generated by H_z, or in other words, block diagonal states. This implies, that inside the PaW universe, these are the only available states. If we start from the space containing all the density operators, what happens if we choose only the states that have a block diagonal form? The result is a measure of the distinguishability from the block diagonal state that is now representing our system and the closest incoherent state $\tau \in \mathcal{I}$

$$\min_{\tau\in\mathcal{I}} S(\mathcal{D}(\rho)||\tau), \tag{4.15}$$

where the dephasing operation is used to guarantee that the state is indeed block diagonal. We can see that this measure also admits a closed form. From the relative entropy of coherence we have

$$\begin{aligned}
\min_{\tau\in\mathcal{I}} S(\mathcal{D}(\rho)||\tau) &= \min_{\tau\in\mathcal{I}} \{\mathrm{Tr}[\mathcal{D}(\rho)\log\mathcal{D}(\rho)] - \mathrm{Tr}[\mathcal{D}(\rho)\log\tau]\} \\
&= \mathrm{Tr}[\mathcal{D}(\rho)\log\mathcal{D}(\rho)] - \max_{\tau\in\mathcal{I}} \mathrm{Tr}[\Delta(\rho)\log\tau] \\
&= \mathrm{Tr}[\mathcal{D}(\rho)\log\mathcal{D}(\rho)] - \mathrm{Tr}[\Delta(\rho)\log\Delta(\rho)] \\
&= S(\Delta(\rho)) - S(\mathcal{D}(\rho)) \\
&= C_r(\mathcal{D}(\rho)),
\end{aligned} \tag{4.16}$$

where in the second line we used the invariance under dephasing of τ and, in the third line, we used the non-negativity of the relative entropy $S(\delta_1||\delta_2) \geq 0$. Then the relative entropy of coherence, when performing the minimization from the block diagonal states to the incoherent states, has an equivalent closed form for the case that we consider a dephased state, $\mathcal{D}(\rho)$,

$$C_r(\mathcal{D}(\rho)) = \min_{\tau\in\mathcal{I}} S(\mathcal{D}(\rho)||\tau). \tag{4.17}$$

This fact gives a very intuitive reasoning for the physical interpretation of such measures in terms of the coherences of a state. From the closed form of the relative entropy of coherence

$$C_r(\rho) = S(\Delta(\rho)) - S(\rho), \tag{4.18}$$

we can apply it to a dephased state $\mathcal{D}(\rho)$. This yields

$$\begin{aligned}
C_r(\mathcal{D}(\rho)) &= S(\Delta(\mathcal{D}(\rho))) - S(\mathcal{D}(\rho)) \\
&= S(\Delta(\rho)) - S(\mathcal{D}(\rho)) \\
&= S(\Delta(\rho)) - S(\mathcal{D}(\rho)) + S(\rho) - S(\rho) \\
&= C_r(\rho) - A_G(\rho),
\end{aligned} \tag{4.19}$$

where in the second line it was used the invariance under dephasing of $\Delta(\rho)$ and

$$A_G(\rho) = S(\mathcal{D}(\rho)) - S(\rho). \tag{4.20}$$

This result can be encapsulated in the following proposition:

Proposition 1 *The relative entropy of coherence defined with regards to the set of incoherent states can be broken down in terms of two types of coherences, the internal coherence and the external coherence as*

$$C_r(\rho) = C_r(\mathcal{D}(\rho)) + A_G(\rho).$$

Therefore the relative entropy of coherence is a measure of the total coherence, while $A_G(\rho)$ is a measure of the external coherence and $C_r(\mathcal{D}(\rho))$ is a measure of the internal coherence of a state.

Coherence is understood as the presence of off-diagonal elements on the density matrix in the energy eigenbasis. Here we take the same definition used in Ref. (91), then, by internal coherence we mean the presence of off-diagonal elements of the density matrix of the state in the energy eigenbasis with the same energy. Therefore, external coherence will be represented by off-diagonal elements with different energies. The quantity $A_G(\rho)$, was presented in subsection 2.7.2 as a recognized measure of asymmetry which was first introduced in Ref. (71), usually referred as asymmetry or Holevo asymmetry. When introduced, this measure was brought up in a very similar context to the PaW mechanism used as a way to quantify the quality of a reference frame. When asymmetry relative to time translation is invoked, it can be seen as coherence in the eigenbasis, therefore it is also a measure of coherence (70), more specifically of external coherence. This type of coherence is always defined with regards to an external frame of reference, necessarily, any measure capable of discerning the effects of said reference frame is going to be a measure for the external coherence. It follows, that the necessary coherence in any task where invariance over time translations is a factor (e.g., quantification of reference frames, quantum speed limits and quantum metrology) is the external coherence.

We notice a separation with regards to the frameworks used to define these measures. While $A_G(\rho)$ is a measure for *unspeakable coherence* (70), $C_r(\mathcal{D}(\rho))$ cannot detect invariance over the dephasing operation. After all it is equivalent to how distinguishable a block diagonal state $\mathcal{D}(\rho)$ is from the closest incoherent state τ. Therefore, it is only a measure of *speakable coherence*. This reinforces the conclusion that this measure should be seen as a measure for relative phases between subsystems, that is a measure for internal coherence. Given both definitions, it is straightforward to justify $C_r(\rho)$ as a quantifier for the total coherence.

Although our interest here lies in general states, which can be entangled, we should note, that for the case where we have a product state for the global state we can achieve equality between the measures of internal and external coherence. If we calculate the internal coherence of a general product state $\rho = \rho_S \otimes \rho_C$, we will have

$$\begin{aligned} C_r(\mathcal{D}(\rho_S \otimes \rho_C)) &= S(\Delta(\rho_S \otimes \rho_C)) - S(\mathcal{D}(\rho_S \otimes \rho_C)) \\ &= S(\Delta(\rho_S)) + S(\Delta(\rho_C)) - S(\mathcal{D}(\rho_S \otimes \rho_C)) \\ &= S(\mathcal{D}(\rho_S)) + S(\mathcal{D}(\rho_C)) - S(\mathcal{D}(\rho_S \otimes \rho_C)), \end{aligned} \tag{4.21}$$

where the last line follows because the action of the dephasing operation on an individual system will be identical to the fully dephasing operation on that individual system. Now we specify a classical-quantum state (92)

$$\rho = \frac{1}{N} \sum_{k=0}^{N-1} |k\rangle \langle k|_C \otimes \rho_S^n, \tag{4.22}$$

where n represents a certain clock time and compute its internal coherence, we start by evaluating $S(\mathcal{D}(\rho_S \otimes \rho_C)) = S(\Omega_{SC})$.

$$\begin{aligned} S(\Omega_{SC}) &= -\mathrm{Tr}\left[\frac{1}{N}\sum_{k=0}^{N-1} |k\rangle \langle k|_C \otimes \rho_S^n \log\left(\frac{1}{N}\sum_{k'=0}^{N-1} |k'\rangle \langle k'|_R \otimes \rho_S^{n'}\right)\right] \\ &= -\mathrm{Tr}\left[\frac{1}{N}\sum_{k=0}^{N-1} |k\rangle \langle k|_C \otimes \rho_S^n \log\left(\sum_{k'=0}^{N-1} |k'\rangle \langle k'|_C \otimes \rho_S^{n'}\right)\right] - \mathrm{Tr}\left[\frac{1}{N}\sum_{k=0}^{N-1} |k\rangle \langle k|_C \otimes \rho_S^n \log\left(\frac{1}{N}\right)\right] \\ &= -\mathrm{Tr}\left[\frac{1}{N}\sum_{k=0}^{N-1} |k\rangle \langle k|_C \otimes \rho_S^n \log\left(\sum_{k'=0}^{N-1} |k'\rangle \langle k'|_C \otimes \rho_S^{n'}\right)\right] - \frac{1}{N}\sum_{k=0}^{N-1} \log\left(\frac{1}{N}\right) \mathrm{Tr}\left[|k\rangle \langle k|_C \otimes \rho_S^n\right] \\ &= -\mathrm{Tr}\left[\frac{1}{N}\sum_{k=0}^{N-1}\sum_x |k\rangle \langle k|_C \otimes p_x^n |x\rangle \langle x|_S \log\left(\sum_{k'=0}^{N-1}\sum_{x'} |k'\rangle \langle k'|_C \otimes p_{x'}^{n'} |x'\rangle \langle x'|_S\right)\right] - \frac{1}{N}\sum_{k=0}^{N-1} \log\left(\frac{1}{N}\right) \\ &= -\frac{1}{N}\sum_{k,k'=0}^{N-1}\sum_{x,x'} p_x^n \log\left(p_{x'}^{n'}\right) \mathrm{Tr}\left[|k\rangle \langle k|_C |k'\rangle \langle k'|_C \otimes |x\rangle \langle x|_S |x'\rangle \langle x'|_S\right] - \frac{1}{N}\sum_{k=0}^{N-1} \log\left(\frac{1}{N}\right) \\ &= -\frac{1}{N}\sum_{k=0}^{N-1}\sum_x p_x^n \log\left(p_x^n\right) - \frac{1}{N}\sum_{k=0}^{N-1} \log\left(\frac{1}{N}\right) \\ &= -\frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_x p_x^n \log\left(p_x^n\right) - \log\left(\frac{1}{N}\right)\right] \\ &= -\frac{1}{N}\sum_{k=0}^{N-1}\left[S(\rho_S^n) - S(\mathcal{D}(\rho_C))\right] \\ &\approx S(\rho_S) + S(\mathcal{D}(\rho_C)). \end{aligned} \tag{4.23}$$

Where we defined $\rho_C = |k\rangle \langle k|_C$ and $S(\mathcal{D}(\rho_C)) = -\log\left(\frac{1}{N}\right)$. From the fourth to the fifth line we used the fact that given a spectral decomposition for a density matrix ρ it follows that $f(\rho) = f(\sum_i p_i |i\rangle \langle i|) = \sum_i f(p_i) |i\rangle \langle i|$, for the last line it was used that for sufficiently large N we have $N - 1 \approx N$ and that the von Neumann entropy is invariant under unitary operations. With this result Eq.(4.21) becomes

$$\begin{aligned} C_r(\mathcal{D}(\rho_S \otimes \rho_C)) &= S(\mathcal{D}(\rho_S)) + S(\mathcal{D}(\rho_C)) - S(\rho_S) - S(\mathcal{D}(\rho_C)) \\ &= S(\mathcal{D}(\rho_S)) - S(\rho_S) = A_G(\rho_S). \end{aligned} \tag{4.24}$$

As discussed the clock state inside the mechanism can be understood as a reference, if the size of that state sufficiently large, the internal coherence appears to be external coherence of ρ_S.

