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Wigner's friend and quantum clocks

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"The problem, as I see it, is that you've been told and not told. You've been told, but none of you really understand, and I dare say, some people are quite happy to leave it that way."

Kazuo Ishiguro

ABSTRACT

ROSSI, V. P. **Wigner's friend and quantum clocks**. 2020. 89p. Dissertation (Master of Science) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2020.

In 1962, Eugene P. Wigner introduced a thought experiment that highlighted the incompatibility in quantum mechanics between unitary evolution and wave function reduction in a measurement. This work resulted in a class of thought experiments often called Wigner's Friend Scenarios, which have been providing insights over many frameworks and interpretations of quantum theory. Recently, a no-go theorem obtained by Daniela Frauchiger and Renato Renner brought attention back to the Wigner's Friend and its potential of putting theories to test. Many answers to this result pointed out how timing in the thought experiment could be yielding a paradox. In this work, we ask what would happen if the isolated friend in a Wigner's Friend Scenario did not share a time reference frame with the outer observer, and time should be tracked by a quantum clock. For this purpose, we recollect concepts provided by the theory of quantum reference frames and the quantum resource theory of asymmetry, to learn how to internalize time in this scenario, and introduce a model for a feasible quantum clock proposed by Mischa P. Woods, Ralph Silva and Jonathan Oppenheim, called the quasi-ideal clock. Our results have shown that no decoherent behavior comes from this approach, and the disagreement between the superobserver and its friend persists even for an imprecise clock, indicating that the source of paradox in a Wigner's Friend Scenario may be elsewhere.

Keywords: Quantum foundations. Wigner's friend. Quantum clocks. Quantum reference frames.

RESUMO

ROSSI, V. P. **Amigo de Wigner e relógios quânticos**. 2020. 89p. Dissertação (Mestrado em Ciências) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2020.

Em 1962, Eugene P. Wigner apresentou um experimento mental que destacava a incompatibilidade na mecânica quântica entre a evolução unitária e a redução da função de onda em uma medição. Esse trabalho resultou em uma classe de experimentos mentais usualmente chamados Cenários do tipo Amigo de Wigner, que têm provido informações sobre várias abordagens e interpretações da teoria quântica. Recentemente, um teorema obtido por Daniela Frauchiger e Renato Renner trouxe de volta a atenção sobre o Amigo de Wigner e seu potencial de colocar teorias à prova. Diversas respostas a este resultado indicaram como a marcação temporal no experimento mental poderia estar produzindo um paradoxo. Neste trabalho, nos perguntamos o que aconteceria se o amigo isolado não compartilhasse um referencial temporal com o observador externo, e o tempo fosse rastreado por um relógio quântico. Para este fim, revisitamos conceitos fornecidos pela teoria de referenciais quânticos e pela teoria quântica de recursos de assimetria, para descobrir como internalizar o tempo neste cenário, e apresentamos um modelo de relógio implementável proposto por Mischa P. Woods, Ralph Silva e Jonathan Oppenheim, chamado de relógio quase ideal. Nossos resultados mostraram que nenhum comportamento decoerente surge dessa abordagem, e o desacordo entre o superobservador e seu amigo persiste mesmo para um relógio impreciso, indicando que a fonte de paradoxo em um Cenário do tipo Amigo de Wigner pode estar em outra parte.

Palavras-chave: Fundamentos da teoria quântica. Amigo de Wigner. Relógios quânticos. Referenciais quânticos.

LIST OF FIGURES

CONTENTS

1 INTRODUCTION

Evolution in quantum theory is governed by two postulates: one of them, given by the well-known Schrödinger equation, describes the time evolution of isolated systems via unitary operators, while the measurement postulate describes how a system is to be described after interacting with a measurement aparatus. The first one tells us that isolated systems suffer no loss of information as time goes by, while the second one provides a discontinuous reduction of information. Even though this formulation has proved a powerful theory, generating predictions good enough to astonishingly enhance the technological power of humankind in the last century, there are some theoretically odd features that blur the interpretation we should have about the universe.

To emphasize the incompatibility between these two descriptions, Eugene P. Wigner proposed in 1962 a thought experiment later called the Wigner's Friend.[1] An observer, inside an isolated lab, measures some physical property of a particle. Measurement is described by the measurement postulate, and thus there will be loss of information. But for an outer observer, the lab is completely isolated, so evolution is described by the Schrödinger equation, and no information is lost. This situation leads to different statistical predictions for the internal observer and the external one, and at least one of them must be wrong if quantum theory is self consistent.

On the other hand, the theory of quantum reference frames tells us how measurements over a system can be done with respect to another physical system, and this is usually the source of many mistakes, since reference frames are often taken for granted as ideal and universally shared.[2] According to this theory, if the internal friend is completely isolated inside the lab, she must not share a quantum reference frame with the outer observer. That is because, if they had access to the same quantum reference frame, the external observer could affect or monitor the evolution inside the lab by correlations between it and the reference frame, and the lab would not be in fact isolated. Taking this into account might be crucial to understanding what is the theoretical flaw that allows this sort of paradox to arise.

In this work, we analyze what would happen in a Wigner's Friend Scenario (WFS) if the outer observer had no direct access on how time was passing inside the lab, but should rather describe its state with respect to a quantum clock. The main idea behind this approach is that the unitarily evolved state described by the external observer may not be converted to the colapsed state described by the internal one, but the insertion of a clock could work as a catalytic process to this conversion. The concept of catalytic conversion has been used in many problems in literature, and brings with it a whole formalism of Quantum Resource Theories that shows its worth everyday.[3–5] The work is organized as follows:

In Chapter 2 we will introduce the Wigner's Friend *Gedankenexperiment* by taking a look on Wigner's original proposal, contextualizing the purpose of that piece and the ideas behind it. The recent work of Frauchiger and Renner,[6] that put Wigner's Friend under the spotlight again, is also summarized and discussed. Implications of this work among the foundations of quantum mechanics community as well as main critics and answers are quickly presented, highlighting the clues that pointed to time as a paradox raiser.

In Chapter 3 we focused on studying how one should deal with time measurements in quantum mechanics. We recollect some remarkable results in the history of time in the quantum realm, from Pauli's theorem to the Page-Wooters mechanism, passing through Dirac's extended phase space. Two questions arise from this section: (i) How is one supposed to obtain the static states of the Page-Wooters mechanism, and (ii) how can one build a time operator that works and still is not forbidden by Pauli's result. The first question is answered by the theory of quantum reference frames and the quantum resource theory of asymmetry, and we introduce the tools and formalism provided by them. The second one is answered with an overview on the history of quantum clocks, culminating in the quasi-ideal clock of Woods, Silva and Oppenheim.[7]

Chapter 4 is our main result. We propose a model for a Wigner's Friend Scenario where the outer observer has no access to the parametric time t, but can keep track of time through his quasi-ideal clock. This model of clock allows for analytic time symmetrization in a good approximation for a given regime of the clock's classical uncertainty σ. We conclude that even with the insertion of a clock, the external observer is not allowed to perform any measurement over the lab without raising a paradox, thus implying that either this model for a clock does not work as a catalyst, or the symmetrization process is not a catalytic operation. A discussion on the clock functioning is made, with interesting results that show how the clock works for a specific entangling hamiltonian that describes the unitary evolution of the lab.

Chapter 5 concludes the work by summarizing our results and discussing what they indicate. Further investigation is sugested concerning information storing and internal entanglement in charge sectors of the WFS.

2 TROUBLESOME FRIENDSHIP: WIGNER'S FRIEND SCENARIOS

"To say that prediction is the purpose of a scientific theory is to confuse means with ends. It is like saying that the purpose of a spaceship is to burn fuel."
- David Deutsch[8]

The advent of quantum theory in the 1900's, though it increased astonishingly the predictive power of Physics, left physicists with conceptual and philosophical problems to be solved. The incompatibility between unitary and non-unitary evolutions is particularly highlighted by Eugene P. Wigner, in 1962, in a proposed *Gedankenexperiment* that seems to state the need of "biological" behavior to solve this loose end.[1] Although Wigner later claimed that he disbeliefed on consciousness as a physical element,[9] his famous early work was never really forgotten, since it yields to an interesting class of *Gedankenexperiment* that allows us to analyze the measurement problem with a keen eye, and has provided great insights on the subject.

It is to introduce and summarize this class of thought experiments, called Wigner's Friend Scenarios, that this chapter is dedicated. First section outlines the arguing made by Wigner in his piece. Later, a recent result inspired in Wigner's work provided by Frauchiger and Renner[6] is introduced and demonstrated. Finally, the repercutions of this result are briefly reviewed, and we introduce the central question of the present work.

2.1 Wigner's Friend *Gedankenexperiment*

In 1962, a collection of 123 articles selected by Irving J. Good were published in a book called "*The Scientist Speculates: An anthology of partly-baked ideas*". Among these works, Eugene P. Wigner published his work *Remarks on the Mind-Body Question*,[1] a digression about the role of consciousness in quantum mechanics. In this work, Wigner questioned the state of art of quantum theory at the time to introduce the measurement problem for laypeople and to argue that consciousness could play a crucial role on solving it. Here we will not focus on this discussion itself, but rather in presenting the *Gedankenexperiment* Wigner proposes to exemplify the oddity of a theory that relies on two seemingly incompatible forms of evolution.

Wigner invites us to think of a simple experiment: an observer is looking towards a defined direction. At time increments of t, she knows she is supposed to detect a flash of light. If this signal is detected, the observer knows that at the time $t + 1$ she will detect another flash in 1/4 of tries, and the flash will be missing in 3/4 of tries. If the signal is missing, she is aware that at time $t+1$ she should detect a flash in 3/4 of tries, and miss a flash in 1/4 of tries. Repeating this experiment for a large number of tries should confirm this prediction.

In terms of a wave function, one could say that every time this observer detects a flash, she describes the system in a state $|yes\rangle$. After a time increment, the system evolves unitarily by the Schrödinger equation into the state

$$|\psi_1\rangle = \frac{1}{2}\left(|yes\rangle + \sqrt{3}\,|no\rangle\right), \tag{2.1}$$

and if she did not detect the flash, then the system is said to be in the state $|no\rangle$ and, after a time increment, it will be described by the state

$$|\psi_2\rangle = \frac{1}{2}(\sqrt{3}\,|yes\rangle - |no\rangle), \tag{2.2}$$

where the minus sign is ensuring the unitarity of the evolution. Notice that this describes exactly the experiment presented in the late paragraph: right after a measurement is performed by the observer, she updates her description of the wave function to a state $|yes\rangle$ or $|no\rangle$, and the system evolves via the unitary

$$U_t = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \tag{2.3}$$

into the state $|\psi_1\rangle$ or $|\psi_2\rangle$ again. This is an adequate mathematical description of what the observer perceives: indeed, a large number of measurements will confirm the frequencies introduced in the verbal form before.

Following his introduction to Quantum Mechanics, Wigner points out the property a perception has of being communicated: if this observer would tell us either she has seen a flash or not at a given time, we could also tell with certainty that the state of the system is either $|yes\rangle$ or $|no\rangle$, and make further predictions with this information.

Working on these two concepts together (the assumption of describing a physical system as a wave function and the communicability of the perception of an outcome), Wigner presents its thought experiment: let us suppose that our friend is going to measure flashes. In this scenario, it makes no sense to describe only the system as a wave function. Let us suppose, for instance, that our friend has measured and communicated to us a result *yes* at a given time t. All we can say about this joint wave function at the time $t+1$ is that it is given by the state

$$|\Psi_1\rangle = \frac{1}{2}\left(|yes\rangle_S\,|yes\rangle_F + \sqrt{3}\,|no\rangle_S\,|no\rangle_F\right), \tag{2.4}$$

where $|yes\rangle_F$ and $|no\rangle_F$ assign for the *states of our friend's consciousness*, and $|yes\rangle_S$ and $|no\rangle_S$ assign for the states of the system. This is appearently nice and fine: there is no chance that our friend is lying to us, since every time we ask her if she has observed the flash or not, her answer will be either *yes* or *no*, and the state of the system will be in agreement with it. In other words, there is no $|yes\rangle_S |no\rangle_F$ or $|no\rangle_S |yes\rangle_F$ states to be measured. Furthermore, probabilities are still compatible with the verbal description: in 1/4 of tries, our friend will tell us that she has seen a flash, and in 3/4 of tries she will tell us she has not.

But what did happen to our friend's mind before we ask her what she has seen? If she tells us *yes*, does it mean the real joint state was $|yes\rangle_S |yes\rangle_F$ all the time? If we ask her what she knew between the moment she performed her measurement and the moment we asked her what she saw, will she tell us "I already told you, I *did* see a flash!"? If this is the case, then we are failing in our description of the joint system, since states $|yes\rangle_S |yes\rangle_F$ and $|no\rangle_S |no\rangle_F$ do not have the same properties $|\Psi_1\rangle$ has.

Wigner emphasizes that this problem would not exist if our friend was a consciousness-lacking quantum system, for instance, an atom or another two-level system. Indeed, in accord to Copenhagen Interpretation (CI), a quantum system causes no colapse over another quantum system, and the state $|\Psi_1\rangle$ would be the right description. For Wigner, this was a *life-emerging* issue, and it could even be possible to detect the existence of life or consciousness by analyzing the vanishing coherences on the density operator. Explicitly, Wigner proposes that for a given state $\alpha |yes\rangle_S |yes\rangle_F + \beta |no\rangle_S |no\rangle_F$, with $|\alpha|^2 + |\beta|^2 = 1$, the density operator would be written in the $\{|yes\rangle_S |yes\rangle_F, |no\rangle_S |no\rangle_F\}$ basis as

$$\rho = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \cos\delta \\ \alpha^*\beta \cos\delta & |\beta|^2 \end{pmatrix}, \tag{2.5}$$

where δ is a measurement of the *classicality* of the friend. When $\delta = 0$, the measurement performed by the friend results in an entanglement, as if the friend was an atom. When $\delta = \pi/2$, this is a full-colapse, and the density operator would be describing a statistical mixture of $|yes\rangle_S |yes\rangle_F$ and $|no\rangle_S |no\rangle_F$.

Besides that, we are not confined in measuring the joint system on the same basis it is written. Freedom of choice allows us to measure any quantity our lab is prepared to detect, and since this is a *Gedankenexperiment*, our lab is prepared to detect any physical property. The problem this Wigner's Friend Scenario is concerned becomes explicit when we decide to perform a measure of a generic observable

$$|ok\rangle = \cos\left(\frac{\theta}{2}\right) |yes\rangle_S |yes\rangle_F + e^{i\phi} \sin\left(\frac{\theta}{2}\right) |no\rangle_S |no\rangle_F. \tag{2.6}$$

If Wigner's description was right and $|\Psi_1\rangle$ was the right way of representing the state of the joint system, then the probability associated to this observable would be given by

$$|\langle ok|\Psi_1\rangle|^2 = \frac{1}{4}(2-\cos\theta+\sqrt{3}\sin\theta\cos\phi), \tag{2.7}$$

while if the friend was right and the system in fact colapsed, this probability would be given by

$$|\langle ok|yes\rangle_S|yes\rangle_F|^2 = \cos^2\left(\frac{\theta}{2}\right); \quad |\langle ok|no\rangle_S|no\rangle_F|^2 = \sin^2\left(\frac{\theta}{2}\right). \tag{2.8}$$

This clearly leads to different probability distributions and reveals the major concern of this *Gedankenexperiment*: the prediction power of this theory is obscured. One may argue that there is no superobservable in the real world. Indeed, we are not capable of measuring the state of a friend and distinguishing every degree of freedom of its body and mind (if there is any difference between saying *body* and *mind*). But this limitation is not axiomatic, and at any moment someone, somewhere around the world, could publish a paper claiming the performance of a superobservation.

Even though Wigner's seminal work was just an speculative piece and was not very rigorous when defining what a wave function exactly is describing,[9] it does touch in a crucial wound of quantum theory, and many physicists dedicated their time trying to understand this scenario. Variations and expansions of this *Gedankenexperiment* constitute a class of thought experiments which are usually called *Wigner's Friend Scenarios* (WFS), and can provide powerful insights about theories and interpretations one adopt to describe it. In 2016, a discussion over an Extended Wigner's Friend Scenario (EWFS) brought Wigner's work back to spotlight. In next section, we shall briefly check on this result.

2.2 Frauchiger-Renner no-go theorem

In 2016, Daniela Frauchiger and Renato Renner, a PhD candidate and her supervisor, developed a non-probabilistic framework for general scientific theories.[10] By adopting a story-plot framework, a set theory where elements of a given universe (countable) set would represent stories to be told about an experiment, a scientific theory consisted of one or more rules constructing a subset of forbidden stories, i.e., stories that could not be told about an experiment when describing it under that given theory. Frauchiger's dissertation is a great contribution to foundations of quantum theory, since this framework of stories is probability-free. Probabilities may emerge later when one takes a decision-theoretic framework: a racional agent making bets about an outcome of an experiment. This characterizes probabilities as subjective entities, and the objective Born rule derived is independent of racional agents or bets.

At the end of her dissertation, Frauchiger derives a no-go theorem, later published as an article,[6] by analyzing an EWFS under this non-probabilistic framework. First appearing under the title "*Single-world interpretations of quantum theory cannot be self-consistent*", this result gathered attention of great part of Quantum Foundations and Quantum Information communities. Here we will follow the development presented in the published version of the article.

Let us assume an observer, Alice, who measures the state of a quantum coin inside her lab. The lab is completely isolated from the external world, except by a quantum channel that allows her to send a qubit to a neighbor lab. After her measurement, Alice will perpare a spin-$\frac{1}{2}$ particle in a state conditioned to her measurement outcome.

In a neighbor lab, Bob is waiting for the spin-$\frac{1}{2}$ particle prepared by Alice. He is supposed to measure the spin state in the same basis it was prepared by Alice, and nothing more. This lab is also completely isolated from the external world except by the quantum channel.

Outside the labs, Ursula and Wigner are waiting with their advanced measurement devices. Ursula has a device capable of determining the state of the whole Alice's lab, i.e., the joint state of the quantum coin and of Alice's device, body and mind. Wigner, on the other hand, has a device capable of doing the same with Bob's lab, i.e., determining the joint state of the spin-$\frac{1}{2}$ particle and of Bob's device, body and mind.

The experiment must go as the following protocol says, where $n = XY$ assigns for the X-th main step and Y-th intermediate step. Figure 1 pictorizes the scenario.

- $n = 00$: Alice receives a quantum coin described by the state $|\psi\rangle_C = \frac{1}{\sqrt{3}}|h\rangle_C + \sqrt{\frac{2}{3}}|t\rangle_C$, where h stands for *heads* and t stands for *tails*, and performs her measurement with outcome $r \in \{h, t\}$ the basis $\{|h\rangle_C, |t\rangle_C\}$. If she gets $r = h$, she shall prepare a spin-$\frac{1}{2}$ particle in a state $|{\downarrow}\rangle_S$; if she gets $r = t$, she shall prepare the spin in a state $|{\rightarrow}\rangle_S = \frac{1}{\sqrt{2}}(|{\uparrow}\rangle_S + |{\downarrow}\rangle_S)$. She then sends the particle through the quantum channel to Bob's lab;

- $n = 10$: Bob performs his measurement of the spin-$\frac{1}{2}$ particle with respect to the basis $\{|{\uparrow}\rangle_S, |{\downarrow}\rangle_S\}$, with outcome $z \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$;

- $n = 20$: Ursula performs a measurement with outcome $u \in \{ok, fail\}$ with respect to the basis $\{|ok\rangle_U, |fail\rangle_U\}$, where $|ok\rangle_U = \frac{1}{\sqrt{2}}(|h\rangle_C |h\rangle_A - |t\rangle_C |t\rangle_A)$, $\{|h\rangle_A, |t\rangle_A\}$ being states representing Alice's device, body and mind. If her device detects this state, her result is said to be $u = ok$. If any other state is detected, then her result is said to be $u = fail$.

- $n = 30$: Wigner performs a measurement with outcome $w \in \{ok, fail\}$ with respect to the basis $\{|ok\rangle_W, |fail\rangle_W\}$, where $|ok\rangle_W = \frac{1}{\sqrt{2}}(|\downarrow\rangle_S |\downarrow\rangle_B - |\uparrow\rangle_S |\uparrow\rangle_B)$, $\{|\uparrow\rangle_B, |\downarrow\rangle_B\}$ being states representing Bob's device, body and mind. If his device detects this state, his result is said to be $w = ok$. If any other state is detected, then his result is said to be $w = fail$.

- $n = 40$: If $u = ok$ and $w = ok$, the experiment is halted. Otherwise, it is restarted.

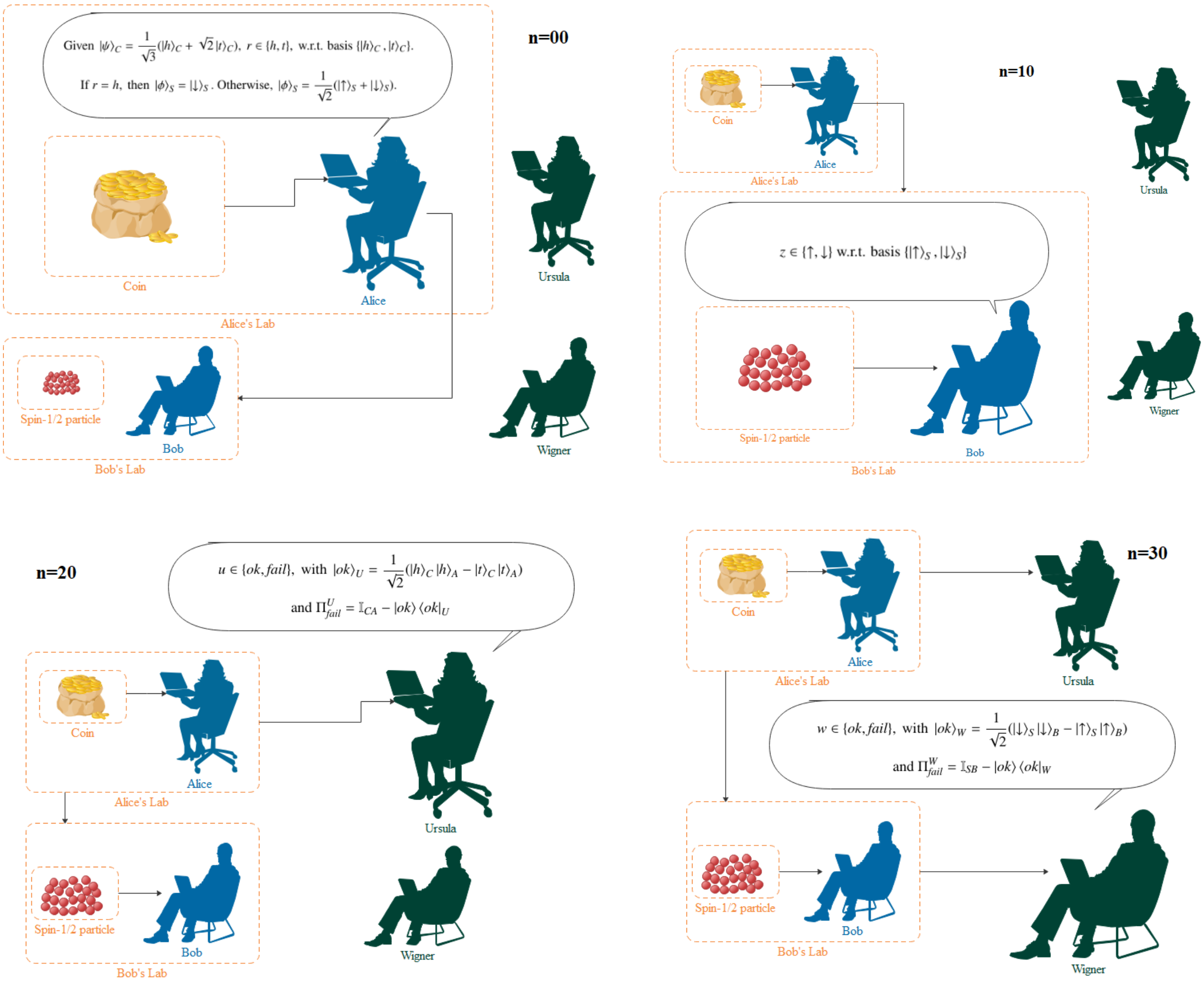

Figure 1 – Schematic representation of the Frauchiger-Renner *Gedankenexperiment*.

Source: By the author.

