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Welfare effects of monetary integration: an analysis for economies with frictions on the foreign exchange market

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RIBEIRÃO PRETO
2024

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Dissertação apresentada ao Programa de Pós-Graduação em Economia da Faculdade de Economia, Administração e Contabilidade de Ribeirão Preto da Universidade de São Paulo, para obtenção do título de Mestre em Ciências. Versão Original.

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RESUMO

CRUZ, Samuel Levi Alves. *Efeitos de bem-estar da integração monetária: uma análise para economias com fricções no mercado de câmbio* 2024. 63f. Manual – Faculdade de Economia, Administração e Contabilidade de Ribeirão Preto, Universidade de São Paulo, Ribeirão Preto, 2024.

Esta pesquisa examina a formação de áreas de moeda comum e busca compreender os efeitos sobre os países membros. Uma abordagem teórica para o problema é proposta, analisando esses efeitos em um modelo monetário com fricções de troca e também fricções no mercado de câmbio estrangeiro. A análise é realizada comparando a formação de uma união monetária entre dois países, utilizando o modelo desenvolvido como base. Determinamos as condições sob as quais uma economia de moeda única gera uma eficiência comercial maior em comparação com o caso de múltiplas moedas. Além disso, descobrimos que, sob políticas monetárias idênticas, o bem-estar em uma economia com uma única moeda supera consistentemente o de uma economia com múltiplas moedas. O resultado desta pesquisa pode ser útil para a tomada de decisões em políticas econômicas e para o desenvolvimento de futuras uniões monetárias

Palavras-chave: Moeda, União Monetária, Mercado de Câmbio, Fricções
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ABSTRACT

CRUZ, Samuel Levi Alves. Welfare effects of monetary integration: an analysis for economies with frictions on the foreign exchange market. 2024. 40f. Manual – Faculdade de Economia, Administração e Contabilidade de Ribeirão Preto, Universidade de São Paulo, Ribeirão Preto, 2024.

This research examines the formation of common currency areas and understand the effects on member countries. A theoretical approach to the problem is proposed by analyzing these effects in a monetary model with trade frictions and also frictions in the foreign exchange market. The analysis is carried out by comparing the formation of a monetary union between two countries, using the developed model as a basis. We determine the conditions under which a single currency economy yields greater trade efficiency compared to the case of multiple currencies. Additionally, it finds that, under identical monetary policies, the welfare in an economy with a single currency consistently surpasses that of an economy with multiple currencies. The result of this research can be useful for decision-making in economic policies and for the development of future currency unions.

Keywords:Money, Monetary Union, Exchange Market, Frictions
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1 INTRODUCTION

The establishment of monetary unions, or common currency areas, involves the adoption of a unified currency by nations previously characterized by distinct currencies, reflecting a collective commitment among participating nations. This type of economic relationship can have significant effects on the economies of these countries, such as the creation of a monetary union, which could influence trade dynamics, financial stability, and economic efficiency in the region. Ögren (2019) highlights that a monetary union has the potential to reduce transaction costs among member countries. However, upon joining a monetary union, a member country surrenders autonomous control over its monetary policy, an important instrument for economic stabilization. Furthermore, adoption of a common currency subjects the member country to the aggregated risks associated with all participating nations. [1]

Since the seminal research of Mundell (1961) about the formation of common currency areas, an extensive theoretical and empirical literature has been forming. In general, most part of these research are looking for understand the benefits and disadvantages of the creation of monetary unions. However, Silva and Terneyro (2010) argues that the existing literature on monetary unions has some limitations. Specifically, there is a shortage of empirical studies that effectively demonstrate the advantages and disadvantages of currency unions. Furthermore, there is a noticeable absence of a unified welfare-based framework for comprehensively assessing the costs and benefits associated with these unions.

The objective of this research is to establish a framework for examining and understanding the effects of participating in a monetary union. To achieve this, we employ a theoretical approach and model based on the works of Lagos and Wright (2005), Geromichalos and Jung (2018). Based on their models, our framework incorporates two distinct economies: one with complete monetary integration, where all countries share the same currency, and another where countries maintain separate currencies. In both economies, only certain agents from each country have the opportunity to engage in foreign trade in each period. However, transactions within each country are exclusively conducted using the national currency, necessitating a FOREX market for currency exchange, modeled as a over-the-counter market similarly to Geromichalos and Jung (2018). Consequently, our model incorporates matching frictions in the goods market as well as frictions in the FOREX market.

Using this framework, we characterize the steady-state equilibrium for both economies and compare the differences in trade volume and welfare between them. This approach allows us to thoroughly examine and understand the effects of participating in a monetary union. Additionally, we conduct an exercise where we introduce fluctuating monetary policy in the economy with multiple currencies and compare it with the economy with a single currency.

We identify that the volume of trades in a economy with multiple currencies surpasses the economy with one currency under specific conditions. Specifically, it depends on the mone-

[1] For instance, we can mention the debt crisis in Greece, which started in the end of 2009 and influenced all countries in the Eurozone

tary policies of both countries,particularly the deviation from Friedman's rule. Additionally, depends on the price paid to exchange currencies in the economy with multiple currencies and in the probability of a meeting between a buyer and a dealer.

We observe that, for the same inflation rates, both the volume of trade and the welfare of the economy with a single currency consistently exceed those of the economy with multiple currencies. Additionally, when we introduce fluctuating monetary policy in the economy with multiple currencies, we find that the output can surpass that of the economy with a single currency. However, in our exercise, when considering the expected volume of trade in the economy with multiple currencies, it is lower than that of the economy with a single currency.

Monetary unions are an important research subject in international economics. These unions manifest in diverse forms, ranging from what are commonly termed as "national" currency unions, where a single country opts to adopt a unified currency which previously had distinct currencies (e.g., the United States and Germany in in the 19th century), to multinational currency unions such as the Eurozone, which continue to expand with new members such as Croatia in 2023 and the West African Monetary Zone, as well as historical examples like the Latin Monetary Union(Ögren, 2019). Due to the increasing integration among the countries, and the arise of *dollarization*[2], this topic becomes even more relevant.

According to Silva and Terneyro (2010), exists benefits and downsides for countries that decide to integrate a monetary union. Among the benefits, one can highlight is the enhanced control over inflation rates that countries with high inflation rates can achieve upon joining a monetary union. That is because, a monetary union with a set of credible anchor countries may be capable of eliminates the inflationary bias resulting from inconsistencies in monetary policy. Besides that, Meller and Nautz (2012), Tillmann (2012) and Adelakun (2020) observes that the inflation persistence tends to reduce in a monetary union (the first two study in the EMU and the last study the West Africa monetary zone).

Another potential benefit, which our research corroborates, is the increase in trade and greater capital integration among the participating countries of the monetary union. Argument reinforced by Rose, Lockwood and Quah (2000), who verified through a cross-sectional panel that two countries with the same currency trade more than countries with separated currencies and more recently Glick and Rose (2016) uses differents empirical gravity models and observed that EMU (Economic and Monetary Union) has increased exportation around 50%.

However, Silva and Terneyro (2010) uses a differences-in-differences specification to compare the trade-flows between some countries of EMU and others similar trade partners. The findings suggest that the impact of the euro is nearly negligible. Besides that, Flandreau (2000) examines bilateral trade patterns for the Scandinavian Currency Union and Latin Monetary Union using gravity tests but fails to uncover significant results.

Furthermore, Ravikumar and Wallace (2002) analyzes a model with two identical countries and $N \geq 3$ perishable goods in each date and a continuum [0,1] of N types of agents for each

[2] Countries adopt a foreign currency, US Dollar in most cases (e.g. Ecuador and Panama)

country. Despite this, there exists two distinct currencies with fixed supply, where a fraction of individuals from each country have an endowment of a unit of currency. Individuals who begins the period with this endowment are not able to produce. In this model the authors found that any equilibrium in which the national and foreign currency have different roles in the economy, is dominated in ex-ante welfare terms by the best single uniform currency equilibrium. However, Chen and Novy (2018) encountered evidence that suggests that the effects of trade resulting from monetary unions vary significantly among countries. This heterogeneity implies that conventional homogeneous estimates regarding monetary unions do not offer useful guidance for countries when deciding whether or not to join such unions.

Silva and Terneyro (2010) affirms that the main argument against monetary unions, in the perspective of a member country, is the lost of his monetary policy independence. In spite of that, Mundell (1961) states that this cost is relatively small when the optimal monetary area criteria are met.[3]

Besides that, Frankel (1999) asserts that two other properties of an optimal monetary area are essential to ensure the benefits of a monetary union: the degree of openness, characterized by the volume of trade with a group of partner countries; and the income correlation among the countries. Consequently, countries that exhibit substantial levels of trade, indicating a high degree of openness and income correlation, stand to benefit from participating in a monetary union. In our research, we observe that the availability of FOREX dealers is also important in deciding whether to join a monetary union or not.

Additionally, Kiyotaki and Moore (2004) argue that in a hypothetical market where there are no specific macroeconomic shocks for the countries, therefore there is no need for independent stabilization policies, this is one of the main arguments to maintain separated currencies, and people can choose a variety of goods they produce, a monetary union may not be preferable.

Kocherlakota and Krueger (1999) corroborates with previous findings, suggesting that in an economy where agents' preferences between consuming national and foreign goods are heterogeneous and this preference information is private, it may be socially optimal for countries to maintain separate currencies, even if these countries cannot independently control their money supply, as currencies serve as signals of preferences. Moreover, based on the model presented in the paper, it is emphasized that the impetus for economic and monetary union in Europe may be an optimal response to increased consumer indifference regarding the nationality of products consumed between these countries.

Araujo and Ferraris (2021) developed an economic model of search in monetary markets based on the model by Lagos and Wright (2005) to study an economy with the presence of multiple currencies and exchange restrictions. The authors utilize this model to explore scenarios in which a economy with multiple currencies leads to a socially superior allocation compared to an economy with monetary integration through a common currency. They find that, in a economy where the FOREX market works as a Walrasian market, foreign currencies enable reallocations

3 Optimal monetary area is a region where the welfare of the countries are larger in a monetary union than with multiple currencies

of poorly allocated domestic liquidity. This occurs because, according to Araujo and Ferraris (2021) model in their Decentralized Market there is a misallocation of liquidity due to randomness in the division of types of individuals, whether they will be buyers or sellers. Therefore, due to uncertainty, all individuals need to carry currency from one period to another. However, sellers do not need currency in the decentralized market. Thus, after the division of types is realized, sellers have currency without a need, and exchanges between currencies fill the gap of the lack of a credit market. Furthermore, another factor pointed out is that the implicit values of currencies are inefficiently low. If an individual holds foreign currency, and it appreciates, their liquidity constraint is relaxed, and since the devaluation of the domestic currency is very low, this implies a Pareto improvement. On the other hand, the present research incorporates some frictions in the FOREX market, considering it as an over-the-counter market rather than a Walrasian market, and examines the impact of these frictions as well.