Proposition 1 shows that the relative entropy of coherence does not always represent the same phenomenon. The physical interpretation of such a measure takes into account how distinguishable the initial state of the system is with regards to the final desired state. Therefore for an arbitrary state ρ this “distance” to the set of incoherent states reflects on how close this arbitrary state is to being incoherent, which is interpreted here as a measure of total coherence. Without a doubt, the set of incoherent states is further than the set of block diagonal states when taking a general state ρ, which is neither incoherent nor block diagonal. This is contained in Proposition 1 as

$$C_r(\rho) \geq A_G(\rho), \tag{4.25}$$

following from the positivity of the relative entropy of coherence. Hence the relative entropy of coherence, representing the total coherence, is going to be an upper bound for the Holevo asymmetry, being also an upper bound for $C_r(\mathcal{D}(\rho))$, something expected for a measure of total coherence. This explains why the Holevo asymmetry is equivalent to the relative entropy, when a minimization is taken over all states that are invariant over a group action. Although any incoherent state is also going to be invariant over a group generated by the total Hamiltonian, the set of block diagonal states, is always closer to an arbitrary state ρ. This implies that the total coherence $C_r(\rho)$ given by how distinguishable a general state is from a group invariant state, is a completely different measure from A_G.

In terms of the PaW mechanism, the division of the total coherence in internal and external tells us what is the quantity responsible for the PaW clock to work. When applying the dephasing operation its action averages over the possible phases of the unknown time reference frame. Hence, this operation is going to eliminate any external coherence, if any, that the global state of the system has. The same cannot be said to the internal coherence, since it is defined between the subsystems as a relational degree of freedom, it is not erased even if we do not have access to a reference frame, therefore:

Proposition 2 *The internal coherence, the relative entropy of coherence with minimization from a block diagonal state $\mathcal{D}(\rho)$ to the set of incoherent states $\mathcal{I}$, is responsible for the proper working of the PaW clock. And it is given by*

$$C_r(\mathcal{D}(\rho)) = S(\Delta(\rho)) - S(\mathcal{D}(\rho)).$$

with $C_r(\mathcal{D}(\rho))$ being the relative entropy of coherence applied to the dephasing operation $\mathcal{D}(\rho)$ and $\Delta(\rho)$ as the closest incoherent state to $\mathcal{D}(\rho)$.

When evaluating this proposition it is important to understand the hypothesis taken with regards to the PaW clock. This mechanism is dependent on the conditional probabilities, as stated before. Thus, we expect these probabilities to give an indication of the performance of the clock in the model, especially in regards of its functioning. Even if the conditional probabilities do not describe all aspects of the model or the way in which we perform the measurements is not clarified, they are related to the principle by which the PaW clock works. Based on this principle it seems reasonable to say that for any state that renders a probability of agreement $\text{Prob}(\text{R}|\text{R}) = 0.5$, regardless of the subtleties of the process, when acquiring the information about clock time, the PaW clock will not work. In the same manner, if a state renders a probability of agreement $\text{Prob}(\text{R}|\text{R}) = 1$, we expect the clock to work near perfection. From this, it follows that any measure that is going to be necessary for the model to work (but not necessarily sufficient) must be zero when the clock does not work and maximum for the best clock.

Granted this, we can see that the internal coherence, as given by $C_r(\mathcal{D}(\rho))$, is necessary for the mechanism through direct calculation. For the Bell-diagonal states this measure admits an analytic form in terms of the triplet $\{c_1, c_2, c_3\}$

$$\begin{aligned} C_r(\mathcal{D}(\rho)) = &-\frac{1-c_3}{2}\log(1-c_3) \\ &+\sum_{i=1}^{2}\frac{(1+(-1)^i c_1+(-1)^i c_2-c_3)}{4}\log(1+(-1)^i c_1+(-1)^i c_2-c_3). \end{aligned} \tag{4.26}$$

When evaluated for the set with $c_1 = -c_2$ and the set $c_1 = c_2 = 0$, which corresponds to the sets with worst probabilities, this measure is always zero. For the case of perfect agreement, that is $c_1 = c_2 = 1$ and $c_3 = -1$, this measures is equal to 1. It turns out that this is not only the expected result, given our hypotheses, for the necessary measure for the PaW clock but also a very good indicator for the states which will improve the performance of PaW clock. However, there are other factors which are of importance.

Proposition 3 *For every family of parameters c_3 that is held constant, the internal coherence $C_r(\mathcal{D}(\rho))$ is greater for greater probability of agreement*[‡]*.*

By family we mean every group of states that are associated with a fixed c_3 for any c_1 and c_2. The only condition is that these parameters are within the range that returns

‡ This proposition is written with the convention that we adopted, which is that the desired probabilities are for agreement. As stated before a good clock can also be made with the convention for disagreement, in that case the with would read "..$C_r(\mathcal{D}(\rho))$ is greater for greater probability of disagreement".

physically acceptable density matrices for both the initial state ρ and the dephased state $\mathcal{D}(\rho)$, with the latter being composed by a different set of values. Fixing the parameter c_3 in a given number, an order where all states with more internal coherence yield the best clock results, in the form of better conditional probabilities, emerges. It is worth noting that there is one family of states, for which $c_3 = 0$, where the agreement between probabilities and values for the internal coherence is in perfect accord with the best and worst results.

We have seen that if the initial states have entanglement between the subsystems this did not affect the performance of the PaW clock, but does this remain valid if there is entanglement on the dephased state? After all it is also shown that the best result is obtained for the maximally entangled Bell state $|\psi^+\rangle$, which is also invariant under the dephasing operation, therefore we have a maximally entangled state as the dephased state. However the answer to this question is not necessarily. This can be seen by examining any state in the family that has $c_3 = 0$. We can calculate the entanglement for this family using the concurrence $\mathcal{C}(\rho)$ (93), a widely known measure for entanglement. For two qubits it has an explicit form

$$\mathcal{C}(\rho) = \max\left\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right\}, \tag{4.27}$$

where each λ_i is an eigenvalue of the matrix $\rho\tilde{\rho}$ in decreasing order and

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y). \tag{4.28}$$

Therefore, we need to obtain the eigenvalues of $\rho\tilde{\rho}$. These are:

$$\begin{aligned}
\lambda_1 &= \frac{c_1^2}{16} + \frac{c_1c_2}{8} - \frac{c_1c_3}{8} + \frac{c_1}{8} + \frac{c_2^2}{16} - \frac{c_2c_3}{8} + \frac{c_2}{8} \\
&+ \frac{c_3^2}{16} - \frac{c_3}{8} + \frac{1}{16}, \\
\lambda_2 &= \frac{c_1^2}{16} + \frac{c_1c_2}{8} + \frac{c_1c_3}{8} - \frac{c_1}{8} + \frac{c_2^2}{16} + \frac{c_2c_3}{8} - \frac{c_2}{8} \\
&+ \frac{c_3^2}{16} - \frac{c_3}{8} + \frac{1}{16}, \\
\lambda_3 &= \lambda_4 = \frac{c_3^2}{16} + \frac{c_3}{8} + \frac{1}{16}.
\end{aligned} \tag{4.29}$$

When setting $c_3 = 0$ we get

$$\begin{aligned}
\lambda_1 &= \frac{c_1^2}{16} + \frac{c_1c_2}{8} + \frac{c_1}{8} + \frac{c_2^2}{16} + \frac{c_2}{8} + \frac{1}{16}, \\
\lambda_2 &= \frac{c_1^2}{16} + \frac{c_1c_2}{8} - \frac{c_1}{8} + \frac{c_2^2}{16} - \frac{c_2}{8} + \frac{1}{16}, \\
\lambda_3 &= \lambda_4 = \frac{1}{16}.
\end{aligned} \tag{4.30}$$

Focusing on the first two eigenvalues, we see that they can be grouped as

$$\lambda_1 = \frac{(c_1 + c_2)^2}{16} + \frac{(c_1 + c_2)}{8} + \frac{1}{16} \tag{4.31}$$

and

$$\lambda_2 = \frac{(c_1+c_2)^2}{16} - \frac{(c_1+c_2)}{8} + \frac{1}{16}, \tag{4.32}$$

forming perfect squares. Therefore,

$$\begin{aligned} \mathcal{C}(\rho) &= \max\left\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right\} \\ &= \max\left\{0, \frac{c_1+c_2+1}{4} - \frac{c_1+c_2-1}{4} - \frac{1}{2}\right\} \\ &= \max\left\{0, \frac{1}{4} + \frac{1}{4} - \frac{1}{2}\right\} = 0. \end{aligned} \tag{4.33}$$

Hence, there is no entanglement between the subsystems of the dephased state and yet this family returns a non-zero probability of agreement, with best than random correlation, that is the PaW clock works without entanglement. Even in the original work, when dealing with density operators, and not wave functions, it is possible to see that the dephased state, which belongs to the same family that have $c_3 = 0$, is not entangled. One could still claim that entanglement is necessary for the case of pure states. Although it may be true, it is hard to justify this reasoning given that the pure states that yield maximum probabilities of agreement are states with maximum internal coherence. It does not seem that one can take the internal coherence out of the picture and still be left with a working PaW clock. Based on these results it really seems that entanglement is neither a sufficient nor necessary condition for the PaW clock to work.