Agents will also make statements about measurement outcomes. It is assumed that they can make statements about measurements of their own and measurements performed by other agents, using the following assumptions:

- Quantum Theory (Q): Let an agent A be capable of certainly state about any given system that "*The state of the system is properly described by* $|\psi\rangle \in \mathcal{H}_S$", and also that "*The value of a physical property* x *is given by a projection of* $|\psi\rangle_S$ *with respect to a family of Heisenberg projectors* $\{\Pi_x(t_0)\}_{x\in\chi}$ *at the time* t_0*, being completed at time* t"[i]. With these two statements, if $\langle\psi|\Pi_\xi(t_0)|\psi\rangle = 1$ for $\xi \in \chi$, then agent A can properly state that

  "*I am certain that* $x = \xi$ *at time* t*.*"

- Self Consistency (C): If an agent A can properly state that "*I am certain that agent B, who is reasoning over the same theory as I am, is certain that* $x = \xi$ *at time* t", then he/she can also state that

  "*I am certain that* $x = \xi$ *at time* t*.*"

- Single-world (S): If an agent A can properly state that "*I am certain that* $x = \xi$ *at time* t", then he/she must **deny** that

  "*I am certain that* $x \neq \xi$ *at time* t*.*"

Let us now see what is happening in the *Gedankenexperiment*. At first, the joint state of the whole system is given by[ii]

$$|\Psi(00)\rangle = \frac{1}{\sqrt{3}}(|h\rangle_C + \sqrt{2}\,|t\rangle_C) \otimes |\bot\rangle_A \otimes |\bot\rangle_S \otimes |\bot\rangle_B\,, \tag{2.9}$$

where $|\bot\rangle$ represents a state of readiness with null outcome if measured. The measurement of Alice is described by a unitary process which results on an entangled state between Alice and the coin, exactly as we did in the last section, i.e.,

$$|\Psi(01)\rangle = \frac{1}{\sqrt{3}}(|h\rangle_C\,|h\rangle_A + \sqrt{2}\,|t\rangle_C\,|t\rangle_A) \otimes |\bot\rangle_S \otimes |\bot\rangle_B\,. \tag{2.10}$$

Then Alice shall prepare the spin-$\frac{1}{2}$ particle as it is said at the protocol, which is described by another unitary evolution entangling Alice+coin with the spin particle:

$$|\Psi(02)\rangle = \frac{1}{\sqrt{3}}(|h\rangle_C\,|h\rangle_A\,|{\downarrow}\rangle_S + \sqrt{2}\,|t\rangle_C\,|t\rangle_A\,|{\rightarrow}\rangle_S) \otimes |\bot\rangle_B\,; \tag{2.11}$$

or writing it with respect to the $\{|{\uparrow}\rangle\,, |{\downarrow}\rangle\}_S$ basis,

$$|\Psi(02)\rangle = \frac{1}{\sqrt{3}}(|h\rangle_C\,|h\rangle_A\,|{\downarrow}\rangle_S + |t\rangle_C\,|t\rangle_A\,|{\uparrow}\rangle_S + |t\rangle_C\,|t\rangle_A\,|{\downarrow}\rangle_S) \otimes |\bot\rangle_B\,. \tag{2.12}$$

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ii Subscript C assigns for coin states, A for Alice's mind states, S for spin-$\frac{1}{2}$ particle states, B for Bob's mind states. Subscript U stands for eigenstates of Ursula's measurement, and W for eigenstates of Wigner's measurement.

After that, Alice would send the spin-$\frac{1}{2}$ particle to Bob, and he would perform his measurement, again described as an entangling unitary evolution, resulting in

$$|\Psi(10)\rangle = \frac{1}{\sqrt{3}}(|h\rangle_C |h\rangle_A |{\downarrow}\rangle_S |{\downarrow}\rangle_B + |t\rangle_C |t\rangle_A |{\uparrow}\rangle_S |{\uparrow}\rangle_B + |t\rangle_C |t\rangle_A |{\downarrow}\rangle_S |{\downarrow}\rangle_A). \tag{2.13}$$

Now, notice that

$$|h\rangle_C |h\rangle_A = \frac{1}{\sqrt{2}}(|fail\rangle_U + |ok\rangle_U); \quad |t\rangle_C |t\rangle_A = \frac{1}{\sqrt{2}}(|fail\rangle_U - |ok\rangle_U); \tag{2.14}$$

$$|{\downarrow}\rangle_S |{\downarrow}\rangle_B = \frac{1}{\sqrt{2}}(|fail\rangle_W + |ok\rangle_W); \quad |{\uparrow}\rangle_S |{\uparrow}\rangle_B = \frac{1}{\sqrt{2}}(|fail\rangle_W - |ok\rangle_W), \tag{2.15}$$

where we adopted $|fail\rangle_U = \frac{1}{\sqrt{2}}(|h\rangle_C |h\rangle_A + |t\rangle_C |t\rangle_A)$ and $|fail\rangle_W = \frac{1}{\sqrt{2}}(|{\downarrow}\rangle_S |{\downarrow}\rangle_B + |{\uparrow}\rangle_S |{\uparrow}\rangle_B)$.[11] Thus, we can write the joint state of both labs as

$$|\Psi(10)\rangle = \sqrt{\frac{3}{4}}\, |fail\rangle_U |fail\rangle_W + \frac{1}{\sqrt{12}}(|fail\rangle_U |ok\rangle_W - |ok\rangle_U |fail\rangle_W + |ok\rangle_U |ok\rangle_W). \tag{2.16}$$

So we see from here that there is a $\frac{1}{12}$ probability of both Ursula and Wigner detecting *ok* on their measurement devices, and thus the halting condition will eventually be achieved due to the overlap $|ok\rangle_U |ok\rangle_W$. Now, agents start to reason about which outcome each other agent has seen. Ursula knows that, at step $n = 10$, the wave function can be written as

$$|\Psi(10)\rangle = \frac{1}{\sqrt{3}}(\sqrt{2}\, |fail\rangle_U |{\downarrow}\rangle_S |{\downarrow}\rangle_B + |t\rangle_C |t\rangle_A |{\uparrow}\rangle_S |{\uparrow}\rangle_B), \tag{2.17}$$

and thus for Ursula to detect *ok* with her device, Bob should have detected $z = +\frac{1}{2}$, since the state $|{\downarrow}\rangle_B$ is uniquely overlapped with the state $|fail\rangle_U$. So, if Ursula can state that "*I am certain that* $u = ok$ *at* $n = 21$", then she can also state by assumption (Q) that

"*I am certain that Bob is certain that* $z = +\frac{1}{2}$ *at* $n = 11$",

and, by assumption (C), she can state that

"*I am certain that* $z = +\frac{1}{2}$ *at time* $n = 11$".

Bob can make the same protocol. If we check on the state at $n = 02$, we can see that the state $|{\uparrow}\rangle_S$ is overlapped only with the state $|t\rangle_A$. It means that if Bob can state "*I am certain that* $z = +\frac{1}{2}$ *at time* $n = 11$", then by (Q) he can also state that

"*I am certain that Alice is certain that* $r = t$ *at time* $n = 01$",

and by assumption (C), he can state that

"*I am certain that* $r = t$ *at time* $n = 01$".

Ursula could make the same statements just by using assumptions (Q) and (C), so Ursula can also state that "*I am certain that* $r = t$ *at time* $n = 01$". Wigner, on his turn, is also capable of deriving the same conclusions, since he and Ursula have access to the same information (joint wave function) and can freely share their measurement results. So Wigner can also state that "*I am certain that* $r = t$ *at time* $n = 01$"

This means that Bob, Ursula and Wigner all agree that Alice must have detected $r = t$ at her measurement for the halting condition to be achieved. Thus, we should reach the halting condition once we assume that Alice indeed detected $r = t$, that is, that Alice describes the wave function as

$$|\Psi(01)\rangle = |t\rangle_C \, |t\rangle_A \, |\bot\rangle_S \, |\bot\rangle_B \,, \tag{2.18}$$

with probability $\frac{2}{3}$, as can be seen from Eq. (2.10). Alice would then prepare the spin-$\frac{1}{2}$ particle, leading to the wave function

$$|\Psi(02)\rangle = \frac{1}{\sqrt{2}}(|t\rangle_C \, |t\rangle_A \, |{\uparrow}\rangle_S + |t\rangle_C \, |t\rangle_A \, |{\downarrow}\rangle_S) \, |\bot\rangle_B \,, \tag{2.19}$$

and send it to Bob. At his lab, Bob will perform his measurement resulting in a wave function

$$|\Psi(10)\rangle = \frac{1}{\sqrt{2}}(|t\rangle_C \, |t\rangle_A \, |{\uparrow}\rangle_S \, |{\uparrow}\rangle_B + |t\rangle_C \, |t\rangle_A \, |{\downarrow}\rangle_S \, |{\downarrow}\rangle_B) = |t\rangle_C \, |t\rangle_A \, |fail\rangle_W \,. \tag{2.20}$$

So, by assumption (Q), if Alice can state that "*I am certain that* $r = t$ *at* $n = 01$", then she can also state that "*I am certain that Wigner is certain that* $w = fail$ *at time* $n = 31$". This leads to a contradiction: if Wigner can state that "*I am certain that* $w = ok$ *at time* $n = 31$", by using assumptions (Q) and (C), he is capable of properly stating "*I am certain that* $w = fail$ *at time* $n = 31$", a statement he must deny by assumption (S). This leads to Frauchiger and Renner famous theorem

Theorem. *(Frauchiger-Renner)* Any theory that satisfies assumptions (Q), (C) and (S) yields contradictory statements when applied to this *Gedankenexperiment.*

On its first version, Frauchiger and Renner paper argued this was an evidence that the single-world assumption (S) should be ruled out, since consistency (C) seems to be a mandatory property in a theory and experiments detecting quantum coherence on macroscopic systems would give assumption (Q) an experimental support, and hence the title "*Single-world interpretations of quantum theory cannot be self-consistent*". The later version, published as "*Quantum theory cannot consistently describe the use of itself*", was less conclusive, and focused simply on presenting the main result and discussing which interpretation would violate which assumption.

2.3 Questions and answers on Frauchiger and Renner's result

Many articles answered Frauchiger and Renner's work with questions and counterarguments. Some readers, in fact, will probably finish the last section with many of these questions and counterarguments, bringing their own favorite interpretations and experiences in different fields of quantum theory to the debate. Hereby, we will cite some of these arguments that particularly helped us to build our approach. We will not stick to explanations about different quantum interpretations, phenomena and other jargons unless it sounds necessary, but complementary reading will be indicated.

First of all, it is relevant to mention an important result which converges to the Frauchiger-Renner theorem. Brukner[12] derived a theorem stating that observer-independent facts, typically known as *facts of the world*, are incompatible with the universal validity of quantum theory and other reasonable assumptions one could make. The theorem can be stated as follows:

Theorem. *(Brukner)* Any theory that satisfies assumptions

- (Q) *Universal validity of quantum theory*: quantum predictions hold at any scale;
- (L) *Locality*: measurements an observer performs have no influence on outcomes of other distant observers;
- (F) *Freedom of choice*: the choice of measurements does not depend on any other random variable described in the experiment;
- (W) *Facts of the world*: truth values atributed to statements A_i made by observers form a Boolean algebra $\mathcal{A}$, which is equiped with a countably additive positive measure $p(A) \geq 0$, $\forall A \in \mathcal{A}$, representing the probability for the statement to be true;

will eventually lead to contradictions.

Brukner also argues that the last assumption is equivalent to the (C) assumption of Frauchiger and Renner. This result is often discussed in recent literature along with Frauchiger and Renner's, and evidence some hidden assumptions made on their work. In fact, the first approach one may take is to question whether assumptions (Q), (C) and (S) are the only assumptions made by agents. At the end of their article, Frauchiger and Renner list a number of interpretations of quantum mechanics and discuss which assumption each interpretation is letting go. But some of them are said to give up an assumption by elimination criteria using the no-go theorem, what may be not necessarily true. It is said that Bohmian mechanics,[13] for instance, do not assume consistency between observations made by different agents, just because it clearly assumes (Q) and (S) as true. This reasoning may sound controversal, and other hidden assumptions may be relevant to this discussion, since (Q), (S) and (C) may be all central hipothesis of a given interpretation, and secondary (protective) hipothesis may be given up instead. Frauchiger and Renner even mention some of them on their work, like the non-probabilistic aspect of the world they are describing; the subjectivity of a measurement, thereby presented as statements made by agents; the definition of key concepts, such as textitagent or *time*. Indeed, great part of the arguing against this controversal result lies on finding hidden assumptions that may be way more questionable than (Q), (C) or (S).[14]

In defense of the Copenhagen interpretation (CI), it was argued that the wave function must have no representational interpretation, but rather a probabilistic one about how the intrinsecally random behavior of microscopic phenomena may be described after interaction with a ultimate measurement device, thus giving up on assumption (Q), that claims universal validity for quantum theory.[15] The discussion is transposed, therefore, to what does make a device the ultimate one. If Alice and Bob are ultimate observers, then a wave function reduction indeed happened. If Wigner and Ursula have the ultimate devices, then it is wrong to say that Alice and Bob in fact observed any outcome. Another argument is that a hidden assumption made by the authors admits subjective consciousness about a measurement outcome even with no colapse, which is fundamentally incompatible with Copenhagen interpretation.[16] Indeed, it can be said that, for CI, to acquire knowledge about a physical property is mathematically described by the reduction of the wave function, so if there is no reduction, then there is no knowledge to be discussed among agents. Similar arguments concerning the misconception of a subjective consciousness and the incorrect use of superposition states or reduction of the wave function are also given by Tausk,[17] although using an *interpretationless* approach.

Given that agents make statements about physical properties of parts of the same joint state at different times, it is interesting to adopt a framework in which the main object to be analyzed is, in fact, a chain of events (where an *event* stands for a set of physical properties measured at a certain instant of time). Frauchiger herself discussed this EWFS within the story-plot framework,[10] and one of its earliest responses used that same framework to argue that no interpretation of quantum mechanics could be in fact ruled out by the *Gedankenexperiment*.[18] An approach within the Theory of Consistent Stories[19] argued that some assumptions made at the given times do not belong to the same family of stories, and thus cannot simultaneously describe the same experiment.[20] Relational interpretation[21] yielded a similar argument, pointing out that agents were unseemly performing self-measurements and making statements which rely on registers from the past that, in fact, do not exist.[22] A timeless formulation of the problem, discribing it in a Page-Wootters universe[iii],[23] showed that the paradox is ruled out for three proposed conditional probabilities, and moved the discussion to whether a definition of conditional probability is the appropriate one for Wigner's Friend Scenarios.[24]

Another class of arguments is that observers completely neglect the uncontrolable and (almost ever) unavoidable degrees of freedom of the environment. The so called Decoherence Program[25,26] emerges to give its analyses of this EWFS in an interpretation-free approach. A first argument is that internal agents (Alice and Bob) cannot properly tell which environmental evolutions had in fact occured, and thus cannot make their statements.[27] A deeper look on this argument yielded the conclusion that, if internal agents in fact observed outcomes, then states describing their minds would change during the measurement performed by superobservers (Ursula and Wigner), due to interaction with the external environment, and if their minds do not change, it must mean that they observed nothing at all.[28] Another interesting work ensures this conclusion, proposing a Bell theorem for pre-measurements which would allow external observers to conclude that Alice and Bob, in fact, did not observe any outcomes if decoherence is not taken into account.[29] This collection of arguments have a strong appeal to the operational feature of quantum theory, and from an information-theoretical point of view, after being presented to the *Gedankenexperiment*, it may be immediate to think that some information has been *stolen* or *hidden* somewhere.

An approach fusing the argument of a proper time registering and the decoherence framework comes from the Montevideo Interpretation.[30] This framework argues that, even though decoherence provides an apparent colapse, it cannot be classified as a proper colapse, and in a first moment any measurement could be undone by an skillful experimentalist with a miraculous evolution-reversing device. Our inability to distinguish between a state that has gone through an unitary evolution and lost its coherences to the environment and a colapsed state comes from the impossibility of measuring time and lenght with infinite precision. This impossibility would axiomatically arise from relativistic arguments, that would demand an infinite energy to build an ideal continuous quantum clock, combined with Heisenberg uncertainty relations. An argument proposes that this approach could completely exterminate the paradox in FR[iv] *Gedankenexperiment*, not only for all practical purposes, as one may say if we take just decoherence in account.[31]

Our work shall follow this path, working on the introduction of a quantum clock which is imprecise by construction, and which will act as the time reference frame in a given

[iii] We shall discuss this particular mechanism in the following chapter.
[iv] From now on, FR will stand for Frauchiger-Renner.

WFS. Associating the limitations of a knowledge one may have over a given experiment with a fundamental uncertainty has a stronger theoretical appeal than associating it to uncontroled degrees of freedom of what one would call an environment, since there is no a priori argument that forbids an observer to build an experiment in a way that most (or all) of these degrees of freedom can be neglected. The main question of our work can be summarized as

> *"What would happen in a Wigner's Friend Scenario if internal friends were in fact isolated, in the sense that superobservers had no access on how time is passing inside the labs? In other words, what are the consequences of the lack of a shared time reference frame in a WFS?"*

This section did not cover the whole of recent literature about Wigner's Friend Scenarios and the FR *Gedankenexperiment*, and did not intented to do so. Interesting works such as one relating the FR *Gedankenexperiment* to Generalized Probability Theories (GPT) can be found in Vilasini, Nurgalieva and del Rio.[32] A disclaimer between the usually mistaken concepts of *formalism* and *interpretation* is made on Baumann and Wolf.[33] Philosphical digressions on the proper language used within the literature about this problem and an analysis of how much this problem could properly contribute to the evolution on the state-of-art of the measurement problem are shown in Hansen and Wolf.[34,35] Durham[36] confronted Brukner's result questioning its validity when special relativity is taken into account, and experimental verifications of it are provided by Proietti *et al.*[37] and Bong *et al.*[38]

In the next chapter, we will introduce the problem of describing time in quantum theory, and also some of the methods used for internalizing reference frames. A candidate for quantum clock will be presented which, although not continuous, can satisfy all conditions an ideal clock must do, with an error that typically decays exponentially with the dimension of the clock. This chapter shall provide all the main tools we use to approach a Wigner's Friend Scenario under the framework of time internalization.

3 TRACKING THE TIME IN QUANTUM THEORY

"He who made eternity out of years remains beyond our reach. His ways remain inscrutable because He not only plays dice with matter but also with time."
— Karel V. Kuchăr[39]

The early years of quantum theory were a fuzzy time for physics, and there is much to be discussed in the field of history and philosophy of science during the first three decades of the 20th century. Particularly, the conceptual transition between classical and quantum mechanics might seem to rely a lot on guesses. Of course, these impressions always arise when one take a look to the past with a mind of the present — it might be easy to say that an abandoned formalism sounds absurd based on 60 years of discussion, but at the edges of the state of art, any try is valid.

Why position in quantum theory takes the role of an operator, while time keeps on being treated as a classical quantity? What could possibly mean the time-energy uncertainty relation if time is a parameter and not an observable, and thus is not subjected to the algebraic boundaries imposed over observables? This chapter is devoted to digress a little about how to deal with time in quantum theory. A brief overview about the history of time within the theory is going to be made in the first part. Following, we recollect some resource-theoretic concepts that might be useful when one deals with quantum reference frames. After that, we introduce a feasible system which is capable of emulating the properties of an ideal clock under suitable conditions.

3.1 Dealing with time in Quantum Mechanics

3.1.1 Classification of *time* and Pauli's argument

What do we mean by *time*? In a first moment, any physicist would answer this question (at least in the context of quantum theory) with the usual "*it is a real parameter in the Schrödinger equation*", or something similar. This is right in some sense, but Pashby[40] invites us to be more careful and replicate with "*which time are we talking about?*". The following tripartite definition of time is fully given by him. A pictorical summary can be found in Figure 2.

When one refers to the real parameter in the Schrödinger equation, one is actually talking about **external time**. It is precisely what it is: a parameter one uses to run over the temporal coordinate of the spatiotemporal inertial reference frame, with respect to which the dynamical equations of motion are going to be described in a simpler form.

However, it is certainly not equivalent to the time one sees a physical property of a system being detected, and certainly not the same time we would be talking about if one asks "*what time is it?*". It is not granted, thus, that an observer has direct access to this parametric time when performing a real experiment.

The values of the external time associated with the existence of proper physical properties of the system are called **event times**. These values can be related with variables which characterize the system, and even other external variables and parameters, to allow us to predict when further events occur. Therefore, given enough data, event times can be *predicted* by a good theory. As an example, Pashby introduce the motion of a classical free particle, and predicts the time when the event "*the particle crosses the point* $x = 0$" occurs given a set of initial data $(q(0), p(0))$.

To keep track of time during an experiment, an experimentalist usually appeals to a **clock**. Clocks are internal systems, and hence subjected to the same dynamics governing the main system. They possess one or more physical properties whose evolution is functionally conected to the external time, and a clock is said to be ideal if it has at least one of these properties which covaries with the external time, i.e., is linearly related to it. In this sense, the external time can be thought as the time measured by an ideal clock, at least in operational terms. The obsevation of this external time is, thus, subjected to the feasibility of an ideal clock.

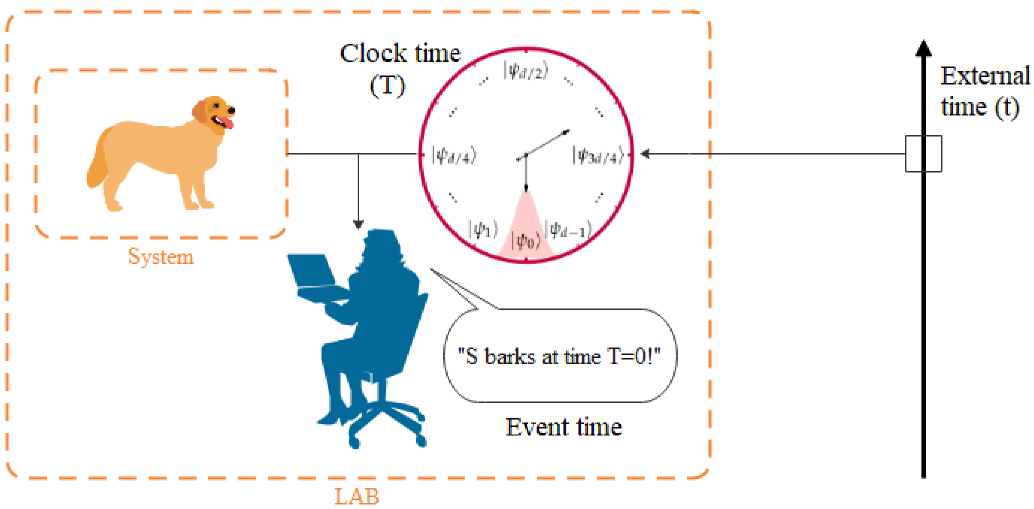

Figure 2 – Schematic representation of Pashby's tripartite classification of time.

Source: By the author.

To guarantee a notion of common time between inertial reference frames, Einstein argues that it takes a colection of distant clocks capable of emiting and receiving light signals. In Einstein's Relativity, thus, one cannot access the external time, but only the time registered by clocks composing the reference frames. The mechanism by which they keep track of the external time, however, is negligible as long as they remain small enought so they do not interact with the system of interest. It is perfectly plausible that these clocks are ideal clocks, keeping track of the external time through variables that covary with it. But the scenario in quantum theory is slightly different, since there is no system small enough that can work as an ideal clock and still not affect the dynamics of the main system. This is the source of the arguments given by Schrödinger and Pauli to forbid the existence of an observable for time.[41,42]

The following discussion will be mostly guided by Martinelli.[41] With $\mathcal{B}(\mathcal{H})$ being the set of operators acting over the Hilbert space $\mathcal{H}$, let us define an operator $T(t) \in \mathcal{B}(\mathcal{H})$, which locally covaries with the external time, i.e.,

$$\frac{dT(t)}{dt} = k, \quad k \in \mathbb{R}, \tag{3.1}$$

meaning that $T(t) = kt\mathbb{I} + T(0)$, where $\mathbb{I}$ is the identity. For practical purposes, we will take $k = 1$ from now on. If the clock also globally covaries with the external time, then $\langle T(t)\rangle = \langle T(0)\rangle + t$, where $\langle T(t)\rangle = \langle\psi|T(t)|\psi\rangle$, $\forall\,|\psi\rangle \in \mathcal{D}(T)$, $\mathcal{D}(T)$ being the domain of T. We can also define a *strongly continuous group* as follows:

Definition 3.1.1. Let $U_t \in \mathcal{B}(\mathcal{H}), \forall t \in \mathbb{R}$ be a family of operators over a given Hilbert space. It is said to be a *one-parameter strongly continuous group* if, for each element of this family,

- $U_{t=0} = \mathbb{I}$;
- $U_{t_1}U_{t_2} = U_{t_1+t_2}, \forall t_1, t_2 \in \mathbb{R}$;
- Given $|\psi\rangle \in \mathcal{H}$, then $\lim_{t\to 0} U_t\,|\psi\rangle = |\psi\rangle$.