Moreover, according to Geromichalos and Jung (2018), a significant part of the international macroeconomic literature assumes that the foreign exchange market is a competitive market. This assumption is often made for facilitates the tractability in the model. Additionally, Geromichalos and Jung (2018) highlights the foreign exchange market's significance as the world's largest over-the-counter market. In response, Geromichalos and Jung (2018) develop a model with multiple economies, each with its own currency, and introduces frictions to better represent the dynamics of the foreign exchange market and they study those dynamics. However, the main focus on Geromichalos and Jung (2018) is to study the dynamism on the international trade, they consider only the case with multiple currencies. In this research, we include an economy with a single currency and aim to compare which scenario is most efficient and under what conditions.

The paper is organized as follows. Section 2 establishes the environment that belongs to both economies, the economy with multiple currencies and the economy with single currency. Section 3 examines the economy with single currency. Section 4 analyzes the economy with multiple currencies. Section 5 presents our key findings regarding the comparison between these distinct forms of money. Section 6 examines the economy with multiple currencies with fluctuations in the monetary policy and compare it with the economy with single currency. Finally, section 7 provides the concluding remarks of this research.

2 BASIC ENVIRONMENT

The model is a version of the economy search model of Lagos and Wright (2005), we modify some structures of the environment and extend the model for multiples countries. Also, the model is similar to the economy model of Geromichalos and Jung (2018). In this way, time is discrete and infinite. There are two identical nations indexed by $i = \{1, 2\}$. There are two types of agents in each country: sellers with measure $1 + \delta$, $\delta \in [0, 1]$ and buyers with unit measure. Additionally, there are other agent without nationality, with measure d, called dealers. All agents have an infinite lifespan and apply a discount rate to the future $\beta \in (0, 1)$.

Each period is divided into three sub-periods: the third involves a frictionless Walrasian Centralized Market (CM) for each country, the second sub-period consists of trades in bilateral random meeting in distincts descentralized markets (DM), where credit is not feasible because agents are anonymous and unable to commit themselves to future actions, and in the first sub-period a FOREX market opens.

In the CM, all agents, including $dealers$, act as both buyers and sellers of the general good X, which is produced using labor, l, through a linear production function. The two CMs are separate from each other: Agents from country i are not eligible to participate in CM_{-i}. Although, because $dealers$ have no nationality, they can participate in both CMs. At the end of the third period, a fraction $\delta \in [0, 1]$ of buyers experience a trade shock, which means they gain the opportunity to trade in foreign DM. Those buyers who experienced the shock are known as T-type, while the rest are called N-type.

In the second sub-period, there are a decentralized market for each country. In the DM_i (DM_{-i}), agents engage in the trade of a special good, denoted as q_i (q_{-i}), which is different to each country. These trades take place through random bilateral meetings between local sellers and buyers, who can be either local residents or foreigners of the T-type. Note that, buyers T-type can consume in both DM's in the same period. Due to the anonymity of agents and their inability to commit to future actions, credit is not a viable option in this market. Therefore, agents require a medium of exchange in this market, which in this model will be fiat currencies. Since, the number of sellers is greater than the number of buyers, assume that every buyer matches with a seller. Given a match, buyers makes a take-it-or-leave-it offer to the seller. At the end of the DM, all meetings are dissolved. During the third sub-period, the FOREX market opens for currency exchange. Further elaboration on this sub-period will follow later.

Now, consider agents' preferences. The utility of a typical buyer is given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \{u(q_t) + u(\tilde{q}_t) + X_t - l_t\},$$

where X_t is the consumption of general good in the CM, l_t is the labor utilized to produce the general good, q_t and $\tilde{q}_t$ are the consumption of the local and foreign special good, respectively. Consider that agents have a logarithmic utility function, that is $u(q_t) = ln(q_t)$.

For a typical seller we have

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \{-q_t/\mu + X_t - l_t\},$$

where, X_t and l_t are as before, and $-q_t/\mu$ [1] is the disutility of producing q_t units of special good in the DM.

And for a typical dealer the utility is give by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \{X_t - l_t\}.$$

Next, we will present specific components of the economy with a single currency an the economy with multiple currencies, along with a formal description of the role of the FOREX market in both of these economies.

[1] Given our logarithmic utility function, if we have the situation where $c(q_t) = q_t$, which is quite common in this literature, we will have $q^* = 1$. In such cases, the gains of trade would be non-positive. Since our objective is to compare welfare levels, we introduce the parameter μ into the disutility function of producing q_t with the aim of ensuring that the gains of trade are positive.

3 SINGLE CURRENCY

Suppose that the countries participate in the same monetary union, because of that they share the same monetary authority. Additionally, as previously mentioned, due to the anonymity and incapacity to commit to future actions, agents require a medium of exchange in the DM. Let's consider that there exists only one perfectly divisible fiat currency, denoted as $m \in \mathbb{R}_+$, with its value in numeraire units represented by ϕ. The monetary authority of the monetary union controls the stock of money M and can alter it with a net growth rate denoted as τ. The introduction or withdrawal of new money occurs through lump-sum transfers to buyers at the conclusion of each period.

Note that since the economy has only one currency, sellers of country 1 and 2 accepts m as medium of exchange in both DMs. Because of that the participation in the FOREX market becomes irrelevant.

Next, let's proceed to introduce the agents' value functions for this economy. First, in the CM_i agents can trade their money for the general good X at the price ϕ, where each unit of currency can be exchanged for ϕ units of goods within the CM_i. Thus, the value function of buyer i, who holds m_i units of money in the CM_i, can be formulated as

$$W_i^b(m_i) = \max_{X,l,m_i'} \{X - l + \beta \mathbb{E}_c\{V_i^c(m_i')\}$$
$$\text{s.t. } X + \phi m_i' = l + \phi m_i + T,$$

where T is the real value of lump-sum monetary transfer made by the monetary authority of the monetary union, m_i' represents the amount of money that the agent of the country i chooses today to carry for tomorrow. Additionally, V_i^c is the expected value function in the type $c = T, N$ of a buyer i in the DM_i. Substituting $(X - l)$ from the constraint, we have:

$$W_i^b(m_i) = \phi m_i + T + \max_{m_i'}\{-\phi m_i' + \beta \mathbb{E}_c\{V_i^c(m_i')\}\}. \tag{3.1}$$

Note that, the buyer's function value in the CM_i is linear in the quantity of money, m_i, brought to the CM_i. Because of that, m_i does not impact the decision of m_i'.

Similarly, the seller's value function in the CM_i is given by

$$W_i^s(m_i) = \max_{X,l}\{X - l + \beta V_i^s(0)\}$$
$$\text{s.t. } X = l + \phi m_i.$$

According to Rocheteau and Wright (2005), the seller never want to leave CM with any money, because of that $V_i^s(0)$ is the seller i value function in the DM. Replacing again $X - l$, we obtain:

$$W_i^s(m_i) = \phi m_i + \beta V_i^s(0).$$

Now, the expected value function for the buyer i who begins the second sub-period with m_i units of currency is given by

$$\mathbb{E}_c\{V_i^c(m_i)\} = \delta V_i^T(m_i) + (1-\delta)V_i^N(m_i), \tag{3.2}$$

where $V_i^T(m_i)$ is the DM value function of a T-type buyer i, and $V_i^N(m_i)$ is the DM value function of a N-type buyer i, where both carry m_i units of money. Additionally, the DM value function of a T-type buyer i satisfies:

$$V_i^T(m_i) = u(q_i) + u(\tilde{q}_i) + W_i^B(m_i - p_i - \tilde{p}_i), \tag{3.3}$$

where q_i and $\tilde{q}_i$ are, respectively, the consumption of local and foreign special good. Furthermore, p_i is the units of m_i that buyer i transfer to seller i to acquire q_i, and and $\tilde{p}_i$ is the quantity of m_i that buyer i transfer to seller $-i$ to acquire $\tilde{q}_i$.

Moreover, for the N-type buyer who goes to second sub-period with m we have the value function in the DM given by

$$V_i^n(m_i) = u(q_i) + W_i^B(m_i - p_i). \tag{3.4}$$

At last, the value function of a seller i who enter the DM with no money, is given by

$$V_i^S(0) = \frac{1}{1+\delta}[-q_i/\mu + W_i^S(p_i)] + \frac{\delta}{1+\delta}[-\tilde{q}_{-i}/\mu + W_i^S(\tilde{p}_{-i})]. \tag{3.5}$$

3.1 Terms of Trade

Consider that the buyer T-type buyer i visits, in the second sub-period, first the local DM, then after visits the foreign DM. Let's study the terms of trade of these markets. In the second sub-period, the problem of the buyer i, who carries $\tilde{m}_i$ and meet with a seller in the DM_{-i} is to maximize his surplus simultaneously ensuring they satisfy the seller's participation constraint, and it can be expressed as:

$$\begin{gathered}\max_{\tilde{q}_i,\tilde{p}_i} u(\tilde{q}_i) + W_i^b(\tilde{m}_i - \tilde{p}_i) - W_i^b(\tilde{m}_i)\\ \text{s.t.}\quad \text{-}\ \tilde{q}_i/\mu + W_{-i}^s(\tilde{p}_i) \geq W_{-i}^s(0),\\ \tilde{p}_i \leq \tilde{m}_i.\end{gathered}$$

Note that if the seller's participation does not hold with equality, the buyer could enhance their surplus by reducing the quantity offered to the seller. Therefore, it's imperative that the seller's participation constraint is satisfied with equality. Because of that and given the linearity of W_i^b e W_{-i}^s, we have

$$\begin{aligned}&\max_{\tilde{q}_i,\tilde{p}_i} u(\tilde{q}_i) - \phi\tilde{p}_i\\ &\text{s.t. } \tilde{q}_i/\mu = \phi\tilde{p}_i,\\ &\qquad \tilde{p}_i \leq \tilde{m}_i.\end{aligned} \tag{3.6}$$

Thus, the solution to the problem in (3.6) is

$$\tilde{q}_i = \begin{cases} \tilde{q}_i^*, & \text{if } \tilde{q}_i^*/\mu \leq \phi\tilde{m}_i \\ \tilde{q}_i, & \text{if } \tilde{q}_i^*/\mu > \phi\tilde{m}_i \end{cases}, \tag{3.7}$$

$$\tilde{p}_i = \begin{cases} \tilde{m}_i^*, & \text{if } \tilde{q}_i^*/\mu \leq \phi\tilde{m}_i \\ \tilde{m}_i, & \text{if } \tilde{q}_i^*/\mu > \phi\tilde{m}_i \end{cases}, \tag{3.8}$$

where $\tilde{m}_i^* = \tilde{q}_i^*/\mu\phi$, and $\tilde{q}_i^* = \{\tilde{q}_i : u'(\tilde{q}_i) = 1/\mu\}$.