There is an interesting feature of treating the PaW mechanism as we did. Let the space of a quantum state be $\left(C^d\right)^{\otimes N}$; this space carries representations of the unitary group $U(d)$ and the symmetric group S_N. By Schur-Weyl duality (94) there will be a decomposition of irreducible representations of these groups as

$$\left(C^d\right)^{\otimes N} \cong \bigoplus_{\lambda} V_\lambda \otimes P_\lambda \tag{4.34}$$

where the sum is taken over Young frames (95) and the spaces $\{V_\lambda\}$ and $\{P_\lambda\}$ are irreducible representations of the unitary and symmetric group respectively. It follows that the representation of each state is going to act irreducibly on their respective basis. For our case the dephasing operation in Eq. (4.4) will impose a $U(1)$-SSR, acting trivially on the elements of the symmetric subspace; therefore, the states on the space will not be perceived as invariant under permutations. For some elements changing the labels of the subsystems will change the global state, this can be translated to the possibility of distinguishing the system and the clock through the use of the symmetric subspace.

4.3 Internal coherence and work

To examine the relation between the proposed measure for internal coherence and extractable work from coherence let us consider the protocol proposed in Refs. (96–97).

These authors give a general protocol that could extract work from n copies of an initial state $\hat{\rho}$ given access to a heat bath composed of an unlimited number of qubits on the thermal state

$$\tau_B = \frac{e^{-\beta H_B}}{\mathcal{Z}}, \tag{4.35}$$

where H_B is the bath Hamiltonian and $\mathcal{Z}$ is the partition function $\mathcal{Z} = \mathrm{Tr}\left(e^{-\beta H_B}\right)$. The process consists in applying a dephasing operation to all the n copies of the state $\hat{\rho}$ yielding n states $\mathcal{D}(\hat{\rho})$ which were then converted, individually, into thermal states τ_B. The work produced in this process, which is shown to be equal to the difference in free energies $F(\rho) = \langle E\rangle_\rho - kTS(\rho)$, where $\langle E\rangle_\rho$ is the average energy, from the initial states $\hat{\rho}$ and the thermal states in the limit that $n \to \infty$, could them be stored in a system with a weight that acted like a battery. Then in the single shot version, using a single copy of the state, the work that could be extracted is given by

$$W_{Tot} = F(\mathcal{D}(\hat{\rho})) - F(\tau_B). \tag{4.36}$$

From this definition it follows that the total work that is extractable from the single shot regime of a state ρ in transforming it to a fully dephased state is given by

$$W(\rho) = F(\mathcal{D}(\rho)) - F(\Delta(\rho)), \tag{4.37}$$

which, under average energy conservation, can be easily demonstrated to be directly correlated with the internal coherence of the state

$$W(\rho) = kT\left(S(\Delta(\rho)) - S(\mathcal{D}(\rho))\right) = kTC_r(\mathcal{D}(\rho)). \tag{4.38}$$

This result agrees with the interpretation given in Ref. (91), where the work extractable from coherence W_{coh}, which can be seen as a lower bound of the internal coherence $C_r(\mathcal{D}(\rho))$, is defined.

Thermal operations, is another interesting framework to analyze. As a strict subset of the time covariant operations, the set of thermal operations only includes those unitaries for which the conservation of energy is guaranteed, hence enforcing an energy SSR. In this set, allowed transformations, that take one state to another, exhibit a phenomenon called *work locking*. (98)

The setting is very similar to what was described above for the protocol that extracts a certain amount of work from a state. For a given initial state $\hat{\rho}$ and a bath ρ_B we wish to perform the transformation

$$\hat{\rho} \otimes \rho_B \to \hat{\sigma} \otimes \rho'_B, \tag{4.39}$$

with the difference being that the aforementioned protocol demands that energy is conserved on average, hence this gives a less strict set of operations where conservation of energy is not demanded at all times. The phenomenon appears when considering the work that can be extracted from the dephased version, $\mathcal{D}(\hat{\rho})$. When this is the case the transformation is given by

$$\mathcal{D}(\hat{\rho}) \otimes \rho_B \rightarrow \mathcal{D}(\hat{\sigma}) \otimes \rho'_B, \tag{4.40}$$

which says that the work that is extractable from $\hat{\rho}$ is the same as that extractable from $\mathcal{D}(\hat{\rho})$, implying that the work from coherence is locked. (98) To "unlock" this work an ancillary coherent state can be used. So given two states with coherence ρ_1 and ρ_2 for which

$$\rho_1 = \rho_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \tag{4.41}$$

their dephased versions are going to output an incoherent $\mathcal{D}(\rho_1) = \mathcal{D}(\rho_2) = \tau_1$, and the work that can be extracted from τ_1 is zero. Now if using as an initial state the product state $\rho_1 \otimes \rho_2$, considering one of the states as a coherent ancilla, the dephased state of the product is

$$\mathcal{D}(\rho_1 \otimes \rho_2) = \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.42}$$

for which clearly $\mathcal{D}(\rho_1 \otimes \rho_2) \neq \tau_1 \otimes \tau_1$, making this state one that can be used to extract nonzero work from coherence.

The role of internal coherence must be clear at this point, and also allows a different physical interpretation with regards to the role of the ancillary state used as a "reference" to unlock the work from ρ_1. When dealing with individual systems $C_r(\mathcal{D}(\rho)) = 0$ and therefore $C_r(\rho) = A_G(\rho)$; the state representing a system will carry only extrinsic properties related to how it was produced. Hence, any coherence present in this individual state is assigned the role of external coherence, which cannot be done to the internal coherence, as it is a relational property. Therefore, when working with thermal operations the information of this external coherence is lost, resulting in an incoherent state. To enable work extraction from coherence the ancilla is used, not as a reference for the initial state, but as a way of generating a relative phase between subsystems. This phase can be viewed as a remnant of the extrinsic information carried by state and ancilla about the reference frame in which they were prepared, which allows a relational phase to be established, therefore generating internal coherence. That is why the work can be unlocked with as few as a two states, which is the minimum necessary to generate internal coherence.

4.4 More than qubits

An interesting approach to recover time from within quantum mechanics, was done by David Pegg. (39) Here we briefly review this approach in the light of what we showed in the present work. He started by considering a vector that would represent what is called the *totality of physical reality*, where this vector was the zero energy eigenstate of the Hamiltonian of the Universe. Such condition restricts the global state to be invariant under the dephasing operation, Eq. (4.4), a condition akin to that of the Page-Wootters model. The zero energy eigenstate was then shown to be proportional to $\sum_{j=0}^{s} |\phi_j\rangle$ where s is related to the dimension, $s+1$, of the Hilbert space, and the states are connected through the Hamiltonian of the Universe

$$|\phi_m\rangle = \exp[-iHm\beta]\,|\phi_0\rangle\,, \tag{4.43}$$

for a constant β. The concept of time is then recovered by splitting each possible state of the Universe $|\phi_m\rangle$ as a system and a clock $|\phi_m\rangle = |C_m\rangle\,|S_m\rangle$. In this construction every possible state of the Universe is then associated with every time as measured internally by the clock. This approach is very similar in nature to what was expected by Wootters (31) for an N particle universe.

The conditions required for Pegg's mechanism were almost identical to the conditions that we used to draw the conclusion with regards to internal coherence relation to the emergence of time. Hence a very similar approach could be used to study this proposal: Extend the possible states to encompass not only a pure global state, project those to the zero energy eigenstate, choose at least a bipartition of the Universe to work as a clock, since there is no requirement set that the states corresponding to the system should be orthogonal to each other, and compute the probabilities. Thanks to the generality of the relative entropy of coherence our proposed measure is not restricted to qubits, it can measure the internal coherence of a system with d dimension (qudits), and could in principle be used to test the best global states for the model.

5 ENTER THE INTERACTION

We discussed the many advances that were achieved in this formulation. Among them, they accomplished to resolve criticisms about the propagators of the formulation (22), showed causal relation on the change in reference frames (41), established limits on the precision on constructing an observable to measure time. (30) A point in common for these advances is the use of idealized quantum clocks. Even though such clocks are usually sufficient to extract the physics of the problem, we can wonder what would be the case for a more realistic scenario where we rely on a non-ideal clock, i.e., a system with finite dimension and states which are not necessarily orthogonal. Although there are several models proposed for the construction of quantum clocks, this work is especially interested in a clock model that mimics several properties of the ideal clock while having finite dimension. Such a clock was introduced in section 3.3, the quasi-ideal clock.