With this definition, it is possible to enunciate Stone's theorem:

Theorem. *(Stone)* Let $|\psi\rangle \in \mathcal{H}$, and let $\{U_t\}_{t\in\mathbb{R}}$ be a one-parameter strongly continuous group in which every element is unitary and bounded. Then, $\{U_t\}_{t\in\mathbb{R}}$ will be associated with a self adjoint generator $H \in \mathcal{B}(\mathcal{H})$ such that

$$\lim_{t\to 0} \frac{(U_t - \mathbb{I})}{it}\,|\psi\rangle = H\,|\psi\rangle\,. \tag{3.2}$$

Conversely, let $H \in \mathcal{B}(\mathcal{H})$ be a self adjoint operator. Then, it generates a one-parameter strongly continous group of unitary operators, each given by[i]

$$U_t = e^{iHt}, \quad t \in \mathbb{R}. \tag{3.3}$$

Now, we can perceive some useful properties of our time operator. In the Heisenberg picture, $T(t) = U_t T(0) U_t^\dagger$. But since global covariance already told us that $T(t) = T(0) + t$, we have

$$U_t T(0) U_t^\dagger = T(0) + t, \tag{3.4}$$

which means that T and $T + t$ are unitarilly equivalent (where $T = T(0)$), for any $t \in \mathbb{R}$, and thus share the same spectrum. This implies that spec$(T) \equiv \mathbb{R}$, and by applying U_t from the right, we get

$$U_t T - T U_t = [U_t, T] = t U_t. \tag{3.5}$$

Also, the commuting relation with the self adjoint generator of $\{U_t\}_{t\in\mathbb{R}}$ yields to

$$\begin{aligned}
[T, H] &= TH - HT & (3.6)\\
&= T \lim_{t\to 0} \frac{(U_t - \mathbb{I})}{it} - \lim_{t\to 0} \frac{(U_t - \mathbb{I})}{it} T & (3.7)\\
&= - \lim_{t\to 0} \frac{[U_t, T]}{it} & (3.8)\\
&= \lim_{t\to 0} i U_t = i, & (3.9)
\end{aligned}$$

where we used Eq.(3.5) and the property that $\lim_{t\to 0} U_t = \mathbb{I}$. These are the properties expected from a time operator T which covaries both locally and globally with the external time t. Pauli claims, however, that there would be no feasible system detaining such physical property.

His claim can be summarized with the Stone theorem. Since T is an observable, it is a self adjoint operador capable of generating a family $\{U_\lambda = e^{i\lambda T}\}_{\lambda\in\mathbb{R}}$ which is a one-parameter strongly continous group with unitary elements. That being said, if $[H, T] = -i$, then

$$[H, U_\lambda] = [H, e^{i\lambda T}] = \left[H, \sum_{n=0}^{\infty} \frac{(i\lambda T)^n}{n!}\right] = \sum_{n=1}^{\infty} \frac{(i\lambda)^n}{n!} [H, T^n], \tag{3.10}$$

since for $n = 0$, $[H, I] = 0$. Then, by the property that $[T, H] = i \Rightarrow [T^n, H] = inT^{n-1}$,

$$[H, U_\lambda] = -\sum_{n=1}^{\infty} \frac{(i\lambda)^n}{(n-1)!} iT^{n-1} = \lambda \sum_{n=1}^{\infty} \frac{(i\lambda T)^{n-1}}{(n-1)!} = \lambda U_\lambda. \tag{3.11}$$

This allows us to make the same process we did with the time operator, i.e., apply $U_\lambda^\dagger$ by the left side of the commutator, obtaining

$$U_\lambda^\dagger H U_\lambda = H + \lambda. \tag{3.12}$$

[i] In this work, $\hbar = 1$.

This relation implies that H and $H + \lambda$ are equivalent up to an unitary transformation, which means they share the same spectrum spec(H). Since λ is any real number, it turns out that spec(H) $\equiv \mathbb{R}$, which would be impossible: an Hamiltonian without a lower bound would allow the extraction of an infinite amount of energy of the system. Such miraculous energy source is unfeasible, an that is why Pauli (and many others after him[43,44]) argues against the existence of a system capable of keeping track of the external time in an ideal way. Summarizing, Pauli's theorem can be enunciated as follows

Theorem. *(Pauli)* Let $\mathcal{H}$ be a separable Hilbert space, and let $H, T \in \mathcal{B}(\mathcal{H})$ be self adjoint operators over this Hilbert space. If T obeys a global covariance relation with every element of the one-parameter strongly continuous group of unitaries generated by H, i.e., if

$$U_t T U_t^\dagger = T + t, \quad \forall t \in \mathbb{R}, \tag{3.13}$$

then spec(H) = spec(T) $\equiv \mathbb{R}$.

This result is often interpreted as a disclaimer for giving up paying too much attention to this foundational concern of the quantum theory: if there is no possible time opertor, then let time be a parameter in Schrödinger Equation. Following Pashby[40] again, what could it even mean to measure the time of a particle? If measuring position means one is asking what is the probability of detecting a particle in a given region of space, there is definetly some oddity in asking what is the probability of detecting a particle in a given region of time. If your outcome is null, then where have the particle gone? Many causal and conservational problems could arise from this sort of scenario.

However, one should not give up so easily on the persecution of a time operator, since for quantum theory every measurable property is represented by an operator acting over the Hilbert space, and the time an event occurs is expected to be measurable. Avoiding here metaphysical discussions that appeal to Pauli's theorem,[44] we will highlight Pashby's argument that everything this argument is telling is that the existence of (i) a bounded self adjoint hamiltonian H, (ii) a self adjoint operator T with $\sigma(T) = \mathbb{R}$ and (iii) a one-parameter strongly continuous group of unitaries generated by H such that $U_t^\dagger T U_t = T + t$, $\forall t \in \mathbb{R}$ are never simultaneously allowed. This still allows some paths to be taken in the search for an operator that tells us how time is passing for a given system, and we shall now take the one followed by Dirac, Wheeler and DeWitt.

3.1.2 Extended Hilbert space

There is a strong connection between Quantum Mechanics and the action-angle formalism of Classical Mechanics. Dirac[45] proposes an extention of the phase space of the canonical variables to include another two coordinates, canonically conjugated to each other, given by t and $-W$, where W would mean a fixed value for the energy. The (quantum) equation of motion thus will be given by the identity

$$(H - W)\psi(x,t) = 0, \tag{3.14}$$

where x represents the many degrees of freedom quantized by the conventional process. Since t and $-W$ are a canonical pair, it implies in quantum mechanical terms that $[W,t] = i$, and following the first quantization for position and momentum, one may atribute the form

$$W = i\frac{\partial}{\partial t}, \tag{3.15}$$

obtaining the expression

$$H\psi(x,t) - i\frac{\partial}{\partial t}\psi(x,t) = 0. \tag{3.16}$$

This might sound like nothing new, but Dirac's technique of expanding the phase space (or similarly, the Hilbert space) removes the double burden from H of simultaneously providing the spectrum of energies and generating time translations. With the inclusion of operators t and $-W$, H is now responsible for providing just the possible outcomes of energy of the system, while time evolution is dictated by $-W$. There would be no problem, at a first moment, that $-W$ had an unbounded spectrum and H had not, since Eq. (3.16) would have solutions just for eigenvalues of $-W$ which were also eigenvalues of H. Pauli's Theorem is, therefore, not forbidding this construction.

Dirac's approach was a first step that led Wheeler and DeWitt, more than 30 years later, to develop their constraint equation for quantum gravity.[46] It was also the impulse for Page and Wootters to develop their model of static universe, with a time given relationally with respect to the variable t of this extended Hilbert space.[23,47] Let us take a look on this formulation more carefully.

Page and Wootters propose a universe described by a bipartite Hilbert space, $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_C$, whose hamiltonian is non-interacting between the parts, i.e.,

$$H = H_S \otimes \mathbb{I}_C + \mathbb{I}_S \otimes H_C. \tag{3.17}$$

Inspired by Wheeler-DeWitt constraint equation, the Page-Wootters mechanism consists of a static universe whose global physical state is a solution of the constraint equation

$$H\,|\Psi\rangle\rangle = 0. \tag{3.18}$$

In this universe, the dynamics of a part is given *relationally* with respect to the other. Therefore, one can define a time operator T_C with $[T_C, H_C] = i$, and however, because H_S is the hamiltonian responsible for providing the energy spectrum of the main system, Pauli's theorem play no role here since $[T_C, H_S] = 0$. T_R is thus keeping track of time passage and still does not imply a system S with unbounded energy spectrum. System C can evidently be thought of as the clock in Figure 2. Indeed, if we take $|\theta_C(t)\rangle$ to be eigenstates of the operator T_C (with eigenvalue t), we can define the state of the system S relationally to the clock as

$$|\psi_S(t)\rangle = \langle\theta_C(t)|\Psi\rangle\rangle. \tag{3.19}$$

Notice that t here is not the external time unless system C is an ideal clock. However, as long as T_C has an spectrum which is isomorph to $\mathbb{R}$, Schrödinger equation can be recovered locally in S:

$$\frac{d}{dt}|\psi_S(t)\rangle = \frac{d}{dt}(\langle\theta_C(t)|\Psi\rangle\rangle) = \frac{d}{dt}(\langle\theta_C(t)|)\,|\Psi\rangle\rangle + \langle\theta_C(t)|\,\frac{d}{dt}(|\Psi\rangle\rangle), \tag{3.20}$$

and since $\frac{d}{dt}|\Psi\rangle\rangle = -iH\,|\Psi\rangle\rangle$ and $\frac{d}{dt}|\theta_C(t)\rangle = -iH_C\,|\theta_C(t)\rangle$, then

$$\begin{aligned} \frac{d}{dt}|\psi_S(t)\rangle &= i\mathbb{I}_S\,\langle\theta_C(t)|H_C|\Psi\rangle\rangle - iH_S\,\langle\theta_C(t)|\mathbb{I}_C|\Psi\rangle\rangle - i\mathbb{I}_S\,\langle\theta_C(t)|H_C|\Psi\rangle\rangle & (3.21)\\ &= -iH_S\,\langle\theta_C(t)|\Psi\rangle\rangle & (3.22)\\ &= -iH_S\,|\psi_S(t)\rangle\,. & (3.23) \end{aligned}$$

So quantum mechanics is working in system S. Nevertheless, t has a new meaning: it is not just the real parameter on the Schrödinger's equation, but the eigenvalues of a time operator T_C of a clock. Martinelli[41] points out how this is not so different from the usual interpretation of the parameter t when we think operationally. In fact, the time of any experiment is given relationally with respect to a clock hanging on a wall inside the lab. The only difference is that this clock is classical, and thus there is no coherences acting between it and the system being measured. When the clock is quantum, however, coherences start to play a crucial role.

Although the Page-Wooters mechanism provides us an example of how Pauli's theorem is avoidable with Dirac's technique of extending the Hilbert space, and push the problem one step closer to the scenario described in Figure 2, there are remaining issues to be dealt with. First, one usually has no access to the physical state $|\Psi\rangle\rangle$. In other words, inside a lab, one is often aware of the descriptions of parties S and C individually, but not of the state $|\Psi\rangle\rangle$ which is solution of Eq. (3.18) and provides the relational dynamics between S and C. So how can one go from ρ_S and ρ_C to $|\Psi\rangle\rangle$? Second, if we are looking for a reasonable description of the world, our whole universe should be feasible. In this sense, we are still stuck with the problem that, while $[T_C, H_S] = 0$, still $[T_C, H_C] = i$, implying $\sigma(H_C) = \mathbb{R}$, which is forbidden. A possible solution to this problem is to search for systems that mimic the clock properties described previously under certain circumstances, despite its hamiltonian being bounded. The following sections aim to answer these questions.

3.2 An overview on Quantum Reference Frames

To answer the first question, suppose that Alice and Bob are astronauts traveling through space in their respective spaceships. They are separated in space, in the sense that they cannot see each other in their radars, but can still communicate with each other through their radios. Suppose also that they do not share any common star or planet in their radars. If Alice is supposed to send Bob instructions so he is capable of reaching her out, what can she tell him? Instructions involving directions will have no meaning, since they do not share any Cartesian reference frame.[2] This simple example pictorizes how not any information can be stored and shared through a string of bits or qubits. If Alice really wants to help Bob, she must send some system capable of *pointing* to some direction, or else Bob's only option is to cover every possible direction to find his fellow explorer.

Information which can be stored and shared through a string of classical or quantum bits is called *speakable*. Lists of instructions or coordinates, for example, are information of this type. However, coordinates mean nothing without a predetermined set of directions in space, and "*meet me at 2 o'clock*" hardly have any signifcance if one has no access to a working clock synchronized to the local timezone. Information that cannot be stored in strings of classical or quantum bits, but rather require some sort of *asymmetric* system, are called *unspeakable*.[2,48]

One may think of a quantum system $|\psi(g)\rangle \in \mathcal{H}$ on which two types of transformation can be performed: the usual one consists of taking $|\psi(g)\rangle \to |\psi'(g)\rangle$ through operations on $|\psi\rangle$. But one could take $g \to g'$ instead, i.e., exchange the adopted reference frame. The set of every $g \in G$ forms a group[ii], and $\{U_g\}_{g\in G}$ is the representation of this group on the Hilbert space $\mathcal{H}$. This is the algebraic description of processes one so often deals with in Physics, such as galilean or lorentzian boosts, and in our previous example, would be the operation Bob would perform over Alice's instructions if they shared a reference system.

But without the knowledge of what is the reference frame g of Alice, everything Bob can do is to cover every direction that might lead him to Alice. In our description, this means to perform an average over every possible transformation between Alice's reference frame and his, an operation called *G-twirling*. So if Alice sends a state $\rho = |\psi(g)\rangle \langle\psi(g)|$, then because of the Superselection Rule (SSR)[49] imposed over the system by the lack of a

[ii] A group G is a set equipped with a binary operation and an identity element e, such that, for every $g, g', g'' \in G$: $U_{g''} = U_{g'+g} = U_{g'}U_g$; $U_{g''g'}U_g = U_{g''}U_{g'g}$; and there is a $g^{-1} \in G$ such that $U_{g^{-1}}U_g = U_gU_{g^{-1}} = e$.

shared reference frame, Bob will describe this state as

$$\mathcal{G}[\rho] = \int_{g\in G} U_g \rho U_g^\dagger dg. \tag{3.24}$$

The G-twirling is precisely the operation that solves the issue of leaving from the individual states ρ_S and ρ_C and arriving at the physical state $|\Psi\rangle\rangle$ from which we can derive the relational state $|\psi_S(t)\rangle$. Notice that, indeed, the state $\rho = \rho_S \otimes \rho_C$[iii] might not be static, since $[\rho, H]$ will hardly vanish for almost every reasonable description of a lab. But if G is the set of time shifts generated by $H = H_S \otimes \mathbb{I}_C + \mathbb{I}_S \otimes H_C$, with H being time-independent, then

$$[\mathcal{G}[\rho], H] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} U_t \rho U_t^\dagger dt H - H \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} U_t \rho U_t^\dagger dt \tag{3.25}$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} U_t [\rho, H] U_t^\dagger dt, \tag{3.26}$$

but since for $\frac{\partial \rho}{\partial t} = 0$ we have

$$i\frac{d\rho(t)}{dt} = i\frac{d(U_t \rho U_t^\dagger)}{dt} \tag{3.27}$$

$$= i\frac{dU_t}{dt}\rho U_t^\dagger + iU_t\rho\frac{dU_t^\dagger}{dt} \tag{3.28}$$

$$= HU_t\rho U_t^\dagger - U_t\rho H U_t^\dagger \tag{3.29}$$

$$= U_t(H\rho - \rho H)U_t^\dagger \tag{3.30}$$

$$= U_t[H, \rho]U_t^\dagger, \tag{3.31}$$

then

$$[\mathcal{G}[\rho], H] = -i \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \frac{d\rho(t)}{dt} dt = -i \lim_{T\to\infty} \frac{\rho(T) - \rho(-T)}{2T}, \tag{3.32}$$

and because the eigenvalues of $\rho(T)$ and $\rho(-T)$ are always non-negative and lower or equal than 1, this limit automatically vanishes, resulting on the static equation

$$[\mathcal{G}[\rho], H] = 0. \tag{3.33}$$

Within this formalism, if Π_t^C are projectors over the eigenspaces of T_C associated with outcomes t, then the relational state of the main system will be given by

$$\rho_S(t) = \mathrm{Tr}_C \left\{ \frac{(\mathbb{I}_S \otimes \Pi_t^C)\mathcal{G}[\rho](\mathbb{I}_S \otimes \Pi_t^C)}{\mathrm{Tr}\{(\mathbb{I}_S \otimes \Pi_t^C)\mathcal{G}[\rho]\}} \right\}. \tag{3.34}$$

The scenario described by Figure 2, that we have been developing hitherto, forbids an observer to directly access the external parametric time t. We are never allowed to describe states such as $U_t\rho U_t^\dagger$, but must instead tell the passage of time relationally to

[iii] Of course, one could describe an entangled initial state. But since this is seldom the case between a system and its clock, we will treat the separable case here.

the time of the lab, given by a clock state. Since time shifts are represented in quantum theory as phase shifts, the lack of access to this external phase reference frame imposes a SSR that forbids states asymmetric with respect to the $U(1)$ group generated by H. Every quantum scenario with restrictions over allowed operations might be conveniently treated by a quantum resource theory, and the structure of this formalism of physical time measurements is an invitation for a resource-theoretic treatment.

3.2.1 Quantum Resource Theory of Asymmetry

The concept of resource is borrowed from the economical concept of *scarcity*. A certain state of things is more or less valuable according to the easiness with which it can be extracted, obtained or implemented.[48,49] It is convenient thus to consider a theory that can keep track of how valuable is a given state or experimental protocol under certain physical restrictions. Structurally, a resource theory is a commutative ordered monoid.[50] This can be defined as follows

Definition 3.2.1. *(Commutative ordered monoids)* Let $\mathcal{A}$ be a set equipped with a binary operation $+$, a distinguish element e, called identity, and a ordering relation $\geq$. Then $\mathcal{A}$ is said to be a commutative ordered monoid if, for any $a, a', a'' \in \mathcal{A}$, we have

- if $a \geq a'$ and $a' \geq a''$, then $a \geq a''$;
- if $a \geq a'$ and $a' \geq a$, then $a = a'$;
- $a + (a' + a'') = (a + a') + a''$ and $a + a' = a' + a$;
- $a + e = a$;
- if $a \geq a'$, then $a + a'' \geq a' + a''$.

In physical terms, one should read the symbol $\geq$ as "*is convertible to*". Whenever $\rho \geq \sigma$, it means that, in the set of operations and descriptions allowed by the restriction imposed over the experimental scenario, there will be ways of transforming ρ into σ. If, otherwise, $\rho \ngeq \sigma$, it must be read as "*ρ is not convertible to σ*", meaning that there is no way of performing this transformation with the knowledge one has access to in the experimental scenario. This is the case, for example, of Bob performing $\rho_A \to \rho_B = U_g \rho_A U_g^\dagger$, since he has no means of accessing Alice's reference frame.

It is thus typical to define a quantum resource theory (QRT) in terms of free operations and free states, such as the following[49]

Definition 3.2.2. *(Quantum Resource Theory)* Let $\mathcal{H}_i$ be Hilbert spaces of subsystems i, and $\mathcal{B}(\mathcal{H})$ the set of bounded operators in $\mathcal{H}$. Let $\mathcal{S}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ the subset of bounded, positive semi-defined operators with unitary trace in $\mathcal{H}$, $\mathcal{O}(\mathcal{H}_A, \mathcal{H}_B) \subset \mathcal{B}(\mathcal{H}_A, \mathcal{H}_B)$ be a set of CPTP maps and $\mathcal{F}(\mathbb{C}, \mathcal{H}) \subset \mathcal{S}(\mathcal{H})$. Then, the tuple $\mathcal{R} = (\mathcal{F}, \mathcal{O})$ is a Quantum Resource Theory if

- $\mathbb{I} \in \mathcal{O}(\mathcal{H})$, $\forall \mathcal{H}$;
- $\Phi \in \mathcal{O}(\mathcal{H}_A, \mathcal{H}_B)$ and $\Theta \in \mathcal{O}(\mathcal{H}_B, \mathcal{H}_C) \rightarrow \Theta \otimes \Phi \in \mathcal{O}(\mathcal{H}_A, \mathcal{H}_C)$, $\forall \mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C$.

The set $\mathcal{F}$ is called the set of *free states*, while the set $\mathcal{O}$ is called the set of *free operations*. The demands for constructing a Quantum Resource Theory are perfectly reasonable: to do nothing is always a free operation, and a sequence of free operations must be free as well. A corolary of this definition is often highlighted due to its interpretational convenience, being known as the Golden Rule of QTR:

Definition 3.2.3. *(Golden Rule of Quantum Resource Theories)* Let $\mathcal{R} = (\mathcal{F}, \mathcal{O})$ be a QRT. If $\Phi \in \mathcal{O}(\mathcal{H}_A, \mathcal{H}_B)$ and $\rho \in \mathcal{F}(\mathcal{H}_A)$, then $\Phi(\rho) \in \mathcal{F}(\mathcal{H}_B)$.

The interpretation is as simple as it seems: acting a free operation over a free state will necessarily lead to a free state. In other words, free operations cannot convert free states into resource states. Bob, indeed, could not turn any instruction given by Alice into something more useful only by performing operations he was allowed without a reference frame.

Quantum Resource Theories are always constructed operationally, taking into account what is the central phenomena to be studied to define the set of free operations or of free states (usually, one starts by defining just one of the sets, and the other is obtained from the Golden Rule).[49] The quantum resource theory of entanglement, for example, starts from the assumption that for bipartite scenarios, operations performed locally over each of the parties and classical comunication between them are always allowed. The set of free states, as consequence, is restricted to separable states. For the quantum resource theory of asymmetry, which we will be dealing with, it is useful to begin by defining the set of free states.

Definition 3.2.4. *(Set of free states for QRT of Asymmetry)*[49] Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ be the set of bounded operators acting on $\mathcal{H}$. Let $\mathcal{S}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ be the set of bounded, positive semi defined operators with unitary trace in this Hilbert space. Let also $\{U_g\}_{g \in G}$ be the group of transformation representations generated by the group G over $\mathcal{S}(\mathcal{H})$. Then, $\mathcal{F}(\mathcal{H}) \subset \mathcal{S}(\mathcal{H})$ such that

$$\mathcal{F}(\mathcal{H}) := \{\mathcal{G}[\rho], \forall \rho \in \mathcal{S}(\mathcal{H})\}, \tag{3.35}$$

is said to be the set of free states for a QRT of asymmetry with respect to the group G, with $\mathcal{G}$ being the G-twirling operation previously defined.

The set of free states for a QRT of asymmetry can be thus defined as the set of states which are invariant over a G-twirl. Within this theory, the G-twirling operation may often be called the *symmetrization* operation, and $\mathcal{F}(\mathcal{H})$ is called the set of *symmetric states*.[48] Any state not living inside this set is, of course, an *asymmetric state*, and therefore carries resource for forbidden operations. It implies, however, a different set $\mathcal{F}(\mathcal{H})$ for each SSR one may impose over a system, i.e., for each group G with respect to there is no way of centainly performing transformations. The G-twirling can be defined as a *resource destructing operation*, since it takes both resource and free states into the set of free states.[49]

Starting from this definition, it is possible to construct a set of free operations for SSR as

Definition 3.2.5. *(Set of free operations for QRT of Asymmetry)* Let $\mathcal{F}(\mathcal{H})$ be the set of free states for a QRT of Asymmetry subjected to a certain SSR. The set $\mathcal{O}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ such that

$$\mathcal{O}(\mathcal{H}) := \{\Phi \in \mathcal{B}(\mathcal{H}) | [\Phi(\cdot), U_g(\cdot)U_g^\dagger] = 0, \forall g \in G\}. \tag{3.36}$$

is said to be the set of free operations for this QRT.