Let's now continue with the examination of the terms of trade in the domestic DM. The problem of the buyer i, who carries m and meet with a seller in the DM is given by

$$\begin{aligned} \max_{q_i, p_i} u(q_i) + u(\tilde{q}_i) + W_i^b(m_i - p_i - \tilde{p}_i) - (u(\bar{\tilde{q}}_i) + W_i^b(m - \bar{\tilde{p}}_i)) \\ \text{s.t.} \quad \text{-}\, q_i/\mu + W_i^s(p_i) \geq W_i^s(0), \\ p_i \leq m_i, \end{aligned}$$

where, $\tilde{m}_i$ is the entering money holdings of a buyer i in the DM_{-i}, so, $\tilde{m}_i = m_i - p_i$. Once more, let $\tilde{p}_i$ and $\tilde{q}_i$ be the terms of trade of that match in DM_{-i} when the buyer i trades in the DM_i as well, and let $\bar{\tilde{p}}_i$ and $\bar{\tilde{q}}_i$ be the terms of trade in DM_{-i} when the buyer i do not trade in the DM_i. Considering again the linearity of W_i^s and W_i^b, we have

$$\begin{aligned} \max_{q_i, p_i} u(q_i) + u(\tilde{q}_i(\tilde{m}_i)) - \phi p_i - \phi\tilde{p}_i(\tilde{m}_i) - u(\bar{\tilde{q}}_i(\tilde{m}_i)) + \phi\bar{\tilde{p}}_i(\tilde{m}_i) \\ \text{s.t. } q_i/\mu = \phi p_i, \\ \tilde{m}_i = m_i - p_i, \\ p_i \leq m_i. \end{aligned} \tag{3.9}$$

The solution to this problem is described in the following lemma.

Lemma 1. *In a economy with a single currency, consider the problem of the buyer* i*, who enter the second sub-period with* m *units of money. We have the following results:*

$$q_i = \begin{cases} q_i^* = \{q_i : u'(q_i) = 1/\mu\}, & \text{if } m_i^* + \tilde{m}_i^* \leq m_i \\ q_i = \{q_i : u'(q_i) = u'(\tilde{q}_i) < 1/\mu\}, & \text{if } m_i < m_i^* + \tilde{m}_i^* \end{cases}, \tag{3.10}$$

$$\tilde{p}_i = \begin{cases} m_i^* = q_i^*/\mu\phi, & \text{if } m_i^* + \tilde{m}_i^* \leq m_i \\ \hat{m}_i = q_i/\mu\phi, & \text{if } m_i < m_i^* + \tilde{m}_i^* \end{cases}, \tag{3.11}$$

Proof. See Appendix. □

Note that, when $m_i < m_i^* + \tilde{m}_i^*$, $q_i = \{q_i : u'(\phi p_i \mu) = u'(\phi\tilde{p}_i\mu) > 1/\mu\}$, and $\tilde{m}_i = m_i - p_i$, this implies that $\tilde{p}_i = \tilde{m}_i$ and also that $p_i = \tilde{p}_i = \frac{m_i}{2}$.

3.2 Optimal Behavior

Now, we analyzes the object function of a $buyer_i$ in the CM_i. Substituting (3.3) and (3.4) into (3.2) and advance it by one period. Then plug the rising expression into (3.1), we have the buyer i's objective function:

$$\begin{aligned} Obj^i = -\phi m_i' + \beta\phi' m_i' + \beta\delta u(\phi' p_i \mu) + \beta\delta u(\phi' \tilde{p}_i \mu) \\ -\beta\delta\phi' p_i - \beta\delta\phi' \tilde{p}_i - \beta\delta u(\bar{q}_i) + \beta\delta\phi' \bar{p}_i + \beta u(\bar{q}_i) - \beta\phi' \bar{p}_i, \end{aligned} \tag{3.12}$$

where, $\bar{q}_i$ and $\bar{p}_i$ are the terms of trade between a buyer i, that did not visited the DM_{-i}, and a seller i.

Consider the three sub-cases of money holdings: I: $m_i^* + \tilde{m}_i^* \leq m_i$; II: $m_i* \leq m_i \leq m_i^* + \tilde{m}_i^*$; and III: $m_i \leq m_i^*$. Assuming an interior solution, the following lemma expresses the first-order conditions for this problem.

Lemma 2. *Define $Obj_s^i(m_i')$ the buyer i's objective function when this agent holds m_i, and s = {1, 2, 3} are the three sub-cases of money holdings. here s = 1 is the case when $m_i^* + \tilde{m}_i^* \leq m_i$, s = 2 happens when $m_i* \leq m_i \leq m_i^* + \tilde{m}_i^*$ and s = 3 when $m_i \leq m_i^*$. Then we have:*

$$\frac{\partial Obj_1^i(m_i')}{\partial m_i'} = 0 = \phi - \beta\phi',$$

$$\frac{\partial Obj_2^i(m_i')}{\partial m_i'} = 0 = -\phi + \beta\phi' + \beta\delta\mu\phi' u'(\mu\phi' p_i)\frac{\partial p_i}{\partial m_i'} + \beta\delta\mu\phi' u'(\mu\phi' \tilde{p}_i)\frac{\partial \tilde{p}_i}{\partial m_i'}$$
$$- \beta\delta\phi' \frac{\partial p_i}{\partial m_i'} - \beta\delta\phi' \frac{\partial \tilde{p}_i}{\partial m_i'},$$

$$\frac{\partial Obj_3^i(m_i')}{\partial m_i'} = 0 = -\phi + \beta\phi' + \beta\delta\mu\phi' u'(\mu\phi' p_i)\frac{\partial p_i}{\partial m_i'} + \beta\delta\mu\phi' u'(\mu\phi' \tilde{p}_i)\frac{\partial \tilde{p}_i}{\partial m_i'}$$
$$- \beta\delta\phi' \frac{\partial p_i}{\partial m_i'} - \beta\delta\phi' \frac{\partial \tilde{p}_i}{\partial m_i'} + \beta(1-\delta)\phi\{\mu u'(\mu\phi m_i') - 1)\}.$$

Proof. Replacing the terms of trade found previously, and obtaining the derivative with respect to m_i' yields the desired result. □

3.3 Steady-State Equilibrium with Multiple Currencies

In this research, we are focusing on sub-case 2, as sub-case 1 only arises with the Friedman Rule, and sub-case 3 would result in an even more inefficient equilibrium than sub-case 2. Note that, from the terms of trade $u'(\mu\phi p_i) = u'(\mu\phi\tilde{p}_i)$. Thus, from Lemma 2, we have:

$$u'(q_i) = \frac{\phi/\phi' + \beta\delta - \beta}{\beta\delta\mu} \tag{3.13}$$

Note that, from equation 3.13 we can observe that the product is the same for both countries. Therefore, we have $q_i = q$.

A steady-state monetary equilibrium is a sequence of $\phi/\phi' = (1 + \tau)$ that solves the difference equation (3.13), where τ is a time invariant monetary authority policy. In the steady-state monetary equilibrium we have $q_t = q_{t+1} = q^{sc}$.

Proposition 1. *Exists a unique steady-state monetary equilibrium for sub-case 2 of money holdings, when* $(1+\tau) > \beta$.

Proof. Note that, from 3.13, and since we are analyzing sub-case 2, we must have:

$$u'(q^{sc}) = \frac{\phi/\phi' + \beta\delta - \beta}{\beta\delta\mu} > 1/\mu,$$

and this case arises only when $(1+\tau) > \beta$.

From our logarithmic utility function, we have $u'(q^{sc}) = \frac{1}{q^{sc}}$, and since 3.13 depends only on invariant parameters, there exists a unique q^{sc} that satisfies 3.13. □

Thus, from (3.13), and given the logarithmic utility, we have

$$q^{sc} = \frac{\beta\delta\mu}{(1+\tau) + \beta\delta - \beta}. \tag{3.14}$$

4 MULTIPLE CURRENCIES

Consider now the case with multiple currencies, where each country has his own monetary authority. Each country have perfectly divisible fiat currency, referred as $m_i \in \mathbb{R}_+$, $i = 1, 2$, whose value in numeraire units is ϕ_i. The monetary authority of each country controls the stock of money M_i and can alter with net growth rate τ_i. The new money m_i still is introduced or withdrawn through lump-sum transfers to buyers i at the conclusion of each period.

In the DM_i, sellers i only accept the local currency, that is m_i. Therefore, T-type buyers $-i$ need to acquire m_i if they want to consume the special good of country i. The FOREX market facilitates this acquisition, allowing T-type buyers i to exchange m_i for m_{-i} with dealers. Let $\alpha_d \in [0, 1]$ be the probability of a dealer contacts with a buyer, and $\alpha_i \in [0, 1]$ denote the probability of a buyer i contacts a dealer. Given a match between a buyer i and a dealer, the buyer can trade m_i for m_{-i} at a mark-up $\kappa > 1$. Additionally, dealers can obtain money from two potential sources. First, they can carry money from the previous CMs. Second, dealers has access to a interdealer market which is perfectly competitive and occur at the same time of FOREX market. In that market, a dealer can acquire m_i, $i = 1, 2$, at market price from others dealers. Table 1 illustrates this dynamic.

Table 1 – Trading Activity

	Subperiods		
	1° subperiod (Forex Market - exchange money)	2° subperiod (DMs - trade special good)	3° subperiod (CMs - trade general good)
Buyer *i* N-type	Do not participate	Trade with seller *i*	Trade only with copatriots and Dealers
Buyer *i* T-type	Can Exchange m_i for m_{-i} with *Dealers*	Trade with seller *i* AND seller -*i*	Trade only with copatriots and Dealers
Seller *i*	Do not participate	Trade with Buyer *i* OR Buyer -*i* T-type	Trade only with copatriots and Dealers
Dealers	Can exchange currency with buyers and in the Walrasian interdealer market with *Dealers*	Do not participate	Can trade in both CMs

Now, we present the value functions of the agents in each market. In the competitive centralized market i, CM_i, agents have the opportunity to trade money for the general good X at the price ϕ_i, where each unit of currency can be exchanged for ϕ_i units of goods within the CM_i. Consequently, the value function of a buyer i who carry m_i units of money in the CM_i can be expressed as:

$$W_i^b(m_i) = \max_{X,l,m_i'} \{X - l + \beta \mathbb{E}_c\{F_i^c(m_i')\}$$

$$\text{s.t. } X + \phi_i m_i' = l + \phi_i m_i + T_i,$$

where, T_i is the real value of lump-sum monetary transfer made by the monetary authority of country i, m_i' represents the amount of money that the agent chooses today to carry for tomorrow. Additionally, F_i^c is the value function of buyer i of the type $c = T, N$ in the FOREX market. Substituting $(X - l)$ from the constraint, we have:

$$W_i^b(m_i) = \phi_i m_i + T_i + \max_{m_i'}\{-\phi_i m_i' + \beta \mathbb{E}_c\{F_i^c(m_i')\}\}. \tag{4.1}$$

Note that, the buyer's function value in the CM_i is linear in the quantity of money, m_i, brought to the CM_i. As a result, m_i does not impact the decision regarding m_i'.