5.1 Time non-local Schrödinger equation

We start by briefly revising previous work (49) describing the effects of interaction in the Page-Wootters model. The question is straightforward, given the inevitability of interaction between the two parts of the universe, what happens if we introduce an interaction? This comes in a change in the Hamiltonian, that now, is going to have an extra term

$$H = H_S + H_C + H_I. \tag{5.1}$$

However it is still demanded that the total Hamiltonian obeys the constraint, hence

$$H \left|\Psi\right\rangle = (H_S + H_C + H_I) \left|\Psi\right\rangle = 0, \tag{5.2}$$

where H_S, H_C and H_I are the system, clock and interaction Hamiltonians, respectively, obtaining the same Wheeler-DeWitt-like equation. Defining an initial clock state $|0\rangle$, demanding a state evolution between these states to be

$$|\tau\rangle = e^{-iH_C\tau} \left|0\right\rangle, \tag{5.3}$$

and conditioning the global state, they demonstrate that we can still have a Schrödinger-like evolution. More specifically the system state will obey a time non-local Schrödinger equation

$$i\frac{d}{d\tau} \left|\psi_S(\tau)\right\rangle = H_S \left|\psi_S(\tau)\right\rangle + \int d\tau' K(\tau,\tau') \left|\psi_S(\tau')\right\rangle, \tag{5.4}$$

were $K(\tau,\tau') = \langle\tau| H_I |\tau'\rangle$ is seen as the kernel connected to the interaction Hamiltonian. Given that this equation is non-local in time, it is implied that to verify the solution for this equation it is required knowledge of the system state at all times. (49)

5.2 Quasi-Ideal clock again

The result obtained in Eq.(5.4), as many others, employ the use of ideal clocks that in the context of the Page-Wootter's mechanism is usually considered to be infinite-dimensional, possessing a distinguishable basis of time states and being associated to a time observable $\mathcal{T}$. Here choose a more realistic model which was presented previously the quasi-ideal clock. For convenience we will, briefly, recapitulate the core idea and definition of this clock model.

The QIC is based on the SWP clock, which can be seen as a quantum rotor (99) having a Hamiltonian

$$H_C = \sum_{j=0}^{d-1} \omega j \left| E_j \right\rangle \left\langle E_j \right| , \tag{5.5}$$

where d is the dimension of its Hilbert space. The SWP clock states form an orthonormal set of states $\{|\theta_k\rangle\}$ for $k = 0, 1, \ldots, d-1$ which is dubbed the time basis. These states are connected to the energy eigenstates through a discrete Fourier transform

$$|\theta_k\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{-i2\pi jk/d} \left| E_j \right\rangle , \tag{5.6}$$

and evolve according to $e^{-iH_C T/d} |\theta_k\rangle = |\theta_{k+1}\rangle$, thus establishing regular, integer, time intervals $t_k = (T/d)k$, such that the periodic condition $e^{-iH_C T} |\theta_k\rangle = |\theta_{\text{ mod }(k,d)}\rangle$ is satisfied.

The QIC is defined as a coherent superposition of time states (79)

$$|\psi_{QI}(k_0)\rangle = \sum_{k \in S_d(k_0)} \psi(k_0; k) \left| \theta_k \right\rangle , \tag{5.7}$$

where $S_d(k_0)$ is a set of d consecutive integers centered about k_0, which is considered the initial time position of the QIC, and ωj_0 is the average energy of the clock and $\psi(k_0; k)$ is the analytic Gaussian function. The equivalent relation to Eq.(5.3) is

$$e^{-iH_C t} |\psi_{QI}(k_0)\rangle = |\psi_{QI}(k_0 + td/T)\rangle + |\varepsilon\rangle , \tag{5.8}$$

where T is the period of the clock, $|\varepsilon\rangle$ is an error term composed of an error in changing the mean position of the clock, i.e., passing from k_0 to $k_0 + 1$ and an error in changing the consecutive integers, i.e., $S(k_0)$ to $S(k_0 + 1)$, whose norm is exponentially small with dimension. Differently from the SWP clock, the quasi-ideal states are continuous, in the sense that its evolution holds for arbitrarily small time intervals, where $t \in \mathbb{R}$.

5.3 Relative state for the Quasi Ideal clock

From now on, without loss of generality, we choose the initial state of the clock to be centered around $k_0 = 0$ and rewrite the evolution, so that Eq.(5.8) become

$$e^{-i\tau H_C} |\psi_{QI}(0)\rangle = |\psi_{QI}(\tau d/T)\rangle + |\varepsilon\rangle . \tag{5.9}$$

We wish to define the relative state of the system as being a slice of the global state in respect to the QIC at time τ

$$|\psi_S(\tau d/T)\rangle_e = \langle\psi_{QI}\,(\tau d/T)\,|\Psi\rangle\,. \tag{5.10}$$

Although we are defining the system state with the parameter $\tau d/T$, from here on we are going to make the change $\tau d/T \to \tau$, so that all results will be written in terms of τ.

The subscript in Eq.(5.10) indicates that the state is an effective state for the system as can be seen in the equation below

$$\begin{aligned} |\psi_S(\tau)\rangle_e &= \frac{1}{T}\int_0^T d\tau' \,\langle\psi_{QI}\,(\tau)\,|\psi_{QI}\,(\tau')\rangle\,|\psi_S(\tau')\rangle \\ &= \frac{1}{T}\int_0^T d\tau' F_{QI}(\tau-\tau')\,|\psi_S(\tau')\rangle\,. \end{aligned} \tag{5.11}$$

We can show that the natural choice for the global state used above is connected to the global state using the SWP clock, which has essentially the same construction that was provided in Ref.(100).

Another aspect of Eq.(5.10) worth noting is that we consider, here on, that the dimension of the clock is large enough that we can ignore the error on the evolution given by Eq.(5.8) considering $\langle\psi_{QI}(\tau)|\Psi\rangle \approx \langle\psi_{QI}(0)|\,e^{i\tau H_C}\,|\Psi\rangle$. Thus, using the relative state definition we can obtain a slightly different form for the effective system state, which is going to be useful

$$\begin{aligned} |\psi_S(\tau)\rangle_e &= \sum_{k\in S_d(\tau)} \psi^*\,(\tau;k)\,\langle\theta_k|\Psi\rangle \\ &= \frac{1}{\sqrt{d}}\sum_{k\in S_d(\tau)} \psi^*\,(\tau;k)\,|\psi_S(k)\rangle\,, \end{aligned} \tag{5.12}$$

where we used that $|\psi_S(k)\rangle = \sqrt{d}\,\langle\theta_k|\Psi\rangle$. We can see that, written in this way, the effective system state is a superposition of the system state conditioned to the time basis $|\psi_S(k)\rangle$. The relative state $|\psi_S(k)\rangle$ is not an effective version as in Eq.(5.10), due to the fact that the time basis form a complete distinguishable basis. Here we will repeat this construction with a more appropriate notation and then present the derivation for our global state.

We start with a generic state for the universe

$$|\Psi\rangle = \sum_{n=0}^{d-1}\sum_{m=0}^{d_S-1} p_{n,m}\,|E_n\rangle\,|E_m\rangle\,, \tag{5.13}$$

that can be written as

$$\begin{aligned}
|\Psi\rangle &= \sum_{m=0}^{d_S-1} \tilde{p}_m \,|E=-E_m\rangle\,|E_m\rangle \\
&= \sum_{m=0}^{d_S-1}\sum_{k=0}^{d-1} \tilde{p}_m \,|\theta_k\rangle\,\langle\theta_k|E=-E_m\rangle\,|E_m\rangle \\
&= \frac{1}{\sqrt{d}}\sum_{m=0}^{d_S-1}\sum_{k=0}^{d-1} \tilde{p}_m e^{-i2\pi jk/d}\,|\theta_k\rangle\,|E_m\rangle \\
&= \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1} |\theta_k\rangle\,|\psi_S(k)\rangle\,, \qquad (5.14)
\end{aligned}$$

where in the second line we used the resolution of the identity for time states. In order to reach our result we use the covariant POVM generated by the Quasi-Ideal states (26), that in the limit of large dimension is approximately $\{P_{QI}(\tau) \coloneqq U(\tau)\,|\psi_{QI}(0)\rangle\,\langle\psi_{QI}(0)|\,U^\dagger(\tau)\}_{\tau\in[0,T]}$, hence ,

$$\begin{aligned}
|\Psi\rangle &= \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}\frac{1}{T}\int_0^T d\tau\,|\psi_{QI}(\tau)\rangle\,\langle\psi_{QI}(\tau)|\theta_k\rangle\,|\psi_S(k)\rangle \\
&= \frac{1}{\sqrt{d}T}\sum_{k=0}^{d-1}\int_0^T d\tau\,|\psi_{QI}(\tau)\rangle\,\psi^*(\tau;k)\,|\psi_S(k)\rangle \\
&= \frac{1}{T}\int_0^T d\tau\,|\psi_{QI}(\tau)\rangle\,\frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}\psi^*(\tau;k)\,|\psi_S(k)\rangle\,, \qquad (5.15)
\end{aligned}$$

using the definition given in Eq.(5.12) we get

$$|\Psi\rangle = \frac{1}{T}\int_0^T d\tau\,|\psi_{QI}(\tau)\rangle\,|\psi_S(\tau)\rangle_e\,. \qquad (5.16)$$

The state we obtain is not exactly $|\psi_S(\tau)\rangle$, the one we would like to prepare, but an approximation that is dependent on how well the QIC can distinguish the clock states at different times. This effect is due to the non-ideal condition of the clock: time will be known with a certain accuracy that depends on the function $F_{QI}(\tau-\tau')$. In order for this function to become sharply peaked around τ, and hence increasing the accuracy of the clock, we need to time-squeeze the QIC, which is accomplished for a regime where $\sigma < \sqrt{d}$. Thereby, we reduce the uncertainty in time readings in exchange for an increased uncertainty in energy, turning the QIC closer to a time state. This seems to come in detriment of quantum control (79): For squeezed states, the QIC becomes more vulnerable to the back-reaction of implementing unitaries, i.e., its errors do not decay exponentially with dimension any longer. This appears to imply that if it were to be implemented a quantum control, an unitary which preserves energy, in this mechanism, it would be necessary to introduce another clock for that purpose only. Here on we are going to refer to the effective system state as only the system state, for simplicity.