Notice that, if Φ is also covariant to global translations,[48] then

$$\mathcal{G}[\Phi(\rho)] = \int_{g\in G} U_g \Phi(\rho) U_g^\dagger dg = \int_{g\in G} \Phi(U_g \rho U_g^\dagger) dg = \Phi(\mathcal{G}[\rho]), \tag{3.37}$$

A map Φ satisfying this condition is often called *G-covariant channel* or *translation-covariant channel*. When the group responsible for generating translations is $U(1)$, these maps receive a specific form. Since $\mathcal{H}$ is a finite separable Hilbert space and $U(1)$ is a compact group, there is a way of representing $\mathcal{H}$ as[48]

$$\mathcal{H} = \bigoplus_{q\in\mathcal{Q}} \mathcal{H}_q, \tag{3.38}$$

where each $\mathcal{H}_q$ is a m_q-dimensional Hilbert space called *charge sector q*. Taking this into account, a general G-covariant channel possess the form[48,51]

$$\Phi(\rho) = \sum_{q\in\mathcal{Q}} \alpha_q \pi_q(\rho) \pi_q^\dagger, \tag{3.39}$$

where π_q is a permutation between the states of a charge sector $\mathcal{H}_q$. In other words, a SSR generated by a lack of phase reference frame confines the information of a system in its charge sectors, so that it cannot flow between them. This will get clearer with a further example, but first, let us analyze how can we quantify the asymmetry of a system.

The most useful feature of a quantum resource theory for our purposes is the possibility of idenitifying *monotones* associated with the resource property. Monotones are functions capable of witnessing or quantifying the convertibility between states, and are mathematically described as homomorphisms between the resource theory $\mathcal{A}$ and $\mathbb{R}_{\geq 0}$, where $\mathbb{R}_{\geq 0}$ is the set of non-negative real numbers.

Definition 3.2.6. *(Homomorphism)* Let $\mathcal{A}$ and $\mathcal{A}'$ be commutative ordered monoids. An ordered map $f : \mathcal{A} \to \mathcal{A}'$ is an homomorphism if, for every $a, a' \in \mathcal{A}$, we have

- $a \geq a' \Rightarrow f(a) \geq f(a')$;
- $f(a + a') = f(a) + f(a')$;
- $f(0) = 0$.

Since $\mathbb{R}_{\geq 0}$ is also a commutative ordered monoid with respect to addition, we want to search for a functional f capable of quantifying the convertibility between elements of $\mathcal{A}$ with real numbers.

A general monotone for asymmetry can be defined as[48]

Definition 3.2.7. *(Monotone for asymmetry)* Let $\mathcal{R} = (\mathcal{F}, \mathcal{O})$ be a QRT of Asymmetry with respect to a group G. A function $f : \mathcal{S}(\mathcal{H}) \to \mathbb{R}$, where $\mathcal{S}(\mathcal{H})$ is the set of bounded, self-adjoint operators on $\mathcal{H}$ with unitary trace, is said to be an asymmetry monotone if

- $f(\rho) \geq 0, \forall \rho \in \mathcal{S}(\mathcal{H})$;
- $f(\rho) = 0, \forall \rho \in \mathcal{F}(\mathcal{H})$;
- f is monotone under G-covariant channels.

The first condition is trivial if f is going to be an homomorphism of the commutative ordered monoid $\mathcal{A}$ which we associate to $\mathcal{R}$. Indeed, $x \geq 0$, $\forall x \in \mathcal{A}$, since any resource state must be freely converted to nothing, i.e., discarded. Addiction of some auxiliary systems and disposal of degrees of freedom must be always allowed operations if we are willing to conserve quantum theory[iv]. The second one is reasonable, but not obvious: free states are not resource states, and thus its resource measure must be null. The last one states that operating with free operations over a system will never generate resource. Vaccaro *et al.*[54] propose a candidate function that satisfies these three conditions:

[iv] Since they are essential processes for fundamental theorems, such as Stinespring dilation.[52,53]

Definition 3.2.8. *(Holevo asymmetry)* Let $\mathcal{H}$ be a Hilbert space and $\mathcal{G}$ be the G-twirling operation definded for some group G. The quantity

$$A_G(\rho) := S(\mathcal{G}[\rho]) - S(\rho), \quad \forall \rho \in \mathcal{S}(\mathcal{H}) \tag{3.40}$$

is called Holevo asymmetry, with $S(\cdot)$ being the von Neumann entropy.

Proposition 3.2.1. The Holevo asymmetry is an asymmetry monotone.

Proof. By the concavity of the von Neumann entropy,[55]

$$S(\mathcal{G}[\rho]) = S\left(\int_{g\in G} U_g \rho U_g^\dagger dg\right) \geq \int_{g\in G} S(U_g \rho U_g^\dagger) dg = \int_{g\in G} S(\rho) dg = S(\rho), \tag{3.41}$$

and thus $S(\mathcal{G}[\rho]) - S(\rho) \geq 0$. The second property is trivial, since if $\rho \in \mathcal{F}(\mathcal{H})$, then $\mathcal{G}[\rho] = \rho$, and therefore

$$A_G(\rho) = S(\mathcal{G}[\rho]) - S(\rho) = 0. \tag{3.42}$$

For the third condition, we can represent the most general G-covariant channel as

$$\Phi(\rho) = \sum_{i=1}^{N} p_i \Phi_i(\rho), \tag{3.43}$$

where $\sum_{i=1}^N p_i = 1$ and $\Phi_i \in \mathcal{O}(\mathcal{H})$. We must now show that

$$A_G(\rho) \geq \sum_{i=1}^{N} p_i A_G(\Phi_i(\rho)). \tag{3.44}$$

This must ensure that any free operation over a state ρ will always decrease the amount of resource measured by the Holevo asymmetry. Explicitly, we have

$$S(\mathcal{G}[\rho]) - S(\rho) \geq \sum_i p_i S(\mathcal{G}[\Phi_i(\rho)]) - \sum_i p_i S(\Phi_i(\rho)), \tag{3.45}$$

or yet

$$S(\mathcal{G}[\rho]) - \sum_i p_i S(\mathcal{G}[\Phi_i(\rho)]) \geq S(\rho) - \sum_i p_i S(\Phi_i(\rho)). \tag{3.46}$$

Starting from the left side,

$$\begin{aligned} S\left(\int_{g\in G} U_g \rho U_g^\dagger dg\right) - \sum_i p_i S\left(\int_{g\in G} U_g \Phi_i(\rho) U_g^\dagger dg\right) \geq \int_{g\in G} S(U_g \rho U_g^\dagger) dg \\ - \sum_i p_i \int_{g\in G} S(U_g \Phi_i(\rho) U_g^\dagger) dg, \end{aligned} \tag{3.47}$$

by the concavity of S. Then, since $S(U_g \rho U_g^\dagger) = \rho$, $\forall g \in G$,

$$S(\mathcal{G}[\rho]) - \sum_i p_i S(\mathcal{G}[\Phi_i(\rho)]) \geq S(\rho) - \sum_i p_i S(\Phi_i(\rho)), \tag{3.48}$$

the same result obtained by Eq. (3.44). □

It is interesting to state also, for a bipartite system such as the Page-Wootters mechanism, a quantification of the local resources available in each of the parties. Such monotone is defined as *local asymmetry*,[54] and is computed as

$$A_{G\otimes G}(\rho_{SC}) = S(\mathcal{G}_{G\otimes G}[\rho_{SC}]) - S(\rho_{SC}), \tag{3.49}$$

where

$$\mathcal{G}_{G\otimes G}[\rho_{SC}] = \int_{g\in G}\int_{g'\in G} U_g^S \otimes U_{g'}^C(\rho_{SC})U_g^{S\dagger} \otimes U_{g'}^{C\dagger} dg dg' \tag{3.50}$$

is called local symmetrization or local G-twirling. This quantity $A_{G\otimes G}$ is also an asymmetry monotone.[48] By combining Holevo and local asymmetries, one can obtain another useful measure for asymmetry of bipartite systems, given by

$$A^{sh}_{G\otimes G}(\rho_{SC}) = A_{G\otimes G}(\rho_{SC}) - A_G(\rho_{SC}). \tag{3.51}$$

This quantity is called *shared asymmetry.*[54] Mendes and Soares-Pinto[56] and Martinelli and Soares-Pinto[57] analyzed this quantity for bipartite systems with product initial states $\rho = \rho_S \otimes \rho_C$, the later deriving the identity

$$A^{sh}_{G\otimes G}(\rho_{SC}) = A_G(\rho_S) + A_G(\rho_C) - A_G(\rho_{SR}), \tag{3.52}$$

which made them name this quantity as *mutual asymmetry*, refering to the informational notion of mutual information.

3.2.2 Catalytic convertibility

Catalytic convertibility is a common concept in Quantum Resource Theories, and is borrowed from the Chemistry concept of catalysis. Mathematically, a Resource Theory that allows catalytic convertibility can be deffined by a *non-cancelative* comutative ordered monoid.[50]

Definition 3.2.9. (*Non-cancelative commutative ordered monoid*) Let $x, y, z \in \mathcal{A}$ be elements of a commutative ordered monoid. $\mathcal{A}$ is said to be *non-cancelative* if

$$x + z \geq y + z \;\not\Longrightarrow\; x \geq y. \tag{3.53}$$

In resource-theoretic terms, it means that x is not convertible in y by itself, but in the presence of z, this process is allowed. Fritz[50] provides a nice example: the conversion of *wood+nails* to *table* is not allowed, but the conversion *wood+nails+hammer* to *table+hammer* is possible. The state z is called the *catalyst* of this conversion. A resource theory that is non-cancelative can be turned into a cancelative one by redefining its ordering relation, such that for any $x, y, z \in \mathcal{A}$,

$$x + z \geq y + z \implies x \succeq y. \tag{3.54}$$

This relation can be read as "*x is catalytic convertible into y*", and a resource theory which is cancelative becomes an abelian ordered group[v]. Resource Theories of this type are the ones that allow borrowing of resources, since the concept of a debt resource $-x$ is included in an abelian ordered group. This is not the case, however, of the QRT of Asymmetry we are going to deal with.

The problem we are willing to solve can be summarized as a catalytic conversion. We are not allowed to perform the conversion $\rho(t) \to |\uparrow\uparrow\rangle\langle\uparrow\uparrow|$ or $|\downarrow\downarrow\rangle\langle\downarrow\downarrow|$ within a Wigner's Friend Scenario, where $\rho(t)$ is the entangled state the superobserver uses to describe the measurement occuring inside the lab. But with the help of a clock, it may be obtained $\rho(t) \otimes \tau_C \to |\uparrow\uparrow\rangle\langle\uparrow\uparrow|$ (or $|\downarrow\downarrow\rangle\langle\downarrow\downarrow|$) $\otimes\, \tau_C$. Indeed, it has already been proved[58] that this conversion is nearly achievable through symmetric operations (in our context, G-covariant channels), which leads to a resulting state whose fidelity with the desired state is lower bounded. We will announce this result here.

Theorem. (*Lower bound for fidelity in catalytic conversion*) Let $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_C$ be a separable Hilbert space subjected to a SSR. Let $\mathcal{R}$ be the QRT associated to this SSR, $\tau_C \in \mathcal{H}_C$ be an accessible state, and $\sigma_S \otimes \tau_C$ be a desired state. Then, there exists $\Phi \in \mathcal{O}(\mathcal{H}_C, \mathcal{H})$ a G-covariant channel such that

$$F(\Phi(\tau_C), \sigma_S \otimes \tau_C) \geq 2^{-\frac{\Delta A}{2}}, \tag{3.55}$$

with $F(\cdot,\cdot)$ being the Uhlmann fidelity[vi] between the desired state and the state obtained through the G-covariant channel, and $\Delta A = A_G(\sigma_S \otimes \tau_C) - A_G(\tau_C)$ is the difference between Holevo asymmetries of the desired state and the available state.

Equipped with these concepts, one is capable of, given a global hamiltonian for a bipartite system and an initial state, obtain a solution to the Page-Wootters mechanism, quantify the amount of resource in this given state and obtain the relational partial states. It is interesting to present a simple example to familiarize the reader.

3.2.3 An example: two qubits

We can think of an universe constituted of two qubits, one of them playing the role of main system S and the other being the clock C. The global hamiltonian will be given by

$$H = H_S \otimes \mathbb{I}_C + \mathbb{I}_S \otimes H_C = \omega(\sigma_z^S \otimes \mathbb{I}_C + \mathbb{I}_S \otimes \sigma_z^C), \tag{3.56}$$

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v i.e., a commutative ordered monoid that has, for every $a \in \mathcal{A}$, an element $a^{-1} \in \mathcal{A}$ such that $aa^{-1} = a^{-1}a = e$.

vi Defined as[55] $F(\rho,\sigma) = \left[\mathrm{Tr}\{\sqrt{\rho^{1/2}\sigma\rho^{1/2}}\}\right]^2$.

σ_z being the typical Pauli matrix. The initial state will be given by the two spins pointing to the same direction

$$\rho = |+\rangle\langle+|_S \otimes |+\rangle\langle+|_C\,, \tag{3.57}$$

where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, and $\{|0\rangle, |1\rangle\}$ is the σ_z eigenbasis. The G-twirling will be thus described by

$$\mathcal{G}[\rho] = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} e^{-iHt}\rho e^{iHt} dt, \tag{3.58}$$

since the evolution is periodic in $\tau = 2\pi/\omega$. The integrand can be explicitly described by

$$e^{-iHt}\rho e^{iHt} = \frac{1}{4}\begin{pmatrix} 1 & e^{i2\omega t} & e^{i2\omega t} & e^{i4\omega t} \\ e^{-i2\omega t} & 1 & 1 & e^{i2\omega t} \\ e^{-i2\omega t} & 1 & 1 & e^{i2\omega t} \\ e^{-i4\omega t} & e^{-i2\omega t} & e^{-i2\omega t} & 1 \end{pmatrix}, \tag{3.59}$$

and performing the G-twirling will lead to

$$\mathcal{G}[\rho] = \frac{1}{4}\begin{pmatrix} 1 & & & \\ & 1 & 1 & \\ & 1 & 1 & \\ & & & 1 \end{pmatrix}. \tag{3.60}$$

Notice, then, that the G-twirling erases any coherence between different charge sectors: there are nonzero elements just inside the subspaces such that $\text{span}(\mathcal{H}_0) = \{|00\rangle\}$, $\text{span}(\mathcal{H}_1) = \{|01\rangle, |10\rangle\}$ and $\text{span}(\mathcal{H}_2) = \{|11\rangle\}$.

Let us assume now that a time operator over the clock state is given by $T = \sigma_x^C$. Evidently this is not an ideal clock, since $[T, H_C] \neq i$, but this will provide us a good insight on how clocks work. This is a two-ticks clock with outcomes "$+$" and "$-$". If we are going to detect the state $|\pm\rangle\langle\pm|_C$, then

$$\frac{(\mathbb{I}_S \otimes |\pm\rangle\langle\pm|_C)\mathcal{G}[\rho](\mathbb{I}_S \otimes |\pm\rangle\langle\pm|_C)}{\text{Tr}\{(\mathbb{I}_S \otimes |\pm\rangle\langle\pm|_C)\mathcal{G}[\rho]\}} = \frac{1}{4}\begin{pmatrix} 1 & \pm1 & \pm\frac{1}{2} & \frac{1}{2} \\ \pm1 & 1 & \frac{1}{2} & \pm\frac{1}{2} \\ \pm\frac{1}{2} & \frac{1}{2} & 1 & \pm1 \\ \frac{1}{2} & \pm\frac{1}{2} & \pm1 & 1 \end{pmatrix}, \tag{3.61}$$

and tracing off the clock degrees of freedom, we finally get the conditional state

$$\rho_S(\text{“}\pm\text{”}) = \frac{1}{2}\begin{pmatrix} 1 & \pm\frac{1}{2} \\ \pm\frac{1}{2} & 1 \end{pmatrix}. \tag{3.62}$$

The amount of information concerning the passage of external time this system is providing is given by

$$A_G(\rho) = S(\mathcal{G}[\rho]) - S(\rho) = \frac{3}{2}, \tag{3.63}$$

since $\mathcal{G}[\rho]$ has eigenvalues $\frac{1}{4}$ with multiplicity 3 and 0 with multiplicity 1, and $S(\rho) = 0$. The local asymmetry, however, is given by

$$\mathcal{G}_{G\otimes G}[\rho] = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} e^{-iH_S t}\rho_S e^{iHt}dt \otimes \frac{\omega}{2\pi}\int_0^{2\pi/\omega} e^{-iH_C t'}\rho_C e^{iH_C t'}dt' = \frac{\mathbb{I}_S\otimes\mathbb{I}_C}{4}, \qquad (3.64)$$

i.e., the maximally mixed state. Its entropy is well known to be $S(\mathcal{G}_{G\otimes G}[\rho]) = \log 4 = 2$, and thus the shared asymmetry is given by

$$A^{sh}_{G\otimes G}(\rho) = A_{G\otimes G}(\rho) - A_G(\rho) = \frac{1}{2}. \qquad (3.65)$$

Given a lemma proposed by Simões,[48] shared asymmetry is bounded for $G = U(1)$ by the relation

$$0 \leq A^{sh}_{G\otimes G}(\rho_{SC}) \leq \min\{\log(\dim\mathcal{H}_S), \log(\dim\mathcal{H}_C)\}, \qquad (3.66)$$

what implies that, for this specific case, $0 \leq A^{sh}_{G\otimes G}(\rho) \leq 1$. This indicates that, even though our clock is not the worst possible, it is not the best either. Indeed, the conditional probabilities for the clock and the system pointing to the same direction are $P(+|+) = P(-|-) = \frac{3}{4}$, while the probabilities of the clock mistracking the dynamics of the system is $P(+|-) = P(-|+) = \frac{1}{4}$. If the clock and the system shared the maximum amount of asymmetry possible, then the former would perfectly track the dynamics of the later.

This was a simple example on how one can deal with these quantities. Generally, the QRT of Asymmetry has been applied to study how much information can be derived from symmetries of hamiltonians, and the main idea of the theory is to analyze how asymmetry beween states or between states and expressions can provide an understanding about some physical processes.[48,59] There are also discussions on how a measurement over one of the systems may affect the dynamics of the other and the distribution of information between parties.[60–62] We are going to use it to identify how well a system is keeping track of the dynamics of the other when we insert a clock in a Wigner's Friend Scenario. Our first question made in the last paragraph of section 3.1 must be satisfyed by now, and we can venture on trying to answer the second one.

3.3 A model of a feasible clock: quasi-ideal clock states

One of the first atempts to build a quantum clock is due to Salecker and Wigner.[63] In this work, they propose a system of k free particles moving between two resting particles A and B, separated in space. Whenever a moving particle reaches a resting particle, it is deflected elastically towards the other resting particle. Each moving particle, however, takes a different period $n_i\tau$ to go back and forth, in a way that particle 1 takes $n_1\tau$ to do so, particle 2 takes $n_2\tau$ and so on, making a parallel with the different pointers of a classical clock (tracking seconds, minutes and hours). With the system working, $k+2$

photons will be sent towards the clockwork, each targeting one of the moving or resting particles, and being scatered back to a detector in the same position as the lightsource, as examplified in Figure 3. Time is tracked based on the ratio between the distances $\overline{A1}/\overline{AB}$, $\overline{A2}/\overline{AB}$, etc, which are invariant under lorentzian boosts.

Authors were concentrated on discussing the feasibility of this clock in operational ways, deriving a lower bound for its mass and mass spread to ensure it works. Even though no mention was made with respect to an operator canonically conjugated to the hamiltonian and the main focus was to quantum mechanically describe relativistic scenarios, this approach sure did leave a clue for the next steps of the construction of a proper clock.

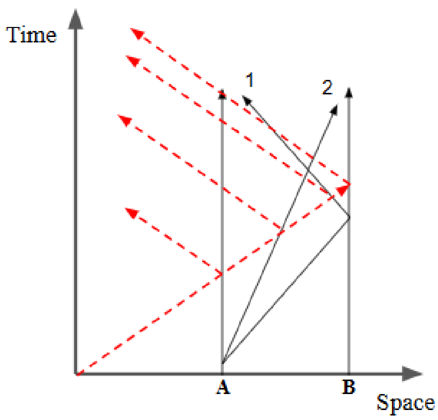

Figure 3 – Model for Salecker-Wigner Clock: full black arrows represent particle worldlines, while dashed red arrows represent photon worldlines.

Source: Adapted from SALECKER;WIGNER.[63]

Peres[64] brings the issue of constructing a clock based on a system with an observable T canonically conjugated to the Hamiltonian. His first proposal is given by a quatization of the classical time of displacement of a free particle,

$$\tau = \frac{m}{p}q \rightarrow T = m(P^{-1}X + XP^{-1}). \tag{3.67}$$

Notice that, if $H = \frac{1}{2m}P^2$, then

$$[T, H] = \frac{1}{2}(P^{-1}XP^2 + XP - PX - P^2XP^{-1}) = i. \tag{3.68}$$

It is relevant to notice[40] that the classical time of displacement is actually given by $\tau = \frac{m}{p}(q(t) - q(0))$, and one usually obtains Eq. (3.67) by a change of reference frame in order to make $q(0) = 0$. This is not allowed in quantum theory, however, since if $X(0) = 0$, then $X(t) = 0, \forall t \in \mathbb{R}$. Setting $q(0) = 0$ corresponds to setting $\langle X(0) \rangle = 0$, which can be done by picking a specific state, but do not imply on the quantum version of Eq. (3.67). This time operator would then be properly given by

$$T = m(U_t^\dagger (P^{-1}X + XP^{-1})U_t - (P^{-1}X + XP^{-1})), \tag{3.69}$$

resulting in $[T, H] = 0$. This would also raise problems such as how this operator could not capture the consequences of the first measurements over the outcomes of the second, just like any two-measurements based quantity[vii]. Furthermore, Peres himself discards this operator as a candidate for a time operator by arguing that its eigenvectors do not have a clear interpretation.

Another model proposed by Peres in the same work, though, is described by a d-dimensional system with hamiltonian

$$H = \omega \sum_{n=0}^{d-1} n \left|n\right\rangle \left\langle n\right| , \tag{3.70}$$

$\{|n\rangle\}$ being the orthonormal basis of energy eigenstates. Another orthornormal basis can be obtained by

$$\left|\theta_k\right\rangle = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} e^{-i2\pi kn/d} \left|n\right\rangle , \tag{3.71}$$

with $k \in [0, d-1] \subset \mathbb{Z}$. This is a sort of "Fourier transform" of the energy basis. Notice that, indeed,

$$\langle \theta_k | \theta_{k'} \rangle = \frac{1}{d} \sum_{n,n'=0}^{d-1} e^{-i2\pi(kn-k'n')/d} \langle n | n' \rangle = \frac{1}{d} \sum_{n=0}^{d-1} e^{-i2\pi n(k-k')/d} = \delta_{k,k'}, \tag{3.72}$$

$\forall k, k' \in [0, d-1]$, and

$$\sum_{k=0}^{d-1} \left|\theta_k\right\rangle \left\langle \theta_k\right| = \frac{1}{d} \sum_{k,n,n'=0}^{d-1} e^{-i2\pi k(n-n')/d} \left|n\right\rangle \left\langle n'\right| = \sum_{n,n'=0}^{d-1} \delta_{nn'} \left|n\right\rangle \left\langle n'\right| = \mathbb{I}. \tag{3.73}$$

Also, these states are discretely shifted by U_t in steps of τ/d, with $\tau = 2\pi/\omega$

$$e^{-iH\tau/d} \left|\theta_k\right\rangle = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} e^{-i2\pi kn/d} e^{-iH\tau/d} \left|n\right\rangle = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} e^{-i\frac{2\pi n}{d}\left(k+\frac{\omega\tau}{2\pi}\right)} \left|n\right\rangle = \left|\theta_{k+1}\right\rangle , \tag{3.74}$$

and for every $k \in [0, d-1]$, $|\theta_k\rangle$ will be completely indistinguishable from $|\theta_{k+md}\rangle$, $m \in \mathbb{Z}$, since

$$\left|\theta_{k+md}\right\rangle = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} e^{-i2\pi n(k+md)/d} \left|n\right\rangle = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} e^{-i2\pi nm} e^{-i2\pi kn/d} \left|n\right\rangle = \left|\theta_k\right\rangle , \tag{3.75}$$

[vii] A typical example of this sort of problem is given by the construction of an work operator based on two energy measurements.[65]

for n and m are both integers.

States $\{|\theta_k\rangle\}_{k=0}^{d-1}$ can be thought of as time markings in a classical clock. Indeed, for an analog clock, the passage of the hour hand by the 1 mark means always "*1 o'clock*", despite the number m of times this passage happened. To remove the degeneracy, one must perform a number of other measurements, such as look through the window to ensure if it is day or night, and check on the calendar to learn what day, month and year is it.