In the same way, the seller's value function in the CM_i is given by

$$W_i^s(m_i) = \max_{X,l}\{X - l + \beta V_i^s(0)\}$$
$$\text{s.t. } X = l + \phi_i m_i,$$

where $V_i^s(0)$ is the seller's value function in the DM_i, as augmented before, he leaves the CM_i with no money. Replacing $X - l$, we obtain:

$$W_i^s(m_i) = \phi_i m_i + \beta V_i^s(0).$$

Now, note that since the dealer can visit both CMs, and participates in the interdealer market this agent can have both currencies in the CM. Let $\mathbf{m} \equiv (m_1, m_2)$, and $\boldsymbol{\phi} \equiv (\phi_1, \phi_2)$. Therefore, the dealer's value function is given by

$$W^D(\boldsymbol{m}) = \max_{X,l,\boldsymbol{m'}}\{X - l + \beta F_d(\boldsymbol{m'})\}$$
$$\text{s.t. } X + \boldsymbol{\phi m'} = l + \boldsymbol{\phi m}$$

where $F_d(\mathbf{m}')$ is the value function of a dealer who starts the FOREX market with $\mathbf{m}'$. Again, substituting (X-l), we have

$$W^D(\boldsymbol{m}) = \boldsymbol{\phi m} + \max_{\boldsymbol{m'}}\{-\boldsymbol{\phi m'} + \beta F_d(\boldsymbol{m'})\}. \tag{4.2}$$

Let, ϵ be the price of m_2 in terms of m_1, that is $\epsilon = \frac{\phi_2}{\phi_1}$, and given a match between a T-type buyer i and a dealer let $\{\bar{m}_i^i, \bar{m}_{-i}^i\}$ and $\{\bar{m}_i^d, \bar{m}_{-i}^d\}$ be the portfolios of money of buyers i and dealers, respectively, after the FOREX market trades. Next we introduce the value function of a dealer who starts the FOREX market with portfolio $\boldsymbol{m}^d$:

$$F_d(\boldsymbol{m}^d) = (1 - \alpha_d)W^D(\boldsymbol{m}^d) + \tfrac{\alpha_d}{2}\int W^D(\bar{\boldsymbol{m}})dH^1(m_1) + \tfrac{\alpha_d}{2}\int W^D(\bar{\boldsymbol{m}})dH^2(m_2),$$

H^i is the cumulative distribution function that pertains to the money holdings of a random buyer the dealer might interact with in the FOREX market.

Now, the expected value function for the buyer i who begins the FOREX market with m_i units of currency is given by

$$\mathbb{E}_c\{F_i^c(m_i)\} = \delta F_i^T(m_i) + (1 - \delta)V_i^N(m_i) \tag{4.3}$$

where $F_i^T(m_i)$ is the FOREX value function of a T-type buyer i, and $V_i^N(m_i)$ is the value function of a buyer who goes to the second sub-period only with local currency m_i. Additionally,

$$F_i^T(m_i) = \alpha_i V_i^t(\bar{m}_i^i, \bar{m}_{-i}^i) + (1-\alpha_i)V_i^N(m_i), \tag{4.4}$$

where, $V_i^t(\bar{m}_i^i, \bar{m}_{-i}^i)$ is the DM value function of a T-type buyer i who matched with a dealer and have acquired foreign money. That value function satisfies:

$$V_i^T(\bar{m}_i^i, \bar{m}_{-i}^i) = u(q_i) + u(\tilde{q}_i) + W_i^B(m_i - p_i - \kappa(\iota_{\{i=1\}}\epsilon + \iota_{\{i=2\}}(1/\epsilon))\bar{m}_{-i}^i), \tag{4.5}$$

where q_i and $\tilde{q}_i$ are, respectively, the consumption of local and foreign special good. Furthermore, p_i is the units of m_i that buyer i transfer to seller i to acquire q_i. Besides that, as Geromichalos and Jung (2018), consider that buyers spend all the foreign money in the foreign Descentralized Market if they have the possibility to visit it. Consequently, if a T-type buyer i decides to consume $\tilde{q}_i$ they must participate in the FOREX as previously mentioned, where they incur a mark-up cost of κ. Additionally, the value of the national currency must be converted into the value of the foreign currency.

Moreover, for the buyer who goes to second sub-period only with m_i we have the value function in the DM given by

$$V_i^n(m_i) = u(q_i) + W_i^B(m_i - p_i). \tag{4.6}$$

Finally, the value function of a seller i who does not carry money to DM_i, is given by

$$V_i^S(0) = \frac{1}{1+\delta}[-q_i + W_i^S(p_i)] + \frac{\delta\alpha_{-i}}{1+\delta}[-\tilde{q}_{-i} + W_i^S(\tilde{p}_{-i})] + \frac{\delta(1-\alpha_{-i})}{1+\delta}W_i^S(0). \tag{4.7}$$

4.1 Terms of Trade

Now, we study the terms of trade of the descentralized markets. The terms of trade are determined in a bilateral meet between buyers and sellers. The buyer selects an offer (q, p) or $(\tilde{q}, \tilde{p})$ that maximizes their surplus while ensuring they meet the seller's participation constraint. As assumed before, buyers spend all their foreign money in the foreign DM. Therefore, the terms of trade of a buyer i, who carries m_{-i} units of foreing money and meet with a seller in the DM_{-i} are defined by the following lemma:

Lemma 3. *Given an encounter between a buyer* i *and a seller* $-i$*, the terms of trade are defined as:* $\tilde{q}_i = \mu\phi_{-i}m_{-i}$ *and* $\tilde{p}_i = m_{-i}$

Proof. The proof is omitted as it is considered trivial. □

Now, let's proceed with the study of the terms of trade in the local DM. The problem of the buyer i, who carries m_i and meet with a seller in the DM_i is given by

$$\max_{q_i, p_i} u(q_i) + u(\tilde{q}_i) + W_i^b(m_i - p_i) - W_i^b(m_i)$$
$$\text{s.t.} \quad - q_i/\mu + W_i^s(p) \geq W_i^s(0),$$
$$p \leq m_i.$$

Considering again the linearity of W_i^s and W_i^b, we have

$$\begin{aligned} \max_{q_i,p_i} u(q_i) + u(\tilde{q}_i(m_{-i})) - \phi_i p_i \\ \text{s.t. } q_i/\mu = \phi_i p_i, \\ p_i \leq m_i. \end{aligned} \tag{4.8}$$

The solution to this problem is described in the following lemma.

Lemma 4. *In a economy with multiple currencies, consider the problem of the buyer i, who enter the second sub-period with m_i units of money. We have the following results:*

$$q_i(m_i) = min\{\mu\phi_i m_i, q_i^*\} \textit{ and } p_i(m_i) = min\{m_i, m_i^*\}.$$

Where, $q^* = \{q : u'(q) = 1/\mu\}$ *and* $m_i^* = q^*/\mu\phi_i$.

Proof. The proof is omitted as it is considered trivial. □

4.2 Optimal Behavior

First, consider a T-type buyer i who meets a dealer in the FOREX market. This agent wants to chose a portfolio $\bar{\mathbf{m}}_\mathbf{i} = [\bar{m}_i^i, \bar{m}_{-i}^i]$ to optimize his value function in the DM restrict to a currency restriction given by $\bar{m}_i^i + \kappa(\iota_{\{i=1\}}\epsilon + \iota_{\{i=2\}}(1/\epsilon))\bar{m}_{-i}^i = m_i$, this restriction arises because the amount of money m_i that the buyer i enter in the FOREX market is equal than the quantity $\bar{m}_i^i$ of local currency plus the quantity $\bar{m}_{-i}^i$ of foreign currency that he left with. However, $\bar{m}_{-i}^i$ is traded at a mark-up κ and the value of foreign currency needs to be converted into value of domestic currency, through the exchange rate ϵ.

Thus, the problem of a buyer i who enter the FOREX market with m_i and meets a dealer is given by

$$\begin{aligned} \max_{\bar{m}_i^i,\bar{m}_{-i}^i} V_i^T(\bar{m}_i^i, \bar{m}_{-i}^i) \\ \text{s.t. } \bar{m}_i^i + \kappa(\iota_{\{i=1\}}\epsilon + \iota_{\{i=2\}}(1/\epsilon))\bar{m}_{-i}^i \leq m_i \end{aligned}$$

The solution of this problem is described in the following lemma

Lemma 5. *Consider the problem of buyer i in the FOREX market. We have the following results:*

$$\bar{m}_i^i = \begin{cases} m_i - \kappa(\iota_{\{i=1\}}(\epsilon) + \iota_{\{i=2\}}(1/\epsilon))\bar{m}_{-i}^{i\ *}, & \textit{if } m_i^* + \tilde{m}_i^* \leq m_i \\ \frac{m_i}{2}, & \textit{if } m_i < m_i^* + \tilde{m}_i^* \end{cases}, \tag{4.9}$$

$$\bar{m}_{-i}^i = \begin{cases} \bar{m}_{-i}^{i\ *}, & \textit{if } m_i^* + \tilde{m}_i^* \leq m_i \\ \frac{m_i}{2\kappa(\iota_{\{i=1\}}\epsilon+\iota_{\{i=2\}}(1/\epsilon))}, & \textit{if } m_i < m_i^* + \tilde{m}_i^* \end{cases}, \tag{4.10}$$

Proof. Since the agent consume all $\bar{m}^i_{-i}$ in the DM_{-i}, and if he have enough m_i to buy the first best in both DMs, he will acquire in the FOREX only $\bar{m}^i_{-i}{}^*$, substituting that in the currency restriction we found $\bar{m}^i_i$.

Furthermore, note that, the currency coinstraint must hold with equality. If the agent can not buy the first best in both DMs and given the terms of trade, from the first-order condition we have:

$$\kappa u'(q) = u'(\tilde{q})$$

and since we have logarithmic utility, we have: $\bar{m}^i_{-i} = \frac{m_i}{2\kappa(\iota_{\{i=1\}}\epsilon + \iota_{\{i=2\}}(1/\epsilon))}$, substituting that in the currency restriction we found $\bar{m}^i_i = m_i/2$. □

Note that, based on the value functions of the dealers and Lemma 5, the optimal strategy for a dealer is to spend all their money in the CM and not carry any money to the next period, as this agent has the opportunity to trade money with other dealers in the Walrasian interdealer market. Thus, suppose that this agent matches with a buyer i, they acquire the m_i and trade it for m_{-i} in the interdealer market with another dealer who matches with a buyer $-i$ and then provide it to the buyer with whom it matched.