5.4 Equation of motion

To obtain the equation of motion for the mixed-state system we first derive an equation of motion for the pure-state system with QIC, which we present in the form of a lemma

Lemma 6 *Given a QIC state evolved to time τ, its time evolution will be given by*

$$\frac{d}{d\tau}\left|\psi_{QI}(\tau)\right\rangle = -i\frac{T}{d}H_C\left|\psi_{QI}(\tau)\right\rangle - \left|\varepsilon'\right\rangle, \tag{5.17}$$

with error decreasing exponentially with dimension

$$\|\left|\varepsilon'\right\rangle\| \leq \mathcal{O}(poly(d)e^{\frac{-\pi d}{4}}). \tag{5.18}$$

Proof: the idea is to use Lemma 3 to obtain the infinitesimal evolution of the QIC, and then employ the definition of derivative

$$\begin{aligned}
\frac{d}{d\tau}\left|\psi_{QI}(\tau)\right\rangle &= \sum_k \frac{d}{d\tau}\psi(\tau;k)\left|\theta_k\right\rangle \\
&= \sum_k \left[\lim_{\delta\to 0}\frac{\psi(\tau+\delta;k)-\psi(\tau;k)}{\delta}\right]\left|\theta_k\right\rangle \\
&= \lim_{\delta\to 0}\frac{\sum_k \psi(\tau+\delta;k)\left|\theta_k\right\rangle - \sum_k \psi(\tau;k)\left|\theta_k\right\rangle}{\delta} \\
&= \lim_{\delta\to 0}\frac{\left|\psi_{QI}(\tau)\right\rangle - i\delta(T/d)H_C\left|\psi_{QI}(\tau)\right\rangle - \left|\varepsilon\right\rangle_i - \left|\psi_{QI}(\tau)\right\rangle}{\delta} \\
&= -\frac{iTH_C}{d}\left|\psi_{QI}(\tau)\right\rangle - \lim_{\delta\to 0}\frac{\left|\varepsilon\right\rangle_i}{\delta}.
\end{aligned} \tag{5.19}$$

The error is will be something like $\left|\varepsilon\right\rangle_i = \sum_{k\in S_d(k_0)}\left[\delta(\varepsilon_1+\varepsilon_2+\varepsilon_3)+\delta^2 C\right]\left|\theta_k\right\rangle$, where the form of each error can be found in section 3.3.1, therefore its limit will be

$$\begin{aligned}
\lim_{\delta\to 0}\frac{\left|\varepsilon\right\rangle_i}{\delta} &= \lim_{\delta\to 0}\sum_{k\in S_d(k_0)}\frac{\left[\delta(\varepsilon_1+\varepsilon_2+\varepsilon_3)+\delta^2 C\right]}{\delta}\left|\theta_k\right\rangle \\
&= \lim_{\delta\to 0}\sum_{k\in S_d(k_0)}\left[(\varepsilon_1+\varepsilon_2+\varepsilon_3)+\delta C\right]\left|\theta_k\right\rangle \\
&= \sum_{k\in S_d(k_0)}(\varepsilon_1+\varepsilon_2+\varepsilon_3)\left|\theta_k\right\rangle \\
&= \sum_{k\in S_d(k_0)}\varepsilon'\left|\theta_k\right\rangle = \left|\varepsilon'\right\rangle.
\end{aligned} \tag{5.20}$$

Hence,

$$\|\left|\varepsilon'\right\rangle\| \leq \begin{cases} 2\pi A\left(2\sqrt{d}\left(\frac{1}{2}+\frac{1}{2\pi d}+\frac{1}{1-e^{-\pi}}\right)e^{-\frac{\pi d}{4}}+\frac{1}{2}+\frac{1}{2\pi d}+\frac{1}{1-e^{-\pi}}\right)e^{-\frac{\pi d}{4}} & \text{if } \sigma=\sqrt{d} \\ 2\pi A\left(2\sigma\left(\frac{\alpha_0}{2}+\frac{1}{2\pi\sigma^2}+\frac{1}{1-e^{-\pi\sigma^2\alpha_0}}\right)e^{-\frac{\pi\sigma^2\alpha_0}{4}}+\left(\frac{1}{2\pi d}+\frac{d}{2\sigma^2}+\frac{1}{1-e^{-\frac{\pi d}{\sigma^2}}}+\frac{1}{1-e^{-\frac{\pi d^2}{\sigma^2}}}\right)e^{-\frac{\pi d^2}{4\sigma^2}}\right) & \text{otherwise} \end{cases} \tag{5.21}$$

which also decreases exponentially with dimension.

Then, the Schrödinger-like equation for the system will be

$$\begin{aligned} i\frac{d}{d\tau}\left|\psi_S(\tau)\right\rangle_e &= i\frac{d}{d\tau}\left\langle\psi_{QI}\left(\tau\right)|\Psi\right\rangle \\ &= \left(-\frac{T}{d}\left\langle\psi_{QI}\left(\tau\right)\right|H_C + i\left\langle\varepsilon'\right|\right)\left|\Psi\right\rangle \\ &= -\frac{T}{d}\left\langle\psi_{QI}\left(\tau\right)\right|H_C\left|\Psi\right\rangle + i\left\langle\varepsilon'|\Psi\right\rangle \\ &= \frac{T}{d}\left\langle\psi_{QI}\left(\tau\right)\right|\left(H_S + H_I - H\right)\left|\Psi\right\rangle + \left|\varepsilon_s\right\rangle \\ &= H_S\frac{T}{d}\left|\psi_S(\tau)\right\rangle_e + \frac{T}{d}\left\langle\psi_{QI}\left(\tau\right)\right|H_I\left|\Psi\right\rangle + \left|\varepsilon_s\right\rangle. \end{aligned} \tag{5.22}$$

Our interest is in obtaining an equation of motion for a mixed system state with a gravitationally induced interaction. This interaction can be obtained from the Newtonian gravitational potential energy $P = -Gm_Cm_S/x$, where we already written it in terms of the mass of the clock and the system, m_C and m_S, respectively. Therefore, the interaction Hamiltonian will be

$$H_I = -\mathbb{G}H_S \otimes H_C, \tag{5.23}$$

where we defined $\mathbb{G} \coloneqq G/c^4x$. We proceed by obtaining a closed form for the Schrödinger-like equation, starting by calculating the second term of the RHS of Eq.(5.22)

$$\begin{aligned} \left\langle\psi_{QI}(\tau)\right|H_I\left|\Psi\right\rangle &= -\mathbb{G}H_S\left\langle\psi_{QI}\left(\tau\right)\right|H_C\left|\Psi\right\rangle \\ &= \mathbb{G}H_S\left[i\frac{d}{T}\frac{d}{d\tau}\left\langle\psi_{QI}\left(\tau\right)\right| + \frac{d}{T}i\left\langle\varepsilon'\right|\right]\left|\Psi\right\rangle \\ &= i\mathbb{G}H_S\left[\frac{d}{T}\sum_k\frac{d}{d\tau}\psi^*\left(\tau;k\right)\left\langle\theta_k|\Psi\right\rangle + \frac{d}{T}i\left\langle\varepsilon'|\Psi\right\rangle\right] \\ &= i\mathbb{G}H_S\left[\frac{d}{T\sqrt{d}}\sum_k\frac{d}{d\tau}\psi^*\left(\tau;k\right)\left|\psi_S(k)\right\rangle + \frac{d}{T}\left|\varepsilon_s\right\rangle\right], \end{aligned}$$

hence,

$$\left\langle\psi_{QI}\left(\tau\right)\right|H_I\left|\Psi\right\rangle = i\mathbb{G}H_S\frac{d}{T}\frac{d}{d\tau}\left|\psi_S(\tau)\right\rangle_e + \mathbb{G}H_S\frac{d}{T}\left|\varepsilon_s\right\rangle. \tag{5.24}$$

With this result we go back to Eq.(5.22), getting

$$\begin{aligned} &i\frac{d}{d\tau}\left|\psi_S(\tau)\right\rangle = H_S\frac{T}{d}\left|\psi_S(\tau)\right\rangle_e + i\mathbb{G}H_S\frac{d}{d\tau}\left|\psi_S(\tau)\right\rangle_e + \mathbb{G}H_S\frac{d}{T}\left|\varepsilon_s\right\rangle + \left|\varepsilon_s\right\rangle \\ &= H_S\frac{T}{d}\left|\psi_S(\tau)\right\rangle_e + \mathbb{G}H_S\left[H_S\frac{T}{d}\left|\psi_S(\tau)\right\rangle_e + i\mathbb{G}H_S\frac{d}{d\tau}\left|\psi_S(\tau)\right\rangle_e + \mathbb{G}H_S\frac{d}{T}\left|\varepsilon_s\right\rangle + \left|\varepsilon_s\right\rangle\right] + \mathbb{G}H_S\frac{d}{T}\left|\varepsilon_s\right\rangle + \left|\varepsilon_s\right\rangle \\ &= H_S\frac{T}{d}\left|\psi_S(\tau)\right\rangle_e + \mathbb{G}H_S^2\frac{T}{d}\left|\psi_S(\tau)\right\rangle_e + \left(\mathbb{G}H_S + \mathbb{G}H_S\frac{d}{T} + 1\right)\left|\varepsilon_s\right\rangle + \mathcal{O}(\mathbb{G}^2) \\ &= H_S\frac{T}{d}\left|\psi_S(\tau)\right\rangle_e + \mathbb{G}H_S^2\frac{T}{d}\left|\psi_S(\tau)\right\rangle_e + \left|\varepsilon_{sg}\right\rangle + \mathcal{O}(\mathbb{G}^2), \end{aligned} \tag{5.25}$$