It is possible to construct a time operator diagonal in this pointer basis, given by

$$T = \sum_{k=0}^{d-1} k\frac{\tau}{d} |\theta_k\rangle \langle\theta_k| , \tag{3.76}$$

which will capture the discrete jumps caused by $U_{\tau/d}$. However, it is not canonically conjugate to the hamilonian, since

$$\langle\theta_k|[T,H]|\theta_k\rangle = \langle\theta_k|TH|\theta_k\rangle - \langle\theta_k|HT|\theta_k\rangle = \frac{k\tau}{d}(\langle H\rangle - \langle H\rangle) = 0, \tag{3.77}$$

and because we want $[T,H] = i$, these diagonal terms should not vanish. Furthermore, for all $k \in [0, d-1]$

$$\langle H\rangle = \langle\theta_k|H|\theta_k\rangle = \sum_{n=0}^{d-1} n\omega| \langle n|\theta_k\rangle |^2 = \sum_{n=0}^{d-1} \frac{n\omega}{d} = \frac{(d-1)}{2}\omega, \tag{3.78}$$

and

$$\langle H^2\rangle = \langle\theta_k|H^2|\theta_k\rangle = \sum_{n=0}^{d-1} n^2\omega^2| \langle n|\theta_k\rangle |^2 = \sum_{n=0}^{d-1} \frac{n^2\omega^2}{d} = \frac{(d-1)(2d-1)}{6}\omega^2, \tag{3.79}$$

leading to an uncertainty on the energy

$$\Delta H = \sqrt{\langle H^2\rangle - \langle H\rangle^2} = \frac{\omega}{2\sqrt{3}}\sqrt{d^2-1} = \frac{\langle H\rangle}{\sqrt{3}}\sqrt{\frac{d+1}{d-1}}. \tag{3.80}$$

For small dimensions, $\Delta H \approx \langle H\rangle$, and for large clocks, $\Delta H \to \langle H\rangle /\sqrt{3}$. Therefore, even though a system with access to high values of energy is picked to be a clock and prepared in a state $|\theta_0\rangle$, its behavior will be strongly dominated by quantum phenomena, not recovering a classical clock.

Anyway, Peres shows how this clock can indeed keep track of time in three classic scenarios: measuring the time of flight of a free particle traveling between two detectors; timing the duration of an atomic decay; controlling the application of a magnetic external field responsible for fliping a spin. These results show, nevertheless, that the resolution of the clock is bounded from above. The more accurate this clock is, the more it will interact with the system which it is timing on, up to the point where the described phenomena could even cease from happening (with a too highly energetic clock, for example, the decaying atom could be constantly fed and this decay could come to a halt, resulting in a quantum Zeno effect[66]). For Peres, this represented a structural scar in the Hamiltonian formulation of quantum theory, for it would imply the impossibility of properly differentiating a state with respect to time, the crucial operation in Schrödinger equation. The last sentence in his work claims: "(...) the Hamiltonian approach to quantum physics carries the seeds of its own demise".[64] As we have properly discussed before, time tracked by a clock is not the same time with respect to which differentiation takes place in the Schrödinger equation, and hitherto the Hamiltonian approach is serving to its purposes. This reservation made, the clock proposed by Peres, often called Salecker-Wigner-Peres (SWP) clock, proved itself worthy of a closer look.

A more complex model for a clock was recently proposed[67] consisting of two structures: a system capable of keeping track of the external time, called a *clockwork*, and a string of classical bits each registering an outcome of the observable T at different instants of time, called *tick registers*. This definition externalizes a concern with the feats a suitable clock must have: measurements of time should not disturb its capability of keeping track of time in the future. A clock that tells time only once does not worth the effort of its preparation. This was actually one of the concerns of Salecker and Wigner when analyzing the minimum mass a quantum clock should have. In operational terms, a clock is a tuple constituted of an initial state of the clockwork, ρ_C, and a set of time-homogeneous markovian dynamics.[68]

Definition 3.3.1. *(Quantum clock)* Let $\rho_C^0 \in \mathcal{H}_C$ be the state of a d-dimensional system, and $\{\mathcal{M}_t\}$ be a family of CPTP linear maps such that $\mathcal{M}_t : \mathcal{H}_C \to \mathcal{H}_C \otimes \mathcal{H}_T$, $t \geq 0$. Then, the tuple $(\rho_0^C, \mathcal{M}_t)$ is called a *quantum clock.*

Practically, one could return to our example of the clock given by a qubit. If it is initially in the state $\rho_0^C = |+\rangle\langle+|$, a good definition of an evolution map is

$$\mathcal{M}_\tau = \sum_{t=0,1} \Pi_t U_{\tau/2}(\cdot) U_{\tau/2}^\dagger \Pi_t \otimes |t\rangle\langle t|\,, \tag{3.81}$$

with $\Pi_0 = |+\rangle\langle+|$ and $\Pi_1 = |-\rangle\langle-|$. Since the evolution U_t is generated by $H_C = \omega\sigma_z^C$, the time evolution flips the qubit between the states $|+\rangle\langle+|$ and $|-\rangle\langle-|$, and we can define the tick register after $t = N\tau$ as

$$\rho_T(N) = \mathrm{Tr}_C\{\bigcirc_{i=1}^N \mathcal{M}_\tau^i(\rho_C)\} = |1\rangle\langle 1|_1 \otimes |0\rangle\langle 0|_2 \otimes |1\rangle\langle 1|_3 \otimes |0\rangle\langle 0|_4 \otimes ... \otimes |t_N\rangle\langle t_N|_N\,, \tag{3.82}$$

where $\mathcal{M}_\tau^i : \mathcal{H}_C \to \mathcal{H}_C \otimes \mathcal{H}_{T_i}$ registers the tick in the i-th entry of the tick register. This tick register can be read by an observer as "*tic-toc-tic-toc-...*", a structure often called two-ticks clock.

This definition of a quantum clock has been recently used to prove how the precision of single time signals is bounded, as Peres was aware of.[68] A singleton generator is a clock capable of generating a tick register given by

$$\rho_T(N) = |0\rangle\langle 0|_1 \otimes |0\rangle\langle 0|_2 \otimes ... \otimes |0\rangle\langle 0|_{j-1} \otimes |1\rangle\langle 1|_j \otimes |0\rangle\langle 0|_{j+1} \otimes ... \otimes |0\rangle\langle 0|_N . \quad (3.83)$$

This tick register can be read as "*silence-...-silence-tic!-silence-...*". Particularly, the sharpness of the singleton signal, defined as $R = \mu^2/\sigma^2$, where μ is the mean time when the time signal is supposed to be generated and σ is the spread of the peak, is confined to be always lower or equal to d^2, for any singleton generator with dimension $d \geq 4$. Thus even though the dimension of the clock is incresed in order to reach higher precision in time tracking, this process will never be infinetly precise as a measurement performed over an ideal clock. This result justifies a choice of clock which intrinsecally considers this gaussian behavior in time signal generation.

It is about time to introduce our chosen model for a clock. It was proposed first by Woods, Silva and Oppenheim[7] based on the Salacker-Wigner-Peres clock, and can be defined as

Definition 3.3.2. *(Quasi-ideal clock states)* Let $\mathcal{H}_C$ be a d-dimensional Hilbert space, with $\{|\theta_k\rangle\}$ being the orthonormal basis of time eigenstates of the Salecker-Wigner-Peres clock. The subspace $\Lambda_{\sigma,n_0} \subseteq \mathcal{H}_C$ is said to be the set of quasi-ideal states, and is given by

$$\Lambda_{\sigma,n_0} := \{|\psi(k_0)\rangle \in \mathcal{H}_C; \quad k_0 \in \mathbb{R}; \sigma \in (0,d) \subset \mathbb{R}; n_0 \in (0, d-1) \subset \mathbb{R}\}, \quad (3.84)$$

where

$$|\psi(k_0)\rangle = \sum_{k \in S_d(k_0)} A e^{-\frac{\pi}{\sigma^2}(k-k_0)^2} e^{i2\pi n_0 (k-k_0)/d} |\theta_k\rangle . \quad (3.85)$$

Here, A is a normalizing factor such that $\langle\psi(k_0)|\psi(k_0)\rangle = 1$, $\forall k_0 \in \mathbb{R}$, and $S_d(k_0)$ is the set of d integers or half-integers centered around k_0, i.e.,

$$S_d(k_0) := \begin{cases} \left\{k \in \mathbb{Z}; -\frac{d}{2} \leq k - k_0 < \frac{d}{2}\right\}, & \text{if } d \text{ is even} \\ \left\{k \in \mathbb{Z} + \frac{1}{2}; -\frac{d}{2} \leq k - k_0 < \frac{d}{2}\right\}, & \text{if } d \text{ is odd} \end{cases} . \quad (3.86)$$

One can think of this state as a clock hand pointing to the time k_0, which, unlike the SWP clock, does not have to be an integer. This hand, however, is not sharp: it is actually spread around k_0 by a gaussian width σ. The n_0 represents the mean occupation of the energy eigenstates $\{|n\rangle\}$, and will be responsible for controling the intereferences between different time marks $|\theta_k\rangle$. The measurment of time will still be described by

Peres' time operator, nevertheless, this state already shows signs of its immense potential when we analyze $\langle T(t)\rangle$. As it is shown in Figure 4, for dimensions as great as $d = 8$, the expectation value for T already covaries with the external time t for most part of the period of the clock with reasonably small deviation. The bigger the size of the clock, the more accurate $\langle T\rangle$ becomes as a measurement of the external time t.

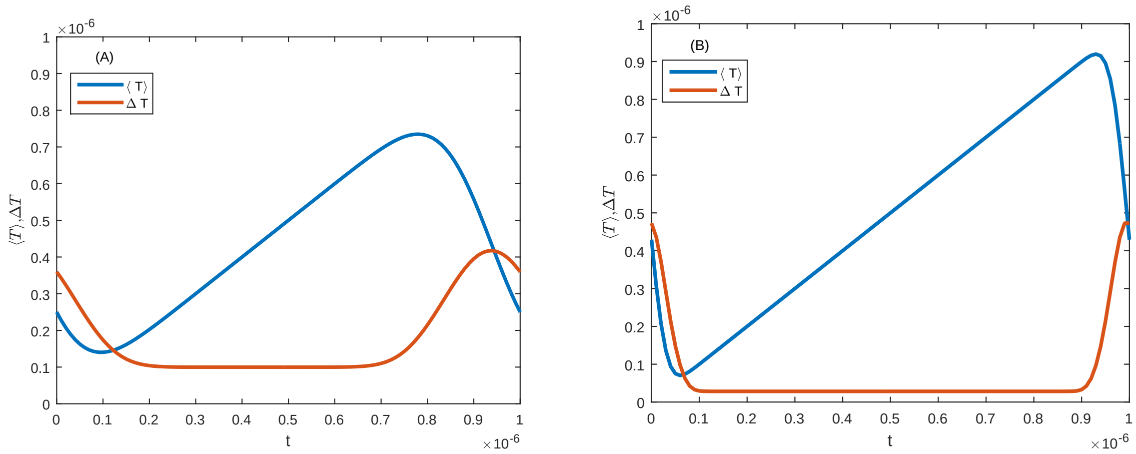

Figure 4 – Expected value and standard deviation of T measurements over quasi-ideal clock states with clock dimension (A) $d = 8$ and (B) $d = 100$. For both plots, $\sigma = \sqrt{d}$; $n_0 = \frac{d-1}{2}$; and $\tau = 1\mu$s.

Source: By the author.

Some useful properties arise from these states, and we will hereby introduce them and discuss its implications. Interesting secondary properties can be found in Appendix A.

Lemma 3.3.1. (*Quasi-continuity*) Let $\mathcal{H}_C$ be a d-dimensional Hilbert space and $|\psi(k_0)\rangle \in \Lambda_{\sigma,n_0}$ a quasi-ideal clock state whose dynamics is generated by the Hamiltonian H_C. Then, for any $t \in \mathbb{R}$,

$$e^{-iH_C t}\,|\psi(k_0)\rangle = \sum_{k\in S_d(k_0+td/\tau)} Ae^{-\frac{\pi}{\sigma^2}(k-k_0+td/\tau)^2}e^{i2\pi n_0(k-k_0+td/\tau)/d}\,|\theta_k\rangle + |\epsilon\rangle\,, \tag{3.87}$$

with

$$|\langle\theta_k|\epsilon\rangle| \leq O\left(t\ \mathrm{poly}(d)e^{-\frac{\pi}{4}d}\right), \quad d \to \infty. \tag{3.88}$$

In other words, differently from the Salecker-Wigner-Peres clock, whose steps have to be quantized, this clock is allowed to track the external time continuously up to an error that should quickly vanish as the size of the clock becomes large enough. This property is crucial for our approach, once a G-twirling demands the knowledge of how infinitesimal time translations occur over the system.

Lemma 3.3.2. (*Quasi-canonical commutation*) Let $\mathcal{H}_C$ be a d-dimensional Hilbert space and $|\psi(k_0)\rangle \in \Lambda_{\sigma,n_0}$ a quasi-ideal clock state whose dynamics is generated by the Hamiltonian H_C. Let T be the previously defined time operator, diagonal in the basis $\{|\theta_k\rangle\}$. Then, for any $k_0 \in \mathbb{R}$,

$$[T, H_C]\,|\psi(k_0)\rangle = i\,|\psi(k_0)\rangle + |\epsilon_c\rangle\,, \tag{3.89}$$

with

$$|\,\langle\epsilon_c|\epsilon_c\rangle\,|^2 = O\left(\text{poly}(d)e^{-\frac{\pi}{4}d}\right), \quad d \to \infty. \tag{3.90}$$

This is an impressive result that demonstrates how weak the restriction imposed by Pauli theorem actually is. As we mentioned before, his argument would only forbid the simultaneous existence of a bounded hamiltonian H responsible for generating continuous shifts on the eigenvalues of a T operator cannonically conjugated to it. The assumption that this would rule out every possibility of quantizing time in quantum theory sustained over not a so solid foundation. We have, therefore, a pair of operators T and H, canonically conjugated, that when measured over a large system described by a quasi-ideal clock state $|\psi(k_0)\rangle$ generate the statistics of a canonical pair, just like a time operator would do with the hamiltonian. Besides that, the expected value of T covaries with the external time with quite a small error. In fact, whichever is the gaussian deviation σ, it is demonstrated[7] that $\Delta H \Delta T = \frac{1}{2}$, saturating the Heisenberg uncertainty principle.

Another feature of this clock is its autonomous control, i.e., how well its dynamics is not affected by the action of a external specific potential. Even though this is not a property upon which we will rely very much, it is still worth mentioning to emphasize how close the quasi-ideal clock states are from an ideal clock. For an ideal clock, with a continuous distinguishable basis $\{|t\rangle\}$ of eigenstates of the operator $T = \int_{-\infty}^{\infty} t\,|t\rangle\langle t|\,dt$, the change of the evolution generator from H_C to $H_C + V(t)$, $V(t)$ being an integrable potential whose entries are functions of the eigenvalues t, leads only to a global phase. Let $|\psi\rangle \in \mathcal{H}_C$, then

$$\langle t_0|e^{-i(H_C+V(t))t}|\psi\rangle = \langle t_0 - t|e^{-iV(t)t}|\psi\rangle = \langle t_0 - t|\mathbb{I} - iV(t)t - \frac{V^2(t)t^2}{2!} + ...|\psi\rangle\,, \tag{3.91}$$

and inserting the completeness relation $\mathbb{I} = \int_{-\infty}^{\infty} dt\,|t\rangle\langle t|$ between the products $V^n(t) = V \cdot V \cdot ...$, we are led to

$$\langle t_0|e^{-i(H_C+V(t))t}|\psi\rangle = e^{-i\int_{t_0-t}^{t_0} V(t')dt'}\,\langle t_0 - t|\psi\rangle\,. \tag{3.92}$$

This leads us to the last lemma.

Lemma 3.3.3. (*Autonomous quasi-control*) Let $\mathcal{H}_C$ be a d-dimensional Hilbert space and $|\psi(k_0)\rangle \in \Lambda_{\sigma,n_0}$ a quasi-ideal clock state whose dynamics is generated by the Hamiltonian

H_C. Let $V(t)$ be a smooth[viii] external potential, diagonal in the time basis, given by

$$V = \frac{2\pi}{\tau} \sum_{k=0}^{d-1} V_0 \left(\frac{2\pi k}{d} \right) |\theta_k\rangle \langle\theta_k| \,. \tag{3.93}$$

Then, the time translation generated by $H_C + V$ is given by

$$\langle\theta_k| e^{-i(H_C+V)t} |\psi(k_0)\rangle = e^{-i \int_{k-td/\tau}^{k} V_d(t')dt'} \langle\theta_k|\psi(k_0 + td/\tau)\rangle + \langle\theta_k|\epsilon_v\rangle \,, \tag{3.94}$$

with

$$|\langle\theta_k|\epsilon_v\rangle| \leq O\left(\text{poly}(d) e^{-\frac{\pi}{d}\frac{d}{\xi}}\right), \quad d \to \infty, \tag{3.95}$$

with ξ being a function inversely proportional to the derivatives of V_0.

This must answer the second question made at the end of Section 3.1. We are now aware of how can we begin from local states ρ_S and ρ_C to obtain the global state $\mathcal{G}[\rho]$ which is a solution for the constraint Eq. (3.33), and inherits a whole theoretical structure from quantum resource theory of asymmetry in this process. We also know how should a quantum clock work, and have access to a model of finite-size clock which can mimic the properties of an ideal clock with a small error. We are able, from now on, to approach to a Wigner's Friend Scenario and observe the consequences of including a quantum clock in its description. This is the main result of this work, and will be done in the next chapter.

viii i.e., an operator whose derivatives are all finite within integration limits.

4 INTERNALIZING TIME IN WIGNER'S FRIEND SCENARIOS

Hitherto we introduced the sort of scenario we are interested in studying, highlighting its relevance to the foundations of quantum theory and listing some interesting approaches that propose more or less convincing answers to the problem. We argued that, if the friend inside the lab is in fact isolated, she must not share a quantum reference frame with the external superobserver, and asked ourselves what could possibly arise from this situation. Then, a brief introduction to some resource theoretical tools was made, and also a short passage over the properties of the model of clock we are willing to insert to the problem.

It is finally time to approach our problem. Section 4.1 will introduce our Wigner's Friend Scenario (WFS) and discuss how each observer is describing the state of the whole lab. Section 4.2 will deal with the problem of internalizing the external time to the joint system lab+clock. Section 4.3 will finally analyze our results and propose some other approaches.

4.1 Model

Let us suppose a WFS constituted of a single lab. Inside the lab, there are at $t = 0$ a qubit with initial state $|+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \in \mathcal{H}_S$; an observer, labeled as Friend and described in the Hilbert space $\mathcal{H}_F$ with $\text{span}(\mathcal{H}_F) = \{|\perp\rangle, |\uparrow\rangle, |\downarrow\rangle\}$, and an ideal clock capable of tracking time in a way that $\mathcal{G}[\rho] = \rho(t)$, with $t \in \mathbb{R}$. At the time $t = t_F$, the Friend is going to perform a measurement of σ_z over the system, describing the final state of the whole lab (system+herself) as either $|\uparrow\uparrow\rangle_{FS}$ or $|\downarrow\downarrow\rangle_{FS}$.

Outside the lab, a superobserver, called Wigner, is going to perform the measurment of the global property

$$|ok\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow\uparrow\rangle_{FS} + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\downarrow\downarrow\rangle_{FS} \tag{4.1}$$

of the lab, at a time $t_W > t_F$. He, however, has no access to how time is passing inside the lab, and can appeal just to a quasi-ideal clock such as the one described in Chapter 3. He then will describe the relational lab state as

$$\rho^W_{SF}(k) = \text{Tr}_C\left\{\frac{(\mathbb{I}_{FS}\otimes\Pi^C_k)\mathcal{G}[\rho_{FS}\otimes\rho_C](\mathbb{I}_{FS}\otimes\Pi^C_k)}{\text{Tr}\{(\mathbb{I}_{FS}\otimes\Pi^C_k)\mathcal{G}[\rho_{FS}\otimes\rho_C]\}}\right\}, \tag{4.2}$$

and for $k > d\frac{t_F}{\tau}$, with τ being the period of the clock, he will calculate

$$P_W(ok|k) = \text{Tr}\{\Pi^{FS}_{ok}\rho^W_{FS}(k)\}. \tag{4.3}$$

From the Friend's viewpoint, the lab state becomes static once her measurement is performed. The system will be thus in one of the product states previously enumerated, and from her perspective, Wigner will observe the outcome *ok* with probability

$$P_F(ok|\uparrow) = |\langle ok|\uparrow\uparrow\rangle_{FS}|^2, \quad P_F(ok|\downarrow) = |\langle ok|\downarrow\downarrow\rangle_{FS}|^2. \tag{4.4}$$

Our main quantities are going to be the Δ's, that represent the difference between the Friend and Wigner's predicions of the probability with which the outcome *ok* occurs. They can be defined as

$$\Delta_0(\theta,\phi) = P_F(ok|\uparrow) - P_W(ok|k); \quad \Delta_1(\theta,\phi) = P_F(ok|\downarrow) - P_W(ok|k), \tag{4.5}$$

and the paradox vanishes when

$$\Delta_0 = \Delta_1 = 0, \tag{4.6}$$

for any $|ok\rangle$ being measured.

A possible model for describing the entangling process between the system and the Friend is such that

$$U_t^{FS} = \begin{pmatrix} 1-\Theta(t-t_F) & 0 & -i\Theta(t-t_F) & 0 & 0 & 0 \\ 0 & 1-\Theta(t-t_F) & 0 & 0 & 0 & -i\Theta(t-t_F) \\ -i\Theta(t-t_F) & 0 & 1-\Theta(t-t_F) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -i\Theta(t-t_F) & 0 & 0 & 0 & 1-\Theta(t-t_F) \end{pmatrix}, \tag{4.7}$$

where $\Theta(t-t_F)$ is Heaviside's step function[i]. This would lead to an instantaneous transition as long as $t > t_F$, but it is a too unrealistic description for what is happening in a measurement.

We propose another model for the description of this measurement, given by $H_{SF} \in \mathcal{B}(\mathcal{H}_{FS})$ such that

$$H_{FS} = \omega_0(|\uparrow\uparrow\rangle\langle\perp\uparrow| + |\perp\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\perp\downarrow| + |\perp\downarrow\rangle\langle\downarrow\downarrow|). \tag{4.8}$$

This hamiltonian generates an evolution inside the lab such that

$$U_t^{FS} = e^{-iH_{FS}t} = \begin{pmatrix} \cos\omega_0 t & 0 & -i\sin\omega_0 t & 0 & 0 & 0 \\ & \cos\omega_0 t & 0 & 0 & 0 & -i\sin\omega_0 t \\ -i\sin\omega_0 t & 0 & \cos\omega_0 t & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -i\sin\omega_0 t & 0 & 0 & 0 & \cos\omega_0 t \end{pmatrix}. \tag{4.9}$$

[i] $\Theta(t-t_F) = 1$ if $t > t_F$, 0, if $t < t_F$ and $\frac{1}{2}$, if $t = t_F$.

This is inspired by De Pasquale *et al.*[69] Even though the authors appeal necessarily to degenerate degrees of freedom of the pointers of the measurement apparatus (here, the Friend) in order to rule out the periodic behavior of the evolution, this is not precisely the appropriate framework for our WFS. Decoherence would introduce a state reduction to Wigner's perspective also, and the insertion of a clock would then become obsolete. Anyway, Wigner here is a superobserver, which means he is capable of distinguishing any degree of freedom of the Friend. So the evolved state of the lab, given $\rho_{FS}(0) = |\perp\rangle\langle\perp|_F \otimes |+\rangle\langle+|_S$, will be given by

$$U_t^{FS}\rho_{FS}(0)U_t^{FS\dagger} = \frac{1}{2}\begin{pmatrix} \cos^2\omega_0 t & \cos^2\omega_0 t & \frac{i}{2}\sin 2\omega_0 t & 0 & 0 & \frac{i}{2}\sin 2\omega_0 t \\ \cos^2\omega_0 t & \cos^2\omega_0 t & \frac{i}{2}\sin 2\omega_0 t & 0 & 0 & \frac{i}{2}\sin 2\omega_0 t \\ -\frac{i}{2}\sin 2\omega_0 t & -\frac{i}{2}\sin 2\omega_0 t & \sin^2\omega_0 t & 0 & 0 & \sin^2\omega_0 t \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{i}{2}\sin 2\omega_0 t & -\frac{i}{2}\sin 2\omega_0 t & \sin^2\omega_0 t & 0 & 0 & \sin^2\omega_0 t \end{pmatrix}, \quad (4.10)$$

which will actually describe the entangling process $|\perp\rangle_F \otimes |+\rangle_S \to |\Phi_+\rangle_{FS}$[ii] only for $t = \left(m + \frac{1}{2}\right)\frac{\pi}{\omega_0}$. So if there is a way that Wigner's clock can detect $k = \left(m + \frac{1}{2}\right)\frac{\pi d}{\omega_0 \tau}$, then he can properly describe this entangling process through his quasi-ideal clock.