Now, we analyzes the objective function of a $buyer_i$ in the CM_i. Substituting (4.5) and (4.6) into (4.4). Then plug the rising expression into (4.3) we have:

$$\delta[\alpha_i(u(q) + u(\tilde{q}) + W^b_i(m_i - p - \kappa(\iota_{\{i=1\}}\epsilon + \iota_{\{i=2\}}(1/\epsilon))\bar{m}^i_{-i})) + (1-\alpha_i)(u(q) + W^b_i(m_i - p))] + (1-\delta)[u(q) + W^b_i(m_i - p)]$$

then advance it by one period and plug it in (4.1). Thus, we have the buyer i's objective function:

$$\begin{aligned} Obj^i = -\phi_i m'_i + \beta\phi'_i m'_i + \beta\delta\alpha_i u(\mu\phi'_i p'_i) + \beta\delta\alpha_i u(\mu\phi'_{-i}\bar{m}^i_{-i}{}') \\ -\beta\delta\alpha_i\phi'_i p - \beta\delta\alpha_i\phi'_i\kappa(\iota_{\{i=1\}}\epsilon + \iota_{\{i=2\}}(1/\epsilon)\bar{m}^i_{-i}{}' \\ -\beta\delta\alpha_i u(\bar{q}) + \beta\delta\alpha_i\phi'_i\bar{p} + \beta u(\bar{q}) - \beta\phi'_i\bar{p}, \end{aligned} \tag{4.11}$$

where, $\bar{q}$ and $\bar{p}$ are the terms of trade between a buyer i, that did not visited the DM_{-i}, and a seller i.

Consider again, three sub-cases: I: $m^*_i + \tilde{m}^*_i \leq m_i$; II: $\tilde{m}^*_i \leq m_i \leq m^*_i + \tilde{m}^*_i$; and III: $m_i \leq m^*_i$. That is, in the first sub-case the buyer i have sufficiently m_i to buys the first best in both DMs, in the second sub-case the agent have enough m_i to acquire the first best just in one DM, and in the third sub-case the agent does not have sufficiently m_i to buy the first best in any DM. Assuming an interior solution, the following lemma expresses the first-order conditions for this problem.

Lemma 6. *Define $Obj^i_s(m'_i)$ the buyer i's objective function when this agent holds m_i, and s = {1, 2, 3} are the three sub-cases of money holdings. Where s = 1 is the case when $m^*_i + \tilde{m}^*_i \leq m_i$, s = 2 happens when $\tilde{m}^*_i \leq m_i \leq m^*_i + \tilde{m}^*_i$ and s = 3 when $m_i \leq m^*_i$. Then we have:*

$$\frac{\partial Obj_1^i(m_i')}{\partial m_i'} = 0 = \phi_i - \beta\phi_i'$$

$$\frac{\partial Obj_2^i(m_i')}{\partial m_i'} = 0 = -\phi_i + \beta\phi_i' + \beta\delta\alpha_i\mu\phi_i' u'(\mu\phi_i'\bar{m}_i^{i\prime})\frac{\partial \bar{m}_i^{i\prime}}{\partial m_i'} + \beta\delta\alpha_i\mu\phi_{-i}' u'(\mu\phi_{-i}'\bar{m}_{-i}^{i}{}')\frac{\partial \bar{m}_{-i}^{i}{}'}{\partial m_i'}$$

$$\text{-}\ \beta\delta\alpha_i\phi_i'\frac{\partial \bar{m}_i^{i\prime}}{\partial m_i'} - \kappa(\iota_{\{i=1\}}\epsilon + \iota_{\{i=2\}}(1/\epsilon))\beta\delta\alpha_i\phi_i'\frac{\partial \bar{m}_{-i}^{i}{}'}{\partial m_i'}$$

$$\frac{\partial Obj_3^i(m_i')}{\partial m_i'} = 0 = -\phi_i + \beta\phi_i' + \beta\delta\alpha_i\mu\phi_i' u'(\mu\phi_i'\bar{m}_i^{i\prime})\frac{\partial \bar{m}_i^{i\prime}}{\partial m_i'} + \beta\delta\alpha_i\mu\phi_{-i}' u'(\mu\phi_{-i}'\bar{m}_{-i}^{i}{}')\frac{\partial \bar{m}_{-i}^{i}{}'}{\partial m_i'}$$

$$\text{-}\ \beta\delta\alpha_i\phi_i'\frac{\partial \bar{m}_i^{i\prime}}{\partial m_i'} - \kappa(\iota_{\{i=1\}}\epsilon + \iota_{\{i=2\}}(1/\epsilon)\beta\delta\alpha_i\phi_i'\frac{\partial \bar{m}_{-i}^{i}{}'}{\partial m_i'} + \beta(1-\delta\alpha_i)\phi_i\{\mu u'(\mu\phi_i m_i') - 1)\}$$

Proof. Replacing the terms of trade found previously, and obtaning the derivative with respect to m_i' yields the desired result. □

4.3 Steady-State Equilibrium with Multiple Currencies

Once again, we are focusing in the sub-case 2. Thus, from Lemmas 5 and 6, we have:

$$u'(\phi_i'\bar{m}_i^{i\prime}) = \frac{\phi_i/\phi_i' + \beta\delta\alpha_i - \beta}{\mu\beta\delta\alpha_i} \tag{4.12}$$

and,

$$u'(\phi_{-i}'\bar{m}_{-i}^{i}{}') = \frac{\kappa(\phi_i/\phi_i' + \beta\delta\alpha_i - \beta)}{\mu\beta\delta\alpha_i} \tag{4.13}$$

A steady-state monetary equilibrium is a sequence of $\phi_i/\phi_i' = (1+\tau_i)$ that solves the difference equation (4.12), where τ_i is a time invariant monetary authority policy of country i. In the steady-state monetary equilibrium we have $q_{i,t} = q_{i,t+1} = q_i^{mc}$, and $\tilde{q}_{i,t} = \tilde{q}_{i,t+1} = \tilde{q}_i^{mc}$.

Proposition 2. *Exists a unique steady-state monetary equilibrium in the sub-case 2 of money holdings, when* $(1+\tau_i) > \beta$*.*

Proof. Note that, from 4.12 and 4.13, and since we are analyzing sub-case 2, we must have:

$$u'(q_i^{mc}) = \frac{\phi_i/\phi_i' + \beta\delta\alpha_i - \beta}{\mu\beta\delta\alpha_i} > 1/\mu,$$

$$u'(\tilde{q}_i^{mc}) = \frac{\kappa(\phi_i/\phi_i' + \beta\delta\alpha_i - \beta)}{\mu\beta\delta\alpha_i} > 1/\mu,$$

and this case arises only when $(1+\tau_i) > \beta$.

From our logarithmic utility function, we have $u'(q_i^{mc}) = \frac{1}{q_i^{mc}}$ and $u'(\tilde{q}_i^{mc}) = \frac{1}{\tilde{q}_i^{mc}}$, and since 4.12 and 4.13 depends only on invariant parameters, there exists a unique q_i^{mc} and a unique $\tilde{q}_i^{mc}$ that satisfies 4.12 and 4.13. □

Thus, from (4.12) and (4.13) and given the logarithmic utility, we have:

$$q^{mc} = \frac{\mu\beta\delta\alpha_i}{(1+\tau_i) + \beta\delta\alpha_i - \beta}, \tag{4.14}$$

and

$$\tilde{q}^{mc} = \frac{\mu\beta\delta\alpha_i}{\kappa[(1+\tau_i) + \beta\delta\alpha_i - \beta]}. \tag{4.15}$$

5 COMPARISON BETWEEN SINGLE CURRENCY AND MULTIPLE CURRENCIES

This section is intended to compare some aspects between the economy with single currency and the economy with multiple currencies. We want to analyzes which economy generates a more efficient allocation. Thus, we shall analyzes the volume of trade goods exchange within each economy.

The following propositions presents a criterion for comparing the level of trade in the economies studied before.

Proposition 3. *The volume of goods traded in country i in the national DM in the economy with a single currency is greater than or equal to the volume of goods traded in country i in the economy with multiple currencies if and only if*

$$\frac{[1+\tau_i]-\beta}{[1+\tau]-\beta} \geq \alpha_i.$$

Proof. From the equation (3.14) and (6.3) we derive the preceding inequality. □

Proposition 4. *The volume of goods traded in country i in the foreign DM in the economy with a single currency is greater than or equal to the volume of goods traded in country i in the foreign DM in the economy with multiple currencies if and only if*

$$\kappa \geq \frac{\alpha_i[(1+\tau)-\beta+\beta\delta]}{(1+\tau_i)-\beta+\beta\delta\alpha_i},$$

Proof. From the equation (3.14) and (6.4) we derive the preceding inequality. □

Note that, in the economy with single currency a T-type buyer consumes $2q^{sc}$ and in the economy with multiple currencies a typical T-type buyer consumes $q^{mc} + \tilde{q}^{mc}$.Therefore,

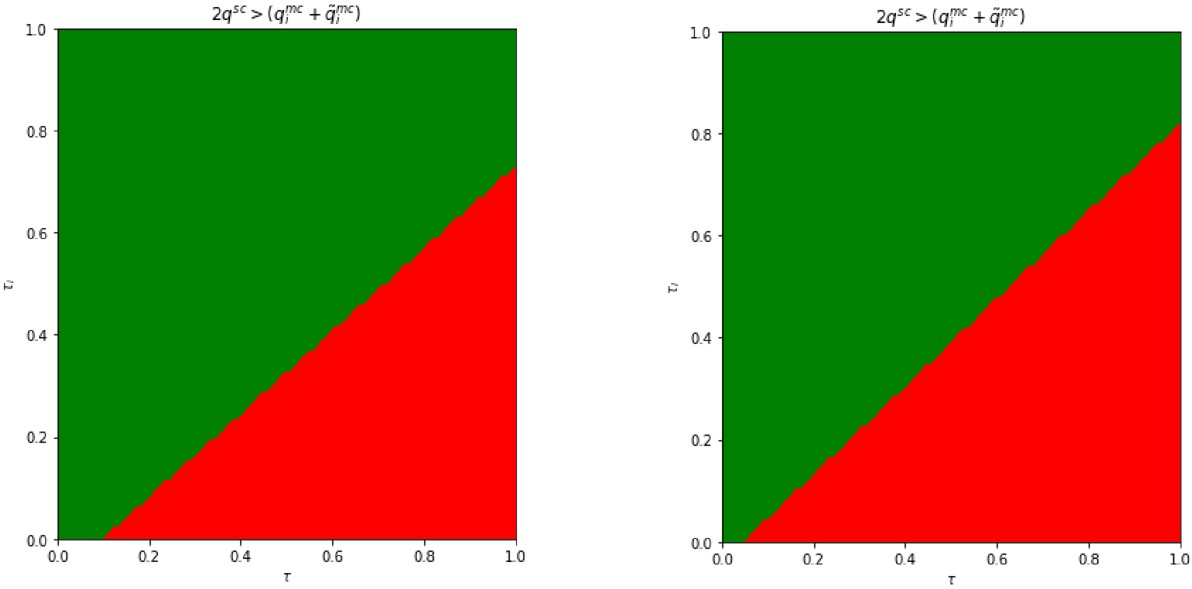
Proposition 5. *The volume of goods traded in the country i in the economy with single currency is greater than or equal to the volume traded in the economy with multiple currencies if and only if*

$$\kappa \geq \frac{\alpha_i[(1+\tau)-\beta+\beta\delta]}{2[(1+\tau_i)-\beta+\beta\delta\alpha_i]-\alpha_i[(1+\tau)-\beta+\beta\delta]}.$$