The error $|\varepsilon_{sg}\rangle$ will be exponentially small in dimension if the Hamiltonian of the system H_S is bounded and that the system state in relation to the time basis is normalized, i.e., $\langle\psi_S(k)|\psi_S(k)\rangle = 1$. Being both reasonable demands, therefore, as discussed previously, with an appropriate dimension size we can neglect it. Disregarding terms of second order and above of $\mathbb{G}$, we obtain our approximate Schrödinger equation

$$i\frac{d}{d\tau}\left|\psi_S(\tau)\right\rangle_e = H_S\frac{T}{d}\left|\psi_S(\tau)\right\rangle_e + \mathbb{G}H_S^2\frac{T}{d}\left|\psi_S(\tau)\right\rangle_e. \tag{5.26}$$

Now we consider a mixture of the pure system states in Eq.(5.11) to be

$$\rho_S(\tau) = \sum_\iota p_\iota \left|\psi_S^\iota(\tau)\right\rangle_e \left\langle\psi_S^\iota(\tau)\right|, \tag{5.27}$$

where $\rho_S(\tau) = U(\tau)\rho_S(0)U^\dagger(\tau) = \sum_\iota p_\iota \left|\psi_S^\iota(\tau)\right\rangle_e \left\langle\psi_S^\iota(\tau)\right|$. By taking the derivative of this state and using Eq.(5.26) we find

$$\frac{d\rho_S(\tau)}{d\tau} = -i\frac{T}{d}[H_S, \rho_S(\tau)] - i\mathbb{G}\frac{T}{d}[H_S^2, \rho_S(\tau)]. \tag{5.28}$$

We readily see that the effect of the gravitational interaction appears in the form of a second term with the system Hamiltonian squared, analogously to the Schrödinger-like equation. Therefore, we can consider the effect of the Hamiltonian $H_d = \frac{T}{d}H_S\left(\mathbb{1} + \mathbb{G}H_S\right)$, which seems to be analogous to a time dilation effect for quantum clocks in the presence of gravity. (101) Here we only considered the interaction mediated by gravity without explicitly introducing time dilation. Then, we see that the interaction between clock and system state does not induce a non-unitary behavior in the evolution of the system state, provided that Eq.(5.26) holds. Therefore, as in the ideal clock case (see Appendix C) there is no decoherence.

To obtain Eq.(5.28), the evolution is considered with negligible clock error. Now we continue by considering the effect of the small contribution given by the error of the QIC which we wish to represent by a certain potential. We start by defining the normalized version of the error $\left|\varepsilon_{sg}\right\rangle = \sqrt{\left\langle\varepsilon_s|\varepsilon_s\right\rangle}\left|\varepsilon'_{sg}\right\rangle$, then, from $\left|\varepsilon\right\rangle_s := i\left\langle\varepsilon'|\Psi\right\rangle$, we have

$$\begin{aligned}
\left|\varepsilon_{sg}\right\rangle &:= \left(1 + \mathbb{G}H_S + \mathbb{G}H_S\frac{d}{T}\right) i \sum_{k\in S_d(0)} \varepsilon' \left\langle\theta_k|\Psi\right\rangle \\
&= \mathbb{G}_H\varepsilon' \sum_{k\in S_d(0)} \left\langle\theta_k|\Psi\right\rangle \\
&= \frac{\mathbb{G}_H\varepsilon'}{\sqrt{d}} \sum_{k\in S_d(0)} \left|\psi_S(k)\right\rangle.
\end{aligned} \tag{5.29}$$

With this, we define

$$V(\tau) := \sqrt{\left\langle\varepsilon_s|\varepsilon_s\right\rangle}\left|\varepsilon_{sg}\right\rangle{}_e\!\left\langle\psi_S(\tau)\right|, \tag{5.30}$$

in a way that Eq.(5.25), disregarding terms of second order of $\mathbb{G}$, becomes

$$i\frac{d}{d\tau}\left|\psi_S(\tau)\right\rangle_e = H_S\frac{T}{d}\left|\psi_S(\tau)\right\rangle_e + \mathbb{G}H_S^2\frac{T}{d}\left|\psi_S(\tau)\right\rangle_e + V(\tau)\left|\psi_S(\tau)\right\rangle_e. \tag{5.31}$$

From Eq.(5.30), we can see that the potential can be written as

$$V(\tau) = \frac{\mathbb{G}_H \varepsilon'}{\sqrt{d}} \sum_{k,k'} \psi(\tau; k') \left|\psi_S(k)\right\rangle \left\langle\psi_S(k')\right| = \frac{\mathbb{G}_H \varepsilon'}{\sqrt{d}} V_{k,k'}(\tau). \tag{5.32}$$

Let us use the following transformations

$$\tilde{\rho}(\tau) = e^{iH_d\tau} \rho_S(\tau) e^{-iH_d\tau} \tag{5.33}$$

and

$$\tilde{V}(\tau) = e^{iH_d\tau} V(\tau) e^{-iH_d\tau}. \tag{5.34}$$

Then, the initial state of the system will coincide with the transformed initial state $\tilde{\rho}(0) = \rho_S(0)$, and we obtain

$$\frac{d\tilde{\rho}(\tau)}{d\tau} = -i[\tilde{V}(\tau), \tilde{\rho}(\tau)]. \tag{5.35}$$

Integrating the above equation and iterating it we get,

$$\frac{d\tilde{\rho}(\tau)}{d\tau} = -i[\tilde{V}(\tau), \tilde{\rho}(0)] - \int_0^\tau [\tilde{V}(\tau), [\tilde{V}(s), \tilde{\rho}(s)]] ds, \tag{5.36}$$

an equation that is only dependent on the density operator of the system of interest. We assume weak coupling between clock and system due to the nature of the assumed potential, i.e., gravitational, and expand the transformed density operator around τ, i.e.,

$$\tilde{\rho}(s) = \tilde{\rho}(\tau) + (s-\tau)\frac{d\tilde{\rho}(\tau)}{d\tau} + \mathcal{O}((s-\tau)^2). \tag{5.37}$$

It is easy to see that the derivatives will contribute with higher orders terms, retaining only terms up to second order we reach

$$\frac{d\tilde{\rho}(\tau)}{d\tau} = -i[\tilde{V}(\tau), \rho(0)] - \int_0^\tau [\tilde{V}(\tau), [\tilde{V}(s), \tilde{\rho}(\tau)]] ds. \tag{5.38}$$

Undoing the transformation, we find

$$\frac{d\rho_S(\tau)}{d\tau} = -i[H_d, \rho_S(\tau)] - i[V(\tau), U^\dagger \rho_S(0) U] - \int_0^\tau [V(\tau), [V(s), \rho(\tau)]_{\tau-s}] ds, \tag{5.39}$$

where

$$[V(s), \rho(\tau)]_{t-s} = e^{-iH_d(\tau-s)}[V(s), \rho(\tau)] e^{iH_d(\tau-s)}.$$

Finally, making use of the Baker-Campbell-Hausdorff formula (102),

$$U^\dagger \rho_S(0) U = \rho_S(0) + i\tau[H_d, \rho_S(0)] - \frac{\tau^2}{2}[H_d, [H_d, \rho_S(0)]] + \mathcal{O}(\tau^3),$$

to expand the second term of the equation above. For a sufficiently small time step τ, we may truncate this series into first order in τ, finding

$$\frac{d\rho_S(\tau)}{d\tau} = -i[H_d, \rho_S(\tau)] - i[V(\tau), \rho_S(0)] + \tau[V(\tau), [H_d, \rho_S(0)]] - \int_0^\tau [V(\tau), [V(s), \rho(\tau)]_{\tau-s}] ds \tag{5.40}$$

This is the equation of motion for the mixed state conditioned on clock time. We can see that apart the first term, which describes unitary evolution with a dilated time Hamiltonian, as seen in Eq.(5.26), we obtain a fourth term associated with decoherence of the system state. As shown in Eq.(5.32) the potential constructed, can also be written as

$$V(\tau) = \frac{\mathbb{G}_H \varepsilon'}{\sqrt{d}} V_{k,k'}(\tau). \tag{5.41}$$

If we expand this potential coefficient in order to show the influence of the error and gravitational constant we get

$$\begin{aligned} V(\tau) &\propto \left(1 + \mathbb{G}H_S + \mathbb{G}H_S \frac{d}{T}\right)^2 \\ &= \left[1 + \mathbb{G}H_S\left(1 + \frac{d}{T}\right) + \left[\mathbb{G}H_S\left(1 + \frac{d}{T}\right)\right]^2\right]. \end{aligned} \tag{5.42}$$

For this potential, the energy given by the Hamiltonian is going to be the energy of the system state $|\psi_S(k)\rangle$, the system state related to the time basis. Then, the decoherence term will have contributions of the order of energy and squared energy, differently to the equation of motion obtained in Ref.(101), which has a dependence only in respect to the square of energy. We also note that in the case that the energy $|\psi_S(k)\rangle$ is negligible we still have decoherence, due to the non-ideal nature of the clock. Moreover, in Eq.(5.40) there are two new terms, the second and the third ones, both dependent on the initial system state $\rho_S(0)$. Given the dependence on initial conditions this shows that this equation of motion is non-linear in nature, differently from other results, in the context of gravitation interaction. (101, 103–104)

We should emphasize that for our case, the non-unitary and non-linear effects arise due to the non-ideal nature of the chosen clock model, and the "idealness" can be related to its dimension, showing the role that the size of the clock plays. The smaller its size the more pronounced these new terms become, while the larger it is, more similar to an ideal clock and therefore less noticeable non-unitary/non-linear behaviour.