4.2 Symmetrization

Let us now assume that Wigner describes the initial state of his whole universe (lab+clock) as the tripartite product state

$$\rho(0) = |\perp\rangle\langle\perp|_F \otimes |+\rangle\langle+|_S \otimes |\psi(0)\rangle\langle\psi(0)|_C\,, \quad (4.11)$$

where $|\psi(0)\rangle$ is the quasi-ideal clock state introduced in the last chapter, with $k_0 = 0$. Global time translations are going to be generated by the non-interacting hamiltonian

$$H = H_{FS} \otimes \mathbb{I}_C + \mathbb{I}_{FS} \otimes H_C, \quad (4.12)$$

between the lab and the clock, where H_{FS} is given by Eq. (4.8) and H_C is the Salecker-Wigner-Peres clock hamiltonian given in equation 3.70. This non-interacting approach allows us to split the unitary U_t in two, each one acting in a part, so that

$$\rho(t) \approx \rho_{FS}(t) \otimes |\psi(td/\tau)\rangle\langle\psi(td/\tau)|_C\,, \quad (4.13)$$

where $\rho_{FS}(t)$ is given by Eq. (4.10), and we assumed a clock large enough so the error given by lemma 3.3.1 can be properly neglected. Explicitly,

$$|\psi(td/\tau)\rangle\langle\psi(td/\tau)| = \sum_{k,k' \in S_d(td/\tau)} |A|^2 e^{-\frac{\pi}{\sigma^2}(k-td/\tau)^2} e^{-\frac{\pi}{\sigma^2}(k'-td/\tau)^2} e^{i2\pi n_0(k-k')/d} |\theta_k\rangle\langle\theta_{k'}|\,. \quad (4.14)$$

ii Here, $|\Phi_\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle)$ are Bell states.

By completing the square, we can write

$$|\psi(td/\tau)\rangle\langle\psi(td/\tau)| = \sum_{k,k'\in S_d(td/\tau)} |A|^2 e^{-\frac{\pi}{2\sigma^2}(k-k')^2} e^{i2\pi n_0(k-k')/d} e^{-\frac{\pi}{\sigma^2}\left(\frac{\sqrt{2}d}{\tau}t-\frac{k+k'}{\sqrt{2}}\right)^2} |\theta_k\rangle\langle\theta_{k'}|\,. \tag{4.15}$$

The G-twirling operation is given by

$$\mathcal{G}[\rho] = \lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}\rho(t)dt, \tag{4.16}$$

and we can write the lab state as $\rho_{FS}(t) = \sum_{i,j} f_{ij}(t)|i\rangle\langle j|$, where $|i\rangle$, $|j\rangle$ are representations of the $\mathcal{H}_{FS}$ basis that we have been using so far, and $f_{ij}(t)$ are the non zero entries of matrix Eq. (4.10), given by linear combinations of the functions 1 and $e^{\pm i2\omega_0 t}$, such that

$$f_{ij}(t) = \cos^2(\omega_0 t) = \frac{1}{2}\left(1+\frac{e^{i2\omega_0 t}+e^{-i2\omega_0 t}}{2}\right); \tag{4.17}$$

$$f_{ij}(t) = \sin^2(\omega_0 t) = \frac{1}{2}\left(1-\frac{e^{i2\omega_0 t}+e^{-i2\omega_0 t}}{2}\right); \tag{4.18}$$

$$f_{ij}(t) = \sin(2\omega_0 t) = \frac{1}{2i}\left(e^{i2\omega_0 t}-e^{-i2\omega_0 t}\right). \tag{4.19}$$

Hence

$$\begin{aligned}\mathcal{G}[\rho]_{ij} \leq \lim_{T\to\infty}\frac{|A|^2\sigma}{2\sqrt{2}T}\int_{-T}^{T}\sum_{k,k'\in S_d(td/\tau)} & e^{-\frac{\pi}{2\sigma^2}(k-k')^2}e^{i2\pi n_0(k-k')/d} \\ &\times\left[e^{-\frac{\pi}{\sigma^2}\left(\frac{\sqrt{2}d}{\tau}t-\frac{k+k'}{\sqrt{2}}\right)^2} f_{ij}(t)\right]|\theta_k\rangle\langle\theta_{k'}|\,dt.\end{aligned} \tag{4.20}$$

where we appealed to the clock property that

$$\langle\psi(k_0)|\psi(k_0)\rangle \leq |A|^2\left(\frac{\sigma}{\sqrt{2}}+\epsilon_1+\epsilon_2\right), \tag{4.21}$$

and both errors ϵ_1 and ϵ_2 are exponentially vanishing with respect to the clock size d as long as $\sigma \geq \sqrt{d}$ (see Appendix A). Indeed, $|A|^2$ is nearly constant in time for a large clock, as it can be seen in Figure 5. Notice also from the plots that $|A|^2 \approx \frac{\sqrt{2}}{\sigma}$ (for $d = 8$, for example, $\sigma = \sqrt{d}$ implies $|A|^2 = \frac{\sqrt{2}}{\sqrt{8}} = 0.5$), and henceforth the ordering relation $\geq$ is going to be treated as a strict equality from now on.

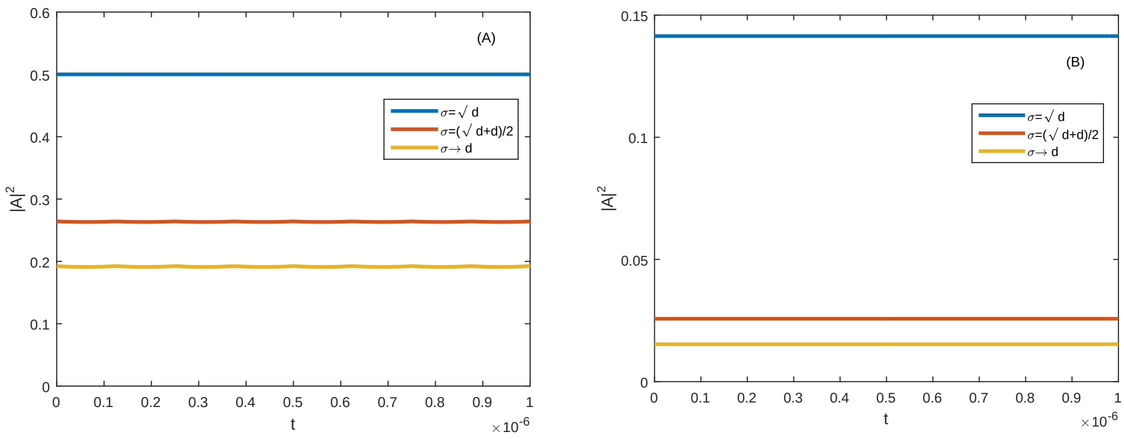

Figure 5 – Normalizing factor values as functions of time for $\sigma \geq d$, given (A) $d = 8$ and (B) $d = 100$.

Source: By the author.

The term between brackets in Eq. (4.20) is appearently the only part of the summation that is time dependent. However, looking closely, the summation boundaries given by $S_d(k_0)$ are time dependent, and thus we cannot simply commute the integration with this summation. We must then analyze how $S_d(td/\tau)$ changes with t.

First, notice that

$$S_d(td/\tau) = \left\{k \in \mathbb{Z}| - \frac{d}{2} \leq k - \frac{td}{\tau} < \frac{d}{2}\right\} = \left\{k \in \mathbb{Z}|t - \frac{\tau}{2} \leq \frac{k\tau}{d} < t + \frac{\tau}{2}\right\}. \quad (4.22)$$

Hence, if $d = 2$, we have for $t = 0$

$$S_2(0) = \{k \in \mathbb{Z}| - \tau/2 \leq k\tau/2 < \tau/2\} = \{k \in \mathbb{Z}| - 1 \leq k < 1\} = \{-1, 0\}. \quad (4.23)$$

If $t = \tau/2$, in its turn,

$$S_2(1) = \{k \in \mathbb{Z}|\tau/2 - \tau/2 \leq k\tau/2 < \tau/2 + \tau/2\} = \{k \in \mathbb{Z}|0 \leq k < 2\} = \{0, 1\}. \quad (4.24)$$

Now notice that, if t is infinitesimally greater than 0, then $t - \tau/2$ will be greater then $-\tau/2$, and so the value $k = -1$ fits no longer in the set $S_2(2t/\tau)$. Withal, the value $t + \tau/2$ will be greater then $\tau/2$, and so the value 1 is now included in the set. Therefore,

$$S_2(dt/\tau) = S_2(0), t \in (-\tau/2, 0]; \quad S_2(dt/\tau) = S_2(1), t \in (0, \tau/2]; \ldots \quad (4.25)$$

In the same way, if $d = 3$, we have $t = 0$

$$S_3(0) = \{k \in \mathbb{Z} + \frac{1}{2}| - \tau/2 \leq k\tau/3 < \tau/2\} = \left\{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}\right\}. \quad (4.26)$$

When $t = \tau/3$, then

$$S_3(1) = \{k \in \mathbb{Z} + \frac{1}{2} | -\tau/6 \leq k\tau/3 < 5\tau/6\} = \left\{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right\}. \tag{4.27}$$

And with the same discussion, it is possible to realize that if $t \to 0^+$, then $S_3(3t/\tau)$ does not include $k = -3/2$, but will include $k = 3/2$. Therefore,

$$S_3(td/\tau) = S_3(0), t \in (-\tau/3, 0]; \quad S_3(td/\tau) = S_3(1), t \in (0, \tau/3]; ... \tag{4.28}$$

Generalizing this discussion for any d,

$$S_d(td/\tau) = S_d(n), \quad t \in \left(\frac{\tau}{d}(n-1), \frac{\tau}{d}n\right], n \in \mathbb{Z}, \tag{4.29}$$

where $S_d(n)$ is the set d integers (if d is even) of half integers (if d is odd) centered around n. That way,

$$\begin{aligned} \mathcal{G}[\rho]_{ij} = \lim_{T\to\infty} \frac{|A|^2\sigma}{2\sqrt{2}T} \sum_{n=-Td/\tau}^{Td/\tau} \int_{\frac{\tau}{d}(n-1)}^{\frac{\tau}{d}n} \sum_{k,k' \in S_d(n)} e^{-\frac{\pi}{2\sigma^2}(k-k')^2} e^{i2\pi n_0 (k-k')/d} \\ \times \left[e^{-\frac{\pi}{\sigma^2}\left(\frac{\sqrt{2}d}{\tau}t - \frac{k+k'}{\sqrt{2}}\right)^2} f_{ij}(t) \right] |\theta_k\rangle \langle\theta_{k'}| \, dt. \end{aligned} \tag{4.30}$$

Now that the boundaries of the summation over k, k' do not depend on time anymore, we can commute the integration with the summation. Defining $N := Td/\tau$, we finally get

$$\begin{aligned} \mathcal{G}[\rho]_{ij} = \lim_{N\to\infty} \frac{|A|^2\sigma d}{2\sqrt{2}\tau N} \sum_{n=-N}^{N} \sum_{k,k' \in S_d(n)} e^{-\frac{\pi}{2\sigma^2}(k-k')^2} e^{i2\pi n_0 (k-k')/d} |\theta_k\rangle \langle\theta_{k'}| \\ \times \int_{\frac{\tau}{d}(n-1)}^{\frac{\tau}{d}n} e^{-\frac{\pi}{\sigma^2}\left(\frac{\sqrt{2}d}{\tau}t - \frac{k+k'}{\sqrt{2}}\right)^2} f_{ij}(t) dt. \end{aligned} \tag{4.31}$$

Let us now transform the elements of the basis which constitutes $f_{ij}(t)$. Begining with $f_{ij}(t) = 1$,

$$\int_{\frac{\tau}{d}(n-1)}^{\frac{\tau}{d}n} e^{-\frac{\pi}{\sigma^2}\left(\frac{\sqrt{2}d}{\tau}t - \frac{k+k'}{\sqrt{2}}\right)^2} dt, \tag{4.32}$$

and with a simple change of variable $u = \frac{\sqrt{\pi}}{\sigma}\left[\frac{\sqrt{2}d}{\tau}t - \frac{k+k'}{\sqrt{2}}\right]$,

$$\begin{aligned} \int_{\frac{\tau}{d}(n-1)}^{\frac{\tau}{d}n} e^{-\frac{\pi}{\sigma^2}\left(\frac{\sqrt{2}d}{\tau}t - \frac{k+k'}{\sqrt{2}}\right)^2} dt = \frac{\sigma\tau}{2\sqrt{2}d} \Bigg\{ \mathrm{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\left(n - \frac{k+k'}{2}\right)\right] \\ - \mathrm{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\left(n - 1 - \frac{k+k'}{2}\right)\right]\Bigg\}, \end{aligned} \tag{4.33}$$

where

$$\mathrm{erf}(x) = \frac{2}{\pi}\int_0^x e^{-t^2} dt \tag{4.34}$$

is the error function.

In the same sense, when $f_{ij}(t) = e^{\pm i2\omega_0 t}$, then we ought to solve

$$\int_{\frac{\tau}{d}(n-1)}^{\frac{\tau}{d}n} e^{-\frac{\pi}{\sigma^2}\left(\frac{\sqrt{2}d}{\tau}t-\frac{k+k'}{\sqrt{2}}\right)^2} e^{\pm i2\omega_0 t}dt. \tag{4.35}$$

By completing the square and performing a change of variable, and adopting

$$\Gamma = \frac{\omega_0 \tau \sigma}{\sqrt{2\pi}d} \tag{4.36}$$

as a substantial value associated to this integration, we have

$$\begin{aligned}\int_{\frac{\tau}{d}(n-1)}^{\frac{\tau}{d}n} e^{-\frac{\pi}{\sigma^2}\left(\frac{\sqrt{2}d}{\tau}t-\frac{k+k'}{\sqrt{2}}\right)^2} e^{\pm i2\omega_0 t}dt = \frac{\sigma\tau e^{-\Gamma^2}}{2\sqrt{2}d} e^{\pm i\sqrt{2\pi}\Gamma(k+k')/\sigma} \Bigg\{ &\mathrm{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\left(n-\frac{k+k'}{2}\right) \pm i\Gamma\right] \\ &-\mathrm{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\left(n-1-\frac{k+k'}{2}\right) \pm i\Gamma\right]\Bigg\}.\end{aligned} \tag{4.37}$$

Since the relative lab state that Wigner describes is associated to a projection of $\mathcal{G}[\rho]$ over a subspace $|\theta_K\rangle\langle\theta_K|$, it is convenient to study what is happening with the diagonal elements of $\mathcal{G}[\rho]$ in the clock space. Let us look first to the behavior of the symmetrization of $f_{ij}(t) = 1$. We have

$$\begin{aligned}\mathcal{G}[f_{ij}(t)=1] = \lim_{N\to\infty} \frac{|A|^2\sigma^2}{8N} \sum_{n=-N}^{N} \sum_{k,k'\in S_d(n)} |\theta_k\rangle\langle\theta_{k'}|\, e^{-\frac{\pi}{2\sigma^2}(k-k')^2} e^{i2\pi n_0(k-k')/d} \\ \times\left\{\mathrm{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\left(n-\frac{k+k'}{2}\right)\right] - \mathrm{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\left(n-1-\frac{k+k'}{2}\right)\right]\right\}.\end{aligned} \tag{4.38}$$

Now, let us assume $d = 2$, and also that we are willing to project our state to the clock subspace associated to $K = 0$, i.e., the projection $|\theta_0\rangle\langle\theta_0|$. There are just two sets which include the pointer $|\theta_0\rangle$, namely, $S_2(0) = \{-1, 0\}$ and $S_2(1) = \{0, 1\}$. In this case, the summation component over the set $S_2(0)$ refering to the entry $|\theta_0\rangle\langle\theta_0|$ will be given by

$$\mathrm{erf}\left[\frac{\sqrt{2\pi}}{\sigma}(0)\right] - \mathrm{erf}\left[\frac{\sqrt{2\pi}}{\sigma}(-1)\right]; \tag{4.39}$$

while the summation component over $S_2(1)$ referring to this entry is

$$\mathrm{erf}\left[\frac{\sqrt{2\pi}}{\sigma}(1)\right] - \mathrm{erf}\left[\frac{\sqrt{2\pi}}{\sigma}(0)\right]. \tag{4.40}$$

Adding these two terms, we get

$$2\mathrm{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\right]. \tag{4.41}$$

If we now look to $K = 1$, we notice that there are again only two sets including the vector $|\theta_1\rangle$: $S_2(1) = \{0,1\}$ and $S_2(2) = \{1,2\}$. Looking to the summation components, we get

$$S_2(1): \quad \text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}(0)\right] - \text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}(-1)\right]; \tag{4.42}$$

$$S_2(2): \quad \text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}(1)\right] - \text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}(0)\right]. \tag{4.43}$$

totalizing

$$2\text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\right]. \tag{4.44}$$

If we now assume $d = 4$, and look for the clock subspace $K = 0$, there will be 4 sets including the vector $|\theta_0\rangle$: $S_4(-1)$, $S_4(0)$, $S_4(1)$ e $S_4(2)$. By repeting the procedure, the summation components result in

$$S_4(-1): \quad \text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}(-1)\right] - \text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}(-2)\right]; \tag{4.45}$$

$$S_4(0): \quad \text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}(0)\right] - \text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}(-1)\right]; \tag{4.46}$$

$$S_4(1): \quad \text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}(1)\right] - \text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}(0)\right]; \tag{4.47}$$

$$S_4(2): \quad \text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}(2)\right] - \text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}(1)\right], \tag{4.48}$$

and they thus add to

$$2\text{erf}\left[2\frac{\sqrt{2\pi}}{\sigma}\right]. \tag{4.49}$$

We can hence conclude that the value associated to the the projection $|\theta_K\rangle\langle\theta_K|$ for $f_{ij}(t) = 1$ does not depend K, but only on the clock size, yielding to

$$2\text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\frac{d}{2}\right]. \tag{4.50}$$

Note also that the component $|\theta_{K+md}\rangle\langle\theta_{K+md}|$ is indistinguishable from the component $|\theta_K\rangle\langle\theta_K|$, and thus for each $K + md$ terms of the summation n, there is a new term given by Eq. (4.50) being added to the total value of the component, summing up to $2N/d$ terms. This results in

$$\Pi_K^C \mathcal{G}[f_{ij}(t) = 1]\Pi_K^C = \lim_{N\to\infty} \frac{|A|^2\sigma^2}{8N}\frac{2N}{d}2\text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\frac{d}{2}\right] = \frac{|A|^2\sigma^2}{2d}\text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\frac{d}{2}\right]. \tag{4.51}$$

Similarly, we can analyze the behavior of the elements $\mathcal{G}[f_{ij}(t) = e^{\pm i2\omega_0 t}]$. Explicitly,

$$\mathcal{G}[f_{ij}(t) = e^{\pm i2\omega_0 t}] = \lim_{N\to\infty} \frac{|A|^2\sigma e^{-\Gamma^2}}{4\sqrt{2}d} e^{\pm i\frac{\sqrt{2\pi}}{\sigma}\Gamma(k+k')} \left\{ \text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\left(n - \frac{k+k'}{2}\right) \pm i\Gamma\right] - \text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\left(n - 1 - \frac{k+k'}{2}\right) \pm i\Gamma\right]\right\}, \tag{4.52}$$

Let us assume again $d = 2$, and that we are going to perform a projetion over the subspace $K = 0$. Once more, only two sets contain terms $|\theta_0\rangle\langle\theta_0|$, which are $S_2(0)$ e $S_2(1)$. By repeting the previous procedure, we get

$$S_2(0): \quad \text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}(0) \pm i\Gamma\right] - \text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}(-1) \pm i\Gamma\right]; \tag{4.53}$$

$$S_2(1): \quad \text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}(1) \pm i\Gamma\right] - \text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}(0) \pm i\Gamma\right], \tag{4.54}$$

that add to

$$\text{erf}\left[\frac{\sqrt{2\pi}}{\sigma} \pm i\Gamma\right] + \text{erf}\left[\frac{\sqrt{2\pi}}{\sigma} \mp i\Gamma\right]. \tag{4.55}$$

Repeting these steps for $d = 2$ e $K = 1$,

$$S_2(1): \quad e^{\pm i\frac{\sqrt{2\pi}}{\sigma}\Gamma(2)}\left\{\text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}(0) \pm i\Gamma\right] - \text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}(-1) \pm i\Gamma\right]\right\}; \tag{4.56}$$

$$S_2(2): \quad e^{\pm i\frac{\sqrt{2\pi}}{\sigma}\Gamma(2)}\left\{\text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}(1) \pm i\Gamma\right] - \text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}(0) \pm i\Gamma\right]\right\}, \tag{4.57}$$

leading to

$$e^{\pm i\frac{\sqrt{2\pi}}{\sigma}2\Gamma}\left\{\text{erf}\left[\frac{\sqrt{2\pi}}{\sigma} \pm i\Gamma\right] + \text{erf}\left[\frac{\sqrt{2\pi}}{\sigma} \mp i\Gamma\right]\right\}. \tag{4.58}$$

We can thus see that $|\theta_K\rangle\langle\theta_K|$ para $f_{ij}(t) = e^{\pm i2\omega_0 t}$ depends on K and on the clock size, such that

$$e^{\pm i2\frac{\sqrt{2\pi}}{\sigma}\Gamma K}\left\{\text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\frac{d}{2} \pm i\Gamma\right] + \text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\frac{d}{2} \mp i\Gamma\right]\right\}. \tag{4.59}$$

Furthermore, we must take into account the indistinguishability between K and $k = K + md$, $m \in \mathbb{Z}$, yielding to a new term being added to $|\theta_K\rangle\langle\theta_K|$ for each m. Therefore

$$\Pi_K^C\mathcal{G}[f_{ij}(t) = e^{\pm i2\omega_0 t}]\Pi_K^C = \lim_{N\to\infty} \frac{|A|^2\sigma^2 e^{-\Gamma^2}}{4N} e^{\pm i2\frac{\sqrt{2\pi}}{\sigma}\Gamma K}\,\text{Re}\left\{\text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\frac{d}{2} \pm i\Gamma\right]\right\} \times \sum_{m=-N/d}^{N/d} e^{\pm i2\frac{\sqrt{2\pi}}{\sigma}\Gamma md}. \tag{4.60}$$

Notice now that the summation on m will vanish when the limit is taken, unless $e^{\pm i2\frac{\sqrt{2\pi}}{\sigma}\Gamma md} = 1$, for any $m \in \mathbb{Z}$. This condition implies that

$$\frac{\sqrt{2\pi}}{\sigma}\Gamma d = q\pi \iff \omega_0\tau = q\pi, \quad q \in \mathbb{Z} \tag{4.61}$$

that is, $\omega_0 = \frac{q}{2}\omega$. Notice that this is precisely the condition presented in the last paragraph of Sec. 4.1: if $\omega_0\tau = q\pi$, then the instant of time that Wigner is supposed to detect the measurement is going to be described as $K = \left(m + \frac{1}{2}\right)\frac{\pi d}{\omega_0\tau} = \left(m + \frac{1}{2}\right)\frac{d}{q}$. Then, for a given experimental setup in which $\omega_0\tau = q\pi$, there will always be m such that K is an integer, and this detection is possible. Out of this resonance, Wigner is never allowed to detect the instant of time for which the lab system completed the entanglement, and will always describe the lab as a mixed state.