Proof. From the equation (3.14), (6.3) and (6.4) we derive the preceding inequality. □

Therefore, note that the difference between the volume of trade in both economies depend on the monetary policy, especifically how distant is the inflation $(1 + \tau)$ from the Friedman's rule. When we have a large inflation in the economy with single currency it may be desirable to countries to adopt his own currency. Besides, the probability of a match between a buyer i and a dealer is also crucial in determining the volume of trades. This probability can be interpreted as the ease of finding a FOREX dealer, which has been facilitated with advancements in technology.

The green region of figure 1 presents the combinations of τ and τ_i where the volume of trades of country i are bigger in the economy with single currency than the economy with multiple currencies. Two scenarios are considered with the mark-up κ assuming values of 1.25 and 1.1, respectively. In the simulations, we set $(\beta, \alpha_i, \delta) = (0.96, 0.9, 0.8)$.



(a) $\kappa = 1.25$ (b) $\kappa = 1.1$

Figure 1 – Comparison of volume of trades

We can observe that the economy with multiple currencies has a larger volume of trade than the economy with a single currency only when the economy with a single currency experiences high inflation. Additionally, note that when the mark-up is higher, the region where it is preferable for a country to have its own currency, rather than participate in a monetary union, decreases. This is because agents need to exchange more money to consume in the foreign market. However, as mentioned before, advancements in technology may increase the number of FOREX dealers, leading to a more competitive market and reducing the mark-up.

Now, we examine the welfare properties of two economies, utilizing average utility as the welfare criterion. Firstly, it's important to note that due to the Take-it-or-leave-it between buyers and sellers in the DM, sellers have no surplus in the trade. Therefore, to study welfare, we only need to consider the value function of the buyer in the DM In the steady-state equilibrium of the economy with single currency, the value function of the buyer is as follows:

$$\mathbb{E}_c\{V_i^c(m_i)\} = \delta V_i^T(m_i) + (1-\delta)V_i^N(m_i).$$

Lemma 7. *The welfare, utilizing average utility as the welfare criterion, in the steady-state equilibrium of the economy with single currency is given by:*

$$\mathcal{W}^{sc} = 2\left(\frac{2\delta(u(q^{sc}) - q^{sc}/\mu) + (1-\delta)(u(q^*) - 1)}{1-\beta}\right).$$

Proof. See Appendix. □

Now, for the economy with multiple currencies, consider $\alpha_i = \alpha_{-i} = \alpha$, and employing the same criteria, but now we have the expected value function of the buyer i as follows:

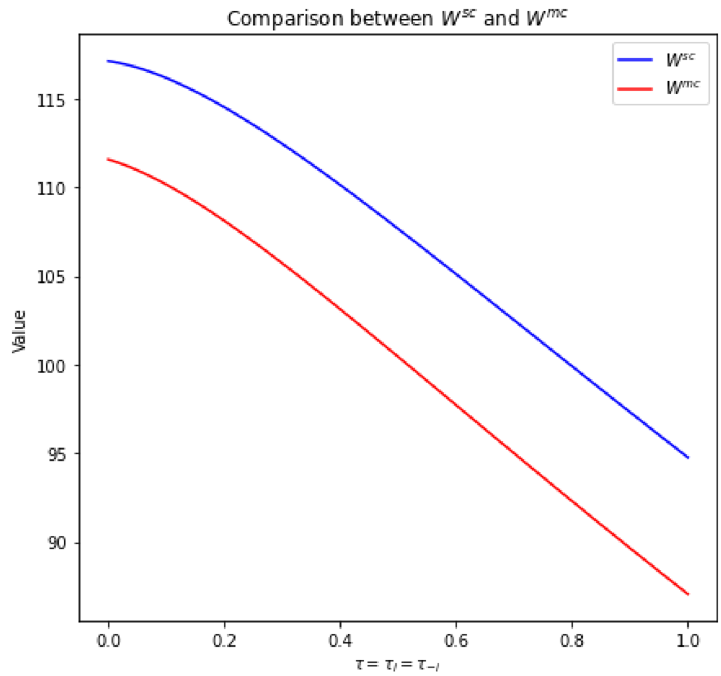


Figure 2 – Comparison between Welfares

$$\delta\alpha V_i^T + (1-\alpha)\delta V_i^N + (1-\delta)V_i^N.$$

Thus, we have the following lemma:

Lemma 8. *The welfare, utilizing average utility as the welfare criterion, in the steady-state equilibrium of the economy with multiple currencies is given by:*

$$\mathcal{W}^{mc} = \mathcal{W}_1 + \mathcal{W}_2,$$

where $\mathcal{W}_i$*, for* $i = 1, 2$ *is:*

$$\mathcal{W}_i = \frac{\delta\alpha(u(q_i^{mc}) - q_i^{mc}/\mu) + \delta\alpha(u(\tilde{q}_i^{mc}) - \tilde{q}_i^{mc}/\mu) + (1-\delta\alpha)(u(q^*) - 1)}{1-\beta}.$$

Proof. See Appendix. □

Note that, from Lemmas 7 and 8 we can deduce that welfare represents the expected gains of trade for each economy. Furthermore, it is worth noting that since the gains of trade increase with q until q^*, the welfare, as previously discussed, depends on the values of τ, τ_1, τ_2, and the probability of a match between a buyer and a dealer.

However, Figure 2 demonstrates that, for a sufficiently high μ and for the same monetary policy in both the economy with multiple currencies and the economy with a single currency, the welfare is always higher in the economy with a single currency. In the simulations, we set $(\beta, \alpha, \delta, \kappa, \mu) = (0.96, 0.9, 0.8, 1.1, 10)$. Indeed, this observation is quite intuitive, as for the same level of inflation, the economy with a single currency will typically have a higher output than the economy with multiple currencies. This is because a single currency may eliminates some frictions in the FOREX market, which can positively impact economic activity and output levels.

6 FLUCTUATING MONETARY POLICY

Monetary policy, in a monetary union, is not customized for any of the member countries. That is one of the most important arguments against joining a monetary union. This argument can be interpreted as meaning that, countries within the union may find that their monetary policy options become more limited compared to those with independent currencies.

Furthermore, numerous studies have shown that inflation in monetary unions becomes less volatile. For instance, Meller and Nautz (2012) and Tillmann (2012) analyzed a reduction in inflation persistence among countries in the EMU following the establishment of the monetary union, a conclusion supported by Adelakun (2020), who examined a similar reduction in the West Africa Monetary Zone. Additionally, Holtemöller (2007) utilize the McCallum and Nelson (2000) model to investigate the effects of joining a monetary union, and they concluded that joining a monetary union decreases the variability of the inflation rate.

Therefore, this section aims to conduct another exercise where the economy with a single currency maintains a constant monetary policy as before, while the economy with multiple currencies adopts a fluctuating monetary policy.

Consider again the economy with multiple currencies. Country 2 remains unchanged, while now the monetary authority of country 1 can adjust the stock of money M_1 with a net growth rate of τ_1^H with probability λ, or τ_1^L with probability $(1-\lambda)$. Agents are only informed about the realized value of τ_1 at the beginning of the period.

Note that, the value functions of the seller do not change. In that way, the value function of a buyer i who carry m_i units of money in the CM_i can be expressed as:

$$W_i^b(m_i) = \phi_i m_i + T_i + \max_{m_i'}\{-\phi_i m_i' + \beta \mathbb{E}_{\phi_1'^p}\{\mathbb{E}_c\{F_i^c(m_i')\}\}\}, \tag{6.1}$$

note that now we have the expectation around the $\phi_1'^p$, where $p = H, L$. We have:

$$\mathbb{E}_{\phi_1'^p}\{\mathbb{E}_c\{F_i^c(m_i')\}\} = \lambda \mathbb{E}_c\{F_i^c(m_i')\} + (1-\lambda)\mathbb{E}_c\{\bar{F}_i^c(m_i')\}, \tag{6.2}$$

where, $F_i^c(m_i')$ is the expected value function for the buyer i who begins the FOREX market with m_i units of currency when $\phi_i^p = \phi_i^H$ and $\bar{F}_i^c(m_i')$ is the same as before but when $\phi_i^p = \phi_i^L$. The other buyers' value functions remain the same.

In the same way, the dealer value function in the CM of a dealer who carry **m** is given by:

$$W^D(\boldsymbol{m}) = \boldsymbol{\phi m} + \max_{\boldsymbol{m'}}\{-\boldsymbol{\phi m'} + \beta \mathbb{E}_{\phi_1'^p}\{F_d(\boldsymbol{m'})\}\},$$

where,

$$\mathbb{E}_{\phi_1'^p}\{F_d(\boldsymbol{m'})\}\} = \lambda F_d(\boldsymbol{m'}) + (1-\lambda)\bar{F}_d(\boldsymbol{m'}).$$

6.1 Terms of Trade

Note that agents are aware of the realization of ϕ_i^p at the beginning of the period. Therefore, when agents enter the DM, they already know this information. As a result, the terms of trade remain the same as in section 4.

6.2 Optimal Behavior

As before, when agents enter the FOREX market, they are already aware of the realization of ϕ_i^p. Consequently, the portfolio of a T-type buyer i remains the same as in section **??**.