6 CONCLUSION

In this thesis we explored the Page-Wootters conditional probability interpretation through modern lens. We showed a split of the total coherence of a state into internal and external, as measured by the relative entropy of coherence. While the external coherence is directly connected to a framework for unspeakable information, and therefore connected to several tasks where the use of asymmetry theory can be employed, the internal coherence seems to be only relative to speakable information, thus it is not useful for such tasks. Nevertheless, internal coherence seems to be a very important quantity. This was shown by investigating the PaW mechanism using two-qubit Bell-diagonal states. Considering the conditional probabilities as indicative of the performance and working of the PaW clock, we put forward evidence that internal coherence is a necessary quantity for the mechanism and not entanglement, as is usually believed. By investigating the parameters of the triplet of correlation $\{c_1, c_2, c_3\}$ we found that the lack of internal coherence in the system is always associated with the worst possible conditional probabilities of agreement between the clock and the system, and a maximum value of internal coherence is associated with the best possible probabilities of agreement. It was noted that internal coherence can also be used as an indicator for which states are going to give the best probabilities of agreement, given that a family of states related by the parameter c_3 is fixed, where greater internal coherence correlates with a greater probability of agreement.

The internal coherence proved important also when dealing with quantum thermodynamics. In this case when the allowed operations and resources are carefully accounted for, in a way that no coherence can be sneaked in and that conservation of energy is always guaranteed, basically the same condition where an energy SSR takes place. The work that can be extracted in a singleshot regime is directly connected to the internal coherence between the states. This observation is useful when examining the phenomena of work locking, because it allows another interpretation of the use of additional copies to unlock the work of coherence. Instead of seeing the additional copies acting as frames of reference that provide an orientation which alleviates the constraints imposed by the SSR, adding copies to the initial state is interpreted here as a means to create internal coherence between the subsystems, which in turn can be used to extract work.

We also further develop on the Page-Wootters formulation by examining how the qualities of the clock, here considered as non-idealness, influenced the dynamics of a general mixed system state when there is an interaction between clock and system. To this end, we utilized the quasi-ideal clock, a finite clock that can approximate an ideal clock really well, being continuous for any arbitrarily small time value.

When the considered interaction is intermediate through gravity, we obtain an equation of motion that in the limit of very large dimension, when we can neglect the errors of the quasi-ideal clock, the evolution is unitary, and interestingly, containing a term that appears to indicate a time dilation effect. We note here that we do not consider the states to be in a superposition of proper times.

This case of high enough dimension is akin to considering that the quasi-ideal clock behaves as an ideal clock. Non-unitarity in the equation of motion is achieved when we do account for those errors, finding terms that resemble decoherence. We also obtained two additional terms dependent on the system state's initial conditions, making the equation to be non-linear. These two additional terms are distinct, one resembling unitary dynamics in relation to the perturbation and the other as decoherence involving the time dilated Hamiltonian. It is noteworthy that non-unitarity here can be connected to the dimension of our finite clock. Meanwhile, an infinite dimension will be akin to have an ideal clock, not having a large enough dimension will contribute to decoherence of the clock and attenuated non-unitary effects on the system state.

There are several ways in which the results presented here can be given continuation, one such venue would be to explore further the introduced measure of internal coherence. In the context of the Page-Wootters formalism it is interesting that can be shown, making use of a connection among relational approaches, that some important quantities, such as the conditional probabilities, are indeed gauge invariant(24), showing that they do not violate the constraints of the theory. However in a similar manner entanglement can be shown to not be gauge invariant, our results seem to indicate that this would be the case. It is interesting to show that internal coherence is gauge invariant, a path that can possibly illuminate the meaning of the families of states for which we get good results. Another path would be to explore different aspects of time inside the mechanism. This can be done by exploring different clock models, such as of a thermodynamic clock. (105) This is a ticking clock which operates with an engine, driving it forward. In this scenario we could evaluate the impact of thermodynamic entropy in the Page-Wootters mechanism and even ask questions about the arrow of time in the formulation.

Lastly, of special interest would be to extend the results to relativistic settings. There are some results (106) showing that there can be a difference in the proper time measured by a good quantum clock (good stopwatches) depending on its state of motion being classical or not. Specially intriguing would be to study the impact on ticking clocks such as the one described above. Exploring these effects in the timeless formulation would be fascinating.

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APPENDIX

APPENDIX A – ERROR BOUNDS

Here we show three results which are used to bound the errors in Chapter 3. These are well known, and for simplicity we choose to follow the presentation in (79). Then, for $\Delta \in \mathbb{R}$ and $b > X \in \mathbb{R}$ the first result is

$$\begin{aligned}\sum_{n=b}^{\infty} e^{-\frac{(n-X)^2}{\Delta^2}} &= \sum_{m=0}^{\infty} e^{-\frac{(m+b-X)^2}{\Delta^2}} \\ &= e^{-\frac{(b-X)^2}{\Delta^2}} \sum_{m=0}^{\infty} e^{-\frac{2m(b-X)}{\Delta^2}} e^{-\frac{m^2}{\Delta^2}} \\ &< e^{-\frac{(b-X)^2}{\Delta^2}} \sum_{m=0}^{\infty} e^{-\frac{2m(b-X)}{\Delta^2}} = \frac{e^{-\frac{(b-X)^2}{\Delta^2}}}{1-e^{-\frac{2(b-X)}{\Delta^2}}},\end{aligned} \tag{A.1}$$

where in the first line it was used a simple change $m = n - b$, and the second line was obtained by completing the squares and the last equality used a result about geometric series. The second result begins with

$$\sum_{n=b}^{\infty}(n-X)e^{-\frac{(n-X)^2}{\Delta^2}} = (b-X)e^{-\frac{(b-X)^2}{\Delta^2}} + \sum_{n=b+1}^{\infty}(n-X)e^{-\frac{(n-X)^2}{\Delta^2}}, \tag{A.2}$$

then, evaluating the second term on the RHS we get

$$\begin{aligned}\sum_{n=b+1}^{\infty}(n-X)e^{-\frac{(n-X)^2}{\Delta^2}} &< \int_b^{\infty} dx(x-X)e^{-\frac{(x-X)^2}{\Delta^2}} \\ &= \frac{\Delta^2}{2}e^{-\frac{(b-X)^2}{\Delta^2}},\end{aligned} \tag{A.3}$$

where $(x-X)e^{-\frac{(x-X)^2}{\Delta^2}}$ is monotonically decreasing for $x > \Delta + X$ and it was used that $\int_b^{\infty} txe^{\frac{x^2}{\Delta^2}}dx = \frac{\Delta^2}{2}e^{\frac{b^2}{\Delta^2}}$, obtaining

$$\sum_{n=b}^{\infty}(n-X)e^{-\frac{(n-X)^2}{\Delta^2}} < \left(b - X\frac{\Delta^2}{2}\right)e^{-\frac{(b-X)^2}{\Delta^2}}. \tag{A.4}$$

Lastly, for $b > \sqrt{2}\Delta + X \in \mathbb{R}$ we have

$$\sum_{n=b}^{\infty}(n-X)^2e^{-\frac{(n-X)^2}{\Delta^2}} = (b-X)^2e^{-\frac{(b-X)^2}{\Delta^2}} + \sum_{n=b+1}^{\infty}(n-X)^2e^{-\frac{(n-X)^2}{\Delta^2}}, \tag{A.5}$$

similarly to the second result we evaluate the sum

$$\sum_{n=b+1}^{\infty}(n-X)^2 e^{-\frac{(n-X)^2}{\Delta^2}} < \int_b^{\infty} dx (x-X)^2 e^{-\frac{(x-X)^2}{\Delta^2}}$$
$$= \frac{\Delta^2}{2}\left[(b-X)e^{-\frac{(b-X)^2}{\Delta^2}} + \int_b^{\infty} dx e^{-\frac{(x-X)^2}{\Delta^2}}\right], \qquad \text{(A.6)}$$

which was obtained since $(x-X)^2 e^{-\frac{(x-X)^2}{\Delta^2}}$ is monotonically decreasing for $x > \sqrt{2}\Delta + X$. Using

$$\int_b^{\infty} dx e^{-\frac{(x-X)^2}{\Delta^2}} < \sum_{n=b}^{\infty} e^{-\frac{(n-X)^2}{\Delta^2}} < \frac{e^{-\frac{(b-X)^2}{\Delta^2}}}{1-e^{-\frac{2(b-X)}{\Delta^2}}}. \qquad \text{(A.7)}$$

we can write

$$\sum_{n=b}^{\infty}(n-X)^2 e^{-\frac{(n-X)^2}{\Delta^2}} < \left[(b-X)^2 + \frac{\Delta^2}{2}\left(b-X+\frac{1}{1-e^{-\frac{2(b-X)}{\Delta^2}}}\right)\right] e^{-\frac{(b-X)^2}{\Delta^2}}. \qquad \text{(A.8)}$$