So, in this resonant regime, the summation becomes $\sum_{m=-N/d}^{N/d} e^{\pm i2\frac{\sqrt{2\pi}}{\sigma}\Gamma md} = \frac{2N}{d} + 1$, and therefore

$$\Pi_K^C\mathcal{G}[f_{ij}(t) = e^{\pm i2\omega_0 t}]\Pi_K^C = \begin{cases} \frac{|A|^2\sigma^2 e^{-\Gamma^2}}{2d} e^{\pm i2\frac{\sqrt{2\pi}}{\sigma}\Gamma K}\mathrm{Re}\left\{\mathrm{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\frac{d}{2} \pm i\Gamma\right]\right\}, & \text{if } \omega_0 = \frac{q}{2}\omega; \\ 0, \text{ otherwise} \end{cases}. \tag{4.62}$$

From these expressions, we can reconstruct the operator $(\mathbb{I}_{FS} \otimes \Pi_K^C)\mathcal{G}[\rho](\mathbb{I}_{FS} \otimes \Pi_K^C)$. The non zero entries of this matrix are either $\cos^2\omega_0 t$, $\sin^2\omega_0 t$ or $\frac{i}{2}\sin 2\omega_0 t$. Hence,

$$\Pi_K^C\mathcal{G}[\cos^2\omega_0 t]\Pi_K^C = \frac{1}{2}\Pi_K^C\mathcal{G}[1]\Pi_K^C + \frac{1}{4}(\Pi_K^C\mathcal{G}[e^{i2\omega_0 t}]\Pi_K^C + \Pi_K^C\mathcal{G}[e^{-i2\omega_0 t}]\Pi_K^C); \tag{4.63}$$

$$\Pi_K^C\mathcal{G}[\sin^2\omega_0 t]\Pi_K^C = \frac{1}{2}\Pi_K^C\mathcal{G}[1]\Pi_K^C - \frac{1}{4}(\Pi_K^C\mathcal{G}[e^{i2\omega_0 t}]\Pi_K^C + \Pi_K^C\mathcal{G}[e^{-i2\omega_0 t}]\Pi_K^C); \tag{4.64}$$

$$\Pi_K^C\mathcal{G}[\sin 2\omega_0 t]\Pi_K^C = \frac{1}{2i}(\Pi_K^C\mathcal{G}[e^{i2\omega_0 t}]\Pi_K^C - \Pi_K^C\mathcal{G}[e^{-i2\omega_0 t}]\Pi_K^C), \tag{4.65}$$

where $\mathbb{I}_{FS} \otimes \Pi_K^C$ was abreviated by Π_K^C. Out of ressonance, then, we got a relative state

$$\rho_{FS}^W(K) = \mathrm{Tr}_C\left\{\frac{(\mathbb{I}_{FS} \otimes \Pi_K^C)\mathcal{G}[\rho](\mathbb{I}_{FS} \otimes \Pi_K^C)}{\mathrm{Tr}\{(\mathbb{I}_{FS} \otimes \Pi_K^C)\mathcal{G}[\rho]\}}\right\} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}. \tag{4.66}$$

So the clock is not properly keeping track of the external time, and everything Wigner can tell is that the measurement inside the lab might have happened or not with equal probabilities. However, when $\omega_0 = \frac{q}{2}\omega$, then

$$\rho^W_{FS}(K) = \frac{1}{4}\begin{pmatrix} 1+\mathcal{R}(\Gamma,K) & 1+\mathcal{R}(\Gamma,K) & i\mathcal{Q}(\Gamma,K) & 0 & 0 & i\mathcal{Q}(\Gamma,K) \\ 1+\mathcal{R}(\Gamma,K) & 1+\mathcal{R}(\Gamma,K) & i\mathcal{Q}(\Gamma,K) & 0 & 0 & i\mathcal{Q}(\Gamma,K) \\ -i\mathcal{Q}(\Gamma,K) & -i\mathcal{Q}(\Gamma,K) & 1-\mathcal{R}(\Gamma,K) & 0 & 0 & 1-\mathcal{R}(\Gamma,K) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -i\mathcal{Q}(\Gamma,K) & -i\mathcal{Q}(\Gamma,K) & 1-\mathcal{R}(\Gamma,K) & 0 & 0 & 1-\mathcal{R}(\Gamma,K) \end{pmatrix}, \quad (4.67)$$

where, using the fact that $\frac{\sqrt{2\pi}}{\sigma}\Gamma = \frac{\omega_0\tau}{d} = \frac{q\pi}{d}$ when in resonance, we define

$$\mathcal{R}(\Gamma,K) = e^{-\Gamma^2}\frac{\text{Re}\left\{\text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\frac{d}{2} + i\Gamma\right]\right\}}{\text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\frac{d}{2}\right]}\cos\left(\frac{2\pi qK}{d}\right); \quad \mathcal{Q}(\Gamma,K) = \mathcal{R}(\Gamma,0)\sin\left(\frac{2\pi qK}{d}\right). \quad (4.68)$$

Notice that, when $K = \left(m + \frac{1}{2}\right)\frac{d}{q}$, the instant of time Wigner sees the entanglement being completed, then

$$\frac{2\pi q}{d}K = \frac{2\pi q}{d}\frac{d}{q}\left(m + \frac{1}{2}\right) = \pi(2m+1), \quad (4.69)$$

and then $\cos(2\pi qK/d) = -1$, for any $m \in \mathbb{Z}$, resulting in

$$\rho^W_{FS}\left(K = \left(m+\frac{1}{2}\right)\frac{d}{q}\right) = \frac{1}{4}\begin{pmatrix} 1-\mathcal{R}(\Gamma,0) & 1-\mathcal{R}(\Gamma,0) & 0 & 0 & 0 & 0 \\ 1-\mathcal{R}(\Gamma,0) & 1-\mathcal{R}(\Gamma,0) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+\mathcal{R}(\Gamma,0) & 0 & 0 & 1+\mathcal{R}(\Gamma,0) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+\mathcal{R}(\Gamma,0) & 0 & 0 & 1+\mathcal{R}(\Gamma,0) \end{pmatrix}. \quad (4.70)$$

We therefore have obtained the relative state of the lab with respect to Wigner's quasi-ideal clock. It will be from this state that he is going to perform the measurement of $|ok\rangle$ and compare with the predictions of the Friend. We emphasize once more that this analytical approach is valid only under some assumptions: d is large enough so that $e^{-\frac{\pi d}{4}}$ is negligible, and $\sigma \geq \sqrt{d}$ so that the normalizing factor can be taken as a constant.

4.3 Results

Now that we have access to the relative state $\rho^W_{FS}(K)$, we can study the behavior of the Δ's, looking for conditions that rule out the paradox. If Wigner is going to measure the observable $|ok\rangle\langle ok|$, with

$$|ok\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow\uparrow\rangle_{FS} + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\downarrow\downarrow\rangle_{FS}, \tag{4.71}$$

then

$$P_F(ok|\uparrow) = \cos^2\left(\frac{\theta}{2}\right); \quad P_F(ok|\downarrow) = \sin^2\left(\frac{\theta}{2}\right), \tag{4.72}$$

the same situation we had in Sec. 4.1. Wigner, by its turn, will predict a probability

$$P_W\left(ok|\left(m+\frac{1}{2}\right)\frac{d}{q}\right) = \frac{1}{4}(1+\sin\theta\cos\phi)(1+\mathcal{R}(\Gamma,0)). \tag{4.73}$$

It is clear, just by taking a quick look, that the condition $\Delta_0 = \Delta_1 = 0$ will not be satisfied for any (θ,ϕ) characterizing $|ok\rangle$. Notice, however, that

$$\Delta_{0(1)} = \frac{1\pm\cos\theta}{2} - \frac{1}{4}\left(1 + e^{-\left(q\sqrt{\frac{\pi}{2}}\frac{\sigma}{d}\right)^2}\frac{\mathrm{Re}\left\{\mathrm{erf}\left[\sqrt{\frac{\pi}{2}}\frac{d}{\sigma} + iq\sqrt{\frac{\pi}{2}}\frac{\sigma}{d}\right]\right\}}{\mathrm{erf}\left[\sqrt{\frac{\pi}{2}}\frac{d}{\sigma}\right]}\right)(1+\sin\theta\cos\phi) \tag{4.74}$$

does not depend on σ directly, but rather on the ratio σ/d, that can vary from $\frac{1}{\sqrt{d}}$ to nearly 1 without violating the conditions for which our results are valid. For $d = 100$, for example, $\frac{\sigma}{d} \in [0.1, 1)$, and the transition can go as slow as $\omega_0 = 0.5\omega$, so that the measurement will be completed at $K = 50$, or as fast as $\omega_0 = 25\omega$, for which the measurement occurs right at $K = 1$. The values of (θ,ϕ) for which Δ_0 and Δ_1 are both null are given, for different ω_0 and σ, in Figure 6.

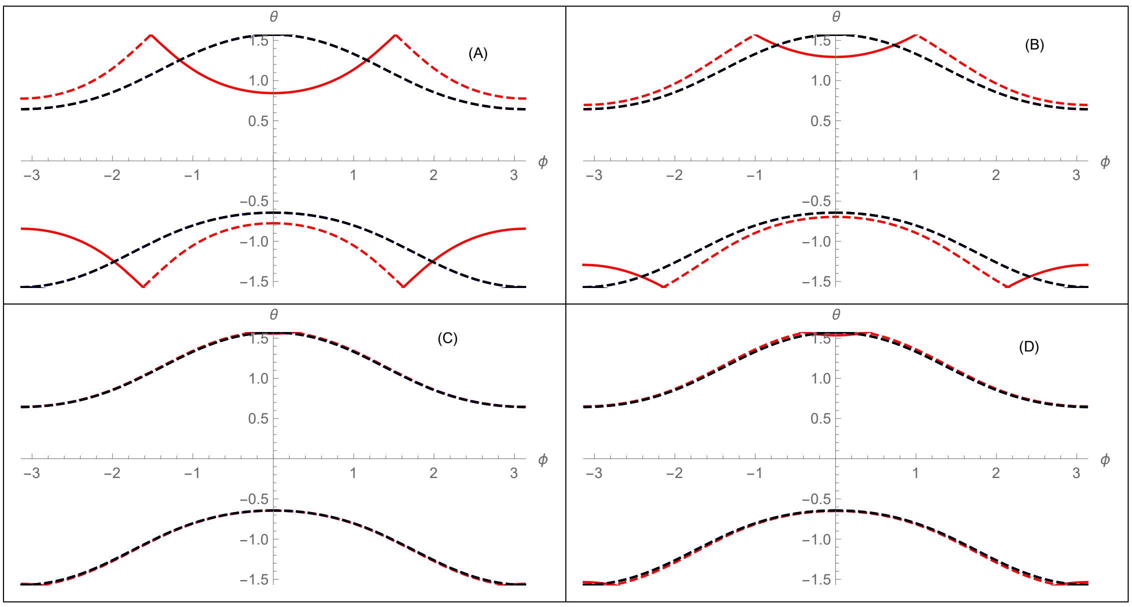

Figure 6 – Values of θ and ϕ for which Δ_0 (solid lines) and Δ_1 (dashed lines) are 0. Red lines refer to $\omega_0 = 0.5\omega$ and black ones to $\omega_0 = 5\omega$, for ratio $\frac{\sigma}{d}$ equal to (A) 0.1, (B) 0.35, (C) 0.7 and (D) 0.999.

Source: By the author.

It is possible to see from Figure 6 that there are few points for which $\Delta_0 = \Delta_1 = 0$. For $\sigma = \sqrt{d}$ and $\omega_0 = 0.5\omega$, for example, those points correspond to the observable

$$|ok\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle_{FS} \pm i\,|\downarrow\downarrow\rangle_{FS}), \tag{4.75}$$

which are indeed the only observables that guarantee a consonant scenario between the Friend and Wigner's predictions with an ideal clock. As long as the measurement described by Wigner becomes faster or the ratio $\frac{\sigma}{d}$ becomes closer to the unit, however, the only observable capable of ruling out the paradox is

$$|ok\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle_{FS} + |\downarrow\downarrow\rangle_{FS}). \tag{4.76}$$

That is because, for a unitary process which is way faster than the clock or a large gaussian spread σ, the function $\mathcal{R}(\Gamma, 0)$ responsible for modulating the consequences of the time tracking exponentially vanishes, resulting on the mixed state given by Eq. (4.66). The paradox for this observable indeed vanishes: while the Friend will predict a probability $\frac{1}{2}$ for the detection of ok because she sees either $|\uparrow\uparrow\rangle_{FS}$ or $|\uparrow\uparrow\rangle_{FS}$, Wigner will predict the same probability for he sees a statistical mixture of the entangled state $|\Phi_+\rangle$ and the initial state $|\perp\rangle_F \otimes |+\rangle_S$.

It is interesting to analyze the shared asymmetry between the lab and the clock. For this purpose, simulations were carried out in a way that we are allowed to work at any value of $\sigma \in (0, d)$ with just numeric errors associated. The values of $A^{sh}_{G\otimes G}(\rho)$ can be seen in Figure 7. It is clear how asymmetry is increased for time-squeezed states ($\sigma < \sqrt{d}$). With these time-squeezed states, the expected value for energy n_0 does not seem to have any influence over how well the clock is tracking the evolution in the lab. For the symmetric state ($\sigma = \sqrt{d}$) and for energy-squeezed states ($\sigma > \sqrt{d}$), however, the population distribution of the clock can initiate a small recovery of shared asymmetry. For $\sigma \to d$, however, the clockwork mechanism is completely lost, which was expected from our previous results, since this results in the mixed state between the entangled state and the initial state.

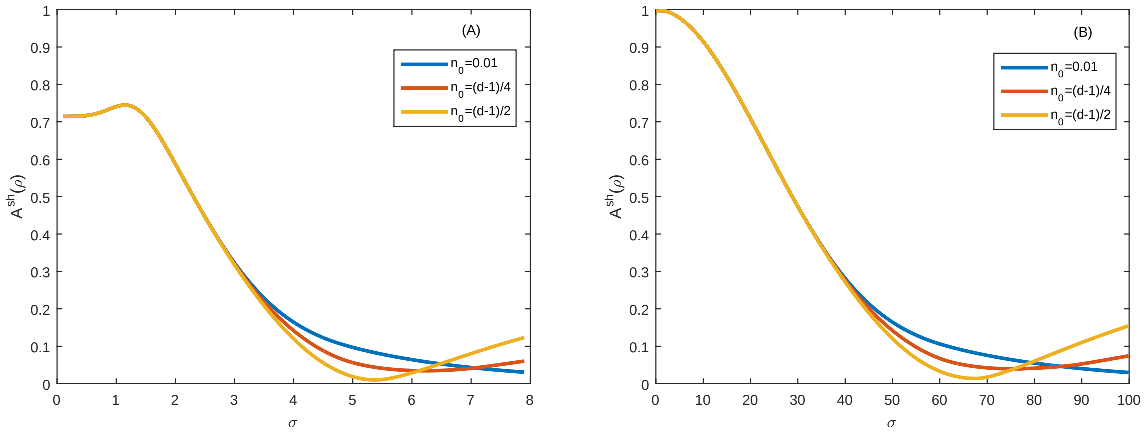

Figure 7 – Shared asymmetry between the lab, evolving periodically, and the clock, as a function of σ for different mean energies n_0. For both plots, $\omega_0 = \omega$, with (A) $d = 8$ and (B) $d = 100$.

Source: By the author.

Furthermore, notice that the maximum amount of asymmetry shared between the lab and the clock is still way bellow the upper bound for shared asymmetry,[48] which is $\log 6 \approx 2.6$ for clock sizes that allow the error in quasi-continuity to be neglected. This could mean that either the quasi-ideal clock is not really working to keep track of time of the lab dynamics, or shared asymmetry is not a good quantifyer for the physical property responsible for making a functional clock. The maximum amount of shared asymmetry slightly grows with the clock size, what does not mean in any sense that this upper bound could be reached for $d \to \infty$. The decaying behavior of the curve clearly do not depend on the dimension d, but rather with the ratio $\frac{\sigma}{d}$ that characterizes the prediction differences Δ_0 and Δ_1. The bigger the clock, the smaller the ratio $\frac{\sigma}{d}$ for a symmetric state, and since this is the most *ideal* quasi-ideal state (in the sense that the decrease in errors associated with lemmas 3.3.1, 3.3.2 and 3.3.3 is the fastest), the possibility of accessing the maximum shared asymmetry available with such a state is a convenient feature.

The oddity with the relaxation that happens when $\sigma \to d$ is not quite clear. In this regime, quasi-ideal states are strongly energy-squeezed, such that for d large enough and integer n_0

$$|\psi(k_0)\rangle = \sum_{k\in S_d(k_0)} A e^{-\frac{\pi}{\sigma^2}(k-k_0)^2} e^{i2\pi n_0(k-k_0)/d} |\theta_k\rangle \tag{4.77}$$

$$\approx e^{-i2\pi n_0 k_0/d} \frac{1}{\sqrt{d}} \sum_{k\in S_d(k_0)} e^{i2\pi n_0 k/d} |\theta_k\rangle \tag{4.78}$$

$$= e^{-i\langle H\rangle t} |n_0\rangle , \tag{4.79}$$

with $k_0 = td/\tau$ (see Apendix A). For states like these, any eigenvalue of T is equiprobable, and it is quite odd to observe that energy-squeezed states can track time as well as states with $\sigma = \frac{d}{2}$, for instance.

One could claim that the transition given by Eq. (4.9) is still too unrealistic for a description of a von Neumann measurement. In a real measurement, indeed, there is no possibility of unmaking this entangled state. Since instantaneous transitions such as the one given by Eq. (4.7) are also unrealistic, for every measurement has a finite duration, one could assume an analytical transition such that

$$U_t^{FS} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1-\tanh(\omega_0 t)}{\sqrt{1+\tanh^2(\omega_0 t)}} & 0 & -i\frac{1+\tanh(\omega_0 t)}{\sqrt{1+\tanh^2(\omega_0 t)}} & 0 & 0 & 0 \\ 0 & \frac{1-\tanh(\omega_0 t)}{\sqrt{1+\tanh^2(\omega_0 t)}} & 0 & 0 & 0 & -i\frac{1+\tanh(\omega_0 t)}{\sqrt{1+\tanh^2(\omega_0 t)}} \\ -i\frac{1+\tanh(\omega_0 t)}{\sqrt{1+\tanh^2(\omega_0 t)}} & 0 & \frac{1-\tanh(\omega_0 t)}{\sqrt{1+\tanh^2(\omega_0 t)}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & -i\frac{1+\tanh(\omega_0 t)}{\sqrt{1+\tanh^2(\omega_0 t)}} & 0 & 0 & 0 & \frac{1-\tanh(\omega_0 t)}{\sqrt{1+\tanh^2(\omega_0 t)}} \end{pmatrix}, \tag{4.80}$$

where ω_0 again defines how instantaneous is the measurement, and it is assumed the instant of time the Friend performs its measurement to be $t_F = 0$. The transition described by Eq. (4.9) imposed a constraint relating ω_0 to ω because it was suposed to halt at a certain instant of time — otherwise, the measurement would be unmade. The transition described above does not allow the unmaking of a measurement, and thus will not impose any constraint over ω_0. This evolution would result in a lab state for arbitrary time t given by

$$\rho_{FS}(t) = \begin{pmatrix} \frac{(1-\tanh(\omega_0 t))^2}{4(1+\tanh^2(\omega_0 t))} & \frac{(1-\tanh(\omega_0 t))^2}{4(1+\tanh^2(\omega_0 t))} & i\frac{\operatorname{sech}^2(\omega_0 t)}{4(1+\tanh^2(\omega_0 t))} & 0 & 0 & i\frac{\operatorname{sech}^2(\omega_0 t)}{4(1+\tanh^2(\omega_0 t))} \\ \frac{(1-\tanh(\omega_0 t))^2}{4(1+\tanh^2(\omega_0 t))} & \frac{(1-\tanh(\omega_0 t))^2}{4(1+\tanh^2(\omega_0 t))} & i\frac{\operatorname{sech}^2(\omega_0 t)}{4(1+\tanh^2(\omega_0 t))} & 0 & 0 & i\frac{\operatorname{sech}^2(\omega_0 t)}{4(1+\tanh^2(\omega_0 t))} \\ -i\frac{\operatorname{sech}^2(\omega_0 t)}{4(1+\tanh^2(\omega_0 t))} & -i\frac{\operatorname{sech}^2(\omega_0 t)}{4(1+\tanh^2(\omega_0 t))} & \frac{(1+\tanh(\omega_0 t))^2}{4(1+\tanh^2(\omega_0 t))} & 0 & 0 & \frac{(1+\tanh(\omega_0 t))^2}{4(1+\tanh^2(\omega_0 t))} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -i\frac{\operatorname{sech}^2(\omega_0 t)}{4(1+\tanh^2(\omega_0 t))} & -i\frac{\operatorname{sech}^2(\omega_0 t)}{4(1+\tanh^2(\omega_0 t))} & \frac{(1+\tanh(\omega_0 t))^2}{4(1+\tanh^2(\omega_0 t))} & 0 & 0 & \frac{(1+\tanh(\omega_0 t))^2}{4(1+\tanh^2(\omega_0 t))} \end{pmatrix} \tag{4.81}$$

Values of shared asymmetry for this model of measurement are almost negligible, with an order of magnitude of 10^{-4} for $d = 8$, even for smooth transitions ($\omega_0 = \omega$), as it can be seen at Figure 8. This leads to the scenario given by the mixed state $\rho_{FS}(K) = \frac{1}{2}\rho_{FS}(0) + \frac{1}{2}\left|\Phi_+\right\rangle\left\langle\Phi_+\right|$ once more, and the only observable *ok* Wigner is allowed to measure without disagreeing from the Friend is $|ok\rangle = |\Phi_+\rangle$. It therefore indicates how the model for the measurement is not the problem, since we are obtaining the same result for different entangling hamiltonians.

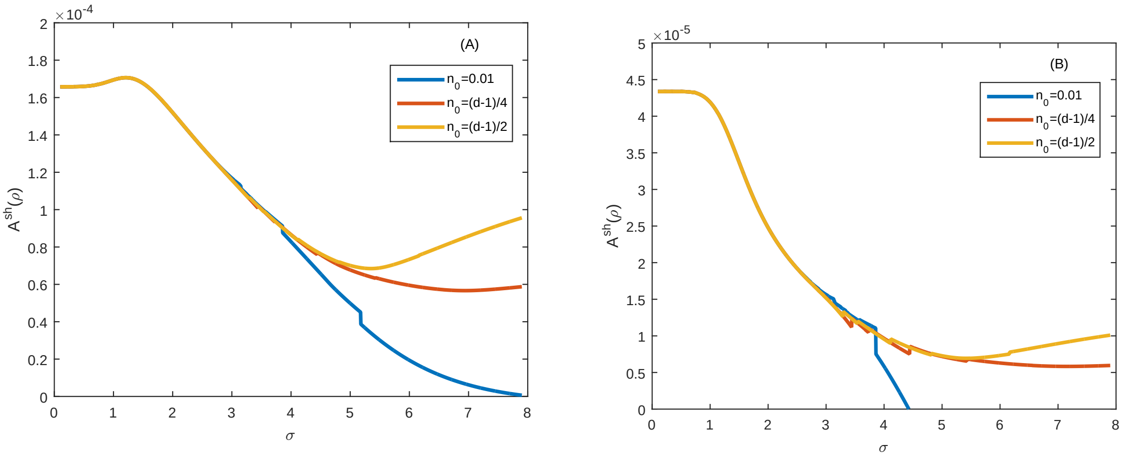

Figure 8 – Shared asymmetry between the lab, evolving non periodically, and the clock, as a function of σ for different mean energies n_0. For both plots, $d = 8$, with (A) $\omega_0 = \omega$ and (B) $\omega_0 = 4\omega$.

Source: By the author.

Still, there is always a possibility of performing G-covariant channels over the clock in order to obtain a final state as close to $|\uparrow\uparrow\rangle\langle\uparrow\uparrow| \otimes \rho_C$ or $|\downarrow\downarrow\rangle\langle\downarrow\downarrow| \otimes \rho_C$ as theorem 3.2.2 provides. However, adding the system $\rho_{FS}(0)$ is not a G-covariant channel, since it is an asymmetric state for any entangling hamiltonian we provide, and thus $\rho_{FS}(0) \otimes \mathcal{G}[\rho_C] \neq \mathcal{G}[\rho_{FS}(0) \otimes \rho_C]$. Furthermore, projections can hardly be defined as free operations. In fact, it has recently been proved that projective measurements cost infinite resources to be performed,[70] so this can also be a source of paradox: introduce uncertainty not only over the measurement of time, but also over any measurement, might rule out the paradox for any observable other than the $|ok(\theta, \phi)\rangle$ highlighted above.

5 CONCLUSIONS AND FURTHER WORK

This work focused on studying how a proper measurement of time by quantum clocks could affect the paradox that arises in a Wigner's Friend Scenario. After adopting a model for describing a WFS constituted of a single lab, where the internal observer would perceive a proper wave reduction, while the external observable described the measurement as a von Neumann measurement, we adopted a model for a feasible quantum clock that can reproduce properties of an ideal clock up to vanishing errors, and with respect to which the outer superobserver is supposed to track time. We internalized time with the formalism provided by the theory of quantum reference frames, quantifying how well the quasi-ideal clock was keeping track of the dynamics in the lab with a monotone provided by quantum resource theory of asymmetry, noticing that the clock operation is strongly dependent on the entangling hamiltonian that generates the von Neumann measurement inside the lab. Shared asymmetry, the quantifyer adopted for telling how well the clock is working, proved to be also dependend on the clock size, optimizing its maximum value for increasing d. The uncertainty of the gaussian spread that characterizes the quasi-ideal clock does not have a direct influence over the clock functioning, but rather the ratio $\frac{\sigma}{d}$. This ratio also has an influence over the possible observables the outter observer can measure without disagreeing with the inner friend with respect to the probability of detecting the outcome ok.