Now, we analyzes the objective function of a $buyer_i$ in the CM_i. Substituting (4.5) and (4.6) into (4.4). Then plug the rising expression into (4.3) and then into (6.2) we have:

$$\lambda(\delta[\alpha_i(u(q)+u(\tilde{q})+W_i^b(m_i-p-\kappa(\iota_{\{i=1\}}\epsilon^H+\iota_{\{i=2\}}(1/\epsilon^H))\bar{m}_{-i}^i))+(1-\alpha_i)(u(q)+W_i^b(m_i-p))]+(1-\delta)[u(q)+W_i^b(m_i-p)])+(1-\lambda)(\delta[\alpha_i(u(q)+u(\tilde{q})+W_i^b(m_i-p-\kappa(\iota_{\{i=1\}}\epsilon^L+\iota_{\{i=2\}}(1/\epsilon^L))\bar{m}_{-i}^i))+(1-\alpha_i)(u(q)+W_i^b(m_i-p))]+(1-\delta)[u(q)+W_i^b(m_i-p)]),$$

then advance it by one period and plug it in (6.1) in that way we have the buyer i's objective function. Assuming again an interior solution and focusing the second sub-case we have the first-order condition to the problem of buyer 1:

$$\frac{\partial Obj_2^1(m_1')}{\partial m_1'}=0=-\phi_1+\lambda(\beta\phi_1^{H\prime}+\beta\delta\alpha_1\mu\phi_1^{H\prime}u'(\mu\phi_1^{H\prime}\bar{m}_1^{1\prime})\frac{\partial\bar{m}_1^{1\prime}}{\partial m_1'}+\beta\delta\alpha_1\mu\phi_2'u'(\mu\phi_2'\bar{m}_2^{1\prime})\frac{\partial\bar{m}_2^{1\prime}}{\partial m_1'}-\beta\delta\alpha_1\phi_1^{H\prime}\frac{\partial\bar{m}_1^{1\prime}}{\partial m_1'}-\kappa\epsilon^H\beta\delta\alpha_1\phi_1^{H\prime}\frac{\partial\bar{m}_2^{1\prime}}{\partial m_1'})+(1-\lambda)(\beta\phi_1^{L\prime}+\beta\delta\alpha_1\mu\phi_1^{L\prime}u'(\mu\phi_1^{L\prime}\bar{m}_1^{1\prime})\frac{\partial\bar{m}_1^{1\prime}}{\partial m_1'}+\beta\delta\alpha_1\mu\phi_2'u'(\mu\phi_2'\bar{m}_2^{1\prime})\frac{\partial\bar{m}_2^{1\prime}}{\partial m_1'}-\beta\delta\alpha_1\phi_1^{L\prime}\frac{\partial\bar{m}_1^{1\prime}}{\partial m_1'}-\kappa\epsilon^L\beta\delta\alpha_1\phi_1^{L\prime}\frac{\partial\bar{m}_2^{1\prime}}{\partial m_1'}),$$

and then for buyer 2:

$$\frac{\partial Obj_2^2(m_2')}{\partial m_2'}=0=-\phi_2+\lambda(\beta\phi_2'+\beta\delta\alpha_2\mu\phi_2'u'(\mu\phi_2'\bar{m}_2^{2\prime})\frac{\partial\bar{m}_2^{2\prime}}{\partial m_2'}+\beta\delta\alpha_2\mu\phi_1^{H\prime}u'(\mu\phi_1^{H\prime}\bar{m}_1^{2\prime})\frac{\partial\bar{m}_1^{2\prime}}{\partial m_2'}-\beta\delta\alpha_2\phi_2'\frac{\partial\bar{m}_2^{2\prime}}{\partial m_2'}-\kappa(1/\epsilon^H)\beta\delta\alpha_2\phi_2'\frac{\partial\bar{m}_1^{2\prime}}{\partial m_2'})+(1-\lambda)(\beta\phi_2'+\beta\delta\alpha_2\mu\phi_2'u'(\mu\phi_2'\bar{m}_2^{2\prime})\frac{\partial\bar{m}_2^{2\prime}}{\partial m_2'}+\beta\delta\alpha_2\mu\phi_1^{L\prime}u'(\mu\phi_1^{L\prime}\bar{m}_1^{2\prime})\frac{\partial\bar{m}_1^{2\prime}}{\partial m_2'}-\beta\delta\alpha_2\phi_2'\frac{\partial\bar{m}_2^{2\prime}}{\partial m_2'}-\kappa\epsilon^L\beta\delta\alpha_1\phi_2'\frac{\partial\bar{m}_1^{2\prime}}{\partial m_2'}).$$

6.3 Comparison with single currency

From the FOC for country 2, from lemma 5 and because we have logarithmic utility, we have:

$$q_2^{mc}=\frac{\mu\beta\delta\alpha_2}{(1+\tau_2)+\beta\delta\alpha_2-\beta}, \tag{6.3}$$

and

$$\tilde{q}_2^{mc}=\frac{\mu\beta\delta\alpha_2}{\kappa[(1+\tau_2)+\beta\delta\alpha_2-\beta]}. \tag{6.4}$$

Note that, the realization of the ϕ_1^p does not impact the output of country 2.

From the FOC for country 1, from lemma 5, and since we have logarithmic utility, we have:

$$p_1 = \frac{\beta\delta\alpha_1\mu}{\phi_1+\lambda(\beta\delta\alpha_1\phi_1^H-\beta\phi_1^H)+(1-\lambda)(\beta\delta\alpha_1\phi_1^L-\beta\phi_1^L)}.$$

In that way, we can have two possibles outcomes:

$$q_1^{mc,H} = \frac{\beta\delta\alpha_1\mu}{(1+\tau^H)+\lambda(\beta\delta\alpha_1-\beta)+(1-\lambda)(\beta\delta\alpha_1\phi_1^L/\phi_1^H-\beta\phi_1^L/\phi_1^H)}, \tag{6.5}$$

$$\tilde{q}_1^{mc,H} = \frac{\beta\delta\alpha_1\mu}{\kappa[(1+\tau^H)+\lambda(\beta\delta\alpha_1-\beta)+(1-\lambda)(\beta\delta\alpha_1\phi_1^L/\phi_1^H-\beta\phi_1^L/\phi_1^H)]}, \tag{6.6}$$

and

$$q_1^{mc,L} = \frac{\beta\delta\alpha_1\mu}{(1+\tau^L)+\lambda(\beta\delta\alpha_1\phi_1^H/\phi_1^L-\beta\phi_1^H/\phi_1^L)+(1-\lambda)(\beta\delta\alpha_1-\beta)}, \tag{6.7}$$

$$\tilde{q}_1^{mc,L} = \frac{\beta\delta\alpha_1\mu}{\kappa[(1+\tau^L)+\lambda(\beta\delta\alpha_1\phi_1^H/\phi_1^L-\beta\phi_1^H/\phi_1^L)+(1-\lambda)(\beta\delta\alpha_1-\beta)]}. \tag{6.8}$$

Now, suppose that $\phi_1^H > \phi_1^L$, and $\lambda = 0.5$, meaning that ϕ_1^H has the same probability of occurring as ϕ_1^L. Also consider that $\delta = 1$, meaning that all buyers have the possibility of visiting the foreign DM.

Let's compare the two cases with the single-currency economy:

1. The volume of trade in country 1 with cyclical monetary policy when $(1+\tau_1^L)$ occurs, assuming that the monetary policy of the single-currency economy is the same as in that economy, i.e., $(1+\tau_1^L) = (1+\tau)$.

2. The volume of trade in country 1 with cyclical monetary policy when $(1+\tau_1^H)$ occurs, assuming that $(1+\tau_1^H) = (1+\tau)$.

The following propositions express these comparisons:

Proposition 6. *Given* $\tau_1^H > \tau_1^L$, $\delta = 1$, $\tau_1^L = \tau$, $E^H \equiv \frac{(1+\tau_1^H)}{(1+\tau_1^L)}$ *and considering* $\frac{2(1+\tau)}{\beta} > (1-\alpha)(1+E^H)$. *The volume of goods traded in country 1 in the national DM in the economy with multiple currencies and cyclical monetary policy when* τ_1^L *occurs is greater than or equal to the volume of goods traded in country i in the economy with a single currency if and only if*

$$\frac{2(1+\tau)}{\beta} \leq 1 + E^H.$$

Proof. From the equations 6.7 and equation 3.14 we found the desired result. □

Proposition 7. *Given* $\tau_1^H > \tau_1^L$, $\delta = 1$, $\tau_1^L = \tau$, $E^H \equiv \frac{(1+\tau_1^H)}{(1+\tau_1^L)}$ *and considering* $\frac{2(1+\tau)}{\beta} > (1-\alpha)(1+E^H)$. *The volume of goods traded in country 1 in the foreign DM in the economy with multiple currencies and fluctuating monetary policy when* τ_1^L *occurs is greater than or equal to the volume of goods traded in country i in the economy with a single currency if and only if*

$$\frac{2(\alpha_1-\kappa)(1+\tau)}{(\alpha_1-1)\beta} \leq 1 + E^H.$$

Proof. From the equations 6.8 and equation 3.14 we found the desired result. □

Proposition 8. *Given $\tau_1^H > \tau_1^L$, $\delta = 1$, $\tau_1^H = \tau$, $E^L \equiv \frac{(1+\tau_1^L)}{(1+\tau_1^H)}$ and considering $\frac{2(1+\tau)}{\beta} > (1-\alpha)(1+E^H)$. The volume of goods traded in country 1 in the national DM in the economy with multiple currencies and cyclical monetary policy when τ_1^L occurs will never be greater than to the volume of goods traded in country i in the economy with a single currency*

Proof. Suppose by contradiction that $q_1^{mc,H} \geq q^{sc}$. From the equations 6.5 and equation 3.14 we have:

$$\frac{\beta\alpha_1\mu}{(1+\tau)+0.5(\beta\alpha_1)-0.5\beta+0.5\beta\alpha_1 E^L-0.5\beta E^L} \geq \frac{\beta\mu}{(1+\tau)}$$

From that we have:

$$\frac{2(1+\tau)}{\beta} \leq E^L + 1$$

Not that we are analyzing sub-case 2 of money holding, because of that $(1+\tau) > \beta$, thus the left side of the inequality is bigger than 2. However, since $E^L < 1$, the right side of the inequality is less than 2. □

Note that, buyers adjust their money holdings based on their expectations regarding monetary policy. They need to be prepared for both scenarios, whether high or low inflation occurs. When low inflation occurs, buyers tend to consume more because they hold more money in anticipation of high inflation. However, since inflation turns out to be low, their money becomes more valuable, allowing them to purchase a greater quantity of goods, which can be expressed in propositions 6 and 7. However, since the scenario where high inflation occur buyers have less money because they made their decision expecting that the low scenario could occur, which corresponds to the case in Proposition 8 where their consumption is reduced.