APPENDIX B – ADDITION OF ERRORS

Lemma 7 *Let* $\{\Gamma_m\}_{m=1}^{N}$ *be a sequence of operators on a finite dimensional Hilbert space* $\mathcal{H}$ *respecting*

$$\|\Gamma_m\|_2 \leq 1, \tag{B.1}$$

and $\{|\phi_m\rangle\}_{m=1}^{N}$ *a sequence of pure states in the same Hilbert space* $\mathcal{H}$*, which are not necessarily normalised, which respect*

$$\|\,|\phi_m\rangle - \Gamma_m\,|\phi_m - 1\rangle\,\|_2 = \varepsilon_m. \tag{B.2}$$

Then, $\forall n \in 1, 2, \ldots, N$

$$\|\,|\phi_n\rangle - \Gamma_n\Gamma_{n-1}\ldots\Gamma_1\,|\phi_0\rangle\,\|_2 \leq \sum_{m=1}^{n} \varepsilon_m. \tag{B.3}$$

Proof: For $n = k+1$ we get

$$\begin{aligned}\|\,|\phi_{k+1}\rangle - \Gamma_{k+1}\Gamma_k\ldots\Gamma_1\,|\phi_0\rangle\,\|_2 &= \|\,|\phi_{k+1}\rangle - \Gamma_{k+1}\,|\phi_k\rangle - \Gamma_{k+1}(|\phi_k\rangle - \Gamma_k\ldots\Gamma_1\,|\phi_0\rangle)\|_2 \\ &\leq \|\,|\phi_{k+1}\rangle - \Gamma_{k+1}\,|\phi_k\rangle\,\|_2 + \|\Gamma_{k+1}(|\phi_k\rangle - \Gamma_k\ldots\Gamma_1\,|\phi_0\rangle)\|_2 \\ &\leq \|\,|\phi_{k+1}\rangle - \Gamma_{k+1}\,|\phi_k\rangle\,\|_2 + \|\Gamma_{k+1}\|_2\|(|\phi_k\rangle - \Gamma_k\ldots\Gamma_1\,|\phi_0\rangle)\|_2 \\ &\leq \|\,|\phi_{k+1}\rangle - \Gamma_{k+1}\,|\phi_k\rangle\,\|_2 + |\,|\phi_k\rangle - \Gamma_k\ldots\Gamma_1\,|\phi_0\rangle\,\|_2,\end{aligned} \tag{B.4}$$

where in the second line it was used that $\|F+G\|_2 \leq \|F\|_2 + \|G\|_2$ with $F, G \in \mathcal{H}$, in the third line it was used that $\|FG\|_2 \leq \|F\|_2\|G\|_2$. The first element is equal to ε_{k+1} by definition, for the second element we can apply the same process obtaining at the end a term associated to k equal to ε_k, therefore by induction

$$\|\,|\phi_{k+1}\rangle - \Gamma_{k+1}\Gamma_k\ldots\Gamma_1\,|\phi_0\rangle\,\|_2 \leq \varepsilon_{k+1} + \sum_{m=1}^{k} \varepsilon_m = \sum_{m=1}^{k+1} \varepsilon_m. \tag{B.5}$$

APPENDIX C – EQUATION OF MOTION FOR THE IDEAL CLOCK

Following the normal prescription to generalize Eq.(5.4), taking the state of the system to be $|\psi_S(\tau)\rangle$ we get

$$\rho_S(0) = \sum_\iota p_\iota |\psi_S^\iota(0)\rangle \langle\psi_S^\iota(0)|\,, \tag{C.1}$$

with $\rho_S(\tau) = U(\tau)\rho_S(0)U^\dagger(\tau) = \sum_\iota p_\iota |\psi_S^\iota(\tau)\rangle \langle\psi_S^\iota(\tau)|$, then

$$\begin{aligned}\frac{d\rho_S(\tau)}{d\tau} &= \sum_\iota p_\iota \left(\frac{d}{d\tau}\{|\psi_S^\iota(\tau)\rangle\} \langle\psi_S^\iota(\tau)| + |\psi_S^\iota(\tau)\rangle \frac{d}{d\tau}\{\langle\psi_S^\iota(\tau)|\}\right)\\ &= -i\sum_\iota p_\iota \left(\{H_S |\psi_S^\iota(\tau)\rangle + H_k |\psi_S^\iota(\tau)\rangle\} \langle\psi_S^\iota(\tau)| - |\psi_S^\iota(\tau)\rangle \{\langle\psi_S^\iota(\tau)| H_S + \langle\psi_S^\iota(\tau)| H_k\}\right),\end{aligned} \tag{C.2}$$

to proceed we first need to obtain the action of the integral operator H_k. Here we consider a gravitational interaction between clock and system, therefore the Hamiltonian of the universe will be

$$H = H_S + H_C - \frac{G}{c^4 x} H_S \otimes H_C, \tag{C.3}$$

where G is the gravitational constant, x is the distance between clock and system and c is the speed of light. We are using the idealized momentum clock $H_C = P_C$, then, the kernel $K(\tau, \tau')$ can be computed as follows

$$\begin{aligned}K(\tau,\tau') &= -\frac{G}{c^4 d} H_S \langle\tau| P_C |\tau'\rangle\\ &= -\frac{G}{c^4 d} H_S \left[\int dpp\, \langle\tau|p\rangle \langle p|\tau'\rangle\right]\\ &= -\frac{G}{c^4 d} H_S \left[\frac{1}{2\pi}\int dppe^{-ip(\tau'-\tau)}\right]\\ &= -\frac{G}{c^4 d} H_S i\dot{\delta}(\tau'-\tau),\end{aligned} \tag{C.4}$$

where the dot on the delta was used in the place of the time derivative. The adjoint kernel $K^\dagger(\tau, \tau')$ is readily obtained. With this

$$\begin{aligned}H_k |\psi_S(\tau)\rangle &= -\int \frac{G}{c^4 d} H_S i\dot{\delta}(\tau'-\tau)\, |\psi_S(\tau')\rangle\, d\tau'\\ &= i\frac{G}{c^4 d} H_S \int \delta(\tau'-\tau) \frac{d}{d\tau'} |\psi_S(\tau')\rangle\, d\tau'\\ &= i\frac{G}{c^4 d} H_S \frac{d}{d\tau} |\psi_S(\tau)\rangle\end{aligned} \tag{C.5}$$

and its adjoint

$$\begin{aligned}\langle\psi_S(\tau)|\, H_k &= i\frac{G}{c^4 d}\int \dot{\delta}(\tau'-\tau)\,\langle\psi_S(\tau')|\, H_S d\tau' \\ &= -i\frac{G}{c^4 d}\int \delta(\tau'-\tau)\frac{d}{d\tau'}\,\langle\psi_S(\tau')|\, H_S d\tau' \\ &= -i\frac{G}{c^4 d}H_S\frac{d}{d\tau}\,\langle\psi_S(\tau)|\, H_S. \end{aligned} \tag{C.6}$$

Then, defining $\mathbb{G} := \frac{G}{c^4 d}$, we get

$$\begin{aligned}\frac{d\rho_S(\tau)}{d\tau} &= -i\sum_\iota p_\iota\left(\{H_S\,|\psi_S^\iota(\tau)\rangle + H_k\,|\psi_S^\iota(\tau)\rangle\}\,\langle\psi_S^\iota(\tau)| - |\psi_S^\iota(\tau)\rangle\,\{\langle\psi_S^\iota(\tau)|\,H_S + \langle\psi_S^\iota(\tau)|\,H_k\}\right) \\ &= -i\sum_\iota p_\iota\left([H_S, |\psi_S^\iota(\tau)\rangle\,\langle\psi_S^\iota(\tau)|] + \mathbb{G}\left\{iH_S\frac{d\,|\psi_S^\iota(\tau)\rangle}{d\tau}\,\langle\psi_S^\iota(\tau)| + i\,|\psi_S^\iota(\tau)\rangle\,\frac{d\,\langle\psi_S^\iota(\tau)|}{d\tau}H_S\right\}\right) \\ &= -i\sum_\iota p_\iota\Big([H_S, |\psi_S^\iota(\tau)\rangle\,\langle\psi_S^\iota(\tau)|] + \mathbb{G}\Big\{H_S^2\,|\psi_S^\iota(\tau)\rangle\,\langle\psi_S^\iota(\tau)| + H_S H_k\,|\psi_S^\iota(\tau)\rangle\,\langle\psi_S^\iota(\tau)| \\ &\qquad - |\psi_S^\iota(\tau)\rangle\,\langle\psi_S^\iota(\tau)|\,H_S^2 - |\psi_S^\iota(\tau)\rangle\,\langle\psi_S^\iota(\tau)|\,H_k H_S\Big\}\Big) \\ &= -i\sum_\iota p_\iota\Big([H_S, |\psi_S^\iota(\tau)\rangle\,\langle\psi_S^\iota(\tau)|] + \mathbb{G}[H_S^2, |\psi_S^\iota(\tau)\rangle\,\langle\psi_S^\iota(\tau)|] + \mathcal{O}(\mathbb{G}^2)\Big)\,, \end{aligned} \tag{C.7}$$

where in the second line we used Eq.(5.4). As all other terms will be of orders superior to $\mathbb{G}$ which is proportional to the inverse of c^4, we assume these terms to be negligible, therefore our equation of motion will be approximately

$$\frac{d\rho_S(\tau)}{d\tau} = -i[H_S, \rho_S(\tau)] - i\mathbb{G}[H_S^2, \rho_S(\tau)]. \tag{C.8}$$