The procedure of internalizing time did not rule out the paradox for any observable $|ok\rangle$ of a global lab property, but rather for specific observables characterized by the angles (θ, ϕ). Under the circumstances for which the quasi-ideal clock is functionally keeping track of the lab's dynamics, for a periodic evolution happening inside the lab, the problem behaves exactly as the WFS with an ideal clock. With the uncertainty over the quasi-ideal clock approaching its upper bound, timing is lost and the outer observable is left with a statistical mixture between the initial state and the entangled state that represents the von Neumann measurement, also not ruling out the paradox for any $|ok\rangle$. In other words, the insertion of a feasible clock with an intrinsic uncertainty is not enough to simulate the decoherence that represents the final step of a measurement. Indeed, as it is clarified by Żukowski and Markiewicz,[29] a complete quantum measurement is divided in

1. **Pre-measurement**, described as the von Neumann measurement;
2. **Decoherence**, where coherences are erased by interaction with uncontrolled degrees of freedom of the environment.

If the insertion of the clock was capable of generating decoherence, the incresing of d would certainly force the paradox to vanish, at least if the internal friend performed a non-selective measurement, and thus described the lab state as a statistical mixture of $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$. It does not seem to be the case, even though the clock has uncontrolled degrees of freedom, represented here by the gaussian uncertainty σ.

There are some issues to be taken into account, however, if decoherence is to be considered a central feature of quantum theory, and not a phenomenon that may occur in some situations. First is the well known discussion of decoherence being a for all practical purposes (FAPP) sort of answer for the measurement problem, once if humankind refine its measurement skills to include every degree of freedom of the environment, no outcome would ever be detected. Under Zurek's decoherence program, the information lost with a wave reduction is actually just stored in the environment, and a skillful experimentalist could be capable of unmaking a measurement. In order to avoid this argument, the Montevideo Interpretation already proposed how uncertainties in time and lenght measurements are fundamental.[30] Indeed, the uncertainty in time measurements in our WFS proved to be capable of controling which measurement Wigner is allowed to perform without disagreeing from his friend. Those uncontrolled degrees of freedom, here represented as a classical error generator, have proved to be relevant for the development of the problem.

The choice of inserting a single quasi-ideal clock, to which just the external observer had access, can be questioned as a source of paradox. This choice reflects the assumption that, inside the lab, any degree of freedom was perfectly controled. If the internal observer had access to another quasi-ideal clock, its uncertainty σ would represent uncontrolled degrees of freedom inside the isolated lab, which was not compatible with the description we intented to make. Any decoherence that could possibly emerge should come from the lack of knowledge Wigner had on how time was passing inside it. Furthermore, we assume Wigner's perspective as the priorized one, and the friend in question would be experimentally described as a second quantum system inside the lab.[29] One could adopt the inverse scenario, where the inner friend had access to a quasi-ideal quantum clock while the outer observer would have access to an ideal one. This might raise the decoherent behavior inside the lab. A scenario for which both the internal and the external observer had access to quasi-ideal clock states would be more complex, since we would be dealing with multipartite states. Any scenario for which both of them had access to the same clock, quasi-ideal or not, or to different clocks synchronized between themselves, would violate the definition of an isolated lab, for Wigner would have ways to percieve and influence what was happening inside the lab by interecting with the shared clock or time reference frame. If the lab is ideally isolated, the only period of time in which Wigner has access to its content is during his measurement.

Another source of paradox could be the global measurement performed by Wigner, or even the clock measurement Π_K^C from which he constructs the relational lab state $\rho_{FS}^W(K)$. As argued before, these measurements are not G-covariant channels, and thus cannot be performed under a SSR over time without access to an asymmetric state. This is indeed reasonable since no measurement inside a lab is infinitely precise.

One way of avoiding the paradox is simply giving up on consistency between predictions provided for the observers in quantum theory. This was an assumption made both by Frauchiger and Renner[6] and by Brukner[12] for the derivation of their no-go theorems. In our discussion, this means that the constraint given by Eq. (4.6) can be abandoned, and therefore Wigner is allowed to perform any measurement $|ok\rangle$ he is capable of. Indeed, it can be argued that both observers have access to different parts of the universe, and thus there is no point on demanding them to provide the same probability distributions. Many current interpretations of Quantum Mechanics in fact give up on self consistency within the theory, and a recent work provides an enforcement on how objectivity of a measurement becomes subjetive with respect to the adopted quantum reference frame.[71] Our results thus might reinforce the no-go theorems of Frauchiger and Renner and of Brukner.

Anyway, the greatest feature of this work is that the quasi-ideal clock in fact proved to be a good model for a quantum clock, that can emulate every property of an ideal clock with sharp precision. Even without infinitesimal control of every degree of freedom of the clock (a classical uncertainty represented by σ), for any time-squeezed and symmetric clock state ($\sigma \leq d$) the periodic Wigner's Friend Scenario evolution would go sharply close to the scenario with an ideal clock, given the correct regime of slow entangling evolution inside the lab. More than that, this behavior was recovered even without saturating the upper bound on the shared asymmetry, which might indicate that a tighter upper bound could be derived from Carmo and Soares-Pinto's result.[48,72] Further analysis of information storage between the parties could also lead to clues on why the ressonant clock works even without maximum shared asymmetry between them, and also a careful study of the charge sectors that arise from the SSR imposed over this universe by the global hamiltonian, and how internal entanglement might be optimizing the clock functioning. These analyses could also explain why the clock would not work for the non periodic scenario, since the charge sectors could change from one scenario to another.

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Appendix

APPENDIX A – USEFUL PROPERTIES OF QUASI-IDEAL CLOCK STATES

This appendix intent to showcase the proofs for secondary properties of quasi-ideal clock states that are most useful to our discussion. Lemmas 3.3.1, 3.3.2 and 3.3.3 are proved in the original article[7] by appealing majorly to the following properties. Derivations are all extracted from the main article, here made in a slower pace.

A.1 Upper bound for gaussian summations

First of all, we want to show that

$$\sum_{n=a}^{\infty} e^{-\frac{(n-X)^2}{\Delta^2}} < \frac{e^{-\frac{(a-X)^2}{\Delta^2}}}{1-e^{-\frac{2(a-X)^2}{\Delta^2}}}, \quad \forall a > X \in \mathbb{R}. \tag{A.1}$$

Changing the dumb index $n \to m = n - a$, we have

$$\sum_{n=a}^{\infty} e^{-\frac{(n-X)^2}{\Delta^2}} = \sum_{m=0}^{\infty} e^{-\frac{(m+a-X)^2}{\Delta^2}} = e^{-\frac{(a-X)^2}{\Delta^2}} \sum_{m=0}^{\infty} e^{-\frac{2(a-X)m}{\Delta^2}} e^{-\frac{m^2}{\Delta^2}}. \tag{A.2}$$

Notice, however, that $e^{-\frac{m^2}{\Delta^2}} < 1$, for $m \geq 0$ e $\Delta \in \mathbb{R}$, so

$$e^{-\frac{2(a-X)m}{\Delta^2}} e^{-\frac{m^2}{\Delta^2}} < e^{-\frac{2(a-X)m}{\Delta^2}}. \tag{A.3}$$

This leads to

$$\sum_{m=0}^{\infty} e^{-\frac{2(a-X)m}{\Delta^2}} e^{-\frac{m^2}{\Delta^2}} < \sum_{m=0}^{\infty} e^{-\frac{2(a-X)m}{\Delta^2}} = \frac{1}{1-e^{-\frac{2(a-X)}{\Delta^2}}}, \tag{A.4}$$

where we appealed in the last step to the the fact that the summation was a telescoping series for any $a > X$. Therefore,

$$\sum_{n=a}^{\infty} e^{-\frac{(n-X)^2}{\Delta^2}} = e^{-\frac{(a-X)^2}{\Delta^2}} \sum_{m=0}^{\infty} e^{-\frac{2(a-X)m}{\Delta^2}} e^{-\frac{m^2}{\Delta^2}} < \frac{e^{-\frac{(a-X)^2}{\Delta^2}}}{1-e^{-\frac{2(a-X)}{\Delta^2}}}, \quad \forall a > X \in \mathbb{R}. \tag{A.5}$$

A.2 Upper bound for normalizing factor

Given

$$|\psi(k_0)\rangle = \sum_{k \in S_d(k_0)} A e^{-\frac{\pi}{\sigma^2}(k-k_0)^2} e^{-i2\pi(k-k_0)/d} |\theta_k\rangle, \tag{A.6}$$

we got

$$\langle\psi(k_0)|\psi(k_0)\rangle = |A|^2 \sum_{k \in S_d(k_0)} e^{-\frac{2\pi}{\sigma^2}(k-k_0)^2}. \tag{A.7}$$

Expanding this summation over the whole set $\mathbb{Z}$,

$$\langle\psi(k_0)|\psi(k_0)\rangle = |A|^2 \left(\sum_{k \in \mathbb{Z}} e^{-\frac{2\pi}{\sigma^2}(k-k_0)^2} + \epsilon_1 \right), \tag{A.8}$$

where

$$|\epsilon_1| = \sum_{k\in\mathbb{Z}/S_d(k_0)} e^{-\frac{2\pi}{\sigma^2}(k-k_0)^2} = 2\sum_{k-k_0=d/2}^{\infty} e^{-\frac{2\pi}{\sigma^2}(k-k_0)^2} < \frac{2e^{-\frac{\pi d^2}{2\sigma^2}}}{1-e^{-\frac{2\pi d}{\sigma^2}}} := \bar{\epsilon_1}, \tag{A.9}$$

and we used the result given in subsection A.1. Let us now appeal to Poisson summation formula, which states that any function f that have a continuous Fourier transform $\hat{f}$ must then satisfy the identity

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{k\in\mathbb{Z}} \hat{f}(k). \tag{A.10}$$

Let us take a look to the function that constitutes the argument in summation A.8. Its continuous Fourier transform is given by

$$\int_{-\infty}^{\infty} e^{-\frac{2\pi}{\sigma^2}(k-k_0)^2} e^{-i2\pi km} dk = e^{-i2\pi k_0 m} \int_{-\infty}^{\infty} e^{-\frac{2\pi}{\sigma^2}(k-k_0)^2} e^{-i2\pi(k-k_0)m} dk = \tag{A.11}$$

$$e^{-i2\pi k_0 m} e^{-\frac{\pi\sigma^2}{2}m^2} \int_{-\infty}^{\infty} e^{-\left(\frac{\sqrt{2\pi}}{\sigma}(k-k_0)+i\sigma\sqrt{\frac{\pi}{2}}m\right)^2} dk,$$

where we completed the square. Making the obvious variable substitution

$$u = \frac{\sqrt{2\pi}}{\sigma}(k-k_0) + i\sigma\sqrt{\frac{\pi}{2}}m; \quad du = \frac{\sqrt{2\pi}}{\sigma}dk, \tag{A.12}$$

this leads to

$$\int_{-\infty}^{\infty} e^{-\frac{2\pi}{\sigma^2}(k-k_0)^2} e^{-i2\pi km} dk = \frac{\sigma}{\sqrt{2}} e^{-\frac{\pi\sigma^2}{2d^2}(md)^2} e^{-i2\pi k_0 (md)/d}. \tag{A.13}$$

Therefore,

$$\begin{aligned}
\langle\psi(k_0)|\psi(k_0)\rangle &= |A|^2 \left(\sum_{k\in\mathbb{Z}} e^{-\frac{2\pi}{\sigma^2}(k-k_0)^2} + \epsilon_1\right) && \text{(A.14)}\\
&= |A|^2 \left(\sum_{m\in\mathbb{Z}} \frac{\sigma}{\sqrt{2}} e^{-\frac{\pi\sigma^2}{2d^2}(md)^2} e^{-i2\pi k_0(md)/d} + \epsilon_1\right) && \text{(A.15)}\\
&= |A|^2 \left(\frac{\sigma}{\sqrt{2}} + \epsilon_2 + \epsilon_1\right), && \text{(A.16)}
\end{aligned}$$

where

$$\begin{aligned}
|\epsilon_2| &= \sum_{m\in\mathbb{Z}-\{0\}} \frac{\sigma}{\sqrt{2}} e^{-\frac{\pi\sigma^2}{2d^2}(md)^2} e^{-i2\pi k_0(md)/d} && \text{(A.17)}\\
&< \frac{\sigma}{\sqrt{2}} \sum_{m\in\mathbb{Z}-\{0\}} e^{-\frac{\pi\sigma^2}{2d^2}(md)^2} && \text{(A.18)}\\
&= 2\frac{\sigma}{\sqrt{2}} \sum_{m=1}^{\infty} e^{-\frac{\pi\sigma^2}{2d^2}(md)^2} && \text{(A.19)}\\
&< \frac{2\sigma}{\sqrt{2}} \frac{e^{-\frac{\pi\sigma^2}{2}}}{1-e^{-\pi\sigma^2}} := \bar{\epsilon_2}. && \text{(A.20)}
\end{aligned}$$

A.3 Definitions for quasi-ideal clock states and representation w.r.t. energy basis

Definition A.3.1. *(Distance of mean energy with respect to spectrum boundaries)* Let $\alpha_0 \in (0,1] \subset \mathbb{R}$. It is a parameter that quantifies how close n_0 is from the spectrum boundaries, 0 or $d-1$, and is given by

$$\alpha_0 = \left(\frac{2}{d-1}\right)\min\{n_0,(d-1)-n_0\} = 1 - \left|1 - n_0\left(\frac{2}{d-1}\right)\right|, \tag{A.21}$$

where we used $\min(a,b) = (a+b-|a-b|)/2$. Notice that, when $n_0 = 0$ ou $n_0 = d-1$, then $\alpha_0 = 0$. When $n_0 = (d-1)/2$, on the other hand, then $\alpha_0 = 1$.

Definition A.3.2. (*Analytical extension for clock states*) For every $|\psi(k_0)\rangle \in \Lambda_{\sigma,n_0}$, it can be defined a function $\psi : \mathbb{R} \to \mathbb{C}$ such that

$$\psi(k_0;k) = \langle\theta_k|\psi(k_0)\rangle = Ae^{-\frac{\pi}{\sigma^2}(k-k_0)^2}e^{i2\pi n_0(k-k_0)/d}, \tag{A.22}$$

with $k \in S_d(k_0)$ and the property $\psi(k_0;k+y) = \psi(k_0-y,k)$.

Definition A.3.3. *(Classification for quasi-ideal clock states)* Let $|\psi(k_0)\rangle \in \Lambda_{\sigma,n_0}$. It is said that $|\psi(k_0)\rangle$ is either

- *symmetric*, for $\sigma = \sqrt{d}$;
- *time squeezed*, for $\sigma < \sqrt{d}$;
- *energy squeezed*, $\sigma > \sqrt{d}$.

Furthermore, if $n_0 = \frac{d-1}{2}$, the adverb *completely* is added to the previous definition.

Definition A.3.4. (*Continuous Fourier transform*) Let $\psi : \mathbb{R} \to \mathbb{C}$ be the analytical extension for a clock state. Then, $\bar{\psi} : \mathbb{R} \to \mathbb{C}$ is the continuous Fourier transform of ψ,

$$\bar{\psi}(k_0,p) := \frac{1}{\sqrt{d}}\int_{-\infty}^{\infty}\psi(k_0,x)e^{-i2\pi px/d}dx = A\frac{\sigma}{\sqrt{d}}e^{-\frac{\pi\sigma^2}{d^2}(p-n_0)^2}e^{-i2\pi pk_0/d}. \tag{A.23}$$

Lemma A.3.1. (*Representation w.r.t. energy basis*) The continuous Fourier transform $\bar{\psi}(k_0;p)$ is an exponentially good aproximation for the projection of $|\psi(k_0)\rangle$ over the energy basis, i.e.,

$$|\langle n|\psi(k_0)\rangle - \bar{\psi}(k_0;n)| < \left(\frac{2^{9/4}d^{-1/4}}{1-e^{-\pi}}\right)e^{-\frac{\pi}{4}d}, \quad n \in [0,d-1] \subset \mathbb{Z}. \tag{A.24}$$

Proof. Notice that the projection over an energy eigenstate results in

$$\begin{aligned}\langle n|\psi(k_0)\rangle &= \langle n|\frac{1}{\sqrt{d}}\sum_{k\in S_d(k_0)}\sum_{n'=0}^{d-1}\psi(k_0;k)e^{-i2\pi n'k/d}|n'\rangle & \text{(A.25)}\\ &= \frac{1}{\sqrt{d}}\sum_{k\in S_d(k_0)}\psi(k_0;k)e^{-i2\pi nk/d}, & \text{(A.26)}\end{aligned}$$

which is nothing more than the discrete Fourier transform of ψ. For to estimate an upper bound for the difference between discrete ($\langle E_n|\psi(k_0)\rangle$) and continuous ($\bar{\psi}$) Fourier transforms, we shall quantify the difference between both and the summation over the whole set $\mathbb{Z}$. To begin with the discrete Fourier transform,

$$\left|\frac{1}{\sqrt{d}}\sum_{k\in S_d(k_0)}\psi(k_0;k)e^{-i2\pi nk/d}-\frac{1}{\sqrt{d}}\sum_{k\in\mathbb{Z}}\psi(k_0;k)e^{-i2\pi nk/d}\right|=\left|\frac{1}{\sqrt{d}}\sum_{k\in\mathbb{Z}/S_d(k_0)}\psi(k_0;k)e^{-i2\pi nk/d}\right|,$$

which is by its turn equal to

$$\begin{aligned}\left|\frac{1}{\sqrt{d}}\sum_{k\in\mathbb{Z}/S_d(k_0)}\psi(k_0;k)e^{-i2\pi nk/d}\right| &= \frac{|A|}{\sqrt{d}}\left|\sum_{k\in\mathbb{Z}/S_d(k_0)}e^{-\frac{\pi}{\sigma^2}(k-k_0)^2}e^{i2\pi n_0(k-k_0)/d}e^{-i2\pi nk/d}\right| \\ &= \frac{|A|}{\sqrt{d}}\left|e^{-i2\pi nk_0/d}\sum_{k\in\mathbb{Z}/S_d(k_0)}e^{-\frac{\pi}{\sigma^2}(k-k_0)^2}e^{i2\pi(n_0-n)(k-k_0)/d}\right| \\ &= \frac{|A|}{\sqrt{d}}\left|\sum_{k\in\mathbb{Z}/S_d(k_0)}e^{-\frac{\pi}{\sigma^2}(k-k_0)^2}e^{i2\pi(n_0-n)(k-k_0)/d}\right| \\ &\leq \frac{|A|}{\sqrt{d}}\left|\sum_{k\in\mathbb{Z}/S_d(k_0)}e^{-\frac{\pi}{\sigma^2}(k-k_0)^2}\right| \\ &= \frac{|A|}{\sqrt{d}}\left|2\sum_{k=d/2}^{\infty}e^{-\frac{\pi}{\sigma^2}(k-k_0)^2}\right|,\end{aligned} \tag{A.27}$$

where we shifted the dumb index to be centered in 0, and since $S_d(0)=\left[-\frac{d}{2},\frac{d}{2}\right]$, we can split the summation in two summations whose absolute value is identical. Applying the result of section A.1, we have

$$\left|\frac{1}{\sqrt{d}}\sum_{k\in S_d(k_0)}\psi(k_0;k)e^{-i2\pi nk/d}-\frac{1}{\sqrt{d}}\sum_{k\in\mathbb{Z}}\psi(k_0;k)e^{-i2\pi nk/d}\right|<\frac{|A|}{\sqrt{d}}\frac{2e^{-\frac{\pi d^2}{4\sigma^2}}}{1-e^{-\frac{\pi d}{\sigma^2}}}, \tag{A.28}$$

and applying the result of section A.2, assuming both errors ϵ_1 and ϵ_2 as negligible (i.e., $\sigma\approx\sqrt{d}$ for an exponentially decaying error ϵ_2),

$$|A|<2^{1/4}d^{-1/4}, \tag{A.29}$$

and therefore

$$\left|\frac{1}{\sqrt{d}}\sum_{k\in S_d(k_0)}\psi(k_0;k)e^{-i2\pi nk/d}-\frac{1}{\sqrt{d}}\sum_{k\in\mathbb{Z}}\psi(k_0;k)e^{-i2\pi nk/d}\right|<2^{5/4}d^{-3/4}\frac{e^{-\frac{\pi d}{4}}}{1-e^{-\pi}}. \tag{A.30}$$

Let us now take a look to the continuous Fourier transform. First, we apply the Poisson summation formula to the summation over all the integers. Notice that

$$\int_{-\infty}^{\infty}\psi(k_0;k)e^{-i2\pi nk/d}e^{-i2\pi mk}dk=e^{-i2\pi(n+md)k_0/d}\int_{-\infty}^{\infty}e^{-\frac{\pi}{\sigma^2}(k-k_0)^2}e^{i2\pi(n_0-n-md)(k-k_0)/d}dk= \tag{A.31}$$

$$= e^{-i2\pi(n+md)k_0/d} e^{-\frac{\pi\sigma^2}{d^2}(n+md-n_0)^2} \int_{-\infty}^{\infty} e^{-\left(\frac{\sqrt{\pi}}{\sigma}(k-k_0)-i\frac{\sqrt{\pi}\sigma}{d}(n_0-n-md)\right)^2} dk = \sqrt{d}\bar{\psi}(k_0; n+md).$$

And thus Poisson summation formula leads to

$$\frac{1}{\sqrt{d}} \sum_{k\in\mathbb{Z}} \psi(k_0; k) e^{-i2\pi nk/d} = \sum_{m\in\mathbb{Z}} \bar{\psi}(k_0; n+md). \tag{A.32}$$

So

$$\left|\bar{\psi}(k_0; n) - \sum_{m\in\mathbb{Z}} \bar{\psi}(k_0, n+md)\right| = \left|\sum_{m\in\mathbb{Z}-\{0\}} \bar{\psi}(k_0; n+md)\right| \tag{A.33}$$

$$= \frac{2|A\sigma|}{\sqrt{d}} \left|\sum_{m=1}^{\infty} e^{-\frac{\pi\sigma^2}{d^2}(n+md-n_0)^2} e^{-i2\pi(n+md)k_0/d}\right| \tag{A.34}$$

$$\leq \frac{2|A\sigma|}{\sqrt{d}} \left|\sum_{m=1}^{\infty} e^{-\frac{\pi\sigma^2}{d^2}(n+md-n_0)^2}\right| \tag{A.35}$$

$$< \frac{2|A|\sigma}{\sqrt{d}} \frac{e^{-\frac{\pi\sigma^2}{d^2}\frac{d^2}{4}}}{1-e^{-\frac{2\pi\sigma^2}{d^2}\frac{d}{2}}}, \tag{A.36}$$

where we applied the result of section A.1. Applying the result of section A.2, assuming ϵ_1 and ϵ_2 both negligible again (i.e., exponentially vanishing with large d, implying $\sigma \approx \sqrt{d}$), results in

$$\left|\bar{\psi}(k_0; n) - \sum_{m\in\mathbb{Z}} \bar{\psi}(k_0; n+md)\right| < 2^{5/4} d^{-1/4} \left(\frac{e^{-\frac{\pi d}{4}}}{1-e^{-\pi}}\right). \tag{A.37}$$

The difference between discrete and continuous Fourier transforms, which we aim to achieve, will be at least the sum of these two upper bounds (since the discrete transform might be distant from the summation over all the integers from bellow, while the continuous transform might be distant from above, or contrariwise). This sum leads to

$$2^{5/4} d^{-3/4} \left(\frac{e^{-\frac{\pi d}{4}}}{1-e^{-\pi}}\right) + 2^{5/4} d^{-1/4} \left(\frac{e^{-\frac{\pi d}{4}}}{1-e^{-\pi}}\right) = 2^{5/4} d^{-1/4} \left(\frac{e^{-\frac{\pi d}{4}}}{1-e^{-\pi}}\right) \frac{1+\sqrt{d}}{\sqrt{d}} \tag{A.38}$$

$$< 2^{9/4} d^{-1/4} \left(\frac{e^{-\frac{\pi d}{4}}}{1-e^{-\pi}}\right), \tag{A.39}$$

which is precisely the upper bound proposed by the lemma. □