In Figure 3, we observe the quantities q^{sc}, $q_1^{mc,L}$, and $q_1^{mc,H}$, along with the expected consumption from Country 1 with fluctuating monetary policy, represented by $0.5q_1^{mc,L} + 0.5q_1^{mc,H}$ for $\tau^L = \tau = 0.1$ and varying values of τ_H. It is noteworthy that $q_1^{mc,L}$ increases as E^L increases, and we can observe when it surpasses the consumption in the single currency. However, $q_1^{mc,H}$ decreases as E^L increases, and since in the Figure 3 τ^H is growing, the resulting is a reduction in the expected consumption. However, in Figure 4, when $\tau^H = \tau^L$, it is observed that, as discussed earlier, if the monetary union has a high inflation the volume of trade in an economy with multiple currencies may be higher, even the expected consumption in the case of fluctuating monetary policy. For the simulation, we set $(\beta, \alpha, \delta, \kappa, \mu, \tau) = (0.96, 0.9, 1, 1.1, 10, 0.1)$.

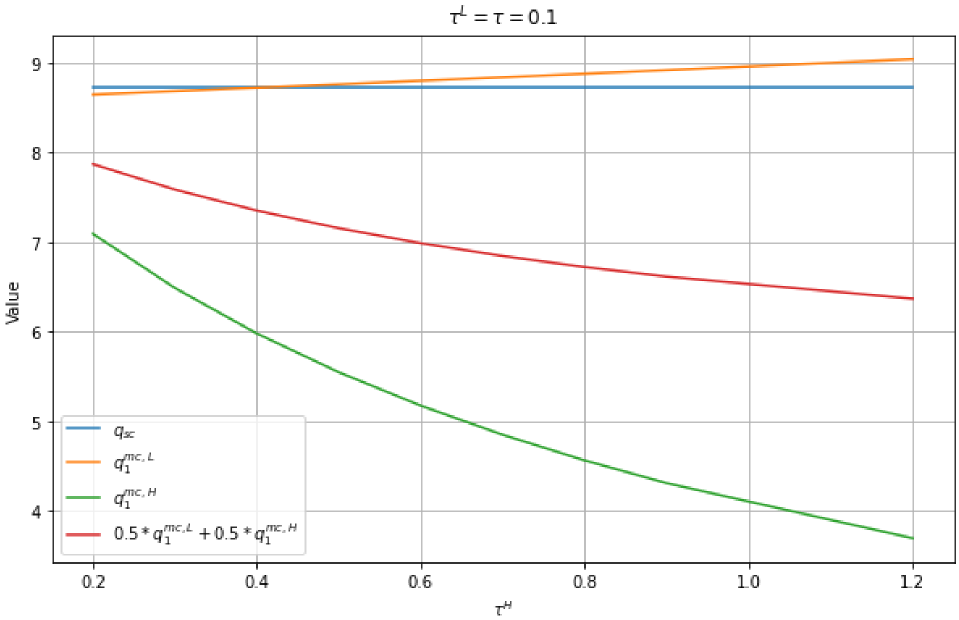


Figure 3 – Volume of Trade $\tau = \tau^L$

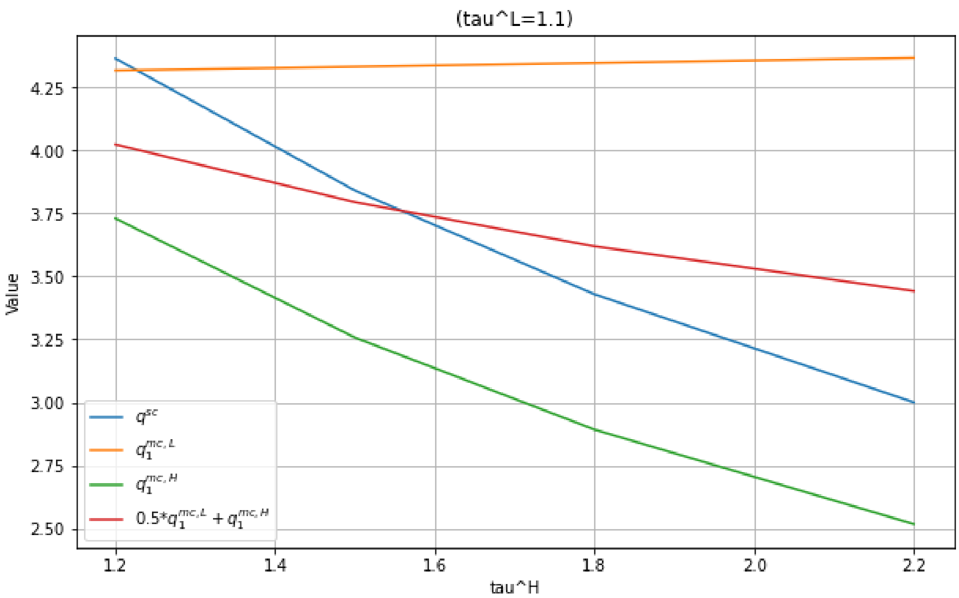


Figure 4 – Volume of Trade $\tau = \tau^H$

7 DISCUSSION AND CONCLUSIONS

This study investigates the implications of joining a monetary union. Using a model based on the monetary search framework of Lagos and Wright (2005) extended to multiple countries, we analyze two scenarios: one where countries share a single currency and another where they maintain separate currencies. Through the analysis of steady-state equilibria in both economies, we investigate the efficiency and welfare implications of adopting a common currency.

We observed that in cases where the economy with a currency union experiences an unstable monetary policy, characterized by higher inflation rates, maintaining separate currencies may be preferable. Specifically, the discrepancy between the monetary policy and the Friedman's Rule plays a crucial role in the decision between maintaining separate currencies or joining a monetary union. Additionally, factors such as the probability of not finding a FOREX dealer and the associated mark-up costs affect the efficiency of the economy with multiple currencies, leading to a reduction in trade volume. However, advancements in technology may increase the probability of finding a FOREX dealer, thereby reducing mark-up costs.

We also observe that under the same monetary policy, the economy with a single currency generates higher welfare compared to the economy with multiple currencies, primarily due to the frictions present in the latter. Additionally, when considering fluctuations in the monetary policy of the economy with a single currency, we find that in some cases, the economy with multiple currencies can generate a higher volume of trade. This is because agents prepare themselves for the possibility of experiencing high inflation. However, when scenarios of low inflation happens, agents have more money and their money can buy more goods.

Our research has find interesting insights into the nuances of economies adopting a common currency. However, it is important to acknowledge the limitations of our model and identify areas for future research. Firstly, while we have incorporated fluctuating monetary policy into our model, it may not capture all the complexities of monetary policy dynamics. Future studies could explore the inclusion of cyclical monetary policy, incorporating shocks to productivity and different monetary policy responses to these shocks. This would allow for a better understanding of how monetary policy operates within a monetary union, where it is not customized for any individual member country.

Additionally, there is potential for empirical research to further investigate parameters related to FOREX dealers and mark-up costs. By conducting simulations with similar real-world data, we can gain a better understanding of these factors and how they impact currency exchange in practice. This empirical analysis could improve the validity and applicability of our research findings.

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APPENDIX

Proof. of Lemma 1:

From 3.9 substituing the restriction $\bar{m} = m_i - p_i$ in the objective, and since we have $q_i/\mu = \phi p_i$ from the other restriction we have the following FOC:

$$\mu\phi u'(\mu\phi_i) - \mu\phi u'(\mu\phi(m - p_i)) - \phi + \phi = 0,$$

from that we have: $u'(q) = u'(\tilde{q})$ □

Proof. of Lemma 7: We have that from equation 3.3:

$$V_i^T(m_i) = u(q_i) + u(\tilde{q}_i) + W_i^B(m_i - p_i - \tilde{p}_i),$$

and from equation 3.4 we have:

$$V_i^n(m_i) = u(q_i) + W_i^B(m_i - p_i)$$

Substituting W_i^B in both equations, we get:

$$V_i^T = u(q_i) + u(\tilde{q}_i) + \phi m_i - \phi p_i - \phi\tilde{p}_i + T - \phi m_i' + \beta\delta V_i^T + \beta(1-\delta)V_i^N,$$

$$V_i^N = u(q^*) + \phi m_i - \phi p_i^* + T - \phi m_i' + \beta\delta V_i^T + \beta(1-\delta)V_i^N,$$

Applying Cramer's Rule, we get:

$$V_i^T = \frac{(1-\beta+\beta\delta)u(q^{sc})+(1-\beta+\beta\delta)u(\tilde{q}^{sc})-(1-\beta+\beta\delta)\phi m_i+(\beta-\beta\delta)u(q^*)-(\beta-\beta\delta)q^*/\mu}{1-\beta},$$

$$V_i^N = \frac{(1-\beta\delta)u(q^*)-(1-\beta\delta)q^*/\mu+\beta\delta(u(q^{sc})+u(\tilde{q}^{sc}))-\beta\delta\phi m_i}{1-\beta}.$$

Substituting these into our welfare criteria, and since both country have the same output, we obtain the desired result. □

Proof. of Lemma 8: We have that from equation 4.5:

$$V_i^T(\bar{m}_i^i, \bar{m}_{-i}^i) = u(q_i) + u(\tilde{q}_i) + W_i^B(m_i - p_i - \kappa(\iota_{\{i=1\}}\epsilon + \iota_{\{i=2\}}(1/\epsilon))\bar{m}_{-i}^i),$$

and from equation 4.6 we have:

$$V_i^n(m_i) = u(q_i) + W_i^B(m_i - p_i)$$

Substituting W_i^B in both equations, we get:

$$V_i^T = u(q_i) + u(\tilde{q}_i) + \phi_i m_i - \phi_i p_i - \phi_i\kappa(\iota_{\{i=1\}}\epsilon + \iota_{\{i=2\}}(1/\epsilon))\bar{m}_{-i}^i) + T - \phi_i m_i' + \beta\delta\alpha V_i^T + \beta(1-\delta)V_i^N + \beta\delta(1-\alpha)V_i^N,$$

$$V_i^N = u(q^*) + \phi_i m_i - \phi_i p_i^* + T - \phi_i m_i' + \beta\delta\alpha V_i^T + \beta(1-\delta)V_i^N + \beta\delta(1-\alpha)V_i^N,$$

Applying Cramer's Rule, we get:

$$V_i^T = \frac{(1-\beta+\beta\delta\alpha)u(q_i^{mc})+(1-\beta+\beta\delta\alpha)u(\tilde{q}_i^{mc})-(1-\beta+\beta\delta\alpha)\phi_i m_i+(\beta-\beta\delta\alpha)u(q^*)-(\beta-\beta\delta\alpha)q^*/\mu}{1-\beta},$$

$$V_i^N = \frac{(1-\beta\delta\alpha)u(q^*)-(1-\beta\delta\alpha)q^*/\mu+\beta\delta\alpha(u(q^{sc})+u(\tilde{q}^{sc}))-\beta\delta\alpha\phi m_i}{1-\beta}.$$

Substituting these into our welfare criteria, we obtain the desired result. □